Technical Appendix to Accompany

"When Do Auctions Ensure the Welfare-Maximizing Allocation of Scarce Inputs?"

by John Mayo and David Sappington

Proposition 4′. If $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$ and $m_1 - m_2 \in (9\,t\,D, 0)$, then firm 1 will not win an auction for the input increment even though consumers' surplus would be higher if it did win the auction. In contrast, if $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$ and $m_1 - m_2 \in (0, 9\,t\,D)$, then firm 1 will win an auction for the input increment even though consumers' surplus would be higher if it did not win the auction.

Proof. From Lemma 1 in the text, for $i, j \in \{1, 2\}$ $(j \neq i)$:

$$\frac{\partial CS}{\partial k_i} = \frac{1}{18\,t}\,\frac{\partial m_i}{\partial k_i}\,[\,9\,t + m_i - m_j\,]$$

$$\Rightarrow \frac{\partial CS}{\partial k_1} - \frac{\partial CS}{\partial k_2} = \frac{1}{2}\left[\frac{\partial m_1}{\partial k_1} - \frac{\partial m_2}{\partial k_2}\right] + \frac{1}{18\,t}\,[\,m_1 - m_2\,]\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]. \tag{1}$$

(1) implies:

$$\frac{\partial CS}{\partial k_1} - \frac{\partial CS}{\partial k_2} > \frac{\partial CS}{\partial k_2} - \frac{\partial CS}{\partial k_1} \Leftrightarrow \frac{\partial CS}{\partial k_1} - \frac{\partial CS}{\partial k_2} > 0$$

$$\Leftrightarrow\ m_1 - m_2 > 9\,t\,D \ \text{ where } D \equiv \left[\frac{\frac{\partial m_1}{\partial k_1} - \frac{\partial m_2}{\partial k_2}}{\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}}\right]. \tag{2}$$

The conclusion in the proposition follows from (2) and from Proposition 1 in the text. ■

The Triopoly Setting

There are three firms located on a circle with unit circumference. Firm i is located at x_i, for $i = 1, 2, 3$ where $0 < x_2 < x_3 < 1$ and x_1 is normalized to 0. Let x_{ij} denote the location of the consumer who is indifferent between purchasing from firm i and firm j. x_{12} is determined by:

$$v_2 - p_2 - t\,[\,x_2 - x_{12}\,] = v_1 - p_1 - t\,[\,x_{12} - 0\,]$$

$$\Rightarrow\ 2\,t\,x_{12} = t\,x_2 + v_1 - v_2 + p_2 - p_1$$

$$\Rightarrow\ x_{12} = \frac{1}{2}\,x_2 + \frac{1}{2\,t}\,[\,v_1 - v_2 + p_2 - p_1\,]. \tag{3}$$

Similarly, x_{23} is determined by:

$$v_3 - p_3 - t\,[\,x_3 - x_{23}\,] = v_2 - p_2 - t\,[\,x_{23} - x_2\,]$$

$$\Rightarrow\ 2\,t\,x_{23} = t\,[\,x_2 + x_3\,] + v_2 - v_3 + p_3 - p_2$$

$$\Rightarrow \qquad x_{23} \;=\; \frac{1}{2}\,[\,x_2 + x_3\,] + \frac{1}{2\,t}\,[\,v_2 - v_3 + p_3 - p_2\,]\,. \tag{4}$$

Also, x_{31} is determined by:

$$v_1 - p_1 - t\,[\,1 - x_{31}\,] \;=\; v_3 - p_3 - t\,[\,x_{31} - x_3\,]$$

$$\Rightarrow \quad 2\,t\,x_{31} \;=\; t\,[\,1 + x_3\,] + v_3 - v_1 + p_1 - p_3$$

$$\Rightarrow \qquad x_{31} \;=\; \frac{1}{2}\,[\,1 + x_3\,] + \frac{1}{2\,t}\,[\,v_3 - v_1 + p_1 - p_3\,]\,. \tag{5}$$

From (3) and (5), firm 1's profit is:

$$\pi_1 \;=\; [\,p_1 - c_1\,]\,[\,x_{12} + 1 - x_{31}\,]$$

$$=\; [\,p_1 - c_1\,]\,\{\,\frac{1}{2}\,x_2 + \frac{1}{2\,t}\,[\,v_1 - v_2 + p_2 - p_1\,] + \frac{1}{2} - \frac{1}{2}\,x_3 - \frac{1}{2\,t}\,[\,v_3 - v_1 + p_1 - p_3\,]\,\}$$

$$=\; [\,p_1 - c_1\,]\,\{\,\frac{1}{2}\,[\,x_2 + 1 - x_3] + \frac{1}{2\,t}\,[\,2\,v_1 - v_2 - v_3 + p_2 + p_3 - 2\,p_1\,]\,\}\,. \tag{6}$$

Differentiating (6) provides:

$$\frac{\partial \pi_1}{\partial p_1} \;=\; [\,p_1 - c_1\,]\left[-\frac{1}{t}\right] + \frac{1}{2}\,[\,x_2 + 1 - x_3\,] + \frac{1}{2\,t}\,[\,2\,v_1 - v_2 - v_3 + p_2 + p_3 - 2\,p_1\,] \;=\; 0$$

$$\Rightarrow \quad 2\,\frac{p_1}{t} \;=\; \frac{c_1}{t} + \frac{1}{2}\,[\,x_2 + 1 - x_3\,] + \frac{1}{2\,t}\,[\,2\,v_1 - v_2 - v_3 + p_2 + p_3\,]$$

$$\Rightarrow \quad 4\,p_1 \;=\; 2\,c_1 + t\,[\,x_2 + 1 - x_3\,] + \,2\,v_1 - v_2 - v_3 + p_2 + p_3$$

$$\Rightarrow \quad p_1 \;=\; \frac{1}{4}\,[\,2\,c_1 + t\,(\,x_2 + 1 - x_3\,) + \,2\,v_1 - v_2 - v_3 + p_2 + p_3\,]\,. \tag{7}$$

From (3) and (4), firm 1's profit is:

$$\pi_2 \;=\; [\,p_2 - c_2\,]\,[\,x_{23} - x_{12}\,]$$

$$=\; [\,p_2 - c_2\,]\,\{\,\frac{1}{2}\,[\,x_2 + x_3\,] + \frac{1}{2\,t}\,[\,v_2 - v_3 + p_3 - p_2\,] - \frac{1}{2}\,x_2 - \frac{1}{2\,t}\,[\,v_1 - v_2 + p_2 - p_1\,]\,\}$$

$$=\; [\,p_2 - c_2\,]\,\{\,\frac{1}{2}\,x_3 + \frac{1}{2\,t}\,[\,2\,v_2 - v_1 - v_3 + p_1 + p_3 - 2\,p_2\,]\,\}\,. \tag{8}$$

Differentiating (8) provides:

$$\frac{\partial \pi_2}{\partial p_2} \;=\; [\,p_2 - c_2\,]\left[-\frac{1}{t}\right] + \frac{1}{2}\,x_3 + \frac{1}{2\,t}\,[\,2\,v_2 - v_1 - v_3 + p_1 + p_3\,] - \frac{1}{t}\;p_2 \;=\; 0$$

$$\Rightarrow \quad p_2\,\frac{2}{t} \;=\; \frac{c_2}{t} + \frac{x_3}{2} + \frac{1}{2\,t}\,[\,2\,v_2 - v_1 - v_3 + p_1 + p_3\,]$$

$$\Rightarrow \quad p_2 \, \frac{4}{2\,t} \;=\; \frac{2\,c_2}{2\,t} + \frac{t\,x_3}{2\,t} + \frac{1}{2\,t}\,[\,2\,v_2 - v_1 - v_3 + p_1 + p_3\,]$$

$$\Rightarrow \quad p_2 \;=\; \frac{1}{4}\,[\,2\,c_2 + t\,x_3 + 2\,v_2 - v_1 - v_3 + p_1 + p_3\,]\;. \tag{9}$$

From (4) and (5), firm 3's profit is:

$$\begin{aligned}
\pi_3 \;&=\; [\,p_3 - c_3\,]\,[\,x_{31} - x_{23}\,] \\
&=\; [\,p_3 - c_3\,]\,\{\,\frac{1}{2}\,[\,1 + x_3\,] + \frac{1}{2\,t}\,[\,v_3 - v_1 + p_1 - p_3\,] \\
&\qquad\qquad - \frac{1}{2}\,[\,x_2 + x_3\,] - \frac{1}{2\,t}\,[\,v_2 - v_3 + p_3 - p_2\,]\,\} \\
&=\; [\,p_3 - c_3\,]\,\{\,\frac{1}{2}\,[\,1 - x_2] + \frac{1}{2\,t}\,[\,2\,v_3 - v_1 - v_2 + p_1 + p_2 - 2\,p_3\,]\,\}\,.
\end{aligned} \tag{10}$$

Differentiating (10) provides:

$$\frac{\partial \pi_3}{\partial p_3} \;=\; [\,p_3 - c_3\,]\left[-\frac{1}{t}\right] + \frac{1}{2}\,[\,1 - x_2\,] + \frac{1}{2\,t}\,[\,2\,v_3 - v_1 - v_2 + p_1 + p_2\,] - \frac{1}{t}\,p_3 \;=\; 0$$

$$\Rightarrow \quad p_3\,\frac{2}{t} \;=\; \frac{c_3}{t} + \frac{t}{2\,t}\,[\,1 - x_2\,] + \frac{1}{2\,t}\,[\,2\,v_3 - v_1 - v_2 + p_1 + p_2\,]$$

$$\Rightarrow \quad 4\,p_3 \;=\; 2\,c_3 + t\,[\,1 - x_2\,] + 2\,v_3 - v_1 - v_2 + p_1 + p_2$$

$$\Rightarrow \quad p_3 \;=\; \frac{1}{4}\,[\,2\,c_3 + t\,(\,1 - x_2\,) + 2\,v_3 - v_1 - v_2 + p_1 + p_2\,]\;. \tag{11}$$

From (9) and (11):

$$\begin{aligned}
p_2 + p_3 \;&=\; \frac{1}{4}\,[\,2\,c_2 + t\,x_3 + 2\,v_2 - v_1 - v_3\,] + \frac{1}{4}\,p_1 + \frac{1}{4}\,p_3 \\
&\quad +\; \frac{1}{4}\,[\,2\,c_3 + t\,(\,1 - x_2\,) + 2\,v_3 - v_1 - v_2\,] + \frac{1}{4}\,p_1 + \frac{1}{4}\,p_2
\end{aligned}$$

$$\Rightarrow \quad \frac{3}{4}\,[\,p_2 + p_3\,] \;=\; \frac{1}{4}\,[\,2\,c_2 + 2\,c_3 + t\,(\,1 - x_2 + x_3\,) + v_2 + v_3 - 2\,v_1\,] + \frac{2}{4}\,p_1$$

$$\Rightarrow \quad p_2 + p_3 \;=\; \frac{1}{3}\,[\,v_2 + v_3 - 2\,v_1 + 2\,c_2 + 2\,c_3 + t\,(\,1 - x_2 + x_3\,)\,] + \frac{2}{3}\,p_1\,. \tag{12}$$

