**Technical Appendix to Accompany**

**"Hotelling Competition with Avoidable Horizontal Product Differentiation: Choosing Between Online Privacy and Disclosure"**

by Shourjo Chakravorty and David E. M. Sappington

Equations and Definitions from the Text

$$G_1 - p_1 - \min\{\, t\,x,\, t\,[\,1-x\,] + K_1 \,\} \;>\; G_2 - p_2 - \min\{\, t\,[\,1-x\,]\,,\, t\,x + K_2 \,\}\,. \tag{1}$$

$$t\,[\,1-x\,] + K_1 \;<\; t\,x \quad\Leftrightarrow\quad x \;>\; \frac{1}{2} + \frac{K_1}{2\,t}\,. \tag{2}$$

$$t\,x + K_2 \;<\; t\,[\,1-x\,] \quad\Leftrightarrow\quad x \;<\; \frac{1}{2} - \frac{K_2}{2\,t}\,. \tag{3}$$

$$r_1 \;\equiv\; \frac{1}{2}\,[\,r_L + r_H\,] - \frac{K_1}{2\,t}\,[\,r_H - r_L\,]\,;\;\; r_2 \;\equiv\; \frac{1}{2}\,[\,r_L + r_H\,] + \frac{K_2}{2\,t}\,[\,r_H - r_L\,] \;>\; r_1\,. \tag{4}$$

$$A \;\equiv\; \frac{1}{3}\,[\,G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,]\,. \tag{5}$$

Additional Lemmas

The following lemmas (Lemmas A1 – A18) are employed to prove the formal conclusions in the text.

**Lemma A1**. A user who buys a phone from Firm 2 will change the default setting on the phone if and only if the user is located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$.

Proof. If a user located at $x$ buys a phone from Firm 2, the user will change the default setting on the phone if and only if:

$$G_2 - t\,x - K_2 \;>\; G_2 - t\,[\,1-x\,] \quad\Leftrightarrow\quad t\,[\,1-2\,x\,] \;>\; K_2 \quad\Leftrightarrow\quad 1 - 2\,x \;>\; \frac{K_2}{t}$$

$$\Leftrightarrow\quad 2\,x \;<\; 1 - \frac{K_2}{t} \quad\Leftrightarrow\quad x \;<\; \frac{1}{2} - \frac{K_2}{2\,t}. \;\blacksquare$$

**Lemma A2**. A user who buys a phone from Firm 1 will change the default setting on the phone if and only if the user is located in $(\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$.

Proof. If a user located at $x$ buys a phone from Firm 1, the user will change the default setting on the phone if:

$$G_1 - t\,[\,1 - x\,] - K_1 \;>\; G_1 - t\,x \quad\Leftrightarrow\quad t\,[\,1 - 2\,x\,] \;<\; -\,K_1 \quad\Leftrightarrow\quad 1 - 2\,x \;<\; -\,\frac{K_1}{t}$$

$$\Leftrightarrow\quad 2\,x \;>\; 1 + \frac{K_1}{t} \quad\Leftrightarrow\quad x \;>\; \frac{1}{2} + \frac{K_1}{2\,t}. \quad\blacksquare$$

**Lemma A3**. A user located in $\left[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,\right]$ will not change the default setting on the phone she purchases.

Proof. The proof follows directly from Lemmas A1 and A2. $\blacksquare$

**Lemma A4**. Suppose a user located at $x_0 \in [\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,]$ is indifferent between buying a phone from Firm 1 and from Firm 2. Then: (i) all users located in $[\,0, x_0\,)$ will buy a phone from Firm 1; and (ii) all users located in $[\,x_0, 1\,]$ will buy a phone from Firm 2.

Proof. Lemma A3 implies that because the user at $x_0$ is indifferent between buying a phone from Firm 1 and from Firm 2:

$$\begin{aligned}
& G_1 - t\,x_0 - p_1 \;=\; G_2 - t\,[\,1 - x_0\,] - p_2 \\
\Leftrightarrow\quad & t\,[\,1 - 2\,x_0\,] \;=\; G_2 - G_1 + p_1 - p_2 \\
\Leftrightarrow\quad & 2\,x_0 \;=\; 1 + \frac{1}{t}\,[\,G_1 - G_2 + p_2 - p_1\,] \\
\Leftrightarrow\quad & x_0 \;=\; \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,]\,.
\end{aligned} \tag{6}$$

(6) and Lemma A3 imply that a user located at $x \in [\,\frac{1}{2} - \frac{K_2}{2\,t}, x_0\,)$ will buy a phone from Firm 1 because:

$$\begin{aligned}
& G_1 - t\,x - p_1 \;>\; G_2 - t\,[\,1 - x\,] - p_2 \\
\Leftrightarrow\quad & t\,[\,1 - 2\,x\,] \;>\; G_2 - G_1 + p_1 - p_2 \\
\Leftrightarrow\quad & 2\,x \;<\; 1 + \frac{1}{t}\,[\,G_1 - G_2 + p_2 - p_1\,] \\
\Leftrightarrow\quad & x \;<\; \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,] \;=\; x_0\,.
\end{aligned}$$

(6) and Lemma A1 imply that a user located at $x \in [\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ will buy a phone from Firm 1 because:

$$G_1 - t\,x - p_1 \;>\; G_2 - t\,x - p_2 - K_2$$

$$\Leftrightarrow\quad G_1 - G_2 + p_2 - p_1 \;>\; -\,K_2 \quad\Leftrightarrow\quad \frac{1}{2\,t}\,[\,G_1 - G_2 + p_2 - p_1\,] \;>\; -\,\frac{K_2}{2\,t}$$

$$\Leftrightarrow \quad \frac{1}{2} + \frac{1}{2t}[\,G_1 - G_2 + p_2 - p_1\,] \;>\; \frac{1}{2} - \frac{K_2}{2t} \quad \Leftrightarrow \quad x_0 \;>\; \frac{1}{2} - \frac{K_2}{2t}\,.$$

(6) and Lemma A3 imply that a user located at $x \in (\,x_0, \frac{1}{2} + \frac{K_1}{2t}\,]$ will buy a phone from Firm 2 because:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,x - p_1$$

$$\Leftrightarrow \quad t\,[\,1 - 2\,x_0\,] \;<\; G_2 - G_1 + p_1 - p_2$$

$$\Leftrightarrow \quad 2\,x \;>\; 1 + \frac{1}{t}[\,G_1 - G_2 + p_2 - p_1\,]$$

$$\Leftrightarrow \quad x \;>\; \frac{1}{2t}[\,t + G_1 - G_2 + p_2 - p_1\,] \;=\; x_0\,.$$

(6) and Lemma A3 imply that a user located at $x \in (\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ will buy a phone from Firm 2 because:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,[\,1 - x\,] - p_1 - K_1$$

$$\Leftrightarrow \quad G_1 - G_2 + p_2 - p_1 \;<\; K_1 \quad \Leftrightarrow \quad \frac{1}{2t}[\,G_1 - G_2 + p_2 - p_1\,] \;<\; \frac{K_1}{2t}$$

$$\Leftrightarrow \quad \frac{1}{2} + \frac{1}{2t}[\,G_1 - G_2 + p_2 - p_1\,] \;<\; \frac{1}{2} + \frac{K_1}{2t} \quad \Leftrightarrow \quad x_0 \;<\; \frac{1}{2} + \frac{K_1}{2t}\,. \quad \blacksquare$$

**Lemma A5**. Suppose a user located at $x_1 \in [\,0, \frac{1}{2} - \frac{K_2}{2t}\,)$ is indifferent between buying a phone from Firm 1 and from Firm 2. Then: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,]$ are similarly indifferent; and (ii) all users located in $(\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ will buy a phone from Firm 2.

Proof. Lemma A1 implies that when a user at $x \in [\,0, \frac{1}{2} - \frac{K_2}{2t}\,)$ is indifferent between buying a phone from Firm 1 and from Firm 2:

$$G_1 - t\,x - p_1 \;=\; G_2 - t\,x - p_2 - K_2$$

$$\Leftrightarrow \quad p_2 \;=\; p_1 + G_2 - G_1 - K_2\,. \tag{7}$$

(7) implies that when the user located at $x_1 \in [\,0, \frac{1}{2} - \frac{K_2}{2t}\,)$ is indifferent between buying a phone from Firm 1 and from Firm 2, the same is true of all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,)$.

Lemma A3 implies that when (7) holds, the user at $\widetilde{x} = \frac{1}{2} - \frac{K_2}{2t}$ is indifferent between buying a phone from Firm 1 and from Firm 2 because:

$$G_1 - t\,\widetilde{x} - p_1 \;=\; G_2 - t\,[\,1 - \widetilde{x}\,] - p_2$$

$$\Leftrightarrow \quad p_2 - (\,p_1 + G_2 - G_1\,) \;=\; -\,t\,[\,1 - 2\,\widetilde{x}\,] \quad \Leftrightarrow \quad -\,K_2 \;=\; -\,t\,[\,1 - 2\,\widetilde{x}\,]$$

$$\Leftrightarrow \quad \frac{K_2}{t} \;=\; 1 - 2\,\widetilde{x} \quad \Leftrightarrow \quad \widetilde{x} \;=\; \frac{1}{2} - \frac{K_2}{2\,t}\,.$$

Lemma A3 implies that when (7) holds, a user located at $x \in (\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ will buy a phone from Firm 2. This is the case because when (7) holds:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,x - p_1$$

$$\Leftrightarrow \quad p_2 - (\,p_1 + G_2 - G_1\,) \;<\; -\,t\,[\,1 - 2\,x\,] \quad \Leftrightarrow \quad -\,K_2 \;<\; -\,t\,[\,1 - 2\,x\,]$$

$$\Leftrightarrow \quad \frac{K_2}{t} \;>\; 1 - 2\,x \quad \Leftrightarrow \quad x \;>\; \frac{1}{2} - \frac{K_2}{2\,t}\,.$$

Lemma A2 implies that when (7) holds, a user located at $x \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ will buy a phone from Firm 2 because:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,[\,1 - x\,] - p_1 - K_1$$

$$\Leftrightarrow \quad p_2 \;<\; p_1 + G_2 - G_1 + K_1\,. \tag{8}$$

(7) ensures that (8) holds. ■

**Lemma A6**. Suppose a user located at $x_2 \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ is indifferent between buying a phone from Firm 1 and from Firm 2. Then: (i) all users located in $[\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ are similarly indifferent; and (ii) all users located in $[\,0, \frac{1}{2} + \frac{K_1}{2t}\,)$ will buy a phone from Firm 1.

Proof. Lemma A2 implies that when a user at $x \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ is indifferent between buying a phone from Firm 1 and from Firm 2:

$$G_1 - t\,[\,1 - x\,] - p_1 - K_1 \;=\; G_2 - t\,[\,1 - x\,] - p_2$$

$$\Leftrightarrow \quad p_2 \;=\; p_1 + G_2 - G_1 + K_1\,. \tag{9}$$

(9) implies that when the user located at $x_2 \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ is indifferent between buying a phone from Firm 1 and from Firm 2, the same is true of all users located in $(\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$.

Lemma A3 implies that when (9) holds, the user at $\widehat{x} = \frac{1}{2} + \frac{K_1}{2t}$ is indifferent between buying a phone from Firm 1 and from Firm 2 because:

$$G_1 - t\,\widehat{x} - p_1 \;=\; G_2 - t\,[\,1 - \widehat{x}\,] - p_2$$

$$\Leftrightarrow \quad p_2 - (\,p_1 + G_2 - G_1\,) \;=\; -\,t\,[\,1 - 2\,\widehat{x}\,] \quad \Leftrightarrow \quad K_1 \;=\; -\,t\,[\,1 - 2\,\widehat{x}\,]$$

$$\Leftrightarrow \quad \frac{K_1}{t} \;=\; 2\,\widehat{x} - 1 \quad \Leftrightarrow \quad \widehat{x} \;=\; \frac{1}{2} + \frac{K_1}{2\,t}\,.$$

Lemma A3 implies that when (9) holds, a user located at $x \in [\,\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}\,)$ will buy a phone from Firm 1. This is the case because when (9) holds:

$$
\begin{aligned}
& G_1 - t\,x - p_1 \;>\; G_2 - t\,[\,1-x\,] - p_2 \\
\Leftrightarrow \quad & p_2 - (\,p_1 + G_2 - G_1\,) \;>\; -t\,[\,1 - 2\,x\,] \quad \Leftrightarrow \quad K_1 \;>\; -t\,[\,1-2\,x\,] \\
\Leftrightarrow \quad & \frac{K_1}{t} \;>\; 2\,x - 1 \quad \Leftrightarrow \quad x \;<\; \frac{1}{2} + \frac{K_1}{2\,t}\,.
\end{aligned}
$$

Lemma A1 implies that when (9) holds, a user located at $x \in [\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ will buy a phone from Firm 1 because:

$$
\begin{aligned}
& G_1 - t\,x - p_1 \;>\; G_2 - t\,x \;- p_2 - K_2 \\
\Leftrightarrow \quad & p_2 - (\,p_1 + G_2 - G_1\,) \;>\; -\,K_2\,. 
\end{aligned} \tag{10}
$$

(9) ensures that (10) holds. ■

**Lemma A7**. If $p_1 \geq p_2 + G_1 - G_2 + K_2$, then all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ (weakly) prefer to buy a phone from Firm 2 than from Firm 1. The preference is strict if the inequality holds strictly.

Proof. Lemma A1 implies that a user located at $x \in [\,0, \frac{1}{2} - \frac{K_2}{2\,t})$ (weakly) prefers to buy a phone from Firm 2 than from Firm 1 if:

$$
G_1 - t\,x - p_1 \;\leq\; G_2 - t\,x - p_2 - K_2 \quad \Leftrightarrow \quad p_1 \;\geq\; p_2 + G_1 - G_2 + K_2\,. \tag{11}
$$

It is apparent from (11) that the preference is strict if the inequality holds strictly. ■

**Lemma A8**. If $p_1 \geq p_2 + G_1 - G_2 + K_2$, then all users located in $[\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ (weakly) prefer to buy a phone from Firm 2 than from Firm 1. The preference is strict if the inequality holds strictly.

Proof. Lemma A3 implies that a user located at $x \in \left[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,\right]$ (weakly) prefers to buy a phone from Firm 2 than from Firm 1 if:

$$
\begin{aligned}
& G_1 - t\,x - p_1 \;\leq\; G_2 - t\,[\,1-x\,] - p_2 \\
\Leftrightarrow \quad & p_1 \;\geq\; p_2 + G_1 - G_2 + t\,[\,1 - 2\,x\,]\,.
\end{aligned} \tag{12}
$$

The maintained assumption ensures the inequality in (12) holds if:

$$
K_2 \;\geq\; t\,[\,1 - 2\,x\,] \quad \Leftrightarrow \quad x \;\geq\; \frac{1}{2} - \frac{K_2}{2\,t}\,. \tag{13}
$$

(13) holds for all users located in $\left[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,\right]$. Furthermore, it is apparent from (12) and (13) that all users located in $\left[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,\right]$ strictly prefer to buy a phone from Firm 2 than from Firm 1 if $p_1 > p_2 + G_1 - G_2 + K_2$.

Lemmas A1 and A2 imply that all users in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ strictly prefer to buy a phone from Firm 2 than from Firm 1 if:

$$G_1 - t[1-x] - p_1 - K_1 \ < \ G_2 - t[1-x] - p_2$$

$$\Leftrightarrow \ \ p_1 \ > \ p_2 + G_1 - G_2 - K_1 \,. \tag{14}$$

The maintained assumption ensures the inequality in (14) holds. ■

**Lemma A9**. If $p_2 \geq p_1 + G_2 - G_1 + K_1$, then all users located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ (weakly) prefer to buy a phone from Firm 1 than from Firm 2. The preference is strict if the inequality holds strictly.

Proof. Lemma A2 implies that a user located at $x \in (\frac{1}{2} + \frac{K_1}{2t}, 1]$ (weakly) prefers to buy a phone from Firm 1 than from Firm 2 if:

$$G_1 - t[1-x] - p_1 - K_1 \ \geq \ G_2 - t[1-x] - p_2 \quad \Leftrightarrow \quad p_2 \ \geq \ p_1 + G_2 - G_1 + K_1 \,. \tag{15}$$

It is apparent from (15) that the preference is strict if the inequality holds strictly. ■

**Lemma A10**. If $p_2 \geq p_1 + G_2 - G_1 + K_1$, then all users located in $[0, \frac{1}{2} + \frac{K_1}{2t}]$ (weakly) prefer to buy a phone from Firm 1 than from Firm 2. The preference is strict if the inequality holds strictly.

Proof. Lemma A3 implies that a user located at $x \in [\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ (weakly) prefers to buy a phone from Firm 1 than from Firm 2 if:

$$G_1 - t\,x - p_1 \ \geq \ G_2 - t[1-x] - p_2$$

$$\Leftrightarrow \ \ p_2 \ \geq \ p_1 + G_2 - G_1 - t[1-2x] \,. \tag{16}$$

The maintained assumption ensures the inequality in (16) holds if:

$$K_1 \ \geq \ -t[1-2x] \quad \Leftrightarrow \quad x \ \leq \ \frac{1}{2} + \frac{K_1}{2t} \,. \tag{17}$$

(17) holds for all users located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$. Furthermore, it is apparent from (16) and (17) that all users located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ strictly prefer to buy a phone from Firm 1 than from Firm 2 if $p_2 > p_1 + G_2 - G_1 + K_1$.

Lemmas A1 and A2 imply that all users in $[0, \frac{1}{2} - \frac{K_2}{2t})$ strictly prefer to buy a phone from Firm 1 than from Firm 2 if:

$$G_1 - t\,x - p_1 \ > \ G_2 - t\,x - p_2 - K_2$$

$$\Leftrightarrow \ \ p_2 \ > \ p_1 + G_2 - G_1 - K_2 \,. \tag{18}$$

The maintained assumption ensures the inequality in (18) holds. ■

**Assumption 1**. $K_1 \in [\,0,\,t\,)$, $K_2 \in [\,0,\,t\,)$, and $(K_1\,,\,K_2\,) \neq (0\,,\,0\,)$.

**Lemma A11**. Suppose Assumption 1 holds. Then in any equilibrium in which $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ strictly prefer to buy a phone from Firm 1 than from Firm 2; (ii) all users located in $(\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ strictly prefer to buy a phone from Firm 2 than from Firm 1; and (iii) some user located in $[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,]$ is indifferent between buying a phone from Firm 1 and from Firm 2.

Proof. Lemmas A1 and A2 imply that all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ strictly prefer to buy a phone from Firm 1 than from Firm 2 if:

$$G_1 - t\,x\, - p_1 \;>\; G_2 - t\,x\, - p_2 - K_2$$

$$\Leftrightarrow \;\; p_1 \;<\; p_2 + G_1 - G_2 + K_2 \;\;\Leftrightarrow\;\; p_1 - p_2 \;<\; G_1 - G_2 + K_2\,. \tag{19}$$

Lemmas A1 and A2 also imply that all users located in $(\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ strictly prefer to buy a phone from Firm 2 than from Firm 1 if:

$$G_1 - t\,[\,1 - x\,] - p_1 - K_1 \;<\; G_2 - t\,[\,1 - x\,] - p_2$$

$$\Leftrightarrow \;\; p_2 \;<\; p_1 + G_2 - G_1 + K_1 \;\;\Leftrightarrow\;\; p_1 - p_2 \;>\; G_1 - G_2 - K_1\,. \tag{20}$$

(19) and (20) imply that conclusions (i) and (ii) in the lemma hold.

To prove conclusion (iii), first suppose all users located in $\left[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,\right]$ strictly prefer to buy a phone from Firm 1 than from Firm 2. Then the user located at $\frac{1}{2} + \frac{K_1}{2\,t}$ weakly prefers to buy a phone from Firm 1 than Firm 2. Consequently, Lemma A3 implies:

$$G_1 - t\left[\frac{1}{2} + \frac{K_1}{2\,t}\right] - p_1 \;\geq\; G_2 - t\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] - p_2$$

$$\Leftrightarrow \;\; p_1 - p_2 \;\leq\; G_1 - G_2 - t\left[\frac{1}{2} + \frac{K_1}{2\,t} - \left(\frac{1}{2} - \frac{K_1}{2\,t}\right)\right]$$

$$\Leftrightarrow \;\; p_1 - p_2 \;\leq\; G_1 - G_2 - K_1\,.$$

(20) implies that this inequality cannot hold. Therefore, it is not the case that all users located in $[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,]$ strictly prefer to buy a phone from Firm 1 than from Firm 2 when $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$.

Now suppose all users located in $\left[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,\right]$ strictly prefer to buy a phone from Firm 2 than from Firm 1. Then the user located at $\frac{1}{2} - \frac{K_2}{2\,t}$ weakly prefers to buy a phone from Firm 1 than Firm 2. Consequently, Lemma A3 implies:

$$G_2 - t\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] - p_2 \;\geq\; G_1 - t\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] - p_1$$

$$\Leftrightarrow \;\; p_1 - p_2 \;\geq\; G_1 - G_2 - t\left[\frac{1}{2} - \frac{K_2}{2\,t} - \left(\frac{1}{2} + \frac{K_2}{2\,t}\right)\right]$$

$$\Leftrightarrow \;\; p_1 - p_2 \;\geq\; G_1 - G_2 + K_2\,.$$

(19) implies that this inequality cannot hold. Therefore, it is not the case that all users located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ strictly prefer to buy a phone from Firm 2 than from Firm 1 when $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$. Consequently, it must be the case that some user located in $[\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$ is indifferent between buying a phone from Firm 1 and from Firm 2. ■

**Lemma A12**. When Assumption 1 holds: (i) $c_1 - r_1$ is the lowest price that Firm 1 can profitably charge when all users buy a phone from Firm 1; and (ii) $c_2 - r_2$ is the lowest price that Firm 2 can profitably set when all users buy a phone from Firm 2.

Proof. Lemmas A1 – A3 imply that when Assumption 1 holds, Firm 1's profit when all users buy a phone from Firm 1 at price $p_1$ is:

$$\overline{\pi}_1 \;=\; [\,p_1 + r_L - c_1\,]\left[\frac{1}{2} + \frac{K_1}{2\,t}\right] + [\,p_1 + r_H - c_1\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]$$

$$= \; p_1 - c_1 + r_L\left[\frac{1}{2} + \frac{K_1}{2\,t}\right] + r_H\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] \;=\; p_1 - c_1 + r_1$$

$$\Rightarrow \;\; \overline{\pi}_1 \;\geq\; 0 \;\;\Leftrightarrow\;\; p_1 \;\geq\; c_1 - r_1\,.$$

Lemmas A1 – A3 also imply that when Assumption 1 holds, Firm 2's profit when all users buy a phone from Firm 2 at price $p_2$ is:

$$\overline{\pi}_2 \;=\; [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right]$$

$$= \; p_2 - c_2 + r_H\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + r_L\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;=\; p_2 - c_2 + r_2$$

$$\Rightarrow \;\; \overline{\pi}_2 \;\geq\; 0 \;\;\Leftrightarrow\;\; p_2 \;\geq\; c_2 - r_2\,. \;\; ■$$

**Lemma A13**. Suppose Assumption 1 holds. Then $c_1 - r_L > c_1 - r_1$ and $c_2 - r_H < c_2 - r_2$.

Proof. (4) implies:

$$c_1 - r_L \;>\; c_1 - r_1 \;\;\Leftrightarrow\;\; r_L \;<\; r_1 \;\;\Leftrightarrow\;\; \frac{1}{2}[\,r_L + r_H\,] - \frac{K_1}{2\,t}[\,r_H - r_L\,] \;>\; r_L$$

$$\Leftrightarrow \ \frac{1}{2}\left[r_H - r_L\right] - \frac{K_1}{2t}\left[r_H - r_L\right] \ > \ 0 \ \ \Leftrightarrow \ \left[\frac{1}{2} - \frac{K_1}{2t}\right]\left[r_H - r_L\right] \ > \ 0\,. \tag{21}$$

The last inequality in (21) holds because $K_1 \in [0, t)$.

(4) also implies:

$$c_2 - r_H \ < \ c_2 - r_2 \ \Leftrightarrow \ \ r_H \ > \ r_2 \ \ \Leftrightarrow \ \ \frac{1}{2}\left[r_L + r_H\right] + \frac{K_2}{2t}\left[r_H - r_L\right] \ < \ r_H$$

$$\Leftrightarrow \ \frac{1}{2}\left[r_H - r_L\right] - \frac{K_2}{2t}\left[r_H - r_L\right] \ > \ 0 \ \ \Leftrightarrow \ \left[\frac{1}{2} - \frac{K_2}{2t}\right]\left[r_H - r_L\right] \ > \ 0\,. \tag{22}$$

The last inequality in (22) holds because $K_2 \in [0, t)$. ■

**Lemma A14**. Suppose Assumption 1 holds. Then: (i) setting $p_1 < c_1 - r_1$ is a weakly dominated strategy for Firm 1; and (ii) setting $p_2 < c_2 - r_H$ is a weakly dominated strategy for Firm 2.

Proof. We first prove that setting $p_1 < c_1 - r_1$ is a weakly dominated strategy for Firm 1. We do so by showing that Firm 1 can always secure at least as much profit by setting $p_1 = c_1 - r_L$. The proof proceeds by analyzing Cases 1A – 1E, which consider five distinct regions for $p_2$.

Case 1A. $p_2 < c_1 - r_L + G_2 - G_1 - K_2$.

First suppose $p_1 = c_1 - r_L$. Lemmas A7 and A8 imply that all users buy a phone from Firm 2 in this case. Therefore, Firm 1's profit is 0.

We now show that Firm 1's profit is non-positive if it sets $p_1 < c_1 - r_1$. There are five subcases to consider. (Lemma A13 implies that $p_1 < c_1 - r_L$ in each subcase.)

Case 1A(i). $p_2 < p_1 + G_2 - G_1 - K_2$. Lemmas A7 and A8 imply that all users buy a phone from Firm 2 in this case. Therefore, Firm 1's profit is 0.

Case 1A(ii). $p_2 = p_1 + G_2 - G_1 - K_2$. Lemmas A7 and A8 imply that in this case: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ buy a phone from Firm 2. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\pi_1 \ = \ \frac{1}{2}\left[p_1 + r_L - c_1\right]\left[\frac{1}{2} - \frac{K_2}{2t}\right] \ < \ 0\,. \tag{23}$$

The inequality in (23) holds because $p_1 < c_1 - r_L$.

Case 1A(iii). $p_2 \in (\,p_1 + G_2 - G_1 - K_2,\, p_1 + G_2 - G_1 + K_1)$. Lemmas A3, A4, and A11 imply that Firm 1's profit in this case is:

$$\pi_1 \ = \ \left[p_1 + r_L - c_1\right] x_0 \ < \ 0 \tag{24}$$

where $x_0 > 0$ is defined in (6). The inequality in (24) holds because $p_1 < c_1 - r_L$.

Case 1A(iv). $p_2 = p_1 + G_2 - G_1 + K_1$. Lemmas A9 and A10 imply that in this case: (i) all users located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[0, \frac{1}{2} + \frac{K_1}{2t}]$ buy a phone from Firm 1. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\pi_1 = \left[\frac{1}{2} + \frac{K_1}{2t}\right][p_1 + r_L - c_1] + \frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2t}\right][p_1 + r_H - c_1] < 0. \tag{25}$$

The inequality in (25) holds if $p_1 + r_H - c_1 \leq 0$ because $p_1 < c_1 - r_L$. The inequality in (25) holds if $p_1 + r_H - c_1 > 0$ because in this case:

$$\pi_1 < \left[\frac{1}{2} + \frac{K_1}{2t}\right][p_1 + r_L - c_1] + \left[\frac{1}{2} - \frac{K_1}{2t}\right][p_1 + r_H - c_1] < 0.$$

The last inequality here follows from Lemma A12 because $p_1 < c_1 - r_1$.

Case 1A(v). $p_2 > p_1 + G_2 - G_1 + K_1$. Lemmas A9 and A10 imply all users buy a phone from Firm 1 in this case. Therefore, Lemma A2 implies Firm 1's profit is:

$$\left[\frac{1}{2} + \frac{K_1}{2t}\right][p_1 + r_L - c_1] + \left[\frac{1}{2} - \frac{K_1}{2t}\right][p_1 + r_H - c_1] < 0. \tag{26}$$

The inequality in (26) follows from Lemma A12 because $p_1 < c_1 - r_1$.

Case 1B. $p_2 = c_1 - r_L + G_2 - G_1 - K_2$.

First suppose $p_1 = c_1 - r_L$. Lemmas A7 and A8 imply that in this case: (i) all users located in $[0, \frac{1}{2} - \frac{K_2}{2t})$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[\frac{1}{2} - \frac{K_2}{2t}, 1]$ buy a phone from Firm 2. Therefore, Lemma A2 implies Firm 1's profit is:

$$\frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2t}\right][p_1 + r_L - c_1] = \frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2t}\right][c_1 - r_L + r_L - c_1] = 0. \tag{27}$$

Now suppose $p_1 < c_1 - r_1$. There are three subcases to consider: (i) $p_2 \in (p_1 + G_2 - G_1 - K_2,\, p_1 + G_2 - G_1 + K_1)$; (ii) $p_2 = p_1 + G_2 - G_1 + K_1$; and (iii) $p_2 > p_1 + G_2 - G_1 + K_1$. Firm 1's profit in these subcases is as specified in (24), (25), and (26), respectively. Each of these profits is negative.

Case 1C. $p_2 \in (c_1 - r_L + G_2 - G_1 - K_2,\, c_1 - r_L + G_2 - G_1 + K_1)$.

If $p_1 = c_1 - r_L$ in this case, then (6) and Lemmas A3, A4, and A11 imply Firm 1's profit is:

$$[p_1 + r_L - c_1]\, x_0 = [c_1 - r_L + r_L - c_1]\, x_0 = 0.$$

If Firm 1 sets $p_1 < c_1 - r_1$, there are three subcases to consider: (i) $p_2 \in (p_1 + G_2 - G_1 - K_2,\, p_1 + G_2 - G_1 + K_1)$; (ii) $p_2 = p_1 + G_2 - G_1 + K_1$; and (iii) $p_2 > p_1 + G_2 - G_1 + K_1$. Firm 1's profit in these subcases is as specified in (24), (25), and (26) respectively. Each of these profits is negative.

Case 1D. $p_2 = c_1 - r_L + G_2 - G_1 + K_1$.

If $p_1 = c_1 - r_L$ in this case, then Lemmas A9 and A10 imply that: (i) all users located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[0, \frac{1}{2} + \frac{K_1}{2t}]$ buy a phone from Firm 1. Therefore, Lemma A2 implies Firm 1's profit is:

$$\left[\frac{1}{2} + \frac{K_1}{2t}\right][p_1 + r_L - c_1] + \frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2t}\right][p_1 + r_H - c_1]$$

$$= \left[\frac{1}{2} + \frac{K_1}{2t}\right][c_1 - r_L + r_L - c_1] + \frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2t}\right][c_1 - r_L + r_H - c_1]$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2t}\right][r_H - r_L] > 0.$$

If Firm 1 instead sets $p_1 < c_1 - r_1$, then $p_2 > p_1 + G_2 - G_1 + K_1$. Firm 1's profit in this case is as specified in (26). This profit is negative.

Case 1E. $p_2 > c_1 - r_L + G_2 - G_1 + K_1$.

If $p_1 = c_1 - r_L$ in this case, then Lemmas A9 and A10 imply that all users buy a phone from Firm 1. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\left[\frac{1}{2} + \frac{K_1}{2t}\right][p_1 + r_L - c_1] + \left[\frac{1}{2} - \frac{K_1}{2t}\right][p_1 + r_H - c_1]$$

$$= \left[\frac{1}{2} + \frac{K_1}{2t}\right][c_1 - r_L + r_L - c_1] + \left[\frac{1}{2} - \frac{K_1}{2t}\right][c_1 - r_L + r_H - c_1]$$

$$= \left[\frac{1}{2} - \frac{K_1}{2t}\right][r_H - r_L] > 0.$$

If Firm 1 instead sets price $p_1 < c_1 - r_1$, then $p_2 > p_1 + G_2 - G_1 + K_1$. Firm 1's profit in this case is as specified in (26). This profit is negative.

We now prove that setting $p_2 < c_2 - r_H$ is a weakly dominated strategy for Firm 2. We do so by showing that Firm 2 can always secure at least as much profit by setting $p_2 = c_2 - r_H$. The proof proceeds by analyzing Cases 2A – 2E, which consider five distinct regions for $p_1$.

Case 2A. $p_1 < c_2 - r_H + G_1 - G_2 - K_1$.

