Technical Appendix to Accompany

"The Impact of Public Ownership in the Lending Sector"

by

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Equations from the Text

$$x_L = \frac{1}{2} + \frac{[\,\gamma_1 - \gamma_2\,]\, p_L V}{2\,t} \in (0,1) \quad \text{and} \quad x_H = \frac{1}{2} + \frac{[\,\gamma_1 - \gamma_2\,]\, p_H V}{2\,t} \in (0,1)\,. \tag{1}$$

$$\begin{aligned}\pi_1(\gamma_1,\gamma_2) &= \phi_L\, q_2\,[1-q_1]\,[p_L\,(1-\gamma_1)\,V - I] + \phi_H\, q_1\,[1-q_2]\,[p_H\,(1-\gamma_1)\,V - I] \\ &+ \phi_L\,[1-q_1]\,[1-q_2]\,x_L\,[p_L\,(1-\gamma_1)\,V - I] + \phi_H\, q_1\, q_2\, x_H\,[p_H\,(1-\gamma_1)\,V - I]\,.\end{aligned} \tag{2}$$

$$\begin{aligned}\pi_2(\gamma_1,\gamma_2) &= \phi_L\, q_1\,[1-q_2]\,[p_L\,(1-\gamma_2)\,V - I] + \phi_H\, q_2\,[1-q_1]\,[p_H\,(1-\gamma_2)\,V - I] \\ &+ \phi_L\,[1-q_1]\,[1-q_2]\,[1-x_L]\,[p_L\,(1-\gamma_2)\,V - I] + \phi_H\, q_1\, q_2\,[1-x_H]\,[p_H\,(1-\gamma_2)\,V - I]\,.\end{aligned} \tag{3}$$

$$\begin{aligned}W_L(\gamma_1,\gamma_2) &= \phi_L \left\{ q_2\,[1-q_1]\left[p_L\,\gamma_1\,V - \frac{t}{2}\right] + q_1\,[1-q_2]\left[p_L\,\gamma_2\,V - \frac{t}{2}\right] \right. \\ &\left. + [1-q_1]\,[1-q_2]\left[[x_L\,\gamma_1 + (1-x_L)\,\gamma_2]\,p_L V - \frac{t\,(x_L)^2}{2} - \frac{t\,(1-x_L)^2}{2}\right]\right\}.\end{aligned} \tag{4}$$

$$\begin{aligned}W_H(\gamma_1,\gamma_2) &= \phi_H \left\{ q_1\,[1-q_2]\left[p_H\,\gamma_1\,V - \frac{t}{2}\right] + q_2\,[1-q_1]\left[p_H\,\gamma_2\,V - \frac{t}{2}\right] \right. \\ &\left. + q_1\,q_2\left[[x_H\,\gamma_1 + (1-x_H)\,\gamma_2]\,p_H V - \frac{t\,(x_H)^2}{2} - \frac{t\,(1-x_H)^2}{2}\right] + [1-q_1]\,[1-q_2]\,R\right\}.\end{aligned} \tag{5}$$

$$W(\gamma_1,\gamma_2) = \pi_1(\gamma_1,\gamma_2) + \pi_2(\gamma_1,\gamma_2) + W_L(\gamma_1,\gamma_2) + W_H(\gamma_1,\gamma_2)\,. \tag{6}$$

$$\widetilde{W}_i(\gamma_1,\gamma_2) = \alpha_i\, W(\gamma_1,\gamma_2) + [1-\alpha_i]\,\pi_i(\gamma_1,\gamma_2)\,. \tag{7}$$

$$\gamma_i \;=\; \left[\frac{1}{2-\alpha_i}\right]\gamma_{-i} + \left[\frac{1-\alpha_i}{2-\alpha_i}\right] D_i \quad \text{for} \quad -i \neq i, \quad i, -i \in \{1, 2\} \tag{8}$$

where:

$$D_i \;=\; 1 - 2\,t\left[\frac{B_i}{A}\right] - [I + t]\,\frac{C}{A}\,; \tag{9}$$

$$A \;=\; [1-q_1]\,[1-q_2]\,\phi_L\,(p_L V)^2 + q_1\,q_2\,\phi_H\,(p_H V)^2\;; \tag{10}$$

$$B_i \;=\; [1-q_i]\,q_{-i}\,\phi_L\,p_L V + q_i\,[1-q_{-i}]\,\phi_H\,p_H V\,; \quad \text{and} \tag{11}$$

$$C \;=\; [1-q_1]\,[1-q_2]\,\phi_L\,p_L V + q_1\,q_2\,\phi_H\,p_H V\,. \tag{12}$$

$$\gamma_1^* \;=\; \frac{[1-\alpha_2]\,D_2 + [1-\alpha_1]\,[2-\alpha_2]\,D_1}{[2-\alpha_1]\,[2-\alpha_2] - 1}\,. \tag{13}$$

$$\gamma_2^* \;=\; \frac{[1-\alpha_1]\,D_1 + [1-\alpha_2]\,[2-\alpha_1]\,D_2}{[2-\alpha_1]\,[2-\alpha_2] - 1}\,. \tag{14}$$

Detailed Proofs of Formal Conclusions in the Text

Proof of Lemma 1.

An L borrower at location x is indifferent between securing financing from the two lenders when lender i offers sharing rate γ_i if:

$$p_L\,\gamma_1 V - t\,x \;=\; p_L\,\gamma_2 V - t\,[1-x] \quad \Leftrightarrow \quad t\,[1-2\,x] \;=\; p_L V\,[\gamma_2 - \gamma_1]$$

$$\Leftrightarrow \quad 2\,t\,x \;=\; t - p_L V\,[\gamma_2 - \gamma_1] \quad \Leftrightarrow \quad x \;=\; \frac{1}{2} + \frac{p_L V\,[\gamma_1 - \gamma_2]}{2\,t}.$$

The corresponding derivation of x_H is analogous and so is omitted. ■

Proof of Lemma 2.

The welfare of the $\phi_L\,q_2\,[1-q_1]$ L borrowers approved only by lender 1 is:

$$\phi_L\,q_2\,[1-q_1]\left[p_L\,\gamma_1 V - \int_0^1 t\,\xi\,d\xi\right] \;=\; \phi_L\,q_2\,[1-q_1]\left[p_L\,\gamma_1\,V - \frac{t}{2}\right]. \tag{15}$$

The welfare of the $\phi_L\,q_1\,[1-q_2]$ L borrowers approved only by lender 2 is:

$$\phi_L\,q_1\,[1-q_2]\left[p_L\,\gamma_2\,V - \int_0^1 t\,(1-\xi)\,d\xi\right] \;=\; \phi_L\,q_1\,[1-q_2]\left[p_L\,\gamma_2\,V - \frac{t}{2}\right]. \tag{16}$$

$\phi_L\,[1-q_2]\,[1-q_1]$ $\;L$ borrowers are approved by both lenders. Of these contested L borrowers, those on $[0,\,x_L]$ secure welfare:

$$\begin{aligned}
&\phi_L\,[1-q_2]\,[1-q_1]\left[x_L\,p_L\,\gamma_1\,V-\int_0^{x_L} t\,\xi\,d\xi\right] \\
&\qquad = \phi_L\,[1-q_2]\,[1-q_1]\left[x_L\,p_L\,\gamma_1\,V-\frac{t\,(x_L)^2}{2}\right]. \qquad (17)
\end{aligned}$$

The corresponding welfare of the contested L type borrowers on $[x_L,\,1]$ is:

$$\begin{aligned}
&\phi_L\,[1-q_2]\,[1-q_1]\left[(1-x_L)\,p_L\,\gamma_2 V-\int_{x_L}^{1} t\,(1-\xi)\,d\xi\right] \\
&\quad = \phi_L\,[1-q_2]\,[1-q_1]\left\{[1-x_L]\,p_L\,\gamma_2 V - t\left[\frac{2\,(1-x_L)-1+(x_L)^2}{2}\right]\right\} \\
&\quad = \phi_L\,[1-q_2]\,[1-q_1]\left[[1-x_L]\,p_L\,\gamma_2 V-\frac{t\,(1-x_L)^2}{2}\right]. \qquad (18)
\end{aligned}$$

(17) and (18) imply that the welfare of all contested L borrowers is:

$$\phi_L\,[1-q_2]\,[1-q_1]\left[[x_L\,p_L\,\gamma_1+(1-x_L)\,p_L\,\gamma_2]\,V-\frac{t\,(x_L)^2}{2}-\frac{t\,(1-x_L)^2}{2}\right]. \qquad (19)$$

(15), (16), and (19) imply that the welfare of L borrowers is as specified in (4). Analogous calculations reveal that the welfare of H borrowers is as specified in (5). (2), (3), (4), and (5) imply that total welfare is:

$$\begin{aligned}
W(\gamma_1,\gamma_2) \;=\;& \phi_L\left\{q_2\,[1-q_1]\left[p_L\,\gamma_1\,V-\frac{t}{2}\right]+q_1\,[1-q_2]\left[p_L\,\gamma_2\,V-\frac{t}{2}\right]\right. \\
&\left.+\,[1-q_2]\,[1-q_1]\left[[x_L\,\gamma_1+(1-x_L)\,\gamma_2]\,p_L\,V-\frac{t\,(x_L)^2}{2}-\frac{t\,(1-x_L)^2}{2}\right]\right\} \\
&+\phi_H\left\{q_1\,[1-q_2]\left[p_H\,\gamma_1\,V-\frac{t}{2}\right]+q_2\,[1-q_1]\left[p_H\,\gamma_2\,V-\frac{t}{2}\right]\right. \\
&\left.+\,q_2\,q_1\left[[x_H\,\gamma_1+(1-x_H)\,\gamma_2]\,p_H\,V-\frac{t\,(x_H)^2}{2}-\frac{t\,(1-x_H)^2}{2}\right]+\,[1-q_2]\,[1-q_1]\,R\right\} \\
&+\phi_L\,q_2\,[1-q_1]\,[p_L\,(1-\gamma_1)\,V-I]+\phi_H\,q_1\,[1-q_2]\,[p_H\,(1-\gamma_1)\,V-I] \\
&+\phi_L\,[1-q_2]\,[1-q_1]\,x_L\,[p_L\,(1-\gamma_1)\,V-I]+\phi_H\,q_2\,q_1\,x_H\,[p_H\,(1-\gamma_1)\,V-I] \\
&+\phi_L\,q_1\,[1-q_2]\,[p_L\,(1-\gamma_2)\,V-I]+\phi_H\,q_2\,[1-q_1]\,[p_H\,(1-\gamma_2)\,V-I] \\
&+\phi_L\,[1-q_2]\,[1-q_1]\,[1-x_L]\,[p_L\,(1-\gamma_2)\,V-I]+\phi_H\,q_2\,q_1\,[1-x_H]\,[p_H\,(1-\gamma_2)\,V-I]
\end{aligned}$$

$$
\begin{aligned}
= \;& \phi_L \left\{ q_2 \left[1-q_1\right] \left[-\frac{t}{2}\right] + q_1 \left[1-q_2\right] \left[-\frac{t}{2}\right] + \left[1-q_2\right]\left[1-q_1\right] \left[-\frac{t\,(x_L)^2}{2} - \frac{t\,(1-x_L)^2}{2}\right] \right\} \\
& + \;\phi_H \left\{ q_1 \left[1-q_2\right] \left[-\frac{t}{2}\right] + q_2 \left[1-q_1\right] \left[-\frac{t}{2}\right] + \; q_2\, q_1 \left[-\frac{t\,(x_H)^2}{2} - \frac{t\,(1-x_H)^2}{2}\right] \right\} \\
& + \phi_L\, q_2 \left[1-q_1\right] \left[p_L V - I\right] + \phi_H\, q_1 \left[1-q_2\right] \left[p_H V - I\right] + \;\phi_H \left[1-q_2\right] \left[1-q_1\right] R \\
& + \phi_L \left[1-q_2\right] \left[1-q_1\right] x_L \left[p_L\, V - I\right] + \phi_H\, q_2\, q_1\, x_H \left[p_H\, V - I\right] \\
& + \phi_L\, q_1 \left[1-q_2\right] \left[p_L V - I\right] + \phi_H\, q_2 \left[1-q_1\right] \left[p_H\, V - I\right] \\
& + \phi_L \left[1-q_2\right] \left[1-q_1\right] \left[1-x_L\right] \left[p_L V - I\right] + \phi_H\, q_2\, q_1 \left[1-x_H\right] \left[p_H\, V - I\right] \\
= \;& \phi_L \left\{ q_2 \left[1-q_1\right] \left[-\frac{t}{2}\right] + q_1 \left[1-q_2\right] \left[-\frac{t}{2}\right] + \left[1-q_2\right]\left[1-q_1\right] \left[-\frac{t\,(x_L)^2}{2} - \frac{t\,(1-x_L)^2}{2}\right] \right\} \\
& + \;\phi_H \left\{ q_1 \left[1-q_2\right] \left[-\frac{t}{2}\right] + q_2 \left[1-q_1\right] \left[-\frac{t}{2}\right] + \; q_2\, q_1 \left[-\frac{t\,(x_H)^2}{2} - \frac{t\,(1-x_H)^2}{2}\right] \right\} \\
& + \phi_L\, q_2 \left[1-q_1\right] \left[p_L V - I\right] + \phi_H\, q_1 \left[1-q_2\right] \left[p_H\, V - I\right] + \;\phi_H \left[1-q_2\right] \left[1-q_1\right] R \\
& + \phi_L \left[1-q_2\right] \left[1-q_1\right] \left[p_L V - I\right] + \phi_H\, q_2\, q_1 \left[p_H\, V - I\right] \\
& + \phi_L\, q_1 \left[1-q_2\right] \left[p_L V - I\right] + \phi_H\, q_2 \left[1-q_1\right] \left[p_H\, V - I\right] \\
= \;& q_2 \left[1-q_1\right] \left[-\frac{t}{2}\right] + q_1 \left[1-q_2\right] \left[-\frac{t}{2}\right] - \phi_L \left[1-q_2\right] \left[1-q_1\right] \left[\frac{t\,(x_L)^2}{2} + \frac{t\,(1-x_L)^2}{2}\right] \\
& - \phi_H\, q_2\, q_1 \left[\frac{t\,(x_H)^2}{2} + \frac{t\,(1-x_H)^2}{2}\right] + \;\phi_H \left[p_H\, V - I\right] \left\{ q_1 \left[1-q_2\right] + q_2\, q_1 + q_2 \left[1-q_1\right] \right\} \\
& + \;\phi_L \left[p_L\, V - I\right] \left\{ q_2 \left[1-q_1\right] + \left[1-q_2\right] \left[1-q_1\right] + q_1 \left[1-q_2\right] \right\} + \;\phi_H \left[1-q_2\right] \left[1-q_1\right] R \\
= \;& -\;\frac{t}{2} \left[q_2\,(1-q_1) + q_1(1-q_2)\right] - \frac{t}{2}\,\phi_L \left[1-q_2\right] \left[1-q_1\right] \left[(x_L)^2 + (1-x_L)^2\right] \\
& - \frac{t}{2}\,\phi_H\, q_2\, q_1 \left[(x_H)^2 + (1-x_H)^2\right] + \phi_L \left[p_L\, V - I\right] \left[1 - q_1\, q_2\right] \\
& + \;\phi_H \left[p_H\, V - I\right] \left[q_1 + q_2 - q_1\, q_2\right] + \;\phi_H \left[1-q_2\right] \left[1-q_1\right] R. \qquad (20)
\end{aligned}
$$

To characterize the equilibrium sharing rates, first observe from (1) and (2) that:

$$
\begin{aligned}
\frac{\partial \pi_1\,(\gamma_1, \gamma_2)}{\partial \gamma_1} \;=\;& -\left[1-q_1\right] q_2\, \phi_L\, p_L V - q_1 \left[1-q_2\right] \phi_H\, p_H V \\
& - \left[1-q_1\right] \left[1-q_2\right] \phi_L\, p_L V x_L - q_1\, q_2\, \phi_H\, p_H V x_H
\end{aligned}
$$

$$+ \; [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\,\frac{\partial x_L}{\partial \gamma_1} + \; q_1\,q_2\,\phi_H\,[p_H V\,(1-\gamma_1) - I]\,\frac{\partial x_H}{\partial \gamma_1}$$

$$\begin{aligned}
= \;\; & -[1-q_1]\,q_2\,\phi_L\,p_L V - q_1\,[1-q_2]\,\phi_H\,p_H V \\
& - [1-q_1]\,[1-q_2]\,\phi_L\,p_L V\left[\frac{1}{2} - \frac{(\gamma_2-\gamma_1)\,p_L V}{2\,t}\right] - \; q_1\,q_2\,\phi_H\,p_H V\left[\frac{1}{2} - \frac{(\gamma_2-\gamma_1)\,p_H V}{2\,t}\right] \\
& + [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\,\frac{p_L V}{2\,t} + \; q_1\,q_2\,\phi_H\,[p_H V\,(1-\gamma_1) - I]\,\frac{p_H V}{2\,t}
\end{aligned}$$

$$\begin{aligned}
= \;\; & -[(1-q_1)\,q_2\,\phi_L\,p_L V + q_1\,(1-q_2)\,\phi_H\,p_H V] - \;\frac{1}{2}\,[(1-q_1)\,(1-q_2)\,\phi_L\,p_L V + q_1\,q_2\,\phi_H\,p_H V] \\
& + \frac{1}{2\,t}\left[(1-q_1)\,(1-q_2)\,\phi_L\,(p_L V)^2 + q_1\,q_2\,\phi_H\,(p_H V)^2\right][\gamma_2-\gamma_1] \\
& + \frac{1}{2\,t}\left[(1-q_1)\,(1-q_2)\,\phi_L\,(p_L V)^2 + q_1\,q_2\,\phi_H\,(p_H V)^2\right][1-\gamma_1] \\
& - \frac{1}{2\,t}\,[(1-q_1)\,(1-q_2)\,\phi_L\,p_L V + q_1\,q_2\,\phi_H\,p_H V]\,I
\end{aligned}$$

$$= \; -B_1 - \frac{C}{2} + \frac{A}{2\,t}\,[1+\gamma_2-2\,\gamma_1] - \frac{C\,I}{2\,t}, \tag{21}$$

where A, B_1, and C are defined in the statement of the lemma. (21) and symmetry provide:

$$\frac{\partial \pi_2}{\partial \gamma_2} \;=\; -\,B_2 - \frac{C}{2} + \frac{A}{2\,t}\,[1+\gamma_1-2\,\gamma_2] - \frac{C\,I}{2\,t}\,, \tag{22}$$

where B_2 is defined in (11). For future reference, notice from (11) that:

$$\begin{aligned}
B_2 - B_1 \;&=\; [q_1\,(1-q_2) - q_2\,(1-q_1)]\,\phi_L\,p_L V + [q_2\,(1-q_1) - q_1\,(1-q_2)]\,\phi_H\,p_H V \\
&=\; [q_2 - q_1]\,[\phi_H\,p_H - \phi_L\,p_L]\,V\,.
\end{aligned} \tag{23}$$