(7) and (12) provide:

$$\begin{aligned}
p_1 \;&=\; \frac{1}{4}\,[\,2\,c_1 + t\,(\,x_2 + 1 - x_3\,) + \,2\,v_1 - v_2 - v_3\,] \\
&\quad +\; \frac{1}{12}\,[\,v_2 + v_3 - 2\,v_1 + 2\,c_2 + 2\,c_3 + t\,(\,1 - x_2 + x_3\,)\,] + \frac{2}{12}\,p_1
\end{aligned}$$

$$\Rightarrow \quad p_1 \left[1 - \frac{2}{12} \right] \;=\; \frac{1}{12} [\, 6\, v_1 - 3\, v_2 - 3\, v_3 + \, 6\, c_1 + t\, (\, 3\, x_2 + 3 - 3\, x_3\,)$$

$$-\;\; 2\, v_1 + v_2 + v_3 + 2\, c_2 + 2\, c_3 + t\, (\, 1 - x_2 + x_3\,)\,]$$

$$\Rightarrow \quad p_1 \;=\; \frac{1}{10} [\, 4\, v_1 - 2\, v_2 - 2\, v_3 + \, 6\, c_1 + 2\, c_2 + 2\, c_3 + t\, (\, 2\, x_2 + 4 - 2\, x_3\,)\,]$$

$$\Rightarrow \quad p_1^* \;=\; \frac{1}{5} [\, 2\, v_1 - v_2 - v_3 + 3\, c_1 + c_2 + c_3 + t\, (\, 2 + \, x_2 - x_3\,)\,]\, . \tag{13}$$

From (9) and (13):

$$p_2 \;=\; \frac{1}{4}\, p_1 + \frac{1}{4}\, p_3 + \frac{1}{4} [\, 2\, c_2 + t\, x_3 + 2\, v_2 - v_1 - v_3\,]$$

$$=\; \frac{1}{4}\, p_3 + \frac{1}{4} [\, 2\, c_2 + t\, x_3 + 2\, v_2 - v_1 - v_3\,]$$

$$+\;\; \frac{1}{20} [\, 2\, v_1 - v_2 - v_3 + 3\, c_1 + c_2 + c_3 + t\, (\, 2 + \, x_2 - x_3\,)\,]$$

$$=\; \frac{1}{4}\, p_3 + \frac{1}{20} [\, 10\, v_2 - 5\, v_1 - 5\, v_3 + 10\, c_2 + 5\, t\, x_3$$

$$+\;\; 2\, v_1 - v_2 - v_3 + 3\, c_1 + c_2 + c_3 + t\, (\, 2 + \, x_2 - x_3\,)\,]$$

$$=\; \frac{1}{4}\, p_3 + \frac{1}{20} [\, 9\, v_2 - 3\, v_1 - 6\, v_3 + 3\, c_1 + 11\, c_2 + c_3 + t\, (\, 2 + x_2 + 4\, x_3\,)\,]\, . \tag{14}$$

From (11) and (13):

$$p_3 \;=\; \frac{1}{4}\, p_2 + \frac{1}{4}\, p_1 + \frac{1}{4} [\, 2\, c_3 + t\, (\, 1 - x_2\,) + 2\, v_3 - v_1 - v_2\,]$$

$$=\; \frac{1}{4}\, p_2 + \frac{1}{4} [\, 2\, c_3 + t\, (\, 1 - x_2\,) + 2\, v_3 - v_1 - v_2\,]$$

$$+\;\; \frac{1}{20} [\, 2\, v_1 - v_2 - v_3 + 3\, c_1 + c_2 + c_3 + t\, (\, 2 + \, x_2 - x_3\,)\,]$$

$$=\; \frac{1}{4}\, p_2 + \frac{1}{20} [\, 10\, v_3 - 5\, v_1 - 5\, v_2 + 10\, c_3 + 5\, t\, (\, 1 - x_2\,)$$

$$+\;\; 2\, v_1 - v_2 - v_3 + 3\, c_1 + c_2 + c_3 + t\, (\, 2 + \, x_2 - x_3\,)\,]$$

$$=\; \frac{1}{4}\, p_2 + \frac{1}{20} [\, 9\, v_3 - 6\, v_2 - 3\, v_1 + 3\, c_1 + c_2 + 11\, c_3 + t\, (\, 7 - 4\, x_2 - x_3\,)]\, . \tag{15}$$

(14) and (15) imply:

$$p_2 \;=\; \frac{1}{20} [\, 9\, v_2 - 3\, v_1 - 6\, v_3 + 3\, c_1 + 11\, c_2 + c_3 + t\, (\, 2 + \, x_2 + 4\, x_3\,)\,]$$

$$+\;\; \frac{1}{16}\, p_2 + \frac{1}{80} [\, 9\, v_3 - 6\, v_2 - 3\, v_1 + 3\, c_1 + c_2 + 11\, c_3 + t\, (\, 7 - 4\, x_2 - x_3\,)\,]$$

$$\Rightarrow \quad \frac{15}{16}\, p_2 \;=\; \frac{1}{80}\,[\, 36\, v_2 - 12\, v_1 - 24\, v_3 + 12\, c_1 + 44\, c_2 + 4\, c_3 + 4\, t\,(\, 2 + \; x_2 + 4\, x_3\,)$$

$$+ \;\; 9\, v_3 - 6\, v_2 - 3\, v_1 + 3\, c_1 + c_2 + 11\, c_3 + t\,(\, 7 - 4\, x_2 - x_3\,)\,]$$

$$= \; \frac{1}{80}\,[\, 30\, v_2 - 15\, v_1 - 15\, v_3 + 15\, c_1 + 45\, c_2 + 15\, c_3 + t\,(\, 15 + 15\, x_3\,)\,]$$

$$\Rightarrow \quad p_2^* \;=\; \frac{1}{5}\,[\, 2\, v_2 - v_1 - v_3 + c_1 + 3\, c_2 + c_3 + t\,(\, 1 + x_3\,)\,]\,. \tag{16}$$

(15) and (16) imply:

$$p_3 \;=\; \frac{1}{20}\,[\, 9\, v_3 - 6\, v_2 - 3\, v_1 + 3\, c_1 + c_2 + 11\, c_3 + t\,(\, 7 - 4\, x_2 - x_3\,)\,]$$

$$+ \;\; \frac{1}{20}\,[\, 2\, v_2 - v_1 - v_3 + c_1 + 3\, c_2 + c_3 + t\,(\, 1 + x_3\,)\,]$$

$$= \; \frac{1}{20}\,[\, 8\, v_3 - 4\, v_2 - 4\, v_1 + 12\, c_3 + 4\, c_1 + 4\, c_2 + t\,(\, 8 - 4\, x_2\,)\,]$$

$$\Rightarrow \quad p_3^* \;=\; \frac{1}{5}\,[\, 2\, v_3 - v_1 - v_2 + c_1 + c_2 + 3\, c_3 + t\,(\, 2 - x_2\,)]\,. \tag{17}$$

Let $m_i \equiv v_i - c_i$ for $i = 1, 2, 3$. Then (13), (16), and (17) imply:

$$p_1^* - c_1 \;=\; \frac{1}{5}\,[\, 2\, v_1 - v_2 - v_3 + c_2 + c_3 - 2\, c_1 + t\,(\, 2 + x_2 - x_3\,)\,]$$

$$= \; \frac{1}{5}\,[\, 2\, m_1 - m_2 - m_3 + t\,(\, 2 + x_2 - x_3\,)\,]\,. \tag{18}$$

$$p_2^* - c_2 \;=\; \frac{1}{5}\,[\, 2\, v_2 - v_1 - v_3 + c_1 + c_3 - 2\, c_2 + t\,(\, 1 + x_3\,)\,]$$

$$= \; \frac{1}{5}\,[\, 2\, m_2 - m_1 - m_3 + t\,(\, 1 + x_3\,)\,]\,. \tag{19}$$

$$p_3^* - c_3 \;=\; \frac{1}{5}\,[\, 2\, v_3 - v_1 - v_2 + c_1 + c_2 - 2\, c_3 + t\,(\, 2 - x_2\,)\,]$$

$$= \; \frac{1}{5}\,[\, 2\, m_3 - m_1 - m_2 + t\,(\, 2 - x_2\,)\,]\,. \tag{20}$$

(13), (16), and (17) also imply:

$$p_2^* - p_1^* \;=\; \frac{1}{5}\,[\, 3\, v_2 - 3\, v_1 + 2\, c_2 - 2\, c_1 - t\,(\, 1 + x_2 - 2\, x_3\,)\,]\,. \tag{21}$$

$$p_3^* - p_2^* \;=\; \frac{1}{5}\,[\, 3\, v_3 - 3\, v_2 + 2\, c_3 - 2\, c_2 - t\,(\, x_2 + x_3 - 1\,)\,]\,. \tag{22}$$

$$p_1^* - p_3^* \;=\; \frac{1}{5}\,[\,3\,v_1 - 3\,v_3 + 2\,c_1 - 2\,c_3 - t\,(\,x_3 - 2\,x_2\,)\,]\,. \tag{23}$$

From (3) and (21):

$$\begin{aligned}
x_{12}^* \;&=\; \frac{1}{2}\,x_2 + \frac{1}{2\,t}\,[\,v_1 - v_2\,] + \frac{1}{10\,t}\,[\,3\,v_2 - 3\,v_1 + 2\,c_2 - 2\,c_1 - t\,(\,1 + x_2 - 2\,x_3\,)\,] \\
&=\; \frac{1}{10\,t}\,[\,2\,v_1 - 2\,v_2 + 2\,c_2 - 2\,c_1] + \frac{1}{10}\,[\,4\,x_2 + 2\,x_3 - 1\,] \\
&=\; \frac{1}{5\,t}\left[\,v_1 - v_2 + c_2 - c_1 + \frac{t}{2}\,(\,4\,x_2 + 2\,x_3 - 1\,)\right] \\
&=\; \frac{1}{5\,t}\left[\,m_1 - m_2 + \frac{t}{2}\,(\,4\,x_2 + 2\,x_3 - 1\,)\right].
\end{aligned} \tag{24}$$

