TECHNICAL APPENDIX TO ACCOMPANY "SIMPLE COST SHARING CONTRACTS"

Proof of Lemma 1.

The choice of an optimal LCSCR contract involves the choice of an x_L such that the supplier chooses the LCS option when $x \le x_L$ and the CR option when $x > x_L$. When he selects the LCS option and his innate cost realization is x, the supplier's choice of cost reduction is determined by:

$$\underset{y}{\text{Maximize}} \;\; [1 - \alpha]y - \frac{1}{4k}y^2 \,. \tag{L1.1}$$

(L1.1) implies:

$$y_L(x) \;=\; 2k[1 - \alpha] \,. \tag{L1.2}$$

(L1.2) implies that when he chooses the LCS option, the supplier's profit under the LCS option at the cut-off value of innate costs, x_L, is:

$$T(x_L) + \alpha[x_L - y(x_L)] - C(y(x_L)) - [x_L - y(x_L)]$$

$$= \;\; T(x_L) - [1 - \alpha][x_L - 2k(1 - \alpha)] - \tfrac{1}{4k}[2k(1 - \alpha)]^2 \,. \tag{L1.3}$$

Because the supplier is indifferent between the two options when his innate cost realization is x_L and since his profit is always zero under the CR option, (L1.3) implies that:

$$T(x_L) \;=\; [1 - \alpha]\, x_L - k[1 - \alpha]^2 . \tag{L1.4}$$

(L1.4) implies that the buyer's procurement cost when $x = x_L$ is:

$$T(x_L) + \alpha[x_L - y(x_L)] \;=\; T(x_L) + \alpha x_L - \alpha 2k[1 - \alpha] \;=\; x_L - k[1 - \alpha^2] . \tag{L1.5}$$

From (L1.2) and (L1.5), the buyer's procurement cost when $x \le x_L$:

$$T(x_L) + \alpha[x - y(x)] \;=\; T(x_L) + \alpha x - \alpha 2k[1 - \alpha]$$

$$= \;\alpha x + [1 - \alpha]x_L - k[1 - \alpha^2]. \tag{L1.6}$$

The reduction in expected procurement costs from the LCSCR contract with cost reimbursement fraction α relative to the CR contract is:

$$G_{L[\alpha]} \;=\; \int_{\underline{x}}^{x_L} [(1 - \alpha)(x - x_L) + k(1 - \alpha^2)]\, dF(x). \tag{L1.7}$$

From (L1.7):

$$\frac{\partial G_{L[\alpha]}}{\partial x_L} = k[1 - \alpha^2] f(x_L) - [1 - \alpha] \int_{\underline{x}}^{x_L} dF(x)$$

$$= [1 - \alpha][(1 + \alpha) k f(x_L) - F(x_L)]. \tag{L1.8}$$

(L1.8) implies $\frac{\partial G_{L[\alpha]}}{\partial x_L} \gtreqless 0$ as $\frac{F(x_L)}{f(x_L)} \lesseqgtr [1 + \alpha]k$. (L1.9)

Because $F(\cdot)/f(\cdot)$ is monotonically increasing, (L1.9) implies the unique value of x_L that maximizes $G_{L[\alpha]}$ is:

$$x_L^*(\alpha) = \text{minimum}\{\underline{x} + [1+\alpha]k\delta, \; \overline{x}\}. \; ■ \tag{L1.10}$$

<u>Proof of Lemma 2</u>.

The buyer's problem of minimizing the expected cost of securing one unit of the good is:

$$\underset{T(x),\, y(x)}{\text{Minimize}} \int_{\underline{x}}^{\overline{x}} T(x) \; dF(x) \tag{L2.1}$$

subject to: $u(x|x) \geq 0$ for all $x \in [\underline{x}, \overline{x}]$; and (L2.2)

$$u(x|x) \geq u(\hat{x}|x) \quad \text{for all } x, \hat{x} \in [\underline{x}, \overline{x}] \; , \tag{L2.3}$$

where $u(\hat{x}|x) \equiv T(\hat{x}) - [\hat{x} - y(\hat{x}|x)] - C(y(\hat{x}|x))$, (L2.4)

$$x - y(\hat{x}|x) = \hat{x} - y(\hat{x}), \text{ and} \tag{L2.5}$$

$$C(y) = \tfrac{1}{4k} y^2.$$

Differentiating (L2.5) with respect to x provides:

$$\left.\frac{dy}{dx}\right|_{\hat{x}=x} = 1. \tag{L2.6}$$

Defining $u(x) \equiv u(x|x)$, it follows from (L2.3), (L2.4), and (L2.6) that at the solution to the buyer's problem:

$$u'(x) \;=\; -\;\frac{dC(y(\hat{x}\,|\,x))}{dx}\,\Big|_{\hat{x}=x} \;=\; -C'(y(x)) \;\leq\; 0\,. \tag{L2.7}$$

Since (L2.7) implies the supplier's utility declines with x, the buyer optimally sets $u(\bar{x}) = 0$. Therefore:

$$u(x) \;=\; \int_{x}^{\bar{x}} C'(\,y(\tilde{x})\,)\,d\,\tilde{x}\,. \tag{L2.8}$$

(L2.7) and (L2.8) imply that the supplier's expected utility under the optimal contract is:

$$\int_{\underline{x}}^{\bar{x}} u(x)\,dF(x) \;=\; u(x)\,F(x)\Big|_{\underline{x}}^{\bar{x}} \;-\; \int_{\underline{x}}^{\bar{x}} F(x)\,du(x) \;=\; \int_{\underline{x}}^{\bar{x}} C'(y(x))\,\frac{F(x)}{f(x)}\,dF(x)\,. \tag{L2.9}$$

The first equality in (L2.9) follows from integration by parts. The second equality in (L2.9) holds because $u(x)\,F(x)\Big|_{\underline{x}}^{\bar{x}} \;=\; u(\bar{x})F(\bar{x}) - u(\underline{x})F(\underline{x}) \;=\; 0$.

Because the agent's utility is the difference between the payment he receives and the costs he incurs:

$$T(x) \;=\; x \;-\; y(x) \;+\; C(y(x)) \;+\; u(x)\,. \tag{L2.10}$$

(L2.10) implies the expected payment to the supplier is:

$$\int_{\underline{x}}^{\bar{x}} T(x)\,dF(x) \;=\; \int_{\underline{x}}^{\bar{x}} \left[\, x \;-\; y(x) \;+\; C(y(x)) \;+\; u(x)\,\right] dF(x)\,. \tag{L2.11}$$

(L2.9) and (L2.11) imply:

$$\int_{\underline{x}}^{\bar{x}} T(x)\,dF(x) \;=\; \int_{\underline{x}}^{\bar{x}} \left[\, x \;-\; y(x) \;+\; C(y(x)) \;+\; C'(y)\,\frac{F(x)}{f(x)}\,\right] dF(x)\,. \tag{L2.12}$$

The derivative of (L2.12) with respect to $y(x)$ is:

$$-1 \;+\; C'(y(x)) \;+\; C''(y(x))\,\frac{F(x)}{f(x)}\;. \tag{L2.13}$$

Because $C(y) \;=\; \frac{1}{4k}y^2$ and $F(x) = \left(\dfrac{x - \underline{x}}{\bar{x} - \underline{x}}\right)^{\delta}$, (L2.13) can be written as:

$$-1 \;+\; \frac{y(x)}{2k} \;+\; \frac{1}{2k}\left[\frac{x - \underline{x}}{\delta}\right] \;=\; \frac{1}{2k\delta}\,[\,-2k\,\delta \;+\; \delta\,y(x) \;+\; x \;-\; \underline{x}\,]\,. \tag{L2.14}$$

Because (L2.14) implies $-1 + C'(y) + C''(y)\,\dfrac{F}{f} \geq 0$ when $x \geq \underline{x} + 2k\delta$, it follows that expected procurement costs are minimized when $y(x) = 0$ for all $x \geq \underline{x} + 2k\delta$.