To characterize lender 2's preferred sharing rate, notice from (7) that:

$$\frac{\partial\,\widetilde{W}_2\,(\gamma_1,\gamma_2)}{\partial \gamma_2} \;=\; [1-\alpha_2]\,\frac{\partial \pi_2}{\partial \gamma_2} + \alpha_2\left[\frac{\partial W}{\partial \gamma_2}\right]. \tag{24}$$

From (20):

$$\begin{aligned}
\frac{\partial\,W\,(\gamma_1,\gamma_2)}{\partial \gamma_2} \;=\; & -\frac{t}{2}\,\phi_L\,[1-q_2]\,[1-q_1]\left[2\,x_L\left(\frac{\partial x_L}{\partial \gamma_2}\right) - 2\,(1-x_L)\left(\frac{\partial x_L}{\partial \gamma_2}\right)\right] \\
& - \;\frac{t}{2}\,\phi_H\,q_2\,q_1\left[2\,x_H\left(\frac{\partial x_H}{\partial \gamma_2}\right) - 2\,(1-x_H)\left(\frac{\partial x_H}{\partial \gamma_2}\right)\right]
\end{aligned}$$

$$= \; -\,t\,\phi_L\,[1-q_2]\,[1-q_1]\,[2\,x_L\,-1]\,\frac{\partial x_L}{\partial \gamma_2} - \,t\,\phi_H\,q_2\,q_1\,[2\,x_H\,-1]\,\frac{\partial x_H}{\partial \gamma_2}$$

$$
\begin{aligned}
&= \; t\,\phi_L\,[1-q_2]\,[1-q_1]\,\frac{p_L V}{2\,t}\,[2\,x_L-1] + \; t\,\phi_H\,q_2\,q_1\frac{p_H V}{2\,t}\,[2\,x_H-1] \qquad (25)\\
&= \; t\,\phi_L\,[1-q_2]\,[1-q_1]\,\frac{p_L V}{2\,t}\left[1-\frac{(\gamma_2-\gamma_1)\,p_L V}{t}-1\right]\\
&\qquad + \; t\,\phi_H\,q_2\,q_1\frac{p_H V}{2\,t}\left[1-\frac{(\gamma_2-\gamma_1)\,p_H V}{t}-1\right]\\
&= \; -\;\frac{1}{2\,t}\left[\phi_L\,(1-q_2)\,(1-q_1)\,(p_L V)^2+\phi_H\,q_2\,q_1\,(p_H V)^2\right][\gamma_2-\gamma_1] \;=\; -\;\frac{A}{2\,t}\,[\gamma_2-\gamma_1]\,. \qquad (26)
\end{aligned}
$$

The equality in (25) follows from (1).

(22), (24), and (26) provide:

$$
\begin{aligned}
\frac{\partial\,\widetilde{W}_2\,(\gamma_1,\gamma_2)}{\partial\gamma_2} \;&=\; [1-\alpha_2]\left[-B_2-\frac{C}{2}+\frac{A}{2\,t}\,[1+\gamma_1-2\,\gamma_2]-\frac{C\,I}{2\,t}\right]-\;\alpha_2\,\frac{A}{2\,t}\,[\gamma_2-\gamma_1] \;=\; 0\\
\Leftrightarrow \quad & [1-\alpha_2]\,\{-B_2\,2\,t+A\,[1+\gamma_1-2\,\gamma_2]-C\,[I+t]\}-\;\alpha_2\,A\,[\gamma_2-\gamma_1] \;=\; 0\\
\Leftrightarrow \quad & A\,[\gamma_1-2\,\gamma_2]\,[1-\alpha_2]-\alpha_2\,A\,[\gamma_2-\gamma_1]+\;[1-\alpha_2]\,[A-2\,t\,B_2-C\,(I+t)] \;=\; 0\\
\Leftrightarrow \quad & A\,\gamma_1\,[1-\alpha_2+\alpha_2]-A\,\gamma_2\,[2\,(1-\alpha_2)+\alpha_2]+[1-\alpha_2]\,[A-2\,t\,B_2-C\,(I+t)] \;=\; 0\\
\Leftrightarrow \quad & A\,\gamma_2\,[2-\alpha_2] \;=\; A\,\gamma_1+[1-\alpha_2]\,[A-2\,t\,B_2-C\,(I+t)]\\
\Leftrightarrow \quad & \gamma_2 \;=\; \left[\frac{1}{2-\alpha_2}\right]\gamma_1+\;\left[\frac{1-\alpha_2}{2-\alpha_2}\right]\left[1-2\,t\left(\frac{B_2}{A}\right)-[I+t]\,\frac{C}{A}\right]. \qquad (27)
\end{aligned}
$$

By symmetry, (27) implies that the lenders' reaction functions can be rewritten as:

$$
\gamma_i \;=\; \left[\frac{1}{2-\alpha_i}\right]\gamma_{-i}+\left[\frac{1-\alpha_i}{2-\alpha_i}\right]D_i,
$$

as specified in the lemma. ■

Proof of Lemma 3.

(8) and (9) imply that in equilibrium, for $-i \neq i$ and for $(\alpha_1,\alpha_2) \neq (1,1)$:

$$
\begin{aligned}
\gamma_i \;&=\; \left[\frac{1}{2-\alpha_i}\right]\left\{\left[\frac{1}{2-\alpha_{-i}}\right]\gamma_i+\left[\frac{1-\alpha_{-i}}{2-\alpha_{-i}}\right]D_{-i}\right\}+\left[\frac{1-\alpha_i}{2-\alpha_i}\right]D_i\\
\Rightarrow \quad \gamma_i\left[1-\frac{1}{(2-\alpha_i)\,(2-\alpha_{-i})}\right] \;&=\; \left[\frac{1}{2-\alpha_i}\right]\left[\frac{1-\alpha_{-i}}{2-\alpha_{-i}}\right]D_{-i}+\left[\frac{(1-\alpha_i)\,(2-\alpha_{-i})}{(2-\alpha_i)\,(2-\alpha_{-i})}\right]D_i\\
\Rightarrow \quad \gamma_i\,[(2-\alpha_i)\,(2-\alpha_{-i})-1] \;&=\; [1-\alpha_{-i}]\,D_{-i}+[1-\alpha_i]\,[2-\alpha_{-i}]\,D_i\\
\Rightarrow \quad \gamma_i^* \;&=\; \frac{[1-\alpha_{-i}]\,D_{-i}+[1-\alpha_i]\,[2-\alpha_{-i}]\,D_i}{[2-\alpha_1]\,[2-\alpha_2]-1}. \qquad ■ \qquad (28)
\end{aligned}
$$

Proof of Corollary 1.

From (11) and (23):

$$D_i - D_{-i} \;=\; \frac{2\,t}{A}\,[B_{-i} - B_i] \;=\; \frac{2\,t}{A}\,[q_{-i} - q_i]\,[\phi_H\,p_H - \phi_L\,p_L]\,V\,. \tag{29}$$

(28) and (29) imply:

$$\begin{aligned}\gamma_i^* - \gamma_{-i}^* \;&=\; \frac{[1-\alpha_i]\,[1-\alpha_{-i}]\,[D_i - D_{-i}]}{[2-\alpha_1]\,[2-\alpha_2]-1} \\ &=\; \left[\frac{(1-\alpha_i)\,(1-\alpha_{-i})}{(2-\alpha_1)\,(2-\alpha_2)-1}\right]\frac{2\,t}{A}\,[q_{-i} - q_i]\,[\phi_H\,p_H - \phi_L\,p_L]\,V\,. \quad \blacksquare\end{aligned} \tag{30}$$

Proof of Proposition 1

The proof follows immediately from (30) and Assumption 1. ■

Proof of Proposition 2.

From (13):

$$\gamma_1^* \;=\; \frac{[1-\alpha_2]\,D_2 + [1-\alpha_1]\,[2-\alpha_2]\,D_1}{[2-\alpha_1]\,[2-\alpha_2]-1} \tag{31}$$

$$\begin{aligned}\Rightarrow\quad \frac{\partial \gamma_1^*}{\partial \alpha_1} \;&\overset{s}{=}\; -\,\{[2-\alpha_1]\,[2-\alpha_2]-1\}\,[2-\alpha_2]\,D_1 \\ &\qquad +\;\{[1-\alpha_2]\,D_2 + [1-\alpha_1]\,[2-\alpha_2]\,D_1\}\,[2-\alpha_2] \\ &=\; D_1\,[2-\alpha_2]\,\{[1-\alpha_1]\,[2-\alpha_2] - [2-\alpha_1]\,[2-\alpha_2] + 1\} + D_2\,[1-\alpha_2]\,[2-\alpha_2] \\ &=\; [1-\alpha_2]\,[2-\alpha_2]\,[D_2 - D_1] \\ &=\; [1-\alpha_2]\,[2-\alpha_2]\,\frac{2\,t}{A}\,[q_1 - q_2]\,[\phi_H\,p_H - \phi_L\,p_L]\,V \;\gtreqless\; 0 \;\text{ as }\; q_1 \gtreqless q_2.\end{aligned} \tag{32}$$

The equality in (32) follows from (29). The inequalities in (32) reflect Assumption 1.

From (14):

$$\begin{aligned}\frac{\partial \gamma_2^*}{\partial \alpha_1} \;&=\; \frac{\partial}{\partial \alpha_1}\left\{\left[\frac{1-\alpha_1}{(2-\alpha_1)\,(2-\alpha_2)-1}\right]D_1 + \left[\frac{(1-\alpha_2)\,(2-\alpha_1)}{(2-\alpha_1)\,(2-\alpha_2)-1}\right]D_2\right\} \\ &=\; -\,\left\{\frac{-\,[(2-\alpha_1)\,(2-\alpha_2)-1] + [1-\alpha_1]\,[2-\alpha_2]}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right\}D_1 \\ &\qquad +\left\{\frac{-\,[(2-\alpha_1)\,(2-\alpha_2)-1]\,[1-\alpha_2] + [1-\alpha_2]\,[2-\alpha_1]\,[2-\alpha_2]}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right\}D_2 \\ &=\; -\,\left[\frac{1-\alpha_2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right]D_1 + \left[\frac{1-\alpha_2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right]D_2\end{aligned}$$

$$= \left[\frac{1-\alpha_2}{[(2-\alpha_1)(2-\alpha_2)-1]^2} \right] [D_2 - D_1]$$

$$= \left[\frac{1-\alpha_2}{[(2-\alpha_1)(2-\alpha_2)-1]^2} \right] \frac{2t}{A} [q_1 - q_2] [\phi_H p_H - \phi_L p_L] V \gtreqless 0 \text{ as } q_1 \gtreqless q_2. \tag{33}$$

The equality in (33) follows from (29). The inequalities in (33) reflect Assumption 1. ■

Proof of Proposition 3.

The proof follows from Proposition 2 as long as the welfare of each borrower declines as γ_1 and γ_2 decline. Consider a reduction in the sharing rate of lender i from $\widehat{\gamma}_i$ to $\widetilde{\gamma}_i$, for $i = 1, 2$. Any borrower that continues to secure financing from the same lender after sharing rates decline experiences a reduction in welfare because his transactions cost is unchanged and his payoff from a successful project is smaller. Now consider the welfare of a $j \in \{L, H\}$ borrower who originally secures financing from lender 1 but secures financing from lender 2 after the sharing rates decline. In this case:

$$p_j V \widehat{\gamma}_1 - t\, x \;\geq\; p_j V \widehat{\gamma}_2 - t\,[1-x] \;>\; p_j V \widetilde{\gamma}_2 - t\,[1-x]\,. \tag{34}$$

The weak inequality in (34) holds because the borrower initially prefers to secure financing from lender 1. The strict inequality in (34) holds because $\widetilde{\gamma}_2 < \widehat{\gamma}_2$. (34) implies that the borrower's welfare declines when the sharing rates of both lenders decline. The analysis for the other relevant cases is analogous. ■

Proof of Proposition 4.

From (28):

$$\gamma_i^* = \frac{[1-\alpha_{-i}]\, D_{-i}}{[2-\alpha_{-i}] - 1} = D_{-i} \text{ when } \alpha_i = 1. \tag{35}$$

The proposition follows from (35), since D_1 and D_2 are independent of α_1 and α_2, from (9) – (12). ■

Proof of Corollary 2.

The proof follows immediately from Propositions 2, 3, and 4. ■

Proof of Lemma 4.

$$\frac{\partial}{\partial \alpha_1} \left\{ \frac{[1-\alpha_2]\,[1-\alpha_1]}{[2-\alpha_1]\,[2-\alpha_2] - 1} \right\} = \frac{-\,[(2-\alpha_1)(2-\alpha_2)-1]\,[1-\alpha_2] + [1-\alpha_2]\,[1-\alpha_1]\,[2-\alpha_2]}{[(2-\alpha_1)(2-\alpha_2)-1]^2}$$

$$= \frac{[1-\alpha_2]\,\{1-(2-\alpha_1)\,(2-\alpha_2)+[2-\alpha_2]\,[1-\alpha_1]\}}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}$$

$$= -\frac{[1-\alpha_2]^2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2} < 0. \quad (36)$$

Analogously:

$$\frac{\partial}{\partial \alpha_2}\left\{\frac{[1-\alpha_2]\,[1-\alpha_1]}{[2-\alpha_1]\,[2-\alpha_2]-1}\right\} = -\frac{[1-\alpha_1]^2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2} < 0. \quad (37)$$

The lemma follows from (30), (36), and (37). ■

Proof of Lemma 5.

From (20):

$$W = G_1 - \frac{t}{2}\,\phi_L\,[1-q_1]\,[1-q_2]\left[(x_L)^2+(1-x_L)^2\right] - \frac{t}{2}\,\phi_H\,q_1\,q_2\left[(x_H)^2+(1-x_H)^2\right], \quad (38)$$

where:

$$G_1 = -\frac{t}{2}\,[q_2\,(1-q_1)+q_1(1-q_2)] + \phi_L\,[p_L\,V-I]\,[1-q_1\,q_2]$$
$$+\ \phi_H\,[p_H\,V-I]\,[q_1+q_2-q_1\,q_2] + \phi_H\,[1-q_2]\,[1-q_1]\,R.$$

Notice that G_1 is independent of α_1, α_2, γ_1, and γ_2. From (1):

$$(x_L)^2+(1-x_L)^2 = \left[\frac{1}{2}-\frac{(\gamma_2-\gamma_1)\,p_L V}{2\,t}\right]^2 + \left[\frac{1}{2}+\frac{(\gamma_2-\gamma_1)\,p_L V}{2\,t}\right]^2$$

$$= \frac{1}{2}+\frac{(\gamma_2-\gamma_1)^2\,(p_L V)^2}{2\,t^2}, \text{ and} \quad (39)$$

$$(x_H)^2+(1-x_H)^2 = \left[\frac{1}{2}-\frac{(\gamma_2-\gamma_1)\,p_H V}{2\,t}\right]^2 + \left[\frac{1}{2}+\frac{(\gamma_2-\gamma_1)\,p_H V}{2\,t}\right]^2$$

$$= \frac{1}{2}+\frac{(\gamma_2-\gamma_1)^2\,(p_H V)^2}{2\,t^2}. \quad (40)$$

(10), (38), (39), and (40) provide:

$$W = G_1 - \frac{t}{2}\,\phi_L\,[1-q_1]\,[1-q_2]\left[\frac{1}{2}+\frac{(\gamma_2-\gamma_1)^2\,(p_L V)^2}{2\,t^2}\right] - \frac{t}{2}\,\phi_H\,q_1\,q_2\left[\frac{1}{2}+\frac{(\gamma_2-\gamma_1)^2\,(p_H V)^2}{2\,t^2}\right]$$

$$= G - \frac{1}{4\,t}\phi_L\,[1-q_1]\,[1-q_2]\,[\gamma_2-\gamma_1]^2\,(p_L V)^2 - \frac{1}{4\,t}\,\phi_H\,q_1\,q_2\,[\gamma_2-\gamma_1]^2\,(p_H V)^2$$

$$= G - \frac{A}{4\,t}\,[\gamma_2-\gamma_1]^2. \quad (41)$$

where:
$$G \;=\; G_1 - \frac{t}{4}\,\phi_L\,[1-q_1]\,[1-q_2] - \frac{t}{4}\,\phi_H\,q_1\,q_2,$$
so G is independent of α_1, α_2, γ_1, and γ_2. ■

Proof of Proposition 5.