From (4) and (22):

$$\begin{aligned}
x_{23}^* \;&=\; \frac{1}{2}\,[\,x_2 + x_3\,] + \frac{1}{2\,t}\,[\,v_2 - v_3\,] + \frac{1}{10\,t}\,[\,3\,v_3 - 3\,v_2 + 2\,c_3 - 2\,c_2 - t\,(\,x_2 + x_3 - 1\,)\,] \\
&=\; \frac{4\,t}{10\,t}\,[\,x_2 + x_3\,] + \frac{1}{10\,t}\,[\,2\,v_2 - 2\,v_3 + 2\,c_3 - 2\,c_2 + t\,] \\
&=\; \frac{1}{5\,t}\left[\,v_2 - v_3 + c_3 - c_2 + \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,)\right] \\
&=\; \frac{1}{5\,t}\left[\,m_2 - m_3 + \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,)\right].
\end{aligned} \tag{25}$$

From (5) and (23):

$$\begin{aligned}
x_{31}^* \;&=\; \frac{1}{2}\,[\,1 + x_3\,] + \frac{1}{2\,t}\,[\,v_3 - v_1\,] + \frac{1}{10\,t}\,[\,3\,v_1 - 3\,v_3 + 2\,c_1 - 2\,c_3 - t\,(\,x_3 - 2\,x_2\,)\,] \\
&=\; \frac{4\,t}{10\,t}\;x_3 + \frac{1}{10\,t}\,[\,2\,v_3 - 2\,v_1 + 2\,c_1 - 2\,c_3\,] + \frac{2\,t}{10\,t}\;x_2 + \frac{5\,t}{10\,t} \\
&=\; \frac{1}{5\,t}\left[\,v_3 - v_1 + c_1 - c_3 + \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5\,)\right] \\
&=\; \frac{1}{5\,t}\left[\,m_3 - m_1 + \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5\,)\right].
\end{aligned} \tag{26}$$

From (24), the x_{12}^* market boundary is well defined if:

$$0 \;<\; \frac{1}{5\,t}\,[\,m_1 - m_2] + \left[\frac{1}{5\,t}\right]\frac{t}{2}\,[\,4\,x_2 + 2\,x_3 - 1\,] \;\;<\;\; x_2$$

$$\Leftrightarrow \quad \frac{t}{2}[1-4x_2-2x_3] \; < \; m_1-m_2 \; < \; \frac{t}{2}[1+6x_2-2x_3]. \tag{27}$$

Similarly, from (25), the x^*_{23} market boundary is well defined if:

$$x_2 \; < \; \frac{1}{5t}[m_2-m_3]+\left[\frac{1}{5t}\right]\frac{t}{2}[4x_2+4x_3+1] \; < \; x_3$$

$$\Leftrightarrow \quad \frac{t}{2}[6x_2-4x_3-1] \; < \; m_2-m_3 \; < \; \frac{t}{2}[6x_3-4x_2-1]. \tag{28}$$

And, from (26), the x^*_{31} market boundary is well defined if:

$$x_3 \; < \; \frac{1}{5t}[m_3-m_1]+\left[\frac{1}{5t}\right]\frac{t}{2}[2x_2+4x_3+5] \; < \; 1$$

$$\Leftrightarrow \quad \frac{t}{2}[6x_3-2x_2-5] \; < \; m_3-m_1 \; < \; \frac{t}{2}[5-2x_2-4x_3]. \tag{29}$$

The inequalities in (27), (28), and (29) are assumed to hold throughout the ensuing analysis.

From (6) and (18):

$$\begin{aligned}\pi^*_1 &= [p^*_1-c_1][x^*_{12}+1-x^*_{31}] \\ &= \frac{1}{5}[2m_1-m_2-m_3+t(2+x_2-x_3)][x^*_{12}+1-x^*_{31}].\end{aligned} \tag{30}$$

From (24) and (26):

$$\begin{aligned}x^*_{12}+1-x^*_{31} &= \frac{1}{5t}[m_1-m_2]+\frac{1}{10}[4x_2+2x_3-1]+1 \\ &\quad -\frac{1}{5t}[m_3-m_1]-\frac{1}{10}[2x_2+4x_3+5] \\ &= \frac{1}{5t}[2m_1-m_2-m_3+t(2+x_2-x_3)].\end{aligned} \tag{31}$$

(30) and (31) provide:

$$\pi^*_1 = \frac{1}{25t}[2m_1-m_2-m_3+t(2+x_2-x_3)]^2. \tag{32}$$

From (8) and (19):

$$\pi^*_2 = [p^*_2-c_2][x^*_{23}-x^*_{12}] = \frac{1}{5}[2m_2-m_1-m_3+t(1+x_3)][x^*_{23}-x^*_{12}]. \tag{33}$$

From (24) and (25):

$$x^*_{23} - x^*_{12} \;=\; \frac{1}{5\,t}\left[m_2 - m_3 + \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,)\right] - \;\frac{1}{5\,t}\left[m_1 - m_2 + \frac{t}{2}\,(\,4\,x_2 + 2\,x_3 - 1\,)\right]$$

$$=\; \frac{1}{5\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,x_3 + 1\,)\,]\,. \tag{34}$$

(33) and (34) provide:

$$\pi^*_2 \;=\; \frac{1}{25\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,1 + x_3\,)\,]^2\;. \tag{35}$$

From (10) and (20):

$$\pi^*_3 \;=\; [\,p^*_3 - c_3\,]\,[\,x^*_{31} - x^*_{23}\,] \;\;=\;\; \frac{1}{5}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]\,[\,x^*_{31} - x^*_{23}\,]\,. \tag{36}$$

From (25) and (26):

$$x^*_{31} - x^*_{23} \;=\; \frac{1}{5\,t}\left[m_3 - m_1 + \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5\,)\right] - \;\frac{1}{5\,t}\left[m_2 - m_3 + \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,)\right]$$

$$=\; \frac{1}{5\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]\,. \tag{37}$$

(36) and (37) provide:

$$\pi^*_3 \;=\; \frac{1}{25\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]^2\;. \tag{38}$$

From (32), (35), and (38):

$$\frac{\partial \pi^*_1}{\partial k_1} \;=\; \frac{4}{25\,t}\,[\,2\,m_1 - m_2 - m_3 + t\,(\,2 + \,x_2 - x_3\,)\,]\;\frac{\partial m_1}{\partial k_1} \;>\; 0\,. \tag{39}$$

$$\frac{\partial \pi^*_1}{\partial k_2} \;=\; -\;\frac{2}{25\,t}\,[\,2\,m_1 - m_2 - m_3 + t\,(\,2 + \,x_2 - x_3\,)\,]\;\frac{\partial m_2}{\partial k_2} \;<\; 0\,. \tag{40}$$

$$\frac{\partial \pi^*_1}{\partial k_3} \;=\; -\;\frac{2}{25\,t}\,[\,2\,m_1 - m_2 - m_3 + t\,(\,2 + \,x_2 - x_3\,)\,]\;\frac{\partial m_3}{\partial k_3} \;<\; 0\,. \tag{41}$$

$$\frac{\partial \pi^*_2}{\partial k_2} \;=\; \frac{4}{25\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,1 + x_3\,)\,]\;\frac{\partial m_2}{\partial k_2} \;>\; 0\,. \tag{42}$$

$$\frac{\partial \pi^*_2}{\partial k_1} \;=\; -\;\frac{2}{25\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,1 + x_3\,)\,]\;\frac{\partial m_1}{\partial k_1} \;<\; 0\,. \tag{43}$$

$$\frac{\partial \pi^*_2}{\partial k_3} \;=\; -\;\frac{2}{25\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,1 + x_3\,)\,]\;\frac{\partial m_3}{\partial k_3} \;<\; 0\,. \tag{44}$$

$$\frac{\partial \pi_3^*}{\partial k_3} = \frac{4}{25\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]\,\frac{\partial m_3}{\partial k_3} > 0\,. \tag{45}$$

$$\frac{\partial \pi_3^*}{\partial k_1} = -\,\frac{2}{25\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]\,\frac{\partial m_1}{\partial k_1} < 0\,. \tag{46}$$

$$\frac{\partial \pi_3^*}{\partial k_2} = -\,\frac{2}{25\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]\,\frac{\partial m_2}{\partial k_2} < 0\,. \tag{47}$$

The inequalities in (39), (40), and (41) hold because, from (27) and (29):

$$\begin{aligned} 2\,m_1 - m_2 - m_3 + t\,[\,2 + x_2 - x_3\,] &= m_1 - m_2 - (m_3 - m_1) + t\,[\,2 + x_2 - x_3\,] \\ &> \frac{t}{2}\,[\,1 - 4\,x_2 - 2\,x_3\,] - \frac{t}{2}\,[\,5 - 2\,x_2 - 4\,x_3\,] + t\,[\,2 + x_2 - x_3\,] \\ &= \frac{t}{2}\,[\,-4 - 2\,x_2 + 2\,x_3 + 4 + 2\,x_2 - 2\,x_3\,] = 0\,. \end{aligned} \tag{48}$$

The inequalities in (42), (43), and (44) hold because, from (27) and (28):

$$\begin{aligned} 2\,m_2 - m_1 - m_3 + t\,[\,1 + x_3\,] &= m_2 - m_3 - (m_1 - m_2) + t\,[\,1 + x_3\,] \\ &> \frac{t}{2}\,[\,6\,x_2 - 4\,x_3 - 1\,] - \frac{t}{2}\,[\,1 + 6\,x_2 - 2\,x_3\,] + t\,[\,1 + x_3\,] \\ &= \frac{t}{2}\,[\,-2\,x_3 - 2\,] + \frac{t}{2}\,[\,2 + 2\,x_3\,] = 0\,. \end{aligned} \tag{49}$$

The inequalities in (45), (46), and(47) hold because, from (28) and (29):

$$\begin{aligned} 2\,m_3 - m_1 - m_2 + t\,[\,2 - x_2\,] &= m_3 - m_1 - (m_2 - m_3) + t\,[\,2 - x_2\,] \\ &> \frac{t}{2}\,[\,6\,x_3 - 2\,x_2 - 5\,] - \frac{t}{2}\,[\,6\,x_3 - 4\,x_2 - 1\,] + t\,[\,2 - x_2\,] \\ &= \frac{t}{2}\,[\,2\,x_2 - 4\,] + \frac{t}{2}\,[\,4 - 2\,x_2\,] = 0\,. \end{aligned} \tag{50}$$