(L2.14) implies that when $x \leq \underline{x} + 2k\delta$, expected procurement costs initially decline and then increase with x. Therefore, the optimal cost reduction is interior, and is determined by

$$-2k\delta + \delta\, y(x) + x - \underline{x} = 0. \tag{L2.15}$$

(L2.15) implies:

$$y(x) = 2k - \frac{x-\underline{x}}{\delta} \quad \text{when } x \leq \underline{x} + 2k\delta. \qquad \blacksquare \tag{L2.16}$$

Proof of Lemma 3.

Let G_O denote the reduction in the buyer's expected procurement costs under the optimal contract relative to the cost reimbursement contract. To analyze G_O, it is necessary to consider two cases. In the first case (case I), $\Delta \equiv \bar{x} - \underline{x}$ is relatively large, and so $\underline{x} + 2k\delta < \bar{x}$. Consequently, the supplier is optimally induced to supply strictly positive effort levels for small x's and to supply no effort for large x's. In the second case (Case II), Δ is relatively small, and so the supplier is induced to supply a strictly positive level of effort for all realizations of Δ.

Case I: $\bar{x} \geq \underline{x} + 2k\delta$ (or $T \equiv \frac{\Delta}{2k\delta} \geq 1$).

$$G_O = \int_{\underline{x}}^{\underline{x}+2k\delta} \left[y(x) - C(y(x)) - C'(y)\,\frac{F(x)}{f(x)} \right] dF(x) \tag{L3.1}$$

$$= \int_{\underline{x}}^{\underline{x}+2k\delta} \left[y(x) - C(y(x)) - \frac{y(x)}{2k}\left[\frac{x-\underline{x}}{\delta}\right]\right] dF(x) \tag{L3.2}$$

$$= \int_0^1 \left[y(2k\delta t + \underline{x}) - C(y(2k\delta t + \underline{x})) - y(2k\delta t + \underline{x})\,t \right] d\left(\frac{2k\delta t}{\Delta}\right)^{\delta} \tag{L3.3}$$

$$= \int_0^1 \left[2k(1-t) - k(1-t)^2 - 2k(1-t)t \right] d\left(\frac{2k\delta t}{\Delta}\right)^{\delta} \tag{L3.4}$$

$$= k\delta\left(\frac{2k\delta}{\Delta}\right)^{\delta} \int_0^1 [1 - 2t + t^2]\, t^{\delta-1} dt \tag{L3.5}$$

$$= \; k\delta \left(\frac{2k\delta}{\Delta}\right)^{\delta} \left(\frac{1}{\delta} - \frac{2}{\delta+1} + \frac{1}{\delta+2}\right) \qquad \text{(L3.6)}$$

$$= \; k \left(\frac{2k\delta}{\Delta}\right)^{\delta} \left(\frac{2}{(\delta+1)(\delta+2)}\right). \qquad \text{(L3.7)}$$

(L3.3) follows from (L3.2) above by letting $t = [x - \underline{x}]/(2k\delta)$. Notice that t varies from 0 to 1 as x varies from $\underline{x}$ to $\underline{x} + 2k\delta$. (L3.4) follows from (L2.16).

Case II: $x < x + 2k\delta$ (or $T < 1$).

$$G_O \; = \; \int_{\underline{x}}^{\bar{x}} \left[y(x) - C(y(x)) - C'(y) \frac{F(x)}{f(x)} \right] dF(x) \qquad \text{(L3.8)}$$

$$= \; \int_{\underline{x}}^{\bar{x}+2k\delta T} \left[y(x) - C(x) - \frac{y(x)}{2k} - \left[\frac{x - \underline{x}}{\delta}\right] \right] dF(x) \qquad \text{(L3.9)}$$

$$= \; \int_{0}^{T} \left[y(2k\delta t + \underline{x}) - y(2k\delta t + \underline{x})\, t \right] d\left(\frac{2k\delta t}{\Delta}\right)^{\delta} \qquad \text{(L3.10)}$$

$$= \; \int_{0}^{T} \left[2k(1-t) - k(1-t)^2 - 2k(1-t)\,t \right] d\left(\frac{2k\delta t}{\Delta}\right)^{\delta} \qquad \text{(L3.11)}$$

$$= \; k\delta T^{-\delta} \int_{0}^{T} \left[1 - 2t + t^2 \right] t^{\delta-1}\, dt \qquad \text{(L3.12)}$$

$$= \; k\delta T^{-\delta} \left(\frac{T^{\delta}}{\delta} - \frac{2T^{\delta+1}}{\delta+1} + \frac{T^{\delta+2}}{\delta+2} \right) \qquad \text{(L3.13)}$$

$$= \; k\delta \left(\frac{1}{\delta} - \frac{2T}{\delta+1} + \frac{T^2}{\delta+2} \right). \qquad \text{(L3.14)}$$

Notice again that (L3.10) follows from (L3.9) by letting $t = [x - \underline{x}]/(2k\delta)$. Also notice that t varies from 0 to T as x varies from $\underline{x}$ to $\bar{x}$. (L3.11) follows from (L2.16).

It follows from (L1.7) and (L1.10) that the expected gain from the optimal FPCR contract relative to the CR contract is:

$$G_F \quad = \quad \int_{\underline{x}}^{x_F^*} [x \; - \; x_F^* \; + \; k] \, dF(x) \, . \tag{L3.15}$$

where $x_F^* \; = \;$ minimum $\{\underline{x} \; + \; k\delta, \; \bar{x}\}$. (L3.16)

To analyze the expected gain in (L3.15), it is convenient to consider two cases. In the first case (Case I), $\Delta \equiv \bar{x} - \underline{x}$ is relatively large, and so $\underline{x} + k\delta < \bar{x}$. Consequently, x_F^* is interior, and so the agent chooses the fixed price contract for low x's and the cost reimbursement contract for high x's. In the second case, Δ is relatively small, and so the agent chooses the fixed price contract for all realizations of x.

Case I: $\bar{x} \;\geq\; \underline{x} + k\delta \quad (\text{or } T \;\geq\; \frac{1}{2})$.

