When Do Auctions Ensure the Welfare-Maximizing Allocation of Scarce Inputs?

by

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Abstract

We determine when an unfettered auction will ensure the welfare-maximizing allocation of a scarce input that enhances product quality and may reduce production costs. A supplier values the input both for its “use value” and for its “foreclosure value,” because once the input is acquired, it is unavailable to rivals. An unfettered auction often ensures the welfare-maximizing allocation of an input increment. However, it can fail to do so when the input would increase relatively rapidly the competitive position of a rival with a moderate competitive disadvantage. Bidder handicapping that ensures auctions generate welfare-maximizing input allocations differ from standard handicapping policies.

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1 Introduction

It is well known that a vertically integrated firm might seek to deny access to an upstream input in order to foreclose downstream rivals from operating in lucrative retail markets. It is also well known that a monopolist typically is willing to pay more than a potential entrant for an essential input because, by foreclosing entry, the monopolist can secure its monopoly profit whereas an entrant can gain at most its share of a smaller duopoly profit.\(^1\)

Recently, foreclosure concerns have expanded to the domain of auctions. For instance, some have questioned whether leading suppliers of wireless communications services might outbid smaller rivals in auctions of scarce radio spectrum primarily to limit the ability of the smaller rivals to develop into effective competitors (U.S. Department of Justice, 2013). Similarly, the Supreme Court has considered the possibility of “predatory bidding,” whereby a firm intentionally bids particularly aggressively for a scarce input in order to limit downstream competition from other potential input purchasers.\(^2\) Because auctions are commonly employed to allocate scarce inputs in practice,\(^3\) these considerations raise important public policy concerns.

Numerous authors have recognized that auctions do not always ensure the welfare-maximizing allocation of scarce inputs.\(^4\) However, to our knowledge, the literature does not provide a clear delineation of the industry conditions under which unfettered input auctions – auctions that allocate inputs to the bidders that value them most highly\(^5\) – will, and

\(^1\)Corresponding considerations explain why a monopolist may engage in preemptive patenting to exclude rivals (e.g., Gilbert and Newbery, 1982). Rey and Tirole (2007) provide a comprehensive review of the literature on foreclosure.


\(^3\)To illustrate, auctions have been employed to allocate billions of dollars of spectrum among suppliers of wireless communications services since the mid-1990s. See McAfee and McMillan (1996), Kwerel and Rosston (2000), Hazlett and Munoz (2009), and Cramton et al. (2011), for example. Timber harvesting and oil drilling rights also are typically allocated to suppliers of wood and oil products via auction. See Hendricks et al. (1994), Haile (2001), and Athey et al. (2013), for example.

\(^4\)To illustrate, Jehiel and Moldovanu (2003, p. 271) observe that “when the assets for sale . . . are inputs that will subsequently be used by the successful bidders in imperfect competition with each other . . . auctions can behave in surprisingly problematic ways.” Eso et al. (2010, p. 542) note that “allocating input(s) through efficient auctions may be misguided when bidders are competing firms.”

\(^5\)For example, first-price and second-price auctions with no bidder subsidies generally have this feature. The
will not, ensure the welfare-maximizing allocation of inputs. The purpose of this research is to provide such a delineation in the context of a common model of industry competition.

The bidders in our model engage in Hotelling price competition after the input auction concludes. The input being auctioned enhances customer valuation of a firm’s product and can reduce the firm’s production cost. To illustrate, the input might be spectrum that enables a supplier of wireless communications service to increase the speed and reliability of its service, which can reduce customer acquisition and retention costs. Each of the firms in our model has an initial endowment of the input, and the incremental amount of the input that is being auctioned is relatively small. Consequently, no firm can preclude the operation of its rivals even if it were to acquire all of the available input.

Each firm in this setting derives both a “use value” and a “foreclosure value” from the input. The use value arises because the firm that acquires the input increment can employ it to enhance its competitive position. The foreclosure value arises because, by acquiring an increment of a scarce input, a firm precludes its rivals from acquiring the increment. Such preclusion does not foreclose the rival in the traditional sense of driving the rival from the market, but rather in the sense of preventing the rival from employing the increment to improve its competitive position.

A firm (“firm 1”) will value highly, and therefore bid aggressively for, an input increment that will substantially enhance its competitive position. However, the firm’s rival will also bid aggressively for the input in this case in an attempt to prevent firm 1 from acquiring the

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6 Formally, a firm’s “competitive position” is the difference between the valuation that customers place on the firm’s product and the firm’s unit cost of production.

7 Cramton et al. (2011, p. S168) note that auctions may fail to promote economic efficiency because “an incumbent will include in its private value not only its use value of the [scarce input] but also the value of keeping [it] from a competitor.” Our terminology parallels that of the U.S. Department of Justice (2013, p. 10) which, in the context of spectrum auctions, observes, “... the private value [of spectrum] for incumbents ... includes not only the revenue from use of the spectrum but also any benefits gained by preventing rivals from improving their services and thereby eroding the incumbents’ existing businesses. The latter might be called ‘foreclosure value’ as distinct from ‘use value’.”
input that will substantially enhance its competitive position. These offsetting valuations of the rate at which the input increases the competitive positions of the duopolists ensures that the input allocation is determined at auction by the relative levels of the firms’ competitive positions. The firm with the strongest competitive position – and thus the largest market share – will win an unfettered auction for the input increment.

The resulting allocation of the input maximizes welfare whenever the input increases the competitive position of the larger supplier at least as rapidly as it increases the competitive position of the smaller supplier. However, an auction can fail to generate the welfare-maximizing allocation of an input increment when the input would increase relatively slowly the competitive position of a firm with a moderately large market share. In this case, the larger firm may acquire the input increment in an unfettered auction even though welfare would be higher if the smaller firm secured the increment.

In principle, a bid credit for the smaller firm could be implemented to ensure the welfare-maximizing allocation of the input increment. (A bid credit reflects the fraction of its bid that a firm is not required to pay if it wins the auction for the input increment.) However, in contrast to its typical design in practice, the appropriate bid credit does not reflect differences in profitabilities or market shares of the bidding firms. Instead, in a duopoly setting it reflects the extent to which the input would enhance the competitive position of the smaller firm more than it would enhance the competitive position of the larger firm.

The input auctions we analyze are a special case of auctions with externalities that have received considerable attention in the literature. Input auctions entail externalities because the assignment of the input affects both the recipient of the input and competitors that do not receive the input. Studies in this literature often assume that agents are privately informed about their exogenous valuations of the object being auctioned. Our study differs in part

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8Katz and Shapiro (1986) provide an early analysis of the allocation of inputs (technology licenses) in the presence of externalities. Brocas (2013a) demonstrates that externality considerations can sometimes lead a supplier to favor the sale of a valuable resource to a non-rivalrous supplier rather than to a direct competitor. Jehiel et al. (1996, 1999), for example, consider settings where the relevant externalities are not systematically linked to the agents’ valuations of the object. Carillo (1998) and Brocas (2013b, 2014), for example, analyze settings where this linkage is present. Although most studies in this literature consider the alloc-
by allowing valuations and externalities to reflect the equilibrium outcomes of competition among sellers of differentiated products.\textsuperscript{10} Our explicit modeling of industry competition allows us to identify industry and supplier characteristics that enhance or limit the propensity of unfettered auctions to generate the welfare-maximizing allocation of inputs.

Our study also differs from the typical analysis in the literature on auctions with externalities by focusing on settings with complete information. This focus allows us to demonstrate most clearly when and why an unfettered auction will fail to ensure the welfare-maximizing allocation of an input.\textsuperscript{11} We also show, though, that the key considerations that arise in settings with complete information remain relevant in settings with incomplete information.

Some related studies assume that bidders engage in Cournot competition after bidding for capacity at auction. McAfee (1998) finds that small, capacity-constrained firms often outbid larger, unconstrained firms in part because each unconstrained bidder cannot capture the full increase in industry profit that arises when small producers are precluded from acquiring additional capacity.\textsuperscript{12,13} In their analysis of auctions that allocate capacity increments to the firms that value them most highly, Eso et al. (2010) find that even when all suppliers are

\textsuperscript{10}Burguet and McAfee (2009, n. 8) observe that “differentiated product models are notoriously challenging to analyze. However, the analysis of such models represents the natural next step.”

\textsuperscript{11}Related studies that consider settings with complete information include: (i) Krishna (1993), who considers sequential auctions of capacity; (ii) Jehiel and Moldovanu (1996), who analyze agents’ decisions to participate in auctions with externalities; and (iii) Pagnozzi and Rosato (2014), who demonstrate that, due to externalities, welfare can be higher when a new supplier enters an industry by acquiring an incumbent supplier through private bilateral negotiation rather than by auction.

\textsuperscript{12}McAfee (1998) also analyzes a model in which firms engage in Cournot competition after bidding to acquire an input that reduces a firm’s total and marginal cost of production. He finds that the firm with the smallest initial endowment of the input will win the auction for the input increment. Hendricks and McAfee (2010) extend models with Cournot competition among users of an input to include competition among suppliers of the homogeneous input. The authors allow firms to be both buyers and sellers of the input. Equilibrium allocations are determined by (strategic, endogenous) supply and demand in their model, rather than by auctions.

\textsuperscript{13}Borenstein (1988) observes that the profit a firm secures from a license to operate upstream can differ systematically from the total surplus it generates downstream, so auctions of licenses can fail to generate welfare-maximizing outcomes. Burguet and McAfee (2009) find that auctions of operating licenses maximize consumers’ surplus when suppliers face binding financing constraints if consumer demand for the homogeneous retail product is sufficiently elastic.
symmetric *ex ante*, capacity increments often are allocated asymmetrically. Consequently, the equilibrium downstream industry configuration entails one large firm with no capacity constraint facing smaller, capacity-constrained rivals.\textsuperscript{14}

Many analyses of auctions with externalities focus on characterizing the properties of auctions that are optimally designed to achieve a specified objective, such as the maximization of the seller’s payoff.\textsuperscript{15,16} Although our primary focus is not on auction design, we do consider how bidding credits can be structured to ensure the welfare-maximizing allocation of inputs when private and social valuations of inputs diverge. Our finding that substantial information is required to ensure this outcome (and so the appropriate design of bidding credits can be highly problematic in practice) is consistent with the literature’s message regarding the complexity of auction design in the presence of externalities.

