Technical Appendix to Accompany
"Procurement Contracts: Theory vs. Practice"
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The buyer's problem [BP] is:

$$\underset{p(\beta),e(\beta)}{Minimize} \quad \int_{\underline{\beta}}^{\overline{\beta}} p(\beta)dF(\beta) \tag{1}$$

subject to: $u(\beta|\beta) \geq 0 \quad \text{for all} \quad \beta \in [\underline{\beta}, \overline{\beta}]$, and (2)

$u(\beta|\beta) \geq u(\hat{\beta}|\beta) \quad \text{for all} \quad \beta, \hat{\beta} \in [\underline{\beta}, \overline{\beta}]$, (3)

where $u(\hat{\beta}|\beta) = p(\hat{\beta}) - [\hat{\beta} - e(\hat{\beta})] - C(e(\hat{\beta}|\beta), \beta)$, and (4)

$\beta - e(\hat{\beta}|\beta) = \hat{\beta} - e(\hat{\beta}).$ (5)

Define [BP]$'$ to be this same problem except that the global incentive compatibility (GIC) constraints (3) are replaced by the local incentive compatibility constraints ($\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}}|_{\hat{\beta}=\beta} = 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$) which simply ensure that the supplier will not misrepresent his innate cost locally.

Differentiating (5) with respect to β provides:

$$1 - \frac{de(\hat{\beta}|\beta)}{d\beta} = 0, \quad \text{which implies} \quad \frac{de(\hat{\beta}|\beta)}{d\beta}|_{\hat{\beta}=\beta} = 1. \tag{6}$$

Define $u(\beta) \equiv u(\beta|\beta)$. Assuming $u(\beta|\beta)$ is differentiable almost everywhere, (4) and (6) imply that the rate at which the supplier's utility increases with β at the solution to [BP]$'$ is:

$$u'(\beta) = -\frac{dC(e(\hat{\beta}|\beta), \beta)}{d\beta}|_{\hat{\beta}=\beta} = -[C_1(e, \beta) + C_2(e, \beta)]. \tag{7}$$

Recall Assumptions 1 - 3:

Assumption 1. $C(e, \beta) = K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e^2$ where $\widetilde{\beta} \leq \underline{\beta}$.

Assumption 2. $K \geq [\overline{\beta} - \widetilde{\beta}]^{-1}$.

Assumption 3. $-2 \leq \gamma \leq -1$.

The text considers the special case of Assumption 1 in which $\tilde{\beta} = \underline{\beta}$. The proof proceeds here for the more general case where $\tilde{\beta} \leq \underline{\beta}$ because some of the conclusions reported in the text reflect this more general analysis.

Unless otherwise noted, Assumptions 1 - 3 are assumed to hold throughout the ensuing analysis. Assumption 1 implies:

$$C_1(e,\beta) = 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} e \quad ; \quad C_2(e,\beta) = \frac{\gamma K}{\beta-\widetilde{\beta}}\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} e^2 \quad ; \tag{8}$$

$$C_{11}(\cdot) = 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} \quad ; \quad C_{12}(\cdot) = \frac{2\gamma K}{\beta-\widetilde{\beta}}\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} e \; ; \tag{9}$$

$$C_{111}(\cdot) = 0, \quad \text{and} \;\; C_{112}(\cdot) = \frac{2\gamma K}{\beta-\widetilde{\beta}}\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} . \tag{10}$$

The structure of the solution to [BP] depends on the sign of $u'(\beta)$, which is considered in Lemma 1 .

Lemma 1 *$u'(\beta) \leq 0$ and $e(\beta) \leq \beta - \tilde{\beta}$ for all $\beta \in \left[\underline{\beta}, \overline{\beta}\right]$ at the solution to $[BP]'$.*

Proof of Lemma 1 . From (7), $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$ is ensured if $C_1(e,\beta) + C_2(e,\beta) \geq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$. From (8), when Assumption 1 holds:

$$C_1(e,\beta) + C_2(e,\beta) \geq 0 \;\Leftrightarrow\; K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} e\left[2 + \frac{\gamma e}{\beta-\widetilde{\beta}}\right] \geq 0$$

$$\Leftrightarrow\; 2 - \frac{|\gamma| e}{\beta-\widetilde{\beta}} \geq 0 \;\Leftrightarrow\; e(\beta) \leq \frac{2[\beta-\widetilde{\beta}]}{|\gamma|} \; . \tag{11}$$

To prove $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$, we consider any feasible contract, and construct an alternative contract that has lower expected payment and satisfies $e(\beta) \leq [\beta - \widetilde{\beta}] \leq \frac{2[\beta-\widetilde{\beta}]}{|\gamma|}$ for each innate cost realization, β (and so $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$, from (11)).

First recall that when Assumption 1 holds, the first-best effort level is:

$$e^*(\beta) = \arg\min\{\beta - e + C(e,\beta)\} = \left[2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\right]^{-1} . \tag{12}$$

Therefore, because $K \geq [\overline{\beta} - \widetilde{\beta}]^{-1}$ and $|\gamma| \geq 1$ from Assumptions 2 and 3, (12) implies:

$$2e^*(\beta) = \frac{1}{K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}} \leq \left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{|\gamma|-1} [\beta-\widetilde{\beta}] \leq \beta - \widetilde{\beta} \; . \tag{13}$$

(13) implies:

$$\widetilde{\beta} \leq \beta - 2e^*(\beta) \ . \tag{14}$$

A contract is a set of cost-payment pairs. For each individually rational contract $\{(c_i, p_i) \mid i \in I\}$, consider a new contract $\{(\beta, \beta) \mid \beta \in [\underline{\beta}, \overline{\beta}]\} \cup \{(c_i, p_i) \mid p_i \leq \overline{\beta}, c_i \geq \widetilde{\beta}, i \in I\}$. Assume that when multiple (c, p) pairs deliver the same expected utility to the supplier under the new contract, the supplier selects the pair that he would choose under the original contract. The new contract is individually rational because when his innate cost is β, the supplier can always secure non-negative utility by choosing effort level 0 and cost level β.

When the buyer offers the supplier the new contract, one of three possibilities arise:

- The supplier always chooses the same (c, p) pair under the new contract that he would choose under the original contract. In this case, there is no change in realized payment under the new contract.

- The supplier with innate cost β chooses (c_0, p_0) under the original contract and (β, β) under the new contract. We will now show that $p_0 \geq \beta$ in this case, and so the buyer will pay the supplier the same amount or less under the new contract.

  Because (c_0, p_0) is not chosen under the new contract, it is not available under this contract. Consequently, either $p_0 \geq \overline{\beta} \geq \beta$, or $c_0 < \widetilde{\beta}$. Since $p_0 \geq \beta$ in the former case, we need only consider the latter case. In this case, $p_0 \geq c_0 + C(\beta - c_0, \beta)$ because the supplier chooses (c_0, p_0) under the original contract, and so secures non-negative utility by doing so.

  (12) implies that when Assumption 1 holds:

$$C(2e^*(\beta), \beta) = K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \underline{\beta}}\right]^{\gamma} \left[2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \underline{\beta}}\right]^{\gamma}\right]^{-2} = \frac{1}{4K} \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \underline{\beta}}\right]^{-\gamma}$$

$$= 2 \left[2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \underline{\beta}}\right]^{\gamma}\right]^{-1} = 2e^*(\beta) \ . \tag{15}$$

  Because $c_0 < \widetilde{\beta}$ in the case presently under consideration, (14) implies:

$$c_0 < \widetilde{\beta} \leq \beta - 2e^*(\beta). \tag{16}$$

  Hence:

$$p_0 \geq c_0 + C(\beta - c_0, \beta) > \beta - 2e^*(\beta) + C(2e^*(\beta), \beta) = \beta. \tag{17}$$

The weak inequality in (17) reflects the case presently under consideration. The strict inequality in (17) holds because $c + C(\beta - c, \beta)$ is a convex, quadratic function of c that achieves its minimum at $\beta - e^*$. Therefore, because c_0 is less than $\beta - 2e^*(\beta)$ which, in turn, is less than $\beta - e^*(\beta)$, it must be the case that $c_0 + C(\beta - c_0, \beta) > [\beta - 2e^*(\beta)] + C(2e^*(\beta), \beta)$ $(> \beta - e^*(\beta) + C(e^*(\beta), \beta))$. The equality in (17) follows from (15). Consequently, $p_0 \geq \beta$.

- The supplier with innate cost β chooses (c_0, p_0) under the original contract, and (c_1, p_1) under the new contract. We will now show that $p_0 \geq p_1$ in this case, and so the buyer will deliver either the same or a smaller payment under the new contract.

  Because (c_0, p_0) is not chosen under the new contract, it is not available under this contract. Consequently, either $p_0 \geq \overline{\beta} \geq p_1$, or $c_0 < \widetilde{\beta}$. ($\overline{\beta} \geq p_1$ because the (c_1, p_1) pair is chosen under the new contract.) In the former case, $p_0 \geq p_1$. Therefore, it only remains to consider the latter case. In this case:

$$p_0 - [c_0 + C(\beta - c_0, \beta)] \geq p_1 - [c_1 + C(\beta - c_1, \beta)] \ . \tag{18}$$

  (18) holds because the supplier chose (c_0, p_0) rather than (c_1, p_1) under the original contract when both pairs were available. Also notice that $c_1 \geq \widetilde{\beta}$ because (c_1, p_1) is one of the pairs offered in the new contract. In addition, since $e(\beta) \geq 0$ for all β, $c_1 \leq \beta$. In summary, since $c_0 < \widetilde{\beta}$ in the case presently under consideration, we have:

$$c_0 < \widetilde{\beta} \leq c_1 \leq \beta \ . \tag{19}$$

  Since $c + C(\beta - c, \beta)$ is a convex, quadratic function of c that achieves its minimum at $\beta - e^*$, (19) implies:

$$\begin{aligned} |(\beta - e^*) - c_1| &\leq \max\{|(\beta - e^*) - \widetilde{\beta}|, \ |\beta - (\beta - e^*)|\} \\ &= |(\beta - e^*) - \widetilde{\beta}| \ < |(\beta - e^*) - c_0| \ . \end{aligned} \tag{20}$$

  (20) implies $c_0 + C(\beta - c_0, \beta) > c_1 + C(\beta - c_1, \beta)$ (since $c + C(\beta - c, \beta)$ is a convex, quadratic function of c that achieves its minimum at $\beta - e^*$). Consequently, (18) implies $p_0 \geq p_1$. (Otherwise, the supplier would secure a higher utility by realizing cost c_1 – which entails lower total production cost for the supplier – and receiving the higher payment $p_{1.}$)

Therefore, in all three cases, the buyer makes the same or a smaller payment under the new contract. Consequently, we can assume without loss of generality that $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$ when Assumptions 1 - 3 hold. ■

When $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$, it is optimal to set $u(\overline{\beta}) = 0$ in order to satisfy (2). Therefore, from (7), $u(\beta) = \int_{\beta}^{\overline{\beta}} [C_1(e, \beta') + C_2(e, \beta')] d\beta'$, and so the supplier's expected utility under the optimal contract is:

$$\int_{\underline{\beta}}^{\overline{\beta}} u(\beta) dF(\beta) = u(\beta)F(\beta)|_{\underline{\beta}}^{\overline{\beta}} - \int_{\underline{\beta}}^{\overline{\beta}} u'(\beta)F(\beta) du = \int_{\underline{\beta}}^{\overline{\beta}} [C_1(e, \beta) + C_2(e, \beta)] \frac{F(\beta)}{f(\beta)} dF(\beta). \quad (21)$$

(The first equality in (21) follows from integration by parts. The second equality in (21) holds because $u(\beta)F(\beta)|_{\underline{\beta}}^{\overline{\beta}} = u(\overline{\beta})F(\overline{\beta}) - u(\underline{\beta})F(\underline{\beta}) = 0$ under the maintained assumptions.)