It follows from (L3.15) and (L3.16) that:

$$G_F \quad = \quad \int_{\underline{x}}^{\bar{x} + k\delta} [x \; - \; \underline{x} \; + \; k(1 \; - \; \delta)] \; dF(x) \tag{L3.17}$$

$$= \quad \int_0^1 \left[k\delta t \; + \; k(1 \; - \; \delta) \right] d\left(\frac{k\delta t}{\Delta} \right)^{\delta} \tag{L3.18}$$

$$= \quad k\,\delta \left(\frac{k\delta}{\Delta} \right)^{\delta} \int_0^1 [\delta t \; + 1 \; - \; \delta] \, t^{\delta - 1} \, dt$$

$$= \quad k\delta \left(\frac{k\,\delta}{\Delta} \right)^{\delta} \left[\frac{\delta}{\delta + 1} \; + \; \frac{1 - \delta}{\delta} \right] \tag{L3.19}$$

$$= \quad k \left(\frac{k\,\delta}{\Delta} \right)^{\delta} \left[\frac{1}{\delta + 1} \right] \quad . \tag{L3.20}$$

(L3.18) follows from (L3.17) by letting $t \; = \; [x - \underline{x}]/(k\,\delta)$. Notice that t varies from 0 to 1 as x varies from $\underline{x}$ to $\underline{x} + k\delta$.

Case II: $\bar{x} \;<\; \underline{x} + k\delta \quad (\text{or } T \;<\; \frac{1}{2})$.

It follows from (L3.15) and (L3.16) that:

$$G_F \quad = \quad \int_{\underline{x}}^{\bar{x}} [x \; - \; \bar{x} \; + \; k] \; dF(x) \tag{L3.21}$$

$$= \int_{\underline{x}}^{\bar{x}+k\delta\hat{T}} [x - \underline{x} - k\delta\hat{T} + k]\, dF(x) \tag{L3.22}$$

$$= \int_{0}^{\hat{T}} [k\delta t - k\delta\hat{T} + k]\, d\left(\frac{k\delta t}{\Delta}\right)^{\delta} \tag{L3.23}$$

$$= k\delta\hat{T}^{-\delta} \int_{0}^{\hat{T}} [\delta t - \delta\hat{T} + 1]\, t^{\delta-1}\, dt \tag{L3.24}$$

$$= k\delta\hat{T}^{-\delta} \left[\frac{\delta\hat{T}^{\delta+1}}{\delta+1} - \hat{T}^{\delta+1} + \frac{\hat{T}^{\delta}}{\delta}\right] \tag{L3.25}$$

$$= k\left[1 - \frac{\delta\hat{T}}{\delta+1}\right], \tag{L3.26}$$

where $\hat{T} = [\bar{x} - \underline{x}]/(k\delta)$.

Again, (L3.23) follows from (L3.22) by letting $t = [x - \underline{x}]/(k\delta)$. Notice that t varies from 0 to $\hat{T}$ as x varies from $\underline{x}$ to $\bar{x}$.

To complete the proof, we analyze separately the three cases implied by the analysis immediately above and the analysis in the proof of Lemma 2.

Case 1. $\Delta \geq 2k\delta$ (or $T \geq 1$).

Case 1 here corresponds to Case I in both the analysis of the optimal contract and the analysis of the FPCR contract. Therefore, from (L3.7) and (L3.20):

$$\frac{G_F}{G_O} = \left[\left[\frac{k}{\delta+1}\right]\left(\frac{k\delta}{\Delta}\right)^{\delta}\right] \Big/ \left[\left(\frac{2k\delta}{\Delta}\right)^{\delta}\left[\frac{2k}{(\delta+1)(\delta+2)}\right]\right] = \frac{\delta+2}{2^{\delta+1}}. \tag{L3.27}$$

Case 2. $\Delta \in [k\delta, 2k\delta]$ (or $T \in [\frac{1}{2}, 1)$).

Case 2 here corresponds to Case II in the analysis of the optimal contract and to Case I in the analysis of the optimal FPCR contract. Therefore, from (L3.14) and (L3.20):

$$\frac{G_F}{G_O} = \left[\left(\frac{k\delta}{\Delta}\right)^{\delta}\frac{k}{\delta+1}\right] \Big/ \left[k\delta\left(\frac{1}{\delta} - \frac{2T}{\delta+1} + \frac{T^2}{\delta+2}\right)\right] \tag{L3.28}$$

$$= \left(\frac{k\delta}{\Delta}\right)^{\delta}\left(\frac{1}{\delta+1}\right) \Big/ \left[\delta\left(\frac{T^{\delta}}{\delta} - \frac{2T^{\delta+1}}{\delta+1} + \frac{T^{\delta+2}}{\delta+2}\right)\right]. \tag{L3.29}$$

(L3.29) reveals that G_F / G_O declines with T in this case if $D(T) \equiv \frac{T^{\delta}}{\delta} - \frac{2T^{\delta+1}}{\delta+1} + \frac{T^{\delta+2}}{\delta+2}$ is an increasing function of T. $D(\cdot)$ is an increasing function of T because:

$$D'(T) = T^{\delta-1} - 2T^{\delta} + T^{\delta+1} = T^{\delta-1}[1 - 2T + T^2] = T^{\delta-1}[T-1]^2 \geq 0. \tag{L3.30}$$

Case 3. $\Delta \in (0, k\delta)$ (or $T \in (0, \frac{1}{2})$).

Case 3 here corresponds to Case II in the analysis of both the optimal contract and the FPCR contract. Therefore, from (L3.14) and (L3.26):

$$\frac{G_F}{G_O} = \left[k\left(1 - \frac{2\delta T}{\delta+1}\right)\right] \Big/ \left[k\delta\left(\frac{1}{\delta} - \frac{2T}{\delta+1} + \frac{T^2}{\delta+2}\right)\right] = \left(1 + \left[\frac{\delta}{\delta+2}\right]\frac{T^2}{\left[1 - \frac{2\delta T}{\delta+1}\right]}\right)^{-1}. \tag{L3.31}$$

Notice that $T^2/[1 - \frac{2\delta T}{\delta+1}]$ increases with T because the numerator of this term increases with T while the denominator decreases with T. Also notice that since $T \in (0, \frac{1}{2})$ in Case 3, $1 - \frac{2\delta T}{\delta+1} > 1 - \frac{\delta}{\delta+1} > 0$. Therefore, it follows from (L3.31) that G_F / G_O is a decreasing function of T in this case. ■

Proof of Lemma 4.

$T = 1$ when $\delta = \Delta/(2k)$. Therefore, from (L3.27):

$$\lim_{\delta\to\infty} \frac{G_F}{G_O} = \lim_{\delta\to\infty} \frac{\delta+2}{2^{\delta+1}} = 0. \tag{L4.1}$$

The last equality in (L4.1) follows from L'Hopital's Rule. ■

Proof of Proposition 1.

(L1.7) and (L1.10) imply that, relative to the cost reimbursement contract, the expected gain from the optimal LCSCR contract (given α), $G_{L[\alpha]}$, is:

$$\int_{\underline{x}}^{x_L^*} [(1 - \alpha)(x - x_L^*) + k(1 - \alpha^2)]\, dF(x). \qquad \text{(P1.1)}$$

To analyze this expected gain, we need to consider two cases. In the first case (Case I), $\Delta \equiv \bar{x} - \underline{x}$ is relatively large, and so the supplier chooses the LCS option for low x's and the CR option for high x's. In the second case (Case II), Δ is relatively small, and so the agent chooses the LCS option for all realizations of x.