From (30) and (41), W can be written as:
$$W \;=\; G - \frac{A}{4\,t}\left\{\left[\frac{[1-\alpha_2]\,[1-\alpha_1]}{[2-\alpha_1]\,[2-\alpha_2]-1}\right] G_3\right\}^2, \tag{42}$$
where:
$$G_3 \;=\; \frac{2\,t}{A}\,[q_1-q_2]\,[\phi_H\,p_H - \phi_L\,p_L]\,V\,. \tag{43}$$
Notice that G_3 is independent of α_1, α_2, γ_1, and γ_2. Therefore, the proposition follows from (36), (37), and (42). ■

Proof of Proposition 6.

From (41), α_1 and α_2 affect W only through $(\gamma_2 - \gamma_1)$. The proposition then holds because $\gamma_2^* - \gamma_1^*$ is symmetric in α_1 and α_2, from (30). ■

Proof of Proposition 7.

From (1) and (2):
$$\begin{aligned}
\frac{\partial \pi_1}{\partial \alpha_1} \;=\; & -\phi_L\,q_2\,[1-q_1]\,p_L V\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] - \phi_H\,q_1\,[1-q_2]\,p_H V\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] \\
& -\;\phi_L\,[1-q_2]\,[1-q_1]\,x_L\,p_L V\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] + \phi_L\,[1-q_2]\,[1-q_1]\,[p_L\,(1-\gamma_1)\,V - I]\,\frac{\partial x_L}{\partial \alpha_1} \\
& -\;\phi_H\,q_2\,q_1\,x_H\,p_H V\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] + \phi_H\,q_2\,q_1\,[p_H\,(1-\gamma_1)\,V - I]\,\frac{\partial x_H}{\partial \alpha_1} \\
=\; & -[\phi_L\,q_2\,(1-q_1)\,p_L V + \phi_H\,q_1\,(1-q_2)\,p_H V]\,\frac{\partial \gamma_1}{\partial \alpha_1} \\
& -\phi_L\,[1-q_2]\,[1-q_1]\,x_L\,p_L V\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] - \phi_H\,q_2\,q_1\,x_H\,p_H V\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] \\
& +\;\phi_L\,[1-q_2]\,[1-q_1]\,p_L V\,[1-\gamma_1]\,\frac{\partial x_L}{\partial \alpha_1} - \phi_L\,[1-q_2]\,[1-q_1]\,I\,\frac{\partial x_L}{\partial \alpha_1} \\
& +\phi_H\,q_2\,q_1\,p_H V\,[1-\gamma_1]\,\frac{\partial x_H}{\partial \alpha_1} - \phi_H\,q_2\,q_1\,I\,\frac{\partial x_H}{\partial \alpha_1}
\end{aligned}$$

$$
\begin{aligned}
= \;& -\left[\phi_L\, q_2\,(1-q_1)\, p_L V + \phi_H\, q_1\,(1-q_2)\, p_H V\right] \frac{\partial \gamma_1}{\partial \alpha_1} \\
& - \phi_L\,[1-q_2]\,[1-q_1] \left[\frac{1}{2} - \frac{(\gamma_2-\gamma_1)\, p_L V}{2t}\right] p_L V \left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] \\
& - \phi_H\, q_2\, q_1 \left[\frac{1}{2} - \frac{(\gamma_2-\gamma_1)\, p_H V}{2t}\right] p_H V \left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] \\
& - \frac{1}{2\,t}\,\phi_L\,[1-q_2]\,[1-q_1]\,(p_L V)^2\,[1-\gamma_1]\,\frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} \\
& + \frac{1}{2\,t}\,\phi_L\,[1-q_2]\,[1-q_1]\, p_L V\, I\, \frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} \\
& - \frac{1}{2\,t}\,\phi_H\, q_2\, q_1\,(p_H V)^2\,[1-\gamma_1]\,\frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} + \frac{1}{2\,t}\,\phi_H\, q_2\, q_1\, p_H V\, I\, \frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} \\
= \;& -\left[\phi_L\, q_2\,(1-q_1)\, p_L V + \phi_H\, q_1\,(1-q_2)\, p_H V\right] \frac{\partial \gamma_1}{\partial \alpha_1} \\
& - \frac{1}{2}\left[\phi_L\,(1-q_2)\,(1-q_1)\, p_L V + \phi_H\, q_2\, q_1\, p_H V\right] \frac{\partial \gamma_1}{\partial \alpha_1} \\
& + \left[\phi_L\,(1-q_2)\,(1-q_1)\,(p_L V)^2 + \phi_H\, q_2\, q_1\,(p_H V)^2\right] \left[\frac{\gamma_2-\gamma_1}{2\,t}\right] \frac{\partial \gamma_1}{\partial \alpha_1} \\
& - \frac{1}{2\,t}\left[\phi_L\,(1-q_2)\,(1-q_1)\,(p_L V)^2 + \phi_H\, q_2\, q_1\,(p_H V)^2\right] [1-\gamma_1]\,\frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} \\
& + \frac{1}{2\,t}\left[\phi_L\,(1-q_2)\,(1-q_1)\, p_L V + \phi_H\, q_2\, q_1\, p_H V\right] I\, \frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} \\
= \;& -B_1 \left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] - \frac{C}{2}\left[\frac{\partial \gamma_1}{\partial \alpha_1}\right] + \frac{A}{2\,t}\,[\gamma_2-\gamma_1]\,\frac{\partial \gamma_1}{\partial \alpha_1} \\
& \qquad - \frac{A}{2\,t}\,[1-\gamma_1]\,\frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1} + \frac{C}{2\,t}\, I\, \frac{\partial(\gamma_2-\gamma_1)}{\partial \alpha_1}. \qquad (44)
\end{aligned}
$$

The last equality in (44) reflects (10), (11), and (12). From (8):

$$
\gamma_1 \;=\; \left[\frac{1}{2-\alpha_1}\right]\gamma_2 + \left[\frac{1-\alpha_1}{2-\alpha_1}\right] D_1 \;\;\Rightarrow\;\; [2-\alpha_1]\,\gamma_1 \;=\; \gamma_2 + [1-\alpha_1]\, D_1
$$

$$
\Rightarrow\;\; [1-\alpha_1]\,\gamma_1 \;=\; \gamma_2-\gamma_1 + [1-\alpha_1]\, D_1 \;\;\Rightarrow\;\; \gamma_1 \;=\; \left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right] + D_1
$$

$$
\Rightarrow\;\; 1-\gamma_1 \;=\; 1 - \left[\frac{\gamma_2-\gamma_1}{1-\alpha_1} + D_1\right] \;=\; 1 - D_1 - \left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right]. \qquad (45)
$$

From (9):

$$
\frac{A}{2\,t}\,[1-D_1] \;=\; \frac{A}{2\,t}\left[\left(\frac{2\,t}{A}\right) B_1 + [I+t]\left(\frac{C}{A}\right)\right] \;=\; B_1 + I\,\frac{C}{2\,t} + \frac{C}{2}. \qquad (46)
$$

(45) and (46) provide:

$$\frac{A}{2\,t}\,[1-\gamma_1] \;=\; -\,\frac{A}{2\,t}\left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right]+\frac{A}{2\,t}\,[1-D_1]$$

$$=\; -\,\frac{A}{2\,t}\left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right]+B_1+I\,\frac{C}{2\,t}+\frac{C}{2}. \qquad (47)$$

From (44) and (47):

$$\frac{\partial\pi_1}{\partial\alpha_1} \;=\; -\,B_1\left[\frac{\partial\gamma_1}{\partial\alpha_1}\right]-\frac{C}{2}\left[\frac{\partial\gamma_1}{\partial\alpha_1}\right]+\frac{A}{2\,t}\,[\gamma_2-\gamma_1]\,\frac{\partial\gamma_1}{\partial\alpha_1}$$

$$-\left[-\frac{A}{2\,t}\left(\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right)+B_1+I\,\frac{C}{2\,t}+\frac{C}{2}\right]\frac{\partial\,(\gamma_2-\gamma_1)}{\partial\alpha_1}+\frac{C}{2\,t}\,I\,\frac{\partial\,(\gamma_2-\gamma_1)}{\partial\alpha_1}$$

$$=\; \frac{A}{2\,t}\,[\gamma_2-\gamma_1]\,\frac{\partial\gamma_1}{\partial\alpha_1}+\frac{A}{2\,t}\left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right]\frac{\partial\,(\gamma_2-\gamma_1)}{\partial\alpha_1}-\left[B_1+\frac{C}{2}\right]\frac{\partial\gamma_2}{\partial\alpha_1}\,. \qquad (48)$$

From (30):

$$\frac{\partial\,(\gamma_2-\gamma_1)}{\partial\alpha_1} \;=\; \left\{\frac{1-\alpha_2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}-\frac{(1-\alpha_2)\,(2-\alpha_2)}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right\}[D_2-D_1]$$

$$=\; -\left[\frac{(1-\alpha_2)^2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right][D_2-D_1]\,. \qquad (49)$$

From (30) and (32):

$$[\gamma_2-\gamma_1]\,\frac{\partial\gamma_1}{\partial\alpha_1} \;=\; \left[\frac{(1-\alpha_2)^2\,(2-\alpha_2)\,(1-\alpha_1)}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}\right][D_2-D_1]^2\,. \qquad (50)$$

(30) and (49) provide:

$$\left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right]\frac{\partial\,(\gamma_2-\gamma_1)}{\partial\alpha_1} \;=\; -\left[\frac{(1-\alpha_2)^3}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}\right][D_2-D_1]^2\,. \qquad (51)$$

(50) and (51) provide:

$$\frac{A}{2\,t}\,[\gamma_2-\gamma_1]\,\frac{\partial\gamma_1}{\partial\alpha_1}+\frac{A}{2\,t}\left[\frac{\gamma_2-\gamma_1}{1-\alpha_1}\right]\frac{\partial\,(\gamma_2-\gamma_1)}{\partial\alpha_1}$$

$$=\; \frac{A}{2\,t}\left\{\frac{(1-\alpha_2)^2\,(2-\alpha_2)\,(1-\alpha_1)}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}-\frac{(1-\alpha_2)^3}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}\right\}[D_2-D_1]^2$$

$$=\; \frac{A}{2\,t}\left[\frac{(1-\alpha_2)^2\,[(2-\alpha_2)\,(1-\alpha_1)-(1-\alpha_2)]}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}\right][D_2-D_1]^2$$

$$= \left[\frac{(1-\alpha_2)^2\,[1-2\,\alpha_1+\alpha_1\,\alpha_2]}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3} \right] \frac{2\,t}{A}\,[q_2-q_1]^2\,[\phi_H\,p_H-\phi_L\,p_L]^2\,V^2. \tag{52}$$

The last equality in (52) reflects (10) and (29). Also, from (33):

$$\left[B_1+\frac{C}{2}\right]\frac{\partial\gamma_2}{\partial\alpha_1} = -\left[B_1+\frac{C}{2}\right]\left[\frac{1-\alpha_2}{[(2-\alpha_1)\,(2-\alpha_2)-1]^2}\right]\frac{2\,t}{A}\,[q_2-q_1]\,[\phi_H\,p_H-\phi_L\,p_L]\,V\,. \tag{53}$$

(48), (52), and (53) provide (using (10), (11), and (12)):

$$\begin{aligned}
\frac{\partial\pi_1}{\partial\alpha_1} &= \left[\frac{(1-\alpha_2)\,(q_2-q_1)\,[\phi_H\,p_H-\phi_L\,p_L]\,V}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}\right]\frac{2\,t}{A} \\
&\quad \cdot\{\,(1-\alpha_2)\,[1-2\,\alpha_1+\alpha_1\,\alpha_2]\,[q_2-q_1]\,[\phi_H\,p_H-\phi_L\,p_L]\,V \\
&\qquad + [(2-\alpha_1)\,(2-\alpha_2)-1]\left[B_1+\frac{C}{2}\right]\} \\
&= \left[\frac{(1-\alpha_2)\,(q_2-q_1)\,[\phi_H\,p_H-\phi_L\,p_L]\,V}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3}\right]\frac{t}{A} \\
&\quad \cdot\{\,2\,[1-\alpha_2]\,[1-2\,\alpha_1+\alpha_1\,\alpha_2]\,[q_2-q_1]\,[\phi_H\,p_H-\phi_L\,p_L]\,V \\
&\qquad + [(2-\alpha_1)\,(2-\alpha_2)-1]\,[2\,B_1+C]\} \\
&= \frac{[1-\alpha_2]\,[q_2-q_1]\,[\phi_H\,p_H-\phi_L\,p_L]\,t\,V}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3\left[\phi_L\,[1-q_2]\,[1-q_1]\,(p_L)^2+\phi_H\,q_2\,q_1\,(p_H)^2\right]V^2} \\
&\quad \cdot\{\,2\,[1-\alpha_2]\,[1-2\,\alpha_1+\alpha_1\,\alpha_2]\,[q_2-q_1]\,[\phi_H\,p_H-\phi_L\,p_L]\,V \\
&\quad + \,2\,[(2-\alpha_1)\,(2-\alpha_2)-1]\,[\phi_L\,p_L\,q_2\,(1-q_1)+\phi_H\,p_H\,q_1\,(1-q_2)]\,V \\
&\quad + \,[(2-\alpha_1)\,(2-\alpha_2)-1]\,[\phi_L\,p_L\,(1-q_2)\,(1-q_1)+\phi_H\,p_H\,q_2\,q_1]\,V\} \\
&= \frac{[1-\alpha_2]\,[q_2-q_1]\,[\phi_H\,p_H-\phi_L\,p_L]\,t\,[\phi_H\,p_H\,M_H+\phi_L\,p_L\,M_L]}{[(2-\alpha_1)\,(2-\alpha_2)-1]^3\left[\phi_L\,(1-q_2)\,(1-q_1)\,(p_L)^2+\phi_H\,q_2\,q_1\,(p_H)^2\right]},
\end{aligned} \tag{54}$$

where:

$$\begin{aligned}
M_H &= 2\,[1-\alpha_2]\,[1-2\,\alpha_1+\alpha_1\,\alpha_2]\,[q_2-q_1]+2\,[(2-\alpha_1)\,(2-\alpha_2)-1]\,q_1\,[1-q_2] \\
&\qquad + [(2-\alpha_1)\,(2-\alpha_2)-1]\,q_2\,q_1, \quad \text{and}
\end{aligned} \tag{55}$$

$$\begin{aligned}
M_L &= -\,2\,[1-\alpha_2]\,[1-2\,\alpha_1+\alpha_1\,\alpha_2]\,[q_2-q_1]+2\,[(2-\alpha_1)\,(2-\alpha_2)-1]\,q_2\,[1-q_1] \\
&\qquad + [(2-\alpha_1)\,(2-\alpha_2)-1]\,[1-q_2]\,[1-q_1]\,.
\end{aligned} \tag{56}$$

(54) and Assumption 1 imply:

$$\frac{\partial\pi_1}{\partial\alpha_1} \stackrel{s}{=} [q_2-q_1]\,[\phi_H\,p_H\,M_H+\phi_L\,p_L\,M_L]\,. \tag{57}$$

Observation A1. $M \equiv \phi_H \, p_H \, M_H + \phi_L \, p_L \, M_L > 0$ if $M_H > 0$.

Proof. If $M_H > 0$, then:

$$\phi_H \, p_H \, M_H + \phi_L \, p_L \, M_L \; > \; \phi_L \, p_L \, M_H \; + \; \phi_L \, p_L \, M_L \; = \; \phi_L \, p_L \left[M_H + M_L \right] . \tag{58}$$

(55) and (56) imply:

$$\begin{aligned} M_H + M_L \; = \; & 2 \left[(2 - \alpha_1)(2 - \alpha_2) - 1 \right] q_1 \left[1 - q_2 \right] + \left[(2 - \alpha_1)(2 - \alpha_2) - 1 \right] q_2 \, q_1 \\ & + 2 \left[(2 - \alpha_1)(2 - \alpha_2) - 1 \right] q_2 \left[1 - q_1 \right] \\ & + \left[(2 - \alpha_1)(2 - \alpha_2) - 1 \right] \left[1 - q_2 \right] \left[1 - q_1 \right] \; > \; 0 \, . \end{aligned} \tag{59}$$

The inequality in (59) holds because $\alpha_1, \, \alpha_2 \in [0, 1]$ and $q_1, \, q_2 \in \left[\frac{1}{2}, 1 \right]$. The Observation follows from (58) and (59). ■

Observation A2. $M > 0$ if $q_1 > q_2$.

Proof. First suppose $\alpha_2 = 1$. It is apparent from (55) that $M_H > 0$, and so $M > 0$, from Observation A1.