Since the supplier's utility is the difference between the payment he receives and the costs he incurs:

$$p(\beta) = \beta - e(\beta) + C(e(\beta), \beta) + u(\beta). \quad (22)$$

(22) implies the expected payment to the supplier is:

$$P \equiv \int_{\underline{\beta}}^{\overline{\beta}} p(\beta) dF(\beta) = \int_{\underline{\beta}}^{\overline{\beta}} [\beta - e(\beta) + C(e(\beta), \beta) + u(\beta)] dF(\beta) \ . \quad (23)$$

Substituting (21) into (23) provides the following expression for expected payments to the supplier under the optimal contract:

$$P = \int_{\underline{\beta}}^{\overline{\beta}} R(e(\beta)) dF(\beta) \ , \qquad \text{where} \quad (24)$$

$$R(e(\beta)) \equiv \beta - e(\beta) + C(e(\beta), \beta) + [C_1(e, \beta) + C_2(e, \beta)] \frac{F(\beta)}{f(\beta)} \ . \quad (25)$$

The buyer seeks to minimize (24). To identify the optimal level of induced effort, differentiate (24) with respect to $e(\beta)$. Doing so reveals that under the optimal contract:

$$\frac{\partial P}{\partial e} = -1 + C_1(e, \beta) + [C_{11}(e, \beta) + C_{12}(e, \beta)] \frac{F(\beta)}{f(\beta)} \geq 0 \ ; \qquad e(\beta)[\cdot] = 0. \quad (26)$$

Differentiating (26) with respect to $e(\beta)$ provides:

$$\frac{\partial^2 P}{\partial e^2} = C_{11}(e, \beta) + [C_{111}(e, \beta) + C_{112}(e, \beta)] \frac{F(\beta)}{f(\beta)} \ . \quad (27)$$

(8) – (10) imply that when Assumption 1 holds, the expressions in (26) and (27) can be written as:

$$\frac{\partial P}{\partial e} = -1 + 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \frac{F(\beta)}{f(\beta)} + e2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \left[1 + \left[\frac{\gamma}{\beta - \widetilde{\beta}} \right] \frac{F(\beta)}{f(\beta)} \right], \text{ and} \tag{28}$$

$$\frac{\partial^2 P}{\partial e^2} = 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \left[1 + \left[\frac{\gamma}{\beta - \widetilde{\beta}} \right] \frac{F(\beta)}{f(\beta)} \right]. \tag{29}$$

Observation 1 *$e(\beta) = 0$ at the solution to $[BP]'$ if and only if $-1 + 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \frac{F(\beta)}{f(\beta)} \geq 0$.*

<u>Proof of Observation 1</u> . Recall from Lemma 1 that $u'(\beta) \leq 0$ and $e(\beta) \leq \beta - \widetilde{\beta}$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$ at the solution to [BP]$'$ when Assumptions 1 – 3 hold. For a fixed β, the term in (29) is independent of e. If this term is non-positive, expected payment is a concave function of e under the optimal contract. Consequently, the optimal e is achieved at a boundary: either zero or $\beta - \widetilde{\beta}$. (Although $e < 0$ is feasible, the supplier always delivers non-negative effort. Negative effort increases final production cost at least as rapidly as it increases the payment from the buyer, and so is not advantageous for the supplier.) We need to check whether zero or $\beta - \widetilde{\beta}$ induces a lower value for $R(e)$ (defined in (25)) when $2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \left[1 + \left(\frac{\gamma}{\beta - \widetilde{\beta}} \right) \frac{F(\beta)}{f(\beta)} \right] < 0$.

(25) and (8) reveal that when Assumption 1 holds, $R(e)$ is a quadratic function of e with $R(0) = \beta$ and

$$R'(0) = \; -1 + 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \frac{F(\beta)}{f(\beta)} > 0. \tag{30}$$

The inequality in (30) holds because, by assumption in the case presently under consideration, $2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \left[1 + \frac{\gamma}{\beta - \widetilde{\beta}} \frac{F(\beta)}{f(\beta)} \right] < 0$. Therefore, $1 + \frac{\gamma}{\beta - \widetilde{\beta}} \frac{F(\beta)}{f(\beta)} < 0$, which implies $\frac{|\gamma|}{\beta - \widetilde{\beta}} \frac{F(\beta)}{f(\beta)} > 1$. Consequently:

$$-1 + 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} \frac{F(\beta)}{f(\beta)} \geq -1 + 2[\overline{\beta} - \widetilde{\beta}]^{-1} \left[\frac{\overline{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}} \right]^{|\gamma|} \frac{F(\beta)}{f(\beta)} \tag{31}$$

$$\geq -1 + |\gamma| [\overline{\beta} - \widetilde{\beta}]^{-1} \left[\frac{\overline{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}} \right]^{1} \frac{F(\beta)}{f(\beta)} = -1 + \frac{|\gamma|}{\beta - \widetilde{\beta}} \frac{F(\beta)}{f(\beta)} > 0 \; . \tag{32}$$

The inequality in (31) holds because $K \geq [\overline{\beta} - \widetilde{\beta}]^{-1}$, by Assumption 2. The weak inequality in (32) holds because $|\gamma| \leq 2$ by Assumption 3, and because $\overline{\beta} - \widetilde{\beta} \geq \beta - \widetilde{\beta}$ for all β.

Since $R'(0) > 0$ and $R(e)$ is a quadratic function of e, we can conclude that $R(0) \leq R(\beta - \widetilde{\beta})$ if $R(\cdot)$ achieves its critical point on $[\frac{(\beta-\widetilde{\beta})}{2}, (\beta - \widetilde{\beta})]$ instead of $[0, \frac{(\beta-\widetilde{\beta})}{2}]$. Notice from (28) that $R'(e) = 0$ at $e = \frac{1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)}}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[1+(\frac{\gamma}{\beta-\widetilde{\beta}})\frac{F(\beta)}{f(\beta)}\right]}$. Therefore, we seek to show:

$$\frac{1 - 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)}}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[1+(\frac{\gamma}{\beta-\widetilde{\beta}})\frac{F(\beta)}{f(\beta)}\right]} \geq \frac{(\beta-\widetilde{\beta})}{2}. \tag{33}$$

Notice that:

$$1 \leq [K(\overline{\beta}-\widetilde{\beta})]\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma+1} \tag{34}$$

$$\Leftrightarrow \quad 1 \leq 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[\frac{(\beta-\widetilde{\beta})}{2}\right]$$

$$\Rightarrow \quad 1 \leq 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[\frac{(\beta-\widetilde{\beta})}{2} + \frac{\gamma}{2}(\frac{F(\beta)}{f(\beta)}) + \frac{F(\beta)}{f(\beta)}\right] \tag{35}$$

$$\Leftrightarrow \quad 1 - 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \leq 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{(\beta-\widetilde{\beta})}{2}\left[1+\frac{\gamma}{2}[\frac{2}{\beta-\widetilde{\beta}}]\frac{F(\beta)}{f(\beta)}\right]$$

$$\Leftrightarrow \quad \frac{1 - 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)}}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[1+(\frac{\gamma}{\beta-\widetilde{\beta}})\frac{F(\beta)}{f(\beta)}\right]} \geq \frac{(\beta-\widetilde{\beta})}{2}\ . \tag{36}$$

(34) holds because $K[\overline{\beta}-\widetilde{\beta}] \geq 1$ from Assumption 2 and because $\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}} \in [0,1)$, and so $\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma+1} \geq 1$, since $\gamma+1 \in [-1,0]$ from Assumption 3. (35) holds because $\frac{2-|\gamma|}{2} \geq 0$ since $|\gamma| \leq 2$, from Assumption 3. The direction of the inequality in (36) follows from the fact that we are considering the case where $2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[1+(\frac{\gamma}{\beta-\widetilde{\beta}})\frac{F(\beta)}{f(\beta)}\right] < 0$.

In summary, we have shown that if the term in (29) is negative, e is 0 at the solution to $[\text{P}]'$. Therefore, $\frac{\partial P}{\partial e} = -1 + 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \geq 0$, from (26) and (28).

If the term in (29) is positive, expected payment is a convex function of e under the optimal contract. In this case, if the expression in (28) is non-negative at $e = 0$, then e is optimally 0. Again, then, $\frac{\partial P}{\partial e} = -1 + 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \geq 0$, from (26) and (28). In contrast, if the expression in (28) is negative at $e = 0$ (so $-1 + 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} < 0$), then effort is optimally positive.

Therefore, effort is optimally 0 if and only if $-1 + 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \geq 0$. ■

Lemma 2 $e_0(\beta) = \frac{(1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)})^+}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[1+(\frac{\gamma}{\beta-\widetilde{\beta}})\frac{F(\beta)}{f(\beta)}\right]}$ *at the solution to* $[BP]'$*, where* $(1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)})^+ = \max\{0, 1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)}\}$.

Proof of Lemma 2 . From Observation 1, $e = 0$ if $1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \leq 0$. Observation 1 also implies that if $1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} > 0$, the optimal e is the value of e at which the expression in (28) is zero. This value is as specified in the Lemma. ■

Lemma 3

$$e_0(\beta) \equiv \frac{(1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)})^+}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\left[1+(\frac{\gamma}{\beta-\widetilde{\beta}})\frac{F(\beta)}{f(\beta)}\right]} \;\leq\; e^*(\beta) \equiv \frac{1}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}} \quad \textit{for all } \beta \in \left[\underline{\beta}, \overline{\beta}\right]. \tag{37}$$

Proof of Lemma 3. From Observation 1, $e_0(\beta) = 0$ if $1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \leq 0$. Therefore, $e_0(\beta) \leq e^*(\beta)$ when $1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \leq 0$.