Case I: $\bar{x} \geq \underline{x} + [1 + \alpha]k\delta$ (or $T' \equiv \frac{\Delta}{(1+\alpha)k\delta} \geq 1$).

$$G_{L[\alpha]} = \int_{\underline{x}}^{\underline{x}+k(1+\alpha)\delta} \Big[(1 - \alpha)[x - (1 + \alpha)k\delta - \underline{x}] + k(1 - \alpha^2)\Big]\, dF(x). \qquad \text{(P1.2)}$$

$$= k\big[1 - \alpha\big] \int_0^1 [(1 + \alpha)\delta(t - 1) + (1 - \alpha)]\, d\left(\left(\frac{k(1+\alpha)\delta t}{\Delta}\right)^{\delta}\right) \qquad \text{(P1.3)}$$

$$= k\big[1 - \alpha^2\big]\delta\left(\frac{k(1+\alpha)\delta}{\Delta}\right)^{\delta} \int_0^1 (\delta t + 1 - \delta)\, t^{\delta-1}\, dt \qquad \text{(P1.4)}$$

$$= k\big[1 - \alpha^2\big]\delta\left(\frac{k(1+\alpha)\delta}{\Delta}\right)^{\delta} \left(\frac{\delta}{\delta + 1} + \frac{1 - \delta}{\delta}\right) \qquad \text{(P1.5)}$$

$$= k\big[1 - \alpha^2\big]\left(\frac{k(1+\alpha)\delta}{\Delta}\right)^{\delta} \frac{1}{\delta + 1}. \qquad \text{(P1.6)}$$

(P1.3) follows from (P1.2) by letting $t = \big[x - \underline{x}\big]/\big(k\delta\big[1 + \alpha\big]\big)$. Notice that t varies from 0 to 1 as x varies from $\underline{x}$ to $\underline{x} + k\delta\big[1+\alpha\big]$.

Case II: $\bar{x} < \underline{x} + [1 + \alpha]k\delta$ (or $T' < 1$).

$$G_{L[\alpha]} = \int_{\underline{x}}^{\bar{x}} \big[(1 - \alpha)(x - x_F^*) + k(1 - \alpha^2)\big]\, dF(x). \qquad \text{(P1.7)}$$

$$= \big[1 - \alpha\big] \int_{\underline{x}}^{\underline{x}+(1+\alpha)k\delta T'} \big[x - \underline{x} - (1 + \alpha)k\delta T' + k(1 + \alpha)\big]\, dF(x) \qquad \text{(P1.8)}$$

$$= \big[1 - \alpha\big] \int_0^{T'} \big[(1 + \alpha)k\delta t - (1 + \alpha)k\delta T' + k(1+\alpha)\big]\, d\left(\left(\frac{(1+\alpha)k\delta t}{\Delta}\right)^{\delta}\right) \qquad \text{(P1.9)}$$

$$= \; k\left[1 \; - \; \alpha^2\right]\delta\,(T')^{-\delta} \int_0^{T'} \left(\delta t \; - \; \delta T' \; + \; 1\right) t^{\delta-1}\, dt \qquad \text{(P1.10)}$$

$$= \; k\left[1 \; - \; \alpha^2\right]\delta\,(T')^{-\delta} \left(\frac{\delta\,(T')^{\delta+1}}{\delta+1} \; - \; (T')^{\delta+1} \; + \; \frac{(T')^{\delta}}{\delta} \right) \qquad \text{(P1.11)}$$

$$= \; k\left[1 \; - \; \alpha^2\right]\left(1 \; - \; \frac{\delta T'}{\delta+1}\right), \qquad \text{(P1.12)}$$

where $T' \;=\; \Delta / \left[(1 + \alpha) k\, \delta\right]$.

Again, (P1.8) follows from (P1.7) by letting $t \;=\; \left[x \,-\, \underline{x}\right] / \left[k\delta(1 \,+\, \alpha)\right]$. Notice that t varies from 0 to $\underline{x} \;+\; \left[k\,\delta(1 \,+\, \alpha)\right]T'$ as x varies from $\underline{x}$ to $\overline{x}$.

For given $\underline{x}$, $\overline{x}$ and δ, the optimal LCSCR contract is derived by choosing α optimally.

Case I: $\Delta \,\geq\, 2k\delta$ (or $T \,\equiv\, \frac{\Delta}{2k\delta} \,\geq\, 1$).

$$G_{L[\alpha]} \;=\; k\left[1 \,-\, \alpha^2\right]\left(\frac{k(1+\alpha)\delta}{\Delta}\right)^{\delta} \frac{1}{\delta \,+\, 1}\,. \qquad \text{(P1.13)}$$

Since $\left(1 \,-\, \alpha^2\right) \;=\; (1 \,-\, \alpha)(1 \,+\, \alpha)$, the sign of the partial derivative of $G_{L[\alpha]}$ with respect to α has the same sign as the derivative of $(1 \,-\, \alpha)(1 \,+\, \alpha)^{\delta+1}$, which is $(1 \,+\, \alpha)^{\delta}\, [\delta \,-\, (\delta \,+\, 2)\alpha]$. Setting this derivative equal to zero reveals that the value of α that uniquely maximizes $G_{L[\alpha]}$ is:

$$\alpha^* \;=\; \frac{\delta}{\delta \,+\, 2}\;. \qquad \text{(P1.14)}$$

Substituting (P1.14) into (P1.13) provides:

$$G_L \;\equiv\; G_{L[\alpha^*]} \;=\; 2k\left(\frac{2k\delta}{\Delta}\right)^{\delta}\left(\frac{\delta+1}{\delta \,+\, 2}\right)^{\delta\,+\,1} \frac{2}{(\delta \,+\, 1)(\delta \,+\, 2)}\,. \qquad \text{(P1.15)}$$

Case II: $\Delta \,\in\, \left[\left(\frac{\delta \,+\, 1}{\delta \,+\, 2}\right)2k\delta,\; 2k\delta\right]$ (or $T \,\in\, \left[\frac{\delta+1}{\delta+2},\, 1\right]$).

Case IIA: $\alpha \;\leq\; \Delta / (k\delta) \;-\; 1$.

$$G_{L[\alpha]} \;=\; k\,[1 \,-\, \alpha\,]^2 \left(\frac{k[1+\alpha\,]\delta}{\Delta}\right)^{\delta} \frac{1}{\delta \,+\, 1}\,. \qquad \text{(P1.16)}$$

Again, the partial derivative with respect to α of the expression in $G_{L[\alpha]}$ has the same sign as the expression $(1 - \alpha)^{\delta} \, [\delta - (\delta + 2)\alpha]$.

When $\alpha = \Delta / (k\delta) - 1$:

$$\delta - (\delta + 2)\alpha = \delta - [\delta + 2][\Delta / (k\delta) - 1]$$

$$\leq \delta - (\delta + 2)\left[\left(\frac{\delta + 1}{\delta + 2}\right)\left(\frac{2k\delta}{k\delta}\right) - 1\right] = 0. \qquad \text{(P1.17)}$$

The inequality in (P1.17) holds because we are in Case II.