## 2 Input Allocations

We consider a setting where two firms engage in price competition after acquiring a key input (e.g., spectrum) at auction. Let $v_i$ denote the value that each consumer derives from purchasing one unit of firm $i \in \{1, 2\}$’s product. This value is an increasing function of the amount of the input ($k_i$) that firm $i$ employs (i.e., $v_i'(k_i) > 0$). The firm’s unit cost of production, $c_i(k_i)$, also may decline as it acquires more of the input (i.e., $c_i'(k_i) \leq 0$). To illustrate, the product in question might be a subscription to the firm’s wireless communications service and the functionality admitted by this subscription. Consumers value this subscription more highly when additional spectrum enables the firm to increase

\textsuperscript{14}Eso et al. (2010) also analyze a model of price competition between capacity-constrained suppliers of differentiated products. Their focus in this analysis, too, is on potential asymmetries in the post-auction size distributions of industry suppliers.

\textsuperscript{15}Jehiel and Moldovanu (2000) is an exception because the authors focus on second-price auctions and consider the optimal design of reserve prices and entry fees. Das Varma (2002) examines how the relative performance of open and sealed-bid auctions varies according to whether the externalities among bidders are reciprocal or non-reciprocal.

\textsuperscript{16}The properties of the optimal auction depend in part on the seller’s powers to compel potential buyers to participate in the auction. See, for example, Jehiel at el. (1999), Brocas (2003), and Figueroa and Skreta (2009).
the speed and reliability of its wireless communications service.\footnote{A corresponding cost saving can arise when more abundant spectrum allows a firm to reduce its use of alternative, less efficient inputs. Increased customer valuation of a firm’s product also can reduce the firm’s customer acquisition and retention costs.}

All consumers value symmetrically the product enhancement that higher levels of the input provide (e.g., faster download speeds and/or fewer dropped calls). However, consumers differ in their valuations of other elements of the firms’ products (e.g., the color and design of telephone handsets or the geographic locations of a firm’s showrooms and service centers). To capture these different valuations, we employ the standard Hotelling model of competition and assume that potential consumers are distributed uniformly on the $[0, 1]$ interval. Consumers travel either to point $0$ to purchase the product from firm $1$ or to point $1$ to purchase the product from firm $2$. Each consumer experiences unit transportation cost $t$ as she travels to purchase the product. Therefore, a consumer who travels distance $d$ to purchase one unit of the product from firm $i \in \{1, 2\}$ and pays price $p_i$ for the good receives net utility $v_i(\cdot) - p_i - t d$.\footnote{Brocas (2008) considers a related model with externalities in which two agents are located at the opposite ends of a Hotelling line. In her model, a principal must decide where on the line to locate an indivisible good. Brocas demonstrates how the optimal policy varies according to whether the agents are privately informed about their valuation of the good or their transportation costs.}

We will refer to the difference between consumer valuation of firm $i$’s product and firm $i$’s production cost as firm $i$’s value margin, $m_i(k_i) \equiv v_i(k_i) - c_i(k_i)$. To ensure that both firms serve consumers in equilibrium, we assume the firms’ value margins are not too disparate.\footnote{We also assume that $v_1(0)$ and $v_2(0)$ are sufficiently large relative to $t$ that all consumers purchase one unit of the product in equilibrium.}

**Assumption 1.** $-3t < m_1(k_1) - m_2(k_2) < 3t$ for all relevant $k_1$ and $k_2$.

Given the input allocation $(k_1, k_2)$ and the resulting production costs and value margins, equilibrium outcomes in this model are readily calculated using standard techniques.

**Lemma 1.** Equilibrium prices, outputs, profits, and consumers’ surplus are, for $i, j \in \{1, 2\}$ $(j \neq i)$: $p_i = c_i + \frac{1}{3} [3t + m_i - m_j]$; $x_i = \frac{1}{6t} [3t + m_i - m_j]$; $\pi_i = \frac{1}{18t} [3t + m_i - m_j]^2$;

\footnote{We also assume that $v_1(0)$ and $v_2(0)$ are sufficiently large relative to $t$ that all consumers purchase one unit of the product in equilibrium.}
and \( CS = \frac{m_1}{6t} [3t + m_1 - m_2] + \frac{m_2}{6t} [3t + m_2 - m_1] - \frac{5t}{4} - \frac{5}[m_1-m_2]^2. \)

When a firm secures more of the input at auction, it both enhances its own value margin and prevents its rival from acquiring the input to increase its value margin. Therefore, the rate at which a firm’s equilibrium profit increases as it acquires more of the input at auction is the sum of the marginal “use value” \((\frac{\partial \pi_i}{\partial k_i})\) and the marginal “foreclosure value” \((-\frac{\partial \pi_j}{\partial k_j})\) of the input.\(^{21}\) From Lemma 1, for \(i, j \in \{1, 2\} \ (j \neq i)\):

\[
B_i = \frac{\partial \pi_i}{\partial k_i} - \frac{\partial \pi_j}{\partial k_j} = \frac{1}{9t} \left[ 3t + m_i - m_j \right] \left[ m_i'(k_i) + m_j'(k_j) \right]. \tag{1}
\]

The firm that will win an unfettered auction for a marginal input increment is the firm that anticipates the largest increase in its equilibrium profit from securing the increment, accounting for relevant use and foreclosure values.\(^{22}\) Therefore, equation (1) provides:

**Proposition 1.** The firm with the highest value margin will win an unfettered auction for the input increment.

**Proof.** Equation (1) implies that \(B_1 > B_2\) if and only if:

\[
\frac{1}{9t} \left[ 3t + m_1 - m_2 \right] \left[ m_1'(k_1) + m_2'(k_2) \right] > \frac{1}{9t} \left[ 3t + m_2 - m_1 \right] \left[ m_2'(k_2) + m_1'(k_1) \right]
\]

\[
\Rightarrow m_1 - m_2 > m_2 - m_1 \quad \Rightarrow \quad m_1 > m_2. \quad \blacksquare
\]

Proposition 1 implies that the identity of the winning bidder in the input auction is not affected by differences in the rates at which the input increases the value margins of the two firms. One might suspect that firm 2, say, often would outbid firm 1 for the input if the

\(^{20}\)The proof of Lemma 1 is presented in the Appendix. The Appendix also presents the proofs of all other formal conclusions that are not proved in the text.

\(^{21}\)See McAfee (1998) for corresponding discussion.

\(^{22}\)We consider marginal increments here and throughout the ensuing analysis. Doing so allows us to analyze the rate at which the input increases key variables (e.g., profit and welfare) rather than the amount by which the input increment increases these variables. This focus streamlines the formal analysis without affecting the qualitative conclusions that would arise from an analysis of discrete input changes that are sufficiently small. The concluding discussion in section 5 notes some of the additional considerations that can arise in the presence of large, discrete input changes.
input increases firm 2’s value margin substantially more rapidly than it increases firm 1’s value margin. However, like firm 2, firm 1 values the input highly when it increases firm 2’s value margin rapidly. This is the case because firm 1 recognizes that firm 2 will become a substantially more formidable competitor if firm 2 secures the input. Consequently, firm 1 is willing to pay relatively handsomely to prevent its rival from becoming considerably more formidable, just as firm 2 is willing to pay relatively handsomely to become more formidable. These offsetting effects ensure that the equilibrium input allocation is determined solely by differences in the value margins of the two competitors. The firm with the largest value margin serves the most customers and enjoys the largest profit margin (recall Lemma 1), and so will profit most from increasing its competitive position by acquiring the marginal input increment.

In contrast, as Proposition 2 reports, the allocation of the input increment that maximizes welfare (which is the sum of consumers’ surplus and industry profit) will depend upon both differences in the relative value margins of the two competitors and differences in the rates at which the input increases these margins.

**Proposition 2.** Welfare is highest when the input increment is allocated: (i) to firm 1 if \( m_1'(k_1) > m_2'(k_2) + \frac{5}{9t} [m_2 - m_1] [m_1'(k_1) + m_2'(k_2)] \); and (ii) to firm 2 if this inequality is reversed.\(^{23}\)

**Proof.** Lemma 1 implies that the rate at which equilibrium welfare \( (W \equiv CS + \pi_1 + \pi_2) \) increases as more of the scarce input is allocated to firm \( i \) (and so is not allocated to firm \( j \neq i \)) is:

\[
\frac{\partial W}{\partial k_i} - \frac{\partial W}{\partial k_j} = \frac{1}{2} \left[ m_i'(k_i) - m_j'(k_j) \right] + \frac{5}{18t} \left[ m_i - m_j \right] \left[ m_i'(k_i) + m_j'(k_j) \right] > 0 \tag{3}
\]

\[
\Leftrightarrow \quad m_i'(k_i) - m_j'(k_j) > \frac{5}{9t} \left[ m_j - m_i \right] \left[ m_i'(k_i) + m_j'(k_j) \right] \quad \blacksquare \tag{4}
\]

\(^{23}\)The increase in welfare is the same whether the input increment is allocated to firm 1 or to firm 2 when \( m_1'(k_1) = m_2'(k_2) + \frac{5}{9t} [m_2 - m_1] [m_1'(k_1) + m_2'(k_2)] \).
Proposition 2 reports that when the input increases, say, firm 1’s value margin more rapidly than it increases firm 2’s value margin (so $m'_1(k_1) > m'_2(k_2)$), welfare tends to be highest when a marginal increment of the input is awarded to firm 1. This allocation tends to increase consumers’ surplus and/or firm 1’s profit relatively rapidly, except when firm 2’s value margin substantially exceeds firm 1’s value margin. In this event, firm 2 serves many more customers than firm 1 serves in equilibrium. (Recall Lemma 1.) Consequently, welfare can increase most rapidly when firm 2 employs the marginal input increment to enhance the value that its substantial customer base derives from its product and/or to reduce the unit cost of serving this large customer base, even though the increment increases firm 1’s value margin more rapidly than it increases firm 2’s value margin.