When $1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} > 0$,

$$e_0(\beta) \leq e^*(\beta) \;\Leftrightarrow\; 1-2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} \leq 1+\left[\frac{\gamma}{\beta-\widetilde{\beta}}\right]\frac{F(\beta)}{f(\beta)} \tag{38}$$

$$\Leftrightarrow\; 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} \geq \frac{|\gamma|}{\beta-\widetilde{\beta}}\,. \tag{39}$$

(39) holds because: (1) $K \geq [\overline{\beta}-\widetilde{\beta}]^{-1}$ by Assumption 2, and so $K\,\frac{[\beta-\widetilde{\beta}]^{\gamma+1}}{[\overline{\beta}-\widetilde{\beta}]^{\gamma}} \geq \left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma+1}$; and (2) $\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma+1} \geq 1 \geq \frac{|\gamma|}{2}$ since $\gamma+1 \leq 0$ and $|\gamma| \leq 2$ by Assumption 3. ■

Observation 2 $\beta - e^*(\beta)$ *is monotonically increasing in* β *on* $[\underline{\beta}, \overline{\beta}]$.

Proof of Observation 2. $e^{*\prime}(\beta) = \frac{1}{2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}}\frac{|\gamma|}{\beta-\widetilde{\beta}}$. Also, $2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} \geq \frac{|\gamma|}{\beta-\widetilde{\beta}}$, from (39). Therefore, $e^{*\prime}(\beta) \leq 1$ and so $\beta - e^*(\beta)$ is monotonically increasing in β. ■

Lemma 4 *If* $e_0'(\beta) \leq 1$ *for all* $\beta \in \left[\underline{\beta}, \overline{\beta}\right]$ *at the solution to* $[BP]'$*, then* $e_0(\beta)$ *is the effort supply at the solution to* $[BP]$*.*

Proof of Lemma 4. We will show that under the maintained conditions:

$$\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}} \geq 0 \;\text{ for }\; \hat{\beta} < \beta \quad \text{and} \quad \frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}} \leq 0 \;\text{ for }\; \hat{\beta} > \beta \tag{40}$$

at the solution to [BP]′. If (40) holds, the GIC constraints will be satisfied at the solution to [BP]′, and so this solution will constitute the solution to [BP].

First consider $\hat{\beta} < \beta$. From (7):

$$\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}} = \frac{\partial u(\hat{\beta}|\hat{\beta})}{\partial \hat{\beta}} + \int_{\hat{\beta}}^{\beta} \frac{\partial^2 u(\hat{\beta}|\beta_0)}{\partial \hat{\beta} \partial \beta} d\beta_0 = \int_{\hat{\beta}}^{\beta} \frac{\partial^2 u(\hat{\beta}|\beta_0)}{\partial \hat{\beta} \partial \beta} d\beta_0 \tag{41}$$

$$= \int_{\hat{\beta}}^{\beta} \frac{-\partial^2 C(e(\hat{\beta}|\beta_0), \beta_0)}{\partial \hat{\beta} \partial \beta} d\beta_0 = \int_{\hat{\beta}}^{\beta} \frac{-\partial e(\hat{\beta}|\beta_0)}{\partial \hat{\beta}} \frac{\partial C_1(e(\hat{\beta}|\beta_0), \beta_0)}{\partial \beta} d\beta_0 \tag{42}$$

$$= [1 - e'(\hat{\beta})] C_1(e(\hat{\beta}|\beta_0), \beta_0)|_{\hat{\beta}}^{\beta} . \tag{43}$$

Because $e_0'(\beta) \leq 1$ (by assumption), $1 - e'(\hat{\beta}) \geq 0$. Therefore, to prove $\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}} \geq 0$, (43) implies we need to show $C_1(e(\hat{\beta}|\beta_0)), \beta_0)|_{\hat{\beta}}^{\beta} \geq 0$. From (8):

$$C_1(e(\hat{\beta}|\beta_0)), \beta_0)|_{\hat{\beta}}^{\beta} \geq 0 \Leftrightarrow 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}|\beta) - 2K \left[\frac{\hat{\beta} - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}|\hat{\beta}) \geq 0 \tag{44}$$

$$\Leftrightarrow \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}) - \left[\frac{\hat{\beta} - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}) + \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} [\beta - \hat{\beta}] \geq 0. \tag{45}$$

(45) follows from (44) because $e(\hat{\beta}|\beta) = e(\hat{\beta}) + \beta - \hat{\beta}$ (and so $e(\hat{\beta}|\hat{\beta}) = e(\hat{\beta})$) from (5).

Because $\hat{\beta} < \beta$ and $\gamma < 0$, the expression in (45) is decreasing in $e(\hat{\beta})$. Since $e(\hat{\beta}) \leq e^*(\hat{\beta})$ by assumption, (45) will hold if:

$$\left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e^*(\hat{\beta}) - \left[\frac{\hat{\beta} - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e^*(\hat{\beta}) + [\beta - \hat{\beta}] \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} \geq 0, \tag{46}$$

$$\Leftrightarrow \quad \left[\frac{\hat{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} - 1 + 2K[\beta - \hat{\beta}] \left[\frac{\overline{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} \geq 0. \tag{47}$$

(47) is derived from (46) by dividing all terms by $\left[\frac{\hat{\beta}-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} e^*(\hat{\beta})$ $\left(= \left[\frac{\hat{\beta}-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} \left[2K \left[\frac{\hat{\beta}-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\right]^{-1}\right.$ from (37)). Because $\hat{\beta} < \beta$ and $K \geq [\overline{\beta} - \widetilde{\beta}]^{-1}$, (47) will hold if:

$$J(\hat{\beta}) \equiv \left[\frac{\hat{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} - 1 + 2[\beta - \hat{\beta}][\overline{\beta} - \widetilde{\beta}]^{-1} \left[\frac{\overline{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} \geq 0. \tag{48}$$

Notice that $J(\beta) = 0$. Furthermore:

$$J'(\hat{\beta}) = \frac{|\gamma|(\hat{\beta} - \widetilde{\beta})^{|\gamma|-1}}{(\beta - \widetilde{\beta})^{|\gamma|}} - \frac{2(\overline{\beta} - \widetilde{\beta})^{|\gamma|-1}}{(\beta - \widetilde{\beta})^{|\gamma|}} \leq 0 \; . \tag{49}$$

The inequality in (49) holds because $|\gamma| \leq 2$, $\hat{\beta} - \widetilde{\beta} \leq \overline{\beta} - \widetilde{\beta}$, and $|\gamma| \geq 1$. Because $J'(\hat{\beta}) \leq 0$ and $J(\beta) = 0$, we know (48) holds for all $\hat{\beta} < \beta$. Therefore, $C_1(e(\hat{\beta}|\beta_0)), \beta_0)|_{\hat{\beta}}^{\beta} \geq 0$.

Now suppose $\hat{\beta} > \beta$. In this case, (7) implies:

$$\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}} = \frac{\partial u(\hat{\beta}|\hat{\beta})}{\partial \hat{\beta}} - \int_{\beta}^{\hat{\beta}} \frac{\partial^2 u(\hat{\beta}|\beta_0)}{\partial \hat{\beta} \partial \beta} d\beta_0 = - \int_{\beta}^{\hat{\beta}} \frac{\partial^2 u(\hat{\beta}|\beta_0)}{\partial \hat{\beta} \partial \beta} d\beta_0 \tag{50}$$

$$= \int_{\beta}^{\hat{\beta}} \frac{\partial^2 C(e(\hat{\beta}|\beta_0), \beta_0)}{\partial \hat{\beta} \partial \beta} d\beta_0 = \int_{\beta}^{\hat{\beta}} \frac{\partial e(\hat{\beta}|\beta_0)}{\partial \hat{\beta}} \frac{\partial C_1(e(\hat{\beta}|\beta_0), \beta_0)}{\partial \beta} d\beta_0 \tag{51}$$

$$= -[1 - e'(\hat{\beta})] C_1(e(\hat{\beta}|\beta_0), \beta_0)|_{\beta}^{\hat{\beta}} \; . \tag{52}$$

Because $e_0'(\beta) \leq 1$ (by assumption), $1 - e'(\hat{\beta}) \geq 0$. Therefore, to prove $\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}} \leq 0$, (50) implies we need to show $C_1(e(\hat{\beta}|\beta_0)), \beta_0)|_{\hat{\beta}}^{\beta} \geq 0$. From (8):

$$C_1(e(\hat{\beta}|\beta_0), \beta_0)|_{\beta}^{\hat{\beta}} \geq 0 \Leftrightarrow 2K \left[\frac{\hat{\beta} - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}|\hat{\beta}) - 2K \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}|\beta) \geq 0 \tag{53}$$

$$\Leftrightarrow \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}) - \left[\frac{\hat{\beta} - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e(\hat{\beta}) + \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} [\beta - \hat{\beta}] \leq 0 \; . \tag{54}$$

(54) follows from (53) because $e(\hat{\beta}|\beta) = e(\hat{\beta}) + \beta - \hat{\beta}$ (and so $e(\hat{\beta}|\hat{\beta}) = e(\hat{\beta})$) from (5).

Because $\hat{\beta} > \beta$ and $\gamma < 0$, the expression in (54) is increasing in $e(\hat{\beta})$. Since $e(\hat{\beta}) \leq e^*(\hat{\beta})$ by assumption, (54) will hold if:

$$\left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e^*(\hat{\beta}) - \left[\frac{\hat{\beta} - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} e^*(\hat{\beta}) + [\beta - \hat{\beta}] \left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} \leq 0, \tag{55}$$

$$\Leftrightarrow \quad \left[\frac{\hat{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} - 1 + 2K[\beta - \hat{\beta}] \left[\frac{\overline{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} \leq 0 \; . \tag{56}$$

(56) is derived from (55) by dividing all terms by $\left[\frac{\hat{\beta}-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} e^*(\hat{\beta})$ $\left(= \left[\frac{\hat{\beta}-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma} \left[2K \left[\frac{\hat{\beta}-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\right]^{-1}\right.$ from (37)). Because $\hat{\beta} > \beta$ and $K \geq [\overline{\beta} - \widetilde{\beta}]^{-1}$, (56) will hold if:

$$J(\beta) \equiv \left[\frac{\hat{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} - 1 + 2[\beta - \hat{\beta}][\overline{\beta} - \widetilde{\beta}]^{-1} \left[\frac{\overline{\beta} - \widetilde{\beta}}{\beta - \widetilde{\beta}}\right]^{|\gamma|} \leq 0. \tag{57}$$

Again, since $J(\beta) = 0$ and $J'(\hat{\beta}) \leq 0$ (from (49)), we know (57) holds for all $\hat{\beta} > \beta$.

In summary, we have shown that (40) holds. Consequently, the GIC constraints are satisfied at the solution to [BP]′ under the identified conditions. ■

We can now prove Findings 1 - 3.

Proof of Finding 1. Recall that in the setting of Finding 1: (i) $\widetilde{\beta} = \underline{\beta}$; (ii) $f(\beta) = \frac{1}{\overline{\beta}-\underline{\beta}}$; (iii) $\gamma = -2$; and (iv) $K = \frac{5}{4}[\overline{\beta} - \underline{\beta}]^{-1}$.