When $\alpha = 0$, $\delta - [\delta + 2]\alpha > 0$. Therefore, the value of α that uniquely maximizes $G_{L[\alpha]}$ in this range is:

$$\alpha^* = \frac{\delta}{\delta + 2} \,. \qquad \text{(P1.18)}$$

Substituting (P1.18) into (P1.16) provides

$$G_L = 2k \left(\frac{2k\delta}{\Delta}\right)^{\delta} \left(\frac{\delta + 1}{\delta + 2}\right)^{\delta + 1} \frac{2}{(\delta + 1)(\delta + 2)} \,. \qquad \text{(P1.19)}$$

<u>Case IIB</u>: $\alpha > \Delta / (k\delta) - 1$.

$$G_{L[\alpha]} = k[1 - \alpha]\left[1 + \alpha - \frac{2\delta T}{\delta + 1}\right]. \qquad \text{(P1.20)}$$

The sign of the partial derivative with respect to α of the expression for $G_{L[\alpha]}$ in (P1.20) is the same as the sign of the derivative of $[1 - \alpha]\left[1 + \alpha - \frac{2\delta T}{\delta + 1}\right]$, which is $-2\alpha + \frac{2\delta T}{\delta + 1}$. This derivative is negative because:

$$-2\alpha + \frac{2\delta T}{\delta + 1} < 0 \qquad \text{(P1.21)}$$

$$\Leftrightarrow \quad T < \left[\frac{\delta + 1}{\delta}\right]\alpha \quad \Leftrightarrow \quad \frac{\Delta}{2k\delta} < \left(\frac{\delta + 1}{\delta}\right)\alpha$$

$$\Leftarrow \quad \frac{\Delta}{2k\delta} \leq \left[\frac{\delta + 1}{\delta}\right]\left[\frac{\Delta}{k\delta} - 1\right] \qquad \text{(P1.22)}$$

$$\Leftrightarrow \quad \frac{\delta + 1}{\delta} \le \left(2\left(\frac{\delta + 1}{\delta}\right) - 1\right)\frac{\Delta}{2k\delta} \quad \Leftrightarrow \quad \delta + 1 \le (\delta + 2)\frac{\Delta}{2k\delta}$$

$$\Leftrightarrow \quad \left[\frac{\delta + 1}{\delta + 2}\right] 2k\delta \le \Delta \,. \tag{P1.23}$$

(P1.23) holds because we are in Case II. (P1.22) holds because we are in Case IIB. The negative derivative implies the optimal α will never lie in this range.

Case III: $\Delta \le \left[\frac{\delta + 1}{\delta + 2}\right] 2k\delta$ (or $T \in \left[0, \frac{\delta + 1}{\delta + 2}\right]$).

Case IIIA: $\alpha < \Delta / (k\delta) - 1$.

$$G_{L[\alpha]} = k[1 - \alpha^2]\left(\frac{k(1 + \alpha)\delta}{\Delta}\right)^{\delta} \frac{1}{\delta + 1}. \tag{P1.24}$$

The partial derivative with respect to α of the expression for $G_{L[\alpha]}$ in (P1.24) has the same sign as the derivative of $(1 - \alpha)(1 + \alpha)^{\delta + 1}$, which is $(1 + \alpha)^{\delta}(\delta - (\delta + 2)\alpha)$. This derivative is positive because:

$$(1 + \alpha)^{\delta}[\delta - (\delta + 2)\alpha] > 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta + 2} > \alpha \quad \Leftarrow \quad \frac{\delta}{\delta + 2} \ge \frac{\Delta}{k\delta} - 1 \tag{P1.25}$$

$$\Leftrightarrow \quad \frac{2\delta + 2}{\delta + 2} \ge \frac{\Delta}{k\delta} \quad \Leftrightarrow \quad \frac{(\delta + 1)}{(\delta + 2)} 2k\delta \ge \Delta \,. \tag{P1.26}$$

The last equivalence in (P1.26) holds, given that we are in Case III. The last equivalence in (P1.25) holds because we are in Case IIIA. The positive derivative implies the optimal α will not lie in the specified range.

Case IIIB: $\alpha \ge \Delta/(k\delta) - 1$.

$$G_{L[\alpha]} = k[1 - \alpha]\left[1 + \alpha - \frac{2\delta T}{\delta + 1}\right]. \tag{P1.27}$$

The partial derivative of the expression for $G_{L[\alpha]}$ with respect to α in (P1.27) has the same sign as the derivative of $(1 - \alpha)\left(1 + \alpha - \frac{2\delta T}{\delta + 1}\right)$, which is $-2\alpha + \frac{2\delta T}{\delta + 1}$.

Notice that when $\alpha = \Delta/(k\delta) - 1$,

$$-2\alpha + \frac{2\delta T}{\delta+1} = -\frac{2\Delta}{k\delta} + 2 + \left(\frac{2\delta}{\delta+1}\right)\frac{\Delta}{k\delta}$$

$$= 2 + \left[\frac{2\delta}{\delta+1} - 4\right]\frac{\Delta}{2k\delta} = 2\left[1 - \Delta\Big/\left(\frac{\delta+1}{\delta+2}[2k\delta]\right)\right] \geq 0. \tag{P1.28}$$

Also notice that $-2\alpha + \frac{2\delta T}{\delta+1} < 0$ when $\alpha = 1$. Consequently, the optimal value of α in this range is the solution to: $-2\alpha + \frac{2\delta T}{\delta+1} = 0$. Therefore, the unique maximizer of $G_{L[\alpha]}$ in this case is:

$$\alpha^* = \Delta/(2k[\delta+1]). \tag{P1.29}$$

Substituting (P1.29) into (P1.27) provides:

$$G_L = k\left[1 - \frac{\Delta}{2k(\delta+1)}\right]^2. \tag{P1.30}$$

It is now useful to prove that for given δ, $\frac{G_L}{G_O}$ is a non-increasing function of Δ. The proof proceeds by examining three cases:

<u>Case I</u>: $\Delta \geq 2k\delta$ (or $T \geq 1$).

In this case:

$$\frac{G_L}{G_O} = \left(\left(\frac{2k\delta}{\Delta}\right)^{\delta}\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1}\frac{4k}{(\delta+1)(\delta+2)}\right)\Big/\left(\left(\frac{2k\delta}{\Delta}\right)^{\delta}\frac{2k}{(\delta+1)(\delta+2)}\right) = 2\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1}. \tag{P1.31}$$

<u>Case II</u>: $\Delta \in \left[[\frac{\delta+1}{\delta+2}][2k\delta],\ 2k\delta\right]$ (or $T \in \left[\frac{\delta+1}{\delta+2}, 1\right]$).

$$\frac{G_L}{G_O} = \left(2k\left(\frac{2k\delta}{\Delta}\right)^{\delta}\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1}\frac{2}{(\delta+1)(\delta+2)}\right)\Big/\left(k\delta\left(\frac{1}{\delta} - \frac{2T}{\delta+1} + \frac{T^2}{\delta+2}\right)\right)$$

$$= \left(2\left(\frac{2k\delta}{\Delta}\right)^{\delta}\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1}\frac{2}{(\delta+1)(\delta+2)}\right)\Big/\left(\delta\left(\frac{T^{\delta}}{\delta} - \frac{2T^{\delta}}{\delta+1} + \frac{T^{\delta+2}}{\delta+2}\right)\right). \tag{P1.32}$$