Now suppose $\alpha_2 < 1$. Observe that:

$$\begin{aligned} 1 - 2 \, \alpha_1 + \alpha_1 \, \alpha_2 \; &= \; 3 - 2\alpha_1 - 2\alpha_2 + \alpha_1 \, \alpha_2 - 2 + 2 \, \alpha_2 \\ &= \; \left[(2 - \alpha_1)(2 - \alpha_2) - 1 \right] - 2 \left[1 - \alpha_2 \right] \; = \; Z - 2 \left[1 - \alpha_2 \right] , \end{aligned} \tag{60}$$

where:

$$Z \; \equiv \; \left[2 - \alpha_1 \right] \left[2 - \alpha_2 \right] - 1 \; \geq \; 0 \, . \tag{61}$$

(55) and (60) imply that when $q_1 > q_2$:

$$\begin{aligned} M_H \; &= \; 2 \left[1 - \alpha_2 \right] \left[Z - 2 \left(1 - \alpha_2 \right) \right] \left[q_2 - q_1 \right] + 2 \, Z \, q_1 \left[1 - q_2 \right] + Z \, q_2 \, q_1 \\ &= \; 2 \left[1 - \alpha_2 \right] Z \left[q_2 - q_1 \right] + 2 \, Z \, q_1 \left[1 - q_2 \right] + Z \, q_2 \, q_1 + 4 \left[1 - \alpha_2 \right]^2 \left[q_1 - q_2 \right] \\ &> \; 2 \left[1 - \alpha_2 \right] Z \left[q_2 - q_1 \right] + 2 \, Z \, q_1 \left[1 - q_2 \right] + Z \, q_2 \, q_1 \\ &> \; 2 \, Z \left[q_2 - q_1 \right] + 2 \, Z \, q_1 \left[1 - q_2 \right] + Z \, q_2 \, q_1 \\ &= \; Z \left[2 \, q_2 - 2 \, q_1 + 2 \, q_1 - 2 \, q_1 \, q_2 + q_2 \, q_1 \right] \; = \; Z \, q_2 \left[2 - q_1 \right] \; \geq \; 0 \, . \end{aligned} \tag{62}$$

The Observation follows from (61), (62), and Observation A1. ■

Observation A3. $M > 0$ if $q_1 < q_2$ and $1 - 2 \, \alpha_1 + \alpha_1 \, \alpha_2 \geq 0$ ($\Leftrightarrow \; \alpha_1 \leq \frac{1}{2 - \alpha_2}$).

Proof. The proof follows immediately from (55) and Observation A1. ■

Observation A4. $M > 0$ if $q_1 < q_2$, $\alpha_1 > \frac{1}{2 - \alpha_2}$, and

$$q_2 - q_1 \; < \; \frac{\left[2 - \alpha_1 \right] \left[2 - \alpha_2 \right] - 1}{4 \left[1 - \alpha_2 \right] \left[2\alpha_1 - 1 - \alpha_1 \alpha_2 \right]} \, . \tag{63}$$

Proof.
$$\alpha_1 \;>\; \frac{1}{2-\alpha_2} \quad\Leftrightarrow\quad 1 - 2\,\alpha_1 + \alpha_1\,\alpha_2 \;=\; Z - 2\,[1-\alpha_2] \;<\; 0$$
$$\Leftrightarrow\quad 2\,[1-\alpha_2] - Z \;>\; 0\,, \tag{64}$$

where Z is defined in (61). From (55), when $q_1 < q_2$ and (64) holds:

$$\begin{aligned}
M_H \;&=\; 2\,[1-\alpha_2]\,[Z - 2\,(1-\alpha_2)]\,[q_2 - q_1] + 2\,Z\,q_1\,[1-q_2] + Z\,q_2\,q_1 \\
&=\; 2\,[1-\alpha_2]\,[Z - 2\,(1-\alpha_2)]\,[q_2 - q_1] + Z\,q_1\,[2-q_2] \\
&>\; 2\,[1-\alpha_2]\,[Z - 2\,(1-\alpha_2)]\,[q_2 - q_1] + \frac{Z}{2} \;>\; 0 \\
&\Leftrightarrow\quad 2\,[1-\alpha_2]\,[2\,(1-\alpha_2) - Z]\,[q_2 - q_1] \;<\; \frac{Z}{2} \\
&\Leftrightarrow\quad q_2 - q_1 \;<\; \frac{Z}{4\,[1-\alpha_2]\,[2\,(1-\alpha_2) - Z]}\,.
\end{aligned} \tag{65}$$

The proposition follows from (57) and Observations A1 – A4. ■

Proof of Proposition 8.

From (2):

$$\begin{aligned}
\pi_1 \;=\; & G_4 + [1-\gamma_1]\,V\,\{\,\phi_L\,p_L\,q_2\,[1-q_1] + \phi_H\,p_H\,q_1\,[1-q_2] \\
& \qquad\qquad + \;\phi_L\,p_L\,[1-q_1]\,[1-q_2]\,x_L + \phi_H\,p_H\,q_1\,q_2\,x_H\,\} \\
& -I\,[\phi_L\,(1-q_1)\,(1-q_2)\,x_L + \phi_H\,q_1\,q_2\,x_H]\,,
\end{aligned} \tag{66}$$

where $G_4 \;=\; -I\,[\phi_L\,q_2\,(1-q_1) + \phi_H\,q_1\,(1-q_2)]$ is independent of α_1, α_2, γ_1, and γ_2. Differentiating (66) provides:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial \alpha_2} \;=\; & \phi_L\,[1-q_1]\,[1-q_1]\,[p_L\,(1-\gamma_1)\,V - I]\,\frac{\partial x_L}{\partial \alpha_2} + \phi_H\,q_1\,q_2\,[p_H\,(1-\gamma_1)\,V - I]\,\frac{\partial x_H}{\partial \alpha_2} \\
& - \frac{\partial \gamma_1}{\partial \alpha_2}\,V\,\{\,\phi_L\,p_L\,q_2\,[1-q_1] + \phi_H\,p_H\,q_1\,[1-q_2] \\
& \qquad\qquad + \;\phi_L\,p_L\,[1-q_1]\,[1-q_2]\,x_L + \phi_H\,p_H\,q_1\,q_2\,x_H\,\}.
\end{aligned} \tag{67}$$

From (1) and Lemma 4:

$$\frac{\partial x_L}{\partial \alpha_2} \;=\; \left[\frac{p_L V}{2\,t}\right] \frac{\partial\,(\gamma_1 - \gamma_2)}{\partial \alpha_2} \;\lesseqqgtr\; 0 \;\text{ as }\; q_2 \gtreqqless q_1,\;\text{ and} \tag{68}$$

$$\frac{\partial x_H}{\partial \alpha_2} \;=\; \left[\frac{p_H V}{2\,t}\right] \frac{\partial\,(\gamma_1 - \gamma_2)}{\partial \alpha_2} \;\lesseqqgtr\; 0 \;\text{ as }\; q_2 \gtreqqless q_1. \tag{69}$$

(67), (68), and (69) imply:

$$\frac{\partial \pi_1}{\partial \alpha_2} = \frac{V}{2\,t} K \frac{\partial (\gamma_1 - \gamma_2)}{\partial \alpha_2} - \frac{\partial \gamma_1}{\partial \alpha_2} V \{ \phi_L \, p_L \, q_2 \, [1 - q_1] + \phi_H \, p_H \, q_1 \, [1 - q_2] + \phi_L \, p_L \, [1 - q_1] \, [1 - q_2] \, x_L + \phi_H \, p_H \, q_1 \, q_2 \, x_H \}, \quad (70)$$

where, using (10) and (12):

$$\begin{aligned} K &\equiv \phi_L \, p_L \, [1 - q_1] \, [1 - q_2] \, [p_L \, (1 - \gamma_1) \, V - I] + \phi_H \, p_H \, q_1 \, q_2 \, [p_H \, (1 - \gamma_1) \, V - I] \\ &= \phi_L \, p_L \, [1 - q_1] \, [1 - q_2] \, p_L \, (1 - \gamma_1) \, V - \phi_L \, p_L \, [1 - q_1] \, [1 - q_2] \, I \\ &\quad + \phi_H \, p_H \, q_1 \, q_2 \, p_H \, [1 - \gamma_1] \, V - \phi_H \, p_H \, q_1 \, q_2 \, I \\ &= [1 - \gamma_1] \, V \left[\phi_L \, p_L^2 \, (1 - q_1) \, (1 - q_2) + \phi_H \, p_H^2 \, q_1 \, q_2 \right] \\ &\quad - [\phi_L \, p_L \, (1 - q_1) \, (1 - q_2) + \phi_H \, p_H \, q_1 \, q_2] \, I = [1 - \gamma_1] \frac{A}{V} - \frac{C}{V} I. \end{aligned} \quad (71)$$

From (29):

$$D_2 - D_1 = -\frac{2\,t}{A} [B_2 - B_1] \quad \Rightarrow \quad D_2 = D_1 - \frac{2\,t}{A} [B_2 - B_1]. \quad (72)$$

Using (9) and (72) in (31) provides:

$$\begin{aligned} \gamma_1^* &= \frac{[1 - \alpha_2] \left[D_1 - \frac{2\,t}{A} (B_2 - B_1) \right] + [1 - \alpha_1] \, [2 - \alpha_2] \, D_1}{[2 - \alpha_1] \, [2 - \alpha_2] - 1} \\ &= \frac{D_1 \, [1 - \alpha_2 + (1 - \alpha_1) \, (2 - \alpha_2)]}{[2 - \alpha_1] \, [2 - \alpha_2] - 1} - \frac{2\,t \, [1 - \alpha_2] \, [B_2 - B_1]}{A \, [(2 - \alpha_1) \, (2 - \alpha_2) - 1]} \\ &= D_1 - \frac{2\,t \, [1 - \alpha_2] \, [B_2 - B_1]}{A \, [(2 - \alpha_1) \, (2 - \alpha_2) - 1]} \quad (73) \\ &= 1 - 2\,t \left[\frac{B_1}{A} \right] - [I + t] \frac{C}{A} - \frac{2\,t \, [1 - \alpha_2] \, [B_2 - B_1]}{A \, [(2 - \alpha_1) \, (2 - \alpha_2) - 1]} \end{aligned}$$

$$\Rightarrow \quad 1 - \gamma_1^* = 2\,t \left[\frac{B_1}{A} \right] + [I + t] \frac{C}{A} + \frac{2\,t \, [1 - \alpha_2] \, [B_2 - B_1]}{A \, [(2 - \alpha_1) \, (2 - \alpha_2) - 1]}. \quad (74)$$

(71) and (74) imply that for $\gamma_1 = \gamma_1^*$:

$$\begin{aligned} K &= \left\{ 2\,t \left[\frac{B_1}{A} \right] + [I + t] \frac{C}{A} + \frac{2\,t \, [1 - \alpha_2] \, [B_2 - B_1]}{A \, [(2 - \alpha_1) \, (2 - \alpha_2) - 1]} \right\} \frac{A}{V} - \frac{C}{V} I \\ &= \left\{ 2\,t \left[\frac{B_1}{A} \right] + t \frac{C}{A} + \frac{2\,t \, [1 - \alpha_2] \, [B_2 - B_1]}{A \, [(2 - \alpha_1) \, (2 - \alpha_2) - 1]} \right\} \frac{A}{V} \\ &= \frac{t}{V} \left\{ 2 \, B_1 + C + \frac{2 \, [1 - \alpha_2] \, [B_2 - B_1]}{[2 - \alpha_1] \, [2 - \alpha_2] - 1} \right\} \end{aligned}$$

$$= \frac{t}{V}\left\{2\,B_1\left[1-\frac{1-\alpha_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right]+C+\frac{2\,[1-\alpha_2]\,B_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right\}$$

$$= \frac{t}{V}\left\{2\,B_1\left[\frac{3-2\,\alpha_1-2\,\alpha_2+\alpha_1\,\alpha_2-1+\alpha_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right]+C+\frac{2\,[1-\alpha_2]\,B_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right\}$$

$$= \frac{t}{V}\left\{2\,B_1\left[\frac{2-2\,\alpha_1-\alpha_2+\alpha_1\,\alpha_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right]+C+\frac{2\,[1-\alpha_2]\,B_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right\}$$

$$= \frac{t}{V}\left\{2\,B_1\left[\frac{2\,(1-\alpha_1)-\alpha_2\,(1-\alpha_1)}{[2-\alpha_1]\,[2-\alpha_2]-1}\right]+C+\frac{2\,[1-\alpha_2]\,B_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right\}$$

$$= \frac{t}{V}\left\{2\,B_1\left[\frac{(1-\alpha_1)\,(2-\alpha_2)}{[2-\alpha_1]\,[2-\alpha_2]-1}\right]+C+\frac{2\,[1-\alpha_2]\,B_2}{[2-\alpha_1]\,[2-\alpha_2]-1}\right\} \;>\; 0. \tag{75}$$

From Proposition 2:

$$\frac{\partial\gamma_1}{\partial\alpha_2} \gtreqless 0 \quad \text{as} \quad q_2 \gtreqless q_1. \tag{76}$$

Since $K > 0$ from (75), (68), (69), (70), and (76) imply: (i) if $q_2 > q_1$, then π_1 decreases as α_2 increases; (ii) if $q_1 > q_2$, then π_1 increases as α_2 increases; and (iii) if $q_1 = q_2$, then π_1 does not change as α_2 increases.

Analogous arguments reveal: (i) if $q_1 > q_2$, then π_2 decreases as α_1 increases; (ii) if $q_2 > q_1$, then π_2 increases as α_1 increases; and (iii) if $q_2 = q_1$, then π_2 does not change as α_1 increases. ■

Proofs of Conclusions 1 and 2 in Appendix B

Let (γ_1^*, γ_2^*) be the equilibrium sharing rates when all captive and contested borrowers undertake their projects and both lenders serve some contested borrowers in equilibrium. Let π_1^* and π_2^* be the corresponding profits and W^* be the corresponding welfare.

Consider lender 1. We will identify conditions that ensure $\widetilde{W}_1(\gamma_1, \gamma_2^*)$ is maximized when $\gamma_1 = \gamma_1^*$. To do so, we will identify conditions such that for $\gamma_1 \geq 0$, (i) $\pi_1(\gamma_1^*, \gamma_2^*)$ is larger than any $\pi_1(\gamma_1, \gamma_2^*)$ where at (γ_1, γ_2^*) lender 1 does not serve any contested H borrower; and (ii) $W(\gamma_1^*, \gamma_2^*)$ is larger than any $W(\gamma_1, \gamma_2^*)$ where at (γ_1, γ_2^*) lender 1 does not serve any contested H borrower.

We first find conditions that ensure $W(\gamma_1^*, \gamma_2^*)$ is larger than any $W(\gamma_1, \gamma_2^*)$ where at (γ_1, γ_2^*) lender 1 does not serve any contested borrowers. Suppose the following conditions hold:

$$p_H V \gamma_2^* - t \left[\frac{p_H}{p_L}\right] \; < \; R \; < \; p_H V \gamma_2^* - 2\,t; \tag{77}$$

$$p_L V \gamma_2^* \; - \; t \; \geq \; 0; \tag{78}$$

$$t \; < \; R; \tag{79}$$

$$\gamma_2^* \; < \; \frac{t}{p_L V} + \frac{t}{p_H V} \, . \tag{80}$$

(77) can hold if $p_H > 2\,p_L$.

We now specify intervals for γ_1 in which the expressions for $W\,(\gamma_1, \gamma_2^*)$ differ. (77) – (80) imply:

(a) $p_H V \; > \; p_L V \;\; \Rightarrow \;\; \frac{t}{p_H V} < \frac{t}{p_L V} \;\; \Rightarrow \;\; \gamma_2^* - \frac{t}{p_L V} \; < \; \gamma_2^* - \frac{t}{p_H V}$.

(b) $p_H V \gamma_2^* - t \left[\frac{p_H}{p_L}\right] \; < \; R \;\; \Rightarrow \;\; \gamma_2^* - \frac{t}{p_L V} \; < \; \frac{R}{p_H V}$.

(c) $R \; < \; p_H V \gamma_2^* - 2\,t \;\; \Rightarrow \;\; \frac{R + t}{p_H V} \; < \; \gamma_2^* - \frac{t}{p_H V}$.

(d) $\gamma_2^* < \frac{t}{p_L V} + \frac{t}{p_H V} \;\; \Rightarrow \;\; \gamma_2^* - \frac{t}{p_H V} \; < \; \frac{t}{p_L V}$.