Because Assumptions 1 - 3 are satisfied, Lemma 1 ensures $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$. Furthermore, $F(\beta) = \frac{\beta-\underline{\beta}}{\overline{\beta}-\underline{\beta}}$ and $\frac{F(\beta)}{f(\beta)} = \beta - \underline{\beta}$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$.

Notice that $2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}[1 + (\frac{\gamma}{\beta-\underline{\beta}})\frac{F(\beta)}{f(\beta)}] = 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{-2}[1-2] = -2(\frac{5}{4})[\frac{1}{\overline{\beta}-\underline{\beta}}][\frac{\overline{\beta}-\underline{\beta}}{\beta-\underline{\beta}}]^{2} = -\frac{5[\overline{\beta}-\underline{\beta}]}{2[\beta-\underline{\beta}]^{2}} < 0$ in this setting. Therefore, (29) implies that expected payment is a concave function of e. Consequently, as shown in the proof of Observation 1, $e(\beta)$ is optimally 0 for all $\beta \in [\underline{\beta}, \overline{\beta}]$. That is, cost reimbursement is optimal in this setting, so the supplier will optimally provide zero effort. Furthermore, it is readily verified that the GIC constraints are satisfied at this solution, so it is indeed the solution to [BP]. ■

Proof of Finding 2. Recall that in the setting of Finding 2: (i) $\widetilde{\beta} = \underline{\beta}$; (ii) $\gamma = -2$; (iii) $f(\beta) = \frac{3(\beta-\underline{\beta})^{2}}{(\overline{\beta}-\underline{\beta})^{3}}$; and (iv) $K = \frac{5}{4}\ [\overline{\beta} - \underline{\beta}]^{-1}$.

By Lemma 1, $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$. Also notice that $F(\beta) = (\frac{\beta-\underline{\beta}}{\overline{\beta}-\underline{\beta}})^{3}$, and $\frac{F(\beta)}{f(\beta)} = \frac{\beta-\underline{\beta}}{3}$. Furthermore, from (29), expected payment is a convex function of e in this setting, because $1 + \frac{\gamma}{\beta-\underline{\beta}}[\frac{F(\beta)}{f(\beta)}] = 1 - \frac{2}{3} > 0$. In addition, $1 - 2K\left[\frac{\beta-\widetilde{\beta}}{\overline{\beta}-\widetilde{\beta}}\right]^{\gamma}\frac{F(\beta)}{f(\beta)} > 0 \Leftrightarrow 1 - \frac{5}{6}[\frac{\overline{\beta}-\underline{\beta}}{\beta-\underline{\beta}}] > 0$.

Therefore, Observation 1 implies that at the solution to [BP]′: (i) no effort is optimally induced on $[\underline{\beta}, \frac{5}{6}\overline{\beta} + \frac{1}{6}\underline{\beta}]$; and (ii) the optimal effort on $[\frac{5}{6}\overline{\beta} + \frac{1}{6}\underline{\beta}, \overline{\beta}]$ is:

$$\begin{aligned} e_0(\beta) &= \frac{1 - \frac{5}{6}[\frac{\overline{\beta}-\underline{\beta}}{\beta-\underline{\beta}}]}{\frac{5}{2}[\overline{\beta} - \underline{\beta}][\beta - \underline{\beta}]^{-2}[1 - \frac{2}{3}]} \\ &= \frac{6[\beta - \underline{\beta}]^{2} - 5[\overline{\beta} - \underline{\beta}][\beta - \underline{\beta}]}{5[\overline{\beta} - \underline{\beta}]} = \frac{6}{5}\frac{[\beta - \underline{\beta}]^{2}}{[\overline{\beta} - \underline{\beta}]} - [\beta - \underline{\beta}]. \end{aligned} \quad (58)$$

However, the GIC constraints are not satisfied at this solution to [BP]′. To see why, notice that the local second order condition is:

$$\left.\frac{\partial^2 u(\hat{\beta}|\beta)}{\partial \hat{\beta}^2}\right|_{\hat{\beta}=\beta} \leq 0 \quad . \tag{59}$$

Recall that at the solution to [BP]$'$:

$$\left.\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}}\right|_{\hat{\beta}=\beta} = 0 \text{ for all } \beta \in [\underline{\beta}, \overline{\beta}] \ . \tag{60}$$

Because (60) holds for all $\beta \in [\underline{\beta}, \overline{\beta}]$:

$$\left.\frac{\partial}{\partial \beta}\left[\frac{\partial u(\hat{\beta}|\beta)}{\partial \hat{\beta}}\right]\right|_{\hat{\beta}=\beta} = 0 \ \Leftrightarrow \ \left.\frac{\partial^2 u(\hat{\beta}|\beta)}{\partial \hat{\beta}^2}\right|_{\hat{\beta}=\beta} = -\left.\frac{\partial^2 u(\hat{\beta}|\beta)}{\partial \hat{\beta}\partial \beta}\right|_{\hat{\beta}=\beta} \quad . \tag{61}$$

(61) implies that (59) holds if and only if:

$$-\left.\frac{\partial^2 u(\hat{\beta}|\beta)}{\partial \hat{\beta}\partial \beta}\right|_{\hat{\beta}=\beta} \leq 0 \ \Leftrightarrow \ \left.\frac{\partial^2 C(e(\hat{\beta}|\beta), \beta)}{\partial \hat{\beta}\partial \beta}\right|_{\hat{\beta}=\beta} \leq 0 \quad . \tag{62}$$

The equivalence in (62) follows from (4). Notice that:

$$\frac{\partial C(e(\hat{\beta}|\beta), \beta)}{\partial \hat{\beta}} = C_1(\cdot)\frac{\partial e(\hat{\beta})}{\partial \hat{\beta}}, \text{ and}$$

$$\frac{\partial^2 C(e(\hat{\beta}|\beta), \beta)}{\partial \hat{\beta}\partial \beta} = C_1(\cdot)\frac{\partial}{\partial \beta}(\frac{\partial e(\cdot)}{\partial \hat{\beta}}) + \frac{\partial e(\cdot)}{\partial \hat{\beta}}\left[C_{11}(\cdot)\frac{\partial e(\hat{\beta}|\beta)}{\partial \beta} + C_{12}(\cdot)\right] \quad . \tag{63}$$

Since $e(\hat{\beta}|\beta) = \beta - \hat{\beta} + e(\hat{\beta})$ from (5), it follows that:

$$\frac{\partial e(\hat{\beta}|\beta)}{\partial \hat{\beta}} = e'(\hat{\beta}) - 1, \quad \frac{\partial e(\hat{\beta}|\beta)}{\partial \beta} = 1, \text{ and } \frac{\partial}{\partial \beta}\left[\frac{\partial e(\cdot)}{\partial \hat{\beta}}\right] = 0 \ . \tag{64}$$

Substituting from (64) into (63) provides:

$$\left.\frac{\partial^2 C(e(\hat{\beta}|\beta), \beta)}{\partial \hat{\beta}\partial \beta}\right|_{\hat{\beta}=\beta} \leq 0 \quad \Leftrightarrow \quad [C_{11}(e, \beta) + C_{12}(e, \beta)][e'(\beta) - 1] \leq 0. \tag{65}$$

From (10):

$$C_{11}(e, \beta) + C_{12}(e, \beta) = 2K\left[\frac{\beta - \underline{\beta}}{\overline{\beta} - \underline{\beta}}\right]^{\gamma}\left[1 - \frac{|\gamma|e}{\beta - \underline{\beta}}\right] > 0. \tag{66}$$

The inequality in (66) holds because when $e(\beta) = e_0(\beta)$:

$$\begin{aligned} \beta - \underline{\beta} - |\gamma| e &= \beta - \underline{\beta} - |\gamma| [\frac{6}{5} \frac{(\beta - \underline{\beta})^2}{(\overline{\beta} - \underline{\beta})} - (\beta - \underline{\beta})] = [\beta - \underline{\beta}][1 + |\gamma|] - \frac{6}{5} |\gamma| \frac{(\beta - \underline{\beta})^2}{(\overline{\beta} - \underline{\beta})} \\ &= 3[\beta - \underline{\beta}] - \frac{12}{5} \frac{(\beta - \underline{\beta})^2}{(\overline{\beta} - \underline{\beta})} \geq 3[\beta - \underline{\beta}] - \frac{12}{5}[\beta - \underline{\beta}] > 0. \end{aligned} \tag{67}$$

The first equality in (67) follows from (58). The last equality in (67) holds because $\gamma = -2$ in the setting under consideration. (67) ensures $1 - \frac{|\gamma| e}{\beta - \underline{\beta}} > 0$, and so the inequality in (66) holds. Therefore, we need $e'(\beta) \leq 1$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$ to ensure (65) is satisfied.

(58) implies that at the identified solution, for $\beta \in (\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}, \overline{\beta}]$:

$$e'(\beta) = \frac{12}{5} \frac{[\beta - \underline{\beta}]}{[\overline{\beta} - \underline{\beta}]} - 1 > \frac{12}{5} \left[\frac{5}{6} \right] - 1 = 1. \tag{68}$$

(66) and (68) imply that (65) does not hold at the identified solution in this setting.

To derive the optimal contract, we consider the following alternative formulation of the buyer's problem, called [AF].

$$\underset{e(\beta)}{Minimize} \quad \int_{\underline{\beta}}^{\overline{\beta}} \{ \beta - e(\beta) + C(e(\beta), \beta) + [C_1(e, \beta) + C_2(e, \beta)] \frac{F(\beta)}{f(\beta)} \} dF(\beta) \tag{69}$$

$$\text{subject to:} \quad \beta_1 - e(\beta_1) \leq \beta_2 - e(\beta_2) \qquad \text{for all} \;\; \beta_1 \leq \beta_2. \tag{70}$$

Constraint (70) implies that realized cost, $c(\beta) = \beta - e(\beta)$, is (weakly) increasing in β at the solution to [AF].

We now characterize the solution to [AF] and prove it is a solution to [BP] in the present setting. The proof proceeds in two steps. Step 1 demonstrates that for any feasible solution to [BP], there is a solution to [AF] that ensures lower expected payment for the buyer. Step 2 solves [AF] and demonstrates that the solution to [AF] satisfies the GIC constraints.

Step 1. For each effort function $\hat{e}(\beta)$ satisfying the GIC constraints, we will construct an effort function $\tilde{e}(\beta)$ such that: (1) $\beta - \tilde{e}(\beta)$ is weakly increasing in β, and (2) $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$ for all β, where $R(e)$, defined in (25), is the expected payment to the supplier under effort supply e.

Consider an effort function $\hat{e}(\beta)$ satisfying the GIC constraints, and let $B = \{\beta \mid \hat{e}(\beta) \leq e^*(\beta)\}$. First, we show that if $\beta_1, \beta_2 \in B$ and $\beta_1 < \beta_2$, then, $\beta_1 - \hat{e}(\beta_1) \leq \beta_2 - \hat{e}(\beta_2)$.