(P1.32) implies it suffices to show that $\frac{T^{\delta}}{\delta} - \frac{2T^{\delta+1}}{\delta+1} + \frac{T^{\delta+2}}{\delta+2}$ is an increasing function of T. This fact follows because the derivative of this function is:

$$T^{\delta-1} - 2T^{\delta} + T^{\delta+1} = T^{\delta-1}\left[1 - 2T + T^2\right] = T^{\delta-1}[1 - T]^2 \geq 0. \qquad \text{(P1.33)}$$

Case III: $\Delta \leq \left(\frac{\delta+1}{\delta+2}\right)2k\delta$ (or $T \in \left[0, \frac{\delta+1}{\delta+2}\right]$).

$$\frac{G_L}{G_O} = \left(k\left(1 - \frac{\delta T}{\delta+1}\right)^2\right) \Big/ \left(k\delta\left(\frac{1}{\delta} - \frac{2T}{\delta+1} + \frac{T^2}{\delta+2}\right)\right)$$

$$= \frac{\left(1 - \frac{\delta}{\delta+1}T\right)^2}{\left(1 - \frac{\delta}{\delta+1}T\right)^2 + \frac{\delta}{(\delta+1)^2(\delta+2)}T^2}. \qquad \text{(P1.34)}$$

Since $\left(1 - \frac{\delta}{\delta+1}T\right)$ is a decreasing function of T while T^2 is an increasing function of T, (P1.34) implies G_L / G_O is a decreasing function of T.

The foregoing analysis implies that G_L / G_O attains its smallest value in Case I immediately above. Consequently, the proof follows if:

$$2\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1} > \frac{2}{e}. \qquad \text{(P1.35)}$$

To show that (P1.35) holds, first recall that $e \equiv \lim_{w \to \infty} \left(1 + \frac{1}{w}\right)^w$. Now it will be useful to show that $\ln\left(\left(1 + \frac{1}{w}\right)^w\right) = w\ln\left(1 + \frac{1}{w}\right)$ is an increasing function of w on $(0, \infty)$. To do so, we will show that its derivative is declining with w, and is non-negative at $w = \infty$.

$$\frac{d}{dw}\{w \ln\left(1 + \frac{1}{w}\right)\} = \ln\left(1 + \frac{1}{w}\right) + w\frac{1}{1+\frac{1}{w}}\left(-\frac{1}{w^2}\right) = \ln\left(1 + \frac{1}{w}\right) - \frac{1}{1+w}. \qquad \text{(P1.36)}$$

$$\text{As } w \to \infty, \ \ln\left(1 + \frac{1}{w}\right) - \frac{1}{1+w} \to \ln(1) - 0 = 0. \qquad \text{(P1.37)}$$

It remains to show that the derivative of $\ln\left(1 + \frac{1}{w}\right) - \frac{1}{1+w}$ is negative on $(0, \infty)$. This derivative is:

$$-\left(\frac{1}{w^2}\right)\left(\frac{1}{1+\frac{1}{w}}\right) + \frac{1}{(1+w)^2} = -\frac{1}{w(1+w)} + \frac{1}{(1+w)^2}$$

$$= \frac{w-(1+w)}{w(1+w)^2} = -\frac{1}{w(1+w)^2} < 0\,. \qquad \text{(P1.38)}$$

(P1.38) implies $\left(1 + \frac{1}{w}\right)^w$ is an increasing function of w, whose limit as $w \to \infty$ is e. Consequently, $\left(1 + \frac{1}{w}\right)^w < e$ for $w \in (0, \infty)$. Thus:

$$\frac{G_L}{G_O} \geq 2\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1} = 2\left(\frac{\delta+1}{\delta+1+1}\right)^{\delta+1} = 2\left(\frac{1}{[1+1/(1+\delta)]^{1+\delta}}\right) > \frac{2}{e}\,. \quad ■$$

<u>Proof of Proposition 2</u>.

We first prove that $[G_L - G_F]/[G_O - G_F]$ is a non-increasing function of Δ. The proof proceeds by analyzing four distinct cases.

<u>Case I</u>: $\Delta \leq k\delta$ (or $T \equiv \frac{\Delta}{2k\delta} \leq 1/2$).

From (L3.14), (P1.30), and (L3.26):

$$G_O = k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2}\right), \quad G_F = k\left(1 - \frac{2\delta T}{\delta+1}\right), \text{ and}$$

$$G_L = k\left[1 - \frac{\Delta}{2k(\delta+1)}\right]^2 = k\left[1 - \frac{\delta T}{\delta+1}\right]^2 = k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta^2 T^2}{(\delta+1)^2}\right).$$

Consequently:

$$\frac{G_L - G_F}{G_O - G_F} = \left(k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta^2 T^2}{(\delta+1)^2}\right) - k\left(1 - \frac{2\delta T}{\delta+1}\right)\right) \Big/ \left(k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2}\right) - k\left(1 - \frac{2\delta T}{\delta+1}\right)\right)$$

$$= \left(\frac{\delta^2 T^2}{(\delta+1)^2}\right) \Big/ \left(\frac{\delta T^2}{\delta+2}\right) = \frac{\delta(\delta+2)}{(\delta+1)^2}\,. \qquad \text{(P2.1)}$$

Case II: $\Delta \in \left[k\delta, \left(\frac{2\delta+2}{\delta+2}\right)k\delta\right]$ (or $T \in \left[\frac{1}{2}, \frac{\delta+1}{\delta+2}\right]$).

From (L3.14), (P1.30), and (L3.20):

$$G_O = k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2}\right), \; G_L = k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta^2 T^2}{(\delta+1)^2}\right), \text{ and } G_F = k(2T)^{-\delta}/(\delta+1).$$

Consequently:

$$\frac{G_L - G_F}{G_O - G_F} = 1 - \frac{G_O - G_L}{G_O - G_F}$$

$$= 1 - \left(k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2}\right) - k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta^2 T}{(\delta+1)^2}\right)\right) \Big/ \left(k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2}\right) - \frac{k(2T)^{-\delta}}{\delta+1}\right)$$

$$= 1 - \left[\frac{\delta T^2}{(\delta+1)^2(\delta+2)}\right] \Big/ \left[1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2} - \frac{(2T)^{-\delta}}{\delta+1}\right]$$

$$= 1 - \frac{\delta}{(\delta+1)^2(\delta+2)}\left(T^{-2} - \frac{2\delta}{\delta+1}T^{-1} + \frac{\delta}{\delta+2} - \frac{2^{-\delta}T^{-\delta-2}}{\delta+1}\right)^{-1}. \quad \text{(P2.2)}$$

$\frac{G_L - G_F}{G_O - G_F}$ is a decreasing function of T if and only if $\frac{G_O - G_L}{G_O - G_F}$ is an increasing function of T. This is the case if and only if $R(T)$ is a decreasing function of T, where

$$R(T) \equiv \left(T^{-2} - \frac{2\delta}{\delta+1}T^{-1} + \frac{\delta}{\delta+2} - \frac{2^{-\delta}T^{-\delta-2}}{\delta+1}\right). \quad \text{(P2.3)}$$

(Notice that $R(T) > 0$ for all relevant T. Otherwise, it would follow from (P2.2) that $EG_L > EG_O$, which cannot be the case.)