3 Comparing Equilibrium and Welfare-Maximizing Allocations

Proposition 3 identifies the conditions under which an unfettered auction will, and will not, generate the welfare-maximizing allocation of a marginal input increment.

Proposition 3. An unfettered auction will secure the welfare-maximizing allocation of an input increment if the increment: (i) increases the value margins of the two firms at the same rate (i.e., if $m'_1(k_1) = m'_2(k_2)$); (ii) increases the largest value margin most rapidly (i.e., if $m'_i(k_i) > m'_j(k_j)$ and $m_i > m_j$); or (iii) increases most rapidly the value margin that is the smallest by at least $\frac{9t}{\delta} |D|$ (i.e., if $m'_i(k_i) > m'_j(k_j)$ and $m_i < m_j - \frac{9t}{\delta} |D|$ for some $i, j \in \{1, 2\}, j \neq i$, where $D \equiv [m'_2(k_2) - m'_1(k_1)] / [m'_1(k_1) + m'_2(k_2)]$.

If $m'_1(k_1) > m'_2(k_2)$ and $m_1 - m_2 \in (\frac{9t}{\delta} D, 0)$, then firm 1 will not win an auction for the input increment even though welfare would be higher if it did win the auction. In contrast, if $m'_2(k_2) > m'_1(k_1)$ and $m_1 - m_2 \in (0, \frac{9t}{\delta} D)$, then firm 1 will win an auction for the input increment even though welfare would be higher if it did not win the auction.

To understand the conclusions in Proposition 3, recall from Proposition 2 that when the input increment increases the value margins of the two firms at the same rate, welfare
is highest when the input is allocated to the firm with the highest value margin, because this firm serves the most customers in equilibrium. Therefore, because an auction awards the increment to the firm with the highest value margin (recall Proposition 1), the auction ensures the welfare-maximizing allocation of the input increment in this case.

Also recall from Proposition 2 that welfare is maximized by allocating the input increment to the firm whose value margin increases most rapidly with the increment, provided the firm’s value margin is not too much smaller than its rival’s value margin. Consequently, if \( m_1'(k_1) > m_2'(k_2) \), say, then the increment should be awarded to firm 1 unless \( m_1 \) is sufficiently far below \( m_2 \). However, from Proposition 1, firm 2 will win the auction for the increment whenever \( m_2 \) exceeds \( m_1 \). Consequently, when \( m_1'(k_1) > m_2'(k_2) \) and \( m_2 - m_1 \) is strictly positive but relatively small, the auction will not secure the welfare-maximizing allocation of the input increment.

Proposition 2 also reports that when the value margin of firm 1, say, is sufficiently small, welfare is maximized by allocating the input increment to firm 2 even when \( m_2'(k_2) < m_1'(k_1) \), so the input increases firm 2’s value margin relatively slowly. Therefore, an auction secures the welfare-maximizing input allocation by delivering the input increment to firm 2 when \( m_1'(k_1) > m_2'(k_2) \) if \( m_1 \) is sufficiently far below \( m_2 \).

Figure 1 illustrates the conclusions drawn in Proposition 3. The Figure identifies the values of \( \Delta \equiv m_1 - m_2 \) and \( D \equiv [m_2'(k_2) - m_1'(k_1)] / [m_1'(k_1) + m_2'(k_2)] \) for which an unfettered auction will, and will not, generate the welfare-maximizing allocation of the input increment. Firm 1 wins the auction in the right-hand portion of the Figure where \( \Delta > 0 \), so \( m_1 > m_2 \). The resulting input allocation maximizes welfare when \( m_1'(k_1) > m_2'(k_2) \) (in the southeast quadrant of Figure 1) and when \( m_2'(k_2) - m_1'(k_1) > 0 \) is not too pronounced (in the lower portion of the northeast quadrant). Similarly, firm 2 wins the auction in the left-hand portion of the Figure where \( m_2 > m_1 \). The resulting input allocation maximizes welfare when \( m_2'(k_2) > m_1'(k_1) \) (in the northwest quadrant of Figure 1) and when \( m_1'(k_1) - m_2'(k_2) > 0 \) is not too pronounced (in the upper portion of the southwest quadrant).
As consumers become less concerned with the horizontal dimensions of product quality (i.e., as \( t \) declines),\(^{24}\) the solid line in Figure 1 rotates counterclockwise.\(^{25}\) Consequently, the regions in the northeast and southwest quadrants of the figure where an unfettered auction does not secure the welfare-maximizing allocation of the input shrink.

**Corollary 1.** For a given \( D \), the range of differences between \( m_1 \) and \( m_2 \) (i.e., \( \Delta \)) for which an unfettered auction fails to implement the welfare-maximizing allocation of an input increment declines as the products’ horizontal differentiation (i.e., \( t \)) declines.

**Proof.** From Proposition 3, the set of \( \Delta \) values for which an unfettered auction fails to implement the welfare-maximizing allocation of an input increment is: (i) \( \Delta \in \left( \frac{-5}{9} D, 0 \right) \) when \( D < 0 \); and (ii) \( \Delta \in \left( 0, \frac{-5}{9} D \right) \) when \( D > 0 \). Both sets of values contract as \( t \) declines because \( D \) does not vary with \( t \) (since \( m_i'(k_i) \) is not a function of \( t )\).\(^{26}\)

To interpret Corollary 1, recall from Proposition 2 that the welfare-maximizing allocation of an input is determined both by the difference in the rates at which the input increases the firms’ value margins (i.e., by \( |D| \)) and by the difference in the levels of the firms’ value margins (i.e., by \( |\Delta| \)). Technically, Corollary 1 holds because as \( t \) declines, the determination of the welfare-maximizing allocation of the input is influenced relatively more heavily by the latter difference (the difference in value margins), which is the same difference that determines which firm wins the auction for the input increment.\(^{27}\) Consequently, for a given level of \( D \), the welfare-maximizing allocation coincides with the allocation generated by the auction for a broader range of \( \Delta \) values. Intuitively, Corollary 1 holds because a reduction in \( t \) reduces the aggregate transportation costs that consumers incur as the asymmetry in the

\(^{24}\)To illustrate, as \( t \) declines, consumers of a wireless communications service would value relatively more highly vertical dimensions of product quality like call clarity while affording less value to horizontal dimensions of product quality such as handset color.

\(^{25}\)The equation of the solid line in Figure 1 is \( D = \frac{5}{9} \Delta \), so the slope of the line increases as \( t \) declines.

\(^{26}\)The changes in \( t \) considered in Corollary 1 are those for which Assumption 1 always holds. The variations do not include those that would cause the equilibrium market structure to change from duopoly to monopoly.

\(^{27}\)Formally, Proposition 2 indicates that the welfare-maximizing allocation of the input increment is determined by the sign of \( \frac{9}{5} D - \Delta \). As \( t \) declines, the weight on \( D \) declines relative to the weight on \( \Delta \).
firms’ market shares become more pronounced. Therefore, as $t$ declines, welfare increases more rapidly as the input is allocated to the firm that serves the most customers, *ceteris paribus*.

A firm’s equilibrium profit increases as it secures more of the input, *ceteris paribus*. In contrast, the profit of the rival duopolist declines. Consequently, an increased supply of the input can reduce welfare if the firm that acquires the input serves substantially fewer customers than its rival. In this case, the reduction in the profit of the large competitor can exceed the sum of the increase in the profit of the small competitor and the increase in consumers’ surplus. Recall from Proposition 1, though, that the duopolist with the smallest equilibrium market share never wins an unfettered auction for an input increment. Consequently, unfettered auctions will avoid allocations of an input increment that reduce welfare below the level that arises in the absence of the increment. This conclusion is stated formally in Proposition 4, which refers to $W_0$, the level of welfare that prevails in the absence of any input increment.

**Proposition 4.** Welfare declines below $W_0$ as firm $i \in \{1, 2\}$ acquires more of the input if and only if firm $i$’s value margin is sufficiently far below firm $j (\neq i)$’s value margin, i.e., $m_i < m_j - \frac{9}{2} t$. Consequently, an unfettered auction will never allocate an input increment in a manner that reduces welfare below the level that prevails in the absence of the increment.

Before considering extensions of our analysis, we briefly consider one possible policy response to the fact that unfettered auctions do not necessarily ensure welfare-maximizing allocations of inputs. In practice, policymakers sometimes grant bid credits to certain potential bidders in order to encourage them to bid more aggressively for scarce inputs. When firm $i$ is awarded a bid credit of $b_i \in [0, 1)$ in a first-price auction, the firm is only required to pay $B_i [1 - b_i]$ for an input increment that it wins with a bid of $B_i$. Proposition 5 characterizes the bid credit that ensures a first-price auction will generate the welfare-maximizing

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28 See Ayres and Cramton (1996), Cramton et al. (2011), and Athey et al. (2013), for example, for additional discussions and analyses of bid credits in auctions.
allocation of an input increment.

**Proposition 5.** Suppose $m_i'(k_i) > m_j'(k_j)$ and $m_j - m_i \in (0, \frac{9t}{5} |D|)$ for $i, j \in \{1, 2\}$ ($j \neq i$). Then a first-price auction will ensure the welfare-maximizing allocation of the input increment if firm $i$ is awarded a bid credit, $b_i \in (0, 1)$, that satisfies $\frac{b_i}{2-b_i} = \frac{3}{5} |D|$. 