Since $\hat{e}(\beta)$ satisfies the GIC constraints:

$$u(\beta_1|\beta_1) \geq u(\beta_2|\beta_1), \text{ and} \tag{71}$$

$$u(\beta_2|\beta_2) \geq u(\beta_1|\beta_2). \tag{72}$$

Using (4), (71) and (72) can be rewritten as:

$$p(\beta_1) - [\beta_1 - \hat{e}(\beta_1)] - C(\hat{e}(\beta_1), \beta_1) \geq p(\beta_2) - [\beta_2 - \hat{e}(\beta_2)] - C(\hat{e}(\beta_2|\beta_1), \beta_1), \text{ and} \tag{73}$$

$$p(\beta_1) - [\beta_1 - \hat{e}(\beta_1)] - C(\hat{e}(\beta_1|\beta_2), \beta_2) \leq p(\beta_2) - [\beta_2 - \hat{e}(\beta_2)] - C(\hat{e}(\beta_2), \beta_2)\ . \tag{74}$$

Subtracting (74) from (73) and rearranging terms provides:

$$C(\hat{e}(\beta_2|\beta_1), \beta_1) - C(\hat{e}(\beta_1), \beta_1) \geq C(\hat{e}(\beta_2), \beta_2) - C(\hat{e}(\beta_1|\beta_2), \beta_2)\ . \tag{75}$$

Recall from (5) that:

$$\hat{e}(\beta_2|\beta_1) = \beta_1 - \beta_2 + \hat{e}(\beta_2) \quad \text{and} \quad \hat{e}(\beta_1|\beta_2) = \beta_2 - \beta_1 + \hat{e}(\beta_1)\ . \tag{76}$$

Assumption 1 implies that (75) can be rewritten as:

$$K \left[\frac{\beta_1 - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} [\hat{e}(\beta_2|\beta_1)^2 - \hat{e}(\beta_1)^2] \geq K \left[\frac{\beta_2 - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}} \right]^{\gamma} [\hat{e}(\beta_2)^2 - \hat{e}(\beta_1|\beta_2)^2]$$

$$\Leftrightarrow [\beta_1 - \widetilde{\beta}]^{\gamma} [\hat{e}(\beta_2|\beta_1) + \hat{e}(\beta_1)][\hat{e}(\beta_2|\beta_1) - \hat{e}(\beta_1)]$$

$$\geq [\beta_2 - \widetilde{\beta}]^{\gamma} [\hat{e}(\beta_2) + \hat{e}(\beta_1|\beta_2)][\hat{e}(\beta_2) - \hat{e}(\beta_1|\beta_2)]\ . \tag{77}$$

Using (76), (77) can be rewritten as:

$$D(\hat{e}(\beta_2)) \equiv\ [\beta_2 - \widetilde{\beta}]^{\gamma}\, [\hat{e}(\beta_2) + \beta_2 - \beta_1 + \hat{e}(\beta_1)]\, [[\beta_1 - \hat{e}(\beta_1)] - [\beta_2 - \hat{e}(\beta_2)]]$$

$$-[\beta_1 - \widetilde{\beta}]^{\gamma}\, [\hat{e}(\beta_1) + \beta_1 - \beta_2 + \hat{e}(\beta_2)]\, [[\beta_1 - \hat{e}(\beta_1)] - [\beta_2 - \hat{e}(\beta_2)]] \leq 0\ . \tag{78}$$

It is apparent from (78) that $D(\hat{e}(\beta_2))$ is a quadratic function of $\hat{e}(\beta_2)$ for given β_1, β_2, and $\hat{e}(\beta_1)$. Therefore, the equation $D(\hat{e}(\beta_2)) = 0$ will have at most two real roots. Denote these roots by $\hat{e}_1$ and $\hat{e}_2$.

Notice that the coefficient of the $\hat{e}(\beta_2)^2$ term in (78) is $-[\beta_1 - \widetilde{\beta}]^{\gamma} + [\beta_2 - \widetilde{\beta}]^{\gamma} < 0$. This coefficient is negative because $\gamma < 0$ and $\beta_1 < \beta_2$. The negative coefficient implies $D(\cdot)$ is a concave function of $\hat{e}(\beta_2)$. Because (78) requires $D(\hat{e}(\beta_2)) \leq 0$, this concavity ensures we have one of two cases: Case A: $\hat{e}(\beta_2)$ is (weakly) less than the smaller of $\hat{e}_1$ and $\hat{e}_2$; or Case B: $\hat{e}(\beta_2)$ is (weakly) greater than the larger of $\hat{e}_1$ and $\hat{e}_2$.

Notice from (78) that $D(\hat{e}(\beta_2)) = 0$ when $\hat{e}(\beta_2) = \hat{e}(\beta_1) - \beta_1 + \beta_2$. Therefore, one of the roots of this equation is $\hat{e}(\beta_1) - \beta_1 + \beta_2$. Consequently, if we are in Case A so that $\hat{e}(\beta_2)$ is (weakly) less than the smaller of $\hat{e}_1$ and $\hat{e}_2$, we know $\hat{e}(\beta_2) \leq \hat{e}(\beta_1) - \beta_1 + \beta_2$, and so $\beta_1 - \hat{e}(\beta_1) \leq \beta_2 - \hat{e}(\beta_2)$.

We now show that $\hat{e}(\beta_2)$ must be less than the greater of $\hat{e}_1$ and $\hat{e}_2$, and so Case B is not relevant. To show that $\hat{e}(\beta_2)$ is less than the greater of $\hat{e}_1$ and $\hat{e}_2$, it suffices to show that $\hat{e}(\beta_2)$ is less than the average of the two roots. $\hat{e}(\beta_2)$ will be less than the average of the two roots if this average exceeds $\frac{\beta_2 - \widetilde{\beta}}{2}$. This is the case because:

$$\hat{e}(\beta_2) \leq e^*(\beta_2) \leq \frac{\beta_2 - \widetilde{\beta}}{2} \ . \tag{79}$$

The first inequality in (79) holds because $e(\beta) \leq e^*(\beta)$ since $\beta_2 \in B$. The second inequality in (79) holds because:

$$e^*(\beta_2) = \frac{[\beta_2 - \widetilde{\beta}]^{|\gamma|}}{2K[\overline{\beta} - \widetilde{\beta}]^{|\gamma|}} \leq \frac{[\beta_2 - \widetilde{\beta}]^{|\gamma|}}{2[\overline{\beta} - \widetilde{\beta}]^{|\gamma|-1}} \leq \frac{[\beta_2 - \widetilde{\beta}]^{|\gamma|}}{2[\beta_2 - \widetilde{\beta}]^{|\gamma|-1}} = \frac{\beta_2 - \widetilde{\beta}}{2} \ . \tag{80}$$

The first equality in (80) follows from (37). The first inequality in (80) follows from Assumption 2. The second inequality in (80) holds because $\beta_2 \leq \overline{\beta}$.

It is readily shown that the two roots of the equation $D(\hat{e}(\beta_2)) = 0$ are $\hat{e}(\beta_1) + \beta_2 - \beta_1$ and $-\hat{e}(\beta_1) + \left[\frac{[\beta_1 - \widetilde{\beta}]^\gamma + [\beta_2 - \widetilde{\beta}]^\gamma}{[\beta_1 - \widetilde{\beta}]^\gamma - [\beta_2 - \widetilde{\beta}]^\gamma}\right][\beta_2 - \beta_1]$. The average of these roots is $\left[\frac{[\beta_1 - \widetilde{\beta}]^\gamma}{[\beta_1 - \widetilde{\beta}]^\gamma - [\beta_2 - \widetilde{\beta}]^\gamma}\right][\beta_2 - \beta_1]$. To conclude that the average of these roots is greater than $\frac{\beta_2 - \widetilde{\beta}}{2}$, we must show:

$$\left[\frac{[\beta_1 - \widetilde{\beta}]^\gamma}{[\beta_1 - \widetilde{\beta}]^\gamma - [\beta_2 - \widetilde{\beta}]^\gamma}\right][\beta_2 - \beta_1] \ > \ \frac{\beta_2 - \widetilde{\beta}}{2} \tag{81}$$

$$\Leftrightarrow \ \frac{[\beta_2 - \beta_1][\beta_2 - \widetilde{\beta}]^{|\gamma|}}{[\beta_2 - \widetilde{\beta}]^{|\gamma|} - [\beta_1 - \widetilde{\beta}]^{|\gamma|}} \ > \ \left[\frac{\beta_2 - \widetilde{\beta}}{2}\right] \tag{82}$$

$$\Leftrightarrow \ 1 \ > \ \left[\frac{\beta_2 - \widetilde{\beta}}{2}\right]\left[\frac{1}{\beta_2 - \beta_1}\right]\left[1 - \left(\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}}\right)^{|\gamma|}\right] \ . \tag{83}$$

Since $\widetilde{\beta} \leq \beta_1 < \beta_2$, $\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}} \leq 1$. Therefore, since $|\gamma| \leq 2$, $\left[\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}}\right]^2 \leq \left[\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}}\right]^{|\gamma|}$, and so $1 - \left[\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}}\right]^{|\gamma|} \leq 1 - \left[\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}}\right]^2$. Consequently, (83) will hold if:

$$1 > \left[\frac{\beta_2 - \widetilde{\beta}}{2}\right]\left[\frac{1}{\beta_2 - \beta_1}\right]\left[1 - \left(\frac{\beta_1 - \widetilde{\beta}}{\beta_2 - \widetilde{\beta}}\right)^2\right] \ . \tag{84}$$

Notice that:

$$\begin{aligned}[\beta_2 - \widetilde{\beta}]^2 - [\beta_1 - \widetilde{\beta}]^2 &= [(\beta_2 - \widetilde{\beta}) - (\beta_1 - \widetilde{\beta})][(\beta_2 - \widetilde{\beta}) + (\beta_1 - \widetilde{\beta})] \\ &= [\beta_2 - \beta_1][\beta_2 + \beta_1 - 2\widetilde{\beta}] \ . \end{aligned} \tag{85}$$

Therefore, (84) holds if and only if:

$$1 > \left[\frac{\beta_2 - \widetilde{\beta}}{2}\right] \left[\frac{1}{\beta_2 - \beta_1}\right] \frac{[\beta_2 - \beta_1][\beta_2 + \beta_1 - 2\widetilde{\beta}]}{(\beta_2 - \widetilde{\beta})^2} \tag{86}$$

$$\Leftrightarrow \ 2[\beta_2 - \widetilde{\beta}] > \beta_2 + \beta_1 - 2\widetilde{\beta} \quad \Leftrightarrow \quad \beta_2 > \beta_1 \ . \tag{87}$$

Since $\beta_2 > \beta_1$ by assumption, we have shown (81) holds, and so the average of the roots the equation $D(\hat{e}(\beta_2)) = 0$ is greater than $\frac{\beta_2 - \widetilde{\beta}}{2}$. Consequently, $\hat{e}(\beta_2)$ must be less than the greater of $\hat{e}_1$ and $\hat{e}_2$, and so Case A is the only relevant case. Therefore, $\beta_1 - \hat{e}(\beta_1) \leq \beta_2 - \hat{e}(\beta_2)$ for all $\beta_1, \beta_2 \in B$ and $\beta_1 < \beta_2$.