The derivative of $R(T)$ with respect to T on $\left[1/2, \frac{\delta+1}{\delta+2}\right]$ is:

$$-2T^{-3} + \frac{2\delta}{\delta+1}T^{-2} + 2^{-\delta}T^{-\delta-3}\frac{\delta+2}{\delta+1} = T^{-3}\left(-2 + \frac{2\delta T}{\delta+1} + 2^{-\delta}T^{-\delta}\frac{\delta+2}{\delta+1}\right) \equiv T^{-3}S(T). \quad \text{(P2.4)}$$

We will now show that $S(T) < 0$ for $T \in \left(1/2, \frac{\delta+1}{\delta+2}\right)$ by showing that it is strictly convex and the values of its ending points are non-positive. Notice, first, that

$$S(T) \;=\; -2 \;+\; \frac{\delta}{\delta+1} \;+\; \frac{\delta+2}{\delta+1} \;=\; 0 \;\text{ when } T \;=\; 1/2 \;. \tag{P2.5}$$

Next observe that $S(T)$ is convex because:

$$S'(T) \;=\; \frac{2\delta}{\delta+1} - \frac{2^{-\delta}\delta(\delta+2)}{\delta+1}T^{-\delta-1}, \text{ and so } S''(T) \;=\; 2^{-\delta}\delta(\delta+2)T^{-\delta-2} \;>\; 0\,. \tag{P2.6}$$

Now note that when $T = \frac{\delta+1}{\delta+2}$:

$S(T) \;=\; -2 + \frac{2\delta}{\delta+2} + \left(\frac{2\delta+2}{\delta+2}\right)^{-\delta}\frac{\delta+2}{\delta+2} \;=\; -\left(\frac{4}{\delta+2}\right) + \left(\frac{\delta+2}{2\delta+2}\right)^{\delta}\frac{\delta+2}{\delta+1}$, which is non-positive because:

$$-\left(\frac{4}{\delta+2}\right) + \left(\frac{\delta+2}{2\delta+2}\right)^{\delta}\left(\frac{\delta+2}{\delta+1}\right) \;\le\; 0 \;\;\Leftrightarrow\;\; \frac{(\delta+2)^{\delta+2}}{(2\delta+2)^{\delta+1}} \;\le\; 2$$

$$\Leftrightarrow\;\; Z(\delta) \equiv [\delta+2]\ln(\delta\;+\;2) \;-\; [\delta+1]\ln(2\delta\;+\;2) \;\le\; \ln 2\,. \tag{P2.7}$$

Notice that $Z(0) = 2\ln(2) - \ln(2) = \ln 2$. Therefore, to show that $Z(\delta) \le \ln 2$ for all $\delta \ge 0$, it suffices to show that $Z'(\delta) \le 0$ for all $\delta \ge 0$. From (P2.7):

$$Z'(\delta) \;=\; \ln(\delta\;+\;2) + 1 - \ln(2\delta\;+\;2) - 1 \;=\; \ln(\delta\;+\;2) - \ln(2\delta\;+\;2) \;<\; 0\,. \tag{P2.8}$$

(P2.8) implies $\frac{G_L - G_F}{G_O - G_F}$ is a decreasing function of T on $\left[\frac{1}{2}, \frac{\delta+1}{\delta+2}\right]$.

<u>Case III</u>: $\varDelta \in \left[\left(\frac{2\delta+2}{\delta+2}\right)k\delta,\; 2k\delta\right]$ (or $T \in \left[\frac{\delta+1}{\delta+2}, 1\right]$).

From (L3.14), (P1.19), and (L3.20):

$$G_O \;=\; k\left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2}\right),\; G_L \;=\; kT^{-\delta}\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1}\frac{4}{(\delta+1)(\delta+2)},\; \text{and } G_F \;=\; \frac{k(2T)^{-\delta}}{\delta+1}.$$

Consequently:

$$\frac{G_L - G_F}{G_O - G_F} = \left(k\,T^{-\delta} \left(\frac{\delta+1}{\delta+2} \right)^{(\delta+1)} \frac{4}{(\delta+1)(\delta+2)} - \frac{k(2T)^{-\delta}}{\delta+1} \right) \Big/ \left(k \left(1 - \frac{2\delta T}{\delta+1} + \frac{\delta T^2}{\delta+2} \right) - \frac{k(2T)^{-\delta}}{\delta+1} \right)$$

$$= \left(\left(\frac{\delta+1}{\delta+2} \right)^{(\delta+1)} \frac{4}{(\delta+1)(\delta+2)} - \frac{2^{-\delta}}{\delta+1} \right) \Big/ \left(T^{\delta} - (\frac{2\delta}{\delta+1}) T^{\delta+1} + (\frac{\delta}{\delta+2}) T^{\delta+2} - \frac{2^{-\delta}}{\delta+1} \right). \quad \text{(P2.9)}$$

To show $\frac{G_L - G_F}{G_O - G_F}$ is a decreasing function of T, it suffices to show $W(T)$ is an increasing function of T, where:

$$W(T) \equiv T^{\delta} - \left(\frac{2\delta}{\delta+1} \right) T^{\delta+1} + \left(\frac{\delta}{\delta+2} \right) T^{\delta+2}. \quad \text{(P2.10)}$$

$$W'(T) = \delta\left[T^{\delta-1} - 2T^{\delta} + T^{\delta+1} \right] = \delta T^{\delta-1}\left[1 - 2T + T^2 \right] = \delta T^{\delta-1}\left[T - 1 \right]^2 \geq 0. \quad \text{(P2.11)}$$

<u>Case IV</u>: $\Delta \geq 2k\delta$ (or $T \geq 1$).

From (L3.7), (P1.15), and (L3.20):

$$G_O = k\,T^{-\delta} \frac{2}{(\delta+1)(\delta+2)}, \; G_L = k\,T^{-\delta} \left(\frac{\delta+1}{\delta+2} \right)^{\delta+1} \frac{4}{(\delta+1)(\delta+2)}, \text{ and } G_F = \frac{k(2T)^{-\delta}}{\delta+1}.$$

Consequently:

$$\frac{G_L - G_F}{G_O - G_F} = \left(\left(\frac{\delta+1}{\delta+2} \right)^{\delta+1} \frac{4k\,T^{-\delta}}{(\delta+1)(\delta+2)} - \frac{k(2T)^{-\delta}}{\delta+1} \right) \Big/ \left(\frac{2k\,T^{-\delta}}{(\delta+1)(\delta+2)} - \frac{k(2T)^{-\delta}}{\delta+1} \right)$$

$$= \left(\left(\frac{\delta+1}{\delta+2} \right)^{\delta+1} \frac{4}{(\delta+1)(\delta+2)} - \frac{2^{-\delta}}{\delta+1} \right) \Big/ \left(\frac{2}{(\delta+1)(\delta+2)} - \frac{2^{-\delta}}{\delta+1} \right)$$

$$= \left(2\left(\frac{\delta+1}{\delta+2} \right)^{\delta+1} - \frac{\delta+2}{2^{\delta+1}} \right) \Big/ \left(1 - \frac{\delta+2}{2^{\delta+1}} \right). \quad \text{(P2.12)}$$

Notice that the expression in (P2.12) is independent of T.