(a) – (d) imply:

$$0 \; < \; \gamma_2^* - \frac{t}{p_L V} \; < \; \frac{R}{p_H V} \; < \; \frac{R\, + \, t}{p_H V} \; < \; \gamma_2^* - \frac{t}{p_H V}$$
$$< \; \frac{t}{p_L V} \; < \; \gamma_2^* + \frac{t}{p_H V} \; < \; \gamma_2^* + \frac{t}{p_L V} \; < \; 1. \tag{81}$$

When $\gamma_1 \in \left[0, \; \gamma_2^* - \frac{t}{p_L V}\right]$, lender 1 serves some of its captive L borrowers but does not serve any H borrowers or any contested L borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[0, \; \gamma_2^* - \frac{t}{p_L V}\right]$ is:

$$\pi_1(\gamma_1,\ \gamma_2^*) \ = \ [1-q_1]\, q_2\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t}. \tag{82}$$

Lender 2's profit in this interval is:

$$\begin{aligned} \pi_2(\gamma_1,\ \gamma_2^*) \ &= \ [1-q_2]\, q_1\, \phi_L\, [p_L V\,(1-\gamma_2^*) - I] + [1-q_1]\, q_2\, \phi_H\, [p_H V\,(1-\gamma_2^*) - I] \\ &+ [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V\,(1-\gamma_2^*) - I] + q_1\, q_2\, \phi_H\, [p_H V\,(1-\gamma_2^*) - I]\,. \end{aligned} \tag{83}$$

The surplus of lender 1's captive L borrowers is:

$$CSL1_{cap} \ = \ [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\,\xi]\, d\xi\,. \tag{84}$$

The surplus of lender 1's captive H borrowers is:

$$CSH1_{cap} \ = \ [1-q_2]\, q_1\, \phi_H \int_0^1 R\, d\xi\,. \tag{85}$$

The surplus of lender 2's captive L borrowers is:

$$CSL2_{cap} \ = \ [1-q_2]\, q_1\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{86}$$

The surplus of lender 2's captive H borrowers is:

$$CSH2_{cap} \ = \ [1-q_1]\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{87}$$

The surplus of lender 2's contested L borrowers is:

$$CSL2_{con} \ = \ [1-q_2]\,[1-q_1]\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{88}$$

The surplus of lender 2's contested H borrowers is:

$$CSH2_{con} \ = \ q_1\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{89}$$

The surplus of the H borrowers who receive no financing offers is:

$$CSH_0 \ = \ [1-q_2]\,[1-q_1]\, \phi_H \int_0^1 R\, d\xi\,. \tag{90}$$

(82) – (90) imply that when $\gamma_1 \in \left[0,\ \gamma_2^* - \frac{t}{p_L V}\right]$:

$$\begin{aligned} W(\gamma_1, \gamma_2^*) \ &= \ [1-q_1]\, q_2\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t} \\ &+ [1-q_2]\, q_1\, \phi_L\, [p_L V\,(1-\gamma_2^*) - I] + [1-q_1]\, q_2\, \phi_H\, [p_H V\,(1-\gamma_2^*) - I] \\ &+ [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V\,(1-\gamma_2^*) - I] + q_1\, q_2\, \phi_H\, [p_H V\,(1-\gamma_2^*) - I] \end{aligned}$$

$$+ [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\,\xi]\, d\xi + [1-q_2]\, q_1\, \phi_H \int_0^1 R\, d\xi$$

$$+ [1-q_2]\, q_1\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi + [1-q_1]\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi$$

$$+ [1-q_2]\,[1-q_1]\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi + q_1\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi$$

$$+ [1-q_2]\,[1-q_1]\, \phi_H \int_0^1 R\, d\xi \tag{91}$$

$$= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi$$

$$+ [1-q_2]\, q_1\, \phi_L\, [p_L V - I] - [1-q_2]\, q_1\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_1]\, q_2\, \phi_H\, [p_H V - I] - [1-q_1]\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V - I] - [1-q_2]\,[1-q_1]\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ q_1\, q_2\, \phi_H\, [p_H V - I] - q_1\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_2]\, q_1\, \phi_H\, R + [1-q_2]\,[1-q_1]\, \phi_H\, R\,. \tag{92}$$

(92) implies:

$$\frac{\partial W(\gamma_1, \gamma_2^*)}{\partial \gamma_1} = [1-q_1]\, q_2\, \phi_L\, \frac{p_L V}{t}\, [p_L V\,(1-\gamma_1) - I] < 0; \quad \text{and} \tag{93}$$

$$\frac{\partial^2 W(\gamma_1, \gamma_2^*)}{\partial\,(\gamma_1)^2} = -\,\frac{1}{t}\, \phi_L\, p_L^2\, [1-q_1]\, q_2\, V^2 < 0. \tag{94}$$

(93) and (94) imply that $W(\gamma_1,\, \gamma_2^*)$ is concave and decreasing in γ_1 on $\left[0,\, \gamma_2^* - \frac{t}{p_L V}\right]$. Therefore, $W(\gamma_1, \gamma_2^*)$ is maximized at $\gamma_1 = 0$ on this interval.

When $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_L V},\, \frac{R}{p_H V}\right]$, lender 1 serves some of its captive and contested L borrowers but does not serve any H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_L V},\, \frac{R}{p_H V}\right]$ is:

$$\pi_1(\gamma_1,\, \gamma_2^*) = [1-q_1]\, q_2\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t}$$

$$+ [1-q_1]\,[1-q_2]\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, x_L. \tag{95}$$

Lender 2's profit is:

$$\pi_2(\gamma_1, \gamma_2^*) = [1-q_2]\, q_1\, \phi_L\, [p_L V(1-\gamma_2^*) - I] + [1-q_1]\, q_2\, \phi_H\, [p_H V(1-\gamma_2^*) - I]$$
$$+ [1-q_2][1-q_1]\, \phi_L\, [p_L V(1-\gamma_2^*) - I][1-x_L] + q_1\, q_2\, \phi_H\, [p_H V(1-\gamma_2^*) - I]\,. \quad (96)$$

The surplus of lender 1's captive L borrowers is:

$$CSL1_{cap} = [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\,\xi]\, d\xi \quad (97)$$

The surplus of lender 1's captive H borrowers is:

$$CSH1_{cap} = [1-q_2]\, q_1\, \phi_H \int_0^1 R\, d\xi\,. \quad (98)$$

The surplus of the contested L borrowers served by lender 1 is:

$$CSL1_{con} = [1-q_1][1-q_2]\, \phi_L \int_0^{x_L} [p_L V \gamma_1 - t\,\xi]\, d\xi\,. \quad (99)$$

The surplus of lender 2's captive L borrowers is:

$$CSL2_{cap} = [1-q_2]\, q_1\, \phi_L \int_0^1 [p_L V \gamma_2^* - t(1-\xi)]\, d\xi\,. \quad (100)$$

The surplus of lender 2's captive H borrowers is:

$$CSH2_{cap} = [1-q_1]\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t(1-\xi)]\, d\xi\,. \quad (101)$$

The surplus of lender 2's contested L borrowers is:

$$CSL2_{con} = [1-q_2][1-q_1]\, \phi_L \int_{x_L}^1 [p_L V \gamma_2^* - t(1-\xi)]\, d\xi\,. \quad (102)$$

The surplus of lender 2's contested H borrowers is:

$$CSH2_{con} = q_1\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t(1-\xi)]\, d\xi\,. \quad (103)$$

The surplus of the H borrowers who receive no financing offers is:

$$CSH_0 = [1-q_2][1-q_1]\, \phi_H \int_0^1 R\, d\xi\,. \quad (104)$$

(95) – (104) imply that when $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_L V},\ \frac{R}{p_H V}\right]$:

$$W(\gamma_1, \gamma_2^*) = [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t}$$

$$+ [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\,x_L$$

$$+ [1-q_2]\,q_1\,\phi_L\,[p_L V\,(1-\gamma_2^*) - I] + [1-q_1]\,q_2\,\phi_H\,[p_H V\,(1-\gamma_2^*) - I]$$

$$+ [1-q_2]\,[1-q_1]\,\phi_L\,[p_L V\,(1-\gamma_2^*) - I]\,[1-x_L] + q_1\,q_2\,\phi_H\,[p_H V\,(1-\gamma_2^*) - I]$$

$$+ [1-q_1]\,q_2\,\phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\,\xi]\,d\xi + [1-q_2]\,q_1\,\phi_H \int_0^1 R\,d\xi$$

$$+ [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} [p_L V \gamma_1 - t\,\xi]\,d\xi + [1-q_2]\,q_1\,\phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\,d\xi$$

$$+ [1-q_1]\,q_2\,\phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\,d\xi + [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^1 [p_L V \gamma_2^* - t\,(1-\xi)]\,d\xi$$

$$+\ q_1\,q_2\,\phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\,d\xi + [1-q_2]\,[1-q_1]\,\phi_H \int_0^1 R\,d\xi \tag{105}$$

$$= \quad [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V \gamma_1}{t} - [1-q_1]\,q_2\,\phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\,d\xi$$

$$+ [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V - I]\,x_L - [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} t\,\xi\,d\xi$$

$$+ [1-q_2]\,q_1\phi_L\,[p_L V - I] - [1-q_2]\,q_1\,\phi_L \int_0^1 t\,[1-\xi]\,d\xi$$

$$+ [1-q_1]\,q_2\,\phi_H\,[p_H V - I] - [1-q_1]\,q_2\,\phi_H \int_0^1 t\,[1-\xi]\,d\xi$$

$$+ [1-q_2]\,[1-q_1]\,\phi_L\,[p_L V - I]\,[1-x_L] - [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^1 t\,[1-\xi]\,d\xi$$

$$+\ q_1\,q_2\,\phi_H\,[p_H V - I] - q_1\,q_2\,\phi_H \int_0^1 t\,[1-\xi]\,d\xi$$

$$+ [1-q_2]\,q_1\,\phi_H \int_0^1 R\,d\xi + [1-q_2]\,[1-q_1]\,\phi_H \int_0^1 R\,d\xi \tag{106}$$

$$= \quad [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V \gamma_1}{t} - [1-q_1]\,q_2\,\phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\,d\xi$$

$$+ [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V - I] - [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} t\,\xi\,d\xi$$

$$+ [1-q_2]\,q_1\,\phi_L\,[p_L V - I] - [1-q_2]\,q_1\,\phi_L \int_0^1 t\,[1-\xi]\,d\xi$$

$$+ [1-q_1]\,q_2\,\phi_H\,[p_H V - I] - [1-q_1]\,q_2\,\phi_H \int_0^1 t\,[1-\xi]\,d\xi$$

$$- [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^{1} t\,[1-\xi]\,d\xi + q_1\,q_2\,\phi_H\,[p_H V - I] - q_1\,q_2\,\phi_H \int_0^1 t\,[1-\xi]\,d\xi$$

$$+ [1-q_2]\,q_1\,\phi_H\,R + [1-q_2]\,[1-q_1]\,\phi_H\,R \tag{107}$$

$$= [1-q_1]\,q_2\,\phi_L\,[p_L V - I] - \frac{p_L V \gamma_1}{t}\,[1-q_1]\,q_2\,\phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\,d\xi$$

$$- [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} t\,\xi\,d\xi - [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^{1} t\,[1-\xi]\,d\xi + K_2, \tag{108}$$

where K_2 does not vary with γ_1.

Differentiating (108) with respect to γ_1 implies that when $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_L V},\ \frac{R}{p_H V}\right]$:

$$\begin{aligned}
\frac{\partial W(\gamma_1, \gamma_2^*)}{\partial \gamma_1} &= [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1-q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t} \\
&\quad - [1-q_1]\,[1-q_2]\,\phi_L\,t\;x_L\,\frac{\partial x_L}{\partial \gamma_1} + [1-q_2]\,[1-q_1]\,\phi_L\,t\,[1-x_L]\,\frac{\partial x_L}{\partial \gamma_1} \\
&= [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1-q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t} \\
&\quad - [1-q_1]\,[1-q_2]\,\phi_L\,t\,x_L\,\frac{p_L V}{2\,t} + [1-q_2]\,[1-q_1]\,\phi_L\,t\,[1-x_L]\,\frac{p_L V}{2\,t} \\
&= [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1-q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t} \\
&\quad - [1-q_1]\,[1-q_2]\,\phi_L\,\frac{p_L V}{2\,t} t\,[2\,x_L - 1] \\
&= [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1-q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t} \\
&\quad - [1-q_1]\,[1-q_2]\,\phi_L\,\frac{p_L V}{t}\,t\left[x_L - \frac{1}{2}\right] \qquad (109) \\
&\leq [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1-q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t} \\
&\quad + [1-q_1]\,[1-q_2]\,\phi_L\,\frac{p_L V}{t}\,\frac{t}{2} \\
&= [1-q_1]\,\phi_L \frac{p_L V}{t}\left[q_2\,(p_L V - I) + (1-q_2)\,\frac{t}{2}\right] - [1-q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t}. \qquad (110)
\end{aligned}$$

(110) implies that $\frac{\partial W(\gamma_1, \gamma_2^*)}{\partial \gamma_1} < 0$ if $q_2\,[p_L V - I] + [1-q_2]\,\frac{t}{2} \,<\, 0$.

When $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$, lender 1 serves some of its captive and contested L borrowers and some of its captive H borrowers, but no contested H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$ is:

$$\pi_1(\gamma_1,\gamma_2^*) = [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t}$$
$$+\; q_1\, [1-q_2]\, \phi_H\, [p_H V(1-\gamma_1) - I]\, \frac{p_H V \gamma_1 - R}{t}$$
$$+ [1-q_1]\,[1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]\, x_L. \qquad (111)$$

Lender 2's profit is:

$$\pi_2(\gamma_1,\gamma_2^*) = [1-q_2]\, q_1\, \phi_L\, [p_L V(1-\gamma_2^*) - I] + [1-q_1]\, q_2\, \phi_H\, [p_H V(1-\gamma_2^*) - I]$$
$$+ [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V(1-\gamma_2^*) - I]\,[1-x_L] + q_1\, q_2\, \phi_H\, [p_H V(1-\gamma_2^*) - I]\,. \qquad (112)$$

The surplus of lender 1's captive L borrowers is:

$$CSL1_{cap} = [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\,\xi]\, d\xi\,. \qquad (113)$$

The surplus of lender 1's captive H borrowers who are served by lender 1:

$$CSH1_{cap}|_{Served} = [1-q_2]\, q_1\, \phi_H \int_0^{\frac{p_H V \gamma_1 - R}{t}} [p_H V \gamma_1 - t\,\xi]\, d\xi\,.$$

The surplus of lender 1's captive H borrowers who are not served by lender 1:

$$CSH1_{cap}|_{Not\ Served} = [1-q_2]\, q_1\, \phi_H \int_{\frac{p_H V \gamma_1 - R}{t}}^{1} R\, d\xi\,. \qquad (114)$$

The surplus of the contested L borrowers served by lender 1 is:

$$CSL1_{con} = [1-q_1]\,[1-q_2]\, \phi_L \int_0^{x_L} [p_L V \gamma_1 - t\,\xi]\, d\xi\,. \qquad (115)$$

The surplus of lender 2's captive L borrowers is:

$$CSL2_{cap} = [1-q_2]\, q_1\, \phi_L \int_0^{1} [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \qquad (116)$$

The surplus of lender 2's captive H borrowers is:

$$CSH2_{cap} = [1-q_1]\, q_2\, \phi_H \int_0^{1} [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \qquad (117)$$

The surplus of lender 2's contested L borrowers is:

$$CSL2_{con} = [1-q_2]\,[1-q_1]\, \phi_L \int_{x_L}^{1} [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \qquad (118)$$

The surplus of lender 2's contested H borrowers is:

$$CSH2_{con} \;=\; q_1\, q_2\, \phi_H \int_0^1 \left[p_H V \gamma_2^* - t\,(1-\xi)\right] d\xi\,. \tag{119}$$

The surplus of the H borrowers who receive no financing offers is:

$$CSH_0 \;=\; [1-q_2]\,[1-q_1]\,\phi_H \int_0^1 R d\xi\,. \tag{120}$$

(95) – (104) imply that when $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$:

$$\begin{aligned}
W\left(\gamma_1, \gamma_2^*\right) \;=\;& [1-q_1]\, q_2\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t} \\
&+ q_1\,[1-q_2]\,\phi_H\,[p_H V\,(1-\gamma_1) - I]\ \frac{p_H V \gamma_1 - R}{t} \\
&+ [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\, x_L \\
&+ [1-q_2]\, q_1\, \phi_L\, [p_L V\,(1-\gamma_2^*) - I] + [1-q_1]\, q_2\, \phi_H\, [p_H V\,(1-\gamma_2^*) - I] \\
&+ [1-q_2]\,[1-q_1]\,\phi_L\,[p_L V\,(1-\gamma_2^*) - I]\,[1-x_L] + q_1 q_2 \phi_H\, [p_H V\,(1-\gamma_2^*) - I] \\
&+ [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\xi]\, d\xi + [1-q_2]\, q_1\, \phi_H \int_0^{\frac{p_H V \gamma_1 - R}{t}} [p_H V \gamma_1 - t\,\xi]\, d\xi \\
&+ [1-q_2]\, q_1\, \phi_H \int_{\frac{p_H V \gamma_1 - R}{t}}^1 R\, d\xi + [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} [p_L V \gamma_1 - t\,\xi]\, d\xi \\
&+ [1-q_2]\, q_1\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi + [1-q_1]\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi \\
&+ [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi \\
&+ q_1\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi + [1-q_2]\,[1-q_1]\,\phi_H \int_0^1 R\, d\xi
\end{aligned} \tag{121}$$

$$\begin{aligned}
=\;& [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\ \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi \\
&+ q_1\,[1-q_2]\,\phi_H\,[p_H V\,(1-\gamma_1) - I]\ \frac{p_H V \gamma_1 - R}{t} \\
&+ [1-q_2]\, q_1\, \phi_H \int_0^{\frac{p_H V \gamma_1 - R}{t}} [p_H V \gamma_1 - t\,\xi]\, d\xi + [1-q_2]\, q_1\, \phi_H \int_{\frac{p_H V \gamma_1 - R}{t}}^1 R\, d\xi \\
&+ [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V - I]\, x_L - [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} t\,\xi\, d\xi \\
&+ [1-q_2]\, q_1\, \phi_L\, [p_L V - I] - [1-q_2]\, q_1\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi
\end{aligned}$$

$$+ [1-q_1]\, q_2\, \phi_H\, [p_H V - I] - [1-q_1]\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_2]\,[1-q_1]\,\phi_L\,[p_L V - I]\,[1-x_L] - [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^1 t\,[1-\xi]\,d\xi$$

$$+ q_1\, q_2\, \phi_H\, [p_H V - I] - q_1\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\,d\xi + [1-q_2]\,[1-q_1]\,\phi_H \int_0^1 R\, d\xi \tag{122}$$

$$\begin{aligned}
= \; & [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi \\
& + q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\, \frac{p_H V \gamma_1 - R}{t} \\
& - [1-q_2]\, q_1\, \phi_H \int_0^{\frac{p_H V \gamma_1 - R}{t}} t\,\xi\, d\xi + [1-q_2]\, q_1\, \phi_H \int_{\frac{p_H V \gamma_1 - R}{t}}^1 R\, d\xi \\
& + [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V - I] - [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} t\,\xi\, d\xi \\
& + [1-q_2]\, q_1\, \phi_L\, [p_L V - I] - [1-q_2]\, q_1\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi \\
& + [1-q_1]\, q_2\, \phi_H\, [p_H V - I] - [1-q_1]\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi \\
& - [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^1 t\,[1-\xi]\, d\xi + q_1 q_2\, \phi_H\, [p_H V - I] \\
& - q_1\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi + [1-q_2]\,[1-q_1]\,\phi_H \int_0^1 R\, d\xi
\end{aligned} \tag{123}$$

$$\begin{aligned}
= \; & [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi \\
& + q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\, \frac{p_H V \gamma_1 - R}{t} \\
& - [1-q_2]\, q_1\, \phi_H \int_0^{\frac{p_H V \gamma_1 - R}{t}} t\,\xi\, d\xi + [1-q_2]\, q_1\, \phi_H \int_{\frac{p_H V \gamma_1 - R}{t}}^1 R\, d\xi \\
& - [1-q_1]\,[1-q_2]\,\phi_L \int_0^{x_L} t\,\xi\, d\xi - [1-q_2]\,[1-q_1]\,\phi_L \int_{x_L}^1 t\,[1-\xi]\, d\xi + K_3\, ,
\end{aligned} \tag{124}$$

where K_3 does not vary with γ_1.