**Proof.** Equation (1) implies that when firm $i$ receives bid credit $b_i \in (0, 1)$ and firm $j$ ($\neq i$) receives no bid credit in a first-price auction, firm $i$ will outbid firm $j$ for the input increment if:

$$\frac{1}{9t} \left[ 3t + m_i - m_j \right] \left[ m_i'(k_i) + m_j'(k_j) \right] > \frac{1}{9t} \left[ 3t + m_j - m_i \right] \left[ m_i'(k_i) + m_j'(k_j) \right]$$

$$\Leftrightarrow \ 3t + m_i - m_j > [1 - b_i] \left[ 3t + m_j - m_i \right] \Leftrightarrow \ m_j - m_i < 3t \left[ \frac{b_i}{2-b_i} \right].$$

Therefore, when $\frac{b_i}{2-b_i} = \frac{3}{5} |D|$, firm $i$ will secure the input if:

$$m_j - m_i < 3t \left[ \frac{3}{5} |D| \right] = \frac{9t}{5} |D|.$$

Expressions (4) and (5) imply that a first-price auction with the identified bid credit will ensure the welfare-maximizing allocation of the input increment. ■

Proposition 5 reports that in order to ensure a first-price auction generates the welfare-maximizing allocation of an input increment, a bid credit can be awarded to the firm with a moderate value margin disadvantage when the input increment would increase its value margin more rapidly than it would increase the rival’s value margin. The magnitude of the bid credit should increase with the extent to which the input increases the firm’s value margin more rapidly than it enhances the rival’s value margin, ceteris paribus.29

In practice, bid credits often are awarded to competitors that serve relatively few retail customers.30 Proposition 5 identifies two ways in which such a policy can fail to ensure

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29Because $\frac{b_i}{2-b_i}$ is increasing in $b_i$, the bid credit that ensures the auction will generate the welfare-maximizing allocation of the input increment increases with $|D|$. Recall that $|D|$ increases linearly with $|m_i'(k_i) - m_j'(k_j)|$, holding $m_i'(k_i) + m_j'(k_j)$ constant, for $i, j \in \{1, 2\}$ ($j \neq i$).

30Cramton et al. (2011, p. S171) observe that “The most common use of bidding credits has been in U.S. spectrum auctions, where they are granted to small businesses.”
the welfare-maximizing allocation of an input increment. First, firms that serve the fewest retail customers may not be the firms whose value margins increase most rapidly as they acquire more of the input. Second, even when an input increment would increase the value margin of a small competitor more than it would increase the value margin of a large competitor, welfare can be highest when the increment is awarded to the large competitor if its value margin (and thus its market share) sufficiently exceeds the value margin of the small competitor.

It should also be noted that substantial information about prevailing industry conditions is required to design the bid credits identified in Proposition 5. One must know both the rates at which the input increases the equilibrium value margins of the industry competitors and the difference between their equilibrium value margins. In practice, this information can be difficult, if not impossible, to obtain.

4 Extensions

Before concluding, we briefly discuss two extensions of our model.

A. The Setting with Asymmetric Information

The first extension allows for incomplete information about a rival’s ultimate competitive position at the time the input is auctioned. Specifically, we assume customer product valuations \( v_i \) are common knowledge, but each firm is privately informed about its production cost \( c_i \) when the auction takes place. Each firm learns its rival’s cost realization after the auction has concluded, just before the two firms set their prices simultaneously and

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31 To illustrate, suppliers of a wireless communications service that serve many customers may face particularly severe spectrum shortages. Consequently, the value margins of these firms may increase relatively rapidly when they acquire additional spectrum.

32 Policies other than bid credits might be considered to enhance auction performance. Cramton et al. (2011) discuss the potential roles of restrictions on auction participation and limits on the amount of the input that particular competitors can acquire in the auction. Milgrom (2004) and Bhattacharya et al. (2014), among others, note the potential benefits of limiting the number of bidders in settings where potential bidders must incur a cost in order to learn their valuations of the objects being auctioned. Of course, considerable information typically is required to ensure that these policies generate welfare-maximizing input allocations.

33 For expositional simplicity, we generally suppress the dependence of \( v_i, c_i, \) and \( m_i \) on \( k_i \) in the ensuing discussion.
non-cooperatively.  

For simplicity, we assume $c_i \in \{c_{iL}, c_{iH}\}$ where $c_{iL} < c_{iH}$ for $i \in \{1, 2\}$.

To illustrate most readily the central new consideration that arises in this setting, suppose the input increases each firm’s value margin at the same rate, regardless of the firm’s realized cost. It can be shown that the firm that wins the auction for the input increment in this case is the firm with the largest expected value margin differential. Formally, firm $i \in \{1, 2\}$ will win the auction for the input increment when its cost is $c_{is}$ (and so its value margin is $m_{is} = v_i - c_{is}$) and firm $j$’s cost is $c_{jz}$ (and so its value margin is $m_{jz} = v_j - c_{jz}$) if:

$$m_{is} - \left[ \frac{\phi_{sL}}{\phi_{s}} m_{jL} + \frac{\phi_{sH}}{\phi_{s}} m_{jH} \right] > m_{jz} - \left[ \frac{\phi_{Lz}}{\phi_{z}} m_{iL} + \frac{\phi_{Hz}}{\phi_{z}} m_{iH} \right],$$  

where $\phi_{sz} \in (0, 1)$ is the probability that $c_i = c_{is}$ and $c_j = c_{jz}$, $\phi_s \equiv \phi_{sL} + \phi_{sH}$, and $\phi_z \equiv \phi_{Lz} + \phi_{Hz}$ for $s, z \in \{L, H\}$.

Suppose firm $i$ has a cost (and margin) advantage in the sense that $m_{iL} > m_{jL}$ and $m_{iH} > m_{jH}$, and thus $\phi_L m_{iL} + \phi_H m_{iH} > \phi_L m_{jL} + \phi_H m_{jH}$. Then expected welfare is highest when the input increment is awarded to firm $i$. Firm $i$ will win the auction for the increment if firm $i$ knows it has a higher margin than firm $j$ and firm $j$ knows it has a lower margin than firm $i$. This will be the case, for instance, if: (i) $m_i = m_{iL}$ and $m_j = m_{jH}$ (so firm $i$ knows it has its highest margin and firm $j$ knows it has its smallest margin); or (ii) $m_{iH} \geq m_{jL}$ (so firm $i$’s smallest margin is at least as large as firm $j$’s largest margin).

More generally, though, firm $j$ may win the auction for the input increment when expected welfare (and even ex post welfare) would be higher if firm $i$ received the increment. This outcome will arise when the inequality in expression (6) is reversed, so firm $j$’s expected value margin advantage exceeds firm $i$’s expected value margin advantage even though firm $i$’s expected value margin (and its actual value margin) exceeds firm $j$’s corresponding mar-

\footnote{This information structure implies that the firms will not alter their bidding strategies in an attempt to signal (or conceal) their private cost information. See Brocas (2013a) for a related analysis in which signaling considerations arise.}

\footnote{Recall from Proposition 3 that, in the presence of complete information, the auction ensures the welfare-maximizing allocation of the input in this case.}

\footnote{See the Appendix for a proof of this conclusion and proofs of the other conclusions drawn in the ensuing discussion.}
gin(s). For example, suppose \( m_{jH} = 1 \), \( m_{iH} = 2 \), \( m_{jL} = 3 \), \( m_{iL} = 4 \), \( \phi_{LL} = .03 \), \( \phi_{LH} = .11 \), \( \phi_{HL} = .60 \), and \( \phi_{HH} = .26 \). Because \( \phi_{HL} \) is large relative to \( \phi_{LH} \) in this setting, firm \( j \) believes it has a value margin advantage over firm \( i \) when \( m_j = m_{jL} < m_i = m_{iL} \), and so firm \( j \) will outbid firm \( i \) for the input increment.\(^{37}\)

This example illustrates two more general conclusions. First, the same considerations that determine when an auction will ensure the welfare-maximizing allocation of an input in the presence of complete information continue to be relevant in its absence. Second, incomplete information introduces additional considerations that can promote some variation in the qualitative conclusions drawn above.

### B. The Triopoly Setting

The second extension admits competition among three firms that are located on the circumference of a circle (e.g., Salop, 1979). Consumers are distributed uniformly on this circumference and incur unit transportation cost \( t \) when traveling to purchase the product from any supplier. It can be shown that when the input increases the value margins of the three competitors at the same rate (so \( m'_1(k_1) = m'_2(k_2) = m'_3(k_3) \)) in this triopoly setting with complete information, a firm will acquire the input increment in an unfettered auction precisely when welfare is highest if it does so.\(^{38}\) Thus, the natural counterpart to conclusion (i) in Proposition 3 holds in the triopoly setting.

However, the triopoly setting introduces an important asymmetry that does not arise in the duopoly setting.\(^{39}\) When firm 1 acquires the input increment in the duopoly setting, it

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\(^{37}\)An auction may fail to generate the allocation of the input increment that maximizes expected welfare even when the firms’ conditional beliefs are symmetric. To illustrate, suppose \( \varepsilon \in (0, 1) \) is the conditional probability that \( m_j = m_{js} \) when \( m_i = m_{is} \) for \( i, j \in \{1, 2\} (j \neq i) \) and \( s \in \{L, H\} \). Then inequality (6) implies that firm \( j \) will win the auction for the input increment when \( m_j = m_{jL} > m_{iL} = m_i \) if \( [2 - \varepsilon] |m_{jL} - m_{iH}| > \varepsilon |m_{iL} - m_{jH}| \). This inequality will hold even when firm \( i \)'s expected value margin exceeds firm \( j \)'s expected value margin (and so expected welfare is highest when the input increment is awarded to firm \( i \), not firm \( j \)) if, for example, \( m_{jH} = 1 \), \( m_{iH} = 2 \), \( m_{jL} = 3 \), \( m_{iL} = 4 \), \( \phi_{LL} = \phi_{HH} = .2 \), and \( \phi_{LH} = \phi_{HL} = .3 \) (so \( \varepsilon = .4 \)).

\(^{38}\)See Mayo and Sappington (2015) for detailed proofs of the conclusions drawn in the ensuing discussion.