In summary, we have shown $\frac{G_L - G_F}{G_O - G_F}$ is a non-increasing function of Δ in all four cases.

Because $\dfrac{G_L - G_F}{G_O - G_F}$ declines as T increases, it suffices to show that $\dfrac{G_L - G_F}{G_O - G_F} > \frac{1}{2}$ in Case IV above. Therefore, from (P2.12), to complete the proof, it suffices to show:

$$\left[2\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1} - \frac{\delta+2}{2^{\delta+1}}\right] \Big/ \left[1 - (\frac{\delta+2}{2^{\delta+1}})\right] > \frac{1}{2} \quad \text{for} \quad \delta \geq 1. \qquad \text{(P2.13)}$$

(P2.13) holds if and only if:

$$2\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1} - \frac{\delta+2}{2^{\delta+1}} > \frac{1}{2}\left[1 - \frac{\delta+2}{2^{\delta+1}}\right] \qquad \text{(P2.14)}$$

$$\Leftrightarrow \quad 4\left(\frac{\delta+1}{\delta+2}\right)^{\delta+1} > 1 + \frac{\delta+2}{2^{\delta+1}} \qquad \text{(P2.15)}$$

$$\Leftrightarrow \quad M(\delta) \equiv \frac{1}{4}\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}\left[1 + \frac{\delta+2}{2^{\delta+1}}\right] < 1. \qquad \text{(P2.16)}$$

From the proof of Proposition 1, $\left(\dfrac{\delta+2}{\delta+1}\right)^{\delta+1}$ is increasing in δ and is bounded above by e. Also notice that:

$$\frac{\partial}{\partial\delta}\left(\frac{\delta+2}{2^{\delta+1}}\right) = 2^{-2(\delta+1)}\left[2^{\delta+1} - (\delta+2)2^{\delta+1}\ln 2\right]$$

$$= 2^{-(\delta+1)}\left[1 - (\delta+2)\ln 2\right] < 2^{-(\delta+1)}\left[1 - (\delta+2)(.69)\right]$$

$$< 2^{-(\delta+1)}\left[1 - 2(.69)\right] < 0. \qquad \text{(P2.17)}$$

(P2.17) implies $\dfrac{\delta+2}{2^{\delta+1}}$ is a decreasing function of δ.

We will now demonstrate that $M(\delta) < 1$ over all relevant intervals in which $\delta \leq 1$. Initially, suppose $\delta \geq 2.2$. Because $\left(\dfrac{\delta+2}{\delta+1}\right)^{\delta+1} < e < 2.718$, and because $\dfrac{\delta+2}{2^{\delta+1}}$ is decreasing in δ, we know that in this range:

$$M(\delta) < \frac{1}{4}\left[1 + \frac{4.2}{2^{3.2}}\right](2.718) < \frac{1}{4}[1.457][2.718] < \frac{1}{4}[3.96] < 1. \qquad \text{(P2.18)}$$

Now suppose $\delta \in [1.3, 2.2)$. Because $\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}$ is increasing in δ, $\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}$ < $\left(\frac{4.2}{3.2}\right)^{3.2}$ < $(1.3125)^{3.2}$ < 2.3874 in this range. Also, since $\frac{\delta+2}{2^{\delta+1}}$ is decreasing in δ, $1 + \frac{\delta+2}{2^{\delta+1}} \leq 1 + \frac{3.3}{2^{2.3}} < 1.67$ in this range. Therefore, $M(\delta) <$ [1.67][2.3874]/4 < 3.99/4 < 1 in this range.

Now suppose $\delta \in [1.1, 1.3)$. Because $\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}$ is increasing in δ, $\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}$ < $\left(\frac{3.3}{2.3}\right)^{2.3}$ < $(1.43478)^{2.3}$ < 2.29407 in this range. Also, since $\frac{\delta+2}{2^{\delta+1}}$ is decreasing in δ, $1 + \frac{\delta+2}{2^{\delta+1}} \leq 1+\frac{3.1}{2^{2.1}} <$ 1.7231 in this range. Therefore, $M(\delta)$ < [2.29407][1.7231]/4 < (3.953)/4 < 1 in this range.

Finally, suppose $\delta \in [1, 1.1)$. Because $\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}$ is increasing in δ, $\left(\frac{\delta+2}{\delta+1}\right)^{\delta+1}$ < $\left(\frac{3.1}{2.1}\right)^{2.1}$ < 2.265 in this range. Also, since $\frac{\delta+2}{2^{\delta+1}}$ is decreasing in δ, $1 + \frac{\delta+2}{2^{\delta+1}} \leq 1 + ¾ = 1.75$ in this range. Therefore, $M(\delta)$ < [2.265][1.75]/4 < (3.964)/4 < 1 in this range. ■

The simulations in Appendix B reflect the following analysis. From (A3) in Appendix A:

$$G_O(y(x)) = \int_{\underline{x}}^{x_O^*} \left[y(x) - \tfrac{1}{4k}[y(x)]^2 - \tfrac{y(x)}{2k}\,\tfrac{F}{f} \right] dF(x) \,. \tag{B1}$$

Maximizing $G_O(y(x))$ with respect to $y(x)$ provides:

$$y_O(x) = 2k - F(x)/f(x)\,. \tag{B2}$$

Substituting $y_O(x)$ into $G_O(\cdot)$ provides:

$$G_0 = \frac{1}{4k} \int_{\underline{x}}^{x_0^*} \left[2k - \frac{F(x)}{f(x)} \right]^2 dF(x)\,, \tag{B3}$$

where, from (B2):

$$2k = F(x_O^*) / f(x_O^*) . \tag{B4}$$

From (A6) in Appendix A:

$$G_F = \int_{\underline{x}}^{x_F^*} [x - x_F^* + k] \, dF(x) . \tag{B5}$$

Maximizing G_F with respect to x_F^* provides:

$$k = F(x_F^*) / f(x_F^*) . \tag{B4}$$

From (A2) in Appendix A:

$$G_{L[\alpha]} = \int_{\underline{x}}^{x_L} [(1 - \alpha)(x - x_L) + k(1 - \alpha^2)] \, dF(x) . \tag{B5}$$

Maximizing $G_{L[\alpha]}$ with respect to x_L provides:

$$k[1 + \alpha] = F(x_L) / f(x_L) . \tag{B6}$$

Solving (B6) for α provides:

$$\alpha = \frac{F(x_L)}{k f(x_L)} - 1 . \tag{B7}$$

Maximizing $G_{L[\alpha]}$ with respect to α provides:

$$2 k \alpha F(x_L) = \int_{\underline{x}}^{x_L} F(x) \, dx . \tag{B8}$$

(B8) follows in part from the fact that $\int_{\underline{x}}^{x_L} [x_L - x] \, dF(x) = \int_{\underline{x}}^{x_L} F(x) \, dx$.

Substituting (B7) into (B8) reveals:

$$k = F(x_L) / f(x_L) - \int_{\underline{x}}^{x_L} F(x) \, dx / [2F(x_L)] . \tag{B9}$$

The simulations identify the optimal values of x_L (from (B9)) and α (from (B7)).