If $p_2 = c_2 - r_H$, then Lemmas A9 and A10 imply that all users buy a phones from Firm 1 in this case. Therefore, Firm 2's profit is 0.

If Firm 2 instead sets $p_2 < c_2 - r_H$, there are five possibilities to consider.

Case 2A(i). $p_1 < p_2 + G_1 - G_2 - K_1$. Lemmas A9 and A10 imply that all users buy a phone from Firm 1 in this case. Therefore, Firm 2's profit is 0.

Case 2A(ii). $p_1 = p_2 + G_1 - G_2 - K_1$. Lemmas A9 and A10 imply that: (i) all users located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[0, \frac{1}{2} + \frac{K_1}{2t}]$ buy a phone from Firm 1. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2t}\right][p_2 + r_H - c_2] < 0. \tag{28}$$

The inequality in (28) holds because $p_2 < c_2 - r_H$.

Case 2A(iii). $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$. Lemmas A3, A4, and A11 imply that Firm 2's profit in this case is:

$$[p_2 + r_H - c_2][1 - x_0] < 0 \tag{29}$$

where $x_0 \in (0, 1)$ is as specified in (6). The inequality in (29) holds because $p_2 < c_2 - r_H$.

Case 2A(iv). $p_1 = p_2 + G_1 - G_2 + K_2$. Lemmas A7 and A8 imply that in this case: (i) all users located in $[0, \frac{1}{2} - \frac{K_2}{2t}]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\frac{1}{2} - \frac{K_2}{2t}, 1]$ buy a phone from Firm 2. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\frac{1}{2}[p_2 + r_L - c_2]\left[\frac{1}{2} - \frac{K_2}{2t}\right] + [p_2 + r_H - c_2]\left[\frac{1}{2} + \frac{K_2}{2t}\right] \tag{30}$$

$$< \frac{1}{2}[c_2 - r_H + r_L - c_2]\left[\frac{1}{2} - \frac{K_2}{2t}\right] + [c_2 - r_H + r_H - c_2]\left[\frac{1}{2} + \frac{K_2}{2t}\right]$$

$$= \frac{1}{2}[r_L - r_H]\left[\frac{1}{2} - \frac{K_2}{2t}\right] < 0. \tag{31}$$

The first inequality in (31) holds because $p_2 < c_2 - r_H$.

Case 2A(v). $p_1 > p_2 + G_1 - G_2 + K_2$. Lemmas A7 and A8 imply all users buy a phone from Firm 2 in this case. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\pi_2 = [p_2 + r_L - c_2]\left[\frac{1}{2} - \frac{K_2}{2t}\right] + [p_2 + r_H - c_2]\left[\frac{1}{2} + \frac{K_2}{2t}\right] < 0. \tag{32}$$

The inequality in (32) follows from Lemma A12 because $p_2 < c_2 - r_H < c_2 - r_2$ (from Lemma A13).

Case 2B. $p_1 = c_2 - r_H + G_1 - G_2 - K_1$.

If Firm 2 sets $p_2 = c_2 - r_H$ in this case, Lemmas A9 and A10 imply that: (i) all users located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ are indifferent between buying a phone from Firm 1 and from Firm

2; and (ii) all users located in $[\,0, \frac{1}{2} + \frac{K_1}{2t}\,]$ buy a phone from Firm 1. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2\,t}\right][\,p_2 + r_H - c_2\,] \;=\; \frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2\,t}\right][\,c_2 - r_H + r_H - c_2\,] \;=\; 0\,.$$

If Firm 2 instead sets price $p_2 < c_2 - r_H$, there are three possibilities to consider: (i) $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2)$; (ii) $p_1 = p_2 + G_1 - G_2 + K_2$; and (ii) $p_1 > p_2 + G_1 - G_2 + K_2$. Firm 1's profit in these cases is as specified in (29), (30), and (32), respectively. Each of these profits is negative.

Case 2C. $p_1 \in (\,c_2 - r_H + G_1 - G_2 - K_1,\, c_2 - r_H + G_1 - G_2 + K_2)$.

If $p_2 = c_2 - r_H$ in this case, (6) and Lemmas A3, A4, and A11 imply that Firm 2's profit is:

$$\bar{\pi}_2 \;=\; [\,p_2 + r_H - c_2\,][\,1 - x_0\,] \;=\; [\,c_2 - r_H + r_H - c_2\,][\,1 - \,x_0\,] \;=\; 0\,.$$

If Firm 2 instead sets $p_2 < c_2 - r_H$, there are three possibilities to consider: (i) $p_1 \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + K_2)$; (ii) $p_1 = p_2 + G_1 - G_2 + K_2$; and (iii) $p_1 > p_2 + G_1 - G_2 + K_2$. Firm 1's profit in these cases is given by (29), (30), and (32), respectively. Each of these profits is negative.

Case 2D. $p_1 = c_2 - r_H + G_1 - G_2 + K_2$.

If Firm 2 sets $p_2 = c_2 - r_H$ in this case, Lemmas A7 and A8 imply that: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ buy a phone from Firm 2. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\begin{aligned}
\bar{\pi}_2 \;&=\; \frac{1}{2}[\,p_2 + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] + [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \\
&=\; \frac{1}{2}[\,c_2 - r_H + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] + [\,c_2 - r_H + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \\
&=\; \frac{1}{2}[\,r_L - r_H\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;<\; 0\,.
\end{aligned} \tag{33}$$

If Firm 2 instead sets price $p_2 < c_2 - r_H$, then $p_1 > p_2 + G_1 - G_2 + K_2$. Therefore, as in (32), Firm 2's profit is:

$$\begin{aligned}
\pi_2 \;&=\; [\,p_2 + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] + [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \\
&<\; [\,c_2 - r_H + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] + [\,c_2 - r_H + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \\
&=\; [\,r_L - r_H\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;<\; \frac{1}{2}[\,r_L - r_H\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;=\; \bar{\pi}_2\,.
\end{aligned} \tag{34}$$

The first inequality in (34) holds because $p_2 < c_2 - r_H$. The last inequality in (34) holds because $[\, r_L - r_H \,] \left[\, \frac{1}{2} - \frac{K_2}{2\,t} \,\right] < 0$.

<u>Case 2E</u>. Suppose $p_1 > c_2 - r_H + G_1 - G_2 + K_2$.

If Firm 2 sets $p_2 = c_2 - r_H$ in this case, Lemmas A7 and A8 imply that all users buy a phone from Firm 2. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\begin{aligned}
\bar{\pi}_2 \;&=\; [\, p_2 + r_L - c_2 \,] \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] + [\, p_2 + r_H - c_2 \,] \left[ \frac{1}{2} + \frac{K_2}{2\,t} \right] \\
&=\; [\, c_2 - r_H + r_L - c_2 \,] \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] + [\, c_2 - r_H + r_H - c_2 \,] \left[ \frac{1}{2} + \frac{K_2}{2\,t} \right] \\
&=\; [\, r_L - r_H \,] \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] \;<\; 0\,. \qquad (35)
\end{aligned}$$

If Firm 2 instead sets $p_2 < c_2 - r_H$, then $p_1 > p_2 + G_1 - G_2 + K_2$. Therefore, Firm 2's profit is $\pi_2$, as specified in (32). Inequality (34) establishes that $\pi_2 < \bar{\pi}_2$. ■

**Lemma A15**. Suppose Assumption 1 holds. Then an equilibrium does not exist in which $p_1 \geq p_2 + G_1 - G_2 + K_2$ and $p_1 > c_1 - r_L$.

<u>Proof</u>. First suppose that $p_1 > p_2 + G_1 - G_2 + K_2 > c_1 - r_L$. Then Lemmas A7 and A8 imply that all users strictly prefer to purchase a phone from Firm 2 than from Firm 1. Firm 1 earns 0 profit. If Firm 1 reduces its price to $p_1 = p_2 + G_1 - G_2 + K_2$, then (7) and Lemma A5 imply that: (i) all users located in $[\, 0, \frac{1}{2} - \frac{K_2}{2\,t} \,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\, \frac{1}{2} - \frac{K_2}{2\,t}, 1 \,]$ buy a phone from Firm 2. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\widetilde{\pi}_1 \;=\; \frac{1}{2} [\, p_1 + r_L - c_1 \,] \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] \;>\; 0 \quad \text{because } p_1 > c_1 - r_L\,.$$

Because Firm 1 thereby strictly increases its profit, an equilibrium in which $p_1 > p_2 + G_1 - G_2 + K_2 > c_1 - r_L$ does not exist.

Now suppose that $p_1 > c_1 - r_L > p_2 + G_1 - G_2 + K_2$. Lemmas A7 and A8 again imply that all users strictly prefer to purchase a phone from Firm 2 than from Firm 1. Firm 2 can increase its profit by increasing its price to ensure $p_2 + G_1 - G_2 + K_2 = c_1 - r_L$. Therefore, no equilibrium exists in which $p_1 > p_2 + G_1 - G_2 + K_2$ and $p_1 > c_1 - r_L$.

Finally, suppose that $p_1 = p_2 + G_1 - G_2 + K_2$ and $p_1 > c_1 - r_L$. Then (7) and Lemma A5 imply that: (i) all users located in $[\, 0, \frac{1}{2} - \frac{K_2}{2\,t} \,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\, \frac{1}{2} - \frac{K_2}{2\,t}, 1 \,]$ buy a phone from Firm 2. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\widetilde{\pi}_1 \;=\; \frac{1}{2}\,[\,p_1 + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;>\; 0\,. \tag{36}$$

The inequality in (36) holds because $p_1 \;>\; c_1 - r_L$. If Firm 1 were to reduce its price marginally to $p_1 - \varepsilon_1$ where $\varepsilon_1 > 0$, all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ would purchase a phone from Firm 1. Consequently, Firm 1's profit would be at least:

$$\begin{aligned}\pi_1 \;&=\; [\,p_1 - \varepsilon_1 + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\ &=\; \widetilde{\pi}_1 + \widetilde{\pi}_1 - \varepsilon_1\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;>\; \widetilde{\pi}_1 \text{ for } \varepsilon_1 \text{ sufficiently small.}\end{aligned}$$

Because Firm 1 could increase its profit by reducing its price marginally, an equilibrium does not exist in which $p_1 \;=\; p_2 + G_1 - G_2 + K_2$ and $p_1 \;>\; c_1 - r_L$. ■

**Lemma A16**. Suppose Assumption 1 holds and $G_2 - G_1 + c_1 - c_2 - K_2 \neq 0$. Then an equilibrium does not exist in which $p_1 \;=\; p_2 + G_1 - G_2 + K_2$ and $p_1 \;=\; c_1 - r_L$.

Proof. We assume $p_1 \;=\; c_1 - r_L$ throughout the ensuing proof. If $p_1 \;=\; p_2 + G_1 - G_2 + K_2$, then:

$$\begin{aligned} p_2 \;&=\; p_1 + G_2 - G_1 - K_2 \;\;\Rightarrow\;\; p_2 \;=\; c_1 - r_L + G_2 - G_1 - K_2 \\ &\Rightarrow\;\; p_2 + r_L - c_2 \;=\; G_2 - G_1 + c_1 - c_2 - K_2\,. \end{aligned} \tag{37}$$

(7) and Lemma A5 imply that when $p_2 \;=\; p_1 + G_2 - G_1 - K_2$: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ buy a phone from Firm 2. Therefore, (37) and Lemmas A1, A3, and A5 imply that Firm 2's profit is:

$$\begin{aligned}\widetilde{\pi}_2 \;&=\; [\,p_2 + r_H - c_2\,]\left[1 - \left(\frac{1}{2} - \frac{K_2}{2\,t}\right)\right] + [\,p_2 + r_L - c_2\,]\,\frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\ &=\; [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + \frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2\,t}\right][\,G_2 - G_1 + c_1 - c_2 - K_2\,]\,. \end{aligned} \tag{38}$$

First suppose that $G_2 - G_1 + c_1 - c_2 - K_2 \;<\; 0$. Further suppose that Firm 2 increases its price to $p_2^{'} \;=\; p_1 + G_2 - G_1 - K_2 + \varepsilon_2$, where $\varepsilon_2 \;>\; 0$ and $p_1 - p_2^{'} \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$. Lemma A4 implies that in this case: (i) there is a user located at $x_0 \in [\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,]$ who is indifferent between buying a phone from Firm 1 and Firm 2; (ii) all users located in $[\,0, x_0\,)$ buy a phone from Firm 1; and (iii) all users located in $(\,x_0, 1\,]$ buy a phone from Firm 2. Therefore, Lemmas A1 and A3 imply that Firm 2's profit is approximately:

$$\pi_2 \;=\; \left[p_2^{'} + r_H - c_2\right][\,1 - x_0\,] \;=\; [\,p_2 + r_H - c_2 + \varepsilon_2\,][\,1 - x_0\,]\,. \tag{39}$$

(6) and (39) imply that because $p_2^{'} = p_1 + G_2 - G_1 - K_2 + \varepsilon_2$:

$$\begin{aligned}
\pi_2 \;&=\; [\,p_2 + r_H - c_2 + \varepsilon_2\,] \left[ \frac{2\,t - t - G_1 + G_2 + p_1 - p_2^{'}}{2\,t} \right] \\
&=\; [\,p_2 + r_H - c_2 + \varepsilon_2\,] \, \frac{1}{2\,t} \, [\,t - G_1 + G_2 + G_1 - G_2 + K_2 - \varepsilon_2\,] \\
&=\; [\,p_2 + r_H - c_2 + \varepsilon_2\,] \, \frac{1}{2\,t} \, [\,t + K_2 - \varepsilon_2\,] \\
&=\; \left[ \frac{t}{2\,t} + \frac{K_2}{2\,t} \right] [\,p_2 + r_H - c_2 + \varepsilon_2\,] - \frac{\varepsilon_2}{2\,t} [\,p_2 + r_H - c_2 + \varepsilon_2\,] \\
\Rightarrow \quad \lim_{\varepsilon_2 \to 0} \; \pi_2 \;&=\; \left[ \frac{1}{2} + \frac{K_2}{2\,t} \right] [\,p_2 + r_H - c_2\,] \;>\; \widetilde{\pi}_2 \; . \qquad (40)
\end{aligned}$$

The inequality in (40) follows from (38), given the maintained assumption that $G_2 - G_1 + c_1 - c_2 - K_2 < 0$. (40) implies that Firm 2 could increase its profit by increasing $p_2$ marginally above $c_1 - r_L + G_2 - G_1 - K_2$. Consequently, an equilibrium does not exist in which $p_1 = p_2 + G_1 - G_2 + K_2$, $p_1 = c_1 - r_L$, and $G_2 - G_1 + c_1 - c_2 - K_2 < 0$.

Now suppose that $G_2 - G_1 + c_1 - c_2 - K_2 > 0$. Lemmas A7 and A8 imply that if Firm 2 reduced $p_2$ below $c_1 - r_L + G_2 - G_1 - K_2$ by $\varepsilon_3 > 0$, it could induce all users to purchase a phone from Firm 2. (38) and Lemma A1 imply that Firm 2's corresponding profit would be:

$$\begin{aligned}
\pi_2 \;&=\; [\,p_2 + r_L - c_2 - \varepsilon_3\,] \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] + [\,p_2 + r_H - c_2 - \varepsilon_3\,] \left[ \frac{1}{2} + \frac{K_2}{2\,t} \right] \\
&=\; [\,p_2 + r_H - c_2\,] \left[ \frac{1}{2} + \frac{K_2}{2\,t} \right] + [\,p_2 + r_L - c_2\,] \, \frac{1}{2} \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] \\
&\qquad + \, [\,p_2 + r_L - c_2\,] \, \frac{1}{2} \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] - \varepsilon_3 \\
&=\; \widetilde{\pi}_2 + [\,p_2 + r_L - c_2\,] \, \frac{1}{2} \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] - \varepsilon_3 \\
&=\; \widetilde{\pi}_2 + \frac{1}{2} \left[ \frac{1}{2} - \frac{K_2}{2\,t} \right] [\,G_2 - G_1 - c_2 + c_1 - K_2\,] - \varepsilon_3 \\
&>\; \widetilde{\pi}_2 \;\; \text{for } \varepsilon_3 \text{ sufficiently small.} \qquad (41)
\end{aligned}$$

The last equality in (41) holds because $p_1 = c_1 - r_L$ and $p_2 = p_1 + G_2 - G_1 - K_2$. The inequality in (41) follows from (38) because $K_2 < t$ and $G_2 - G_1 + c_1 - c_2 - K_2 > 0$, by assumption. (41) implies that Firm 2 could increase its profit by reducing $p_2$ marginally below $c_1 - r_L + G_2 - G_1 - K_2$. Consequently, an equilibrium does not exist in which $p_1 =$

$p_2 + G_1 - G_2 + K_2$, $p_1 = c_1 - r_L$, and $G_2 - G_1 + c_1 - c_2 - K_2 > 0$. ■

**Lemma A17**. Suppose Assumption 1 holds. An equilibrium does not exist in which $p_1 = p_2 + G_1 - G_2 + K_2$ and $p_1 \leq c_1 - r_1$.

Proof. Lemma A14 implies that setting $p_1 < c_1 - r_1$ is a dominated strategy for Firm 1. Therefore, by assumption, Firm 1 never sets price $p_1 < c_1 - r_1$ in equilibrium.

If $p_1 = p_2 + G_1 - G_2 + K_2$ and $p_1 = c_1 - r_1$, then (7) and Lemma A5 imply that: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,]$ are indifferent between buying a phone from Firm 1 and buying a phone from Firm 2; and (ii) all users located in $(\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ buy a phone from Firm 2. Therefore, Lemma A2 implies that Firm 1's expected profit is:

$$\pi_1 \;=\; \frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2\,t}\right][\,c_1 - r_1 + r_L - c_1\,] \;=\; \frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2\,t}\right][\,r_L - r_1\,] \;<\; 0\,. \tag{42}$$

The inequality in (42) holds because $K_2 \in [0\,, t\,)$ and $r_L < r_1$, from (21). (42) implies that an equilibrium does not exist in which $p_1 = p_2 + G_1 - G_2 + K_2$ and $p_1 = c_1 - r_1$. ■

**Lemma A18**. Suppose Assumption 1 holds. Then an equilibrium does not exist in which $p_2 \geq p_1 + G_2 - G_1 + K_1$ and $p_2 \neq c_2 - r_H$.

Proof. Lemma A14 implies that $p_2 < c_2 - r_H$ is a dominated strategy for Firm 2. Therefore, by assumption, Firm 2 never sets price $p_2 < c_2 - r_H$.

Consider a putative equilibrium in which $p_2 > p_1 + G_2 - G_1 + K_1$. Lemmas A9 and A10 imply that all users purchase a phone from Firm 1, so Firm 2 secures 0 profit in this putative equilibrium. (9) and Lemma A6 imply that if Firm 2 reduces its price to $p_2 = p_1 + G_2 - G_1 + K_1$, then: (i) all users located in $[\,\frac{1}{2} + \frac{K_1}{2t}, 1\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[\,0, \frac{1}{2} + \frac{K_1}{2t}\,)$ buy a phone from Firm 1. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\widehat{\pi}_2 \;=\; \frac{1}{2}\,[\,p_2 + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]. \tag{43}$$

If $p_2 > c_2 - r_H$, then the expression in (43) is strictly positive (because $K_1 \in [0, t)$). Therefore, Firm 2 strictly increases its profit by reducing $p_2$. Consequently, the original putative equilibrium in which $p_2 > p_1 + G_2 - G_1 + K_1$ cannot be an equilibrium.

Now consider a putative equilibrium in which $p_2 = p_1 + G_2 - G_1 + K_1$. Then (9) and Lemma A6 imply that Firm 2's profit is $\widehat{\pi}_2$, as specified in (43).

Again, by assumption, Firm 2 does not set price $p_2 < c_2 - r_H$ because, as implied by Lemma A14, it is a dominated strategy.

If $p_2 > c_2 - r_H$, then $\widehat{\pi}_2 > 0$. If Firm 2 were to reduce its price to $p_2 - \varepsilon_4$ where $\varepsilon_4 > 0$, all users located in $[\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ would purchase a phone from Firm 2. Consequently, Firm 2's profit would be at least:

$$\begin{aligned} \pi_2 \;&=\; [\,p_2 - \varepsilon_4 + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] \\ &=\; \widehat{\pi}_2 + \widehat{\pi}_2 - \varepsilon_4\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] \;>\; \widehat{\pi}_2 \;\text{ for } \varepsilon_4 \text{ sufficiently small.} \end{aligned} \tag{44}$$

(44) implies that Firm 2 could increase its profit by reducing its price marginally below $p_1 + G_2 - G_1 + K_1$. Therefore, the original putative equilibrium cannot be an equilibrium. ■

Formal Conclusions in the Text

**Lemma 1.** *Suppose the default PD setting cannot be changed and $t > |\,A\,|$. Then in equilibrium: (i) the consumer located at $x_0 \equiv \frac{1}{2} - \frac{A}{2\,t} \in (0,1)$ is indifferent between buying a phone from Firm 1 and from Firm 2; (ii) all consumers located in $[\,0, x_0\,)$ buy a phone from Firm 1; and (iii) all consumers located in $(\,x_0, 1\,]$ buy a phone from Firm 2. Furthermore: $p_1 = c_1 - r_L + t - A$; $p_2 = c_2 - r_H + t + A$; $\pi_1 = \frac{1}{2t}\,[\,t - A\,]^2$; and $\pi_2 = \frac{1}{2t}\,[\,t + A\,]^2$.*

Proof. The proof follows from the following lemmas (Lemmas A1.1 – A1.6).

**Lemma A1.1**. *Suppose the default privacy setting cannot be changed. Then: (i) all users buy a phone from Firm 1 if $p_2 - p_1 > G_2 - G_1 + t$; and (ii) all users buy a phone from Firm 2 if $p_2 - p_1 < G_2 - G_1 - t$.*

Proof. All users buy a phone from Firm 1 if, for all $x \in [\,0, 1\,]$:

$$\begin{aligned} &G_1 - t\,x - p_1 \;>\; G_2 - t\,[\,1 - x\,] - p_2 \;\Leftrightarrow\; t\,[\,1 - 2\,x\,] \;>\; G_2 - G_1 - p_2 + p_1 \\ &\Leftrightarrow\; 1 - 2\,x \;>\; \frac{1}{t}\,[\,G_2 - G_1 - p_2 + p_1\,] \;\Leftrightarrow\; x \;<\; \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 - p_1 + p_2\,]. \end{aligned} \tag{45}$$

(45) holds for all $x \in [\,0, 1\,]$ if:

$$\begin{aligned} 1 \;<\; \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 - p_1 + p_2\,] \;&\Leftrightarrow\; \frac{t}{2\,t} \;<\; \frac{1}{2\,t}\,[\,G_1 - G_2 - p_1 + p_2\,] \\ \Leftrightarrow\; t \;<\; G_1 - G_2 - p_1 + p_2 \;&\Leftrightarrow\; p_2 - p_1 \;>\; G_2 - G_1 + t\,. \end{aligned}$$

All users buy a phone from Firm 2 if, for all $x \in [\,0, 1\,]$:

$$G_2 - t\,[\,1 - x\,] - p_2 \;>\; G_1 - t\,x - p_1 \;\Leftrightarrow\; t\,[\,1 - 2\,x\,] \;<\; G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow \quad 1 - 2\,x \;<\; \frac{1}{t}\,[\,G_2 - G_1 - p_2 + p_1\,] \quad \Leftrightarrow \quad x \;>\; \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 - p_1 + p_2\,]\,. \tag{46}$$

(46) holds for all $x \in [\,0, 1\,]$ if:

$$0 \;>\; \frac{1}{2} + \frac{1}{2\,t}\,[\,G_1 - G_2 - p_1 + p_2\,] \quad \Leftrightarrow \quad \frac{1}{2\,t}\,[\,t + G_1 - G_2 - p_1 + p_2\,] < \;0$$

$$\Leftrightarrow \quad p_2 - p_1 \;<\; G_2 - G_1 - t\,. \quad \square$$

**Lemma A1.2**. Suppose the default privacy setting cannot be changed and $t \;>\; 3\,|\,A\,|$. Then no equilibrium exists in which one firm serves all users.

<u>Proof</u>. First suppose Firm 1 serves all users. Then Lemma A1.1 implies that for all $p_2$ that generate nonnegative profit for Firm 2:

$$p_1 \;\leq\; p_2 + G_1 - G_2 - t\,. \tag{47}$$

(47) holds for all such $p_2$ if:

$$p_1 \;\leq\; c_2 - r_H + G_1 - G_2 - t\,. \tag{48}$$

Firm 1's profit when it serves all users at a price that satisfies (48) is:

$$\pi_1 \;=\; p_1 + r_L - c_1 \;\leq\; c_2 - r_H + G_1 - G_2 - t + r_L - c_1$$

$$=\; G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,) - t \;<\; 0 \;\text{ when }\; t \;>\; 3\,|\,A\,|\,. \tag{49}$$

(49) implies that no equilibrium exists in which Firm 1 serves all users.

Now suppose Firm 2 serves all users. Then Lemma A1.1 implies that for all $p_1$ that generate nonnegative profit for Firm 1:

$$p_2 \;\leq\; p_1 + G_2 - G_1 - t\,. \tag{50}$$

(50) holds for all such $p_1$ if:

$$p_2 \;\leq\; c_1 - r_L + G_2 - G_1 - t\,. \tag{51}$$

Firm 2's profit when it serves all users at a price that satisfies (51) is:

$$\pi_2 \;=\; p_2 + r_H - c_2 \;\leq\; c_1 - r_L + G_2 - G_1 - t + r_H - c_2$$

$$=\; G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) - t \;<\; 0 \;\text{ when }\; t \;>\; 3\,A\,. \tag{52}$$

(52) implies that no equilibrium exists in which Firm 2 serves all users. $\square$

**Lemma A1.3**. Suppose the default privacy setting cannot be changed and $p_2 - p_1 \in [G_2 - G_1 - t,\, G_2 - G_1 + t]$. Then: (i) a user located at $x_0 \equiv \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \in [0,1]$ is indifferent between purchasing a phone from Firm 1 and from Firm 2; (ii) if $x_0 > 0$, all users located in $[0, x_0)$ buy a phone from Firm 1; and (iii) if $x_0 < 1$, all users located in $(x_0, 1]$ buy a phone from Firm 2.

Proof. A user located at $x$ is indifferent between purchasing a phone from Firm 1 and from Firm 2 if:

$$
\begin{aligned}
& G_1 - t\,x - p_1 \;=\; G_2 - t[1-x] - p_2 \;\;\Leftrightarrow\;\; t[1-2x] \;=\; G_2 - G_1 - p_2 + p_1 \\
\Leftrightarrow\;\; & 1 - 2x \;=\; \frac{1}{t}[G_2 - G_1 - p_2 + p_1] \;\;\Leftrightarrow\;\; x \;=\; \frac{1}{2} + \frac{1}{t}[G_1 - G_2 - p_1 + p_2] \\
\Leftrightarrow\;\; & x \;=\; \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \;\equiv\; x_0 \\
& \qquad \in [0,1] \;\;\Leftrightarrow\;\; t + G_1 - G_2 + p_2 - p_1 \;\in\; [0, 2t] \\
& \qquad\qquad\qquad\;\; \Leftrightarrow\;\; p_2 - p_1 \;\in\; [G_2 - G_1 - t,\, G_2 - G_1 + t].
\end{aligned}
$$

If $x_0 > 0$, then a user located at $x \in [0, x_0)$ buys a phone from Firm 1 because:

$$
\begin{aligned}
& G_1 - t\,x - p_1 \;>\; G_2 - t[1-x] - p_2 \\
\Leftrightarrow\;\; & t[1-2x] \;>\; G_2 - G_1 + p_1 - p_2 \\
\Leftrightarrow\;\; & 2x \;<\; 1 + \frac{1}{t}[G_1 - G_2 + p_2 - p_1] \\
\Leftrightarrow\;\; & x \;<\; \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \;=\; x_0\,.
\end{aligned}
$$

If $x_0 < 1$, then a user located at $x \in (x_0, 1]$ buys a phone from Firm 2 because:

$$
\begin{aligned}
& G_2 - t[1-x] - p_2 \;>\; G_1 - t\,x - p_1 \\
\Leftrightarrow\;\; & t[1-2x_0] \;<\; G_2 - G_1 + p_1 - p_2 \\
\Leftrightarrow\;\; & 2x \;>\; 1 + \frac{1}{t}[G_1 - G_2 + p_2 - p_1] \\
\Leftrightarrow\;\; & x \;>\; \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \;=\; x_0\,. \;\; \square
\end{aligned}
$$

**Lemma A1.4**. Suppose the default privacy setting cannot be changed and $t > 3\,|\,A\,|$. Then in equilibrium, there exists a $x_0 \in [\,0,1\,]$ such that: (i) a user located at $x_0$ is indifferent between buying a phone from Firm 1 and from Firm 2; (ii) all users located in $[\,0, x_0\,)$ buy a phone from Firm 1; and (iii) all users located in $(\,x_0, 1\,]$ buy a phone from Firm 2. Furthermore: $p_1 = c_1 - r_L + t - A$; $p_2 = c_2 - r_H + t + A$; $\pi_1 = \frac{1}{2t}[\,t - A\,]^2$; and $\pi_2 = \frac{1}{2t}[\,t + A\,]^2$.

<u>Proof</u>. Lemma A1.2 implies that Firm 1 and Firm 2 both serve some users in equilibrium. Therefore, Lemma A1.1 implies that $p_2 - p_1 \in [\,G_2 - G_1 - t,\, G_2 - G_1 - t\,]$. Consequently, Lemma A1.3 implies that a user located at

$$x_0 \equiv \frac{1}{2\,t}[\,t + G_1 - G_2 + p_2 - p_1\,] \in [\,0, 1\,] \tag{53}$$

is indifferent between purchasing a phone from Firm 1 and from Firm 2. Furthermore, all users located in $[\,0, x_0\,)$ buy a phone from Firm 1, and all users located in $(\,x_0, 1\,]$ buy a phone from Firm 2. Therefore, (53) implies that Firm 1's profit is:

$$\pi_1 = [\,p_1 + r_L - c_1\,]\,\frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,]\,. \tag{54}$$

The unique value of $p_1$ that maximizes $\pi_1$ in (54) is given by:

$$\frac{\partial \pi_1}{\partial p_1} = 0 \;\Leftrightarrow\; -[\,p_1 + r_L - c_1\,] + t + G_1 - G_2 + p_2 - p_1 = 0$$

$$\Leftrightarrow\; p_1 = \frac{1}{2}[\,t + c_1 - r_L + G_1 - G_2 + p_2\,]\,. \tag{55}$$

(53) and Lemma A1.3 imply that Firm 2's profit is:

$$\pi_2 = [\,p_2 + r_H - c_2\,]\,\frac{1}{2\,t}\,[\,t + G_2 - G_1 + p_1 - p_2\,]\,. \tag{56}$$

The unique value of $p_2$ that maximizes $\pi_2$ in (56) is given by:

$$\frac{\partial \pi_2}{\partial p_2} = 0 \;\Leftrightarrow\; -[\,p_2 + r_H - c_2\,] + t + G_2 - G_1 + p_1 - p_2 = 0$$

$$\Leftrightarrow\; p_2 = \frac{1}{2}[\,t + c_2 - r_H + G_2 - G_1 + p_1\,]\,. \tag{57}$$

(55) and (57) imply:

$$p_1 = \frac{1}{2}[\,t + c_1 - r_L + G_1 - G_2\,] + \frac{1}{4}[\,t + c_2 - r_H + G_2 - G_1 + p_1\,]$$

$$\Rightarrow\; \frac{3}{4}\,p_1 = \frac{1}{4}[\,2\,t + 2\,c_1 - 2\,r_L + 2\,G_1 - 2\,G_2 + t + c_2 - r_H + G_2 - G_1\,]$$

$$\Rightarrow \quad p_1 = \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,]$$

$$= \frac{1}{3}\,[\,3\,t + G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,) - 3\,r_L + 3\,c_1\,]$$

$$= \frac{1}{3}\,[\,3\,t - 3\,A + 3\,(\,c_1 - r_L\,)\,] \;=\; c_1 - r_L + t - A\,. \tag{58}$$

(57) and (58) imply:

$$p_2 = \frac{1}{2}\,[\,t + c_2 - r_H + G_2 - G_1\,] + \frac{1}{6}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,]$$

$$= \frac{1}{6}\,[\,3\,t + 3\,c_2 - 3\,r_H + 3\,G_2 - 3\,G_1 + 3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,]$$

$$= \frac{1}{6}\,[\,6\,t + 4\,c_2 + 2\,c_1 + 2\,G_2 - 2\,G_1 - 4\,r_H - 2\,r_L\,]$$

$$= \frac{1}{3}\,[\,3\,t + G_2 + r_H - (\,G_1 + r_L - c_1\,) - 3\,r_H + 3\,c_2\,]$$

$$= \frac{1}{3}\,[\,3\,t + 3\,A + 3\,(\,c_2 - r_H\,)\,] \;=\; c_2 - r_H + t + A\,. \tag{59}$$

(58) implies that Firm 1's profit margin is positive because:

$$p_1 + r_L - c_1 \;=\; t - A \;>\; 0\,. \tag{60}$$

(59) implies that Firm 2's profit margin is positive because:

$$p_2 + r_H - c_2 \;=\; t + A \;>\; 0\,. \tag{61}$$

(58) and (59) imply:

$$p_2 - p_1 \;=\; \frac{1}{3}\,[\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,]\,. \tag{62}$$

(58), (62), and Lemma A1.3 imply:

$$\pi_1 = \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H + 3\,r_L - 3\,c_1\,]$$

$$\cdot\ \frac{1}{2\,t}\left[\,t + G_1 - G_2 + \frac{1}{3}\,(\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,)\right]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_2 - c_1 + G_1 - G_2 + r_L - r_H\,]$$

$$\cdot\ [\,3\,t + 3\,G_1 - 3\,G_2 + c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_2 - c_1 + G_1 - G_2 + r_L - r_H\,]^2$$

$$= \frac{1}{18\,t}\,[\,3\,t + G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,)\,]^2$$

$$= \frac{1}{18\,t}\,[\,3\,t - 3\,A\,]^2 \;=\; \frac{1}{2\,t}\,[\,t - A\,]^2\,.$$

(59), (62), and Lemma A1.3 imply:

$$\pi_2 \;=\; \frac{1}{3}\,[\,3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 2\,r_H - r_L + 3\,r_H - 3\,c_2\,]$$

$$\cdot\;\frac{1}{2\,t}\left[\,t + G_2 - G_1 + \frac{1}{3}\,(\,c_1 - c_2 + 2\,G_1 - 2\,G_2 + r_H - r_L\,)\,\right]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_1 - c_2 + G_2 - G_1 + r_H - r_L\,]$$

$$\cdot\;[\,3\,t + 3\,G_2 - 3\,G_1 + c_1 - c_2 + 2\,G_1 - 2\,G_2 + r_H - r_L\,]$$

$$= \frac{1}{18\,t}\,[\,3\,t + c_1 - c_2 + G_2 - G_1 + r_H - r_L\,]^2$$

$$= \frac{1}{18\,t}\,[\,3\,t + G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,]^2$$

$$= \frac{1}{18\,t}\,[\,3\,t + 3\,A\,]^2 \;=\; \frac{1}{2\,t}\,[\,t + A\,]^2\,. \;\square$$

**Lemma A1.5**. Suppose the default privacy setting cannot be changed, $A < 0$, and $t \in (\,|\,A\,|\,, 3\,|\,A\,|\,)$. Then at the unique equilibrium, both firms sell phones, Firm 1's profit is $\pi_1 = \frac{1}{2\,t}\,[\,t - A\,]^2 > 0$ and Firm 2's profit is $\pi_2 = \frac{1}{2\,t}\,[\,t + A\,]^2 > 0$.