\(^{39}\)The triopoly setting also introduces the possibility of free-riding by individual suppliers. McAfee (1998) demonstrates that when multiple industry suppliers do not face capacity constraints, each supplier may be unable to capture the full increase in the profit of the unconstrained suppliers that arises when additional capacity is withheld from capacity-constrained suppliers. The relatively small foreclosure value that each
necessarily precludes its only rival, firm 2, from acquiring the increment. Firm 1 thereby derives: (i) a marginal use value that is proportional to its equilibrium price-cost margin \((p_1 - c_1)\) and \(m'_1(k_1)\); and (ii) a marginal foreclosure value that is proportional to \(p_1 - c_1\) and \(m'_2(k_2)\), for a combined marginal value that is proportional to \([p_1 - c_1] [m'_1(k_1) + m'_2(k_2)]\).

Similarly, firm 2 derives a marginal value from the input increment in the duopoly setting that is proportional to \([p_2 - c_2] [m'_2(k_2) + m'_1(k_1)]\). The symmetry in each rival’s perceived gain from enhancing its own value margin and precluding its only rival from enhancing its value margin implies that the firm that secures the input increment at auction is the firm with the highest equilibrium price-cost margin, which is the firm with the highest value margin.

When firm \(i\) secures the input increment at auction in the triopoly setting, though, the firm derives no foreclosure value with regard to the rival that would not have obtained the input even if firm \(i\) had not secured the increment. The resulting asymmetry in the sum of marginal use and foreclosure values implies that the allocation of the input in the triopoly setting typically will depend upon both the relative magnitudes of the firms’ value margins and the relative rates at which these value margins increase as a firm acquires more of the critical input.

To illustrate this more general conclusion, suppose the firms are equally spaced around the circle, firm 1’s value margin exceeds the value margins of its symmetric rivals (i.e., \(m_1 > m_2 = m_3\)), and the input increases firm 1’s value margin more slowly than it increases the value margins of firm 1’s rivals (i.e., \(m'_1(k_1) < m'_2(k_2) = m'_3(k_3)\)). When \(m_1 - m_2\) is large relative to \(m'_2(k_2) - m'_1(k_1)\), firm 1 will value the input increment most highly, and so will be the firm whose access to the increment is foreclosed should firm 2 or firm 3 submit the highest bid for the input increment. It can be shown that firm 1 will win the auction.

\(\text{unconstrained supplier anticipates from securing additional capacity at auction implies that the capacity-constrained firms (e.g., new industry entrants) may secure auctioned capacity increments.}\)

\(40\) See Jehiel and Moldovanu (2000), for example, for further discussion of the complications that can arise in designing auctions in the presence of externalities with more than two buyers.
for the input increment under these conditions if \( m'_1(k_1)/m'_2(k_2) > \frac{5t-12|m_1-m_2|}{5t+15|m_1-m_2|} \equiv r_f. \)

Therefore, the identity of the firm that wins the auction depends upon both relative value margins and relative rates at which value margins vary with the input. It can be verified that welfare is highest when firm 1 acquires the increment under the specified conditions if \( m'_1(k_1)/m'_2(k_2) > \frac{25t-24|m_1-m_2|}{25t+48|m_1-m_2|} > r_f. \) Therefore, as in the duopoly setting, firm 1’s relatively high value margin may lead it to acquire the input increment when \( m'_1(k_1) \) is relatively low even though welfare would be higher if one of the firm’s rivals acquired the input increment.

5 Conclusions

We have shown that unfettered auctions tend to ensure the welfare-maximizing allocation of a scarce input when the input increases the value margins of the competing suppliers symmetrically. However, auctions can fail to allocate scarce inputs so as to maximize welfare when the input increases relatively rapidly the value margin of a firm that serves a moderately small share of the market.

Our findings suggest that the insights from the foreclosure literature may require some modification when considering settings where inputs have foreclosure value but cannot be employed to fully exclude competitors. We have found that when two firms compete in such a setting, the competitor with the larger market share will win an unfettered auction for an input increment, as the foreclosure literature might suggest. However, the resulting allocation of the input increment will increase welfare when the increment increases the value margin of the large firm at least as rapidly as it increases the value margin of the smaller firm. Therefore, when predicting the welfare implications of input allocations or when designing policies that affect the allocation of scarce inputs, it is important to assess the levels of prevailing value margins (and associated market shares), the relative rates at which firms’ value margins change as they acquire more of the input, and whether the amount of the input being auctioned is sufficiently large to admit complete foreclosure of a rival.

Our findings imply that, in principle, policies that favor small firms in securing access
to scarce inputs can enhance welfare when additional units of the input would increase the value margins of (moderately) small firms relatively rapidly. However, in practice, the design and implementation of such policies are problematic for at least two reasons. First, detailed information about the value of the input to individual industry suppliers is required to know when favoring selected suppliers will enhance welfare. Such information typically is difficult, if not possible, for policymakers to obtain. Second, even if policymakers are somehow able to secure the requisite information, implementing the appropriate favoritism can be challenging.41

The Pioneer’s Preference program that the U.S. Federal Communications Commission (FCC) established in 1991 illustrates the practical difficulties that can arise in attempting to identify the value of a scarce input – spectrum – to individual entities. The program provided “a means of extending preferential treatment in the FCC’s licensing process to parties that demonstrated their responsibility for developing new spectrum-using communications services and technologies” (FCC, 2000).42 Between 1991 and 1997, the Commission received more than 1,500 applications for a pioneer’s preference. The five such preferences the Commission awarded ultimately enjoyed “only limited deployment” whereas many technologies that were not awarded a pioneer’s preference ultimately achieved widespread commercial application (Fusco, 2000).43

The FCC’s experience with spectrum auctions illustrates the problems that can arise when attempting to favor particular industry competitors. The Commission’s spectrum auction in 1996 provided special financing (i.e., relatively small down payments and low interest rates) to small firms. Although the special financing may have helped small firms acquire spectrum in the auction, several of the firms either failed to pay for the spectrum they

41 As Cramton (2002, p. 635) observes, “Gauging the right level of set-asides or bidding credits is extremely difficult. Also, it is nearly impossible to target the favor to the desired group.”

42 See Singh (1996) for additional discussion of the purpose and implementation of the FCC’s Pioneer’s Preference program.

43 QUALCOMM successfully sued the FCC for failing to award the company a pioneer’s preference for its code division multiple access (CDMA) technology (Fusco, 2000).
won or subsequently failed to supply services using the spectrum. In both cases, the favored treatment of small firms led to substantial, costly delays in utilizing the scarce spectrum (Congressional Budget Office, 2005). More recently, two firms that received bidding credits due to their small size outbid other firms for large amounts of spectrum. Due to their status as “very small firms,” these entities received a 25 percent ($3.25 billion) discount on the spectrum they secured at auction. It was later discovered that Dish Network (a company with nearly $14.5 billion in revenue in 2014) was the majority owner of these firms (Solomon, 2015; Ayotte and Pai, 2015). Experiences like these illustrate the practical difficulties associated with designing and implementing policies that can improve upon the performance of unfettered auctions.

Attempts to favor particular bidders for scarce inputs also can conflict with other policy objectives. In particular, such favoritism can reduce the revenue derived from auctions of scarce inputs.\textsuperscript{44} Policies that favor particular suppliers on the basis of endogenous characteristics also can invite welfare-reducing strategic behavior. For instance, if favorable treatment is afforded to firms with small market shares and high marginal valuations of the input, then firms may find it profitable to reduce their market shares (perhaps by reducing the service quality they deliver to their customers) and to inflate their marginal valuations of the input (perhaps by installing relatively few substitute inputs). The welfare implications of such strategic behavior warrant careful study.\textsuperscript{45}

We close by mentioning three extensions of our analysis. First, we have only considered marginal input increments whereas increments of substantial magnitude often are auctioned in practice. Standard foreclosure concerns can emerge when large input increments are

\textsuperscript{44}To illustrate, in evaluating the FCC’s policy of awarding preferences to small bidders in auctions for spectrum, the Congressional Budget Office (2005, preface) concludes, “partly because of their potentially less favorable commercial prospects, small bidders may not pay as much at auction for their licenses as larger bidders pay. As a result, by offering preferences at auction, the government may forgo auction receipts otherwise available to it.”

\textsuperscript{45}Mayo and Sappington (2015) extend the present analysis to settings where firms can supply costly effort to enhance their value margins. In such settings, the identity of the firm that secures an input increment via auction can vary with both prevailing value margins and the manner in which the input alters the marginal productivities of the firms’ efforts.
auctioned. Furthermore, the welfare-maximizing allocation of the increments may vary with such factors as the magnitudes of the increments and the relative post-auction (as well as pre-auction) market shares of the industry suppliers.

Second, although our analysis has focused on the role of auctions in ensuring the welfare-maximizing allocation of scarce inputs, policymakers may pursue other objectives in practice. It can be shown that the key qualitative conclusions drawn above continue to hold if the social objective is to maximize consumers’ surplus rather than welfare. However, there is a broader set of conditions under which an auction fails to implement the input allocation that maximizes consumers’ surplus.\textsuperscript{46} Other social objectives are also possible. For example, as suggested above, policymakers may value the revenue derived from the sale of inputs and/or seek to promote industry participation by small businesses and minority business owners.\textsuperscript{47} The potential for auctions to allocate inputs efficiently and achieve these alternative objectives awaits additional research.

Third, although our analysis has focused on the extent to which auctions ensure the welfare-maximizing allocation of scarce inputs, the basic forces at play in our model likely are relevant more generally. To illustrate, the expenditures competitors devote to securing patents on technologies that enhance product quality or reduce production costs seem likely to be driven largely by the same relative value margin considerations that are central in our analysis. Furthermore, welfare-maximizing expenditures in these settings seem likely to reflect both relative value margins and the relative rates at which the patented technologies would increase the value margins of industry competitors. Explicit investigation of such related considerations awaits further research.

\textsuperscript{46}Mayo and Sappington (2015) show that if \( m_1'(k_1) > m_2'(k_2) \) and \( m_1 - m_2 \in (9tD, 0) \), then firm 1 will not win an auction for the input increment even though consumers’ surplus would be higher if it did win the auction. In contrast, if \( m_2'(k_2) > m_1'(k_1) \) and \( m_1 - m_2 \in (0, 9tD) \), then firm 1 will win an auction for the input increment even though consumers’ surplus would be higher if it did not win the auction.