From (18), (39), (40), and (41), the rate at which firm 1's profit increases as it acquires the input increment and thereby precludes firm 2 from acquiring the increment with probability α_{12} and precludes firm 3 from acquiring the increment with probability α_{13} is:

$$\begin{aligned} B_1 &\equiv \frac{\partial \pi_1^*}{\partial k_1} - \alpha_{12}\,\frac{\partial \pi_1^*}{\partial k_2} - \alpha_{13}\,\frac{\partial \pi_1^*}{\partial k_3} \\ &= \frac{2}{25\,t}\,[\,2\,m_1 - m_2 - m_3 + t\,(\,2 + x_2 - x_3\,)\,]\left[\,2\,\frac{\partial m_1}{\partial k_1} + \alpha_{12}\,\frac{\partial m_2}{\partial k_2} + \alpha_{13}\,\frac{\partial m_3}{\partial k_3}\,\right] \end{aligned} \tag{51}$$

$$= \frac{2}{5\,t}\,[\,p_1^* - c_1\,]\left[\,2\,\frac{\partial m_1}{\partial k_1} + \alpha_{12}\,\frac{\partial m_2}{\partial k_2} + \alpha_{13}\,\frac{\partial m_3}{\partial k_3}\,\right]. \tag{52}$$

From (19), (42), (43), and (44), the rate at which firm 2's profit increases as it acquires the input increment and thereby precludes firm 1 from acquiring the increment with probability α_{21} and precludes firm 3 from acquiring the increment with probability α_{23} is:

$$B_2 \equiv \frac{\partial \pi_2^*}{\partial k_2} - \alpha_{21}\,\frac{\partial \pi_2^*}{\partial k_1} - \alpha_{23}\,\frac{\partial \pi_2^*}{\partial k_3}$$

$$= \frac{2}{25\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,1 + x_3\,)\,]\left[\,2\,\frac{\partial m_2}{\partial k_2} + \alpha_{21}\,\frac{\partial m_1}{\partial k_1} + \alpha_{23}\,\frac{\partial m_3}{\partial k_3}\,\right] \tag{53}$$

$$= \frac{2}{5\,t}\,[\,p_2^* - c_2\,]\left[\,2\,\frac{\partial m_2}{\partial k_2} + \alpha_{21}\,\frac{\partial m_1}{\partial k_1} + \alpha_{23}\,\frac{\partial m_3}{\partial k_3}\,\right]. \tag{54}$$

From (20), (45), (46), and (47), the rate at which firm 3's profit increases as it acquires the input increment and thereby precludes firm 1 from acquiring the increment with probability α_{31} and precludes firm 2 from acquiring the increment with probability α_{33} is:

$$B_3 \equiv \frac{\partial \pi_3^*}{\partial k_3} - \alpha_{31}\,\frac{\partial \pi_3^*}{\partial k_1} - \alpha_{32}\,\frac{\partial \pi_3^*}{\partial k_2}$$

$$= \frac{2}{25\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,2 - x_2\,)\,]\left[\,2\,\frac{\partial m_3}{\partial k_3} + \alpha_{31}\,\frac{\partial m_1}{\partial k_1} + \alpha_{32}\,\frac{\partial m_2}{\partial k_2}\,\right] \tag{55}$$

$$= \frac{2}{5\,t}\,[\,p_3^* - c_3\,]\left[\,2\,\frac{\partial m_3}{\partial k_3} + \alpha_{31}\,\frac{\partial m_1}{\partial k_1} + \alpha_{32}\,\frac{\partial m_2}{\partial k_2}\,\right]. \tag{56}$$

Observe that:

$$2 + x_2 - x_3 \;=\; 1 + (\,1 - x_3\,) + (\,x_2 - 0\,) \;=\; 1 + L_1\,; \tag{57}$$

$$1 + x_3 \;=\; 1 + (\,x_2 - 0\,) + (\,x_3 - x_2\,) \;=\; 1 + L_2\,; \quad \text{and} \tag{58}$$

$$2 - x_2 \;=\; 1 + (\,1 - x_3\,) + (\,x_3 - x_2\,) \;=\; 1 + L_3\,, \tag{59}$$

where L_i is the sum of the distances between firm i and each of its rivals along the circle circumference.

Using (57) – (59), (51) – (55) can be written as:

$$B_1 = \frac{2}{25\,t}\,[\,2\,m_1 - m_2 - m_3 + t\,(\,1 + L_1\,)\,]\left[\,2\,\frac{\partial m_1}{\partial k_1} + \alpha_{12}\,\frac{\partial m_2}{\partial k_2} + \alpha_{13}\,\frac{\partial m_3}{\partial k_3}\,\right], \tag{60}$$

$$B_2 = \frac{2}{25\,t}\,[\,2\,m_2 - m_1 - m_3 + t\,(\,1 + L_2\,)\,]\left[\,2\,\frac{\partial m_2}{\partial k_2} + \alpha_{21}\,\frac{\partial m_1}{\partial k_1} + \alpha_{23}\,\frac{\partial m_3}{\partial k_3}\,\right], \text{ and} \tag{61}$$

$$B_3 \;=\; \frac{2}{25\,t}\,[\,2\,m_3 - m_1 - m_2 + t\,(\,1 + L_3\,)\,]\left[\,2\,\frac{\partial m_3}{\partial k_3} + \alpha_{31}\,\frac{\partial m_1}{\partial k_1} + \alpha_{32}\,\frac{\partial m_2}{\partial k_2}\,\right]. \tag{62}$$

Observe that if a firm acquires an input increment, it precludes one of its rivals from acquiring the increment with probability 1 (provided the increment is always sold, as we assume to be the case). Therefore, $\alpha_{12} + \alpha_{13} \;=\; \alpha_{21} + \alpha_{23} \;=\; \alpha_{31} + \alpha_{32} \;=\; 1$.

$$\begin{aligned}
W^* \;=\; & \int_{x_{31}^*}^{1} [\,v_1 - c_1 - t\,(\,1 - x\,)\,]\,dx \;+\; \int_{0}^{x_{12}^*} [\,v_1 - c_1 - t\,x\,]\,dx \\
& + \int_{x_{12}^*}^{x_2} [\,v_2 - c_2 - t\,(\,x_2 - x\,)\,]\,dx \;+\; \int_{x_2}^{x_{23}^*} [\,v_2 - c_2 - t\,(\,x - x_2\,)\,]\,dx \\
& + \int_{x_{23}^*}^{x_3} [\,v_3 - c_3 - t\,(\,x_3 - x\,)\,]\,dx \;+\; \int_{x_3}^{x_{31}^*} [\,v_3 - c_3 - t\,(\,x - x_3\,)\,]\,dx
\end{aligned} \tag{63}$$

$$\begin{aligned}
= \;& [\,v_1 - c_1\,]\,[\,1 - x_{31}^* + x_{12}^*\,] \,-\, t\;x\Big|_{x_{31}^*}^{1} \,+\, \frac{t}{2}\;x^2\Big|_{x_{31}^*}^{1} \,-\, \frac{t}{2}\;x^2\Big|_{0}^{x_{12}^*} \\
& + \;[\,v_2 - c_2\,]\,[\,x_{23}^* - x_{12}^*\,] \,-\, t\;x_2\;x\Big|_{x_{12}^*}^{x_2} \,+\, \frac{t}{2}\;x^2\Big|_{x_{12}^*}^{x_2} \,-\, \frac{t}{2}\;x^2\Big|_{x_2}^{x_{23}^*} \,+\, t\;x_2\;x\Big|_{x_2}^{x_{23}^*} \\
& + \;[\,v_3 - c_3\,]\,[\,x_{31}^* - x_{23}^*\,] \,-\, t\;x_3\;x\Big|_{x_{23}^*}^{x_3} \,+\, \frac{t}{2}\;x^2\Big|_{x_{23}^*}^{x_3} \,-\, \frac{t}{2}\;x^2\Big|_{x_3}^{x_{31}^*} \,+\, t\;x_3\;x\Big|_{x_3}^{x_{31}^*}
\end{aligned}$$

$$\begin{aligned}
= \;& [\,v_1 - c_1\,]\,[\,1 - x_{31}^* + x_{12}^*\,] - t\,[\,1 - x_{31}^*\,] + \frac{t}{2}\left[\,1 - (x_{31}^*)^2\,\right] - \frac{t}{2}\,(x_{12}^*)^2 \\
& + \;[\,v_2 - c_2\,]\,[\,x_{23}^* - x_{12}^*\,] - t\;x_2\,[\,x_2 - x_{12}^*\,] + \frac{t}{2}\left[\,(x_2)^2 - (x_{12}^*)^2\,\right] \\
& \qquad - \frac{t}{2}\left[\,(x_{23}^*)^2 - (x_2)^2\,\right] + t\;x_2\,[\,x_{23}^* - x_2\,] \\
& + \;[\,v_3 - c_3\,]\,[\,x_{31}^* - x_{23}^*\,] \,-\, t\;x_3\,[\,x_3 - x_{23}^*\,] \,+\, \frac{t}{2}\left[\,(x_3)^2 - (x_{23}^*)^2\,\right] \\
& \qquad - \frac{t}{2}\;\left[\,(x_{31}^*)^2 - (x_3)^2\,\right] + \,t\;x_3\;[\,x_{31}^* - x_3\,]
\end{aligned}$$

$$= \;[\,v_1 - c_1\,]\,[\,1 - x_{31}^* + x_{12}^*\,] + [\,v_2 - c_2\,]\,[\,x_{23}^* - x_{12}^*\,] + [\,v_3 - c_3\,]\,[\,x_{31}^* - x_{23}^*\,]$$

$$
\begin{aligned}
&- \; t + t\, x_{31}^* + \frac{t}{2} - \frac{t}{2}\,(x_{31}^*)^2 - \frac{t}{2}\,(x_{12}^*)^2 - t\,(x_2)^2 + t\; x_2\, x_{12}^* + \frac{t}{2}\,(x_2)^2 \\
&- \; \frac{t}{2}\,(x_{12}^*)^2 - \frac{t}{2}\,(x_{23}^*)^2 + \frac{t}{2}\,(x_2)^2 + t\; x_2\, x_{23}^* - t\,(x_2)^2 - t\,(x_3)^2 + t\; x_3\, x_{23}^* \\
&+ \; \frac{t}{2}\,(x_3)^2 - \frac{t}{2}\,(x_{23}^*)^2 - \frac{t}{2}\,(x_{31}^*)^2 + \frac{t}{2}\,(x_3)^2 + t\; x_3\, x_{31}^* - t\,(x_3)^2 \\
= \;& [\, v_1 - c_1\,]\,[\, 1 - x_{31}^* + x_{12}^*\,] + [\, v_2 - c_2\,]\,[\, x_{23}^* - x_{12}^*\,] + [\, v_3 - c_3\,]\,[\, x_{31}^* - x_{23}^*\,] \\
&- \; \frac{t}{2} + t\, x_{31}^* - t\,(x_{31}^*)^2 - t\,(x_{12}^*)^2 + t\; x_2\, x_{12}^* - t\,(x_{23}^*)^2 + t\; x_2\, x_{23}^* \\
&- \; t\,(x_2)^2 - t\,(x_3)^2 + t\; x_3\, x_{23}^* + t\; x_3\, x_{31}^* \\
= \;& [\, v_1 - c_1\,]\,[\, 1 - x_{31}^* + x_{12}^*\,] + [\, v_2 - c_2\,]\,[\, x_{23}^* - x_{12}^*\,] + [\, v_3 - c_3\,]\,[\, x_{31}^* - x_{23}^*\,] \\
&- \; t\left[\frac{1}{2} + (x_2)^2 + (x_3)^2\right] + t\; x_{31}^*\,[\, 1 - x_{31}^* + x_3\,] \\
&+ \; t\; x_{12}^*\,[\, x_2 - x_{12}^*\,] + t\; x_{23}^*\,[\, x_2 + x_3 - x_{23}^*\,]\,. \qquad (64)
\end{aligned}
$$