Proof. First suppose that an equilibrium exists in which all users buy a phone from Firm 2. Then because the user located at 0 buys a phone from Firm 2:

$$G_2 - p_2 - t \;>\; G_1 - p_1 \quad\Leftrightarrow\quad p_2 \;<\; p_1 + G_2 - G_1 - t\,. \tag{63}$$

(63) must hold for all $p_1$ for which Firm 1's profit margin is positive. Therefore:

$$p_2 \;<\; c_1 - r_L + G_2 - G_1 - t \;\equiv\; \widehat{p}_2\,. \tag{64}$$

Firm 2's profit when it sets a price marginally below $\widehat{p}_2$ is nearly:

$$\pi_2 \;=\; \widehat{p}_2 + r_H - c_2 \;=\; c_1 - r_L + G_2 - G_1 - t + r_H - c_2$$

$$=\; G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) - t \;=\; 3\,A - t \;<\; 0\,. \tag{65}$$

(65) implies that an equilibrium in which all users buy a phone from Firm 2 does not exist under the specified conditions.

Now suppose that an equilibrium exists in which all users buy a phone from Firm 1. Then because the user located at 1 buys a phone from Firm 1:

$$G_1 - p_1 - t \; > \; G_2 - p_2 \;\; \Leftrightarrow \;\; p_1 \; < \; p_2 + G_1 - G_2 - t \, . \tag{66}$$

(66) must hold for all $p_2$ for which Firm 2's profit margin is positive. Therefore:

$$p_1 \; < \; c_2 - r_H + G_1 - G_2 - t \; \equiv \; \widehat{p}_1 \, . \tag{67}$$

Firm 1's profit when it sets a price marginally below $\widehat{p}_1$ is nearly:

$$\begin{aligned} \pi_1 \; &= \; \widehat{p}_1 + r_L - c_1 \; = \; c_2 - r_H + G_1 - G_2 - t + r_L - c_1 \\ &= \; G_1 + r_L - c_1 - (\, G_2 + r_H - c_2 \,) - t \; = \; -\, 3\, A - t \; > \; 0 \, . \end{aligned} \tag{68}$$

If a user located at $x \in [\, 0, 1 \,]$ is indifferent between purchasing a phone from Firm 1 and from Firm 2, then:

$$\begin{aligned} & G_1 - t\, x - p_1 \; = \; G_2 - t\,[\, 1 - x \,] - p_2 \;\; \Leftrightarrow \;\; t\,[\, 1 - 2\, x \,] \; = \; G_2 - G_1 - p_2 + p_1 \\ & \Leftrightarrow \;\; 1 - 2\, x \; = \; \frac{1}{t}\,[\, G_2 - G_1 - p_2 + p_1 \,] \;\; \Leftrightarrow \;\; x \; = \; \frac{1}{2} + \frac{1}{t}\,[\, G_1 - G_2 - p_1 + p_2 \,] \\ & \Leftrightarrow \;\; x \; = \; \frac{1}{2\, t}\,[\, t + G_1 - G_2 + p_2 - p_1 \,] \, . \end{aligned} \tag{69}$$

(69) implies that when $p_2 = c_2 - r_H$ and $p_1 \in (\, \widehat{p}_1, \, \widehat{p}_1 + 2\, t \,)$, users located in $[\, 0, \, \widehat{x}_0 \,)$ purchase a phone from Firm 1 and users located in $(\, \widehat{x}_0, 1 \,]$ purchase a phone from Firm 2, where:

$$\widehat{x}_0 \; = \; \frac{1}{2\, t} \; [\, t + G_1 - G_2 + c_2 - r_H - p_1 \,] \; \in \; (\, 0, 1 \,) \, .$$

Firm 1's corresponding profit is:

$$\pi_1(p_1) \; = \; [\, p_1 + r_L - c_1 \,] \; \frac{1}{2\, t} \; [\, t + G_1 - G_2 + c_2 - r_H - p_1 \,] \, . \tag{70}$$

Differentiating (70) provides:

$$\begin{aligned} \pi_1'(p_1) \; &= \; \frac{1}{2\, t} \; [\, t + G_1 - G_2 + c_2 - r_H - p_1 - (\, p_1 + r_L - c_1 \,) \,] \\ &= \; \frac{1}{2\, t} \; [\, t + G_1 - G_2 + c_2 - r_H - r_L + c_1 - 2\, p_1 \,] \;\; \Rightarrow \;\; \pi_1''(p_1) \; = \; -\frac{1}{t} \; < \; 0 \, . \end{aligned} \tag{71}$$

(67) and (71) imply:

$$\pi_1'(p_1)\,\big|_{p_1 = \widehat{p}_1} \; = \; \frac{1}{2\, t} \; [\, t + G_1 - G_2 + c_2 - r_H - r_L + c_1 - 2\, (\, c_2 - r_H + G_1 - G_2 - t \,) \,]$$

$$= \frac{1}{2t} [\, 3t - G_1 + G_2 - c_2 + r_H - r_L + c_1 \,]$$

$$= \frac{1}{2t} [\, 3t + G_2 + r_H - c_2 - (\, G_1 + r_L - c_1 \,) \,]$$

$$= \frac{1}{2t} [\, 3t + 3A \,] \; = \; \frac{3}{2t} [\, t + A \,] \; > \; 0 \,. \tag{72}$$

(71) and (72) imply that when $p_2 = c_2 - r_H$, Firm 1 will increase $p_1$ above $\widehat{p}_1$, thereby ensuring that both firms sell phones. Consequently, the analysis in the proof of Lemma A1.4 implies that at the unique equilibrium, both firms sell phones, Firm 1's profit is $\pi_1 = \frac{1}{2t} [\, t - A \,]^2 > 0$ and Firm 2's profit is $\pi_2 = \frac{1}{2t} [\, t + A \,]^2 > 0$. □

**Lemma A1.6**. Suppose the default privacy setting cannot be changed, $A > 0$, and $t \in (\, | \, A \, | \, , 3 \, | \, A \, | \,)$. Then at the unique equilibrium, both firms sell phones, Firm 1's profit is $\pi_1 = \frac{1}{2t} [\, t - A \,]^2 > 0$ and Firm 2's profit is $\pi_2 = \frac{1}{2t} [\, t + A \,]^2 > 0$.

Proof. First suppose that an equilibrium exists in which all users buy a phone from Firm 1. Then because the user located at 1 buys a phone from Firm 1:

$$G_1 - p_1 - t \; > \; G_2 - p_2 \;\; \Leftrightarrow \;\; p_1 \; < \; p_2 + G_1 - G_2 - t \,. \tag{73}$$

(73) must hold for all $p_2$ for which Firm 2's profit margin is positive. Therefore:

$$p_1 \; < \; c_2 - r_H + G_1 - G_2 - t \; \equiv \; \widetilde{p}_1 \,. \tag{74}$$

Firm 1's profit when it sets a price marginally below $\widetilde{p}_1$ is nearly:

$$\pi_1 \; = \; \widetilde{p}_1 + r_L - c_1 \; = \; c_2 - r_H + G_1 - G_2 - t + r_L - c_1$$

$$= \; G_1 + r_L - c_1 - (\, G_2 + r_H - c_2 \,) - t \; = \; -3A - t \; < \; 0 \,. \tag{75}$$

(75) implies that an equilibrium in which all users buy a phone from Firm 1 does not exist under the specified conditions.

Now suppose that an equilibrium exists in which all users buy a phone from Firm 2. Then because the user located at 0 buys a phone from Firm 2:

$$G_2 - p_2 - t \; > \; G_1 - p_1 \;\; \Leftrightarrow \;\; p_2 \; < \; p_1 + G_2 - G_1 - t \,. \tag{76}$$

(76) must hold for all $p_1$ for which Firm 1's profit margin is positive. Therefore:

$$p_2 \; < \; c_1 - r_L + G_2 - G_1 - t \; \equiv \; \widetilde{p}_2 \,. \tag{77}$$

Firm 2's profit when it sets a price marginally below $\widetilde{p}_2$ is nearly:

$$\pi_2 \; = \; \widetilde{p}_2 + r_H - c_2 \; = \; c_1 - r_L + G_2 - G_1 - t + r_H - c_2$$

$$= \ G_2 + r_H - c_2 - (G_1 + r_L - c_1) - t \ = \ 3A - t \ > \ 0\,. \tag{78}$$

(69) implies that when $p_1 = c_1 - r_L$ and $p_2 \in (\widetilde{p}_2, \widetilde{p}_2 + 2t)$, users located in $[0, \widetilde{x}_0)$ purchase a phone from Firm 1 and users located in $(\widetilde{x}_0, 1]$ purchase a phone from Firm 2, where:

$$\widetilde{x}_0 \ = \ \frac{1}{2t}\,[\,t + G_1 - G_2 + p_2 - c_1 + r_L\,] \ \in \ (0, 1)\,.$$

Firm 2's corresponding profit is:

$$\begin{aligned} \pi_2(p_2) \ &= \ [\,p_2 + r_H - c_2\,]\,[\,1 - \widetilde{x}_0\,] \\ &= \ \frac{1}{2t}\,[\,p_2 + r_H - c_2\,]\,[\,t + G_2 - G_1 + c_1 - r_L - p_2\,]\,. \end{aligned} \tag{79}$$

Differentiating (79) provides:

$$\begin{aligned} \pi_2'(p_2) \ &= \ \frac{1}{2t}\,[\,t + G_2 - G_1 + c_1 - r_L - p_2 - (p_2 + r_H - c_2)\,] \\ &= \ \frac{1}{2t}\,[\,t + G_2 - G_1 + c_1 - r_L - r_H + c_2 - 2\,p_2\,] \ \Rightarrow \ \pi_2''(p_2) \ = \ -\frac{1}{t} \ < \ 0\,. \end{aligned} \tag{80}$$

(77) and (80) imply:

$$\begin{aligned} \pi_2'(p_2)\,\big|_{p_2 = \widetilde{p}_2} \ &= \ \frac{1}{2t}\,[\,t + G_2 - G_1 + c_1 - r_L - r_H + c_2 - 2\,(c_1 - r_L + G_2 - G_1 - t)\,] \\ &= \ \frac{1}{2t}\,[\,3t - G_2 + G_1 - c_1 + r_L - r_H + c_2\,] \\ &= \ \frac{1}{2t}\,[\,3t + G_1 + r_L - c_1 - (G_2 + r_H - c_2)\,] \\ &= \ \frac{1}{2t}\,[\,3t - 3A\,] \ = \ \frac{3}{2t}\,[\,t - A\,] \ > \ 0\,. \end{aligned} \tag{81}$$

(80) and (81) imply that when $p_1 = c_1 - r_L$, Firm 2 will increase $p_2$ above $\widetilde{p}_2$, thereby ensuring that both firms sell phones. Consequently, the analysis in the proof of Lemma A1.4 implies that at the unique equilibrium, both firms sell phones, Firm 1's profit is $\pi_1 = \frac{1}{2t}\,[\,t - A\,]^2 > 0$ and Firm 2's profit is $\pi_2 = \frac{1}{2t}\,[\,t + A\,]^2 > 0$. □ ■

**Lemma 2.** *Suppose $K_1 = K_2 = 0$. Then in equilibrium, all consumers purchase a phone from Firm 1 if $G_1 - c_1 > G_2 - c_2$. In contrast, all consumers purchase a phone from Firm 2 if $G_2 - c_2 > G_1 - c_1$. Consumers located in $[0, \frac{1}{2})$ implement the privacy setting on the phone they purchase, whereas consumers located in $(\frac{1}{2}, 1]$ implement the disclosure setting. When all consumers purchase a phone from Firm $i$, the firm's profit is (nearly) $G_i - c_i - (G_j - c_j)$*

*for* $i, j \in \{1, 2\}$ *(*$j \neq i$*).*

<u>Proof</u>. The proof follows directly from the following lemmas (Lemmas A2.1 – A2.4).

**Lemma A2.1**. Suppose $K_1 = K_2 = 0$. Then: (i) a user located in $[0, \frac{1}{2})$ will change the default setting on the phone she purchases if and only if she purchases the phone from Firm 2; (ii) a user located in $(\frac{1}{2}, 1]$ will change the default setting on the phone she purchases if and only if she purchases the phone from Firm 1; and (iii) a user located at $\frac{1}{2}$ will not change the default setting on the phone she purchases.

<u>Proof</u>. The conclusions follow directly from the proofs of Lemmas A1 – A3. □

**Lemma A2.2**. Suppose $K_1 = K_2 = 0$. Then: (i) all users buy a phone from Firm 1 if $p_2 > p_1 + G_2 - G_1$; (ii) all users buy a phone from Firm 2 if $p_2 < p_1 + G_2 - G_1$; and (iii) all users are indifferent between buying a phone from Firm 1 and from Firm 2 if $p_2 = p_1 + G_2 - G_1$.

<u>Proof</u>. Lemma A2.1 implies that a user located at $x_1 \in [0, \frac{1}{2})$ will buy a phone from Firm 1 if:

$$G_1 - t\,x_1 - p_1 \;>\; G_2 - t\,x_1 - p_2 \quad \Leftrightarrow \quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

Lemma A2.1 also implies that a user located at $x_2 \in (\frac{1}{2}, 1]$ will buy a phone from Firm 1 if:

$$G_1 - t\,[\,1 - x_2\,] - p_1 \;>\; G_2 - t\,[\,1 - x_2\,] - p_2 \quad \Leftrightarrow \quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

Lemma A2.1 further implies that a user located at $\frac{1}{2}$ will buy a phone from Firm 1 if:

$$G_1 - \frac{1}{2}\,t - p_1 \;>\; G_2 - \frac{1}{2}\,t - p_2 \quad \Leftrightarrow \quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

The proofs of the remaining conclusions are analogous, and so are omitted. □

**Lemma A2.3**. Suppose $K_1 = K_2 = 0$ and $G_2 - c_2 > G_1 - c_1$. Then in equilibrium, all users purchase a phone from Firm 2 at a price just below $c_1 - \frac{1}{2}\,[\,r_H + r_L\,] + G_2 - G_1$. Firm 1's profit is 0. Firm 2's profit is (nearly) $G_2 - c_2 - (\,G_1 - c_1\,)$.

<u>Proof</u>. Lemmas A2.1 and A2.2 imply that for $\varepsilon_5 > 0$, Firm 2's expected profit is:

$$\pi_2 \;=\; \begin{cases} 0 & \text{if } p_2 \,>\, p_1 + G_2 - G_1 \\ \frac{1}{2}\left[\,p_1 + \frac{r_L + r_H}{2} + G_2 - G_1 - c_2\,\right] & \text{if } p_2 \,=\, p_1 + G_2 - G_1 \\ p_1 + \frac{r_L + r_H}{2} + G_2 - G_1 - c_2 - \varepsilon_5 & \text{if } p_2 \,=\, p_1 + G_2 - G_1 - \varepsilon_5\,. \end{cases} \tag{82}$$

Firm 1 must secure nonnegative profit in equilibrium. Therefore, in any equilibrium in which all users either strictly prefer to purchase a phone from Firm 1 or are indifferent between purchasing a phone from Firm 1 and Firm 2, it must be the case that $p_1 \ge c_1 - \frac{1}{2}[\, r_H + r_L\,]$. Consequently, in any such equilibrium:

$$p_1 + \frac{r_L + r_H}{2} + G_2 - G_1 - c_2 \;\ge\; G_2 - c_2 - (\, G_1 - c_1\,) \;>\; 0\,. \tag{83}$$

(82) and (83) imply that for $\varepsilon_5$ sufficiently small, Firm 2 secures strictly higher profit by setting $p_2 = p_1 + G_2 - G_1 - \varepsilon_5$ than by setting $p_2 \ge p_1 + G_2 - G_1$. Therefore, in equilibrium, Firm 2 will set $p_2$ just below $c_1 - \frac{1}{2}[\, r_H + r_L\,] + G_2 - G_1$ to ensure that Firm 1 cannot profitably attract any users. Consequently, Firm 1's profit is 0 and Firm 2's profit is nearly:

$$c_1 - \frac{1}{2}[\, r_H + r_L\,] + G_2 - G_1 + \frac{1}{2}[\, r_H + r_L\,] - c_2 \;=\; G_2 - c_2 - (\, G_1 - c_1\,)\,. \quad \square$$

**Lemma A2.4**. Suppose $K_1 = K_2 = 0$ and $G_1 - c_1 > G_2 - c_2$. Then in equilibrium, all users purchase a phone from Firm 1 at a price just below $p_1 = c_2 - \frac{1}{2}[\, r_H + r_L\,] + G_1 - G_2$. Firm 2's profit is 0. Firm 1's profit is (nearly) $G_1 - c_1 - (\, G_2 - c_2\,)$.

<u>Proof</u>. Lemmas A2.1 and A2.2 imply that for $\varepsilon_6 > 0$, Firm 1's expected profit is:

$$\pi_1 \;=\; \begin{cases} 0 & \text{if } p_1 \;>\; p_2 + G_1 - G_2 \\ \frac{1}{2}\left[\, p_2 + \frac{r_L + r_H}{2} + G_1 - G_2 - c_1\,\right] & \text{if } p_1 \;=\; p_2 + G_1 - G_2 \\ p_2 + \frac{r_L + r_H}{2} + G_1 - G_2 - c_1 - \varepsilon_6 & \text{if } p_1 \;=\; p_2 + G_1 - G_2 - \varepsilon_6\,. \end{cases} \tag{84}$$

Firm 2 must secure nonnegative profit in equilibrium. Therefore, in any equilibrium in which all users either strictly prefer to purchase a phone from Firm 2 or are indifferent between purchasing a phone from Firm 1 and Firm 2, it must be the case that $p_2 \ge c_2 - \frac{1}{2}[\, r_H + r_L\,]$. Consequently, in any such equilibrium:

$$p_2 + \frac{r_L \,+\, r_H}{2} + G_1 - G_2 - c_1 \;\ge\; G_1 - c_1 - (\, G_2 - c_2\,) \;>\; 0\,. \tag{85}$$

(84) and (85) imply that for $\varepsilon_6$ sufficiently small, Firm 1 secures strictly higher profit by setting $p_1 = p_2 + G_1 - G_2 - \varepsilon_6$ than by setting $p_1 \ge p_2 + G_1 - G_2$. Therefore, in equilibrium, Firm 1 will set $p_1$ just below $c_2 - \frac{1}{2}[\, r_H + r_L\,] + G_1 - G_2$ to ensure that Firm 2 cannot profitably attract any users. Consequently, Firm 2's profit is 0, and Firm 1's profit is nearly:

$$c_2 - \frac{1}{2}[\, r_H + r_L\,] + G_1 - G_2 + \frac{1}{2}[\, r_H + r_L\,] - c_1 \;=\; G_1 - c_1 - (\, G_2 - c_2\,)\,. \quad \square \; \blacksquare$$

**Condition 1A**. $t \geq \max \{ r_H - c_2 - A, \, r_L - c_1 + A \}$.

**Condition 1B**. $\frac{1}{2t} [t - A]^2 > -2A + \frac{t - K_1}{2t} [2t + r_H - r_L]$ if $A \leq 0$;

$\frac{1}{2t} [t + A]^2 > 2A + \frac{t - K_2}{2t} [2t - (r_H - r_L)]$ if $A \geq 0$.

**Proposition 1.** *Suppose: (i) $|A| < t$; (ii) $K_2 \in (A, t)$ if $A > 0$; and (iii) $K_1 \in (|A|, t)$ if $A < 0$. Further suppose that Conditions 1A and 1B hold. Then an equilibrium exists in which the outcomes identified in Lemma 1 all prevail and no consumer changes the default setting on the phone she purchases.*

Proof. (6) and Lemmas A2 – A4 and A11 imply that in any equilibrium with the identified properties, Firm 1's profit is:

$$\pi_1 = [p_1 + r_L - c_1] \frac{1}{2t} [t + G_1 - G_2 + p_2 - p_1]. \tag{86}$$

The unique value of $p_1$ that maximizes $\pi_1$ in (86) is given by:

$$\frac{\partial \pi_1}{\partial p_1} = 0 \Leftrightarrow -[p_1 + r_L - c_1] + t + G_1 - G_2 + p_2 - p_1 = 0$$

$$\Leftrightarrow p_1 = \frac{1}{2} [t + c_1 - r_L + G_1 - G_2 + p_2]. \tag{87}$$

(6) and Lemmas A1, A3, A4, and A11 imply that in any equilibrium with the identified properties, Firm 2's profit is:

$$\pi_2 = [p_2 + r_H - c_2] \frac{1}{2t} [t + G_2 - G_1 + p_1 - p_2]. \tag{88}$$

The unique value of $p_2$ that maximizes $\pi_2$ in (88) is given by:

$$\frac{\partial \pi_2}{\partial p_2} = 0 \Leftrightarrow -[p_2 + r_H - c_2] + t + G_2 - G_1 + p_1 - p_2 = 0$$

$$\Leftrightarrow p_2 = \frac{1}{2} [t + c_2 - r_H + G_2 - G_1 + p_1]. \tag{89}$$

(87) and (89) imply that in any equilibrium:

$$p_1 = \frac{1}{2} [t + c_1 - r_L + G_1 - G_2] + \frac{1}{4} [t + c_2 - r_H + G_2 - G_1 + p_1]$$

$$\Rightarrow \frac{3}{4} p_1 = \frac{1}{4} [2t + 2c_1 - 2r_L + 2G_1 - 2G_2 + t + c_2 - r_H + G_2 - G_1]$$

$$\Rightarrow p_1 = \frac{1}{3} [3t + 2c_1 + c_2 + G_1 - G_2 - 2r_L - r_H]$$

$$= \frac{1}{3}\,[\,3\,t + G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,) - 3\,r_L + 3\,c_1\,]$$

$$= \frac{1}{3}\,[\,3\,t - 3\,A + 3\,(\,c_1 - r_L\,)\,] \;=\; c_1 - r_L + t - A\,. \tag{90}$$

(89) and (90) imply:

$$p_2 \;=\; \frac{1}{2}\,[\,t + c_2 - r_H + G_2 - G_1\,] + \frac{1}{6}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,]$$

$$= \frac{1}{6}\,[\,3\,t + 3\,c_2 - 3\,r_H + 3\,G_2 - 3\,G_1 + 3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,]$$

$$= \frac{1}{6}\,[\,6\,t + 4\,c_2 + 2\,c_1 + 2\,G_2 - 2\,G_1 - 4\,r_H - 2\,r_L\,]$$

$$= \frac{1}{3}\,[\,3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 2\,r_H - r_L\,]$$

$$= \frac{1}{3}\,[\,3\,t + G_2 + r_H - (\,G_1 + r_L - c_1\,) - 3\,r_H + 3\,c_2\,]$$

$$= \frac{1}{3}\,[\,3\,t + 3\,A + 3\,(\,c_2 - r_H\,)\,] \;=\; c_2 - r_H + t + A\,. \tag{91}$$

(90) and the maintained assumptions imply that Firm 1's profit margin is positive because:

$$p_1 + r_L - c_1 \;=\; t - A \;>\; 0\,.$$

(91) and the maintained assumptions imply that Firm 2's profit margin is positive because:

$$p_2 + r_H - c_2 \;=\; t + A \;>\; 0\,.$$

(90) and Condition 1A imply that $p_1 \geq 0$ because:

$$p_1 \;\geq\; 0 \quad\Leftrightarrow\quad 3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H \;\geq\; 0$$

$$\Leftrightarrow\quad G_2 + r_H - c_2 - (\,G_1 - 2\,r_L + 2\,c_1\,) \;\leq\; 3\,t$$

$$\Leftrightarrow\quad G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) \;\leq\; 3\,t - 3\,r_L + 3\,c_1$$

$$\Leftrightarrow\quad A \;=\; \frac{1}{3}\,[\,G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,] \;\leq\; t - r_L + c_1\,. \tag{92}$$

(91) and Condition 1A imply that $p_2 \geq 0$ because:

$$p_2 \;\geq\; 0 \quad\Leftrightarrow\quad 3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 2\,r_H - r_L \;\geq\; 0$$

$$\Leftrightarrow\quad G_2 - 2\,r_H + 2\,c_2 - (\,G_1 + r_L - c_1\,) \;\geq\; 3\,t$$

$$\Leftrightarrow\quad G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) \;\geq\; 3\,r_H - 3\,c_2 - 3\,t$$

$$\Leftrightarrow \quad A \;=\; \frac{1}{3}\,[\,G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,] \;\geq\; r_H - c_2 - t\,. \tag{93}$$

(90) and (91) imply:

$$p_2 - p_1 \;=\; \frac{1}{3}\,[\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,]\,. \tag{94}$$

(6), (5), and (94) imply that the user who is indifferent between purchasing a phone from Firm 1 and Firm 2 is located at:

$$\begin{aligned} x_0 \;&=\; \frac{1}{2\,t}\left[\,t + G_1 - G_2 + \frac{1}{3}\,(\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,)\right] \\ &=\; \frac{1}{6\,t}\,[\,3\,t + 3\,G_1 - 3\,G_2 + c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,] \\ &=\; \frac{1}{6\,t}\,[\,3\,t + G_1 - G_2 + c_2 - c_1 + r_L - r_H\,] \\ &=\; \frac{1}{2} - \frac{1}{6\,t}\,[\,G_2 + r_H - c_2 - (G_1 + r_L - c_1)\,] \;=\; \frac{1}{2} - \frac{A}{2\,t}\,. \end{aligned} \tag{95}$$

(95) and the maintained assumptions imply that $x_0 \in (\,\frac{1}{2} - \frac{K_2}{2\,t}\,,\,\frac{1}{2} + \frac{K_1}{2\,t}\,)$, so no user changes the default setting on the phone she purchases (from Lemmas A1 – A4).

(95) and (90) imply:

$$\pi_1 \;=\; [\,p_1 + r_L - c_1\,]\,x_0 \;=\; [\,t - A\,]\left[\frac{t - A}{2\,t}\right] \;=\; \frac{1}{2\,t}\,[\,t - A\,]^2\,. \tag{96}$$

(95) and (91) imply:

$$\pi_2 \;=\; [\,p_2 + r_H - c_2\,]\,[\,1 - x_0\,] \;=\; [\,t + A\,]\left[\frac{t + A}{2\,t}\right] \;=\; \frac{1}{2\,t}\,[\,t + A\,]^2\,. \tag{97}$$

(94) implies:

$$\begin{aligned} & p_1 - p_2 \;>\; G_1 - G_2 - K_1 \\ \Leftrightarrow\;\; & \frac{1}{3}\,[\,c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H\,] \;>\; G_1 - G_2 - K_1 \\ \Leftrightarrow\;\; & c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H \;>\; 3\,G_1 - 3\,G_2 - 3\,K_1 \\ \Leftrightarrow\;\; & c_1 - c_2 - r_L + r_H \;>\; G_1 - G_2 - 3\,K_1 \\ \Leftrightarrow\;\; & K_1 \;>\; \frac{1}{3}\,[\,G_1 - G_2 + c_2 - c_1 + r_L - r_H\,] \;=\; -\,A\,. \end{aligned} \tag{98}$$

(94) also implies:

$$p_1 - p_2 \;<\; G_1 - G_2 + K_2$$

$$\Leftrightarrow \quad \frac{1}{3}\,[\,c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H\,] \;<\; G_1 - G_2 + K_2$$

$$\Leftrightarrow \quad c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H \;<\; 3\,G_1 - 3\,G_2 + 3\,K_2$$

$$\Leftrightarrow \quad c_1 - c_2 - r_L + r_H \;<\; G_1 - G_2 + 3\,K_2$$

$$\Leftrightarrow \quad K_2 \;>\; \frac{1}{3}\,[\,G_2 - G_1 + c_1 - c_2 + r_H - r_L\,] \;=\; A\,. \tag{99}$$

(98), (99), and the maintained assumptions imply:

$$p_1 - p_2 \;\in\; (\,G_1 - G_2 - K_1, G_1 - G_2 + K_2\,). \tag{100}$$

The foregoing analysis and Lemma A11 imply that the identified putative equilibrium is unique among equilibria in which (100) holds. It remains to verify that neither firm can increase its profit by unilaterally changing its price so that (100) does not hold. We first show this is the case for Firm 1.

Lemmas A7 and A8 imply that if Firm 1 sets $p_1 > p_2 + G_1 - G_2 + K_2$, then no users purchase a phone from Firm 1. Therefore, Firm 1's profit (0) is less than the profit specified in (96).

If Firm 1 sets $p_1 = p_2 + G_1 - G_2 + K_2$ when $p_2$ is as specified in (91), then:

$$\begin{aligned}
p_1 \;&=\; p_2 + G_1 - G_2 + K_2 \;=\; c_2 - r_H + t + A + G_1 - G_2 + K_2 \\
&=\; G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,) + c_1 - r_L + t + A + K_2 \\
&=\; -3\,A + c_1 - r_L + t + A + K_2 \;=\; c_1 - r_L - 2\,A + t + K_2 \\
&>\; c_1 - r_L \quad \Leftrightarrow \quad t + K_2 - 2\,A \;>\; 0\,.
\end{aligned}$$

The last inequality holds here because $t > A$ and $K_2 > A$, by assumption. Because $p_1 = p_2 + G_1 - G_2 + K_2$ and $p_1 > c_1 - r_L$, the proof of Lemma A15 implies that Firm 1 can increase its profit when $p_2$ is as specified in (91) by choosing $p_1$ to ensure $p_1 - p_2 \in (\,G_1 - G_2 - K_1, G_1 - G_2 + K_2\,)$. Therefore, Firm 1 cannot increase its profit above the profit specified in (96) by setting $p_1 = p_2 + G_1 - G_2 + K_2$ when $p_2$ is as specified in (91).

