Technical Appendix to Accompany

"On the Performance of Linear Contracts"

by

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This technical appendix to accompany "On the Performance of Linear Contracts" consists of two parts. Appendix A states and proves conclusions that supplement the formal conclusions reported in the text of the paper. Appendix B provides detailed proofs of the formal conclusions in the paper.

Appendix A. Additional Conclusions

Lemma A1 provides the equations that characterize the solution to [P]. Finding A1 extends Finding 3 in the text to the case where $f(x|a)$ is given by the two-parameter gamma density:

$$f(x|a) = \frac{x^{p-1}\, e^{-x/a}}{a^p\, \Gamma(p)} \quad \text{for} \quad x \in [0, \infty), \quad \text{where} \quad \Gamma(p) = \int_0^\infty e^{-u}\, u^{p-1}\, du. \tag{1}$$

Proposition A1 extends Proposition 1 in the text to allow for $\delta \in (1, 2)$. Propositions A2 and A3 provide the corresponding extension of Proposition 2 in the text.

Lemma A1. The solution to [P] is characterized by the solution to the following equations:

$$w(x) = \begin{cases} 0 & \text{if} \quad x < \widehat{x} \\ \left[2\theta \left(\lambda + \mu \left[\frac{f_a(x|a)}{f(x|a)} \right] \right) \right]^{\frac{1}{1-\theta}} & \text{if} \quad x \geq \widehat{x}\ ; \end{cases} \tag{2}$$

$$\int_{\widehat{x}}^{\infty} 2[w(x)]^{\theta}\, f(x|a)dx - a^{\delta} = 0\ ; \tag{3}$$

$$\int_{\widehat{x}}^{\infty} 2[w(x)]^{\theta}\, f_a(x|a)dx - \delta a^{\delta-1} = 0\ ; \qquad \text{and} \tag{4}$$

$$\int_0^{\widehat{x}} x\ f_a(x|a)dx + \int_{\widehat{x}}^{\infty} [x - w(x)]\ f_a(x|a)dx + \mu \left[\int_{\widehat{x}}^{\infty} 2[w(x)]^{\theta}\, f_{aa}(x|a)dx - \delta\, [\delta - 1]\, a^{\delta-2} \right] = 0\ , \tag{5}$$

$$\text{where} \quad \widehat{x} \;=\; \min\left\{ x \geq 0 \mid \lambda + \mu \left[\frac{f_a(x|a)}{f(x|a)} \right] \geq 0 \right\}. \tag{6}$$

Finding A1. Suppose $f(x|a)$ is as specified in (1). Then at the solution to [P-L]:

$$a \;=\; \left[\frac{p\,\theta}{\delta} \frac{[g(c_3^*)]^{\frac{1}{\theta}}}{g_2(c_3^*)} \right]^{\frac{\theta}{\delta-\theta}} ; \;\; \pi^L \;=\; a\,p\left[1 - \frac{\theta}{\delta}\right]; \;\; \beta \;=\; \left[\frac{a^{\delta-\theta}}{g(c_3^*)} \right]^{\frac{1}{\theta}} ; \;\; \text{and} \;\; x_0 = ac_3^*, \tag{7}$$

$$\text{where} \quad g(c) \;=\; \frac{2\theta}{\delta} \int_c^\infty [y-c]^{\theta-1} y\,\varphi(y)\,dy; \quad g_2(c) \;=\; \int_c^\infty [y-c]\,\varphi(y)\,dy; \tag{8}$$

$$\varphi(y) \;=\; \frac{e^{-y}\,y^{p-1}}{\Gamma(p)} \quad \text{for} \;\; p > 0 \;\; \text{and} \;\; y > 0; \quad \text{and} \tag{9}$$

c_3^* is the point at which $\rho(c_3) \equiv \left[\frac{\delta\, g_2(c_3)}{p\;\theta\,[g(c_3)]^{\frac{1}{\theta}}} \right]^{\frac{\delta\theta}{\delta-\theta}}$ attains its minimum value in the range $[0,\ \widehat{c}_3]$, where $\hat{c}_3$ is the value of c that solves:

$$[\delta - \theta] \int_0^\infty e^{-t}\,t^\theta\,[t+c]^{p-1}\,dt \;=\; c\,\theta \int_0^\infty e^{-t}\,t^{\theta-1}\,[t+c]^{p-1}\,dt. \tag{10}$$