\textsuperscript{47}Cramton et al. (2011, p. S169) report that they “consider the primary goal of the regulator to be economic efficiency.” However, the authors also note other goals of spectrum auctions, including revenue generation.
This Appendix provides the proofs of formal conclusions that were not proved in the text. This Appendix also proves the conclusions cited in section 4A and outlines the proofs of the conclusions drawn in section 4B.\textsuperscript{48}

**Proof of Lemma 1.**

The location \((\hat{x} \in [0, 1])\) of the consumer who is indifferent between purchasing the product from the two firms when consumers place value \(v_i\) on firm \(i \in \{1, 2\}\)'s product and when firm \(i\) charges price \(p_i\) for its product is determined by:

\[ v_1 - p_1 - t \hat{x} = v_2 - p_2 - t [1 - \hat{x}] \quad \Rightarrow \quad \hat{x} = \frac{1}{2t} \left[ t + v_1 - p_1 - (v_2 - p_2) \right]. \tag{7} \]

(7) implies that firm 1’s profit is:

\[ \pi_1 = [p_1 - c_1] \hat{x} = \frac{1}{2t} [p_1 - c_1] \left[ t + v_1 - p_1 - (v_2 - p_2) \right]. \tag{8} \]

(8) implies that firm 1’s profit-maximizing price is determined by:

\[ \frac{\partial \pi_1}{\partial p_1} = 0 \quad \Leftrightarrow \quad p_1 = \frac{1}{2} \left[ t + c_1 + v_1 - v_2 + p_2 \right]. \tag{9} \]

Corresponding calculations reveal that firm 2’s profit-maximizing price is determined by:

\[ p_2 = \frac{1}{2} \left[ t + c_2 + v_2 - v_1 + p_1 \right]. \tag{10} \]

(9) and (10) imply that the prices the firms will charge in equilibrium and the firms’ corresponding price-cost margins are, for \(i, j \in \{1, 2\} \ (j \neq i)\):

\[ p_i = \frac{1}{3} \left[ 3t + v_i - v_j + 2c_i + c_j \right] \quad \Rightarrow \quad p_i - c_i = \frac{1}{3} \left[ 3t + m_i - m_j \right]. \tag{11} \]

These prices are readily employed to show that the equilibrium outputs of the firms are:

\[ x_i = \frac{1}{6t} \left[ 3t + m_i - m_j \right] \quad \text{for} \quad i, j \in \{1, 2\} \ (j \neq i). \tag{12} \]

(11) and (12) are readily employed to show that the firms’ equilibrium profits (given \(v_1, v_2, c_1,\) and \(c_2\)) are:

\[ \pi_i = [p_i - c_1] x_i = \frac{1}{18t} \left[ 3t + m_i - m_j \right]^2 \quad \text{for} \quad i, j \in \{1, 2\} \ (j \neq i). \tag{13} \]

Let \(\Delta \equiv m_1 - m_2\) denote the difference between the value margins of firms 1 and 2. Also let \(\pi \equiv \pi_1 + \pi_2\) denote equilibrium industry profit. Then (13) provides:

\[ \pi = \frac{1}{18t} \left[ 3t + \Delta \right]^2 + \frac{1}{18t} \left[ 3t - \Delta \right]^2 = t + \frac{\Delta^2}{9t}. \tag{14} \]

(11) and (12) imply that equilibrium consumers’ surplus is:

\[ \text{See Mayo and Sappington (2015) for additional detail.} \]
\[ CS = \int_{0}^{x_1} [v_1 - p_1 - t x] \, dx + \int_{x_1}^{1} [v_2 - p_2 - t (1 - x)] \, dx \]

\[ = \frac{m_1}{6t} [3t + \Delta] + \frac{m_2}{6t} [3t - \Delta] - \frac{5t}{4} - \frac{5\Delta^2}{36t} = \frac{1}{2} [m_1 + m_2] + \frac{\Delta^2}{36t} - \frac{5t}{4}. \]  
\[ \text{ (15) } \]

**Proof of Proposition 3.**

Propositions 1 and 2 imply that when \( m'_1(k_1) = m'_2(k_2) \), welfare increases most rapidly when the input increment is allocated to the firm with the highest value margin, which an unfettered auction ensures. The propositions also imply that: (1) firm 1 will win the auction for the input increment if \( \Delta > 0 \); and (2) welfare is highest when firm 1 wins the auction if \( \Delta > \frac{9t}{5} D \). Therefore, the auction ensures the welfare-maximizing allocation of the input increment if \( m'_1(k_1) > m'_2(k_2) \) and either \( m_1 > m_2 \) (so firm 1 wins the auction) or \( m_1 < m_2 - \frac{9t}{5} | D | \) (so firm 2 has a substantial value margin advantage and wins the auction). The auction also ensures the welfare-maximizing allocation of the input increment if \( m'_2(k_2) > m'_1(k_1) \) and either \( m_2 > m_1 \) (so firm 2 wins the auction) or \( m_2 < m_1 - \frac{9t}{5} | D | \) (so firm 1 has a substantial value margin advantage and wins the auction).

From Proposition 1, firm 2 wins the auction if \( \Delta < 0 \). From Proposition 2, welfare is highest when firm 1 wins the auction if \( \Delta > \frac{9t}{5} D < 0 \). Therefore, firm 1 does not win the auction when \( \Delta \in (\frac{9t}{5} D, 0) \), even though welfare would be higher if firm 1 did win the auction.

From Proposition 1, firm 1 wins the auction if \( \Delta > 0 \). From Proposition 2, welfare is highest when firm 2 wins the auction if \( \Delta < \frac{9t}{5} D > 0 \). Therefore, firm 1 wins the auction when \( \Delta \in (0, \frac{9t}{5} D) \), even though welfare would be highest if firm 2 won the auction.

**Proof of Proposition 4.**

From (14) and (15), equilibrium welfare is:

\[ W = CS + \pi = \frac{1}{2} [m_1 + m_2] + \frac{5\Delta^2}{36t} - \frac{t}{4} \]
\[ \Rightarrow \frac{\partial W}{\partial k_1} = \frac{1}{18t} m'_1(k_1) [9t + 5\Delta] \gtrless 0 \quad \Leftrightarrow \quad \Delta \gtrless - \frac{9t}{5}. \]  
\[ \text{ (16) } \]

The proof that \( \frac{\partial W}{\partial k_2} \gtrless 0 \Leftrightarrow m_2 \gtrless m_1 - \frac{9t}{5} \) is analogous, and so is omitted.

(16) and Proposition 1 imply that if \( \Delta > 0 \), then firm 1 wins the auction and \( \frac{\partial W}{\partial k_1} = m'_1(k_1) [\frac{1}{2} + \frac{5\Delta}{18t}] > 0. \) (16) and Proposition 1 imply that if \( \Delta < 0 \), then firm 2 wins
the auction and \( \frac{\partial W}{\partial k_2} = m'_2(k_2) \left[ \frac{1}{2} - \frac{5A}{18t} \right] > 0. \) If \( \Delta = 0, \) then (16) implies that \( \frac{\partial W}{\partial k_i} = \frac{1}{2} m'_i(k_i) > 0 \) for \( i = 1, 2. \) ■

The Setting with Asymmetric Information

Let \( \pi_i(c_i, c_j) \) denote firm \( i \)'s profit when its cost is \( c_i \) and firm \( j \)'s cost is \( c_j. \) Also let \( E_i(c_i) \) denote firm \( i \)'s ex ante expected profit when its cost is \( c_i. \) Then Lemma 1 implies that for \( i, j \in \{1, 2\} \) (\( j \neq i \)) and for \( s, z \in \{L, H\} \) (\( z \neq s \)):

\[
\pi_i(c_i, c_j) = \frac{1}{18t} \left[ 3t + m_i - m_j \right]^2
\]

\[
E_i(c_is) = \frac{1}{18t} \left\{ \frac{\phi ss}{\phi s} \left[ 3t + m_is - m_js \right]^2 + \frac{\phi sz}{\phi s} \left[ 3t + m_is - m_jz \right]^2 \right\}. \quad (17)
\]

Differentiating (17) provides:

\[
\frac{\partial E_i(c_is)}{\partial k_i} = \frac{1}{9t \phi s} \left\{ \phi ss \left[ 3t + m_is - m_js \right] m'_is(k_i) + \phi sz \left[ 3t + m_is - m_jz \right] m'_is(k_i) \right\} \quad (18)
\]

and

\[
-\frac{\partial E_i(c_is)}{\partial k_j} = \frac{1}{9t \phi s} \left\{ \phi ss \left[ 3t + m_is - m_js \right] m'_js(k_j) + \phi sz \left[ 3t + m_is - m_jz \right] m'_js(k_j) \right\}. \quad (19)
\]

(18) and (19) imply that the rate at which firm \( i \)'s expected profit increases as it secures more of the input (and so firm \( j \) secures less of the input) when firm \( i \) has cost \( c_is \) is, for \( z \neq s: \)

\[
B_i(c_is) = \frac{\partial E_i(c_is)}{\partial k_i} - \frac{\partial E_i(c_is)}{\partial k_j} = \frac{1}{9t \phi s} \left\{ \phi ss \left[ 3t + m_is - m_js \right] m'_is(k_i) + \phi sz \left[ 3t + m_is - m_jz \right] m'_is(k_i) \right\} + \phi sz \left[ 3t + m_is - m_jz \right] \left[ m'_is(k_i) + m'_js(k_j) \right]. \quad (20)
\]

(18) and (19) also imply that the rate at which firm \( j \)'s expected profit increases as it secures more of the input when firm \( j \) has cost \( c_jz \) is, for \( s \neq z: \)

\[
B_j(c_jz) = \frac{\partial E_j(c_jz)}{\partial k_j} - \frac{\partial E_j(c_jz)}{\partial k_i} = \frac{1}{9t \phi z} \left\{ \phi zz \left[ 3t + m_jz - m_iz \right] m'_jz(k_j) + \phi iz \left[ 3t + m_jz - m_iz \right] m'_iz(k_i) \right\} + \phi iz \left[ 3t + m_jz - m_iz \right] \left[ m'_jz(k_j) + m'_iz(k_i) \right]. \quad (21)
\]
Equation (6) in the text follows from (22).