Lemmas A9 and A10 imply that if Firm 1 sets $p_1 < p_2 + G_1 - G_2 - K_1$, then all users purchase a phone from Firm 1. (4) and (5) imply that the maximum profit Firm 1 can secure by setting such a price when $p_2$ is as specified in (91) is nearly:

$$\begin{aligned}
\pi_{1D} \;&=\; p_2 + G_1 - G_2 - K_1 + r_1 - c_1 \;=\; c_2 - r_H + t + A + G_1 - G_2 - K_1 + r_1 - c_1 \\
&=\; G_1 - c_1 - (\,G_2 + r_H - c_2\,) + A + t + r_1 - K_1
\end{aligned}$$

$$
\begin{aligned}
&= G_1 + r_L - c_1 - (G_2 + r_H - c_2) + A + t + r_1 - r_L - K_1 \\
&= -3A + A + t + \frac{1}{2}[r_L + r_H] - \frac{K_1}{2t}[r_H - r_L] - r_L - K_1 \\
&= -2A + t - K_1 + \frac{1}{2}[r_H - r_L] - \frac{K_1}{2t}[r_H - r_L] \\
&= -2A + t - K_1 + \frac{r_H - r_L}{2t}[t - K_1] = -2A + \frac{t - K_1}{2t}[2t + r_H - r_L]. \qquad (101)
\end{aligned}
$$

(96) and (101) imply that Firm 1 cannot increase its profit by setting $p_1 < p_2 + G_1 - G_2 - K_1$ when $p_2$ is as specified in (91) if Condition 1B holds.

If Firm 1 sets $p_1 = p_2 + G_1 - G_2 - K_1$ when $p_2$ is as specified in (91), then (9) and Lemma A6 imply that: (i) all users located in $[\frac{1}{2} + \frac{K_1}{2t}, 1]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[0, \frac{1}{2} + \frac{K_1}{2t})$ buy a phone from Firm 1. Therefore, Lemma A2 implies that Firm 1's profit is:

$$
\pi_1 = [p_1 + r_L - c_1]\left[\frac{1}{2} + \frac{K_1}{2t}\right] + [p_1 + r_H - c_1]\frac{1}{2}\left[\frac{1}{2} - \frac{K_1}{2t}\right]. \qquad (102)
$$

$$
\begin{aligned}
&< [p_1 + r_L - c_1]\left[\frac{1}{2} + \frac{K_1}{2t}\right] + [p_1 + r_H - c_1]\left[\frac{1}{2} - \frac{K_1}{2t}\right] \\
&= p_1 + r_1 - c_1 = \pi_{1D} < \frac{1}{2t}[t - A]^2. \qquad (103)
\end{aligned}
$$

The first inequality in (103) holds because $p_1 + r_H - c_1$ must be strictly positive if Firm 1 is to secure positive profit in this case. The last inequality in (103) reflects (101) and Condition 1B. (96) and (103) imply that Firm 1 will not set $p_1 = p_2 + G_1 - G_2 - K_1$ when $p_2$ is as specified in (91).

Now we show that Firm 2 cannot increase its profit by unilaterally changing its price so that (100) does not hold when $p_1$ is as specified in (90).

Lemmas A9 and A10 imply that if Firm 2 sets $p_2 > p_1 + G_2 - G_1 + K_1$, then no users purchase a phone from Firm 2, so Firm 2's profit (0) is less than the profit specified in (97).

If Firm 2 sets $p_2 = p_1 + G_2 - G_1 + K_1$ when $p_1$ is as specified in (90), then:

$$
\begin{aligned}
p_2 &= p_1 + G_2 - G_1 + K_1 = c_1 - r_L + t - A + G_2 - G_1 + K_1 \\
&= G_2 + r_H - c_2 - G_1 - r_L + c_2 + t - A + K_1 + c_2 - r_H \\
&= 3A + t - A + K_1 + c_2 - r_H = 2A + t + K_1 + c_2 - r_H > c_2 - r_H.
\end{aligned}
$$

The last inequality holds here because $K_1 > -A$ and $t > -A$, by assumption. Because $p_2 = p_1 + G_2 - G_1 + K_1 > c_2 - r_H$ when $p_1$ is as specified in (90), the proof of Lemma A20 implies that Firm 2 can increase its profit by setting $p_2$ to ensure $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$. Therefore, Firm 2 cannot increase its profit by setting $p_2 = p_1 + G_2 - G_1 + K_1$ when $p_1$ is as specified in (90).

Lemmas A7 and A8 imply that if Firm 2 sets $p_2 < p_1 + G_2 - G_1 - K_2$, then all users purchase a phone from Firm 2. (4) and (5) imply that the maximum profit Firm 2 can secure by setting such a price when $p_1$ is as specified in (90) is nearly:

$$\begin{aligned}
\pi_{2D} &= p_1 + G_2 - G_1 - K_2 + r_2 - c_2 \;=\; c_1 - r_L + t - A + G_2 - G_1 - K_2 + r_2 - c_2 \\
&= G_2 - c_2 - (G_1 + r_L - c_1) - A + t + r_2 - K_2 \\
&= G_2 + r_H - c_2 - (G_1 + r_L - c_1) - A + t + r_2 - r_H - K_2 \\
&= 3A - A + t + r_2 - r_H - K_2 \\
&= 2A + t + \frac{1}{2}[r_H + r_L] + \frac{K_2}{2t}[r_H - r_L] - r_H - K_2 \\
&= 2A + t - K_2 - \frac{1}{2}[r_H - r_L] + \frac{K_2}{2t}[r_H - r_L] \\
&= 2A + t - K_2 - \frac{1}{2t}[r_H - r_L][t - K_2] \qquad (104) \\
&= 2A + [t - K_2]\left[1 - \frac{1}{2t}(r_H - r_L)\right] \;=\; 2A + \frac{t - K_2}{2t}[2t - r_H + r_L]. \qquad (105)
\end{aligned}$$

(97) and (105) imply that Firm 2 cannot increase its profit by setting $p_2 < p_1 + G_2 - G_1 - K_2$ when $p_1$ is as specified in (90) if Condition 1B holds.

If Firm 2 sets $p_2 = p_1 + G_2 - G_1 - K_2$ when $p_1$ is as specified in (90), (7) and Lemma A5 imply that: (i) all users located in $[0, \frac{1}{2} - \frac{K_2}{2t}]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\frac{1}{2} - \frac{K_2}{2t}, 1]$ buy a phone from Firm 2. Therefore, (37) and Lemmas A1, A3, and A5 imply that Firm 2's profit is:

$$\pi_2 = [p_2 + r_H - c_2]\left[\frac{1}{2} + \frac{K_2}{2t}\right] + [p_2 + r_L - c_2]\frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2t}\right]. \qquad (106)$$

If $p_2 = p_1 + G_2 - G_1 - K_2$ and $p_1$ is as specified in (90):

$$\begin{aligned}
p_2 + r_H - c_2 &= p_1 + G_2 - G_1 - K_2 + r_H - c_2 \\
&= c_1 - r_L + t - A + G_2 - G_1 - K_2 + r_H - c_2 \\
&= G_2 + r_H - c_2 - G_1 - r_L + c_1 + t - A - K_2 \\
&= 3A + t - A - K_2 \;=\; 2A + t - K_2 \qquad (107)
\end{aligned}$$

$$\Rightarrow \quad p_2 + r_L - c_2 \;=\; p_2 + r_H - c_2 - r_H + r_L \;=\; 2\,A + t - K_2 - r_H + r_L\,. \tag{108}$$

(107) implies that if Firm 2 is to secure positive profit under the presumed deviation, it must be the case that $2\,A + t - K_2 > 0$.

Initially suppose $p_2 + r_L - c_2 \;=\; 2\,A + t - K_2 - r_H + r_L > 0$. Then (104) – (108) imply:

$$\begin{aligned}
\pi_2 \;&=\; [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 + r_L - c_2\,]\,\frac{1}{2}\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&<\; [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&=\; [\,2\,A + t - K_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,2\,A + t - K_2 - r_H + r_L\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&=\; 2\,A + t - K_2 \; - [\,r_H - r_L\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&=\; 2\,A + t - K_2 - \frac{1}{2\,t}\,[\,r_H - r_L\,]\,[\,t - K_2\,] \;=\; \pi_{2D} \;<\; \frac{1}{2\,t}\,[\,t + A\,]^2\,.
\end{aligned} \tag{109}$$

(109) implies that Firm 2 cannot increase its profit by undertaking the proposed deviation in this case.

Now suppose $p_2 + r_L - c_2 \;=\; 2\,A + t - K_2 - r_H + r_L \;\leq\; 0$. (106) implies that Firm 2's profit is maximized in this case when $p_2 + r_L - c_2 \;=\; 2\,A + t - K_2 - r_H + r_L = 0$. (106) and (107) imply that Firm 2's maximum profit in this case is:

$$\begin{aligned}
\pi_2 \;&=\; [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \;=\; [\,2A + t - K_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] \\
&=\; \frac{1}{2\,t}\,[\,2A + t - K_2\,]\,[\,t + K_2\,]\,.
\end{aligned}$$

Observe that:

$$\begin{aligned}
&\frac{1}{2\,t}\,[\,t + A\,]^2 \;>\; \frac{1}{2\,t}\,[\,2A + t - K_2\,]\,[\,t + K_2\,] \\
\Leftrightarrow\;\; & t^2 + 2\,A\,t + A^2 \;>\; 2\,A\,t + t^2 - K_2\,t + 2\,A\,K_2 + K_2\,t - K_2^2 \\
\Leftrightarrow\;\; & A^2 - 2\,A\,K_2 + K_2^2 \;>\; 0 \;\;\Leftrightarrow\;\; [\,A - K_2\,]^2 \;>\; 0\,.
\end{aligned} \tag{110}$$

(97) and (110) imply that Firm 2 cannot increase its profit by undertaking the proposed deviation in this case. ■

**Condition 2A**. $c_2 > r_H$. **Condition 2B**. $\Omega_1 \equiv \frac{1}{2}[r_H - r_L] - 3A - \frac{1}{8t}[t - A]^2 > 0$.

**Condition 2C**. $K_1 \leq \frac{2t}{2t + r_H - r_L} \Omega_1$. **Condition 2D**. $K_1 < c_1 - r_L - 3A$.

**Proposition 2.** *Suppose $K_1 \in (0, t)$, $K_2 \in (0, t)$, $A < 0$, and Conditions 2A – 2D hold. Then an equilibrium exists in which all consumers purchase a phone from Firm 1. At this equilibrium, all consumers located in $(\frac{1}{2} + \frac{K_1}{2t}, 1]$ (and only these consumers) change the default setting on the phone they purchase. Furthermore, $p_1$ is marginally below $c_1 - r_L - 3A - K_1 > 0$; $p_2 = c_2 - r_H > 0$; $\pi_1 \approx \frac{t - K_1}{2t}[r_H - r_L] - 3A - K_1 > 0$; and $\pi_2 = 0$.*

Proof. We first show that the identified $p_1$ maximizes Firm 1's profit when $p_2 = c_2 - r_H$.

Lemmas A9 and A10 imply that if $p_2 = c_2 - r_H$, then among all $p_1 \leq p_2 + G_1 - G_2 - K_1$, the profit-maximizing $p_1$ for Firm 1 is marginally below:

$$
\begin{aligned}
c_2 - r_H + G_1 - G_2 - K_1 &= G_1 + r_L - c_1 - (G_2 + r_H - c_2) + c_1 - r_L - K_1 \\
&= c_1 - r_L - 3A - K_1 > 0 .
\end{aligned}
$$

The inequality here reflects Condition 2D. Lemma A2 implies that Firm 1's corresponding profit is approximately:

$$
\begin{aligned}
\pi_1 &= [p_1 + r_L - c_1]\left[\frac{1}{2} + \frac{K_1}{2t}\right] + [p_1 + r_H - c_1]\left[\frac{1}{2} - \frac{K_1}{2t}\right] \\
&= p_1 - c_1 + r_L\left[\frac{1}{2} + \frac{K_1}{2t}\right] + r_H\left[\frac{1}{2} - \frac{K_1}{2t}\right] \\
&= p_1 - c_1 + \frac{1}{2}[r_L + r_H] - \frac{K_1}{2t}[r_H - r_L] \\
&= G_1 - G_2 + c_2 - r_H - K_1 - c_1 + \frac{1}{2}[r_L + r_H] - \frac{K_1}{2t}[r_H - r_L] \\
&= G_1 - G_2 + c_2 - c_1 - \frac{1}{2}[r_H - r_L] - K_1\left[\frac{2t + r_H - r_L}{2t}\right] \qquad (111) \\
&= G_1 + r_L - c_1 - (G_2 + r_H - c_2) + \frac{1}{2}[r_H - r_L] - K_1 - K_1\left[\frac{r_H - r_L}{2t}\right] \\
&= \left[\frac{t - K_1}{2t}\right][r_H - r_L] - 3A - K_1 .
\end{aligned}
$$

The expression in (111) is strictly positive because $K_1 \leq \frac{2t}{2t + r_H - r_L} \Omega_1$ (from Condition 2C), and because Condition 2B implies:

$$
G_1 - G_2 + c_2 - c_1 - \frac{1}{2}[r_H - r_L] = G_1 + r_L - c_1 - (G_2 + r_H - c_2) + \frac{1}{2}[r_H - r_L]
$$

$$= \frac{1}{2}[r_H - r_L] - 3A \geq \Omega_1 > 0.$$

We now show that Firm 1 cannot increase its profit by setting $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$ or $p_1 \geq p_2 + G_1 - G_2 + K_2$ when $p_2 = c_2 - r_H$.

(87) implies that when $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$, the price that maximizes Firm 1's profit when $p_2 = c_2 - r_H$ is:

$$p_1 = \frac{1}{2}[t + c_1 - r_L + G_1 - G_2 + p_2] = \frac{1}{2}[t + G_1 - G_2 + c_1 + c_2 - r_H - r_L]. \quad (112)$$

(86) and (112) Firm 1's corresponding profit is:

$$\begin{aligned}
\pi_1^{'} &= [p_1 + r_L - c_1]\frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \\
&= \frac{1}{2}[t + G_1 - G_2 + c_1 + c_2 - r_H - r_L + 2r_L - 2c_1] \\
&\qquad \cdot \frac{1}{4t}[2t + 2G_1 - 2G_2 - G_1 + G_2 - c_1 + c_2 - r_H + r_L - t] \\
&= \frac{1}{8t}[t + G_1 - G_2 + c_2 - c_1 - r_H + r_L]^2. \quad (113)
\end{aligned}$$

(111) and (113) imply that Firm 1 cannot increase its profit by setting $p_1 \in (p_2 + G_1 - G_2 - K_1, p_2 + G_1 - G_2 + K_2)$ because:

$$\begin{aligned}
\pi_1 \geq \pi_1^{'} \;\Leftrightarrow\; & G_1 - G_2 + c_2 - c_1 - \frac{1}{2}[r_H - r_L] - K_1\left[\frac{2t + r_H - r_L}{2t}\right] \\
& \geq \frac{1}{8t}[t + G_1 - G_2 + c_2 - c_1 - r_H + r_L]^2 \\
\Leftrightarrow\; K_1 \leq\; & \frac{2t}{2t + r_H - r_L}[G_1 - G_2 + c_2 - c_1 - \frac{1}{2}(r_H - r_L) \\
& \qquad - \frac{1}{8t}(t + G_1 - G_2 + c_2 - c_1 - r_H + r_L)^2] \\
\Leftrightarrow\; K_1 \leq\; & \frac{2t}{2t + r_H - r_L}\Omega_1. \quad (114)
\end{aligned}$$

Lemmas A7 and A8 imply if Firm 1 sets price $p_1 > p_2 + G_1 - G_2 + K_2$, it will sell no phones, and so will make 0 profit. Therefore, among all $p_1 \geq p_2 + G_1 - G_2 + K_2$, the price that maximizes Firm 1's profit is $p_1 = p_2 + G_1 - G_2 + K_2$. When $p_2 = c_2 - r_H$, this price is $p_1 = c_2 - r_H + G_1 - G_2 + K_2$. (7) and Lemma A5 imply that when $p_1 = p_2 + G_1 - G_2 + K_2$: (i) all users located in $[0, \frac{1}{2} - \frac{K_2}{2t}]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\frac{1}{2} - \frac{K_2}{2t}, 1]$ buy a phone from Firm 2. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\begin{aligned}
\widetilde{\widetilde{\pi}}_1 &= \frac{1}{2}\,[\,p_1 + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&= \frac{1}{2}\,[\,G_1 - G_2 - c_1 + c_2 - r_H + r_L + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right]. \qquad (115)
\end{aligned}$$

If $G_1 - G_2 - c_1 + c_2 - r_H + r_L + K_2 \leq 0$, then $\widetilde{\widetilde{\pi}}_1 \leq 0$. Therefore, Firm 1 will never set $p_1 \geq p_2 + G_1 - G_2 + K_2$ when $p_2 = c_2 - r_H$ in this case.

If $G_1 - G_2 - c_1 + c_2 - r_H + r_L + K_2 > 0$, then $\widetilde{\widetilde{\pi}}_1 > 0$. In this case, if Firm 1 were to reduce its price to $p_1 = c_2 - r_H + G_1 - G_2 + K_2 - \varepsilon_7$ where $\varepsilon_7 > 0$, all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ would purchase a phone from Firm 1. Consequently, Firm 1's profit would be at least:

$$\begin{aligned}
\pi_1 &= [\,p_1 - \varepsilon_7 + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&= [\,G_1 - G_2 - c_1 + c_2 - r_H + r_L + K_2 - \varepsilon_7\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&= [\,G_1 - G_2 - c_1 + c_2 - r_H + r_L + K_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] - \varepsilon_7\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&= \widetilde{\widetilde{\pi}}_1 + \widetilde{\widetilde{\pi}}_1 - \varepsilon_7\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] > \widetilde{\widetilde{\pi}}_1 \quad \text{for sufficiently small } \varepsilon_7. \qquad (116)
\end{aligned}$$

(116) implies that Firm 1 could increase its profit by reducing its price marginally below $p_2 + G_1 - G_2 + K_2$ when $p_2 = c_2 - r_H$. Therefore, Firm 1 will never set $p_1 \geq p_2 + G_1 - G_2 + K_2$ when $p_2 = c_2 - r_H$.

In summary, we have established that when $p_2 = c_2 - r_H$, Firm 1 maximizes its profit by setting $p_1$ marginally below $c_2 - r_H + G_1 - G_2 - K_1$.

We now show that when Firm 1 sets $p_1$ marginally below $c_2 - r_H + G_1 - G_2 - K_1$, Firm 2 maximizes its profit by setting $p_2 = c_2 - r_H$. Lemmas A9 and A10 imply that when Firm 1 sets $p_1$ marginally below $c_2 - r_H + G_1 - G_2 - K_1$, Firm 2 attracts no users (and so secures no profit) if it sets $p_2 = c_2 - r_H$. Firm 2 continues to attract no users (and so continues to secure no profit) if it sets $p_2 > c_2 - r_H$. Firm 2 incurs negative profit if it sets $p_2 < c_2 - r_H$. Therefore, Firm 2 cannot increase its profit by setting $p_2 \neq c_2 - r_H$ when Firm 1 sets $p_1$ marginally below $c_2 - r_H + G_1 - G_2 - K_1$.

Finally, Lemma A2 implies that all users located in the interval $\left[\,\frac{1}{2} + \frac{K_1}{2\,t},\, 1\,\right]$ (and only these users) change the default setting on the phone they purchase from Firm 1. ■

**Condition 3A**. $c_1 > r_1$. **Condition 3B**. $K_2 < G_2 - G_1 + c_1 - r_1$.

**Condition 3C**. $K_2 \left[ \frac{2t - r_H + r_L}{2t} \right] < \Omega_2(p_1)$

for all $p_1 \in [\, c_1 - r_1, \, min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + K_2 \, \} \, ]$,

where $\Omega_2(p_1) \equiv p_1 + G_2 - G_1 - c_2 + \frac{1}{2} [\, r_H + r_L \,] - x_2(p_1)$, and

$x_2(p_1) \equiv \frac{1}{8t} [\, t + G_2 - G_1 - c_2 + r_H + p_1 \,]^2 \geq 0$.

**Proposition 3.** *Suppose $K_1 \in (0, t)$, $K_2 \in (0, t)$, $A > 0$, and Conditions 3A – 3C hold. Then a family of equilibria exist in which all consumers purchase a phone from Firm 2. In each of these equilibria, all consumers located in $[\, 0, \, \frac{1}{2} - \frac{K_2}{2t} \,)$ (and only these consumers) change the default setting on the phone they purchase. Furthermore, $p_1 \in [\, c_1 - r_1, \, min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + K_2 \, \} \, ] > 0$; $p_2$ is marginally below $p_1 + G_2 - G_1 - K_2 > 0$; $\pi_1 = 0$; and $\pi_2 \approx p_1 - c_2 + G_2 - G_1 - K_2 + \frac{1}{2} [\, r_H + r_L \,] + \frac{K_2}{2t} [\, r_H - r_L \,] > 0$.*

<u>Proof</u>. Lemmas A7 and A8 imply that if $p_2$ is marginally below $p_1 + G_2 - G_1 - K_2$, then all users buy a phone from Firm 2. Therefore, Firm 1's profit is 0.

Among all values of $p_2$ below $p_1 + G_2 - G_1 - K_2$, the value of $p_2$ that is most profitable for Firm 2 is marginally below:

$$p_2 = p_1 + G_2 - G_1 - K_2 \,. \tag{117}$$

We first show that if Firm 2 sets $p_2$ marginally below $p_1 + G_2 - G_1 - K_2$, then Firm 1 will set $p_1 \in [\, c_1 - r_1, \, min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + K_2 \, \} \, ]$ in equilibrium. We do so by first explaining why it cannot be the case that $p_1 < c_1 - r_1$ or $p_1 > min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + K_2 \, \}$ in equilibrium. Then we explain why, when $p_1 \in [\, c_1 - r_1, \, min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + K_2 \, \} \, ]$ and Firm 2 sets $p_2$ is marginally below $p_1 + G_2 - G_1 - K_2$, Firm 1 cannot strictly increase its profit by setting a different price.

Lemma A14 implies that setting $p_1 < c_1 - r_1$ is a dominated strategy for Firm 1.

Consider a putative equilibrium in which $p_1 > c_1 - r_1 + K_1 + K_2$ and Firm 2 sets $p_2$ marginally below the $p_2$ in (117). (117) implies:

$$p_2 > c_1 - r_1 + K_1 + K_2 + G_2 - G_1 - K_2 = G_2 - G_1 + K_1 + c_1 - r_1 \,. \tag{118}$$

Lemmas A9 and A10 imply that if Firm 1 sets $p_1$ marginally below $p_2 + G_1 - G_2 - K_1$, all users will purchase a phone from Firm 1. Consequently, (117) and (118) imply that Firm 1's profit will be nearly:

$$\pi_1 = p_1 + r_1 - c_1 = p_2 + G_1 - G_2 - K_1 + r_1 - c_1$$

$$> G_2 - G_1 + K_1 + c_1 - r_1 + G_1 - G_2 - K_1 + r_1 - c_1 = 0 \,.$$

Because Firm 1 can secure strictly positive profit by deviating from its strategy in the putative equilibrium, the putative equilibrium cannot be an equilibrium.

Next, consider a putative equilibrium in which $p_1 > c_1 - r_L$ and Firm 2 sets $p_2$ marginally below the $p_2$ in (117). (117) implies:

$$p_2 > c_1 - r_L + G_2 - G_1 - K_2 \, . \tag{119}$$

Lemmas A7 and A8 imply all users buy a phone from Firm 2 when (119) holds. Therefore, Firm 1's profit is 0.

Suppose Firm 1 reduces its price to:

$$p_1' = p_2 + G_1 - G_2 + K_2 > c_1 - r_L + G_2 - G_1 - K_2 + G_1 - G_2 + K_2 = c_1 - r_L .$$

(7) and Lemma A5 imply that when Firm 1 sets price $p_1'$: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $(\,\frac{1}{2} - \frac{K_2}{2\,t}, 1\,]$ buy a phone from Firm 2. Therefore, Lemma A2 implies that Firm 1's profit is:

$$\pi_1' = \frac{1}{2}\,[\,p_1' + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] > 0 \text{ (because } p_1' > c_1 - r_L).$$

Because Firm 1 can secure strictly positive profit by deviating from its strategy in the putative equilibrium, the putative equilibrium cannot be an equilibrium.

We now show that when $p_1 \in [\,c_1 - r_1, \, c_1 - r_L\,]$ and Firm 2 sets $p_2$ marginally below the $p_2$ in (117), Firm 1 cannot increase its profit by setting a price $p_1' \in (\,p_2 + G_1 - G_2 - K_1, \; p_2 + G_1 - G_2 + K_2\,)$. Lemma A11 implies that when Firm 1 sets such a price: (i) all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2\,t}\,)$ strictly prefer to buy a phone from Firm 1 than from Firm 2; (ii) all users located in $(\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ strictly prefer to buy a phone from Firm 2 than from Firm 1; and (iii) some user located in $[\,\frac{1}{2} - \frac{K_2}{2\,t}, \frac{1}{2} + \frac{K_1}{2\,t}\,]$ is indifferent between buying a phone from Firm 1 and from Firm 2. Furthermore, Lemmas A3 and A4 imply that Firm 1's profit is:

$$\pi_1 = [\,p_1' + r_L - c_1\,]\, x_0 \leq 0 \, , \text{ where } x_0 \text{ is given by (6).} \tag{120}$$

The inequality in (120) holds because $p_1' \leq c_1 - r_L$ and because (6) implies that $x_0 > 0$ in the present setting. (120) implies that Firm 1 cannot secure strictly positive profit by setting $p_1' \in (\,p_2 + G_1 - G_2 - K_1, \; p_2 + G_1 - G_2 + K_2\,)$ under the maintained conditions.

Next we show that Firm 1 cannot increase its profit by setting a price $p_1' \leq p_2 + G_1 - G_2 - K_1$. Observe that under the maintained conditions:

$$c_1 - r_1 + K_1 + K_2 \geq p_1 > p_2 + G_1 - G_2 + K_2 \tag{121}$$

$$\Rightarrow \quad c_1 - r_1 > p_2 + G_1 - G_2 - K_1 \tag{122}$$

The weak inequality in (121) holds because $p_1 \in [\,c_1 - r_1, \, min\,\{\,c_1 - r_L, \, c_1 - r_1 + K_1 + K_2\,\}\,]$. The strict inequality in (121) holds because $p_2$ is marginally below $p_1 + G_2 - G_1 - K_2$. (122) implies that $p_1' < c_1 - r_1$ if Firm 1 sets $p_1' \leq p_2 + G_1 - G_2 - K_1$. Lemma A14 implies that this is a dominated strategy for Firm 1.

Now we show that Firm 1 cannot increase its profit by setting a price $p_1' > p_2 + G_1 - G_2 + K_2$. Lemmas A7 and A8 imply that no user will purchase a phone from Firm 1 in this case. Consequently, Firm 1's profit is 0.

It remains to show that Firm 1 cannot increase its profit by setting a price $p_1' = p_2 + G_1 - G_2 + K_2$. Because Firm 2 sets $p_2$ marginally below the $p_2$ in (117):

$$p_1' \;\approx\; p_1 + G_2 - G_1 - K_2 + G_1 - G_2 + K_2 \;=\; p_1\,. \tag{123}$$

Lemmas A1, A2, and A5 imply that when Firm 1 sets $p_1' = p_2 + G_1 - G_2 + K_2$, all users located in $[\,0, \frac{1}{2} - \frac{K_2}{2t}\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2, whereas all users located in $(\,\frac{1}{2} - \frac{K_2}{2t}, 1\,]$ purchase a phone from Firm 2. Consequently, (123) implies that Firm 1's profit in this case is:

$$\pi_1 \;=\; \frac{1}{2}\,[\,p_1' + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \;\approx\; \frac{1}{2}\,[\,p_1 + r_L - c_1\,]\left[\frac{1}{2} - \frac{K_2}{2t}\right] \;\leq\; 0\,. \tag{124}$$

The inequality in (124) holds because $p_1 \leq c_1 - r_L$.

In summary, we have shown that if Firm 2 sets $p_2$ marginally below $p_1 + G_2 - G_1 - K_2$, then Firm 1 will set $p_1 \in [\,c_1 - r_1, \, min\,\{\,c_1 - r_L, \, c_1 - r_1 + K_1 + K_2\,\}\,]$ in equilibrium.

We now show that Firm 2 maximizes its profit by setting $p_2$ marginally below $p_1 + G_2 - G_1 - K_2$ when Firm 1 sets $p_1 \in [\,c_1 - r_1\,, \, min\,\{\,c_1 - r_L, \, c_1 - r_1 + K_1 + K_2\,\}\,]$. Observe first that this value of $p_2$ is positive because:

$$p_1 + G_2 - G_1 - K_2 \;\geq\; c_1 - r_1 + G_2 - G_1 - K_2 \;>\; 0\,.$$

The inequality here reflects Condition 3B.

When Firm 2 sets $p_2$ marginally below $p_1 + G_2 - G_1 - K_2$, all users purchase a phone from Firm 2. Lemma A1 implies that Firm 2's profit is:

$$\begin{aligned}
\pi_2 \;&=\; [\,p_2 + r_H - c_2\,]\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + [\,p_2 + r_L - c_2\,]\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&=\; p_2 - c_2 + r_H\left[\frac{1}{2} + \frac{K_2}{2\,t}\right] + r_L\left[\frac{1}{2} - \frac{K_2}{2\,t}\right] \\
&=\; p_2 - c_2 + \frac{1}{2}\,[\,r_L + r_H\,] + \frac{K_2}{2\,t}\,[\,r_H - r_L\,] \\
&\approx\; p_1 + G_2 - G_1 - K_2 - c_2 + \frac{1}{2}\,[\,r_L + r_H\,] + \frac{K_2}{2\,t}\,[\,r_H - r_L\,]\,.
\end{aligned} \tag{125}$$

In equilibria in which $p_1 \in [\,c_1 - r_1\,, \, min\,\{\,c_1 - r_L, \, c_1 - r_1 + K_1 + K_2\,\}\,]$ and Firm 2 sets $p_2$ is marginally below $p_1 + G_2 - G_1 - K_2$, Firm 2 secures the least profit when $p_1 = c_1 - r_1$.

Consequently, (4) and (125) imply that Firm 2 earns positive profit in all such equilibria if:

$$\begin{aligned}
\pi_2^{min} &= c_1 - r_1 + G_2 - G_1 - K_2 - c_2 + \frac{1}{2}\left[ r_L + r_H \right] + \frac{K_2}{2\,t}\left[ r_H - r_L \right] \\
&= G_2 - G_1 + c_1 - \frac{1}{2}\left[ r_L + r_H \right] + \frac{K_1}{2\,t}\left[ r_H - r_L \right] \\
&\qquad - \; K_2 - c_2 + \frac{1}{2}\left[ r_L + r_H \right] + \frac{K_2}{2\,t}\left[ r_H - r_L \right] \\
&= G_2 - G_1 + c_1 - c_2 - K_2 + \left[ \frac{K_1 + K_2}{2\,t} \right]\left[ r_H - r_L \right] \\
&= G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\left[ r_H - r_L \right] - K_2 \left[ \frac{2\,t - (r_H - r_L)}{2\,t} \right]. \qquad (126)
\end{aligned}$$

(4) and Condition 3C imply that when $p_1 = c_1 - r_1$:

$$\begin{aligned}
x_2 &= \frac{1}{8\,t}\left[ t + G_2 - G_1 - c_2 + r_H + c_1 - r_1 \right]^2 \\
&= \frac{1}{8\,t}\left[ t + G_2 - G_1 - c_2 + r_H + c_1 - \frac{1}{2}\left( r_H + r_L \right) + \frac{K_1}{2\,t}\left( r_H - r_L \right) \right]^2 \\
&= \frac{1}{8\,t}\left[ t + G_2 - G_1 + c_1 - c_2 + \left( \frac{1}{2} + \frac{K_1}{2\,t} \right)\left( r_H - r_L \right) \right]^2. \qquad (127)
\end{aligned}$$

(4) and Condition 3C imply that when $p_1 = c_1 - r_1$:

$$\begin{aligned}
\Omega_2(\cdot) &= c_1 - r_1 + G_2 - G_1 - c_2 + \frac{1}{2}\left[ r_H + r_L \right] - x_2 \\
&= c_1 - \frac{1}{2}\left[ r_H + r_L \right] + \frac{K_1}{2\,t}\left[ r_H - r_L \right] + G_2 - G_1 - c_2 + \frac{1}{2}\left[ r_H + r_L \right] - x_2 \\
&= G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\left[ r_H - r_L \right] - x_2 \,. \qquad (128)
\end{aligned}$$

(126) – (128) imply that when $p_1 = c_1 - r_1$ and Condition 3C holds:

$$\begin{aligned}
\pi_2^{min} &> G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\left[ r_H - r_L \right] - \Omega_2 \\
&= G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\left[ r_H - r_L \right] \\
&\qquad - \left( G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\left[ r_H - r_L \right] - x_2 \right) = x_2 \;\geq\; 0\,. \qquad (129)
\end{aligned}$$

We now show that Firm 2 cannot increase its profit by setting $p_2 \in (p_1 + G_2 - G_1 - K_2,\ p_1 + G_2 - G_1 + K_1)$ or $p_2 \geq p_1 + G_2 - G_1 + K_1$ when Firm 1 sets $p_1 \in [c_1 - r_1,\ min\{c_1 - r_L,\ c_1 - r_1 + K_1 + K_2\}]$.

