Technical Appendix to Accompany

"On the Design of Piece-Rate Contracts"

by

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Let $\overline{U}_2 = \overline{U}_1 \left(\frac{\delta}{\theta}\right)^{\frac{\theta\delta}{\delta-\theta}}$ denote the largest value of $\overline{U}$ for which the principal's expected payoff at the solution to [P-R] is non-negative.[1] Then we have the following Conclusion.

Conclusion. Suppose $\overline{U} \in (\overline{U}_1, \overline{U}_2]$. Then $\beta = \frac{\theta}{\delta}\left[\frac{\overline{U}}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\theta\delta}}$ at the solution to [P].

Proof. The proof of Proposition 1 in the text reveals that constraint (3) binds at the solution to [P] when $\overline{U} > \overline{U}_1$. Equations (2) – (4) in the text reveal that, the principal's problem in this case is to:

$$\underset{\beta,\, \widetilde{a}}{Maximize} \quad \int_0^{\infty} [x - \beta x]\, f\,(x|\widetilde{a})\, dx \;=\; [1-\beta]\,\widetilde{a}\, p\,, \tag{A1}$$

$$\text{where} \quad \widetilde{a} \;=\; \left[\frac{2\,\theta\,(\beta)^{\theta}}{\delta}\left(\frac{\Gamma\,(p+\theta)}{\Gamma\,(p)}\right)\right]^{\frac{1}{\delta-\theta}}, \quad \text{and} \tag{A2}$$

$$2\,(\beta)^{\theta}\left[\widetilde{a}^{\theta}\right]\frac{\Gamma\,(p+\theta)}{\Gamma\,(p)} - \widetilde{a}^{\delta} \;=\; \overline{U}. \tag{A3}$$

Equations (A2) and (A3) imply that at the solution to [P-R]:

$$\widetilde{a}^{\delta}\left[2\,(\beta)^{\theta}\left[\widetilde{a}^{\theta-\delta}\right]\frac{\Gamma\,(p+\theta)}{\Gamma\,(p)} - 1\right] \;=\; \overline{U} \quad\Rightarrow\quad \widetilde{a}^{\delta}\left[\frac{2\,(\beta)^{\theta}\left(\frac{\Gamma(p+\theta)}{\Gamma(p)}\right)}{\frac{2\,\theta(\beta)^{\theta}}{\delta}\left(\frac{\Gamma(p+\theta)}{\Gamma(p)}\right)} - 1\right] \;=\; \overline{U}$$

$$\Rightarrow\quad \widetilde{a}^{\delta}\left[\frac{\delta}{\theta} - 1\right] \;=\; \overline{U} \quad\Rightarrow\quad \widetilde{a} \;=\; \left[\frac{\theta\,\overline{U}}{\delta-\theta}\right]^{\frac{1}{\delta}}. \tag{A4}$$

Equations (A2) and (A4) imply:

[1] The following Conclusion reveals that $\beta = \frac{\theta}{\delta}\left[\frac{\overline{U}}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\theta\delta}}$ when constraint (3) binds at the solution to [P]. It is never profitable for the principal to increase β above 1. Therefore, $\left(\frac{\theta}{\delta}\right)\left[\frac{\overline{U}_2}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\theta\delta}} = 1 \;\Rightarrow\; \overline{U}_2 = \overline{U}_1\left(\frac{\delta}{\theta}\right)^{\frac{\theta\delta}{\delta-\theta}}$.

$$\left[\frac{2\,\theta\,(\beta)^{\theta}}{\delta}\left(\frac{\Gamma\,(p+\theta)}{\Gamma\,(p)}\right)\right]^{\frac{1}{\delta-\theta}} \;=\; \left[\frac{\theta\,\overline{U}}{\delta-\theta}\right]^{\frac{1}{\delta}} \quad\Rightarrow\quad \frac{2\theta\,(\beta)^{\theta}}{\delta}\left(\frac{\Gamma\,(p+\theta)}{\Gamma\,(p)}\right) \;=\; \left[\frac{\theta\,\overline{U}}{\delta-\theta}\right]^{\frac{\delta-\theta}{\delta}}$$

$$\Rightarrow \quad (\beta)^{\theta} \;=\; \frac{\delta\,\Gamma\,(p)}{2\,\theta\,\Gamma\,(p+\theta)}\left[\frac{\theta}{\delta-\theta}\right]^{\frac{\delta-\theta}{\delta}}\left[\,\overline{U}\,\right]^{\frac{\delta-\theta}{\delta}}. \tag{A5}$$

From the definition of $\overline{U}_1$:

$$\left[\,\overline{U}_1\right]^{\frac{\theta-\delta}{\delta}} \;=\; \left[\frac{\delta-\theta}{\theta}\right]^{\frac{\theta-\delta}{\delta}}\left[\frac{\Gamma(p)}{2\,\Gamma(p+\theta)}\left(\frac{\delta}{\theta}\right)\right]\left(\frac{\delta}{\theta}\right)^{\theta}$$

$$\Rightarrow \quad \left[\frac{1}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\delta}} \;=\; \left[\frac{\theta}{\delta-\theta}\right]^{\frac{\delta-\theta}{\delta}}\left[\frac{\delta\,\Gamma(p)}{2\,\theta\,\Gamma(p+\theta)}\right]\left(\frac{\delta}{\theta}\right)^{\theta}$$

$$\Rightarrow \quad \left[\frac{\theta}{\delta-\theta}\right]^{\frac{\delta-\theta}{\delta}}\left[\frac{\delta\,\Gamma(p)}{2\,\theta\,\Gamma(p+\theta)}\right] \;=\; \left(\frac{\theta}{\delta}\right)^{\theta}\left[\frac{1}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\delta}}. \tag{A6}$$

Equations (A5) and (A6) imply:

$$(\beta)^{\theta} \;=\; \left(\frac{\theta}{\delta}\right)^{\theta}\left[\frac{1}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\delta}}\left[\,\overline{U}\,\right]^{\frac{\delta-\theta}{\delta}} \quad\Rightarrow\quad \beta \;=\; \left(\frac{\theta}{\delta}\right)\left[\frac{\overline{U}}{\overline{U}_1}\right]^{\frac{\delta-\theta}{\theta\delta}}. \quad \blacksquare$$