From (24):

$$
\begin{aligned}
x_2 - x_{12}^* \;&=\; \frac{1}{5\,t}\left[5\,t\,x_2 - v_1 + v_2 - c_2 + c_1 - \frac{t}{2}\,(\,4\,x_2 + 2\,x_3 - 1\,)\right] \\
&=\; \frac{1}{5\,t}\left[v_2 - v_1 + c_1 - c_2 + \frac{t}{2}\,(\,1 + 6\,x_2 - 2\,x_3\,)\right]. \qquad (65)
\end{aligned}
$$

From (25):

$$
\begin{aligned}
x_2 + x_3 - x_{23}^* \;&=\; \frac{1}{5\,t}\left[5\,t\,x_2 + 5\,t\,x_3 - v_2 + v_3 - c_3 + c_2 - \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,)\right] \\
&=\; \frac{1}{5\,t}\left[v_3 - v_2 + c_2 - c_3 + \frac{t}{2}\,(\,6\,x_2 + 6\,x_3 - 1\,)\right]. \qquad (66)
\end{aligned}
$$

From (26):

$$
\begin{aligned}
1 - x_{31}^* + x_3 \;&=\; \frac{1}{5\,t}\left[5\,t + 5\,t\,x_3 - v_3 + v_1 - c_1 + c_3 - \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5\,)\right] \\
&=\; \frac{1}{5\,t}\left[v_1 - v_3 + c_3 - c_1 - \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5 - 10 - 10\,x_3\,)\right] \\
&=\; \frac{1}{5\,t}\left[v_1 - v_3 + c_3 - c_1 + \frac{t}{2}\,(\,5 + 6\,x_3 - 2\,x_2\,)\right]. \qquad (67)
\end{aligned}
$$

(24), (25), (26), (31), (34), (37), (64), (67), (65), and (66) provide:

$$
\begin{aligned}
W^* \;=\; & [\, v_1 - c_1 \,] \, \frac{1}{5\,t} \, [\, 2\,v_1 - v_2 - v_3 + c_2 + c_3 - 2\,c_1 + t\,(\,2 + \, x_2 - x_3\,)\,] \\
& + \; [\, v_2 - c_2 \,] \, \frac{1}{5\,t} \, [\, 2\,v_2 - v_1 - v_3 + c_1 + c_3 - 2\,c_2 + t\,(\,x_3 + 1\,)\,] \\
& + \; [\, v_3 - c_3 \,] \, \frac{1}{5\,t} \, [\, 2\,v_3 - v_1 - v_2 + c_1 + c_2 - 2\,c_3 + t\,(\,2 - x_2\,)\,] - \; t \left[\frac{1}{2} + (x_2)^2 + (x_3)^2 \right] \\
& + \; t \; \frac{1}{5\,t} \left[v_3 - v_1 + c_1 - c_3 + \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5\,) \right] \\
& \qquad \cdot \frac{1}{5\,t} \left[v_1 - v_3 + c_3 - c_1 + \frac{t}{2}\,(\,5 + 6\,x_3 - 2\,x_2\,) \right] \\
& + \; t \; \frac{1}{5\,t} \left[v_1 - v_2 + c_2 - c_1 + \frac{t}{2}\,(\,4\,x_2 + 2\,x_3 - 1\,) \right] \\
& \qquad \cdot \frac{1}{5\,t} \left[v_2 - v_1 + c_1 - c_2 + \frac{t}{2}\,(\,1 + 6\,x_2 - 2\,x_3\,) \right] \\
& + \; t \; \frac{1}{5\,t} \left[v_2 - v_3 + c_3 - c_2 + \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,) \right] \\
& \qquad \cdot \frac{1}{5\,t} \left[v_3 - v_2 + c_2 - c_3 + \frac{t}{2}\,(\,6\,x_2 + 6\,x_3 - 1\,) \right] .
\end{aligned} \tag{68}
$$

Recall that $m_i \equiv v_i - c_i$. Consequently, from (68):

$$
\begin{aligned}
W^* \;=\; & \frac{1}{5\,t} \, \{\, m_1\,[\,2\,m_1 - m_2 - m_3\,] + m_2\,[\,2\,m_2 - m_1 - m_3\,] + m_3\,[\,2\,m_3 - m_1 - m_2\,]\,\} \\
& + \; \frac{1}{5} \, \{\, m_1\,[\,2 + \, x_2 - x_3\,] + m_2\,[\,x_3 + 1\,] + m_3\,[\,2 - x_2\,]\,\} - \; t \left[\frac{1}{2} + (x_2)^2 + (x_3)^2 \right] \\
& + \; \frac{1}{25\,t} \, \Big\{ \left[m_3 - m_1 + \frac{t}{2}\,(\,2\,x_2 + 4\,x_3 + 5\,) \right] \left[m_1 - m_3 + \frac{t}{2}\,(\,5 + 6\,x_3 - 2\,x_2\,) \right] \\
& \qquad + \left[m_1 - m_2 + \frac{t}{2}\,(\,4\,x_2 + 2\,x_3 - 1\,) \right] \left[m_2 - m_1 + \frac{t}{2}\,(\,1 + 6\,x_2 - 2\,x_3\,) \right] \\
& \qquad + \left[m_2 - m_3 + \frac{t}{2}\,(\,4\,x_2 + 4\,x_3 + 1\,) \right] \left[m_3 - m_2 + \frac{t}{2}\,(\,6\,x_2 + 6\,x_3 - 1\,) \right] \Big\} \\
=\; & \frac{2}{5\,t} \, \{\, (m_1)^2 + (m_2)^2 + (m_3)^2 - m_1\,m_2 - m_1\,m_3 \, - m_2\,m_3 \,\} \\
& + \; \frac{1}{5} \, \{\, m_1\,[\,2 + \, x_2 - x_3\,] + m_2\,[\,x_3 + 1\,] + m_3\,[\,2 - x_2\,]\,\} - \; t \left[\frac{1}{2} + (x_2)^2 + (x_3)^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{25\,t}\,\{\,(\,m_3 - m_1\,)^2 + (\,m_2 - m_1\,)^2 + (\,m_3 - m_2\,)^2\,\} \\
&+ \frac{1}{50}\,\{\,[\,m_3 - m_1\,]\,[\,5 + 6\,x_3 - 2\,x_2\,] + [\,m_1 - m_3\,]\,[\,2\,x_2 + 4\,x_3 + 5\,] \\
&\qquad + [\,m_1 - m_2]\,[\,1 + 6\,x_2 - 2\,x_3\,] + [\,m_2 - m_1]\,[\,4\,x_2 + 2\,x_3 - 1\,] \\
&\qquad + [\,m_2 - m_3]\,[\,6\,x_2 + 6\,x_3 - 1\,] + [\,m_3 - m_2]\,[\,4\,x_2 + 4\,x_3 + 1\,]\,\} \\
&+ \frac{t}{100}\,\{\,[\,5 + 6\,x_3 - 2\,x_2\,]\,[\,2\,x_2 + 4\,x_3 + 5\,] + [\,1 + 6\,x_2 - 2\,x_3\,]\,[\,4\,x_2 + 2\,x_3 - 1\,] \\
&\qquad\qquad + [\,6\,x_2 + 6\,x_3 - 1\,]\,[\,4\,x_2 + 4\,x_3 + 1\,]\,\}
\end{aligned}
$$

$$
\begin{aligned}
= \; &\frac{2}{5\,t}\,\{\,(m_1)^2 + (m_2)^2 + (m_3)^2 - m_1\,m_2 - m_1\,m_3 - m_2\,m_3\,\} - t\left[\frac{1}{2} + (x_2)^2 + (x_3)^2\right] \\
&+ \frac{1}{5}\,\{\,2\,m_1 + m_1\,x_2 - m_1\,x_3 + m_2\,x_3 + m_2 + 2\,m_3 - m_3\,x_2\,\} \\
&- \frac{2}{25\,t}\,\{\,(m_1)^2 + (m_2)^2 + (m_3)^2 - m_1\,m_3 - m_1\,m_2 - m_2\,m_3\,\} \\
&+ \frac{1}{50}\,\{\,[\,m_3 - m_1\,]\,[\,5 + 6\,x_3 - 2\,x_2 - 2\,x_2 - 4\,x_3 - 5\,] \\
&\qquad + [\,m_2 - m_1\,]\,[\,4\,x_2 + 2\,x_3 - 1 - 1 - 6\,x_2 + 2\,x_3\,] \\
&\qquad + [\,m_3 - m_2\,]\,[\,4\,x_2 + 4\,x_3 + 1 - 6\,x_2 - 6\,x_3 + 1\,]\,\} \\
&+ \frac{t}{100}\,\{\,4\,x_2\,x_3 - 4\,(x_2)^2 + 50\,x_3 + 24\,(x_3)^2 + 25 \\
&\qquad\qquad - 2\,x_2 + 24\,(x_2)^2 + 4\,x_2\,x_3 + 4\,x_3 - 4\,(x_3)^2 - 1 \\
&\qquad\qquad + 24\,(x_2)^2 + 48\,x_2\,x_3 + 24\,(x_3)^2 + 2\,x_2 + 2\,x_3 - 1\,\}
\end{aligned}
$$