(89) and Lemma A11 imply that when $p_2 \in (p_1 + G_2 - G_1 - K_2,\ p_1 + G_2 - G_1 + K_1)$, the price that maximizes Firm 2's profit is:

$$p_2 = \frac{1}{2}[t + c_2 - r_H + G_2 - G_1 + p_1]. \tag{130}$$

(88) and (130) imply that Firm 2's corresponding profit is:

$$\begin{aligned}
\pi_2^{'} &= [p_2 + r_H - c_2]\frac{1}{2t}[t + G_2 - G_1 + p_1 - p_2] \\
&= \left[\frac{1}{2}(t + c_2 - r_H + G_2 - G_1 + p_1) + r_H - c_2\right] \\
&\qquad \cdot \frac{1}{2t}\left[t + G_2 - G_1 + p_1 - \frac{1}{2}(t + c_2 - r_H + G_2 - G_1 + p_1)\right] \\
&= \frac{1}{8t}[t + G_2 - G_1 - c_2 + r_H + p_1]^2. \qquad (131)
\end{aligned}$$

(129) and (131) imply that Firm 2 cannot increase its profit by setting $p_2 \in (p_1 + G_2 - G_1 - K_2,\ p_1 + G_2 - G_1 + K_1)$ when Condition 3C holds because:

$$\begin{aligned}
\pi_2 \geq \pi_2^{'} \;\Leftrightarrow\; & p_1 + G_2 - G_1 - K_2 - c_2 + \frac{1}{2}[r_L + r_H] + \frac{K_2}{2t}[r_H - r_L] \\
& \geq \frac{1}{8t}[t + G_2 - G_1 - c_2 + r_H + p_1]^2 \\
\Leftrightarrow\; K_2\left[\frac{2t - r_H + r_L}{2t}\right] \leq\; & p_1 + G_2 - G_1 - c_2 + \frac{1}{2}[r_L + r_H] \\
& - \frac{1}{8t}[t + G_2 - G_1 - c_2 + r_H + p_1]^2 = \Omega_2.
\end{aligned}$$

Lemmas A9 and A10 imply that if Firm 2 sets $p_2 > p_1 + G_2 - G_1 + K_1$, it will sell no phones and so will secure 0 profit. Therefore, $p_2 = p_1 + G_2 - G_1 + K_1$ is the profit-maximizing price for Firm 2 among all $p_2 \geq p_1 + G_2 - G_1 + K_1$. (9) and Lemma A6 imply that when $p_2 = p_1 + G_2 - G_1 + K_1$: (i) all users located in $[\frac{1}{2} + \frac{K_1}{2t}, 1]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[0, \frac{1}{2} + \frac{K_1}{2t})$ buy a phone from Firm 1. Therefore, Lemma A1 implies that Firm 2's profit is:

$$\pi_2^{''} = \frac{1}{2}[p_2 + r_H - c_2]\left[\frac{1}{2} - \frac{K_1}{2t}\right]$$

$$= \frac{1}{2}\,[\, p_1 + G_2 - G_1 + K_1 + r_H - c_2 \,] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \;>\; 0\,. \tag{132}$$

The inequality in (132) holds because (4) implies that the minimum value of $\pi_2^{''}$, which occurs when $p_1 = c_1 - r_1$, is:

$$\pi_2^{''min} \;=\; \frac{1}{2}\,[\, c_1 - r_1 + G_2 - G_1 + K_1 + r_H - c_2 \,] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right]$$

$$= \frac{1}{2} \left[ G_2 - G_1 + c_1 - c_2 + K_1 + \left( \frac{1}{2} + \frac{K_1}{2\,t} \right) (\, r_H - r_L \,) \right] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \;>\; 0\,. \tag{133}$$

The inequality in (133) holds because: (i) $\frac{K_1}{2\,t} < \frac{1}{2}$, by assumption; and (ii) the term in the first square brackets in (133) is positive. (ii) holds because Condition 3B ensures this term exceeds:

$$K_2 + K_1 + \left[ \frac{1}{2} + \frac{K_1}{2\,t} \right] [\, r_H - r_L \,] \;>\; 0\,.$$

(133) ensures that (132) holds.

If Firm 2 were to reduce its price to $p_2 = p_1 + G_2 - G_1 + K_1 - \varepsilon_8$ where $\varepsilon_8 > 0$, all users located in $[\, \frac{1}{2} + \frac{K_1}{2\,t}, 1 \,]$ would purchase a phone from Firm 2. Consequently, (132) implies that Firm 2's profit would be at least:

$$\begin{aligned}
\pi_2 \;&=\; [\, p_2 - \varepsilon_8 + r_H - c_2 \,] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \\
&=\; [\, p_1 + G_2 - G_1 + K_1 - \varepsilon_8 + r_H - c_2 \,] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \\
&=\; [\, p_1 + G_2 - G_1 + K_1 + r_H - c_2 \,] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] - \varepsilon_8 \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \\
&=\; \pi_2^{''} + \pi_2^{''} - \varepsilon_8 \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \;>\; \pi_2^{''} \quad \text{for sufficiently small } \varepsilon_8\,.
\end{aligned} \tag{134}$$

(134) implies that Firm 2 could increase its profit by reducing its price marginally below $p_1 + G_2 - G_1 + K_1$. Therefore, Firm 2 will never set $p_2 \geq p_1 + G_2 - G_1 + K_1$ when $p_1 \in [\, c_1 - r_1\,,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + K_2 \,\}\,]$.

In summary, we have shown that when $p_1 \in [\, c_1 - r_1\,,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + K_2 \,\}\,]$, Firm 2 maximizes its profit by setting $p_2$ marginally below $p_1 + G_2 - G_1 - K_2$.

Finally, Lemma A1 implies that all users located in the interval $[\, 0, \frac{1}{2} - \frac{K_2}{2\,t} \,)$ (and only these users) change the default setting on the phone they purchase from Firm 2. ■

**Proposition 4.** *(i)* $\frac{\partial \pi_1^{D1}(\cdot)}{\partial K_1} < 0$ *and* $\frac{\partial \pi_1^{D1}(\cdot)}{\partial K_2} = 0$ *in the MD1 equilibrium characterized in Proposition 2; (ii)* $\frac{\partial \underline{\pi}_2^{D2}(\cdot)}{\partial K_2} \gtreqless 0 \Leftrightarrow r_H - r_L \gtreqless 2t$ *and* $\frac{\partial \underline{\pi}_2^{D2}(\cdot)}{\partial K_1} > 0$ *in the MD2 equilibria characterized in Proposition 3; and (iii)* $\frac{\partial \pi_1^S(\cdot)}{\partial K_1} = \frac{\partial \pi_1^S(\cdot)}{\partial K_2} = \frac{\partial \pi_2^S(\cdot)}{\partial K_2} = \frac{\partial \pi_2^S(\cdot)}{\partial K_1} = 0$ *in the MS equilibrium characterized in Proposition 1.*

Proof. Conclusion (i) follows from Proposition 2, which implies that:

$$\pi_1^{D1}(\cdot) \;\approx\; G_1 - G_2 + c_2 - c_1 - \frac{1}{2}\left[ r_H - r_L \right] - K_1 \left[ \frac{2t + r_H - r_L}{2t} \right]$$

$$\Rightarrow \quad \frac{\partial \pi_1^{D1}(\cdot)}{\partial K_1} \;=\; -1 - \left[ \frac{r_H - r_L}{2t} \right] \;<\; 0 \;\text{ and }\; \frac{\partial \pi_1^{D1}(\cdot)}{\partial K_2} \;=\; 0\,. \tag{135}$$

Conclusion (ii) follows from Proposition 3, which states that:

$$\underline{\pi}_2^{D2}(\cdot) \;\approx\; c_1 - r_1 - c_2 + G_2 - G_1 - K_2 + \frac{1}{2}\left[ r_L + r_H \right] + \frac{K_2}{2t}\left[ r_H - r_L \right]$$

$$\Rightarrow \quad \frac{\partial \underline{\pi}_2^{D2}(\cdot)}{\partial K_2} \;=\; -1 + \frac{r_H - r_L}{2t} \;\gtreqless\; 0 \;\Leftrightarrow\; r_H - r_L \;\gtreqless\; 2t\,; \;\text{ and} \tag{136}$$

$$\frac{\partial \underline{\pi}_2^{D2}(\cdot)}{\partial K_1} \;=\; -\frac{\partial r_1}{\partial K_1} \;=\; \frac{1}{2t}\left[ r_H - r_L \right] \;>\; 0\,. \tag{137}$$

Conclusion (iii) follows directly from Lemma 1 and Proposition 1. ■

Definitions.

$$|A|_{1L} \;\equiv\; 2t - \sqrt{3t^2 + 2t\Delta}\,. \qquad |A|_{1H} \;\equiv\; 2t + \sqrt{3t^2 + 2t\Delta}\,.$$

$$A_{2L} \;\equiv\; 2t - \sqrt{3t^2 - 2t\Delta}\,. \qquad A_{2H} \;\equiv\; 2t + \sqrt{3t^2 - 2t\Delta}\,.$$

$$t_{1L} \;\equiv\; \Delta + 2|A| - \sqrt{3|A|^2 + 4|A|\,\Delta + \Delta^2}\,.\; t_{1H} \;\equiv\; \Delta + 2|A| + \sqrt{3|A|^2 + 4|A|\,\Delta + \Delta^2}\,.$$

$$t_{2L} \;\equiv\; 2A - \Delta - \sqrt{3A^2 - 4A\Delta + \Delta^2}\,.\; t_{2H} \;\equiv\; 2A - \Delta + \sqrt{3A^2 - 4A\Delta + \Delta^2}\,. \tag{138}$$

**Lemma 3.** *Suppose the conditions in Proposition 1 hold. Then* $\pi_1^S > \pi_1^{D1}(0,0)$ *if* $t > t_{1H}$, *whereas* $\pi_1^S < \pi_1^{D1}(0,0)$ *if* $t \in (K_1, t_{1H})$. *Furthermore, if* $t > 2\Delta$, *then* $\pi_1^S > \pi_1^{D1}(0,0)$ *if* $|A| < |A|_{1L}$, *whereas* $\pi_1^S < \pi_1^{D1}(0,0)$ *if* $|A| \in (|A|_{1L}, t)$. *If* $t \leq 2\Delta$, *then* $\pi_1^S < \pi_1^{D1}(0,0)$ *for all* $|A| \in (0,t)$.

Proof. From (5): $\quad A \;<\; 0 \;\Rightarrow\; G_1 + r_L - c_1 \;>\; G_2 + r_H - c_2$

$$\Rightarrow \quad G_1 - c_1 - (\, G_2 - c_2 \,) \; > \; r_H - r_L \; > \; 0\,.$$

Therefore, (5) and Lemma 2 imply:

$$\pi_1^{D1}(0,0) \;=\; G_1 - c_1 - (\, G_2 - c_2 \,) \;=\; G_1 + r_L - c_1 - (\, G_2 + r_H - c_2 \,) + r_H - r_L$$

$$= \; -\,3\,A + \Delta\,. \tag{139}$$

(139) and Proposition 1 imply:

$$\pi_1^S \;\gtreqless\; \pi_1^{D1}(0,0) \;\;\Leftrightarrow\;\; \frac{1}{2\,t}\,[\,t + |\,A\,|\,]^2 \;\gtreqless\; 3\,|\,A\,| + \Delta$$

$$\Leftrightarrow \;\; t^2 + 2\,t\,|\,A\,| + |\,A\,|^2 \;\gtreqless\; 6\,t\,|\,A\,| + 2\,t\,\Delta$$

$$\Leftrightarrow \;\; t^2 - 2\,[\,\Delta + 2\,|\,A\,|\,]\,t + |\,A\,|^2 \;\gtreqless\; 0\,. \tag{140}$$

(138) implies that the ("$t$") roots of the quadratic equation in (140) are given by:

$$t \;=\; \frac{1}{2}\left\{ 2\,[\,\Delta + 2\,|\,A\,|\,] \pm \sqrt{4\,[\,\Delta + 2\,|\,A\,|\,]^2 - 4\,|\,A\,|^2}\,\right\}$$

$$= \; \Delta + 2\,|\,A\,| \pm \sqrt{[\,\Delta + 2\,|\,A\,|\,]^2 - |\,A\,|^2}$$

$$= \; \Delta + 2\,|\,A\,| \pm \sqrt{3\,|\,A\,|^2 + 4\,|\,A\,|\;\Delta + \Delta^2} \;\in\; \{\, t_{1L}, t_{1H} \,\}\,. \tag{141}$$

$t_{1L}$, the smaller root in (141), satisfies the maintained assumption that $t > |\,A\,|$ if and only if:

$$t_{1L} \;>\; |\,A\,| \quad\Leftrightarrow\quad \Delta + 2\,|\,A\,| - \sqrt{3\,|\,A\,|^2 + 4\,|\,A\,|\;\Delta + \Delta^2} \;>\; |\,A\,|$$

$$\Leftrightarrow \quad \Delta + |\,A\,| \;>\; \sqrt{3\,|\,A\,|^2 + 4\,|\,A\,|\;\Delta + \Delta^2}$$

$$\Leftrightarrow \quad \Delta^2 + 2\,|\,A\,|\;\Delta + |\,A\,|^2 \;>\; 3\,|\,A\,|^2 + 4\,|\,A\,|\,\Delta + \Delta^2 \quad\Leftrightarrow\quad 2\,|\,A\,|^2 + 2\,|\,A\,|\,\Delta \;<\; 0\,. \tag{142}$$

This inequality does not hold.

(138) implies that $t_{1H}$ satisfies the maintained assumption that $t > |\,A\,|$ because:

$$t_{1H} \;>\; |\,A\,| \quad\Leftrightarrow\quad \Delta + 2\,|\,A\,| + \sqrt{3\,|\,A\,|^2 + 4\,|\,A\,|\;\Delta + \Delta^2} \;>\; |\,A\,|$$

$$\Leftrightarrow \quad \Delta + |\,A\,| + \sqrt{3\,|\,A\,|^2 + 4\,|\,A\,|\;\Delta + \Delta^2} \;>\; 0\,. \tag{143}$$

(140) – (143) imply that because $t > |\,A\,|$ by assumption, $\pi_1^S - \pi_1^{D1}(0,0)$ is: (i) negative for $t < t_{1H}$; and (ii) positive for $t > t_{1H}$. Formally:

$$\pi_1^S \; > \; \pi_1^{D1}(0,0) \;\text{ if }\; t \; > \; t_{1H} \;\text{ and }\; \pi_1^S \; < \; \pi_1^{D1}(0,0) \;\text{ if }\; t \in (K_1, t_{1H}) \; . \tag{144}$$

The lower bound on $t$ in the second conclusion in (144) (i.e., $K_1$) reflects the maintained assumption that $K_1 < t$.

(139) and Proposition 1 imply:

$$\pi_1^S - \pi_1^{D1}(0,0) \; = \; \frac{1}{2\,t} \, [\, t + |\, A \,| \,]^2 - 3\,|\, A \,| - \Delta \, . \tag{145}$$

(145) implies:

$$\frac{\partial \left( \pi_1^S - \pi_1^{D1}(0,0) \right)}{\partial \,|\, A \,|} \; = \; \frac{1}{t} \, [\, t + |\, A \,| \,] - 3 \; = \; \frac{|\, A \,|}{t} - 2 \; \lesseqgtr \; 0 \;\; \Leftrightarrow \;\; |\, A \,| \; < \; 2\,t$$

$$\Rightarrow \;\; \frac{\partial \left( \pi_1^S - \pi_1^{D1}(0,0) \right)}{\partial \,|\, A \,|} \; < \; 0 \;\text{ for all }\; |\, A \,| \; < \; t \, . \tag{146}$$

The expression in (140) can be written as:

$$|\, A \,|^2 - 4\,t\,|\, A \,| + t^2 - 2\,\Delta\,t \; \gtreqless \; 0 \, .$$

(138) implies that the ("$|\, A \,|$") roots of the associated quadratic equation are:

$$\begin{aligned} |\, A \,| \; &= \; \frac{1}{2} \left[ 4\,t \pm \sqrt{16\,t^2 - 4\,t\,[\,t - 2\,\Delta\,]} \; \right] \; = \; 2\,t \pm \sqrt{4\,t^2 - t\,[\,t - 2\,\Delta\,]} \\ &= \; 2\,t \pm \sqrt{3\,t^2 + 2\,t\,\Delta} \; \in \{\, |\, A \,|_{1L} \,, \, |\, A \,|_{1H} \,\} \, . \end{aligned} \tag{147}$$

(138) implies that:

$$\begin{aligned} |\, A \,|_{1L} \; > \; 0 \;\; &\Leftrightarrow \;\; 2\,t \; > \; \sqrt{3\,t^2 + 2\,t\,\Delta} \;\; \Leftrightarrow \;\; 4\,t^2 \; > \; 3\,t^2 + 2\,t\,\Delta \\ &\Leftrightarrow \;\; t^2 \; > \; 2\,t\,\Delta \;\; \Leftrightarrow \;\; t \; > \; 2\,\Delta \, . \end{aligned} \tag{148}$$

(138) further implies that $|\, A \,|_{1L} < \frac{t}{2}$ because:

$$\begin{aligned} |\, A \,|_{1L} \; < \; \frac{t}{2} \;\; &\Leftrightarrow \;\; 2\,t - \sqrt{3\,t^2 + 2\,t\,\Delta} \; < \; \frac{t}{2} \;\; \Leftrightarrow \;\; \frac{3}{2}\,t \; < \; \sqrt{3\,t^2 + 2\,t\,\Delta} \\ &\Leftrightarrow \;\; \frac{9}{4}\,t^2 \; < \; 3\,t^2 + 2\,t\,\Delta \;\; \Leftrightarrow \;\; \frac{3}{4}\,t^2 + 2\,t\,\Delta \; > \; 0 \, . \end{aligned} \tag{149}$$

(138) also implies that $|\, A \,|_{1H} > t$ because:

$$|\, A \,|_{1H} \; > \; t \;\; \Leftrightarrow \;\; 2\,t + \sqrt{3\,t^2 + 2\,t\,\Delta} \; > \; t \;\; \Leftrightarrow \;\; t + \sqrt{3\,t^2 + 2\,t\,\Delta} \; > \; 0. \tag{150}$$

(140) and (146) – (150) imply that $\pi_1^S - \pi_1^{D1}(0,0)$ is a decreasing function of $|A|$ for $|A| < t$. Furthermore, this function is: (i) positive when $|A| < |A|_{1L}$ (which can occur

if and only if $t > 2\Delta$); and (ii) negative for $|A| \in (|A|_{1L}, t)$ (because $t > |A|_{1H}$, by assumption). Formally:

$$\text{If } t > 2\Delta, \text{ then } \pi_1^S \begin{cases} > \pi_1^{D1}(0,0) & \text{if } |A| < |A|_{1L} \\ < \pi_1^{D1}(0,0) & \text{if } |A| \in (|A|_{1L}, t). \end{cases}$$

$$\text{If } t \le 2\Delta, \text{ then } \pi_1^S < \pi_1^{D1}(0,0) \text{ for all } |A| \in (0, t). \ \blacksquare \tag{151}$$

**Lemma 4.** *Suppose the conditions in Proposition 1 hold and* $A > \Delta$. *Then: (i)* $\pi_2^S > \pi_2^{D2}(0,0)$ *if* $t > t_{2H}$ *or* $A < A_{2L}$ *; and (ii)* $\pi_2^S < \pi_2^{D2}(0,0)$ *if* $t \in (K_2, t_{2H})$ *or* $A \in (A_{2L}, t)$.

Proof. From (5):

$$\begin{aligned} A > \Delta \ &\Rightarrow \ \frac{1}{3}[G_2 + r_H - c_2 - (G_1 + r_L - c_1)] > \Delta \\ &\Rightarrow \ G_2 + r_H - c_2 - (G_1 + r_L - c_1) > 3\Delta \\ &\Rightarrow \ G_2 - c_2 - (G_1 - c_1) > 2\Delta > 0. \end{aligned}$$

Therefore, (5) and Lemma A2.3 imply:

$$\begin{aligned} \pi_2^{D2}(0,0) &= G_2 - c_2 - (G_1 - c_1) = G_2 + r_H - c_2 - (G_1 + r_L - c_1) - (r_H - r_L) \\ &= 3A - \Delta. \end{aligned} \tag{152}$$

(152) and Proposition 1 imply:

$$\begin{aligned} \pi_2^S \gtreqless \pi_2^{D2}(0,0) \ &\Leftrightarrow \ \frac{1}{2t}[t + A]^2 \gtreqless 3A - \Delta \\ &\Leftrightarrow \ t^2 + 2tA + A^2 \gtreqless 6tA - 2t\Delta \\ &\Leftrightarrow \ t^2 - 2[2A - \Delta]t + A^2 \gtreqless 0. \end{aligned} \tag{153}$$

(138) implies that the ("$t$") roots of the quadratic equation in (153) are given by:

$$\begin{aligned} t &= \frac{1}{2}\left\{2[2A - \Delta] \pm \sqrt{4[2A - \Delta]^2 - 4A^2}\right\} \\ &= 2A - \Delta \pm \sqrt{[2A - \Delta]^2 - A^2} \\ &= 2A - \Delta \pm \sqrt{3A^2 - 4A\Delta + \Delta^2} \in \{t_{2L}, t_{2H}\}. \end{aligned} \tag{154}$$

The roots in (154) are real because, since $A > \Delta$ by assumption:

$$3A^2 - 4A\Delta + \Delta^2 = [3A - \Delta][A - \Delta] > 0.$$

(138) implies that $t_{2L}$, the smaller root in (154), does not satisfy the maintained assumption that $t > A$ because:

$$t_{2L} \; < \; A \;\; \Leftrightarrow \;\; 2\,A - \Delta \; < \; \sqrt{3\,A^2 - 4\,A\,\Delta + \Delta^2} + A$$

$$\Leftrightarrow \;\; [\,A - \Delta\,]^2 \; < \; 3\,A^2 - 4\,A\,\Delta + \Delta^2 \;\; \Leftrightarrow \;\; A^2 - 2\,A\,\Delta + \Delta^2 \; < \; 3\,A^2 - 4\,A\,\Delta + \Delta^2$$

$$\Leftrightarrow \;\; 2\,A^2 - 2\,A\,\Delta \; > \; 0 \;\; \Leftrightarrow \;\; A \; > \; \Delta\,. \tag{155}$$

(138) implies that $t_{2H}$, the larger root in (154), satisfies the maintained assumption that $t > A$ because:

$$t_{2H} \; > \; A \;\; \Leftrightarrow \;\; 2\,A - \Delta + \sqrt{3\,A^2 - 4\,A\,\Delta + \Delta^2} \; > \; A$$

$$\Leftrightarrow \;\; \sqrt{3\,A^2 - 4\,A\,\Delta + \Delta^2} \; > \; \Delta - A\,. \tag{156}$$

The last inequality in (156) holds because $\Delta - A < 0$, by assumption.

(153) – (156) imply that for $t > A$:

$$\pi_2^S \; \begin{cases} > \; \pi_2^{D2}(0,0) \;\; \text{if } \; t \; > \; t_{2H} \\ < \; \pi_2^{D2}(0,0) \;\; \text{if } \; t \; \in (\,K_2, \; t_{2H}\,)\,. \end{cases} \tag{157}$$

The lower bound of the open interval in (157) (i.e., $K_2$) reflects the maintained assumption that $K_2 < t$.

(153) can be written as:

$$\pi_2^S \; \gtreqless \; \pi_2^{D2}(0,0) \;\; \Leftrightarrow \;\; A^2 - \; 4\,t\,A + t\,[\,t + 2\,\Delta\,] \; \gtreqless \; 0\,. \tag{158}$$

(138) implies that the ("$A$") roots of the quadratic equation in (158) are given by:

$$A \; = \; \frac{1}{2}\left[\,4\,t \; \pm \; \sqrt{16\,t^2 - 4\,t\,[\,t + 2\,\Delta\,]}\,\right] \; = \; 2\,t \; \pm \; \sqrt{4\,t^2 - t\,[\,t + 2\,\Delta\,]}$$

$$= \; 2\,t \; \pm \; \sqrt{3\,t^2 - 2\,t\,\Delta} \;\; \in \; \{\,A_{2L}, \; A_{2H}\,\}\;. \tag{159}$$

The roots in (159) are real because:

$$3\,t^2 \; > \; 2\,t\,\Delta \;\; \Leftrightarrow \;\; t \; > \; \frac{2}{3}\,\Delta\,. \tag{160}$$

The inequality in (160) holds because the maintained assumptions that $K_2 < t$, $K_2 > A$, and $A > \Delta$ imply:

$$t \; > \; K_2 \; > \; A \; > \; \Delta \; > \; \frac{2}{3}\,\Delta\,. \tag{161}$$

(138) implies that $A_{2L} > 0$ because:

$$A_{2L} \; > \; 0 \;\; \Leftrightarrow \;\; 2\,t - \sqrt{3\,t^2 - 2\,t\,\Delta} \; > \; 0 \;\; \Leftrightarrow \;\; 2\,t \; > \; \sqrt{3\,t^2 - 2\,t\,\Delta}$$

$$\Leftrightarrow \quad 4\,t^2 \;>\; 3\,t^2 - 2\,t\,\Delta \quad \Leftrightarrow \quad t^2 + 2\,t\,\Delta \;>\; 0\,.$$

(138) implies that $A_{2L} < t$ because:

$$A_{2L} \;<\; t \quad \Leftrightarrow \quad 2\,t - \sqrt{3\,t^2 - 2\,t\,\Delta} \;<\; t \quad \Leftrightarrow \quad t \;<\; \sqrt{3\,t^2 - 2\,t\,\Delta}$$

$$\Leftrightarrow \quad t^2 \;<\; 3\,t^2 - 2\,t\,\Delta \quad \Leftrightarrow \quad 2\,t^2 - 2\,t\,\Delta \;>\; 0 \quad \Leftrightarrow \quad t \;>\; \Delta\,. \tag{162}$$

(161) implies that the last inequality in (162) holds.

(138) implies that $A_{2H} > t$ because:

$$A_{2H} \;>\; t \quad \Leftrightarrow \quad 2\,t + \sqrt{3\,t^2 - 2\,t\,\Delta} \;>\; t \quad \Leftrightarrow \quad t + \sqrt{3\,t^2 - 2\,t\,\Delta} \;>\; 0\,. \tag{163}$$

(158) – (163) imply:

$$\pi_2^S \begin{cases} > \;\pi_2^{D2}(0,0) \;\text{ if } A \;<\; A_{2L} \\ < \;\pi_2^{D2}(0,0) \;\text{ if } A \;\in\; (A_{2L},\, t\,)\,. \quad \blacksquare \end{cases} \tag{164}$$

The following Corollaries to Lemmas 3 and 4 explain how the range of settings in which default-switching costs increase the profit of an advantaged firm by inducing a MS equilibrium varies with model parameters.

**Corollary 3.1**. $\frac{dt_{1H}}{d|A|} > 0$ and $\frac{dt_{1H}}{d\Delta} > 0$, so *the range of $t$ realizations for which default-switching costs increase Firm 1's profit in the setting of Lemma 3 (i.e., $t > t_{1H}$)* contracts *as $|A|$ increases or as $\Delta$ increases.*

<u>Proof</u>. From (138):

$$\frac{d\,t_{1H}}{d\,|A|} \;=\; 2 + \frac{1}{2}\left[\,3\,|A\,|^2 + 4\,|A\,|\;\Delta + \Delta^2\right]^{-\frac{1}{2}}\left[\,6\,|A\,| + 4\,\Delta\,\right] \;>\; 0 \;\text{ and}$$

$$\frac{d\,t_{1H}}{d\Delta} \;=\; 1 + \frac{1}{2}\left[\,3\,|A\,|^2 + 4\,|A\,|\;\Delta + \Delta^2\right]^{-\frac{1}{2}}\left[\,4\,|A\,| + 2\,\Delta\,\right] \;>\; 0\,. \quad \blacksquare$$

**Corollary 3.2**. $\frac{d|A|_{1L}}{d\Delta} < 0$. *Furthermore,* $\frac{d|A|_{1L}}{dt} > 0$ *if* $\Delta < \frac{t}{2}$. Therefore, *the range of $|A|$ realizations for which default-switching costs increase Firm 1's profit in the setting of Lemma 3 (i.e., $|A\,| < |A\,|_{1L}$): (i) contracts as $\Delta$ increases; and (ii) expands as $t$ increases if $\Delta < \frac{t}{2}$.*

<u>Proof</u>. From (138):

$$\frac{d\;|A\,|_{1L}}{d\Delta} \;=\; -\,t\left[\,3\,t^2 + 2\,t\,\Delta\,\right]^{-\frac{1}{2}} \;<\; 0 \;\text{ and}$$

$$\frac{d\,|A\,|_{1L}}{dt} \;=\; 2 - \frac{6\,t + 2\,\Delta}{2\,\sqrt{3\,t^2 + 2\,t\,\Delta}} \;\overset{s}{=}\; 4\,\sqrt{3\,t^2 + 2\,t\,\Delta} - (\,6\,t + 2\,\Delta\,)$$

$$\overset{s}{=}\; 2\,\sqrt{3\,t^2 + 2\,t\,\Delta} - (\,3\,t + \Delta\,) \;=\; \sqrt{12\,t^2 + 8\,t\,\Delta} - \sqrt{9\,t^2 + 6\,t\,\Delta + \Delta^2}$$

$$>\; 0 \;\;\Leftrightarrow\;\; 3\,t^2 + 2\,t\,\Delta - \Delta^2 \;>\; 0 \;\;\Leftrightarrow\;\; [\,3\,t - \Delta\,]\,[\,t + \Delta\,] \;>\; 0 \;\;\Leftrightarrow\;\; t \;>\; \frac{\Delta}{3}\,.$$

The last inequality here holds because $t \;>\; 2\,\Delta$, by assumption. ■

**Corollary 4.1**. $\frac{dt_{2H}}{d\Delta} < 0$ *and* $\frac{dt_{2H}}{dA} > 0$*, so the range of* $t$ *realizations for which default-switching costs increase Firm 2's profit in the setting of Lemma 4 (i.e.,* $t > t_{2H}$*) expands as* $\Delta$ *increases or as* $A$ declines.