Now define $\tilde{e}(\beta)$ such that $\beta - \tilde{e}(\beta) = \max\{\beta - e^*(\beta), \sup_{\beta' \leq \beta, \beta' \in B}\{\beta' - \hat{e}(\beta')\}\}$. Since both $\beta - e^*(\beta)$ and $\sup_{\beta' \leq \beta, \beta' \in B}\{\beta' - \hat{e}(\beta')\}$ are (weakly) increasing functions, $\beta - \tilde{e}(\beta)$ is weakly increasing in β.

Recall that $B = \{\beta \mid \hat{e}(\beta) \leq e^*(\beta)\}$, and for all $\beta_1, \beta_2 \in B$ and $\beta_1 < \beta_2$, $\beta_1 - \hat{e}(\beta_1) \leq \beta_2 - \hat{e}(\beta_2)$. Therefore, $\tilde{e}(\beta) = \hat{e}(\beta)$ for $\beta \in B$ and $\tilde{e}(\beta) < \hat{e}(\beta)$ otherwise.

Finally, we will show that $\tilde{e}(\beta)$ ensures lower expected payment than $\hat{e}(\beta)$ by verifying that $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$ for each $\beta \in [\underline{\beta}, \overline{\beta}] \backslash B$, where, recall from (25):

$$R(e) = \beta - e(\beta) + C(e(\beta), \beta) + [C_1(e, \beta) + C_2(e, \beta)]\frac{F(\beta)}{f(\beta)}. \tag{88}$$

Recall that in the setting of Finding 2, $F(\beta) = (\frac{\beta - \underline{\beta}}{\overline{\beta} - \underline{\beta}})^3$, and $\frac{F(\beta)}{f(\beta)} = \frac{\beta - \underline{\beta}}{3}$. Furthermore, from (29), expected payment is a convex, quadratic function of e, because $1 + \frac{\gamma}{\beta - \underline{\beta}}[\frac{F(\beta)}{f(\beta)}] = 1 - \frac{2}{3} > 0$. Therefore, to show $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$, it suffices to show $|\tilde{e} - e_0| \leq |\hat{e} - e_0|$.

There are three cases to consider:

<u>Case 1</u>. $\tilde{e}(\beta) = \hat{e}(\beta)$.

In this case, $R(\tilde{e}(\beta)) = R(\hat{e}(\beta))$.

<u>Case 2</u>. $\tilde{e}(\beta) = e^*(\beta) < \hat{e}(\beta)$.

Because $e_0 \leq e^*(\beta)$ from Lemma 3, $e_0(\beta) \leq \tilde{e}(\beta) < \hat{e}(\beta)$ in this case. Therefore, $|\tilde{e} - e_0| \leq |\hat{e} - e_0|$, and so $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$.

Case 3. $\tilde{e}(\beta) < e^*(\beta) < \hat{e}(\beta)$.

We will show that $\frac{1}{2}[\tilde{e}(\beta) + \hat{e}(\beta)] > e^*(\beta)$ in this case. Consequently, since $e_0(\beta) < e^*(\beta)$ from Lemma 3, $|\tilde{e} - e_0| \leq |\hat{e} - e_0|$ in this case, and so $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$.

To begin, let $\beta_1 = \sup\{\hat{\beta} \mid \hat{\beta} < \beta, \hat{\beta} \in B\}$, and e_1 such that $\beta_1 - e_1 = \sup\{\hat{\beta} - \hat{e}(\hat{\beta}) | \hat{\beta} < \beta, \hat{\beta} \in B\}$. Notice that $\beta_1 < \beta$ since $\tilde{e}(\beta) < e^*(\beta)$. Also recall that inequality (78) holds for $\hat{e}(\beta)$. The two roots of $D(\hat{e}(\beta)) = 0$ are $\hat{e}(\beta_1) + \beta - \beta_1$ and $-\hat{e}(\beta_1) + \left[\frac{[\beta_1 - \widetilde{\beta}]^\gamma + [\beta - \widetilde{\beta}]^\gamma}{[\beta_1 - \widetilde{\beta}]^\gamma - [\beta - \widetilde{\beta}]^\gamma}\right][\beta - \beta_1]$. There are two cases to consider: Case (i) $\beta_1 \in B$; and Case (ii) $\beta_1 \notin B$. In Case (i), $e_1 = \hat{e}(\beta_1)$ and $\tilde{e}(\beta) = \hat{e}(\beta_1) + \beta - \beta_1$. Because $\hat{e}(\beta) > \tilde{e}(\beta)$, $\hat{e}(\beta)$ is greater than both roots in this case. Because the average of the roots exceeds $e^*(\beta)$, $\frac{1}{2}[\tilde{e}(\beta) + \hat{e}(\beta)] > e^*(\beta)$. An analogous argument holds in Case (ii), where a series of β's converging to β_1 is considered instead of a single β_1.

Therefore, since $|\tilde{e} - e_0| \leq |\hat{e} - e_0|$ in all relevant cases in this setting, $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$.

Step 2. We now characterize the solution to [AF] and demonstrate that it satisfies the GIC constraints. In particular, we will show that at the solution to [AF], there exists a β^* such that:

$$e(\beta) = \begin{cases} 0 & \text{for} \quad \beta \in [\underline{\beta}, \beta^*) \\ \beta - \beta^* & \text{for} \quad \beta \in [\beta^*, \overline{\beta}]. \end{cases} \tag{89}$$

Notice that $\beta - e(\beta)$ is a strictly increasing function of β on $[\underline{\beta}, \beta^*)$ and does not vary with β for $\beta \in [\beta^*, \overline{\beta}]$. Therefore, the solution identified in (89) satisfies (70).

We now show that for any feasible solution to [AF], a solution of the form identified in (89) ensures the same or lower expected payment. Recall from (58) that if constraint (70) is ignored, the expression in (69) is minimized in the present setting when

$$e_0(\beta) = \begin{cases} 0 & \text{for} \quad \beta \in [\underline{\beta},\ \frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}) \\ \frac{6}{5}\left[\frac{\beta - \underline{\beta}}{\overline{\beta} - \underline{\beta}}\right]^2 - (\beta - \underline{\beta}) & \text{for} \quad \beta \in [\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \overline{\beta}]. \end{cases} \tag{90}$$

(90) implies $e_0'(\beta) \geq 1$ and so $\beta - e_0(\beta)$ is strictly decreasing on $(\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \overline{\beta}]$. Thus, this solution is not a feasible solution to [AF].

Consider a feasible solution to [AF], $\hat{e}(\beta)$, where $\hat{e}(\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}) \geq 0 = e_0(\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta})$. Define $\beta_0 \in (\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \overline{\beta}]$ as the realization of β such that:

$$\beta - \hat{e}(\beta) \geq \beta_0 - e_0(\beta_0) > \beta - e_0(\beta) \qquad \text{for} \quad \beta \in \left(\beta_0, \overline{\beta}\right] \tag{91}$$

and

$$\beta - \hat{e}(\beta) \leq \beta_0 - e_0(\beta_0) < \beta - e_0(\beta_0) \qquad \text{for} \quad \beta \in (\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \beta_0)\ . \tag{92}$$

Such a β_0 exists as long as $\hat{e}(\overline{\beta}) \leq e_0(\overline{\beta})$, since $\hat{c}(\beta) \equiv \beta - \hat{e}(\beta)$ is (weakly) increasing in β (by (70), since $\hat{e}(\cdot)$ is a feasible solution to [AF]) and $c_0(\beta) \equiv \beta - e_0(\beta)$ is decreasing on $\left(\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \overline{\beta}\right)$ (since $e_0'(\beta) \geq 1$ in this region). If $\hat{e}(\overline{\beta}) > e_0(\overline{\beta})$ (so that $c_0(\overline{\beta}) = \overline{\beta} - e_0(\overline{\beta}) > \hat{c}(\overline{\beta}) = \overline{\beta} - \hat{e}(\overline{\beta})$), define β_0 as $\overline{\beta}$.

Now consider the following feasible solution to [AF]:

$$\tilde{e}(\beta) = \begin{cases} 0 & \text{for} \quad \beta < \beta_0 - e(\beta_0) \\ \beta - [\beta_0 - e_0(\beta_0)] & \text{for} \quad \beta \geq \beta_0 - e(\beta_0) \end{cases} \tag{93}$$

We will show that $\tilde{e}(\beta)$ ensures lower expected payment than $\hat{e}(\beta)$ by verifying that $R(\tilde{e}(\beta)) \leq R(\hat{e}(\beta))$ for each $\beta \in [\underline{\beta}, \overline{\beta}]$, where, recall from (25):

$$R(e) = \beta - e(\beta) + C(e(\beta), \beta) + [C_1(e, \beta) + C_2(e, \beta)] \frac{F(\beta)}{f(\beta)} \ . \tag{94}$$

Recall that $R(\cdot)$ is either an increasing function or a convex, quadratic function of $e(\cdot)$ that attains its minimum value at $e_0(\beta)$. Consequently, to demonstrate that $\tilde{e}(\cdot)$ secures a lower value of $R(\cdot)$ than $\hat{e}(\cdot)$, it suffices to show that:

$$|\hat{e}(\beta) - e_0(\beta)| \geq |\tilde{e}(\beta) - e_0(\beta)| \quad \text{for all} \ \ \beta \in \left[\underline{\beta}, \overline{\beta}\right] . \tag{95}$$

Because $e_0(\beta) = 0$ for $\beta \in \left[\underline{\beta},\ \frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}\right]$ and $e_0'(\beta) \geq 1$ for $\beta \in \left[\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \overline{\beta}\right]$, we know $\beta_0 - e_0(\beta_0) \leq \frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}$. Consequently, we have five relevant regions to consider.

Region 1. $\beta \in [\underline{\beta},\ \beta_0 - e_0(\beta_0)]$.

$R(\cdot)$ is minimized at $e_0(\beta) = 0$ for all β in this region. Furthermore, $\tilde{e}(\beta)$ equals 0 for all β in this region. Consequently, $|\hat{e}(\beta) - e_0(\beta)| \geq |\tilde{e}(\beta) - e_0(\beta)| \ = \ 0$ for all β in this region.

Region 2. $\beta \in (\beta_0 - e_0(\beta_0),\ \frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}]$.