$$
\begin{aligned}
= \; &\frac{8}{25\,t}\,\{\,(m_1)^2 + (m_2)^2 + (m_3)^2 - m_1\,m_2 - m_1\,m_3 - m_2\,m_3\,\} - t\left[\frac{1}{2} + (x_2)^2 + (x_3)^2\right] \\
&+ \frac{1}{50}\,\{\,20\,m_1 + 10\,m_1\,x_2 - 10\,m_1\,x_3 + 10\,m_2\,x_3 + 10\,m_2 + 20\,m_3 - 10\,m_3\,x_2 \\
&\qquad + [\,m_3 - m_1\,]\,[\,2\,x_3 - 4\,x_2\,] + [\,m_2 - m_1\,]\,[\,4\,x_3 - 2\,x_2 - 2\,] \\
&\qquad + [\,m_3 - m_2\,]\,[\,2 - 2\,x_2 - 2\,x_3\,]\,\} \\
&+ \frac{t}{100}\,[\,56\,x_2\,x_3 + 44\,(x_2)^2 + 48\,(x_3)^2 + 50\,x_3 + 56\,x_3 + 23\,]
\end{aligned}
$$

$$= \frac{8}{25\,t}\,\{\,(m_1)^2 + (m_2)^2 + (m_3)^2 - m_1\,m_2 - m_1\,m_3\; - m_2\,m_3\,\} - \;t\left[\frac{1}{2} + (x_2)^2 + (x_3)^2\right]$$

$$+ \;\frac{1}{50}\,\{\,20\,m_1 + 10\,m_1\,x_2 - 10\,m_1\,x_3\; + 10\,m_2\,x_3 + 10\,m_2 + 20\,m_3 - 10\,m_3\,x_2$$

$$+ \;2\,m_3\,x_3 - 4\,m_3\,x_2 - 2\,m_1\,x_3 + 4\,m_1\,x_2 + 4\,m_2\,x_3 - 2\,m_2\,x_2 - 2\,m_2$$

$$- \;4\,m_1\,x_3 + 2\,m_1\,x_2 + 2\,m_1 + 2\,m_3 - 2\,m_3\,x_2$$

$$- \;2\,m_3\,x_3 - 2\,m_2 + 2\,m_2\,x_2 + 2\,m_2\,x_3\,\}$$

$$+ \;\frac{t}{100}\,[\,56\,x_2\,x_3 + 44\,(x_2)^2 + 48\,(x_3)^2 + 50\,x_3 + 56\,x_3 + 23\,]$$

$$= \frac{1}{25\,t}\,[\,8\,(m_1)^2 + 8\,(m_2)^2 + 8\,(m_3)^2 - 8\,m_1\,m_2 - 8\,m_1\,m_3\; - 8\,m_2\,m_3\,]$$

$$+ \;\frac{1}{50}\,[\,22\,m_1 + 6\,m_2 + 22\,m_3 + 16\,m_1\,x_2 - 16\,m_1\,x_3 + 16\,m_2\,x_3 - 16\,m_3\,x_2\,]$$

$$+ \;\frac{t}{100}\,[\,56\,x_2\,x_3 + 44\,(x_2)^2 + 48\,(x_3)^2 + 50\,x_3 + 56\,x_3 + 23\,] - \;t\left[\frac{1}{2} + (x_2)^2 + (x_3)^2\right]$$

$$\Rightarrow \;\; W^* \;=\; \frac{1}{25\,t}\,[\,8\,(m_1)^2 + 8\,(m_2)^2 + 8\,(m_3)^2 - 8\,m_1\,m_2 - 8\,m_1\,m_3\; - 8\,m_2\,m_3\,]$$

$$+ \;t\,[\,11\,m_1 + 3\,m_2 + 11\,m_3 + 8\,m_1\,x_2 - 8\,m_1\,x_3 + 8\,m_2\,x_3 - 8\,m_3\,x_2\,]$$

$$+ \;\frac{t}{100}\,[\,56\,x_2\,x_3 + 44\,(x_2)^2 + 48\,(x_3)^2 + 50\,x_3 + 56\,x_3 + 23\,]$$

$$- \;t\left[\frac{1}{2} + (x_2)^2 + (x_3)^2\right]. \tag{69}$$

Differentiating (69) provides:

$$\frac{\partial W^*}{\partial k_1} \;=\; \frac{1}{25\,t}\,[\,16\,m_1 - 8\,(\,m_2 + m_3\,) + t\,(\,11 + 8\,x_2 - 8\,x_3\,)\,]\,\frac{\partial m_1}{\partial k_1}\,. \tag{70}$$

$$\frac{\partial W^*}{\partial k_2} \;=\; \frac{1}{25\,t}\,[\,16\,m_2 - 8\,(\,m_1 + m_3\,) + t\,(\,3 + 8\,x_3\,)\,]\,\frac{\partial m_2}{\partial k_2}\,. \tag{71}$$

$$\frac{\partial W^*}{\partial k_3} \;=\; \frac{1}{25\,t}\,[\,16\,m_3 - 8\,(\,m_1 + m_2\,) + t\,(\,11 - 8\,x_2\,)\,]\,\frac{\partial m_3}{\partial k_3}\,. \tag{72}$$

(70), (71), and (72) imply that the rate at which welfare increases when firm 1 acquires the input increment and thereby precludes firm 2 from acquiring the increment with probability

α_{12} and precludes firm 3 from acquiring the increment with probability α_{13} is:

$$\begin{aligned} G_1 \equiv & \frac{\partial W^*}{\partial k_1} - \alpha_{12} \frac{\partial W^*}{\partial k_2} - \alpha_{13} \frac{\partial W^*}{\partial k_3} \\ = & \frac{1}{25\,t} [\, 16\, m_1 - 8\,(\, m_2 + m_3\,) + t\,(\, 11 + 8\, x_2 - 8\, x_3\,)\,] \frac{\partial m_1}{\partial k_1} \\ & - \frac{\alpha_{12}}{25\,t} [\, 16\, m_2 - 8\,(\, m_1 + m_3\,) + t\,(\, 3 + 8\, x_3\,)\,] \frac{\partial m_2}{\partial k_2} \\ & - \frac{\alpha_{13}}{25\,t} [\, 16\, m_3 - 8\,(\, m_1 + m_2\,) + t\,(\, 11 - 8\, x_2\,)\,] \frac{\partial m_3}{\partial k_3} \,. \end{aligned} \tag{73}$$

Similarly, the rate at which welfare increases when firm 2 acquires the input increment and thereby precludes firm 1 from acquiring the increment with probability α_{21} and precludes firm 3 from acquiring the increment with probability α_{23} is:

$$\begin{aligned} G_2 \equiv & \frac{\partial W^*}{\partial k_2} - \alpha_{21} \frac{\partial W^*}{\partial k_1} - \alpha_{23} \frac{\partial W^*}{\partial k_3} \\ = & \frac{1}{25\,t} [\, 16\, m_2 - 8\,(\, m_1 + m_3\,) + t\,(\, 3 + 8\, x_3\,)\,] \frac{\partial m_2}{\partial k_2} \\ & - \frac{\alpha_{21}}{25\,t} [\, 16\, m_1 - 8\,(\, m_2 + m_3\,) + t\,(\, 11 + 8\, x_2 - 8\, x_3\,)\,] \frac{\partial m_1}{\partial k_1} \\ & - \frac{\alpha_{23}}{25\,t} [\, 16\, m_3 - 8\,(\, m_1 + m_2\,) + t\,(\, 11 - 8\, x_2\,)\,] \frac{\partial m_3}{\partial k_3} \,. \end{aligned} \tag{74}$$

Also, the rate at which welfare increases when firm 3 acquires the input increment and thereby precludes firm 1 from acquiring the increment with probability α_{31} and precludes firm 2 from acquiring the increment with probability α_{32} is:

$$\begin{aligned} G_3 \equiv & \frac{\partial W^*}{\partial k_3} - \alpha_{31} \frac{\partial W^*}{\partial k_1} - \alpha_{32} \frac{\partial W^*}{\partial k_2} \\ = & \frac{1}{25\,t} [\, 16\, m_3 - 8\,(\, m_1 + m_2\,) + t\,(\, 11 - 8\, x_2\,)\,] \frac{\partial m_3}{\partial k_3} \\ & - \frac{\alpha_{31}}{25\,t} [\, 16\, m_1 - 8\,(\, m_2 + m_3\,) + t\,(\, 11 + 8\, x_2 - 8\, x_3\,)\,] \frac{\partial m_1}{\partial k_1} \\ & - \frac{\alpha_{32}}{25\,t} [\, 16\, m_2 - 8\,(\, m_1 + m_3\,) + t\,(\, 3 + 8\, x_3\,)\,] \frac{\partial m_2}{\partial k_2} \,. \end{aligned} \tag{75}$$

Special Case: $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2} = \frac{\partial m_3}{\partial k_3} > 0$

(60), (61), and (62) imply that in this case:

$$B_1 = \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[m_1 - \frac{1}{2}(m_2 + m_3) + \frac{t}{2}(1 + L_1)\right], \tag{76}$$

$$B_2 = \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[m_2 - \frac{1}{2}(m_1 + m_3) + \frac{t}{2}(1 + L_2)\right], \text{ and} \tag{77}$$

$$B_3 = \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[m_3 - \frac{1}{2}(m_1 + m_2) + \frac{t}{2}(1 + L_3)\right]. \tag{78}$$

(76) and (77) imply that in this case:

$$\begin{aligned} B_1 - B_2 &= \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}[\, m_1 - \frac{1}{2}(m_2 + m_3) + \frac{t}{2}(1 + L_1) \\ &\qquad - \; m_2 + \frac{1}{2}(m_1 + m_3) - \frac{t}{2}(1 + L_2)\,] \\ &= \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[\frac{3}{2}(m_1 - m_2) + \frac{t}{2}(L_1 - L_2)\right] \\ &= \frac{18}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[m_1 + \frac{t}{3}L_1 - \left(m_2 + \frac{t}{3}L_2\right)\right]. \end{aligned} \tag{79}$$

(76) and (78) imply that in this case:

$$\begin{aligned} B_1 - B_3 &= \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}[\, m_1 - \frac{1}{2}(m_2 + m_3) + \frac{t}{2}(1 + L_1) \\ &\qquad - \; m_3 + \frac{1}{2}(m_1 + m_2) - \frac{t}{2}(1 + L_3)\,] \\ &= \frac{12}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[\frac{3}{2}(m_1 - m_3) + \frac{t}{2}(L_1 - L_3)\right] \\ &= \frac{18}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[m_1 + \frac{t}{3}L_1 - \left(m_3 + \frac{t}{3}L_3\right)\right]. \end{aligned} \tag{80}$$

(79) and (80) imply that in this case:

$$B_2 - B_3 = B_1 - B_3 - (B_1 - B_2) = \frac{18}{25\,t}\,\frac{\partial m_i}{\partial k_i}\left[m_2 + \frac{t}{3}L_2 - \left(m_3 + \frac{t}{3}L_3\right)\right]. \tag{81}$$