Differentiating (23) provides:

\[
\frac{\phi_{ss}}{\phi_s} \left[ m_{is} - m_{js} \right] \left[ m_{is}'(k_i) + m_{js}'(k_j) \right] + \frac{\phi_{sy}}{\phi_s} \left[ m_{is} - m_{jy} \right] \left[ m_{is}'(k_i) + m_{jy}'(k_j) \right] \\
> \frac{\phi_{zz}}{\phi_z} \left[ m_{jz} - m_{iz} \right] \left[ m_{jz}'(k_j) + m_{iz}'(k_i) \right] + \frac{\phi_{wz}}{\phi_z} \left[ m_{jz} - m_{iw} \right] \left[ m_{jz}'(k_j) + m_{iw}'(k_i) \right].
\]

Equation (6) in the text follows from (22).

Let \( W(c_i, c_j) \) denote welfare when firm \( i \)'s cost is \( c_i \) and firm \( j \)'s cost is \( c_j \). Also let \( E W \) denote \( ex \ ant\) expected welfare. Then Lemma 1 implies that for \( i, j \in \{1, 2\} \) \( (j \neq i) \) and for \( s, z \in \{L, H\} \):

\[
W(c_i, c_j) = \frac{m_i}{6t} \left[ 3t + m_i - m_j \right] + \frac{m_j}{6t} \left[ 3t + m_j - m_i \right] - \frac{t}{4} - \frac{\left[ m_i - m_j \right]^2}{36t}
\]

\[
\Rightarrow \ E W = \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} \left\{ \frac{m_{is}}{6t} \left[ 3t + m_{is} - m_{jz} \right] + \frac{m_{jz}}{6t} \left[ 3t + m_{jz} - m_{is} \right] \\
- \frac{\left[ m_{is} - m_{jz} \right]^2}{36t} \right\} - \frac{t}{4}.
\]

Differentiating (23) provides:

\[
\frac{\partial E W}{\partial k_i} = \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} \left\{ \frac{1}{6t} \left[ 3t + 2m_{is} - 2m_{jz} \right] m_{is}'(k_i) - \frac{m_{is} - m_{jz}}{18t} m_{is}'(k_i) \right\}
\]

\[
= \frac{1}{18t} \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} \left[ 9t + 5m_{is} - 5m_{jz} \right] m_{is}'(k_i), \quad \text{and}
\]

\[
\frac{\partial E W}{\partial k_j} = \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} \left\{ \frac{1}{6t} \left[ 3t + 2m_{jz} - 2m_{is} \right] m_{jz}'(k_j) + \frac{m_{is} - m_{jz}}{18t} m_{jz}'(k_j) \right\}
\]

\[
= \frac{1}{18t} \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} \left[ 9t + 5m_{jz} - 5m_{is} \right] m_{jz}'(k_j).
\]

(24) and (25) imply that the rate at which expected welfare increases as the input increment is awarded to firm \( i \) (and not to firm \( j \)) is:

\[
\frac{\partial E W}{\partial k_i} - \frac{\partial E W}{\partial k_j} = \frac{1}{18t} \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} \left\{ \left[ 9t + 5m_{is} - 5m_{jz} \right] m_{is}'(k_i) \\
- \left[ 9t + 5m_{jz} - 5m_{is} \right] m_{jz}'(k_j) \right\}.
\]
Assumption A1. \( m'_i(k_i) = m'_j(k_i) > 0 \) and \( m'_i(k_i) = m'_j(k_j) > 0 \) for \( i, j \in \{1, 2\} \) \((j \neq i)\) and \( s, z \in \{L, H\}\) \((z \neq s)\).

**Observation 1.** Suppose Assumption A1 holds. Then expected welfare increases most rapidly when the input is awarded to the firm with the highest expected value margin.

**Proof.** (26) implies that when assumption A1 holds, expected welfare increases more rapidly when the input is awarded to \( i \) than when it is awarded to \( j \) if:

\[
\frac{\partial E}{\partial k_i} - \frac{\partial E}{\partial k_j} = \frac{5}{9} t m'_i(k_i) \sum_{s=L}^{H} \sum_{z=L}^{H} \phi_{sz} [m_{is} - m_{jz}] > 0
\]

\[
\Leftrightarrow \phi_L m_i + \phi_H m_i > \phi_L m_j + \phi_H m_j. \quad \blacksquare
\]

Observations 2–4 consider the case where \( m_{is} > m_{js} \) for \( i, j \in \{1, 2\} \) \((j \neq i)\) and \( s \in \{L, H\}\).

**Observation 2.** Suppose Assumption A1 holds, \( c_i = c_{iL} \), and \( c_j = c_{jH} \). Then firm \( i \) will win the auction for the input increment.

**Proof.** From (22), firm \( i \) will win the auction for the input increment in this setting if:

\[
\phi_{LL} [m_i - m_j] + \phi_{LH} [m_i - m_H] > \phi_{HH} [m_H - m_i] + \phi_{LH} [m_H - m_L]
\]

\[
\Leftrightarrow [m_i - m_H] \left[ \frac{\phi_{LL}}{\phi_H} + \frac{\phi_{LH}}{\phi_H} \right] > \phi_{HH} [m_H - m_i] - \phi_{LL} [m_i - m_L]
\]

\[
\Leftrightarrow [m_i - m_H] \phi_{LH} [\phi_{LL} + \phi_{LH} + \phi_{HH}]
\]

\[
> \phi_{HH} [\phi_{LL} + \phi_{LH}] [m_H - m_i] - \phi_{LL} [\phi_{HH} + \phi_{LH}] [m_i - m_L]
\]

\[
\Leftrightarrow m_i - m_H > \frac{\phi_{HH} \phi_L}{\phi_{LH} [\phi_L + \phi_H]} [m_H - m_i] - \frac{\phi_{LL} \phi_H}{\phi_{LH} [\phi_L + \phi_H]} [m_i - m_L].
\]

The left-hand side of this inequality is strictly positive and the right-hand side is strictly negative, so the inequality holds. \( \blacksquare \)

**Observation 3.** Suppose Assumption A1 holds and \( m_{iH} \geq m_{jL} \). Then firm \( i \) will win the auction for the input increment.

**Proof.** (22) implies that firm \( i \) will win the auction for the input increment when Assumption A1 holds if, for \( s, z \in \{L, H\}\) \((z \neq s)\):

\[
m_{is} - \left[ \frac{\phi_{ss}}{\phi_{s}} m_{js} + \frac{\phi_{sz}}{\phi_{s}} m_{jz} \right] > m_{jz} - \left[ \frac{\phi_{zz}}{\phi_{z}} m_{iz} + \frac{\phi_{sz}}{\phi_{z}} m_{is} \right].
\]
\( \Leftrightarrow m_{is} + \left[ \frac{\phi_{zz}}{\phi_z} m_{iz} + \frac{\phi_{sz}}{\phi_s} m_{is} \right] > m_{jz} + \left[ \frac{\phi_{ss}}{\phi_s} m_{js} + \frac{\phi_{sz}}{\phi_s} m_{jz} \right]. \) \hspace{1cm} (27)

The inequality in (27) holds because \( m_{iL} > m_{iH} \geq m_{jL} > m_{jH}. \)

**Observation 4.** Suppose Assumption A1 holds, \( c_i = c_{iL}, c_j = c_{jL}, \) and \( m_{jL} > m_{iH}. \) Then there are conditions under which firm \( j \) will win the auction even though both ex ante expected welfare and ex post welfare would be greater if the input increment were allocated to firm \( i. \)

**Proof.** Suppose \( m_{jH} = 1, m_{iH} = 2, m_{jL} = 3, m_{iL} = 4, \phi_{LL} = .03, \phi_{LH} = .11, \phi_{HL} = .60, \) and \( \phi_{HH} = .26. \) In this setting, ex post welfare is highest when the input increment is allocated to firm \( j \) because \( m_{jL} > m_{iH} \) and Assumption A1 holds. Firm \( i \) has a higher expected value margin than firm \( j \) in this setting because:

\[
\phi_L m_{iL} + \phi_H m_{iH} = .14[4] + .86[2] = .56 + 1.72 = 2.28
\]

\[
> 2.26 = 1.89 + .37 = .63[3] + .37[1] = \phi_L m_{jL} + \phi_H m_{jH}.
\]

From (22), firm \( j \) will win the auction for the input increment in this setting when \( m_i = m_{iL} \) and \( m_j = m_{jL} \) if:

\[
\frac{\phi_{LL}}{\phi_L} [m_{iL} - m_{jL}] + \frac{\phi_{LH}}{\phi_L} [m_{iL} - m_{jH}] < \frac{\phi_{HL}}{\phi_L} [m_{jL} - m_{iH}] + \frac{\phi_{LL}}{\phi_L} [m_{jL} - m_{iL}]
\]

\[
Leftrightarrow \frac{2\phi_{LL}}{\phi_L} [m_{iL} - m_{jL}] < \frac{\phi_{HL}}{\phi_L} [m_{jL} - m_{iH}] - \frac{\phi_{LH}}{\phi_L} [m_{iL} - m_{jH}]
\]

\[
\Leftrightarrow m_{iL} - m_{jL} + \frac{\phi_{LH}}{2\phi_{LL}} [m_{iL} - m_{jH}] < \frac{\phi_{HL}}{2\phi_{LL}} [m_{jL} - m_{jH}]
\]

\[
\Leftrightarrow 4 - 3 + \frac{11}{.06} [4 - 1] < \frac{60}{.06} [3 - 2] \Leftrightarrow 1 + 5.5 < 10. \]