Differentiating (124) with respect to γ_1 reveals that when $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$:

$$\frac{\partial W(\gamma_1, \gamma_2^*)}{\partial \gamma_1} = [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} - [1-q_1]\, q_2\, \phi_L\, p_L V \gamma_1\, \frac{p_L V}{t}$$

$$
\begin{aligned}
&+\; q_1\,[1-q_2]\,\phi_H\,[p_HV - I]\;\frac{p_HV}{t} - [1-q_2]\,q_1\,\phi_H\,[p_HV\gamma_1 - R]\;\frac{p_HV}{t} \\
&- [1-q_2]\,q_1\,\phi_H\,R\,\frac{p_HV}{t} \\
&- [1-q_1]\,[1-q_2]\,\phi_L\,t\,x_L\frac{\partial x_L}{\partial \gamma_1} + [1-q_2]\,[1-q_1]\,\phi_L\,t\,[1-x_L]\,\frac{\partial x_L}{\partial \gamma_1} \qquad (125)
\end{aligned}
$$

$$
\begin{aligned}
=\;& [1-q_1]\,q_2\,\phi_L\,[p_LV - I]\,\frac{p_LV}{t} - [1-q_1]\,q_2\,\phi_L\,p_LV\gamma_1\,\frac{p_LV}{t} \\
&+\; q_1\,[1-q_2]\,\phi_H\,[p_HV - I - R]\,\frac{p_HV}{t} - q_1\,[1-q_2]\,\phi_H\,[p_HV\gamma_1 - R]\,\frac{p_HV}{t} \\
&- [1-q_1]\,[1-q_2]\,\phi_L t x_L\frac{p_LV}{2t} + [1-q_2]\,[1-q_1]\,\phi_L t\,[1-x_L]\,\frac{p_LV}{2\,t} \qquad (126)
\end{aligned}
$$

$$
\begin{aligned}
=\;& [1-q_1]\,q_2\,\phi_L\,[p_LV - I]\,\frac{p_LV}{t} - [1-q_1]\,q_2\,\phi_L\,p_LV\gamma_1\,\frac{p_LV}{t} \\
&+\; q_1\,[1-q_2]\,\phi_H\,[p_HV - I - R]\;\frac{p_HV}{t} - q_1\,[1-q_2]\,\phi_H\,[p_HV\gamma_1 - R]\;\frac{p_HV}{t} \\
&- [1-q_1]\,[1-q_2]\,\phi_L\,\frac{p_LV}{2}\,[2\,x_L - 1] \qquad (127)
\end{aligned}
$$

$$
\begin{aligned}
=\;& [1-q_1]\,q_2\,\phi_L\,[p_LV\,(1-\gamma_1) - I]\,\frac{p_LV}{t} + q_1\,[1-q_2]\,\phi_H\,[p_HV\,(1-\gamma_1) - I]\,\frac{p_HV}{t} \\
&- [1-q_1]\,[1-q_2]\,\phi_L\frac{p_LV}{2}\,[2\,x_L - 1]\,. \qquad (128)
\end{aligned}
$$

(128) implies that $W(\gamma_1, \gamma_2^*)$ is concave in γ_1 when $\gamma_1 \in \left[\frac{R}{p_HV},\ \frac{R+t}{p_HV}\right]$.

When $\gamma_1 \in \left[\frac{R+t}{p_HV},\ \gamma_2^* - \frac{t}{p_HV}\right]$, lender 1 serves some of its captive and contested L borrowers and all of its captive H borrowers, but does not serve any contested H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\frac{R+t}{p_HV},\ \gamma_2^* - \frac{t}{p_HV}\right]$ is:

$$
\begin{aligned}
\pi_1(\gamma_1, \gamma_2^*) \;=\;& [1-q_1]\,q_2\,\phi_L\,[p_LV\,(1-\gamma_1) - I]\,\frac{p_LV\gamma_1}{t} +\; q_1\,[1-q_2]\,\phi_H\,[p_HV\,(1-\gamma_1) - I] \\
&+ [1-q_1]\,[1-q_2]\,\phi_L\,[p_LV\,(1-\gamma_1) - I]\,x_L\;. \qquad (129)
\end{aligned}
$$

Lender 2's profit is:

$$
\begin{aligned}
\pi_2(\gamma_1, \gamma_2^*) \;=\;& [1-q_2]\,q_1\,\phi_L\,[p_LV\,(1-\gamma_2^*) - I] + (1-q_1)\,q_2\,\phi_H\,[p_HV\,(1-\gamma_2^*) - I] \\
&+ [1-q_2]\,[1-q_1]\,\phi_L\,[p_LV\,(1-\gamma_2^*) - I]\,[1-x_L] + q_1\,q_2\,\phi_H\,[p_HV\,(1-\gamma_2^*) - I]\,. \qquad (130)
\end{aligned}
$$

The surplus of lender 1's captive L borrowers is:

$$
CSL1_{cap} \;=\; [1-q_1]\,q_2\,\phi_L\int_0^{\frac{p_LV\gamma_1}{t}} [p_LV\gamma_1 - t\,\xi]\,d\xi\,. \qquad (131)
$$

The surplus of lender 1's captive H borrowers who are served by lender 1:

$$CSH1_{cap} = [1-q_2]\, q_1\, \phi_H \int_0^1 [p_H V \gamma_1 - t\xi]\, d\xi\,. \tag{132}$$

The surplus of the contested L borrowers served by lender 1 is:

$$CSL1_{con} = [1-q_1]\,[1-q_2]\, \phi_L \int_0^{x_L} [p_L V \gamma_1 - t\,\xi]\, d\xi\,. \tag{133}$$

The Surplus of lender 2's captive L borrowers is:

$$CSL2_{cap} = [1-q_2]\, q_1\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{134}$$

The surplus of lender 2's captive H borrowers is:

$$CSH2_{cap} = [1-q_1]\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{135}$$

The surplus of lender 2's contested L borrowers is:

$$CSL2_{con} = [1-q_2]\,[1-q_1]\, \phi_L \int_{x_L}^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{136}$$

The surplus of lender 2's contested H borrowers is:

$$CSH2_{con} = q_1\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi\,. \tag{137}$$

The surplus of the H borrowers who receive no financing offers is:

$$CSH_{ign} = [1-q_2]\,[1-q_1]\, \phi_H \int_0^1 R\, d\xi\,. \tag{138}$$

(95) – (104) imply that when $\gamma_1 \in \left[\frac{R+t}{p_H V},\ \gamma_2^* - \frac{t}{p_H V}\right]$:

$$\begin{aligned} W(\gamma_1, \gamma_2^*) \;=\; & [1-q_1]\, q_2 \phi_L\, [p_L V (1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t} \\ & + [1-q_1]\,[1-q_2]\, \phi_L\, [p_L V (1-\gamma_1) - I]\, x_L + q_1\, [1-q_2]\, \phi_H\, [p_H V (1-\gamma_1) - I] \\ & + [1-q_2]\, q_1\, \phi_L\, [p_L V (1-\gamma_2^*) - I] + [1-q_1]\, q_2\, \phi_H\, [p_H V (1-\gamma_2^*) - I] \\ & + [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V (1-\gamma_2^*) - I]\, [1-x_L] + q_1\, q_2\, \phi_H\, [p_H V (1-\gamma_2^*) - I] \\ & + [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} [p_L V \gamma_1 - t\,\xi]\, d\xi + [1-q_2]\, q_1\, \phi_H \int_0^1 [p_H V \gamma_1 - t\,\xi]\, d\xi \\ & + [1-q_1]\,[1-q_2]\, \phi_L \int_0^{x_L} [p_L V \gamma_1 - t\,\xi]\, d\xi + [1-q_2]\, q_1\, \phi_L \int_0^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi \end{aligned}$$

$$+ [1-q_1]\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi + [1-q_2]\,[1-q_1]\, \phi_L \int_{x_L}^1 [p_L V \gamma_2^* - t\,(1-\xi)]\, d\xi$$

$$+ \; q_1\, q_2\, \phi_H \int_0^1 [p_H V \gamma_2^* - t\,(1-\xi)]\, d\xi + [1-q_2]\,[1-q_1]\, \phi_H \int_0^1 R\, d\xi \tag{139}$$

$$= \; [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi$$

$$+ \; q_1\,[1-q_2]\, \phi_H\, [p_H V\,(1-\gamma_1) - I] + [1-q_2]\, q_1\, \phi_H \int_0^1 [p_H V \gamma_1 - t\,\xi]\, d\xi$$

$$+ [1-q_1]\,[1-q_2]\, \phi_L\, [p_L V - I]\,[x_L] - [1-q_1]\,[1-q_2]\, \phi_L \int_0^{x_L} t\,\xi\, d\xi$$

$$+ [1-q_2]\, q_1\, \phi_L\, [p_L V - I] - [1-q_2]\, q_1\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_1]\, q_2\, \phi_H\, [p_H V - I] - [1-q_1]\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V - I]\,[1-x_L] - [1-q_2]\,[1-q_1]\, \phi_L \int_{x_L}^1 t\,[1-\xi]\, d\xi$$

$$+ \; q_1\, q_2\, \phi_H\, [p_H V - I] - q_1\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi + [1-q_2]\,[1-q_1]\, \phi_H \int_0^1 R\, d\xi \tag{140}$$

$$= \; [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi$$

$$+ \; q_1\,[1-q_2]\, \phi_H\, [p_H V - I] - [1-q_2]\, q_1\, \phi_H \int_0^1 t\,\xi\, d\xi$$

$$+ [1-q_1]\,[1-q_2]\, \phi_L\, [p_L V - I] - [1-q_1]\,[1-q_2]\, \phi_L \int_0^{x_L} t\,\xi\, d\xi$$

$$+ [1-q_2]\, q_1\, \phi_L\, [p_L V - I] - [1-q_2]\, q_1\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi$$

$$+ [1-q_1]\, q_2\, \phi_H\, [p_H V - I] - [1-q_1]\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi$$

$$- [1-q_2]\,[1-q_1]\, \phi_L \int_{x_L}^1 t\,[1-\xi]\, d\xi$$

$$+ \; q_1\, q_2\, \phi_H\, [p_H V - I] - q_1\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi + [1-q_2]\,[1-q_1]\, \phi_H \int_0^1 R\, d\xi \tag{141}$$

$$= \; [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L \int_0^{\frac{p_L V \gamma_1}{t}} t\,\xi\, d\xi$$

$$- [1 - q_1]\,[1 - q_2]\,\phi_L \int_0^{x_L} t\,\xi\,d\xi - [1 - q_2]\,[1 - q_1]\,\phi_L \int_{x_L}^1 t\,[1 - \xi]\,d\xi + K_4, \tag{142}$$

where K_4 does not vary with γ_1.

Differentiating (142) with respect to γ_1 reveals that when $\gamma_1 \in \left[\frac{R+t}{p_H V},\ \gamma_2^* - \frac{t}{p_H V}\right]$:

$$\begin{aligned}
\frac{\partial W(\gamma_1,\ \gamma_2^*)}{\partial \gamma_1} &= [1 - q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1 - q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t} \\
&\quad - [1 - q_1]\,[1 - q_2]\,\phi_L\,t\,x_L\,\frac{\partial x_L}{\partial \gamma_1} + [1 - q_2]\,[1 - q_1]\,\phi_L\,t\,[1 - x_L]\,\frac{\partial x_L}{\partial \gamma_1} \qquad (143) \\
&= [1 - q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1 - q_1]\,q_2\,\phi_L\,p_L V \gamma_1 \frac{p_L V}{t} \\
&\quad - [1 - q_1]\,[1 - q_2]\,\phi_L \frac{p_L V}{2\,t} t\,[2\,x_L - 1] \qquad (144) \\
&\le [1 - q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V}{t} - [1 - q_1]\,q_2\,\phi_L\,p_L V \gamma_1 \frac{p_L V}{t} \\
&\quad + [1 - q_1]\,[1 - q_2]\,\phi_L\,\frac{p_L V}{t}\,\frac{t}{2} \\
&= [1 - q_1]\,\phi_L\,\frac{p_L V}{t}\left[q_2\,(p_L V - I) + (1 - q_2)\,\frac{t}{2}\right] - [1 - q_1]\,q_2\,\phi_L\,p_L V \gamma_1\,\frac{p_L V}{t}. \qquad (145)
\end{aligned}$$

(145) implies that $\frac{\partial W(\gamma_1,\ \gamma_2^*)}{\partial \gamma_1} < 0$ if $q_2\,[p_L V - I] + [1 - q_2]\,\frac{t}{2} < 0$.

The preceding discussion implies that if $\gamma_1 \in \left[0,\ \gamma_2^* - \frac{t}{p_H V}\right]$, then the maximum of $W(\gamma_1, \gamma_2^*)$ is attained either at $\gamma_1 = 0$ or in the interval $\left[\frac{R}{p_H V},\ \frac{R + t}{p_H V}\right]$.

We now derive sufficient conditions for $W(\gamma_1^*,\ \gamma_2^*)$ to exceed any $W(\gamma_1,\ \gamma_2^*)$ for $\gamma_1 \in \left[0,\ \gamma_2^* - \frac{t}{p_H V}\right]$. In this region, lender 1 does not serve any of the contested H borrowers. Note that in addition to the conditions specified earlier, the following will guarantee that welfare is larger when lender 1 serves all of its captive L and H borrowers and some of the the contested L and H borrowers.

$$q_2\,[p_L V - I] + [1 - q_2]\,\frac{t}{2} \ \le\ 0; \tag{146}$$

$$W(\gamma_1^*, \gamma_2^*) \ >\ W(\gamma_1,\ \gamma_2^*)\big|_{\gamma_1 = 0}; \quad \text{and} \tag{147}$$

$$\pi_1(\gamma_1^*, \gamma_2^*) \ >\ \max_{\gamma_1} \left\{\pi_1(\gamma_1, \gamma_2^*) \mid \gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R + t}{p_H V}\right]\right\}. \tag{148}$$

Let us first simplify (147). Substituting $\gamma_1 = 0$ in (92) provides:

$$
\begin{aligned}
W(\gamma_1, \gamma_2^*)|_{\gamma_1 = 0} \;=\;& [1-q_2]\, q_1\, \phi_L\, [p_L V - I] - [1-q_2]\, q_1\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi \\
&+ [1-q_1]\, q_2\, \phi_H\, [p_H V - I] - [1-q_1]\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi \\
&+ [1-q_2]\,[1-q_1]\, \phi_L\, [p_L V - I] - [1-q_2]\,[1-q_1]\, \phi_L \int_0^1 t\,[1-\xi]\, d\xi \\
&+ q_1\, q_2\, \phi_H\, [p_H V - I] - q_1\, q_2\, \phi_H \int_0^1 t\,[1-\xi]\, d\xi \\
&+ [1-q_2]\, q_1\, \phi_H\, R + [1-q_2]\,[1-q_1]\, \phi_H\, R \qquad (149) \\
=\;& [1-q_2]\, \phi_L\, [p_L V - I] + q_2\, \phi_H\, [p_H V - I] - [1-q_2]\, \phi_L\, \frac{t}{2} - q_2\, \phi_H\, \frac{t}{2} \\
&+ [1-q_2]\, q_1\, \phi_H\, R + [1-q_2]\,[1-q_1]\, \phi\, R \\
=\;& [1-q_2]\, \phi_L \left[p_L V - I - \frac{t}{2}\right] + q_2\, \phi_H \left[p_H V - I - \frac{t}{2}\right] + [1-q_2]\, \phi_H\, R. \qquad (150)
\end{aligned}
$$

Now, consider (148). Recall from (128) that when $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$, $W(\gamma_1,\ \gamma_2^*)$ is concave in γ_1. Hence, if $\frac{\partial W(\gamma_1, \gamma_2^*)}{\partial \gamma_1} > 0$ at $\gamma_1 = \frac{R+t}{p_H V}$, then $W(\gamma_1, \gamma_2^*)$ is maximized at $\gamma_1 = \frac{R+t}{p_H V}$.