(57) – (59) and (73) – (75) imply that in this case:

$$\begin{aligned}
G_1 \;&=\; \frac{1}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,16\,m_1 - 8\,(\,m_2 + m_3\,) + t\,(\,3 + 8\,L_1\,) \\
&\qquad - \alpha_{12}\,[\,16\,m_2 - 8\,(\,m_1 + m_3\,) + t\,(\,3 + 8\,L_2\,)\,] \\
&\qquad - \alpha_{13}\,[\,16\,m_3 - 8\,(\,m_1 + m_2\,) + t\,(\,3 + 8\,L_3\,)\,]\,\} \\
&= \frac{8}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,2\,m_1 - m_2 - m_3 + t\,L_1 - \alpha_{12}\,[\,2\,m_2 - m_1 - m_3 + t\,L_2\,] \\
&\qquad - \alpha_{13}\,[\,2\,m_3 - m_1 - m_2 + t\,L_3\,]\,\} \\
&= \frac{8}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,m_1\,[\,2 + \alpha_{12} + \alpha_{13}\,] - m_2\,[\,1 + 2\,\alpha_{12} - \alpha_{13}\,] \\
&\qquad - m_3\,[\,1 - \alpha_{12} + 2\,\alpha_{13}\,] + t\,[\,L_1 - \alpha_{12}\,L_2 - \alpha_{13}\,L_3\,]\,\} \\
&= \frac{8}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,3\,m_1 - m_2\,[\,1 - (\,\alpha_{12} + \alpha_{13}\,) + 3\,\alpha_{12}\,] \\
&\qquad - m_3\,[\,1 - (\,\alpha_{12} + \alpha_{13}\,) + 3\,\alpha_{13}\,] + t\,[\,L_1 - (\,\alpha_{12}\,L_2 + \alpha_{13}\,L_3\,)\,]\,\} \\
&= \frac{24}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,m_1 - (\,\alpha_{12}\,m_2 + \alpha_{13}\,m_3\,) + \frac{t}{3}\,[\,L_1 - (\,\alpha_{12}\,L_2 + \alpha_{13}\,L_3\,)\,]\,\}\,. \qquad (82)
\end{aligned}$$

Analogous calculations reveal that in this case:

$$G_2 \;=\; \frac{24}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,m_2 - (\,\alpha_{21}\,m_1 + \alpha_{23}\,m_3\,) + \frac{t}{3}\,[\,L_2 - (\,\alpha_{21}\,L_1 + \alpha_{23}\,L_3\,)\,]\,\}\,, \text{ and} \qquad (83)$$

$$G_3 \;=\; \frac{24}{25\,t}\,\frac{\partial m_i}{\partial k_i}\,\{\,m_3 - (\,\alpha_{31}\,m_1 + \alpha_{32}\,m_2\,) + \frac{t}{3}\,[\,L_3 - (\,\alpha_{31}\,L_1 + \alpha_{32}\,L_2\,)\,]\,\}\,. \qquad (84)$$

Case 1. $m_1 + \frac{t}{3}\,L_1 \;>\; m_2 + \frac{t}{3}\,L_2 \;>\; m_3 + \frac{t}{3}\,L_3\,.$

(79), (80), and (81) imply that in this case, firm 1 has the highest marginal valuation of the input, firm 2 has an intermediate marginal valuation of the input, and firm 3 has the lowest marginal valuation of the input. Therefore, when firm 1 secures the input increment at auction, it precludes firm 2 (not firm 3) from acquiring the increment, and so $\alpha_{12} = 1$ and $\alpha_{13} = 0$. Also, if firm 2 were to increase its bid to the point where it won the auction for the input increment, it would preclude firm 1 (not firm 3) from acquiring the increment, and so $\alpha_{21} = 1$ and $\alpha_{23} = 0$. Similarly, if firm 3 were to increase its bid to the point where it won the auction for the input increment, it would preclude firm 1 (not firm 2) from acquiring the increment, and so $\alpha_{31} = 1$ and $\alpha_{32} = 0$.

Therefore, (82) – (84) imply that in Case 1:

$$G_1 = \frac{24}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_1 - m_2 + \frac{t}{3}\,(L_1 - L_2)\right], \tag{85}$$

$$G_2 = \frac{24}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_2 - m_1 + \frac{t}{3}\,(L_2 - L_1)\right], \text{ and} \tag{86}$$

$$G_3 = \frac{24}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_3 - m_1 + \frac{t}{3}\,(L_3 - L_1)\right]. \tag{87}$$

(85) and (86) provide:

$$\begin{aligned} G_1 - G_2 &= \frac{24}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_1 - m_2 + \frac{t}{3}\,(L_1 - L_2) - m_2 + m_1 - \frac{t}{3}\,(L_2 - L_1)\right] \\ &= \frac{48}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_1 + \frac{t}{3}\,L_1 - \left(m_2 + \frac{t}{3}\,L_2\right)\right] > 0. \end{aligned} \tag{88}$$

Similarly, (85) and (87) provide:

$$\begin{aligned} G_1 - G_3 &= \frac{24}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_1 - m_2 + \frac{t}{3}\,(L_1 - L_2) - m_3 + m_1 - \frac{t}{3}\,(L_3 - L_1)\right] \\ &= \frac{24}{25\,t}\frac{\partial m_i}{\partial k_i}\left[2\,m_1 - m_2 - m_3 + \frac{t}{3}\,(2\,L_1 - L_2 - L_3)\right] \\ &= \frac{48}{25\,t}\frac{\partial m_i}{\partial k_i}\left[m_1 + \frac{t}{3}\,L_1 - \frac{1}{2}\left(m_2 + \frac{t}{3}\,L_2\right) - \frac{1}{2}\left(m_3 + \frac{t}{3}\,L_3\right)\right] > 0\,. \end{aligned} \tag{89}$$

(79) and (80) imply that firm 1 will win an unfettered auction for the input increment in Case 1. (88) and (89) imply that welfare is highest when firm 1, rather than firm 2 or firm 3, wins the auction in this case. Analogous arguments reveal that the same is true when $m_1 + \frac{t}{3}\,L_1 > m_3 + \frac{t}{3}\,L_3 > m_2 + \frac{t}{3}\,L_2$.

Corresponding arguments and calculations for settings in which firm 2 or firm 3 wins the auction for the input increment provide the following conclusion.

Observation. Suppose $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2} = \frac{\partial m_3}{\partial k_3}$. Then a firm wins the auction for an input increment if and only if welfare is highest when the firm wins the auction.

Special Case: $x_2 = \frac{1}{3},\ x_3 = \frac{2}{3},\ \ m_1 > m_2 = m_3,\ \ \frac{\partial m_1}{\partial k_1} < \frac{\partial m_2}{\partial k_2} = \frac{\partial m_3}{\partial k_3}$.

Suppose that in this case, $m_1 - m_2$ is sufficiently large relative to $\frac{\partial m_2}{\partial k_2} - \frac{\partial m_1}{\partial k_1}$ that $\alpha_{21} = \alpha_{31} = 1$ and $\alpha_{23} = \alpha_{32} = 0$. Then from (51) – (55) and (73) – (75):

$$B_1 \;=\; \frac{2}{25\,t}\left[2\,(\,m_1 - m_2\,) + \frac{5}{3}\,t\right]\left[2\,\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]; \tag{90}$$

$$B_2 \;=\; B_3 \;=\; \frac{2}{25\,t}\left[m_2 - m_1 + \frac{5}{3}\,t\right]\left[2\,\frac{\partial m_2}{\partial k_2} + \frac{\partial m_1}{\partial k_1}\right]; \tag{91}$$

$$G_1 \;=\; \frac{1}{25\,t}\left\{\left[16\,(\,m_1 - m_2\,) + \frac{25}{3}\,t\right]\frac{\partial m_1}{\partial k_1} - \left[8\,(\,m_2 - m_1\,) + \frac{25}{3}\,t\right]\frac{\partial m_2}{\partial k_2}\right\}; \tag{92}$$

and

$$G_2 \;=\; G_3 \;=\; -\,G_1. \tag{93}$$

(90) and (91) imply that firm 1 will win the auction if:

$$\left[2\,(\,m_1 - m_2\,) + \frac{5}{3}\,t\right]\left[2\,\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right] > \left[m_2 - m_1 + \frac{5}{3}\,t\right]\left[2\,\frac{\partial m_2}{\partial k_2} + \frac{\partial m_1}{\partial k_1}\right]$$

$$\Leftrightarrow\;\; \frac{\partial m_1}{\partial k_1}\left[5\,(\,m_1 - m_2\,) + \frac{5}{3}\,t\right] > \frac{\partial m_2}{\partial k_2}\left[4\,(\,m_2 - m_1\,) + \frac{5}{3}\,t\right]$$

$$\Leftrightarrow\;\; \frac{\frac{\partial m_1}{\partial k_1}}{\frac{\partial m_2}{\partial k_2}} \;>\; \left[\frac{\frac{5}{3}\,t - 4\,(\,m_1 - m_2\,)}{\frac{5}{3}\,t + 5\,(\,m_1 - m_2\,)}\right] \;=\; \frac{5\,t - 12\,\Delta}{5\,t + 15\,\Delta} \;\equiv\; r_f\,. \tag{94}$$

(92) and (93) imply that welfare is highest when firm 1 acquires the input increment if:

$$G_1 \;>\; G_2 \;=\; G_3 \;=\; -\,G_1 \;\;\Leftrightarrow\;\; G_1 \;>\; 0 \;\;\Leftrightarrow$$

$$\left[16\,(\,m_1 - m_2\,) + \frac{25}{3}\,t\right]\frac{\partial m_1}{\partial k_1} \;>\; \left[8\,(\,m_2 - m_1\,) + \frac{25}{3}\,t\right]\frac{\partial m_2}{\partial k_2}$$

$$\Leftrightarrow\;\; \frac{\frac{\partial m_1}{\partial k_1}}{\frac{\partial m_2}{\partial k_2}} \;>\; \left[\frac{\frac{25}{3}\,t - 8\,(\,m_1 - m_2\,)}{\frac{25}{3}\,t + 16\,(\,m_1 - m_2\,)}\right] \;=\; \frac{25\,t - 24\,\Delta}{25\,t + 48\,\Delta} \;\equiv\; r_w\,. \tag{95}$$

Observation. $r_w > r_f$.