Since $\beta < \frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta}$, $R(\cdot)$ is minimized at $e_0(\beta) = 0$ for all β in this region. Therefore, $\tilde{e}(\beta) \equiv \beta - [\beta_0 - e_0(\beta_0)] = \beta - \beta_0$. Furthermore, because $\beta < \beta_0$ and $\tilde{e}(\beta) \geq 0$, $\beta - \hat{e}(\beta) \leq \beta_0 - e_0(\beta_0)$. Therefore, $0 \leq \tilde{e}(\beta) = \beta - \beta_0 \leq \hat{e}(\beta) - e_0(\beta_0) = \hat{e}(\beta)$. In summary, we have $e_0(\beta) = 0 \leq \tilde{e}(\beta) \leq \hat{e}(\beta)$. Consequently, $|\hat{e}(\beta) - e_0(\beta)| \geq |\tilde{e}(\beta) - e_0(\beta)|$ for all β in this region.

Region 3. $\beta \in (\frac{1}{6}\underline{\beta} + \frac{5}{6}\overline{\beta},\ \beta_0)$.

(92) implies $\beta - \hat{e}(\beta) \leq \beta_0 - e_0(\beta_0)$ in this region. Therefore, $\tilde{e}(\beta) = \beta - [\beta_0 - e_0(\beta_0)] \leq \hat{e}(\beta)$. Furthermore, since $e_0'(\beta) \geq 1$ in this region, $e_0(\beta_0) - e_0(\beta) \geq \beta_0 - \beta$ for all β in this

region. Consequently, $e_0(\beta) \leq \beta - [\beta_0 - e_0(\beta_0)] = \tilde{e}(\beta)$. In summary, we have $e_0(\beta) \leq \tilde{e}(\beta) \leq \hat{e}(\beta)$. Consequently, $|\hat{e}(\beta) - e_0(\beta)| \geq |\tilde{e}(\beta) - e_0(\beta)|$ for all β in this region.

Region 4. $\beta = \beta_0$.

$\tilde{e}(\beta) \equiv \beta - [\beta_0 - e_0(\beta_0)] = e_0(\beta_0)$ in this region. Therefore, $|\hat{e}(\beta) - e_0(\beta)| \geq |\tilde{e}(\beta) - e_0(\beta)| = 0$ in this region.

Region 5. $\beta \in (\beta_0, \overline{\beta}]$.

(91) implies $\beta - \hat{e}(\beta) \geq \beta_0 - e_0(\beta_0)$ in this region. Consequently, $\tilde{e}(\beta) = \beta - [\beta_0 - e_0(\beta_0)] \geq \hat{e}(\beta)$. Furthermore, since $e_0'(\beta) \geq 1$ in this region, $e_0(\beta_0) - e_0(\beta) \geq \beta_0 - \beta$, which implies $e_0(\beta) \geq \beta - [\beta_0 - e_0(\beta_0)] = \tilde{e}(\beta)$. In summary, we have $e_0(\beta) \geq \tilde{e}(\beta) \geq \hat{e}(\beta)$. Consequently, $|\hat{e}(\beta) - e_0(\beta)| \geq |\tilde{e}(\beta) - e_0(\beta)|$ for all β in this region.

Therefore, we have shown that $\tilde{e}(\beta)$ secures lower expected payment than $\hat{e}(\beta)$ in all five regions, and thus for all $\beta \in [\underline{\beta}, \overline{\beta}]$. Under the optimal contract, no effort is induced on $[\underline{\beta}, \beta^*)$, and effort $\beta - \beta^*$ is induced on $[\beta^*, \overline{\beta}]$.

To derive the value of β^*, notice that expected payment in the present setting is:

$$[\beta^* + K(\overline{\beta} - \beta^*)^2][1 - F(\beta^*)] + \int_{\underline{\beta}}^{\beta^*} [\beta + K(\overline{\beta} - \beta^*)^2] dF(\beta). \tag{96}$$

Letting $x^* \equiv \frac{\beta^* - \underline{\beta}}{\overline{\beta} - \underline{\beta}}$, the expression in (96) is proportional to

$$[x^* + \frac{5}{4}(1 - x^*)^2][1 - x^3] + \int_0^{x^*} [t + \frac{5}{4}(1 - x^*)^2] d(t^3). \tag{97}$$

To see why, notice that $F(\beta^*) = [x^*]^3$ in this example. Also:

$$\beta^* + K[\overline{\beta} - \beta^*]^2 = \beta^* + \frac{5}{4}[\overline{\beta} - \underline{\beta}]^{-1} \left[\frac{\overline{\beta} - \underline{\beta} - (\beta^* - \underline{\beta})}{\overline{\beta} - \underline{\beta}} \right]^2 [\overline{\beta} - \underline{\beta}]^2$$

$$= [\overline{\beta} - \underline{\beta}] \left[\frac{\beta^*}{\overline{\beta} - \underline{\beta}} + \frac{5}{4} [1 - x^*]^2 \right] = [\overline{\beta} - \underline{\beta}] \left[x^* + \frac{5}{4} [1 - x^*]^2 \right] + \underline{\beta} \, . \tag{98}$$

Furthermore, let $\Delta \equiv \overline{\beta} - \underline{\beta}$ and $t = \frac{\beta - \underline{\beta}}{\Delta}$, so that t varies from 0 to x^* as β varies from

$\underline{\beta}$ to β^*. Notice that $d\beta = \Delta dt$. Consequently:

$$\int_{\underline{\beta}}^{\beta^*} [\beta + K(\overline{\beta} - \beta^*)^2] dF(\beta)$$

$$= \int_{\underline{\beta}}^{\beta^*} \{\beta + \frac{5}{4}[\overline{\beta} - \underline{\beta}]^{-1}[\overline{\beta} - \beta^*]^2\} 3 \frac{[\beta - \underline{\beta}]^2}{[\overline{\beta} - \underline{\beta}]^3} d\beta$$

$$= \int_{\underline{\beta}}^{\beta^*} \{\beta + \frac{5}{4}\Delta^{-1}\left[\frac{\overline{\beta} - \underline{\beta} - (\beta^* - \underline{\beta})}{\Delta}\right]^2 \Delta^2\} 3 \frac{[\beta - \underline{\beta}]^2}{\Delta^2}\left[\frac{1}{\Delta}\right] d\beta$$

$$= \int_0^{x^*} \{\beta + \frac{5}{4}[1 - x^*]^2 \Delta\} \frac{1}{\Delta} 3t^2 \Delta dt \quad = \quad \Delta \int_0^{x^*} \{\frac{\beta}{\Delta} + \frac{5}{4}[1 - x^*]^2\} d(t^3)$$

$$= \Delta \int_0^{x^*} \{t + \frac{\underline{\beta}}{\Delta} + \frac{5}{4}[1 - x^*]^2\} d(t^3) = \Delta \int_0^{x^*} \{t + \frac{5}{4}[1 - x^*]^2\} d(t^3) + \underline{\beta} \int_0^{x^*} d(t^3). \quad (99)$$

(98) and (99) imply:

$$[\beta^* + K(\overline{\beta} - \beta^*)^2][1 - F(\beta^*)] + \int_{\underline{\beta}}^{\beta^*} [\beta + K(\overline{\beta} - \beta^*)^2] dF(\beta)$$

$$= \Delta \left[x^* + \frac{5}{4}[1 - x^*]^2\right][1 - (x^*)^3] + \underline{\beta}[1 - (x^*)^3] + \Delta \int_0^{x^*} \{t + \frac{5}{4}[1 - x^*]^2\} d(t^3) + \underline{\beta}[x^*]^3$$

$$= \Delta \left[x^* + \frac{5}{4}[1 - x^*]^2\right][1 - (x^*)^3] + \int_0^{x^*} \{t + \frac{5}{4}[1 - x^*]^2\} d(t^3)\} + \underline{\beta} \,. \quad (100)$$

To identify the optimal β^*, we can find the value of x^* that minimizes (97). Notice that:

$$\frac{\partial}{\partial x}\{[x + \frac{5}{4}(1 - x)^2][1 - x^3] + \int_0^x [t + \frac{5}{4}(1 - x)^2] d(t^3)\} =$$

$$[1 - \frac{5}{2}(1 - x)][1 - x^3] + [x + \frac{5}{4}(1 - x)^2][-3x^2] + [x + \frac{5}{4}(1 - x)^2] 3x^2 + \int_0^x [-\frac{5}{2}(1 - x)] d(t^3) = 0 \quad (101)$$

$$\Leftrightarrow \quad [1 - \frac{5}{2}(1 - x)][1 - x^3] - \frac{5}{2}[1 - x]x^3 = 0 \quad (102)$$

$$\Leftrightarrow \quad 1 - x^3 = \frac{5}{2}[1 - x][1 - x^3] + \frac{5}{2}[1 - x]x^3 \quad (103)$$

$$\Leftrightarrow \quad 1 - x^3 = \frac{5}{2}[1 - x] \,. \quad (104)$$

For $\beta^* \in (\underline{\beta}, \overline{\beta})$, $x^* \in (0, 1)$. Therefore, dividing both sides of (104) by $1 - x$ provides:

$$1 + x + x^2 = \frac{5}{2}, \quad \text{or} \quad x = \frac{\sqrt{7} - 1}{2} \,. \quad (105)$$

Since $x^* = \frac{\beta^* - \underline{\beta}}{\overline{\beta} - \underline{\beta}}$, $\beta^* = \underline{\beta} + [\overline{\beta} - \underline{\beta}]x^*$, and so:

$$\beta^* = \underline{\beta} + [\overline{\beta} - \underline{\beta}] \left[\frac{\sqrt{7} - 1}{2} \right] = \left[\frac{\sqrt{7} - 1}{2} \right] \overline{\beta} + \left[\frac{2 - (\sqrt{7} - 1)}{2} \right] \underline{\beta} \, . \tag{106}$$

(106) implies:

$$\beta^* = \beta_0 - e_0(\beta_0) = [\frac{3 - \sqrt{7}}{2}]\underline{\beta} + [\frac{\sqrt{7} - 1}{2}]\overline{\beta} \, . \tag{107}$$

It remains to verify that the GIC constraints are satisfied at the identified solution. To begin, notice that with innate cost $\beta \in \left[\underline{\beta}, \beta^*\right]$ the supplier will not deliver any cost-reducing effot because realized costs are fully reimbrused in this range. When $\beta \in (\beta^*, \overline{\beta}]$, the supplier will not deliver effort in excess of $\beta - \beta^*$. Such effort would simply reduce realized cost below β^*, and thereby reduce the supplier's payment. Thus, it only remains to show that for any $\beta \in (\beta^*, \overline{\beta}]$, the supplier will deliver effort $\beta - \beta^*$.