Substituting $x_L = \frac{1}{2} - \frac{[\gamma_2^* - \gamma_1] p_L V}{2t}$ and $\gamma_1 = \frac{R+t}{p_H V}$ in (128) provides:

$$
\begin{aligned}
\left.\frac{\partial W(\gamma_1,\ \gamma_2^*)}{\partial \gamma_1}\right|_{\gamma_1 = \frac{R+t}{p_H V}} \;=\;& [1-q_1]\, q_2\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, \frac{p_L V}{t} \\
&+\ q_1\, [1-q_2]\, \phi_H\, [p_H V\,(1-\gamma_1) - I]\, \frac{p_H V}{t} \\
&- [1-q_1]\,[1-q_2]\, \phi_L\, \frac{p_L V}{2t} \left[1 - \frac{1}{t}\,(\gamma_2^* - \ \gamma_1)\, p_L V - 1\right] \qquad (151) \\
=\;& [1-q_1]\, q_2\, \phi_L\, [p_L V\,(1-\gamma_1) - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V\,(1-\gamma_1) - I]\, \frac{p_H V}{t} \\
&+ [1-q_1]\,[1-q_2]\, \phi_L\, \frac{p_L V}{2\,t}\, [\gamma_2^* - \ \gamma_1]\, p_L V \qquad (152) \\
=\;& [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} - [1-q_1]\, q_2\, \phi_L\, \frac{(p_L V)^2}{t}\, \gamma_1 \\
&+ q_1\, [1-q_2]\, \phi_H \left[p_H V \left(1 - \frac{R+t}{p_H V}\right) - I\right] \frac{p_H V}{t} \\
&+ [1-q_1]\,[1-q_2]\, \phi_L\, \frac{(p_L V)^2}{2t}\, \gamma_2^* - [1-q_1]\,[1-q_2]\, \phi_L\, \frac{(p_L V)^2}{2\,t}\, \gamma_1
\end{aligned}
$$

$$
\begin{aligned}
&= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V - R - t - I]\, \frac{p_H V}{t} \\
&\quad - [1-q_1]\, q_2\, \phi_L\, \frac{(p_L V)^2}{t}\, \gamma_1 - [1-q_1]\, [1-q_2]\, \phi_L\, \frac{(p_L V)^2}{2t}\, \gamma_1 \\
&\quad + [1-q_1]\, [1-q_2]\, \phi_L\, \frac{(p_L V)^2}{2t}\, \gamma_2^* \\
&= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V - R - t - I]\, \frac{p_H V}{t} \\
&\quad - [1-q_1]\, \phi_L\, \frac{(p_L V)^2}{t}\, \gamma_1 \left[q_2 + \frac{1-q_2}{2} \right] + [1-q_1]\, [1-q_2]\, \phi_L\, \frac{(p_L V)^2}{2t}\, \gamma_2^* \\
&= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V - R - t - I]\, \frac{p_H V}{t} \\
&\quad - [1-q_1]\, [1+q_2]\, \phi_L\, \frac{(p_L V)^2}{2\,t}\, \gamma_1 + [1-q_1]\, [1-q_2]\, \phi_L\, \frac{(p_L V)^2}{2t}\, \gamma_2^* \\
&= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V - R - t - I]\, \frac{p_H V}{t} \\
&\quad - [1-q_1]\, \phi_L\, \frac{(p_L V)^2}{2\,t}\, [\gamma_1\, (1+q_2) - (1-q_2)\, \gamma_2^*] \\
&= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V - R - t - I]\, \frac{p_H V}{t} \\
&\quad - [1-q_1]\, \phi_L\, \frac{(p_L V)^2}{2\,t} \left[\left(\frac{R\ +\ t}{p_H V} \right) (1+q_2) - (1-q_2)\, \gamma_2^* \right] \qquad (153) \\
&= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V}{t} + q_1\, [1-q_2]\, \phi_H\, [p_H V - R - t - I]\, \frac{p_H V}{t} \\
&\quad - [1-q_1]\, \phi_L\, \frac{p_L V}{2\,t} \left[\frac{p_L}{p_H} \right] [(R+t)\,(1+q_2) - p_H V\,(1-q_2)\, \gamma_2^*] \qquad (154) \\
&> 0 \quad \Leftrightarrow \quad [1-q_1]\, q_2\, \phi_L\, p_L\, [p_L V - I] + q_1\, [1-q_2]\, \phi_H\, p_H\, [p_H V - R - t - I] \\
&\quad > \frac{1}{2}\, [1-q_1]\, \phi_L\, p_L \left[\frac{p_L}{p_H} \right] [(R\ +\ t)\,(1+q_2) - p_H V\,(1-q_2)\, \gamma_2^*]\,. \qquad (155)
\end{aligned}
$$

From (123):

$$
\begin{aligned}
W(\gamma_1,\ \gamma_2^*)\big|_{\gamma_1 = \frac{R+t}{p_H V}} &= [1-q_1]\, q_2\, \phi_L\, [p_L V - I]\, \frac{p_L V \gamma_1}{t} - [1-q_1]\, q_2\, \phi_L\, \frac{t}{2} \left[\frac{p_L V \gamma_1}{t} \right]^2 \\
&\quad + q_1\, [1-q_2]\, \phi_H\, [p_H V - I] - [1-q_2]\, q_1\, \phi_H\, \frac{t}{2} \left[\frac{p_H V \gamma_1 - R}{t} \right]^2 \\
&\quad + [1-q_2]\, q_1\, \phi_H\, R \left[1 - \frac{p_H V \gamma_1 - R}{t} \right] + [1-q_1]\, [1-q_2]\, \phi_L\, [p_L V - I]
\end{aligned}
$$

$$
\begin{aligned}
& - [1-q_1]\,[1-q_2]\,\phi_L\,\frac{t}{2}\,(x_L)^2 + [1-q_2]\,q_1\,\phi_L\,[p_L V - I] - [1-q_2]\,q_1\,\phi_L\,\frac{t}{2} \\
& + [1-q_1]\,q_2\,\phi_H\,[p_H V - I] - [1-q_1]\,q_2\,\phi_H\,\frac{t}{2} - [1-q_2]\,[1-q_1]\,\phi_L\,\frac{t}{2}\,[1-x_L]^2 \\
& \quad + q_1\,q_2\,\phi_H\,[p_H V - I] - q_1\,q_2\,\phi_H\,\frac{t}{2} + [1-q_2]\,[1-q_1]\,\phi_H R \qquad (156)
\end{aligned}
$$

$$
\begin{aligned}
= \; & [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V \gamma_1}{t} - [1-q_1]\,q_2\,\phi_L\,\frac{t}{2}\left[\frac{p_L V \gamma_1}{t}\right]^2 \\
& + q_1\,[1-q_2]\,\phi_H\,[p_H V - I] - [1-q_2]\,q_1\,\phi_H\,\frac{t}{2} + [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V - I] \\
& - [1-q_1]\,[1-q_2]\,\phi_L\,\frac{t\,(x_L)^2}{2} + [1-q_2]\,q_1\,\phi_L\,[p_L V - I] - [1-q_2]\,q_1\,\phi_L\,\frac{t}{2} \\
& + [1-q_1]\,q_2\,\phi_H\,[p_H V - I] - [1-q_1]\,q_2\,\phi_H\,\frac{t}{2} - [1-q_2]\,[1-q_1]\,\phi_L\,\frac{t\,[1-x_L]^2}{2} \\
& + q_1\,q_2\,\phi_H\,[p_H V - I] - q_1\,q_2\,\phi_H\,\frac{t}{2} + [1-q_2]\,[1-q_1]\,\phi_H\,R \qquad (157)
\end{aligned}
$$

$$
\begin{aligned}
= \; & [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V \gamma_1}{t} - [1-q_1]\,q_2\,\phi_L\,\frac{t}{2}\left[\frac{p_L V \gamma_1}{t}\right]^2 \\
& + q_1\,[1-q_2]\,\phi_H\,[p_H V - I] + [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V - I] \\
& + [1-q_2]\,q_1\,\phi_L\,[p_L V - I] + [1-q_1]\,q_2\,\phi_H\,[p_H V - I] + q_1\,q_2\,\phi_H\,[p_H V - I] \\
& - [1-q_2]\,[1-q_1]\,\phi_L\,\frac{t\,[1-x_L]^2}{2} - [1-q_1]\,[1-q_2]\,\phi_L\,\frac{t\,(x_L)^2}{2} - [1-q_2]\,q_1\,\phi_H\,\frac{t}{2} \\
& - [1-q_2]\,q_1\,\phi_L\,\frac{t}{2} - [1-q_1]\,q_2\,\phi_H\,\frac{t}{2} - q_1\,q_2\,\phi_H\,\frac{t}{2} + [1-q_2]\,[1-q_1]\,\phi_H\,R \qquad (158)
\end{aligned}
$$

$$
\begin{aligned}
= \; & [1-q_1]\,q_2\,\phi_L\,[p_L V - I]\;\frac{p_L V \gamma_1}{t} - [1-q_1]\,q_2\,\phi_L\,\frac{t}{2}\left[\frac{p_L V \gamma_1}{t}\right]^2 \\
& - [1-q_2]\,[1-q_1]\,\phi_L\left[\frac{t\,(1-x_L)^2}{2} + \frac{t\,(x_L)^2}{2}\right] \\
& + q_1\,[1-q_2]\,\phi_H\,[p_H V - I] + q_2\,\phi_H\left[p_H V - I - \frac{t}{2}\right] \\
& + [1-q_2]\,\phi_L\,[p_L V - I] - q_1\,[1-q_2]\,\frac{t}{2} + [1-q_2]\,[1-q_1]\,\phi_H\,R \qquad (159)
\end{aligned}
$$

$$
= \; [1-q_1]\,q_2\,\phi_L\,\frac{p_L}{p_H}\left[1+\frac{R}{t}\right]\left[p_L V - I - \frac{1}{2}\left(\frac{p_L}{p_H}\right)(R+t)\right]
$$

$$- [1-q_2]\,[1-q_1]\,\phi_L \left[\frac{t}{4} + \frac{(p_L V)^2}{4\,t} \left(\gamma_2^* - \frac{R+t}{p_H V} \right)^2 \right]$$

$$+ \; q_1\,[1-q_2]\,\phi_H\,[p_H V - I] + q_2\,\phi_H \left[p_H V - I - \frac{t}{2} \right]$$

$$+ [1-q_2]\,\phi_L\,[p_L V - I] - q_1\,[1-q_2]\,\frac{t}{2} + [1-q_2]\,[1-q_1]\,\phi_H\,R \tag{160}$$

$$= \; [1-q_1]\,q_2\,\phi_L\,\frac{p_L}{p_H} \left[1 + \frac{R}{t} \right] \left[p_L V - I - \frac{1}{2} \left(\frac{p_L}{p_H} \right) (R+t) \right]$$

$$- [1-q_2]\,[1-q_1]\,\phi_L \left[\frac{t}{4} + \frac{1}{4\,t} \left(\frac{p_L}{p_H} \right)^2 (p_H V \gamma_2^* - R - t)^2 \right]$$

$$+ \; q_1\,[1-q_2]\,\phi_H\,[p_H V - I] + q_2 \phi_H \left[p_H V - I - \frac{t}{2} \right]$$

$$+ [1-q_2]\,\phi_L\,[p_L V - I] - q_1\,[1-q_2]\,\frac{t}{2} + [1-q_2]\,[1-q_1]\,\phi_H\,R. \tag{161}$$

(77), (78), (79), (80), (146), (147), (150), (155), and (161), along with the corresponding conditions for lender 2, provide Conclusion 1.

We now derive conditions that ensure $\pi_1(\gamma_1, \gamma_2^*)$ is maximized when $\gamma_1 = \gamma_1^*$, given the conditions specified in (77) – (80).

When $\gamma_1 \in \left[0,\ \gamma_2^* - \frac{t}{p_L V} \right]$, lender 1 serves some $(p_L\,V \gamma_1 / t)$ of its captive L borrowers, but no H borrowers or contested L borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[0,\ \gamma_2^* - \frac{t}{p_L V} \right]$ is:

$$\pi_1\,(\gamma_1, \gamma_2^*) \;\; = \;\; [1-q_1]\,q_2\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\,\frac{p_L\,V \gamma_1}{t}. \tag{162}$$

When $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_L V},\ \frac{R}{p_H V} \right]$, lender 1 serves some $(p_L\,V \gamma_1 / t)$ of its captive L borrowers and some (x_L) of its contested L borrowers, but no H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_L V},\ \frac{R}{p_H V} \right]$ is:

$$\begin{aligned} \pi_1\,(\gamma_1, \gamma_2^*) \;\; &= \;\; [1-q_1]\,q_2\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\,\frac{p_L\,V \gamma_1}{t} \\ &+ \;\; [1-q_1]\,[1-q_2]\,\phi_L\,[p_L V\,(1-\gamma_1) - I]\,x_L\;. \end{aligned} \tag{163}$$

When $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V} \right]$, lender 1 serves some of its captive and contested L borrowers and some $([p_H\,V \gamma_1 - R]\,/t)$ of its captive H borrowers, but no contested H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V} \right]$ is:

$$
\begin{aligned}
\pi_1(\gamma_1, \gamma_2^*) &= [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t} \\
&\quad + [1-q_1][1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]\, x_L \\
&\quad + q_1 [1-q_2]\, \phi_H\, [p_H V(1-\gamma_1) - I]\, \frac{p_H V \gamma_1 - R}{t}. \qquad (164)
\end{aligned}
$$

When $\gamma_1 \in \left[\frac{R+t}{p_H V},\ \gamma_2^* - \frac{t}{p_H V}\right]$, lender 1 serves some of its captive and contested L borrowers and all of its captive H borrowers, but no contested H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\frac{R+t}{p_H V},\ \gamma_2^* - \frac{t}{p_H V}\right]$ is:

$$
\begin{aligned}
\pi_1(\gamma_1, \gamma_2^*) &= [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t} \\
&\quad + [1-q_1][1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]\, x_L + q_1 [1-q_2]\, \phi_H\, [p_H V(1-\gamma_1) - I]. \qquad (165)
\end{aligned}
$$

When $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_H V},\ \frac{t}{p_L V}\right]$, lender 1 serves some of its captive and contested L borrowers, and all of its captive and some of its contested H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\gamma_2^* - \frac{t}{p_H V},\ \frac{t}{p_L V}\right]$ is:

$$
\begin{aligned}
\pi_1(\gamma_1, \gamma_2^*) &= [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I]\, \frac{p_L V \gamma_1}{t} + q_1 [1-q_2]\, \phi_H\, [p_H V(1-\gamma_1) - I] \\
&\quad + [1-q_1][1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]\, x_L + q_1\, q_2\, \phi_H\, [p_H V(1-\gamma_1) - I]\, x_H. \qquad (166)
\end{aligned}
$$

When $\gamma_1 \in \left[\frac{t}{p_L V},\ \gamma_2^* + \frac{t}{p_H V}\right]$, lender 1 serves all of its captive and some of its contested L borrowers, and all of its captive and some of its contested H borrowers. Therefore, lender 1's profit for $\gamma_1 \in \left[\frac{t}{p_L V},\ \gamma_2^* + \frac{t}{p_H V}\right]$ is:

$$
\begin{aligned}
\pi_1(\gamma_1, \gamma_2^*) &= [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I] + q_1 (1-q_2)\, \phi_H\, [p_H V(1-\gamma_1) - I] \\
&\quad + [1-q_1][1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]\, x_L + q_1\, q_2\, \phi_H\, [p_H V(1-\gamma_1) - I]\, x_H. \qquad (167)
\end{aligned}
$$

When $\gamma_1 \in \left[\gamma_2^* + \frac{t}{p_H V},\ \gamma_2^* + \frac{t}{p_L V}\right]$, lender 1 serves all of its captive and some of its contested L borrowers, and all of its captive and all of its contested H borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\gamma_2^* + \frac{t}{p_H V},\ \gamma_2^* + \frac{t}{p_L V}\right]$ is:

$$
\begin{aligned}
\pi_1(\gamma_1, \gamma_2^*) &= [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I] + [1-q_1][1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]\, x_L \\
&\quad + q_1 [1-q_2]\, \phi_H\, [p_H V(1-\gamma_1) - I] + q_1\, q_2\, \phi_H\, [p_H V(1-\gamma_1) - I]. \qquad (168)
\end{aligned}
$$

When $\gamma_1 \in \left[\gamma_2^* + \frac{t}{p_L V},\ 1\right]$, lender 1 serves all of its captive and contested borrowers. Therefore, lender 1's profit when $\gamma_1 \in \left[\gamma_2^* + \frac{t}{p_L V},\ 1\right]$ is:

$$
\pi_1(\gamma_1,\ \gamma_2^*) = [1-q_1]\, q_2\, \phi_L\, [p_L V(1-\gamma_1) - I] + [1-q_1][1-q_2]\, \phi_L\, [p_L V(1-\gamma_1) - I]
$$

$$+\ q_1\,[1-q_2]\,\phi_H\,[p_H V\,(1-\gamma_1)-I]+q_1\,q_2\,\phi_H\,[p_H V\,(1-\gamma_1)-I]\,. \tag{169}$$

(162) and (163) imply that $\pi_1(\gamma_1,\ \gamma_2^*)=0$ at $\gamma_1=0$. Also, $\pi_1(\gamma_1,\ \gamma_2^*)$ is a decreasing function of γ_1 for $\gamma_1 \in \left[0,\ \gamma_2^*-\frac{t}{p_L V}\right] \cup \left[\gamma_2^*-\frac{t}{p_L V},\ \frac{R}{p_H V}\right]$ in part because attracting additional L borrowers reduces profit. Therefore, $\pi_1(\gamma_1,\ \gamma_2^*) < \pi_1(\gamma_1^*,\ \gamma_2^*)$ for all γ_1 in these two intervals.