Proof. From (138):

$$\frac{dt_{2H}}{d\Delta} \;=\; -\,1 + \frac{1}{2}\left[\,3\,A^2 - 4\,A\,\Delta + \Delta^2\,\right]^{-\frac{1}{2}}[\,2\,\Delta - 4\,A\,]$$

$$=\; -\,1 - \left[\,3\,A^2 - 4\,A\,\Delta + \Delta^2\,\right]^{-\frac{1}{2}}[\,2\,A - \Delta\,] \;<\; 0 \;\; \text{and}$$

$$\frac{d\,t_{2H}}{d\,A} \;=\; 2 + \frac{1}{2}\left[\,3\,A^2 - 4\,A\,\Delta + \Delta^2\,\right]^{-\frac{1}{2}}[\,6\,A - 4\,\Delta\,]$$

$$=\; 2 + 3\left[\,3\,A^2 - 4\,A\,\Delta + \Delta^2\,\right]^{-\frac{1}{2}}\left[\,A - \frac{2}{3}\Delta\,\right] \;>\; 0\,.$$

Both inequalities hold because $A > \Delta$, by assumption. ■

**Corollary 4.2**. $\frac{dA_{2L}}{dt} > 0$ *and* $\frac{dA_{2L}}{d\Delta} > 0$*, so the range of* $A$ *realizations for which default-switching costs increase Firm 2's profit in the setting of Lemma 4 (i.e.,* $A < A_{2L}$*) expands as* $t$ *increases or as* $\Delta$ *increases.*

Proof. From (138):

$$\frac{dA_{2L}}{d\Delta} \;=\; t\left[\,3\,t^2 - 2\,t\,\Delta\,\right]^{-\frac{1}{2}} \;>\; 0 \;\; \text{and}$$

$$\frac{dA_{2L}}{dt} \;=\; 2 - \frac{1}{2}\left[\,3\,t^2 - 2\,t\,\Delta\,\right]^{-\frac{1}{2}}[\,6\,t - 2\,\Delta\,] \;=\; 2 - \frac{6\,t - 2\,\Delta}{2\,\sqrt{3\,t^2 - 2\,t\,\Delta}}$$

$$\overset{s}{=}\; 4\,\sqrt{3\,t^2 - 2\,t\,\Delta} - (\,6\,t - 2\,\Delta\,) \;\overset{s}{=}\; 2\,\sqrt{3\,t^2 - 2\,t\,\Delta} - (\,3\,t - \Delta\,)$$

$$=\; \sqrt{12\,t^2 - 8\,t\,\Delta} - \sqrt{9\,t^2 - 6\,t\,\Delta + \Delta^2}$$

$$\stackrel{s}{=}\ 3\,t^2 - 2\,t\,\Delta - \Delta^2 \ = \ [\,3\,t + \Delta\,]\,[\,t - \Delta\,] \ > \ 0\,.$$

The inequalities here reflect the maintained assumptions that $t > A$ and $A > \Delta$. ■

**Proposition 5.** *Suppose: (i)* $G_2 - c_2 \ > \ G_1 - c_1$*; (ii)* $\max\{\,\Delta,\, r_H - c_2 - t\,\} \ < \ A \ < \ \min\{\,t,\, t + c_1 - r_L\,\}$*;*[1] *(iii)* $t \ > \ t_{2H}$*; and (iv)* $K_1 = 0$*. Then Firm 2 and Firm 1 both secure strictly greater profit in equilibrium when* $K_2 \in (\,A,\, t\,)$ *than when* $K_2 = 0$*.*

Proof. (5) and condition (i) in the proposition imply:

$$A \ \equiv \ \frac{1}{3}\,[\,G_2 + r_H - c_2 - (G_1 + r_L - c_1)\,] \ = \ \frac{1}{3}\,[\,G_2 - c_2 - (G_1 - c_1)\,] + \frac{1}{3}\,[\,r_H - r_L\,] \ > \ 0\,.$$

Therefore, $K_1 > -A$ when $K_1 = 0$. Furthermore, condition (ii) ensures that $t > |\,A\,| = A$ and $r_H - c_2 - t \ < \ A \ < \ t + c_1 - r_L$. Therefore, all the maintained assumptions in Proposition 1 hold if $K_2 \in (\,A,\, t\,)$.

Lemma A2.3 implies that when $K_1 = K_2 = 0$, Firm 2's profit is nearly:

$$\begin{aligned} \pi_{2A} \ &= \ G_2 - c_2 - (\,G_1 - c_1\,) \ = \ G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) - (\,r_H - r_L\,) \\ &= \ 3\,A - (\,r_H - r_L\,)\,. \end{aligned} \tag{165}$$

Proposition 1 implies that when $K_2 \in (\,A,\, t\,)$, there exists an equilibrium in which Firm 2's profit is:

$$\pi_{2B} \ = \ \frac{[\,t + A\,]^2}{2\,t}\,. \tag{166}$$

(165) and (166) imply:

$$\begin{aligned} \pi_{2B} \ > \ \pi_{2A} \ &\Leftrightarrow \ \frac{[\,t + A\,]^2}{2\,t} \ > \ 3\,A - (\,r_H - r_L\,) \\ &\Leftrightarrow \ t^2 + 2\,A\,t + A^2 \ > \ 2\,t\,[\,3\,A - (\,r_H - r_L\,)\,] \\ &\Leftrightarrow \ t^2 - 4\,A\,t + A^2 + 2\,t\,[\,r_H - r_L\,] \ > \ 0 \\ &\Leftrightarrow \ t^2 + 2\,t\,[\,r_H - r_L - 2\,A\,] + A^2 \ > \ 0\,. \end{aligned} \tag{167}$$

The roots of the quadratic equation associated with (167) are:

$$t^* \ = \ \frac{1}{2}\left\{ -2\,[\,r_H - r_L - 2\,A\,] \pm \sqrt{4\,[\,r_H - r_L - 2\,A\,]^2 - 4\,A^2}\,\right\}$$

[1] Recall that $\Delta \equiv r_H - r_L > 0$.

$$= \; 2\,A - (\,r_H - r_L\,) \; \pm \; \sqrt{[\,r_H - r_L - 2\,A\,]^2 - A^2}$$

$$= \; 2\,A - (\,r_H - r_L\,) \; \pm \; \sqrt{[\,r_H - r_L\,]^2 - 4\,A\,[\,r_H - r_L\,] + 4\,A^2 - A^2}$$

$$= \; 2\,A - (\,r_H - r_L\,) \; \pm \; \sqrt{3\,A^2 - 4\,A\,[\,r_H - r_L\,] + [\,r_H - r_L\,]^2}$$

$$= \; 2\,A - \Delta \; \pm \; \sqrt{[\,3\,A - \Delta\,]\,[\,A - \Delta\,]}\,. \tag{168}$$

Condition (ii) ensures that $A > \Delta$, which implies $3\,A > \Delta$, so the roots in (168) are real. (138), (167), and (168) imply that $\pi_{2B} > \pi_{2A}$ if $t > 2\,A - \Delta + \sqrt{[\,3\,A - \Delta\,]\,[\,A - \Delta\,]} = t_{2H}$.

Finally, Lemma A2.3 implies that Firm 1's equilibrium profit is 0 when $K_1 = K_2 = 0$. Firm 1's profit in the equilibrium identified in Proposition 1 is $\frac{1}{18\,t}\,[\,3\,t - A\,]^2 > 0$ when $K_1 = 0$ and $K_2 \in (\,A,\, t\,)$. ■

**Proposition 6.** *Suppose: (i) $A < 0$; (ii) $r_H - c_2 - t \, < \, A \, < \, t + c_1 - r_L$; (iii) $|\,A\,| < t$; (iv) $t > t_{1H}$; and (v) $K_2 = 0$. Then Firm 1 and Firm 2 both secure strictly greater profit in equilibrium when $K_1 \in (\,|\,A\,|\,,\, t\,)$ than when $K_1 = 0$.*

Proof. Condition (i) implies that $K_2 > A$ when $K_2 = 0$. Therefore, conditions (i) – (iii) ensure that all the conditions in Proposition 1 hold if $K_1 \in (\,|\,A\,|\,,\, t\,)$.

Lemma 2 implies that when $K_1 = K_2 = 0$, Firm 1's profit is nearly:

$$\pi_{1A} \; = \; G_1 - c_1 - (\,G_2 - c_2\,) \; = \; -\,[\,G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,] + r_H - r_L$$

$$= \; r_H - r_L + 3\,|\,A\,|\,. \tag{169}$$

Proposition 1 implies that when $K_1 \in (\,|\,A\,|\,,\, t\,)$, there exists an equilibrium in which Firm 1's profit is:

$$\pi_{1B} \; = \; \frac{[\,t + |\,A\,|\,]^2}{2\,t}\,. \tag{170}$$

(169) and (170) imply:

$$\pi_{1B} \; > \; \pi_{1A} \;\; \Leftrightarrow \;\; \frac{[\,t + |\,A\,|\,]^2}{2\,t} \; > \; r_H - r_L + 3\,|\,A\,|$$

$$\Leftrightarrow \;\; t^2 + 2\,|\,A\,|\,t + |\,A\,|^2 \; > \; 2\,t\,[\,r_H - r_L + 3\,|\,A\,|\,]$$

$$\Leftrightarrow \;\; t^2 - 4\,|\,A\,|\,t + A^2 - 2\,t\,[\,r_H - r_L\,] \; > \; 0$$

$$\Leftrightarrow \;\; t^2 - 2\,t\,[\,r_H - r_L + 2\,|\,A\,|\,] + A^2 \; > \; 0\,. \tag{171}$$

The roots of the quadratic equation associated with (171) are:

$$\begin{aligned} t_1^* &= \frac{1}{2}\left\{ 2\,[\,r_H - r_L + 2\,|\,A\,|\,] \pm \sqrt{4\,[\,r_H - r_L + 2\,|\,A\,|\,]^2 - 4\,A^2}\,\right\} \\ &= r_H - r_L + 2\,|\,A\,| \pm \sqrt{[\,r_H - r_L + 2\,|\,A\,|\,]^2 - A^2} \\ &= r_H - r_L + 2\,|\,A\,| \pm \sqrt{[\,r_H - r_L\,]^2 + 4\,|\,A\,|\,[\,r_H - r_L\,] + 4\,A^2 - A^2} \\ &= r_H - r_L + 2\,|\,A\,| \pm \sqrt{3\,A^2 + 4\,|\,A\,|\,[\,r_H - r_L\,] + [\,r_H - r_L\,]^2} \\ &= \Delta + 2\,|\,A\,| \pm \sqrt{3\,A^2 + 4\,|\,A\,|\,\Delta + \Delta^2}\,. \end{aligned} \tag{172}$$

(138), (171), and (172) imply that if $t > \Delta + 2\,|\,A\,| \pm \sqrt{3\,A^2 + 4\,|\,A\,|\,\Delta + \Delta^2} = t_{1H}$, then $\pi_{1B} > \pi_{1A}$.

Finally, Lemma 2 implies that Firm 2's equilibrium profit is 0 when $K_1 = K_2 = 0$. Firm 2's profit in the equilibrium identified in Proposition 1 is $\frac{1}{18\,t}\,[\,3\,t - |\,A\,|\,]^2 > 0$ when $K_2 = 0$ and $K_1 \in (\,|\,A\,|\,, t\,)$. ■

**Proposition 7.** *Suppose: (i) $(K_1^*, K_2^*)$ are such that conditions (ii) and (iii) in Proposition 1 hold; (ii) $t > |\,A\,|$; (iii) Condition 1A holds; (iv) Condition 1B holds when $K_1 = K_2 = 0$; and (v) $\Delta < 2\,t$. Then $(K_1^*, K_2^*)$, along with $p_1 = c_1 - r_L + t - A$ and $p_2 = c_2 - r_H + t + A$ constitute an equilibrium in the setting with endogenous $K$.*

Proof. The proof proceeds by showing that neither firm can increase its profit by unilaterally changing its default-switching cost, regardless of the nature of the ensuing equilibrium.

To begin, observe that the proof of Proposition 1 establishes that the prices identified in the present Proposition are the unique prices that arise in a MS equilibrium. Furthermore, the equilibrium profits identified in Proposition 1 ($\pi_1^S > 0$ and $\pi_2^S > 0$) do not vary with $K_1$ and $K_2$. Therefore, Firm $i \in \{1, 2\}$ cannot increase its profit by choosing $K_i \neq K_i^*$ if the resulting $\left(K_i, K_j^*\right)$ default-switching costs induce a MS equilibrium.

Next suppose that Firm $i \in \{1, 2\}$ chooses a $K_i \neq K_i^*$ such that the resulting $\left(K_i, K_j^*\right)$ default-switching costs induce a MD$j$ equilibrium (where $j \neq i,\ i, j \in \{1, 2\}$). Then Firm $i$'s profit will decline to 0. Consequently, Firm $i$ cannot increase its profit by setting $K_i \neq K_i^*$ if the resulting $\left(K_i, K_j^*\right)$ default-switching costs induce a MD$j$ equilibrium.

Now suppose that Firm 1 sets $K_1 \neq K_1^*$ such that the resulting $(K_1, K_2^*)$ default-switching costs induce a MD1 equilibrium. The proof of Proposition 1 establishes that the maximum profit Firm 1 can secure in a MD1 equilibrium is

$$-\,2\,A + \frac{t - K_1}{2\,t}\,[\,2\,t + \Delta\,] \;\leq\; -\,2\,A + \frac{1}{2}\,[\,2\,t + \Delta\,]\,.$$

Condition (ii) in the present Proposition ensures that Firm 1's profit in the MS equilibrium identified in Proposition 1 exceeds $-\,2\,A + \frac{1}{2}\,[\,2\,t + \Delta\,]$. Therefore, Firm 1 cannot increase its profit by setting $K_1 \neq K_1^*$ if the resulting $(K_1, K_2^*)$ default-switching costs induce a MD1 equilibrium.

Finally, suppose that Firm 2 sets a $K_2 \neq K_2^*$ such that the resulting $(K_1^*, K_2)$ default-switching costs induce a MD2 equilibrium. The proof of Proposition 1 establishes that the maximum profit Firm 2 can secure in a MD2 equilibrium is

$$2\,A + \frac{t - K_2}{2\,t}\,[\,2\,t - \Delta\,] \;\leq\; 2\,A + \frac{1}{2}\,[\,2\,t - \Delta\,]\,.$$

The inequality here reflects condition (iii) in the present Proposition. Condition (ii) in this Proposition ensures that Firm 2's profit in the MS equilibrium identified in Proposition 1 exceeds $2\,A + \frac{1}{2}\,[\,2\,t - \Delta\,]$. Therefore, Firm 2 cannot increase its profit by setting $K_2 \neq K_2^*$ if the resulting $(K_1^*, K_2)$ default-switching costs induce a MD2 equilibrium. ■

**Proposition 8.** *Suppose $A < 0$, $G_1 - c_1 - (G_2 - c_2) > \max\{\,\frac{\Delta}{2},\, \frac{1}{2\,t}\,[\,t + |\,A\,|\,]^2\,\}$, $c_2 > \frac{1}{2}\,[\,r_L + r_H\,]$, and $c_1 > r_L$. Then in the setting with endogenous $K$, there exists a MD1 equilibrium in which: (i) $K_1 = K_2 = 0$; (ii) $p_2 = c_2 - \frac{1}{2}\,[\,r_L + r_H\,]$; and (iii) $p_1$ is marginally below $c_2 - r_H + G_1 - G_2$.*

<u>Proof</u>. The proof proceeds by showing that neither firm can increase its profit by unilaterally increasing its default-switching cost above 0, regardless of the nature of the ensuing equilibrium.

We first show that Firm 1 cannot strictly increase its profit by setting $K_1 > 0$ if the resulting $(K_1, 0)$ default-switching costs induce a MD1 equilibrium. The logic employed in the proof of Proposition 2 implies that Firm 1's profit in a MD1 equilibrium, given $K_1 > 0$ and $K_2 \geq 0$ is:

$$\pi_1^{D1}(K_1, K_2) \;=\; G_1 - G_2 + c_2 - c_1 - \frac{\Delta}{2} - K_1\left[\frac{2\,t + \Delta}{2\,t}\right]$$

$$\Rightarrow\quad \frac{\partial \pi_1^{D1}(\cdot)}{\partial K_1} \;=\; -\,1 - \frac{\Delta}{2\,t} \;<\; 0\,. \tag{173}$$

(173) implies that Firm 1 cannot strictly increase its profit setting $K_1 > 0$ if a MD1 equilibrium ensues.

(173) further implies that when $K_1 = K_2 = 0$, Firm 1's profit is:

$$\pi_1^{D1}(0,0) \;=\; G_1 - c_1 - (G_2 - c_2) - \frac{\Delta}{2} \;>\; 0\,. \tag{174}$$

Firm 1's profit is 0 in any MD2 equilibrium. Therefore, (174) implies that Firm 1 cannot increase its profit by setting $K_1 > 0$ if the resulting $(K_1, 0)$ default-switching costs induce a MD2 equilibrium.

Proposition 1 establishes that Firm 1's profit in a MS equilibrium when $A < 0$ is $\pi_1^S = \frac{1}{2t}\left[\, t + |\, A \,|\,\right]^2$. Therefore, (174) and the maintained assumptions ensure that $\pi_1^{D1}(0,0) > \pi_1^S$. Consequently, Firm 1 cannot increase its profit by setting $K_1 > 0$ if the resulting $(K_1, 0)$ default-switching costs induce a MS equilibrium.

To initiate the demonstration that Firm 2 cannot increase its profit by unilaterally increasing $K_2$, observe that Firm 2's profit is 0 in all MD1 equilibria. Consequently, Firm 2 cannot increase its profit by implementing a $K_2 > 0$ that induces a MD1 equilibrium.

Next we establish that a MD2 equilibrium does not exist when $K_1 = 0$ and $A < 0$. To do so, suppose such an equilibrium exists. Then the consumer located at 0 prefers to buy a phone from Firm 2 than from Firm 1. Consequently:

$$G_2 - p_2 - \min\{\, K_2,\, t\,\} \;\geq\; G_1 - p_1 \quad \Rightarrow \quad p_2 \;\leq\; p_1 + G_2 - G_1 - \min\{\, K_2,\, t\,\}\,. \tag{175}$$

(175) reflects the fact that the consumer located at 0 who purchases a phone from Firm 2 will change the default PD setting on the phone if and only if $K_2 < t$.

Rather than serve no customers, Firm 1 will reduce its price at least to $c_1 - r_L$. Therefore, (175) implies that, to attract all consumers, Firm 2's price must satisfy:

$$p_2 \;\leq\; c_1 - r_L + G_2 - G_1 - \min\{\, K_2,\, t\,\}. \tag{176}$$

In any MD2 equilibrium in which (176) holds, Firm 2's profit is:

$$\begin{aligned}
\pi_2 \;&\leq\; c_1 - r_L + G_2 - G_1 - \min\{\, K_2, t\,\} - c_2 + r_H \\
&<\; c_1 - r_L + G_2 - G_1 - c_2 + r_H \\
&=\; G_2 + r_H - c_2 - (G_1 + r_L - c_1) \;=\; 3\,A \;\, < 0\,.
\end{aligned} \tag{177}$$

The first inequality in (177) reflects the fact that Firm 2's revenue from advertisers cannot exceed $r_H$. The second inequality in (177) holds because $K_2 > 0$ (and $t > 0$), by assumption. The last inequality in (177) holds because $A < 0$, by assumption. (177) implies that a MD2 equilibrium does not exist under the maintained assumptions because Firm 2 must secure nonnegative profit in a MD2 equilibrium.

Finally, we establish that a MS equilibrium does not exist when $K_1 = 0$ and $A < 0$. To do so, suppose a MS equilibrium exists. Then there exists a consumer located at $x_0 \in (0, 1)$ who is indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2. Furthermore, the proof of Proposition 1 implies:

$$p_1 \;=\; c_1 - r_L + t - A, \quad p_2 \;=\; c_2 - r_H + t + A, \;\text{ and} \tag{178}$$

$$x_0 \;=\; \frac{1}{2} - \frac{A}{2\,t} \;>\; \frac{1}{2}\,. \tag{179}$$

The inequality in (179) holds because $A < 0$, by assumption.

Because $x_0 > \frac{1}{2}$ and $K_1 = 0$, the consumer located at $x_0$ will change the default PD setting on the phone he purchases if and only if he buys the phone from Firm 1. Therefore, because the consumer located at $x_0$ is indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2:

$$G_1 - p_1 - t\,[\,1 - x_0\,] \;=\; G_2 - p_2 - t\,[\,1 - x_0\,]$$

$$\Rightarrow \;\; p_2 - p_1 \;=\; G_2 - G_1\,. \tag{180}$$

(178) implies:

$$p_2 - p_1 \;=\; c_2 - c_1 - r_H + r_L + 2\,A\,. \tag{181}$$

(180) and (181) imply:

$$G_2 - G_1 \;=\; c_2 - c_1 - r_H + r_L + 2\,A$$

$$\Rightarrow \;\; G_2 + r_H - c_2 - (G_1 + r_L - c_1) \;=\; 2\,A$$

$$\Rightarrow \;\; 3\,A \;=\; 2\,A \quad \Rightarrow \quad A \;=\; 0\,. \tag{182}$$

(182) cannot hold because $A < 0$, by assumption. Therefore, by contradiction, a MS equilibrium does not exist when $K_1 = 0$ and $A < 0$. ■

**Proposition 9.** *Suppose $A > 0$, $G_2 - c_2 - (G_1 - c_1) > \frac{1}{2t}\,[\,t + A\,]^2$, $c_2 > r_H$, $c_1 > \frac{1}{2}\,[\,r_L + r_H\,]$, and $\Delta < 2\,t$. Then in the setting with endogenous $K$, there exists a MD2 equilibrium in which: (i) $K_1 = K_2 = 0$; (ii) $p_1 = c_1 - \frac{1}{2}\,[\,r_L + r_H\,]$; and (iii) $p_2$ is marginally below $c_1 - \frac{1}{2}\,[\,r_L + r_H\,] + G_2 - G_1$.*

Proof. The proof proceeds by showing that neither firm can increase its profit by unilaterally increasing its default-switching cost above 0, regardless of the nature of the ensuing equilibrium.

We first show that Firm 2 cannot strictly increase its profit by setting $K_2 > 0$ if the resulting $(0, K_2)$ default-switching costs induce a MD2 equilibrium. The logic employed in the proof of Proposition 3 implies that Firm 2's profit in a MD2 equilibrium in which $p_1 = c_1 - \frac{1}{2}\,[\,r_L + r_H\,]$ is nearly:

$$\pi_2 \;=\; c_1 - c_2 + G_2 - G_1 - K_2 + K_2\,\frac{\Delta}{2t} \;\;\Rightarrow\;\; \frac{\partial \pi_2}{\partial K_2} \;=\; -\,1 + \frac{\Delta}{2\,t} \;<\; 0\,. \tag{183}$$

The inequality in (183) holds because $\Delta < 2\,t$, by assumption. (183) implies that Firm 2 cannot strictly increase its profit by setting $K_2 > 0$ if a MD2 equilibrium ensues.

(183) further implies that when $K_1 = K_2 = 0$, Firm 2's profit is:

$$\pi_2^{D2}(0,0) \ = \ G_2 - c_2 - (G_1 - c_1) \ > \ 0 \, . \tag{184}$$

Firm 2's profit is 0 in any MD1 equilibrium. Therefore, (184) implies that Firm 2 cannot increase its profit by setting $K_2 > 0$ if the resulting $(0, K_2)$ default-switching costs induce a MD1 equilibrium.

Proposition 1 establishes that Firm 2's profit in a MS equilibrium when $A > 0$ is $\pi_2^S = \frac{1}{2t}[\, t + A\,]^2$. Therefore, (184) and the maintained assumptions ensure that $\pi_2^{D2}(0,0) > \pi_2^S$. Consequently, Firm 2 cannot increase its profit by setting $K_2 > 0$ if the resulting $(0, K_2)$ default-switching costs induce a MS equilibrium.

To initiate the demonstration that Firm 1 cannot increase its profit by unilaterally increasing $K_1$, observe that Firm 1's profit is 0 in all MD2 equilibria. Consequently, Firm 1 cannot increase its profit by implementing a $K_1 > 0$ that induces a MD2 equilibrium.

Next we establish that a MD1 equilibrium does not exist when $A > 0$. To do so, suppose such an equilibrium exists. Then the consumer located at 1 (weakly) prefers to buy a phone from Firm 1 than from Firm 2. Consequently:

$$G_1 - p_1 - \min\{\, K_1, \, t \,\} \ \geq \ G_2 - p_2 \quad \Rightarrow \quad p_1 \ \leq \ p_2 + G_1 - G_2 - \min\{\, K_1, \, t \,\} \, . \tag{185}$$

(185) reflects the fact that the consumer located at 1 who purchases a phone from Firm 1 will change the default PD setting on the phone if and only if $K_1 < t$.

Rather than serve no customers, Firm 2 will reduce its price to $c_2 - r_H$. Therefore, (185) implies that, to attract all consumers, Firm 1's price must satisfy:

$$p_1 \ \leq \ c_2 - r_H + G_1 - G_2 - \min\{\, K_1, \, t \,\}. \tag{186}$$

In any MD1 equilibrium in which (186) holds, Firm 1's profit is:

$$\begin{aligned} \pi_1 \ &\leq \ c_2 - r_H + G_1 - G_2 - \min\{\, K_1, t \,\} - c_1 + r_H \\ &< \ c_2 + G_1 - G_2 - c_1 \ = \ G_1 - c_1 - (G_2 - c_2) \ < \ 0 \, . \end{aligned} \tag{187}$$

The first inequality in (187) reflects the fact that Firm 1's revenue from advertisers cannot exceed $r_H$. The second inequality in (187) holds because $K_1 > 0$ (and $t > 0$), by assumption. The last inequality in (187) holds because the maintained assumptions include $G_1 - c_1 - (\, G_2 - c_2 \,) \ < \ -\frac{1}{2t}[\, t + A\,]^2 \ < \ 0$. (187) implies that a MD1 equilibrium does not exist under the maintained assumptions because Firm 1's profit must be nonnegative in a MD1 equilibrium.

Finally, we establish that a MS equilibrium does not exist when $K_2 = 0$ and $A > 0$. To do so, suppose an MS equilibrium exists. Then there exists a consumer located at $x_0 \in (0, 1)$

who is indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2. Furthermore, the proof of Proposition 1 implies:

$$p_1 \;=\; c_1 - r_L + t - A, \quad p_2 \;=\; c_2 - r_H + t + A, \;\text{ and} \tag{188}$$

$$x_0 \;=\; \frac{1}{2} - \frac{A}{2\,t} \;<\; \frac{1}{2}\,. \tag{189}$$

The inequality in (189) holds because $A > 0$, by assumption.

Because $x_0 < \frac{1}{2}$ and $K_2 = 0$, the consumer located at $x_0$ will change the default PD setting on the phone he purchases if and only if he buys the phone from Firm 2. Therefore, because the consumer located at $x_0$ is indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2:

$$G_2 - p_2 - t\,x_0 \;=\; G_1 - p_1 - t\,x_0 \;\;\Rightarrow\;\; p_2 - p_1 \;=\; G_2 - G_1\,. \tag{190}$$

(188) implies:

$$p_2 - p_1 \;=\; c_2 - c_1 - r_H + r_L + 2\,A\,. \tag{191}$$

(190) and (191) imply:

$$\begin{aligned} & G_2 - G_1 \;=\; c_2 - c_1 - r_H + r_L + 2\,A \\ \Rightarrow\;\; & G_2 + r_H - c_2 - (G_1 + r_L - c_1) \;=\; 2\,A \\ \Rightarrow\;\; & 3\,A \;=\; 2\,A \;\;\Rightarrow\;\; A \;=\; 0\,. \end{aligned} \tag{192}$$

(192) cannot hold because $A > 0$, by assumption. Therefore, by contradiction, a MS equilibrium does not exist when $K_2 = 0$ and $A > 0$. ■

Supplemental Findings

The following supplemental findings (Lemmas A19 – A27 and Propositions A1 – A2) are employed to prove Proposition 10 (below).

**Lemma A19**. Suppose $K_1 = K_2 = 0$. Then: (i) a user located in $[\,0, \frac{1}{2}\,)$ will change the default setting on the phone she purchases if and only if she purchases the phone from Firm 2; (ii) a user located in $(\,\frac{1}{2}, 1\,]$ will change the default setting on the phone she purchases if and only if she purchases the phone from Firm 1; and (iii) a user located at $\frac{1}{2}$ will not change the default setting on the phone she purchases.

Proof. The proof parallels the proofs of Lemmas A1 – A3. ■

**Lemma A20**. Suppose $K_1 = K_2 = 0$. Then: (i) all users buy a phone from Firm 1 if $p_2 > p_1 + G_2 - G_1$; (ii) all users buy a phone from Firm 2 if $p_2 < p_1 + G_2 - G_1$; and (iii) all users are indifferent between buying a phone from Firm 1 and from Firm 2 if $p_2 = p_1 + G_2 - G_1$.

<u>Proof</u>. Lemma A19 implies that a user located at $x_1 \in [0, \frac{1}{2})$ will buy a phone from Firm 1 if:

$$G_1 - t\,x_1 - p_1 \;>\; G_2 - t\,x_1 - p_2 \quad \Leftrightarrow \quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

Lemma A19 also implies that a user located at $x_2 \in (\frac{1}{2}, 1\,]$ will buy a phone from Firm 1 if:

$$G_1 - t\,[\,1 - x_2\,] - p_1 \;>\; G_2 - t\,[\,1 - x_2\,] - p_2 \quad \Leftrightarrow \quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

Lemma A19 further implies that a user located at $\frac{1}{2}$ will buy a phone from Firm 1 if:

$$G_1 - \frac{1}{2}\,t - p_1 \;>\; G_2 - \frac{1}{2}\,t - p_2 \quad \Leftrightarrow \quad p_2 \;>\; p_1 + G_2 - G_1\,.$$

The proofs of the remaining conclusions are analogous, and so are omitted. ■

**Lemma A21**. Suppose $K_1 = K_2 = 0$ and $G_2 - c_2 > G_1 - c_1$. Then in equilibrium, all users purchase a phone from Firm 2 at a price just below $c_1 - \frac{1}{2}\,[\,r_H + r_L\,] + G_2 - G_1$. Firm 1's profit is 0. Firm 2's profit is (nearly) $G_2 - c_2 - (\,G_1 - c_1\,)$.

<u>Proof</u>. Lemmas A19 and A20 imply that for $\varepsilon_9 > 0$, Firm 2's expected profit is:

$$\pi_2 \;=\; \begin{cases} 0 & \text{if } \; p_2 \;>\; p_1 + G_2 - G_1 \\ \frac{1}{2}\left[\,p_1 + \frac{r_L + r_H}{2} + G_2 - G_1 - c_2\,\right] & \text{if } \; p_2 \;=\; p_1 + G_2 - G_1 \\ p_1 + \frac{r_L + r_H}{2} + G_2 - G_1 - c_2 - \varepsilon_9 & \text{if } \; p_2 \;=\; p_1 + G_2 - G_1 - \varepsilon_9\,. \end{cases} \tag{193}$$

(193) reflects the fact that when $K_1 = K_2 = 0$, each consumer perceives the two phones to exhibit no horizontal product differentiation. Consequently, in equilibrium, either: (i) all consumers strictly prefer to purchase a phone from Firm 1; (ii) all consumers strictly prefer to purchase a phone from Firm 2; or (iii) all consumers are indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2. Therefore, if Firm 1 sells any phones in equilibrium with strictly positive probability, it secures nonnegative expected profit only if $p_1 \geq c_1 - \frac{1}{2}\,[\,r_H + r_L\,]$. Consequently, in any such equilibrium:

$$p_1 + \frac{r_L + r_H}{2} + G_2 - G_1 - c_2 \;\geq\; G_2 - c_2 - (\,G_1 - c_1\,) \;>\; 0\,. \tag{194}$$

(193) and (194) imply that for $\varepsilon_9$ sufficiently small, Firm 2 secures strictly higher profit by setting $p_2 = p_1 + G_2 - G_1 - \varepsilon_9$ than by setting $p_2 \geq p_1 + G_2 - G_1$. Therefore, in equilibrium, Firm 2 will set $p_2$ just below $c_1 - \frac{1}{2}\,[\,r_H + r_L\,] + G_2 - G_1$ to ensure that Firm 1

cannot profitably attract any users. Consequently, Firm 1's profit is 0 and Firm 2's profit is nearly:

$$c_1 - \frac{1}{2}[r_H + r_L] + G_2 - G_1 + \frac{1}{2}[r_H + r_L] - c_2 \;=\; G_2 - c_2 - (G_1 - c_1)\,. \;\blacksquare$$

**Lemma A22**. Suppose $\min\{K_1, K_2\} \geq t$. Then no user changes the default setting on the phone she purchases.

Proof. A user located at $x$ will change the default setting on the phone she purchases from Firm 1 if and only if:

$$G_1 - t[1-x] - p_1 - K_1 \;>\; G_1 - t\,x - p_1 \;\Leftrightarrow\; t[1-2x] \;<\; -K_1$$

$$\Leftrightarrow\; 1 - 2x \;<\; -\frac{K_1}{t} \;\Leftrightarrow\; 2x \;>\; 1 + \frac{K_1}{t} \;\Leftrightarrow\; x \;>\; \frac{1}{2} + \frac{K_1}{2t} \;\geq\; 1\,. \tag{195}$$

(195) implies that no user located in $[0,1]$ will change the default setting on a phone she purchase from Firm 1.