The supplier is offered a fixed payment, β^*, for any cost realization $c \in \left[\beta^*, \overline{\beta}\right]$. Therefore, the supplier will maximize his utility by delivering effort equal to the minimum of $e^*(\beta)$ and $\beta - \beta^*$ for all $\beta \in (\beta^*, \overline{\beta}]$. Consequently, the GIC constraints will be satisfied if $e^*(\beta) > \beta - \beta^*$ or $\beta^* > \beta - e^*(\beta)$ for all $\beta \in (\beta^*, \overline{\beta}]$. This will be the case if: (1) $\beta^* > \overline{\beta} - e^*(\overline{\beta})$; and (2) $\overline{\beta} - e^*(\overline{\beta}) > \beta - e^*(\beta)$ for all $\beta \in (\beta^*, \overline{\beta}]$

The first condition holds because, from (37):

$$\overline{\beta} - e^*(\overline{\beta}) \;=\; \overline{\beta} - \frac{1}{2K \left[\frac{\overline{\beta} - \underline{\beta}}{\overline{\beta} - \underline{\beta}} \right]^\gamma} = \overline{\beta} - \frac{1}{2K} = \overline{\beta} - \frac{4[\overline{\beta} - \underline{\beta}]}{(2)5} \tag{108}$$

$$= \;\overline{\beta} - \frac{2}{5}\overline{\beta} + \frac{2}{5}\underline{\beta} = \frac{3}{5}\overline{\beta} + \frac{2}{5}\underline{\beta} \, . \tag{109}$$

(107) and (109) imply:

$$\beta^* \;>\; \overline{\beta} - e^*(\overline{\beta}) \Leftrightarrow [\frac{3 - \sqrt{7}}{2}]\underline{\beta} + [\frac{\sqrt{7} - 1}{2}]\overline{\beta} > \frac{3}{5}\overline{\beta} + \frac{2}{5}\underline{\beta} \tag{110}$$

$$\Leftrightarrow \;\left[\frac{\sqrt{7}}{2} - \frac{1}{2} - \frac{3}{5} \right] \overline{\beta} - \left[\frac{\sqrt{7}}{2} + \frac{2}{5} - \frac{3}{2} \right] \underline{\beta} > 0 \Leftrightarrow \left[5\sqrt{7} - 11 \right] [\overline{\beta} - \underline{\beta}] > 0 \, . \tag{111}$$

The last inequality in (111) holds because $5\sqrt{7} - 11 > 13 - 11 > 0$. Consequently,

$\beta^* > \overline{\beta} - e^*(\overline{\beta})$. To show that the second condition holds, notice that (37) and (109) imply:

$$\overline{\beta} - e^*(\overline{\beta}) > \beta - e^*(\beta) \Leftrightarrow \overline{\beta} - \frac{1}{2K\left[\frac{\overline{\beta}-\underline{\beta}}{\overline{\beta}-\underline{\beta}}\right]^{\gamma}} > \beta - \frac{1}{2K\left[\frac{\beta-\underline{\beta}}{\overline{\beta}-\underline{\beta}}\right]^{\gamma}} \tag{112}$$

$$\Leftrightarrow \quad \overline{\beta} - \frac{2[\overline{\beta}-\underline{\beta}]}{5} > \beta - \frac{2[\overline{\beta}-\underline{\beta}]}{5}\left[\frac{\beta-\underline{\beta}}{\overline{\beta}-\underline{\beta}}\right]^2 = \beta - \frac{2[\beta-\underline{\beta}]^2}{5[\overline{\beta}-\underline{\beta}]} \tag{113}$$

$$\Leftrightarrow \quad \overline{\beta} - \beta > \frac{2}{5[\overline{\beta}-\underline{\beta}]}\left[[\overline{\beta}-\underline{\beta}]^2 - [\beta-\underline{\beta}]^2\right] \tag{114}$$

$$\Leftrightarrow \quad \overline{\beta} - \beta > \frac{2}{5[\overline{\beta}-\underline{\beta}]}\left[\overline{\beta}^2 - \beta^2 - 2\underline{\beta}(\overline{\beta}-\beta)\right] \tag{115}$$

$$\Leftrightarrow \quad \frac{5}{2}[\overline{\beta}-\beta] > \left[\frac{\overline{\beta}-\beta}{\overline{\beta}-\underline{\beta}}\right][\overline{\beta}+\beta-2\underline{\beta}] \Leftrightarrow \frac{5}{2}[\overline{\beta}-\underline{\beta}] > \overline{\beta}-\underline{\beta}+\beta-\underline{\beta} \tag{116}$$

$$\Leftrightarrow \quad \frac{3}{2}[\overline{\beta}-\underline{\beta}] > \beta-\underline{\beta} \Leftrightarrow \beta < \underline{\beta} + \frac{3}{2}[\overline{\beta}-\underline{\beta}] = \overline{\beta} + \frac{1}{2}[\overline{\beta}-\underline{\beta}]\ . \tag{117}$$

(117) implies that $\overline{\beta} - e^*(\overline{\beta}) > \beta - e^*(\beta)$ for all relevant β, since $\beta \leq \overline{\beta} < \overline{\beta} + \frac{1}{2}[\overline{\beta}-\underline{\beta}]$.

In summary, we have shown that $e(\beta) \leq e^*(\beta)$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$, and so the GIC constraints are satisfied in the present setting.

To ensure $\tilde{e}(\beta) = 0$ for all $\beta < \beta_0 - e_0(\beta_0)$, it suffices to reimburse the supplier fully for realized cost in the range $[\underline{\beta}, \beta^*)$. One way to ensure effort $\tilde{e}(\beta) = \beta - [\beta_0 - e_0(\beta_0)]$ for $\beta \in [\beta^*, \overline{\beta}]$ is to pay the supplier β^* for any cost realization $\beta \in [\beta^*, \overline{\beta}]$. Under this reward structure, the supplier will never deliver more effort than is required to ensure cost realization β^*. Additional effort would secure a smaller realized cost, and thereby ensure a lower payment for the supplier. Since $\beta^* > \beta - e^*(\beta)$ for $\beta > \beta^*$, the supplier with high cost will deliver sufficient effort to ensure cost β^*. ■

Proof of Finding 3. Recall that in the setting of Finding 3: (i) $\widetilde{\beta} = \underline{\beta}$; (ii) $\gamma = -2$; (iii) $f(\beta) = \frac{30[\beta-\underline{\beta}]^4[\overline{\beta}-\beta]}{[\overline{\beta}-\underline{\beta}]^6}$; and (iv) $K = [\overline{\beta}-\underline{\beta}]^{-1}$.

By Lemma 1, $u'(\beta) \leq 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$. Letting $\Delta \equiv \overline{\beta} - \underline{\beta}$, notice that $F(\beta) = 6(\frac{\beta-\underline{\beta}}{\overline{\beta}-\underline{\beta}})^5 - 5(\frac{\beta-\underline{\beta}}{\overline{\beta}-\underline{\beta}})^6 = \frac{[\beta-\underline{\beta}]^5}{\Delta^6}\left[6\Delta - 5[\beta-\underline{\beta}]\right]$. Therefore, $\frac{F(\beta)}{f(\beta)} = \frac{6[\beta-\underline{\beta}][\overline{\beta}-\underline{\beta}]-5[\beta-\underline{\beta}]^2}{30[\overline{\beta}-\beta]}$.

Define $x \equiv \frac{\beta-\underline{\beta}}{\Delta} \in [0,1]$. In this setting:

$$1 - 2K\left[\frac{\beta - \widetilde{\beta}}{\overline{\beta} - \widetilde{\beta}}\right]^{\gamma} \frac{F(\beta)}{f(\beta)} > 0 \tag{118}$$

$$\Leftrightarrow \;\; 1 - \frac{2}{\Delta}\left[\frac{\beta - \underline{\beta}}{\Delta}\right]^2 \left[\frac{6\Delta(\beta - \underline{\beta}) - 5[\beta - \underline{\beta}]^2}{30[\overline{\beta} - \beta]}\right] > 0$$

$$\Leftrightarrow \;\; 1 - 2\left[\frac{1}{x^2}\right]\left[\frac{6(\beta - \underline{\beta}) - 5x^2\Delta}{30[\overline{\beta} - \underline{\beta} - (\beta - \underline{\beta})]}\right] \Leftrightarrow 1 - 2\left[\frac{1}{x^2}\right]\left[\frac{6x - 5x^2}{30(1-x)}\right] > 0$$

$$\Leftrightarrow \;\; 1 - \frac{1}{15}\left[\frac{6 - 5x}{x(1-x)}\right] > 0 \;\; \Leftrightarrow \;\; 15x^2 - 20x + 6 < 0 \tag{119}$$

$$\Leftrightarrow \;\; \frac{20 - \sqrt{40}}{30} < x < \frac{20 + \sqrt{40}}{30}$$

$$\Leftrightarrow \;\; \underline{\beta} + \left[\frac{20 - \sqrt{40}}{30}\right][\overline{\beta} - \underline{\beta}] \;\leq\; \beta \;\leq\; \underline{\beta} + \left[\frac{20 + \sqrt{40}}{30}\right][\overline{\beta} - \underline{\beta}] \tag{120}$$

$$\Leftrightarrow \left[\frac{5 + \sqrt{10}}{15}\right]\underline{\beta} + \left[\frac{10 - \sqrt{10}}{15}\right]\overline{\beta} < \;\beta\; < \left[\frac{5 - \sqrt{10}}{15}\right]\underline{\beta} + \left[\frac{10 + \sqrt{10}}{15}\right]\overline{\beta}. \tag{121}$$

(121) and Observation 2 imply that at the solution to [BP]$'$, no effort is optimally induced for the lower and the higher innate cost realizations where the probability density is low. Lemma 2 also implies that on the interval $\left[\left[\frac{5+\sqrt{10}}{15}\right]\underline{\beta} + \left[\frac{10-\sqrt{10}}{15}\right]\overline{\beta}, \left[\frac{5-\sqrt{10}}{15}\right]\underline{\beta} + \left[\frac{10+\sqrt{10}}{15}\right]\overline{\beta}\right]$:

$$e_0(\beta) = \left[\frac{1 - \frac{1}{15}\left[\frac{6-5x}{x(1-x)}\right]}{2\frac{1}{x^2}[1 - (\frac{2}{x})(\frac{6x-5x^2}{30(1-x)})]}\right][\overline{\beta} - \underline{\beta}] = \left[\frac{\frac{15x(1-x)-(6-5x)}{15x(1-x)}}{\frac{2}{x^2}\left[\frac{15x(1-x)-(6x-5x^2)}{15x(1-x)}\right]}\right] \tag{122}$$

$$\left[\frac{[-15x^2 + 20x - 6]\,[x^2]}{2[-10x^2 + 9x]}\right][\overline{\beta} - \underline{\beta}] = \left[\frac{[-15x^2 + 20x - 6]x}{2[9 - 10x]}\right][\overline{\beta} - \underline{\beta}]\;. \tag{123}$$

It is tedious but straightforward to verify that $e_0'(\beta) < 1$. Therefore, by Lemmas 3 and 4, the GIC constraints are satisfied, and $e_0(\beta)$ is the solution to [BP]. ■