Proof. From (94) and (95):

$$r_w \;>\; r_f \;\;\Leftrightarrow\;\; [\,25\,t - 24\,\Delta\,]\,[\,5\,t + 15\,\Delta\,] \;>\; [\,25\,t + 48\,\Delta\,]\,[\,5\,t - 12\,\Delta\,]$$

$$\Leftrightarrow\;\; 25\,(15)\,t\,\Delta - 24\,(5)\,t\,\Delta - 24\,(15)\,\Delta^2 \;>\; -\,25\,(12)\,t\,\Delta + 48\,(5)\,t\,\Delta - 48\,(12)\,\Delta^2$$

$$\Leftrightarrow\;\; [375 - 120 + 300 - 240]\,t\,\Delta \;>\; [360 - 576]\,\Delta^2 \;\;\Leftrightarrow\;\; 315\,t\,\Delta \;>\; -\,216\,\Delta^2\,. \quad \blacksquare$$

The Setting with Value-Enhancing Effort.

In this setting, $v_i(k_i, e_i)$ will denote the value that consumers place on firm i's product when the firm employs k_i units of the input and devotes effort e_i to reducing its unit production cost and enhancing consumer valuation of its product. Furthermore, $c_i(k_i, e_i)$ will denote firm i's corresponding unit cost of production. We assume that $v_i(\cdot)$ is increasing in e_i and $c_i(\cdot)$ is decreasing in e_i. The firms choose their effort supplies (simultaneously and independently) after the conclusion of the input auction, prior to setting prices. Firm $i \in \{1, 2\}$ incurs personal cost $E_i(e_i)$ when it delivers effort e_i. $E_i(\cdot)$ is an increasing, convex function of e_i.

From (32), firm 1's equilibrium profit in this setting is of the form:

$$\pi_1^*(\cdot) \;=\; \frac{1}{18\,t}\left[\,3\,t + \Delta\,\right]^2 - E_1(e_1)\,. \tag{96}$$

Differentiating (96) provides:

$$\frac{\partial \pi_1^*(\cdot)}{\partial k_1} \;=\; \frac{1}{9\,t}\left[\,3\,t + \Delta\,\right]\frac{d\Delta}{dk_1} - E_1'(e_1)\,\frac{de_1}{dk_1} \quad \text{and}$$

$$\frac{\partial \pi_1^*(\cdot)}{\partial k_2} \;=\; \frac{1}{9\,t}\left[\,3\,t + \Delta\,\right]\frac{d\Delta}{dk_2} - E_1'(e_1)\,\frac{de_1}{dk_2}\,. \tag{97}$$

(97) implies that the rate at which firm 1's profit increases as it, rather than firm 2, secures more of the input is:

$$\widehat{B}_1 \;=\; \frac{\partial \pi_1^*(\cdot)}{\partial k_1} - \frac{\partial \pi_1^*(\cdot)}{\partial k_2} \;=\; \frac{1}{9\,t}\left[\,3\,t + \Delta\,\right]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right]. \tag{98}$$

Analogous calculations for firm 2 reveal that the rate at which firm 2's profit increases as it, rather than firm 1, secures more of the input is:

$$\widehat{B}_2 \;=\; \frac{\partial \pi_2^*(\cdot)}{\partial k_2} - \frac{\partial \pi_2^*(\cdot)}{\partial k_1} \;=\; \frac{1}{9\,t}\left[\,3\,t - \Delta\,\right]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right]. \tag{99}$$

(98) and (99) imply that firm 1 will outbid firm 2 for the input increment in an unfettered auction if and only if:

$$\widehat{B}_1 \;>\; \widehat{B}_2 \;\Leftrightarrow\; \frac{1}{9\,t}\left[\,3\,t + \Delta\,\right]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right]$$

$$>\; \frac{1}{9\,t}\left[\,3\,t - \Delta\,\right]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right]$$

$$\Leftrightarrow\; \frac{2\,\Delta}{9\,t}\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] \;>\; E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right]$$

$$\Leftrightarrow \quad \Delta \; > \; \frac{9\,t}{2}\left[\frac{E_1'(e_1)\left(\frac{de_1}{dk_1}-\frac{de_1}{dk_2}\right)-E_2'(e_2)\left(\frac{de_2}{dk_2}-\frac{de_2}{dk_1}\right)}{\frac{d\Delta}{dk_1}-\frac{d\Delta}{dk_2}}\right]. \tag{100}$$

Aggregate welfare in this setting is:

$$\widehat{W}^* \;=\; \frac{m_1}{6\,t}\,[\,3\,t+\Delta\,]+\frac{m_2}{6\,t}\,[\,3\,t-\Delta\,]-\frac{t}{4}-\frac{\Delta^2}{36\,t}-E_1(e_1)-E_2(e_2)$$

$$\begin{aligned}
\Rightarrow \quad \frac{\partial\widehat{W}^*}{\partial k_1} \;=\;& \frac{m_1}{6\,t}\,\frac{d\Delta}{dk_1}+\left[\frac{3\,t+\Delta}{6\,t}\right]\frac{dm_1}{dk_1}-\frac{m_2}{6\,t}\,\frac{d\Delta}{dk_1}+\left[\frac{3\,t-\Delta}{6\,t}\right]\frac{dm_2}{dk_1}-\frac{\Delta}{18\,t}\,\frac{d\Delta}{dk_1}\\
&-\,E_1'(e_1)\,\frac{de_1}{dk_1}-E_2'(e_2)\,\frac{de_2}{dk_2}\\
=\;& \frac{\Delta}{6\,t}\,\frac{d\Delta}{dk_1}-\frac{\Delta}{18\,t}\,\frac{d\Delta}{dk_1}+\frac{1}{2}\left[\frac{dm_1}{dk_1}+\frac{dm_2}{dk_1}\right]+\frac{\Delta}{6\,t}\left[\frac{dm_1}{dk_1}-\frac{dm_2}{dk_1}\right]\\
&-\,E_1'(e_1)\,\frac{de_1}{dk_1}-E_2'(e_2)\,\frac{de_2}{dk_2}\\
=\;& \frac{5\,\Delta}{18\,t}\,\frac{d\Delta}{dk_1}+\frac{1}{2}\left[\frac{dm_1}{dk_1}+\frac{dm_2}{dk_1}\right]-E_1'(e_1)\,\frac{de_1}{dk_1}-E_2'(e_2)\,\frac{de_2}{dk_2}\,.
\end{aligned} \tag{101}$$

Similarly:

$$\begin{aligned}
\frac{\partial\widehat{W}^*}{\partial k_2} \;=\;& \frac{m_1}{6\,t}\,\frac{d\Delta}{dk_2}+\left[\frac{3\,t+\Delta}{6\,t}\right]\frac{dm_1}{dk_2}-\frac{m_2}{6\,t}\,\frac{d\Delta}{dk_2}+\left[\frac{3\,t-\Delta}{6\,t}\right]\frac{dm_2}{dk_2}-\frac{\Delta}{18\,t}\,\frac{d\Delta}{dk_2}\\
&-\,E_1'(e_1)\,\frac{de_1}{dk_2}-E_2'(e_2)\,\frac{de_2}{dk_2}\\
=\;& \frac{\Delta}{6\,t}\,\frac{d\Delta}{dk_2}-\frac{\Delta}{18\,t}\,\frac{d\Delta}{dk_2}+\frac{1}{2}\left[\frac{dm_1}{dk_2}+\frac{dm_2}{dk_2}\right]+\frac{\Delta}{6\,t}\left[\frac{dm_1}{dk_2}-\frac{dm_2}{dk_2}\right]\\
&-\,E_1'(e_1)\,\frac{de_1}{dk_2}-E_2'(e_2)\,\frac{de_2}{dk_2}\\
=\;& \frac{5\,\Delta}{18\,t}\,\frac{d\Delta}{dk_2}+\frac{1}{2}\left[\frac{dm_1}{dk_2}+\frac{dm_2}{dk_2}\right]-E_1'(e_1)\,\frac{de_1}{dk_2}-E_2'(e_2)\,\frac{de_2}{dk_2}\,.
\end{aligned} \tag{102}$$

(101) and (102) imply that the rate at which welfare increases as firm 1, rather than firm 2, acquires more of the input is:

$$\widehat{G}_1 \;=\; \frac{\partial W^*(\cdot)}{\partial k_1}-\frac{\partial W^*(\cdot)}{\partial k_2} \;=\; \frac{5\,\Delta}{18\,t}\left[\frac{d\Delta}{dk_1}-\frac{d\Delta}{dk_2}\right]+\frac{1}{2}\left[\frac{d\,(m_1+m_2)}{dk_1}-\frac{d\,(m_1+m_2)}{dk_2}\right]$$

$$- \; E_1'(e_1) \left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2} \right] - E_2'(e_2) \left[\frac{de_2}{dk_1} - \frac{de_2}{dk_2} \right] . \tag{103}$$

Similarly, the rate at which welfare increases as firm 2, rather than firm 1, acquires more of the input is:

$$\widehat{G}_2 \; = \; \frac{\partial W^*(\cdot)}{\partial k_2} - \frac{\partial W^*(\cdot)}{\partial k_1} \; = \; - \, \widehat{G}_1 \, . \tag{104}$$

(103) and (104) imply that welfare increases more rapidly when firm 1, rather than firm 2, acquires more of the input if and only if:

$$\widehat{G}_1 \; > \; \widehat{G}_2 \;\; \Leftrightarrow \;\; \widehat{G}_1 \; > \; - \, \widehat{G}_1 \;\; \Leftrightarrow \;\; \widehat{G}_1 \; > \; 0$$

$$\Leftrightarrow \;\; \frac{5\,\Delta}{18\,t} \left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2} \right] \; > \; \frac{1}{2} \left[\frac{d\,(m_1+m_2)}{dk_2} - \frac{d\,(m_1+m_2)}{dk_1} \right] + \; E_1'(e_1) \left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2} \right] + E_2'(e_2) \left[\frac{de_2}{dk_1} - \frac{de_2}{dk_2} \right]$$

$$\Leftrightarrow \;\; \Delta \left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2} \right] \; > \; \frac{9\,t}{5} \left[\frac{d\,(m_1+m_2)}{dk_2} - \frac{d\,(m_1+m_2)}{dk_1} \right] + \; \frac{18\,t}{5} \left[E_1'(e_1) \left(\frac{de_1}{dk_1} - \frac{de_1}{dk_2} \right) + E_2'(e_2) \left(\frac{de_2}{dk_1} - \frac{de_2}{dk_2} \right) \right]$$

$$\Leftrightarrow \;\; \Delta \; > \; \frac{9\,t}{5} \left[\frac{\frac{d(m_1+m_2)}{dk_2} - \frac{d(m_1+m_2)}{dk_1}}{\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}} \right] + \; \frac{18\,t}{5} \left[\frac{E_1'(e_1) \left(\frac{de_1}{dk_1} - \frac{de_1}{dk_2} \right) + E_2'(e_2) \left(\frac{de_2}{dk_1} - \frac{de_2}{dk_2} \right)}{\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}} \right] . \tag{105}$$