A user located at $x$ will change the default setting on the phone she purchases from Firm 2 if and only if:

$$G_2 - t\,x - p_2 - K_2 \;>\; G_2 - t[1-x] - p_2 \;\Leftrightarrow\; t[1-2x] \;>\; K_2$$

$$\Leftrightarrow\; 1 - 2x \;>\; \frac{K_2}{t} \;\Leftrightarrow\; 2x \;<\; 1 - \frac{K_2}{t} \;\Leftrightarrow\; x \;<\; \frac{1}{2} - \frac{K_2}{2t} \;\leq\; 0\,. \tag{196}$$

(196) implies that no user located in $[0,1]$ will change the default setting on a phone she purchase from Firm 2. $\blacksquare$

**Lemma A23**. Suppose $\min\{K_1, K_2\} \geq t$. Then: (i) all users buy a phone from Firm 1 if $p_2 - p_1 > G_2 - G_1 + t$; and (ii) all users buy a phone from Firm 2 if $p_2 - p_1 < G_2 - G_1 - t$.

Proof. Lemma A22 implies that no user changes the default setting on the phone she purchases. Therefore, all users buy a phone from Firm 1 if, for all $x \in [0,1]$:

$$G_1 - t\,x - p_1 \;>\; G_2 - t[1-x] - p_2 \;\Leftrightarrow\; t[1-2x] \;>\; G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow\; 1 - 2x \;>\; \frac{1}{t}[G_2 - G_1 - p_2 + p_1] \;\Leftrightarrow\; x \;<\; \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2]\,. \tag{197}$$

(197) holds for all $x \in [0,1]$ if:

$$1 \;<\; \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2] \;\Leftrightarrow\; \frac{t}{2t} \;<\; \frac{1}{2t}[G_1 - G_2 - p_1 + p_2]$$

$$\Leftrightarrow\; t \;<\; G_1 - G_2 - p_1 + p_2 \;\Leftrightarrow\; p_2 - p_1 \;>\; G_2 - G_1 + t\,.$$

Similarly, because no user changes the default setting on the phone she purchases (Lemma A22), all users buy a phone from Firm 2 if, for all $x \in [0,1]$:

$$G_2 - t[1-x] - p_2 > G_1 - tx - p_1 \Leftrightarrow t[1-2x] < G_2 - G_1 - p_2 + p_1$$

$$\Leftrightarrow 1 - 2x < \frac{1}{t}[G_2 - G_1 - p_2 + p_1] \Leftrightarrow x > \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2]. \quad (198)$$

(198) holds for all $x \in [0,1]$ if:

$$0 > \frac{1}{2} + \frac{1}{2t}[G_1 - G_2 - p_1 + p_2] \Leftrightarrow \frac{1}{2t}[t + G_1 - G_2 - p_1 + p_2] < 0$$

$$\Leftrightarrow p_2 - p_1 < G_2 - G_1 - t. \quad \blacksquare$$

**Lemma A24**. Suppose $\min\{K_1, K_2\} \geq t > 3|A|$. Then no equilibrium exists in which one firm serves all users.

Proof. First suppose Firm 1 serves all users. Then Lemma A23 implies that for all $p_2$ that generate nonnegative profit for Firm 2:

$$p_1 \leq p_2 + G_1 - G_2 - t. \quad (199)$$

(199) holds for all such $p_2$ if:

$$p_1 \leq c_2 - r_H + G_1 - G_2 - t. \quad (200)$$

Firm 1's profit when it serves all users at a price that satisfies (200) is:

$$\pi_1 = p_1 + r_L - c_1 \leq c_2 - r_H + G_1 - G_2 - t + r_L - c_1$$

$$= G_1 + r_L - c_1 - (G_2 + r_H - c_2) - t < 0 \text{ when } t > 3|A|. \quad (201)$$

(201) implies that no equilibrium exists in which Firm 1 serves all users.

Now suppose Firm 2 serves all users. Then Lemma A23 implies that for all $p_1$ that generate nonnegative profit for Firm 1:

$$p_2 \leq p_1 + G_2 - G_1 - t. \quad (202)$$

(202) holds for all such $p_1$ if:

$$p_2 \leq c_1 - r_L + G_2 - G_1 - t. \quad (203)$$

Firm 2's profit when it serves all users at a price that satisfies (203) is:

$$\pi_2 = p_2 + r_H - c_2 \leq c_1 - r_L + G_2 - G_1 - t + r_H - c_2$$

$$= G_2 + r_H - c_2 - (G_1 + r_L - c_1) - t < 0 \text{ when } t > 3A. \quad (204)$$

(204) implies that no equilibrium exists in which Firm 2 serves all users. ■

**Lemma A25**. Suppose $\min\{K_1, K_2\} \geq t$ and $p_2 - p_1 \in [G_2 - G_1 - t, G_2 - G_1 + t]$. Then: (i) no user changes the default setting on the phone she purchases; (ii) a user located at $x_0 \equiv \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \in [0, 1]$ is indifferent between purchasing a phone from Firm 1 and from Firm 2; (iii) if $x_0 > 0$, all users located in $[0, x_0)$ buy a phone from Firm 1; and (iv) if $x_0 < 1$, all users located in $(x_0, 1]$ buy a phone from Firm 2.

Proof. Lemma A22 implies that no user changes the default setting on the phone she purchases. Therefore, a user located at $x$ is indifferent between purchasing a phone from Firm 1 and from Firm 2 if:

$$
\begin{aligned}
& G_1 - t\,x - p_1 = G_2 - t[1-x] - p_2 \;\Leftrightarrow\; t[1-2x] = G_2 - G_1 - p_2 + p_1 \\
\Leftrightarrow\;& 1 - 2x = \frac{1}{t}[G_2 - G_1 - p_2 + p_1] \;\Leftrightarrow\; x = \frac{1}{2} + \frac{1}{t}[G_1 - G_2 - p_1 + p_2] \\
\Leftrightarrow\;& x = \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] \equiv x_0 \\
& \in [0,1] \;\Leftrightarrow\; t + G_1 - G_2 + p_2 - p_1 \in [0, 2t] \\
& \Leftrightarrow\; p_2 - p_1 \in [G_2 - G_1 - t,\, G_2 - G_1 + t].
\end{aligned}
$$

If $x_0 > 0$, then a user located at $x \in [0, x_0)$ buys a phone from Firm 1 because:

$$
\begin{aligned}
& G_1 - t\,x - p_1 > G_2 - t[1-x] - p_2 \\
\Leftrightarrow\;& t[1-2x] > G_2 - G_1 + p_1 - p_2 \\
\Leftrightarrow\;& 2x < 1 + \frac{1}{t}[G_1 - G_2 + p_2 - p_1] \\
\Leftrightarrow\;& x < \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] = x_0 .
\end{aligned}
$$

If $x_0 < 1$, then a user located at $x \in (x_0, 1]$ buys a phone from Firm 2 because:

$$
\begin{aligned}
& G_2 - t[1-x] - p_2 > G_1 - t\,x - p_1 \\
\Leftrightarrow\;& t[1-2x_0] < G_2 - G_1 + p_1 - p_2 \\
\Leftrightarrow\;& 2x > 1 + \frac{1}{t}[G_1 - G_2 + p_2 - p_1] \\
\Leftrightarrow\;& x > \frac{1}{2t}[t + G_1 - G_2 + p_2 - p_1] = x_0 . \;\blacksquare
\end{aligned}
$$

**Lemma A26**. Suppose $\min\{K_1, K_2\} \geq t > 3\,|\,A\,|$. Then in equilibrium, no user changes the default setting on the phone she purchases. Furthermore, there exists a $x_0 \in [\,0, 1\,]$ such that: (i) a user located at $x_0$ is indifferent between buying a phone from Firm 1 and from Firm 2; (ii) all users located in $[\,0, x_0\,)$ buy a phone from Firm 1; and (iii) all users located in $(\,x_0, 1\,]$ buy a phone from Firm 2. In addition, $p_1 = c_1 - r_L + t - A$; $p_2 = c_2 - r_H + t + A$; $\pi_1 = \frac{1}{2t}\,[\,t - A\,]^2$; and $\pi_2 = \frac{1}{2t}\,[\,t + A\,]^2$.

Proof. Lemma A22 implies that no user changes the default setting on the phone she purchases. Furthermore, Lemma A24 implies that Firm 1 and Firm 2 both serve some users in equilibrium. Therefore, Lemma A23 implies that $p_2 - p_1 \in [\,G_2 - G_1 - t, G_2 - G_1 - t\,]$. Consequently, Lemma A25 implies that a user located at

$$x_0 \equiv \frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,] \in [\,0, 1\,] \tag{205}$$

is indifferent between purchasing a phone from Firm 1 and from Firm 2. Furthermore, all users located in $[\,0, x_0\,)$ buy a phone from Firm 1, and all users located in $(\,x_0, 1\,]$ buy a phone from Firm 2. Therefore, (205) implies that Firm 1's profit is:

$$\pi_1 = [\,p_1 + r_L - c_1\,]\,\frac{1}{2\,t}\,[\,t + G_1 - G_2 + p_2 - p_1\,]\,. \tag{206}$$

The unique value of $p_1$ that maximizes $\pi_1$ in (206) is given by:

$$\frac{\partial \pi_1}{\partial p_1} = 0 \;\Leftrightarrow\; -\,[\,p_1 + r_L - c_1\,] + t + G_1 - G_2 + p_2 - p_1 = 0$$

$$\Leftrightarrow\; p_1 = \frac{1}{2}\,[\,t + c_1 - r_L + G_1 - G_2 + p_2\,]\,. \tag{207}$$

(205) also implies that Firm 2's profit is:

$$\pi_2 = [\,p_2 + r_H - c_2\,]\,\frac{1}{2\,t}\,[\,t + G_2 - G_1 + p_1 - p_2\,]\,. \tag{208}$$

The unique value of $p_2$ that maximizes $\pi_2$ in (208) is given by:

$$\frac{\partial \pi_2}{\partial p_2} = 0 \;\Leftrightarrow\; -\,[\,p_2 + r_H - c_2\,] + t + G_2 - G_1 + p_1 - p_2 = 0$$

$$\Leftrightarrow\; p_2 = \frac{1}{2}\,[\,t + c_2 - r_H + G_2 - G_1 + p_1\,]\,. \tag{209}$$

(207) and (209) imply:

$$p_1 = \frac{1}{2}\,[\,t + c_1 - r_L + G_1 - G_2\,] + \frac{1}{4}\,[\,t + c_2 - r_H + G_2 - G_1 + p_1\,]$$

$$\Rightarrow\; \frac{3}{4}\,p_1 = \frac{1}{4}\,[\,2\,t + 2\,c_1 - 2\,r_L + 2\,G_1 - 2\,G_2 + t + c_2 - r_H + G_2 - G_1\,]$$

$$\begin{aligned}
\Rightarrow \quad p_1 \;&=\; \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,] \\
&=\; \frac{1}{3}\,[\,3\,t + G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,) - 3\,r_L + 3\,c_1\,] \\
&=\; \frac{1}{3}\,[\,3\,t - 3\,A + 3\,(\,c_1 - r_L\,)\,] \;=\; c_1 - r_L + t - A\,.
\end{aligned} \tag{210}$$

(209) and (210) imply:

$$\begin{aligned}
p_2 \;&=\; \frac{1}{2}\,[\,t + c_2 - r_H + G_2 - G_1\,] + \frac{1}{6}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,] \\
&=\; \frac{1}{6}\,[\,3\,t + 3\,c_2 - 3\,r_H + 3\,G_2 - 3\,G_1 + 3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,] \\
&=\; \frac{1}{6}\,[\,6\,t + 4\,c_2 + 2\,c_1 + 2\,G_2 - 2\,G_1 - 4\,r_H - 2\,r_L\,] \\
&=\; \frac{1}{3}\,[\,3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 2\,r_H - r_L\,] \\
&=\; \frac{1}{3}\,[\,3\,t + G_2 + r_H - (\,G_1 + r_L - c_1\,) - 3\,r_H + 3\,c_2\,] \\
&=\; \frac{1}{3}\,[\,3\,t + 3\,A + 3\,(\,c_2 - r_H\,)\,] \;=\; c_2 - r_H + t + A\,.
\end{aligned} \tag{211}$$

(210) implies that Firm 1's profit margin is positive because:

$$p_1 + r_L - c_1 \;=\; t - A \;>\; 0\,. \tag{212}$$

(211) implies that Firm 2's profit margin is positive because:

$$p_2 + r_H - c_2 \;=\; t + A \;>\; 0\,. \tag{213}$$

(210) and (211) imply:

$$p_2 - p_1 \;=\; \frac{1}{3}\,[\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,]\,. \tag{214}$$

(210), (214), and Lemma A25 imply:

$$\begin{aligned}
\pi_1 \;&=\; \frac{1}{3}\,[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H + 3\,r_L - 3\,c_1\,] \\
&\quad\cdot\; \frac{1}{2\,t}\left[\,t + G_1 - G_2 + \frac{1}{3}\,(\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,)\,\right] \\
&=\; \frac{1}{18\,t}\,[\,3\,t + c_2 - c_1 + G_1 - G_2 + r_L - r_H\,]
\end{aligned}$$

$$\cdot \left[ 3t + 3G_1 - 3G_2 + c_2 - c_1 + 2G_2 - 2G_1 + r_L - r_H \right]$$

$$= \frac{1}{18t} \left[ 3t + c_2 - c_1 + G_1 - G_2 + r_L - r_H \right]^2$$

$$= \frac{1}{18t} \left[ 3t + G_1 + r_L - c_1 - (G_2 + r_H - c_2) \right]^2$$

$$= \frac{1}{18t} \left[ 3t - 3A \right]^2 = \frac{1}{2t} \left[ t - A \right]^2. \tag{215}$$

(211), (214), and Lemma A25 imply:

$$\pi_2 = \frac{1}{3} \left[ 3t + 2c_2 + c_1 + G_2 - G_1 - 2r_H - r_L + 3r_H - 3c_2 \right]$$

$$\cdot \frac{1}{2t} \left[ t + G_2 - G_1 + \frac{1}{3} ( c_1 - c_2 + 2G_1 - 2G_2 + r_H - r_L ) \right]$$

$$= \frac{1}{18t} \left[ 3t + c_1 - c_2 + G_2 - G_1 + r_H - r_L \right]$$

$$\cdot \left[ 3t + 3G_2 - 3G_1 + c_1 - c_2 + 2G_1 - 2G_2 + r_H - r_L \right]$$

$$= \frac{1}{18t} \left[ 3t + c_1 - c_2 + G_2 - G_1 + r_H - r_L \right]^2$$

$$= \frac{1}{18t} \left[ 3t + G_2 + r_H - c_2 - (G_1 + r_L - c_1) \right]^2$$

$$= \frac{1}{18t} \left[ 3t + 3A \right]^2 = \frac{1}{2t} \left[ t + A \right]^2. \blacksquare$$

**Lemma A27**. Suppose $K_1 \in [\, 0, t\, )$ and $K_2 \geq t$. Then: (i) a user who buys a phone from Firm 1 will change the default setting on the phone if and only if the user is located in $( \frac{1}{2} + \frac{K_1}{2t}, 1 ]$; and (ii) a user who buys a phone from Firm 2 will not change the default setting on the phone.

Proof. Result (i) follows from Lemma A2. Result (ii) follows from Lemma A24. ■

Proposition A1 refers to the following assumptions.

**Assumption A1.1**. $K_1 \in [\, 0, t\, )$ and $K_2 \geq t$.

**Assumption A1.2**. $K_1 > -A$.

**Assumption A1.3**. $t > |A|$.

**Assumption A1.4**. $t \geq \max \{ r_H - c_2 - A, \, r_L - c_1 + A \}$.

**Assumption A1.5**. $\frac{1}{2t} [t - A]^2 > -2A + \frac{t - K_1}{2t} [2t + r_H - r_L]$ if $A \leq 0$.

**Proposition A1**. Suppose Assumptions A1.1 – A1.5 hold. Then an equilibrium exists in which all users located in $[0, x_0)$ purchase a phone from Firm 1, and all users located in $(x_0, 1]$ purchase a phone from Firm 2, where $x_0 = \frac{1}{2} - \frac{A}{2t} \in [\frac{1}{2} - \frac{K_2}{2t}, \frac{1}{2} + \frac{K_1}{2t}]$. Furthermore, no user changes the default setting on the phone she buys. In addition, $p_1 = c_1 - r_L + t - A$, $p_2 = c_2 - r_H + t + A$, $\pi_1 = \frac{1}{2t} [t - A]^2$, and $\pi_2 = \frac{1}{2t} [t + A]^2$.

Proof. It is readily verified that in any equilibrium with the identified properties, Firm 1's profit is:

$$\pi_1 = [p_1 + r_L - c_1] \frac{1}{2t} [t + G_1 - G_2 + p_2 - p_1]. \tag{216}$$

The unique value of $p_1$ that maximizes $\pi_1$ in (216) is given by:

$$\frac{\partial \pi_1}{\partial p_1} = 0 \Leftrightarrow -[p_1 + r_L - c_1] + t + G_1 - G_2 + p_2 - p_1 = 0$$

$$\Leftrightarrow p_1 = \frac{1}{2} [t + c_1 - r_L + G_1 - G_2 + p_2]. \tag{217}$$

It is also readily verified that in any equilibrium with the identified properties, Firm 2's profit is:

$$\pi_2 = [p_2 + r_H - c_2] \frac{1}{2t} [t + G_2 - G_1 + p_1 - p_2]. \tag{218}$$

The unique value of $p_2$ that maximizes $\pi_2$ in (218) is given by:

$$\frac{\partial \pi_2}{\partial p_2} = 0 \Leftrightarrow -[p_2 + r_H - c_2] + t + G_2 - G_1 + p_1 - p_2 = 0$$

$$\Leftrightarrow p_2 = \frac{1}{2} [t + c_2 - r_H + G_2 - G_1 + p_1]. \tag{219}$$

(217) and (219) imply that in any equilibrium:

$$p_1 = \frac{1}{2} [t + c_1 - r_L + G_1 - G_2] + \frac{1}{4} [t + c_2 - r_H + G_2 - G_1 + p_1]$$

$$\Rightarrow \frac{3}{4} p_1 = \frac{1}{4} [2t + 2c_1 - 2r_L + 2G_1 - 2G_2 + t + c_2 - r_H + G_2 - G_1]$$

$$\Rightarrow p_1 = \frac{1}{3} [3t + 2c_1 + c_2 + G_1 - G_2 - 2r_L - r_H]$$

$$= \frac{1}{3} [3t + G_1 + r_L - c_1 - (G_2 + r_H - c_2) - 3r_L + 3c_1]$$

$$= \frac{1}{3} [3t - 3A + 3(c_1 - r_L)] = c_1 - r_L + t - A. \tag{220}$$

(219) and (220) imply:

$$
\begin{aligned}
p_2 &= \frac{1}{2}[\,t + c_2 - r_H + G_2 - G_1\,] + \frac{1}{6}[\,3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,] \\
&= \frac{1}{6}[\,3\,t + 3\,c_2 - 3\,r_H + 3\,G_2 - 3\,G_1 + 3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H\,] \\
&= \frac{1}{6}[\,6\,t + 4\,c_2 + 2\,c_1 + 2\,G_2 - 2\,G_1 - 4\,r_H - 2\,r_L\,] \\
&= \frac{1}{3}\,[\,3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 2\,r_H - r_L\,] \\
&= \frac{1}{3}\,[\,3\,t + G_2 + r_H - (\,G_1 + r_L - c_1\,) - 3\,r_H + 3\,c_2\,] \\
&= \frac{1}{3}\,[\,3\,t + 3\,A + 3\,(\,c_2 - r_H\,)\,] \;=\; c_2 - r_H + t + A\,. \qquad (221)
\end{aligned}
$$

(220) and Assumption A1.3 imply that Firm 1's profit margin is positive because:

$$p_1 + r_L - c_1 \;=\; t - A \;>\; 0\,.$$

(221) and Assumption A1.3 imply that Firm 2's profit margin is positive because:

$$p_2 + r_H - c_2 \;=\; t + A \;>\; 0\,.$$

(220) and Assumption A1.4 imply that $p_1 \geq 0$ because:

$$
\begin{aligned}
& p_1 \;\geq\; 0 \quad\Leftrightarrow\quad 3\,t + 2\,c_1 + c_2 + G_1 - G_2 - 2\,r_L - r_H \;\geq\; 0 \\
\Leftrightarrow\quad & G_2 + r_H - c_2 - (\,G_1 - 2\,r_L + 2\,c_1\,) \;\leq\; 3\,t \\
\Leftrightarrow\quad & G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) \;\leq\; 3\,t - 3\,r_L + 3\,c_1 \\
\Leftrightarrow\quad & A \;=\; \frac{1}{3}[\,G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,] \;\leq\; t - r_L + c_1\,. \qquad (222)
\end{aligned}
$$

(221) and Assumption A1.4 imply that $p_2 \geq 0$ because:

$$
\begin{aligned}
& p_2 \;\geq\; 0 \quad\Leftrightarrow\quad 3\,t + 2\,c_2 + c_1 + G_2 - G_1 - 2\,r_H - r_L \;\geq\; 0 \\
\Leftrightarrow\quad & G_2 - 2\,r_H + 2\,c_2 - (\,G_1 + r_L - c_1\,) \;\geq\; 3\,t \\
\Leftrightarrow\quad & G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) \;\geq\; 3\,r_H - 3\,c_2 - 3\,t \\
\Leftrightarrow\quad & A \;=\; \frac{1}{3}[\,G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,)\,] \;\geq\; r_H - c_2 - t\,. \qquad (223)
\end{aligned}
$$

(220) and (221) imply:

$$p_2 - p_1 \;=\; \frac{1}{3}\,[\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,]\,. \tag{224}$$

(5) and (224) imply that the user who is indifferent between purchasing a phone from Firm 1 and Firm 2 is located at:

$$\begin{aligned} x_0 \;&=\; \frac{1}{2\,t}\left[\,t + G_1 - G_2 + \frac{1}{3}\,(\,c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,)\right] \\ &=\; \frac{1}{6\,t}\,[\,3\,t + 3\,G_1 - 3\,G_2 + c_2 - c_1 + 2\,G_2 - 2\,G_1 + r_L - r_H\,] \\ &=\; \frac{1}{6\,t}\,[\,3\,t + G_1 - G_2 + c_2 - c_1 + r_L - r_H\,] \\ &=\; \frac{1}{2} - \frac{1}{6\,t}\,[\,G_2 + r_H - c_2 - (G_1 + r_L - c_1)\,] \;=\; \frac{1}{2} - \frac{A}{2\,t}\,. \end{aligned} \tag{225}$$

(225) and Assumption A1.2 imply that $x_0 \in [\,0, \frac{1}{2} + \frac{K_1}{2\,t}\,)$, so no user changes the default setting on the phone she purchases.

(220) and (225) imply:

$$\pi_1 \;=\; [\,p_1 + r_L - c_1\,]\;x_0 \;=\; [\,t - A\,]\left[\frac{t - A}{2\,t}\right] \;=\; \frac{1}{2\,t}\;[\,t - A\,]^2\,. \tag{226}$$

(221) and (225) imply:

$$\pi_2 \;=\; [\,p_2 + r_H - c_2\,]\,[\,1 - x_0\,] \;=\; [\,t + A\,]\left[\frac{t + A}{2\,t}\right] \;=\; \frac{1}{2\,t}\;[\,t + A\,]^2\,. \tag{227}$$

(224) implies:

$$\begin{aligned} & p_1 - p_2 \;>\; G_1 - G_2 - K_1 \\ \Leftrightarrow\;\; & \frac{1}{3}\,[\,c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H\,] \;>\; G_1 - G_2 - K_1 \\ \Leftrightarrow\;\; & c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H \;>\; 3\,G_1 - 3\,G_2 - 3\,K_1 \\ \Leftrightarrow\;\; & c_1 - c_2 - r_L + r_H \;>\; G_1 - G_2 - 3\,K_1 \\ \Leftrightarrow\;\; & K_1 \;>\; \frac{1}{3}\,[\,G_1 - G_2 + c_2 - c_1 + r_L - r_H\,] \;=\; -\,A\,. \end{aligned} \tag{228}$$

(224) also implies:

$$\begin{aligned} & p_1 - p_2 \;<\; G_1 - G_2 + t \\ \Leftrightarrow\;\; & \frac{1}{3}\,[\,c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H\,] \;<\; G_1 - G_2 + t \\ \Leftrightarrow\;\; & c_1 - c_2 - 2\,G_2 + 2\,G_1 - r_L + r_H \;<\; 3\,G_1 - 3\,G_2 + 3\,t \end{aligned}$$

$$\Leftrightarrow \quad c_1 - c_2 - r_L + r_H \ < \ G_1 - G_2 + 3\,t$$

$$\Leftrightarrow \quad t \ > \ \frac{1}{3}\,[\,G_2 - G_1 + c_1 - c_2 + r_H - r_L\,] \ = \ A\,. \tag{229}$$

(228), (229), and Assumptions A1.2 – A1.3 imply:

$$p_1 - p_2 \ \in \ (\,G_1 - G_2 - K_1, G_1 - G_2 + t\,). \tag{230}$$

The foregoing analysis implies that the identified putative equilibrium is unique among equilibria in which (230) holds. It remains to verify that neither firm can increase its profit by unilaterally changing its price so that (230) does not hold. We first show this is the case for Firm 1.

If Firm 1 sets $p_1 \geq p_2 + G_1 - G_2 + t$, then no users purchase a phone from Firm 1. Therefore, Firm 1's profit (0) is less than the profit specified in (226).

If Firm 1 sets $p_1 < p_2 + G_1 - G_2 - K_1$, then all users purchase a phone from Firm 1. (4) and (5) imply that the maximum profit Firm 1 can secure by setting such a price when $p_2$ is as specified in (221) is nearly:

$$\begin{aligned}
\pi_{1D} \ &= \ p_2 + G_1 - G_2 - K_1 + r_1 - c_1 \ = \ c_2 - r_H + t + A + G_1 - G_2 - K_1 + r_1 - c_1 \\
&= \ G_1 - c_1 - (\,G_2 + r_H - c_2\,) + A + t + r_1 - K_1 \\
&= \ G_1 + r_L - c_1 - (\,G_2 + r_H - c_2\,) + A + t + r_1 - r_L - K_1 \\
&= \ -3\,A + A + t + \frac{1}{2}\,[\,r_L + r_H\,] - \frac{K_1}{2\,t}\,[\,r_H - r_L\,] - r_L - K_1 \\
&= \ -2\,A + t - K_1 + \frac{1}{2}\,[\,r_H - r_L\,] - \frac{K_1}{2\,t}\,[\,r_H - r_L\,] \\
&= \ -2\,A + t - K_1 + \frac{r_H - r_L}{2\,t}\,[\,t - K_1\,] \ = \ -2\,A + \frac{t - K_1}{2\,t}\,[\,2\,t + r_H - r_L\,]\,.
\end{aligned} \tag{231}$$

(226) and (231) imply that Firm 1 cannot increase its profit by setting $p_1 < p_2 + G_1 - G_2 - K_1$ when $p_2$ is as specified in (221) if Assumption A1.5 holds.

If Firm 1 sets $p_1 = p_2 + G_1 - G_2 - K_1$ when $p_2$ is as specified in (221), then: (i) all users located in $[\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[\,0, \frac{1}{2} + \frac{K_1}{2\,t}\,)$ buy a phone from Firm 1. Therefore, Firm 1's profit is:

$$\begin{aligned}
\pi_1 \ &= \ [\,p_1 + r_L - c_1\,] \left[ \frac{1}{2} + \frac{K_1}{2\,t} \right] + [\,p_1 + r_H - c_1\,]\,\frac{1}{2} \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \qquad (232) \\
&< \ [\,p_1 + r_L - c_1\,] \left[ \frac{1}{2} + \frac{K_1}{2\,t} \right] + [\,p_1 + r_H - c_1\,] \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right]
\end{aligned}$$

$$= \; p_1 + r_1 - c_1 \; = \; \pi_{1D} \; < \; \frac{1}{2\,t}\,[\,t - A\,]^2 \,. \tag{233}$$

The first inequality in (233) holds because $p_1 + r_H - c_1$ must be strictly positive if Firm 1 is to secure positive profit in this case. The last inequality in (233) reflects (231) and Assumption A1.5. (226) and (233) imply that Firm 1 will not set $p_1 = p_2 + G_1 - G_2 - K_1$ when $p_2$ is as specified in (221).

Now we show that Firm 2 cannot increase its profit by unilaterally changing its price so that (230) does not hold when $p_1$ is as specified in (220).

If Firm 2 sets $p_2 > p_1 + G_2 - G_1 + K_1$, then no users purchase a phone from Firm 2, so Firm 2's profit (0) is less than the profit specified in (227).

If Firm 2 sets $p_2 = p_1 + G_2 - G_1 + K_1$ when $p_1$ is as specified in (220), then:

$$\begin{aligned} p_2 \; &= \; p_1 + G_2 - G_1 + K_1 \; = \; c_1 - r_L + t - A + G_2 - G_1 + K_1 \\ &= \; G_2 + r_H - c_2 - G_1 - r_L + c_2 + t - A + K_1 + c_2 - r_H \\ &= \; 3\,A + t - A + K_1 + c_2 - r_H \; = \; 2\,A + t + K_1 + c_2 - r_H \; > \; c_2 - r_H \,. \end{aligned}$$

The last inequality holds here because $K_1 > -A$ and $t > -A$, by assumption. Because $p_2 = p_1 + G_2 - G_1 + K_1 \; > \; c_2 - r_H$ when $p_1$ is as specified in (220), Firm 2 can increase its profit by setting $p_2$ to ensure $p_1 - p_2 \in (G_1 - G_2 - K_1, G_1 - G_2 + K_2)$. Therefore, Firm 2 cannot increase its profit by setting $p_2 = p_1 + G_2 - G_1 + K_1$ when $p_1$ is as specified in (220).

If Firm 2 sets $p_2 \leq p_1 + G_2 - G_1 - t$, then all users purchase a phone from Firm 2. (4) and (5) imply that the maximum profit Firm 2 can secure by setting such a price when $p_1$ is as specified in (220) is:

$$\begin{aligned} \pi_{2D} \; &= \; p_1 + G_2 - G_1 - t + r_H - c_2 \; = \; c_1 - r_L + t - A + G_2 - G_1 - t + r_H - c_2 \\ &= \; G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) - A \; = \; 3\,A - A \; = \; 2\,A \,. \end{aligned} \tag{234}$$

(234) implies:

$$\begin{aligned} \pi_1 \; > \; \pi_{2D} \;\; &\Leftrightarrow \;\; \frac{1}{2\,t}\,[\,t + A\,]^2 \; > \; 2\,A \;\; \Leftrightarrow \;\; t^2 + 2\,A\,t + A^2 \; > \; 4\,A\,t \\ &\Leftrightarrow \;\; t^2 - 2\,A\,t + A^2 \; > \; 0 \;\; \Leftrightarrow \;\; [\,t - A\,]^2 \; > \; 0 \,. \end{aligned} \tag{235}$$

(235) implies that Firm 2 cannot increase its profit by setting $p_2 \leq p_1 + G_2 - G_1 - K_2$ when $p_1$ is as specified in (220). ■

Proposition A2 refers to the following assumptions.

**Assumption A2.1**. $K_1 \in [\,0,\, t\,)$ and $K_2 \geq t$.

**Assumption A2.2**. $c_1 > r_1$.

**Assumption A2.3**. $t < G_2 - G_1 + c_1 - r_1$.

**Assumption A2.4**. $\frac{2t - r_H + r_L}{2} < \Omega_2(p_1)$

for all $p_1 \in [\,c_1 - r_1,\, min\,\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$.[2]

**Proposition A2**. Suppose Assumptions A2.1 – A2.4 hold. Then a family of equilibria exist in which all users purchase a phone from Firm 2. In these equilibria, no user changes the default setting on the phone they purchase. Furthermore, $p_1 \in [\,c_1 - r_1,\, min\,\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,] > 0$; $p_2 = p_1 + G_2 - G_1 - t > 0$; $\pi_1 = 0$; and $\pi_2 = p_1 + r_H - c_2 + G_2 - G_1 - t > 0$.

Proof. If $p_2 = p_1 + G_2 - G_1 - t$, then all users buy a phone from Firm 2. Therefore, Firm 1's profit is 0.

We first show that if Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$, then Firm 1 will set $p_1 \in [\,c_1 - r_1,\, min\,\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$ in equilibrium. We do so first by explaining why it cannot be the case that $p_1 < c_1 - r_1$ or $p_1 > min\,\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}$ in equilibrium. Then we explain why, when $p_1 \in [\,c_1 - r_1,\, min\,\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$ and Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$, Firm 1 cannot strictly increase its profit by setting a different price.