**The Triopoly Setting**

Let \( x_i \) denote the location of firm \( i \in \{1, 2, 3\} \) on a circle with unit circumference, where \( 0 = x_1 < x_2 < x_3 < 1. \) Also let \( x_{ij} \) denote the location of the consumer who is indifferent between purchasing from firm \( i \) and firm \( j. \) It is readily verified that:

\[
x_{ij} = \frac{1}{2} [x_i + x_j] + \frac{1}{2t} [v_i - v_j + p_j - p_i]. \hspace{1cm} (28)
\]

These market boundaries permit a specification of each firm’s profit as a function of prevailing prices. When each firm chooses its price to maximize its profit, given the prices set by its competitors, the resulting equilibrium prices are, for \( i, j, y \in \{1, 2, 3\} \ (j \neq i \) and
Then (31) and (32) imply that when Assumption A2 holds:

\[ p_i = \frac{1}{5} \left[ 2v_i - v_j - v_y + 3c_i + c_j + c_y + t \left( 1 + L_i \right) \right], \]

where \( L_i \) denotes the sum of the distances between firm \( i \) and each of its rivals along the circle circumference. (28) and (29) can be employed to demonstrate that equilibrium profits are, for \( i, j, y \in \{1, 2, 3\} \) \((j \neq i \text{ and } y \neq i)\):

\[ \pi_i = \frac{1}{25t} \left[ 2m_i - m_j - m_y + t \left( 1 + L_i \right) \right]^2. \]

Let \( B_i = \frac{\partial W}{\partial k_i} - \alpha_{ij} \frac{\partial W}{\partial k_j} - \alpha_{iy} \frac{\partial W}{\partial k_y} \) denote the rate at which firm \( i \)'s equilibrium profit increases as it acquires the input increment and thereby precludes firm \( j \) from acquiring the increment with probability \( \alpha_{ij} \) and precludes firm \( y \) from acquiring the increment with probability \( \alpha_{iy} \). (30) implies:

\[ B_i = \frac{2}{25t} \left[ 2m_i - m_j - m_y + t \left( 1 + L_i \right) \right] \left[ 2m'_i(k_i) + \alpha_{ij} m'_j(k_j) + \alpha_{iy} m'_y(k_y) \right]. \]

Straightforward but tedious calculations reveal that welfare in this setting is:

\[
W = \int_{x_{31}}^{1} [v_1 - c_1 - t(1 - x)] dx + \int_{0}^{x_{12}} [v_1 - c_1 - t x] dx + \int_{x_{12}}^{x_2} [v_2 - c_2 - t(x_2 - x)] dx \\
+ \int_{x_2}^{x_{23}} [v_2 - c_2 - t(x - x_2)] dx + \int_{x_{23}}^{x_3} [v_3 - c_3 - t(x_3 - x)] dx + \int_{x_3}^{x_{31}} [v_3 - c_3 - t(x - x_3)] dx \\
= \frac{1}{25t} \left[ 8(m_1)^2 + 8(m_2)^2 + 8(m_3)^2 - 8m_1 m_2 - 8m_1 m_3 - 8m_2 m_3 \right] \\
+ t \left[ 11m_1 + 3m_2 + 11m_3 + 8m_1 x_2 - 8m_1 x_3 + 8m_2 x_3 - 8m_3 x_2 \right] \\
+ \frac{t}{100} \left[ 56x_2 x_3 + 44(x_2)^2 + 48(x_3)^2 + 50x_3 + 56x_3 + 23 \right] - t \left[ \frac{1}{2} + (x_2)^2 + (x_3)^2 \right].
\]

Assumption A2. \( m'_1(k_1) = m'_2(k_2) = m'_3(k_3) \equiv m'_i(k_i) \), a strictly positive constant.

Let \( G_i = \frac{\partial W}{\partial k_i} - \alpha_{ij} \frac{\partial W}{\partial k_j} - \alpha_{iy} \frac{\partial W}{\partial k_y} \) denote the rate at which equilibrium welfare increases as the input increment is awarded to firm \( i \), which precludes firm \( j \) from acquiring the increment with probability \( \alpha_{ij} \) and precludes firm \( y \) from acquiring the increment with probability \( \alpha_{iy} \). Then (31) and (32) imply that when Assumption A2 holds:

\[ B_i = \frac{12}{25t} m'_i(k_i) \left[ m_i - \frac{1}{2} (m_j + m_y) + \frac{t}{2} \left( 1 + L_i \right) \right], \text{ and} \]

\[ G_i = \frac{24}{25t} m'_i(k_i) \left[ m_i - (\alpha_{ij} m_j + \alpha_{iy} m_y) + \frac{t}{3} (L_i - [\alpha_{ij} L_j + \alpha_{iy} L_y]) \right]. \]
\textbf{Case 1.} \(m_1 + \frac{t}{3} L_1 \succ m_2 + \frac{t}{3} L_2 \succ m_3 + \frac{t}{3} L_3\) and Assumption A2 holds.

It can be verified that \(B_1 \succ B_2 \succ B_3\) in Case 1. Therefore, when firm 1 secures the input increment at auction, it precludes firm 2 (not firm 3) from acquiring the increment, and so \(\alpha_{12} = 1\) and \(\alpha_{13} = 0\). Also, if firm 2 were to increase its bid to the point where it won the auction for the input increment, it would preclude firm 1 (not firm 3) from acquiring the increment, and so \(\alpha_{21} = 1\) and \(\alpha_{23} = 0\). Similarly, if firm 3 were to increase its bid to the point where it won the auction for the input increment, it would preclude firm 1 (not firm 2) from acquiring the increment, and so \(\alpha_{31} = 1\) and \(\alpha_{32} = 0\).

(33) and (34) then imply that firm 1 will win an unfettered auction for the input increment and welfare is highest when firm 1 wins the auction in Case 1. Analogous arguments reveal that the same is true when \(m_1 + \frac{t}{3} L_1 \succ m_3 + \frac{t}{3} L_3 \succ m_2 + \frac{t}{3} L_2\). Corresponding conclusions for settings in which firm 2 or firm 3 wins the auction for the input increment imply that when Assumption A2 holds, a firm wins the auction for an input increment if and only if welfare is highest when the firm wins the auction.

\textbf{Case 2.} \(x_2 = \frac{1}{3}, x_3 = \frac{2}{3}, m_1 > m_2 = m_3, m'_1(k_1) < m'_2(k_2) = m'_2(k_3)\).

Suppose that \(m_1 - m_2\) is sufficiently large relative to \(m'_2(k_2) - m'_1(k_1)\) that \(\alpha_{21} = \alpha_{31} = 1\) and \(\alpha_{23} = \alpha_{32} = 0\) in this case. Then it is readily verified that:

\begin{align}
B_1 &= \frac{2}{25 t} \left[ 2 \left( m_1 - m_2 \right) + \frac{5}{3} t \right] \left[ 2 m'_1(k_1) + m'_2(k_2) \right]; \\
B_2 &= B_3 = \frac{2}{25 t} \left[ m_2 - m_1 + \frac{5}{3} t \right] \left[ 2 m'_2(k_2) + m'_1(k_1) \right]; \\
G_1 &= \frac{1}{25 t} \left\{ \left[ 16 \left( m_1 - m_2 \right) + \frac{25}{3} t \right] m'_1(k_1) - \left[ 8 \left( m_2 - m_1 \right) + \frac{25}{3} t \right] m'_2(k_2) \right\}; \\
G_2 &= G_3 = -G_1.
\end{align}

(35) and (36) imply that firm 1 will win the auction if:

\[2 \left( m_1 - m_2 \right) + \frac{5}{3} t \left[ 2 m'_1(k_1) + m'_2(k_2) \right] \succ \left[ m_2 - m_1 + \frac{5}{3} t \right] \left[ 2 m'_2(k_2) + m'_1(k_1) \right]\]

\[\Leftrightarrow m'_1(k_1) \left[ 5 \left( m_1 - m_2 \right) + \frac{5}{3} t \right] \succ m'_2(k_2) \left[ 4 \left( m_2 - m_1 \right) + \frac{5}{3} t \right]
\]

\[\Leftrightarrow \frac{m'_1(k_1)}{m'_2(k_2)} > \frac{\frac{5}{3} t - 4 \left[ m_1 - m_2 \right]}{\frac{5}{3} t + 5 \left[ m_1 - m_2 \right]} = \frac{5 t - 12 \Delta}{5 t + 15 \Delta} \equiv r_f.
\]

(37) and (38) imply that welfare is highest when firm 1 acquires the input increment if:

\[G_1 > G_2 = G_3 = -G_1 \Leftrightarrow G_1 > 0 \Leftrightarrow \]
\[
\left[ 16 \left( m_1 - m_2 \right) + \frac{25}{3} t \right] m'_1(k_1) > \left[ 8 \left( m_2 - m_1 \right) + \frac{25}{3} t \right] m'_2(k_2)
\]

\[\Leftrightarrow \frac{m'_1(k_1)}{m'_2(k_2)} > \frac{\frac{25}{3} t - 8 \left[ m_1 - m_2 \right]}{\frac{25}{3} t + 16 \left[ m_1 - m_2 \right]} = \frac{25 t - 24 \Delta}{25 t + 48 \Delta} \equiv r_w. \quad (40)\]

**Observation.** \( r_w > r_f. \)

**Proof.** From (39) and (40):

\[r_w > r_f \Leftrightarrow \left[ 25 t - 24 \Delta \right][5 t + 15 \Delta] > [25 t + 48 \Delta][5 t - 12 \Delta]
\]

\[\Leftrightarrow 25 \left[ 15 \right] t \Delta - 24 \left[ 5 \right] t \Delta - 24 \left[ 15 \right] \Delta^2 > -25 \left[ 12 \right] t \Delta + 48 \left[ 5 \right] t \Delta - 48 \left[ 12 \right] \Delta^2
\]

\[\Leftrightarrow \left[ 375 - 120 + 300 - 240 \right] t \Delta > \left[ 360 - 576 \right] \Delta^2 \Leftrightarrow 315 t \Delta > -216 \Delta^2. \quad \blacksquare\]
Figure 1. Outcomes under Unfettered Input Auctions.
References