Also, (168) and (169) imply that $\pi_1(\gamma_1,\gamma_2^*)$ is a decreasing function of γ_1 for $\gamma_1 \in \left[\gamma_2^*+\frac{t}{p_H V},\ \gamma_2^*+\frac{t}{p_L V}\right] \cup \left[\gamma_2^*+\frac{t}{p_L V},\ 1\right]$. Therefore, $\pi_1(\gamma_1,\gamma_2^*) < \pi_1(\gamma_1^*,\gamma_2^*)$ for all γ_1 in these two intervals.

Furthermore, from (165), $\pi_1(\gamma_1,\gamma_2^*)$ is a decreasing function of γ_1 for $\gamma_1 \in \left[\frac{R+t}{p_H V},\ \gamma_2^*-\frac{t}{p_H V}\right]$.

Hence, $\max\{\pi_1(\gamma_1,\ \gamma_2^*)\}$ for $\gamma_1 \in \left[\frac{R+t}{p_H V},\ \gamma_2^*-\frac{t}{p_H V}\right]$ is less than $\max\{\pi_1(\gamma_1,\ \gamma_2^*)\}$ for $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$.

These observations imply that lender 1 maximizes its profit by choosing $\gamma_1=\gamma_1^*$ if and only if:

$$\max_{\gamma_1}\left\{\pi_1(\gamma_1,\gamma_2^*)\ \mid\ \gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]\right\}\ <\ \pi_1(\gamma_1^*,\ \gamma_2^*)\,. \tag{170}$$

Although (170) provides a necessary and sufficient condition for lender 1 to choose $\gamma_1=\gamma_1^*$, the left hand side of (170) is algebraically cumbersome. So we seek alternative conditions that are sufficient for (170) to hold.

If lender 1 does not serve any contested H borrowers, the maximum profit it can earn is:

$$\max_{\gamma_1}\left\{\widetilde{\pi}_1(\gamma_1,\ \gamma_2^*)\ \mid\ \gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]\right\} \tag{171}$$

where

$$\widetilde{\pi}_1(\gamma_1,\ \gamma_2^*) = \frac{1}{t}\,[1-q_1]\,q_2\,\phi_L\,[p_L V\,(1-\gamma_1)-I]\,p_L V\,\gamma_1 + \frac{1}{t}\,q_1\,[1-q_2]\,\phi_H\,[p_H V\,(1-\gamma_1)-I]\,[p_H V\,\gamma_1-R]\,. \tag{172}$$

$$\Rightarrow\ \frac{\partial\widetilde{\pi}_1(\gamma_1,\ \gamma_2^*)}{\partial\gamma_1} = \frac{1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L V\,[p_L V\,(1-2\,\gamma_1)-I] + \frac{1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H V\,[p_H V\,(1-2\,\gamma_1)+R-I] \tag{173}$$

$$\Rightarrow\ \frac{\partial^2\widetilde{\pi}_1(\gamma_1,\ \gamma_2^*)}{\partial\gamma_1^2} = -\ \frac{2}{t}\,\phi_H\,p_H^2\,V^2\,q_1\,[1-q_2]-\frac{2}{t}\,\phi_L\,p_L^2 V^2\,q_2\,[1-q_1]\,. \tag{174}$$

(174) implies that $\widetilde{\pi}_1(\gamma_1,\gamma_2^*)$ is concave in γ_1. Also, since $p_L V - I < 0$, (172) implies:

$$\widetilde{\pi}_1(\gamma_1,\gamma_2^*)|_{\gamma_1\,=\,\frac{R}{p_H V}}\ <\ 0.$$

Therefore, if $\left.\frac{\partial\widetilde{\pi}_1(\gamma_1,\,\gamma_2^*)}{\partial\gamma_1}\right|_{\gamma_1\,=\,\frac{R}{p_H V}} < 0$, then $\widetilde{\pi}_1(\gamma_1,\ \gamma_2^*) < 0$ for all $\gamma_1 \in \left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$ and $\max\{\widetilde{\pi}_1(\gamma_1,\gamma_2^*)\} < 0$. Hence, lender 1 maximizes its profit by choosing $\gamma_1 = \gamma_1^*$. From (173):

$$\left.\frac{\partial\widetilde{\pi}_1(\gamma_1,\gamma_2^*)}{\partial\gamma_1}\right|_{\gamma_1\,=\,\frac{R}{p_H V}} = \frac{1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L V\left[p_L V\left(1-2\left[\frac{R}{p_H V}\right]\right)-I\right]$$

$$+\ \frac{1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H V\left[p_H V\left(1-2\left[\frac{R}{p_H V}\right]\right)-I\right]+\frac{1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H V R$$

$$= \frac{V}{p_H\,t}\left\{\phi_H\,p_H^2\,[p_H V-I-R]\,q_1\,[1-q_2]+\ \phi_L\,p_L^2\left[p_H V-I\left(\frac{p_H}{p_L}\right)-2\,R\right][1-q_1]\,q_2\right\}. \tag{175}$$

(175) implies that lender 1 maximizes its profit by choosing $\gamma_1 = \gamma_1^*$ if

$$\phi_H\,p_H^2\,[p_H V-I-R]\,q_1\,[1-q_2]+\phi_L\,p_L^2\left[p_H V-I\left(\frac{p_H}{p_L}\right)-2\,R\right][1-q_1]\,q_2\ <\ 0. \tag{176}$$

When (175) does not hold, an alternative sufficient condition for (170) can be derived as follows. Since $\widetilde{\pi}_1(\gamma_1,\gamma_2^*)$ is concave on $\left[\frac{R}{p_H V},\ \frac{R+t}{p_H V}\right]$, if $\left.\frac{\partial\widetilde{\pi}_1(\gamma_1,\gamma_2^*)}{\partial\gamma_1}\right|_{\gamma_1\,=\,\frac{R+t}{p_H V}} > 0$, then $\widetilde{\pi}_1(\gamma_1,\ \gamma_2^*)$ reaches its maximum at $\gamma_1 = \frac{R+t}{p_H V}$. In this case, lender 1 maximizes its profit by choosing $\gamma_1 = \gamma_1^*$ if:

$$\widetilde{\pi}_1(\gamma_1,\gamma_2^*)|_{\gamma_1\,=\,\frac{R+t}{p_H V}}\ <\ \pi_1(\gamma_1^*,\gamma_2^*)\,.$$

Substituting $\gamma_1 = \frac{R\,+\,t}{p_H V}$ in (173) provides:

$$\left.\frac{\partial\widetilde{\pi}_1(\gamma_1,\gamma_2^*)}{\partial\gamma_1}\right|_{\gamma_1\,=\,\frac{R+t}{p_H V}} = \frac{1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L V\left[p_L V\left(\frac{p_H V-2\,R-2\,t}{p_H V}\right)-I\right]$$

$$+\ \frac{1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H V\left[p_H V\left(\frac{p_H V-2\,R-2\,t}{p_H V}\right)+R-I\right]$$

$$= \frac{1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L V\left[p_L V-2\,\frac{p_L}{p_H}\,(R+t)-I\right]$$

$$+\ \frac{1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H V\,[p_H V-2\,(R+t)+R-I]$$

$$= \frac{V}{p_H\,t}\{\,\phi_H\,p_H^2\,q_1\,[1-q_2]\,[p_H V-I-R-2\,t]$$

$$+\ \phi_L\,p_L^2\,[1-q_1]\,q_2\,[\,p_H V-I\,\frac{p_H}{p_L}-2\,R-2\,t\,]\,\}. \tag{177}$$

Also, substituting $\gamma_1 = \frac{R+t}{p_H V}$ in (172) provides:

$$\begin{aligned}
\widetilde{\pi}_1(\gamma_1, \gamma_2^*)\big|_{\gamma_1 = \frac{R+t}{p_H V}} &= \frac{1}{t}\,\phi_L\, p_L\,[1-q_1]\, q_2\, V \left[\frac{R+t}{p_H V}\right]\left[p_L V\left(\frac{p_H V - R - t}{p_H V}\right) - I\right] \\
&+ \frac{1}{t}\,\phi_H\, q_1\,[1-q_2]\left[p_H V\left(\frac{p_H V - R - t}{p_H V}\right) - I\right][R+t-R] \\
&= \phi_H\, q_1\,[1-q_2]\,[p_H V - R - t - I] \\
&+ \frac{1}{p_H\, t}\,\phi_L\, p_L\,[1-q_1]\, q_2\,[R+t]\left[p_L V - \frac{p_L}{p_H}(R+t) - I\right]. \qquad (178)
\end{aligned}$$

(177) and (178) imply that lender 1 maximizes its profit by choosing $\gamma_1 = \gamma_1^*$ if the expression in (177) is positive and:

$$\begin{aligned}
\pi_1(\gamma_1^*, \gamma_2^*) &> \phi_H\, q_1\,[1-q_2]\,[p_H V - R - t - I] \\
&+ \frac{1}{p_H\, t}\,\phi_L\, p_L\,[1-q_1]\, q_2\,[R+t]\left[p_L V - \frac{p_L}{p_H}(R+t) - I\right]. \qquad (179)
\end{aligned}$$

If neither (175) nor (177) holds, then $\widetilde{\pi}_1(\gamma_1, \gamma_2^*)$ is maximized at $\widetilde{\gamma}_1$ where $\left.\frac{\partial \widetilde{\pi}_1(\gamma_1,\, \gamma_2^*)}{\partial \gamma_1}\right|_{\widetilde{\gamma}_1} = 0$. From (173):

$$\begin{aligned}
\left.\frac{\partial \widetilde{\pi}_1(\gamma_1, \gamma_2^*)}{\partial \gamma_1}\right|_{\widetilde{\gamma}_1} = 0 \;&\Leftrightarrow\; 2\,\widetilde{\gamma}_1\{[1-q_1]\, q_2\,\phi_L\, p_L^2 V + q_1\,[1-q_2]\,\phi_H\, p_H^2\} \\
&= \phi_H\, p_H\, q_1\,[1-q_2]\,[p_H V + R - I] + \phi_L\, p_L\, q_2\,[1-q_1]\,[p_H V - I] \\
\Leftrightarrow\; \widetilde{\gamma}_1 &= \frac{\phi_H\, p_H\, q_1\,[1-q_2]\,[p_H V + R - I] + \phi_L\, p_L\, q_2\,[1-q_1]\,[p_L V - I]}{2\,V\,[\phi_H\, p_H^2\, q_1\,(1-q_2) + \phi_L\, p_L^2\, q_2\,(1-q_1)]}. \qquad (180)
\end{aligned}$$

From (172):

$$\begin{aligned}
\widetilde{\pi}_1(\gamma_1, \gamma_2^*) &= [1-q_1]\, q_2\,\phi_L\,[p_L V - p_L V\gamma_1 - I]\,\frac{p_L V\gamma_1}{t} \\
&+ q_1\,[1-q_2]\,\phi_H\,[p_H V - p_H V\gamma_1 - I]\,\frac{p_H V\gamma_1 - R}{t} \\
&= [1-q_1]\, q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V\,\gamma_1}{t} - [1-q_1]\, q_2\,\phi_L\, p_L V\gamma_1\,\frac{p_L V\gamma_1}{t} \\
&+ q_1\,[1-q_2]\,\phi_H\,[p_H V - p_H V\gamma_1 - I + R - R]\,\frac{p_H V\gamma_1 - R}{t} \\
&= [1-q_1]\, q_2\,\phi_L\,[p_L V - I]\,\frac{p_L V\,\gamma_1}{t} - [1-q_1]\, q_2\,\phi_L\left[\frac{(p_L)^2\, V^2\,(\gamma_1)^2}{t}\right]
\end{aligned}$$

$$+\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I + R]\,\frac{p_H V\,\gamma_1 - R}{t} -\ q_1\,[1-q_2]\,\phi_H\,[p_H V \gamma_1 + R]\,\frac{p_H V\,\gamma_1 - R}{t}$$

$$= \ \frac{V\gamma_1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L\,[p_L V - I] - \frac{V^2\,(\gamma_1)^2}{t}\,[1-q_1]\,q_2\,\phi_L\,(p_L)^2$$

$$+\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I + R]\,\frac{p_H V\,\gamma_1}{t} -\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I + R]\,\frac{R}{t}$$

$$-\ q_1\,[1-q_2]\,\phi_H\left[\frac{(p_H V\,\gamma_1)^2 - R^2}{t}\right]$$

$$= \ \frac{V\gamma_1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L\,[p_L V - I] - \frac{V^2\,(\gamma_1)^2}{t}\,[1-q_1]\,q_2\,\phi_L\,(p_L)^2$$

$$+\frac{V\gamma_1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H\,[p_H V - I + R] - q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t}$$

$$-\ q_1\,[1-q_2]\,\phi_H\,\frac{R^2}{t} - \frac{(V\,\gamma_1)^2}{t}\,q_1\,[1-q_2]\,\phi_H\,(p_H)^2 + q_1\,[1-q_2]\,\phi_H\,\frac{R^2}{t}$$

$$= \ \frac{V\gamma_1}{t}\,[1-q_1]\,q_2\,\phi_L\,p_L\,[p_L V - I] - \frac{V^2\,(\gamma_1)^2}{t}\,[1-q_1]\,q_2\,\phi_L\,(p_L)^2$$

$$+\frac{V\,\gamma_1}{t}\,q_1\,[1-q_2]\,\phi_H\,p_H\,[p_H V - I + R]$$

$$-\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t} - \frac{(V\,\gamma_1)^2}{t}\,q_1\,[1-q_2]\,\phi_H\,(p_H)^2$$

$$= \ \frac{V\gamma_1}{t}\,\{[1-q_1]\,q_2\,\phi_L\,p_L\,[p_L V - I] + q_1\,[1-q_2]\,\phi_H\,p_H\,[p_H V - I + R]\}$$

$$-\frac{V^2\,(\gamma_1)^2}{t}\,\left[(1-q_1)\,q_2\,\phi_L\,(p_L)^2 + q_1\,(1-q_2)\,\phi_H\,(p_H)^2\right] -\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t}. \quad (181)$$

Substituting from (180) into (181) provides:

$$\widetilde{\pi}_1\,(\widetilde{\gamma}_1, \gamma_2^*) \ = \ \frac{V\,(\widetilde{\gamma}_1)^2}{t}\,2\,V\,\left[\phi_L\,p_L^2\,q_2\,(1-q_1) + \phi_H\,p_H^2\,q_1\,(1-q_2)\right]$$

$$-\ \frac{V^2\,(\widetilde{\gamma}_1)^2}{t}\,\left[(1-q_1)\,q_2\,\phi_L\,(p_L)^2 + q_1\,(1-q_2)\,\phi_H\,(p_H)^2\right] -\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t}$$

$$= \ \frac{V^2\,(\widetilde{\gamma}_1)^2}{t}\,\left[(1-q_1)\,q_2\,\phi_L\,(p_L)^2 + q_1\,(1-q_2)\,\phi_H\,(p_H)^2\right] -\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t}$$

$$= \ \frac{V^2}{t}\left\{\frac{\phi_H\,p_H\,q_1\,[1-q_2]\,[p_H V + R - I] + \phi_L\,p_L\,q_2\,[1-q_1]\,[p_L V - I]}{2\,V\,[\phi_H\,p_H^2\,q_1\,(1-q_2) + \phi_L\,p_L^2\,q_2\,(1-q_1)]}\right\}^2$$

$$\cdot\ \left[(1-q_1)\,q_2\,\phi_L\,(p_L)^2 + q_1\,(1-q_2)\,\phi_H\,(p_H)^2\right] -\ q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t}$$

$$= \frac{\left[\phi_H\, p_H\, q_1\,(1-q_2)\,(p_H V + R - I) + \phi_L\, p_L\, q_2\,(1-q_1)\,(p_L V - I)\right]^2}{4\,t\left[\phi_L\,(p_L)^2\,(1-q_1)\,q_2 + \phi_H\,(p_H)^2\,q_1\,(1-q_2)\right]} - q_1\,[1-q_2]\,\phi_H\,[p_H V - I]\,\frac{R}{t}. \quad (182)$$

Therefore, lender 1 maximizes its profit by choosing $\gamma_1 = \gamma_1^*$ if neither (175) nor (177) holds and

$$\pi_1\,(\gamma_1^*, \gamma_2^*) \;>\; \widetilde{\pi}_1\,(\widetilde{\gamma}_1, \gamma_2^*)\,.$$

These observations provide Conclusion 2.