Recall that setting $p_1 < c_1 - r_1$ is a dominated strategy for Firm 1.

Consider a putative equilibrium in which $p_1 > c_1 - r_1 + K_1 + t$ and Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$. In this case:

$$p_2 > c_1 - r_1 + K_1 + t + G_2 - G_1 - t = G_2 - G_1 + K_1 + c_1 - r_1\,. \tag{236}$$

If Firm 1 sets $p_1$ marginally below $p_2 + G_1 - G_2 - K_1$, all users will purchase a phone from Firm 1. Consequently, when $p_2 = p_1 + G_2 - G_1 - t$, (236) implies that Firm 1's profit will be nearly:

$$\begin{aligned}\pi_1 &= p_1 + r_1 - c_1 = p_2 + G_1 - G_2 - K_1 + r_1 - c_1\\ &> G_2 - G_1 + K_1 + c_1 - r_1 + G_1 - G_2 - K_1 + r_1 - c_1 = 0\,.\end{aligned}$$

[2]Recall that $\Omega_2(\cdot)$ is defined in Condition 3C.

Because Firm 1 can secure strictly positive profit by deviating from its strategy in the putative equilibrium, the putative equilibrium cannot be an equilibrium.

Next, consider a putative equilibrium in which $p_1 > c_1 - r_L$ and Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$. This implies:

$$p_2 > c_1 - r_L + G_2 - G_1 - t\,. \tag{237}$$

All users buy a phone from Firm 2 when (237) holds. Therefore, Firm 1's profit is 0.

Suppose Firm 1 reduces its price marginally below $p_1'$ where $p_1'$ is equal to:

$$p_1' = p_2 + G_1 - G_2 + t > c_1 - r_L + G_2 - G_1 - t + G_1 - G_2 + t = c_1 - r_L.$$

Firm 1's profit in this case is nearly:

$$\pi_1' = [\,p_1' + r_L - c_1] > 0\,.$$

This inequality holds because $p_1' > c_1 - r_L$ and $x_0 > 0$ (since $p_1' - p_2 < t + G_1 - G_2$). Because Firm 1 can secure strictly positive profit by deviating from its strategy in the putative equilibrium, the putative equilibrium cannot be an equilibrium.

We now show that when $p_1 \in [\,c_1 - r_1,\, c_1 - r_L\,]$ and Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$ , Firm 1 cannot increase its profit by setting a price $p_1' \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + t\,)$. When Firm 1 sets such a price, all users located in $[\,0, x_0)$ buy a phone from Firm 1 and Firm secures profit:

$$\pi_1 = [\,p_1' + r_L - c_1\,]\, x_0 \leq 0. \tag{238}$$

The inequality in (238) holds because $p_1' \leq c_1 - r_L$ and because $x_0 > 0$ in the present setting. (238) implies that Firm 1 cannot secure strictly positive profit by setting $p_1' \in (\,p_2 + G_1 - G_2 - K_1,\, p_2 + G_1 - G_2 + t\,)$ under the maintained conditions.

Next we show that Firm 1 cannot increase its profit by setting a price $p_1' \leq p_2 + G_1 - G_2 - K_1$. Observe that under the maintained conditions:

$$c_1 - r_1 + K_1 + t \geq p_1 = p_2 + G_1 - G_2 + t \tag{239}$$

$$\Rightarrow \quad c_1 - r_1 \geq p_2 + G_1 - G_2 - K_1. \tag{240}$$

The weak inequality in (239) holds because $p_1 \in [\,c_1 - r_1,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$. The equality in (239) holds because $p_2 = p_1 + G_2 - G_1 - t$. (240) implies that $p_1' < c_1 - r_1$ if Firm 1 sets $p_1' \leq p_2 + G_1 - G_2 - K_1$ and $c_1 - r_1 \neq p_2 + G_1 - G_2 - K_1$. This is a dominated strategy for Firm 1.

Now we show that Firm 1 cannot increase its profit by setting a price $p_1' \geq p_2 + G_1 - G_2 + t$. No user will purchase a phone from Firm 1 in this case. Consequently, Firm 1's profit is 0.

In summary, we have shown that if Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$, then Firm 1 will set $p_1 \in [\,c_1 - r_1,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + K_2\,\}\,]$ in equilibrium.

We now show that Firm 2 maximizes its profit by setting $p_2 = p_1 + G_2 - G_1 - t$ when Firm 1 sets $p_1 \in [\, c_1 - r_1 \,, \, min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + t \, \} \,]$. Observe first that this value of $p_2$ is positive because:

$$p_1 + G_2 - G_1 - t \;\geq\; c_1 - r_1 + G_2 - G_1 - t \;>\; 0\,.$$

The inequality here reflects Assumption A2.3.

When Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$, all users purchase a phone from Firm 2, and Firm 2's profit is:

$$\pi_2 \;=\; p_2 + r_H - c_2 \;=\; p_1 + G_2 - G_1 - t - c_2 + r_H. \tag{241}$$

In equilibria in which $p_1 \in [\, c_1 - r_1 \,, \, min \, \{ \, c_1 - r_L, \, c_1 - r_1 + K_1 + t \, \} \,]$ and Firm 2 sets $p_2 = p_1 + G_2 - G_1 - t$, Firm 2 secures the least profit when $p_1 = c_1 - r_1$. Consequently, (4) and (241) implies that Firm 2 earns positive profit in all such equilibria if:

$$\begin{aligned}
\pi_2^{min} \;&=\; c_1 - r_1 + G_2 - G_1 - t - c_2 + r_H \\
&=\; G_2 - G_1 + c_1 - \frac{1}{2}\,[\, r_L + r_H \,] + \frac{K_1}{2\,t}\,[\, r_H - r_L \,] - t - c_2 + r_H \\
&=\; G_2 - G_1 + c_1 - c_2 - t + \frac{1}{2}\,[\, r_H - r_L \,] + \frac{K_1}{2\,t}\,[\, r_H - r_L \,] \\
&=\; G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\,[\, r_H - r_L \,] - \left[ \frac{2\,t - (r_H - r_L)}{2} \right].
\end{aligned} \tag{242}$$

(4) and Condition 3C imply that when $p_1 = c_1 - r_1$:

$$\begin{aligned}
x_2 \;&=\; \frac{1}{8\,t}\,[\, t + G_2 - G_1 - c_2 + r_H + c_1 - r_1 \,]^2 \\
&=\; \frac{1}{8\,t}\left[ t + G_2 - G_1 - c_2 + r_H + c_1 - \frac{1}{2}\,(\, r_H + r_L \,) + \frac{K_1}{2\,t}\,(\, r_H - r_L \,) \right]^2 \\
&=\; \frac{1}{8\,t}\left[ t + G_2 - G_1 + c_1 - c_2 + \left( \frac{1}{2} + \frac{K_1}{2\,t} \right)(\, r_H - r_L \,) \right]^2.
\end{aligned} \tag{243}$$

(4) and Condition 3C also imply that when $p_1 = c_1 - r_1$:

$$\begin{aligned}
\Omega_2(\cdot) \;&=\; c_1 - r_1 + G_2 - G_1 - c_2 + \frac{1}{2}\,[\, r_H + r_L \,] - x_2 \\
&=\; c_1 - \frac{1}{2}\,(\, r_H + r_L \,) + \frac{K_1}{2\,t}\,(\, r_H - r_L \,) + G_2 - G_1 - c_2 + \frac{1}{2}\,[\, r_H + r_L \,] - x_2 \\
&=\; G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\,[\, r_H - r_L \,] - x_2\,.
\end{aligned} \tag{244}$$

(242) – (244) imply that when $p_1 = c_1 - r_1$ and Assumption A2.4 holds:

$$\begin{aligned}\pi_2^{min} &> G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\,[\,r_H - r_L\,] - \Omega_2 \\ &= G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\,[\,r_H - r_L\,] \\ &\quad - \left( G_2 - G_1 + c_1 - c_2 + \frac{K_1}{2\,t}\,[\,r_H - r_L\,] - x_2 \right) = x_2 \geq 0\,. \end{aligned} \tag{245}$$

We now show that Firm 2 cannot increase its profit by setting $p_2 \in (\,p_1 + G_2 - G_1 - t,$ $p_1 + G_2 - G_1 + K_1\,)$ or $p_2 \geq p_1 + G_2 - G_1 + K_1$ when Firm 1 sets $p_1 \in [\,c_1 - r_1\,,\, min\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$.

(219) implies that when $p_2 \in (\,p_1 + G_2 - G_1 - t\,,\, p_1 + G_2 - G_1 + K_1\,)$, the price that maximizes Firm 2's profit is:

$$p_2 = \frac{1}{2}\,[\,t + c_2 - r_H + G_2 - G_1 + p_1\,]\,. \tag{246}$$

(218) and (246) imply that Firm 2's corresponding profit is:

$$\begin{aligned}\pi_2^{'} &= [\,p_2 + r_H - c_2\,]\,\frac{1}{2\,t}\,[\,t + G_2 - G_1 + p_1 - p_2\,] \\ &= \left[\frac{1}{2}\,(\,t + c_2 - r_H + G_2 - G_1 + p_1\,) + r_H - c_2\right] \\ &\quad \cdot \frac{1}{2\,t}\left[\,t + G_2 - G_1 + p_1 - \frac{1}{2}\,(\,t + c_2 - r_H + G_2 - G_1 + p_1\,)\right] \\ &= \frac{1}{8\,t}\,[\,t + G_2 - G_1 - c_2 + r_H + p_1\,]^2\,. \end{aligned} \tag{247}$$

Observe that:

$$\begin{aligned}\pi_2 &= p_1 + r_H - c_2 + G_2 - G_1 - t \\ &= p_1 + r_H + \frac{1}{2}\,[\,r_L + r_H\,] - \frac{1}{2}\,[\,r_L + r_H\,] - c_2 + G_2 - G_1 - t \\ &= p_1 + \frac{1}{2}\,[\,r_L + r_H\,] + G_2 - G_1 - c_2 - t + \frac{1}{2}\,[\,r_H - r_L\,]\,. \end{aligned} \tag{248}$$

(245) and (248) imply that Firm 2 cannot increase its profit by setting $p_2 \in (\,p_1 + G_2 - G_1 - t\,,\, p_1 + G_2 - G_1 + K_1\,)$ when Assumption A2.4 holds because:

$$\pi_2 \geq \pi_2^{'} \;\Leftrightarrow\; p_1 + \frac{1}{2}\,[\,r_L + r_H\,] + G_2 - G_1 - c_2 - t + \frac{1}{2}\,[\,r_H - r_L\,]$$

$$\geq \frac{1}{8\,t}\,[\,t + G_2 - G_1 - c_2 + r_H + p_1\,]^2$$

$$\Leftrightarrow \quad \frac{2\,t - r_H + r_L}{2} \;\leq\; p_1 + G_2 - G_1 - c_2 + \frac{1}{2}\,[\,r_L + r_H\,]$$

$$- \; \frac{1}{8\,t}\,[\,t + G_2 - G_1 - c_2 + r_H + p_1\,]^2 \;=\; \Omega_2\,.$$

If Firm 2 sets $p_2 > p_1 + G_2 - G_1 + K_1$, it will sell no phones and so will secure 0 profit. Therefore, $p_2 = p_1 + G_2 - G_1 + K_1$ is the profit-maximizing price for Firm 2 among all $p_2 \geq p_1 + G_2 - G_1 + K_1$. When $p_2 = p_1 + G_2 - G_1 + K_1$: (i) all users located in $[\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ are indifferent between buying a phone from Firm 1 and from Firm 2; and (ii) all users located in $[\,0, \frac{1}{2} + \frac{K_1}{2\,t}\,)$ buy a phone from Firm 1. Firm 2's profit in this case is:

$$\pi_2^{''} \;=\; \frac{1}{2}\,[\,p_2 + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]$$

$$=\; \frac{1}{2}\,[\,p_1 + G_2 - G_1 + K_1 + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] \;>\; 0\,. \tag{249}$$

The inequality in (249) holds because (4) implies that the minimum value of $\pi_2^{''}$, which occurs when $p_1 = c_1 - r_1$, is:

$$\pi_2^{''min} \;=\; \frac{1}{2}\,[\,c_1 - r_1 + G_2 - G_1 + K_1 + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]$$

$$=\; \frac{1}{2}\left[G_2 - G_1 + c_1 - c_2 + K_1 + \left(\frac{1}{2} + \frac{K_1}{2\,t}\right)(\,r_H - r_L\,)\right]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] \;>\; 0\,. \tag{250}$$

The inequality in (250) holds because: (i) $K_1 < t$, by assumption; and (ii) the term in the first square brackets in (250) is positive. (ii) holds because Assumption A2.3 ensures this term exceeds:

$$t + K_1 + \left[\frac{1}{2} + \frac{K_1}{2\,t}\right][\,r_H - r_L\,] \;>\; 0\,.$$

(250) ensures that (249) holds.

If Firm 2 were to reduce its price to $p_2 = p_1 + G_2 - G_1 + K_1 - \varepsilon_{10}$ where $\varepsilon_{10} > 0$, all users located in $[\,\frac{1}{2} + \frac{K_1}{2\,t}, 1\,]$ would purchase a phone from Firm 2. Consequently, (249) implies that Firm 2's profit would be at least:

$$\pi_2 \;=\; [\,p_2 - \varepsilon_{10} + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]$$

$$=\; [\,p_1 + G_2 - G_1 + K_1 - \varepsilon_{10} + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]$$

$$=\; [\,p_1 + G_2 - G_1 + K_1 + r_H - c_2\,]\left[\frac{1}{2} - \frac{K_1}{2\,t}\right] - \varepsilon_{10}\left[\frac{1}{2} - \frac{K_1}{2\,t}\right]$$

$$= \pi_2^{''} + \pi_2^{''} - \varepsilon_{10} \left[ \frac{1}{2} - \frac{K_1}{2\,t} \right] \;>\; \pi_2^{''} \quad \text{for sufficiently small } \varepsilon_{10}\,. \tag{251}$$

(251) implies that Firm 2 could increase its profit by reducing its price marginally below $p_1 + G_2 - G_1 + K_1$. Therefore, Firm 2 will never set $p_2 \geq p_1 + G_2 - G_1 + K_1$ when $p_1 \in [\, c_1 - r_1\,,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$.

In summary, we have shown that when $p_1 \in [\, c_1 - r_1\,,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$, Firm 2 maximizes its profit by setting $p_2 = p_1 + G_2 - G_1 - t$.

Finally, observe that no user changes the default setting on the phone they purchase from Firm 2 in the present setting. ■

**Proposition 10.** *Suppose: (i) $K_1 = 0$ and $K_2 = K_2^* \geq t$; (ii) Condition 3A holds; (iii) $t < G_2 - G_1 + c_1 - r_1$; (iv) $\frac{2t-\Delta}{2} < \Omega_2(p_1)$ for all $p_1 \in [\, c_1 - r_1, min\,\{\, c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$;*[3] *(v) $G_2 - c_2 > G_1 - c_1$; and (vi) $\Delta > 2\,t$. Then in the setting with endogenous $K$, $(0, K_2^*)$ and $(p_1, p_2)$ prices such that $p_1 \in [\, c_1 - r_1,\, min\,\{\, c_1 - r_L,\, c_1 - r_1 + t\,\}\,]$ and $p_2$ is marginally below $p_1 + G_2 - G_1 - t$ constitute a MD2 equilibrium.*

<u>Proof</u>. The proof proceeds by showing that neither firm can strictly increase its profit by unilaterally changing its default-switching cost, regardless of the nature of the ensuing equilibrium.

First consider Firm 2. Under the specified conditions, Firm 2's profit in the MD2 equilibrium identified in Proposition A2 is:

$$\pi_2^* \;=\; p_1 + r_H - c_2 + G_2 - G_1 - t \;>\; 0\,. \tag{252}$$

It is apparent from (252) that this profit does not vary with $K_2$ when $K_2 \geq t$.

Proposition 3 implies that if Firm 2 reduces $K_2$ to $K_2^{'} \in (0, t\,)$ and a MD2 equilibrium prevails, then Firm 2's profit is nearly:

$$\pi_2^{'} \;=\; p_1 - c_2 + G_2 - G_1 - K_2^{'} + \frac{1}{2}\,[\, r_H + r_L\,] + \frac{K_2^{'}}{2\,t}\,[\, r_H - r_L\,]\,. \tag{253}$$

(252) and (253) imply:

$$\begin{aligned} \pi_2^* - \pi_2^{'} \;&=\; p_1 + r_H - c_2 + G_2 - G_1 - t \\ &\quad - \left( p_1 - c_2 + G_2 - G_1 - K_2^{'} + \frac{1}{2}\,[\, r_H + r_L\,] + K_2^{'}\,\frac{\Delta}{2\,t} \right) \\ &=\; r_H - t + K_2^{'} - \frac{1}{2}\,[\, r_H + r_L\,] - K_2^{'}\,\frac{\Delta}{2\,t}\,. \end{aligned} \tag{254}$$

[3] Recall that $\Omega_2$ is defined by Condition 3C.

(254) implies:

$$\left(\pi_2^* - \pi_2'\right)\Big|_{K_2'=t} \;=\; r_H - \frac{1}{2}\left[\, r_H + r_L \,\right] - \frac{\Delta}{2} \;=\; \frac{\Delta}{2} - \frac{\Delta}{2} \;=\; 0 \;\; \text{and}$$

$$\frac{\partial\left(\pi_2^* - \pi_2'\right)}{\partial K_2'} \;=\; 1 - \frac{\Delta}{2\,t} \;<\; 0\,. \tag{255}$$

(255) implies that $\pi_2^* > \pi_2'$ for all $K_2' \in (0, t\,)$. Therefore, Firm 2 cannot strictly increase its profit by setting $K_2 \in (0, t\,)$ if a MD2 equilibrium prevails.

Lemma A21 implies that if Firm 2 reduces $K_2$ to 0, its profit in the resulting MD2 equilibrium is nearly:

$$\pi_2^0 \;=\; G_2 - c_2 - (\, G_1 - c_1\,)\,. \tag{256}$$

(252) and (256) imply:

$$\pi_2^* \;\geq\; \pi_2^0 \;\;\Leftrightarrow\;\; p_1 + r_H - c_2 + G_2 - G_1 - t \;\geq\; G_2 - c_2 - (\, G_1 - c_1\,)$$

$$\Leftrightarrow\;\; p_1 + r_H - t \;\geq\; c_1\,. \tag{257}$$

Proposition A2 implies that when $K_1 = 0$, the lowest price that Firm 1's will set in a MD2 equilibrium is $c_1 - \frac{1}{2}\left[\, r_H + r_L\,\right]$. Therefore, (257) implies:

$$\pi_2^* \;\geq\; \pi_2^0 \;\; \text{if} \;\; c_1 - \frac{1}{2}\left[\, r_H + r_L\,\right] + r_H - t \;\geq\; c_1 \;\;\Leftrightarrow\;\; \frac{1}{2}\Delta \;\geq\; t \;\;\Leftrightarrow\;\; \Delta \;\geq\; 2\,t\,. \tag{258}$$

The last inequality in (258) holds, by assumption. Therefore, (258) implies that Firm 2 cannot increase its profit by reducing $K_2$ to 0 (which induces a MD2 equilibrium when $K_1 = 0$ and $G_2 - c_2 > G_1 - c_1$).

Firm 2's profit is 0 in any MD1 equilibrium. Therefore, Proposition A2 implies that Firm 2 cannot increase its profit by setting a $K_2$ for which the resulting $(0, K_2)$ default-switching costs induce a MD1 equilibrium.

Finally, the proof of Proposition A2 establishes that Firm 2 cannot increase its profit by setting a $K_2 \neq K_2^*$ that induces a MS equilibrium under the specified conditions.

To initiate the demonstration that Firm 1 cannot increase its profit by unilaterally increasing $K_1$, observe that Firm 1's profit is 0 in all MD2 equilibria. Consequently, Firm 1 cannot increase its profit by implementing a $K_1 > 0$ that induces a MD2 equilibrium.

Next we establish that a MD1 equilibrium does not exist when $G_2 - c_2 > G_1 - c_1$. To do so, suppose such an equilibrium exists. Then the consumer located at 1 (weakly) prefers to buy a phone from Firm 1 than from Firm 2. Consequently:

$$G_1 - p_1 - \min\{\, K_1,\, t\,\} \;\geq\; G_2 - p_2 \;\;\Rightarrow\;\; p_1 \;\leq\; p_2 + G_1 - G_2 - \min\{\, K_1,\, t\,\}\,. \tag{259}$$

(259) reflects the fact that the consumer located at 1 who purchases a phone from Firm 1

will change the default PD setting on the phone if and only if $K_1 < t$.

Rather than serve no customers, Firm 2 will reduce its price to $c_2 - r_H$. Therefore, (259) implies that, to attract all consumers, Firm 1's price must satisfy:

$$p_1 \ \leq \ c_2 - r_H + G_1 - G_2 - \min\{\, K_1,\, t\,\}. \tag{260}$$

In any MD1 equilibrium in which (260) holds, Firm 1's profit is:

$$\begin{aligned} \pi_1 \ &\leq \ c_2 - r_H + G_1 - G_2 - \min\{\, K_1, t\,\} - c_1 + r_H \\ &< \ c_2 + G_1 - G_2 - c_1 \ = \ G_1 - c_1 - (G_2 - c_2) \ < \ 0\,. \end{aligned} \tag{261}$$

The first inequality in (261) reflects the fact that Firm 1's revenue from advertisers cannot exceed $r_H$. The second inequality in (261) holds because $K_1 > 0$ (and $t > 0$), by assumption. The last inequality in (261) holds because $G_2 - c_2 > G_1 - c_1$, by assumption. (261) implies that a MD1 equilibrium does not exist under the maintained assumptions because Firm 1's profit must be nonnegative in a MD1 equilibrium.

Finally, we establish that Firm 1 cannot increase its profit by setting a $K_1 > 0$ that induces a MS equilibrium when $K_2 \geq t$ and $G_2 - c_2 > G_1 - c_1$. In a MS equilibrium, a consumer located at $x_0 \in (0,1)$ is indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2. The proofs of Lemma A25 and Proposition A1 imply that at this equilibrium:

$$p_1 \ = \ c_1 - r_L + t - A, \quad p_2 \ = \ c_2 - r_H + t + A, \ \text{ and} \tag{262}$$

$$x_0 \ = \ \frac{1}{2} - \frac{A}{2\,t} \ < \ \frac{1}{2} \ \text{ and } \ \pi_1^S \ = \ \frac{1}{2\,t}\,[\,t - A\,]^2\,. \tag{263}$$

The inequality in (263) holds because $A > 0$ when $G_2 - c_2 > G_1 - c_1$.

Because $x_0 < \frac{1}{2}$ and $K_2 \geq t$, $x_0 < \frac{1}{2} + \frac{K_1}{2\,t}$ when $K_1 > 0$. Therefore, Lemma A26 implies the consumer located at $x_0$ will not change the default PD setting on any phone he purchases. Consequently, because the consumer located at $x_0$ is indifferent between purchasing a phone from Firm 1 and purchasing a phone from Firm 2, and because $x_0 = \frac{1}{2} - \frac{A}{2\,t}$:

$$\begin{aligned} G_1 - p_1 - t\,x_0 \ = \ G_2 - p_2 - t\,[\,1 - x_0\,] \ &\Rightarrow \ p_2 - p_1 \ = \ G_2 - G_1 - t + t\,x_0 \\ &\Rightarrow \ p_2 - p_1 \ = \ G_2 - G_1 - t + t\left[\frac{1}{2} - \frac{A}{2\,t}\right] \\ &\Rightarrow \ p_2 - p_1 \ = \ G_2 - G_1 - \frac{t}{2} - \frac{A}{2}\,. \end{aligned} \tag{264}$$

(262) implies:

$$p_2 - p_1 \ = \ c_2 - c_1 - r_H + r_L + 2\,A\,. \tag{265}$$

(264) and (265) imply:

$$
\begin{aligned}
& G_2 - G_1 - \frac{t}{2} - \frac{A}{2} \; = \; c_2 - c_1 - r_H + r_L + 2\,A \\
\Rightarrow \;\; & G_2 + r_H - c_2 - (\,G_1 + r_L - c_1\,) \; = \; \frac{t}{2} + \frac{A}{2} + 2\,A \\
\Rightarrow \;\; & 3\,A \; = \; \frac{t}{2} + \frac{5}{2}\,A \;\; \Rightarrow \;\; \frac{A}{2} \; = \; \frac{t}{2} \;\; \Rightarrow \;\; t \; = \; A\,. \qquad (266)
\end{aligned}
$$

(263) and (266) imply that $\pi_1^S = 0$ at this equilibrium. ■

**Proposition 11.** *Suppose (i)* $A < 0$*; (ii)* $t < |A|$*; (iii)* $G_1 - c_1 - (G_2 - c_2) > \frac{\Delta}{2}$*; (iv) Conditions 2A and 2B hold; (v)* $t < c_1 - r_L - 3\,A$*; (vi) and (vii)* $t \leq \frac{2\,t}{2\,t + r_H - r_L}\,\Omega_1$*.*[4] *Then a MD1 equilibrium in which* $K_1 > 0$ *and* $K_2 \geq 0$ *does not exist in the setting with endogenous* $K$*.*

Proof. First consider a putative MD1 equilibrium in which $K_1 \in (0,\, t\,)$ and $K_2 \geq 0$. Arguments analogous to those employed in the proof of Proposition 2 reveal that Firm 1 can increase its profit by reducing $K_1$ marginally. Therefore, the putative equilibrium cannot constitute an equilibrium.

Next consider a putative MD1 equilibrium in which $K_1 \geq t$ and $K_2 \geq 0$. It is readily verified that Firm 1's profit in this equilibrium is $\pi_1^{D1} = -3\,A - t$. Proposition 2 implies that if Firm 1 reduces $K_1$ below $t$, it can secure a profit of nearly $\pi_1^{D1'} = \frac{t - K_1}{2\,t}\,\Delta - 3\,A - K_1$. Observe that when $K_1 \in (0, t\,)$:

$$
\begin{aligned}
& \pi_1^{D1'} \; > \; \pi_1^{D1} \;\; \Leftrightarrow \;\; \frac{t - K_1}{2\,t}\,\Delta - 3\,A - K_1 \; > \; -3\,A - t \\
\Leftrightarrow \;\; & -\,[\,t - K_1\,] \; < \; \frac{t - K_1}{2\,t}\,\Delta \;\; \Leftrightarrow \;\; -1 \; < \; \frac{\Delta}{2\,t} \;\; \Leftrightarrow \;\; \frac{\Delta}{2\,t} + 1 \; > \; 0\,. \qquad (267)
\end{aligned}
$$

Because the last inequality in (267) always holds, (267) implies that the putative equilibrium cannot constitute an equilibrium. ■

**Proposition 12.** *Suppose (i)* $G_2 - c_2 - (G_1 - c_1) > \frac{\Delta}{2}$*; (ii)* $2\,t > \Delta$*; (iii)* $c_1 > \frac{1}{2}\,[\,r_H + r_L]$*; (iv)* $t < G_2 - G_1 + c_1 - \frac{1}{2}\,[\,r_H + r_L]$*; (v) Condition 3C holds if* $t < K_2$*; and (vi)* $\frac{1}{2}\,[\,2\,t - \Delta\,] < \Omega_2(p_1)$ *for all* $p_1 \in [\,c_1 - r_1,\, \min\{\,c_1 - r_L,\, c_1 - r_1 + K_1 + t\,\}\,]$ *if* $t \geq K_2$*.*[5] *Then a MD2 equilibrium in which* $K_1 \in [\,0, t\,)$*,* $K_2 > 0$*, and* $p_1 = c_1 - r_1$ *does not exist in the setting with endogenous* $K$*.*

Proof. First consider a putative MD2 equilibrium in which $K_1 \in [\,0,\, t\,)$ and $K_2 \in (0,\, t\,)$, $p_1 =$

[4] Recall that $\Omega_1$ is defined in Condition 2B.

[5] Recall that $\Omega_2(p_1)$ is defined in Condition 3C.

$c_1 - r_1$, and $p_2$ is marginally below $c_1 - r_1 + G_2 - G_1 - K_2$. The expression for $\pi_2$ in Proposition 3 implies that Firm 2 can increase its MD2 equilibrium profit by reducing $K_2$ marginally. Consequently, the identified putative equilibrium cannot constitute an equilibrium under the maintained conditions.

Now consider a putative MD2 equilibrium in which $K_1 \in [0, t)$, $K_2 \geq t$, $p_1 = c_1 - r_1$, and $p_2$ is marginally below $c_1 - r_1 + G_2 - G_1 - K_2$. Arguments analogous to those employed in the proof of Proposition 3 reveal that Firm 2's profit in this equilibrium is $\pi_2^{D2} = c_1 - r_1 + r_H - c_2 + G_2 - G_1 - t$. Proposition 3 implies that if Firm 2 reduces $K_2$ below $t$, its profit is nearly $\pi_2^{D2'} = c_1 - r_1 - c_2 + G_2 - G_1 - K_2 + \frac{1}{2}[r_H + r_L] + \frac{K_2}{2t}\Delta$. Observe that that when $K_2 \in (0, t)$:

$$\begin{aligned}
\pi_2^{D2} < \pi_2^{D2'} \;&\Leftrightarrow\; c_1 - r_1 + r_H - c_2 + G_2 - G_1 - t \\
&\qquad < c_1 - r_1 - c_2 + G_2 - G_1 - K_2 + \frac{1}{2}[r_H + r_L] + \frac{K_2}{2t}\Delta \\
&\Leftrightarrow\; r_H - t < -K_2 + \frac{1}{2}[r_H + r_L] + \frac{K_2}{2t}\Delta \;\Leftrightarrow\; \frac{\Delta}{2} - t < -K_2 + \frac{K_2}{2t}\Delta \\
&\Leftrightarrow\; \frac{\Delta}{2t}[t - K_2] < t - K_2 \;\Leftrightarrow\; 2t > \Delta. \qquad (268)
\end{aligned}$$

The last inequality in (268) holds, by assumption. Therefore, (268) implies that the identified putative equilibrium cannot constitute an equilibrium under the maintained conditions. ■

**Proposition 13.** *Suppose (i) $G_2 - c_2 > G_1 - c_1$; (ii) $\Delta > 2t$; and (iii) Conditions 3A – 3C hold. Then a MD2 equilibrium in which $K_1 \in [0, t)$, $K_2 \in [0, t)$, and $p_1 = c_1 - r_1$ does not exist in the setting with endogenous $K$.*

Proof. First consider a putative equilibrium in which: (i) $K_1 \in [0, t)$ and $K_2 \in [0, t)$, where $(K_1, K_2) \neq (0, 0)$; (ii) $p_1 = c_1 - r_1$; and (iii) $p_2$ is marginally below $c_1 - r_1 + G_2 - G_1 - K_2$. The expression for $\pi_2$ in Proposition 3 implies that Firm 2 can increase its MD2 equilibrium profit by reducing $K_2$ marginally. Therefore, the identified putative equilibrium cannot constitute an equilibrium.

Now consider a putative equilibrium in which $K_1 = K_2 = 0$, $p_1 = c_1 - r_1$, and $p_2 = c_1 - \frac{1}{2}[r_L + r_H] + G_2 - G_1$. Arguments analogous to those employed in the proof of Proposition 3 reveal that Firm 2's profit in this equilibrium is nearly $\pi_2^{D2} = G_2 - c_2 - (G_1 - c_1)$. Proposition 3 implies that if Firm 2 increases $K_2$ marginally and reduces its price to $p_2 = c_1 - \frac{1}{2}[r_L + r_H] + G_2 - G_1 - \varepsilon_{11}$ (where $\varepsilon_{11} > 0$ is arbitrarily small), then its profit would be nearly $\pi_2^{D2'} = G_2 - G_1 + c_1 - c_2 - \varepsilon_{11} + \frac{\varepsilon_{11}}{2t}\Delta$. Observe that:

$$\begin{aligned}
\pi_2^{D2} < \pi_2^{D2'} \;&\Leftrightarrow\; G_2 - c_2 - (G_1 - c_1) < G_2 - G_1 + c_1 - c_2 - \varepsilon_{11} + \frac{\varepsilon_{11}}{2t}\Delta \\
&\Leftrightarrow\; -\varepsilon_{11} + \frac{\varepsilon_{11}}{2t}\Delta > 0 \;\Leftrightarrow\; \frac{\Delta}{2t} > 1 \;\Leftrightarrow\; \Delta > 2t. \qquad (269)
\end{aligned}$$

The last inequality in (269) holds, by assumption. Therefore, (269) implies that the identified putative equilibrium cannot constitute an equilibrium under the maintained conditions. ■