Proof. It is readily verified that the first-order approach to solving [P-L] is valid under the maintained assumptions. Consequently, [P-L] can be written as:

$$\underset{x_0,\,a,\,\beta}{\text{Maximize}} \;\; \widetilde{L} \;=\; \int_0^{x_0} x\,f(x|a)\,dx + \int_{x_0}^\infty [x - (x - x_0)\beta] f(x|a)\,dx \tag{11}$$

$$\text{subject to:} \quad \int_{x_0}^\infty 2\,[w(x)]^\theta f(x|a)\,dx - a^\delta \;\geq\; 0, \quad \text{and} \tag{12}$$

$$\int_{x_0}^\infty 2\,[w(x)]^\theta f_a(x|a)\,dx - \delta\,a^{\delta-1} \;=\; 0. \tag{13}$$

$$\text{Define} \quad \alpha(c) \;=\; \int_c^\infty [y-c]^\theta\,\varphi(y)\,dy \quad \text{and} \quad c_3 \;=\; \frac{x_0}{a}. \tag{14}$$

(8), (9), and (14) imply that when $y = \frac{x}{a}$, (12) can be written as:

$$2\,a^\theta \beta^\theta \int_{c_3}^\infty [y - c_3]^\theta \varphi(y)\,dy \;\geq\; a^\delta \;\Leftrightarrow\; 2\,a^\theta \beta^\theta\,\alpha(c_3) \;\geq\; a^\delta \;\Leftrightarrow\; \beta^\theta \;\geq\; \frac{a^\delta}{2\,a^\theta \alpha(c_3)}. \tag{15}$$

(1) and (9) imply:

$$f(x|a) \;=\; \left(\frac{1}{a}\right)\varphi\left(\frac{x}{a}\right). \tag{16}$$

Letting $\varphi'(x) = \frac{\partial\varphi(x)}{\partial x}$, (16) implies:

$$\frac{\partial f(x|a)}{\partial a} \;=\; f_a(x|a) \;=\; -\frac{1}{a^2}\varphi\left(\frac{x}{a}\right) - \frac{x}{a^3}\varphi'\left(\frac{x}{a}\right). \tag{17}$$

(17) implies that (13) can be written as:

$$\int_{x_0}^{\infty} 2[(x-x_0)\beta]^{\theta}\left[-\frac{1}{a^2}\varphi\left(\frac{x}{a}\right) - \left[\frac{x}{a^3}\right]\varphi'\left(\frac{x}{a}\right)\right]dx \;=\; \delta\, a^{\delta-1}. \tag{18}$$

Since $y = \frac{x}{a}$, (18) can be written as:

$$\int_{\frac{x_0}{a}}^{\infty} 2[(ay-x_0)\beta]^{\theta}\left[-\frac{1}{a^2}\varphi(y) - \left[\frac{ay}{a^3}\right]\varphi'(y)\right][a]\,dy \;=\; \delta\, a^{\delta-1}$$

$$\Leftrightarrow\quad \beta^{\theta}\int_{c_3}^{\infty}[y-c_3]^{\theta}[-1]\left[\varphi(y) + y\,\varphi'(y)\right]dy \;=\; \frac{\delta\, a^{\delta-\theta}}{2}. \tag{19}$$

Integrating by parts and using the fact that $\varphi(y)$ decays exponentially, (19) can be written as:

$$\beta^{\theta}\,\theta\int_{c_3}^{\infty}[y-c_3]^{\theta-1}y\,\varphi(y)\,dy \;=\; \frac{\delta\, a^{\delta-\theta}}{2} \quad\Leftrightarrow\quad \beta^{\theta}\,g(c_3) \;=\; a^{\delta-\theta} \quad\Leftrightarrow\quad \beta \;=\; \left[\frac{a^{\delta-\theta}}{g(c_3)}\right]^{\frac{1}{\theta}}. \tag{20}$$

Since $y = \frac{x}{a}$ and $c_3 = \frac{x_0}{a}$ from (14), (11) can be written as:

$$\widetilde{L} \;=\; ap - \int_{c_3}^{\infty}[ay-x_0]\beta\,\varphi(y)\,dy \;=\; a[p - \beta\, g_2(c_3)] \;=\; a\left[p - \left(\frac{g_2(c_3)}{g(c_3)^{\frac{1}{\theta}}}\right)a^{\frac{\delta-\theta}{\theta}}\right]. \tag{21}$$

(15), (20), and (21) imply that [P-L] can be written as:

$$\underset{a,\,\beta_1,\,c_3}{Maximize}\quad a\left[p - \left(\frac{g_2(c_3)}{g(c_3)^{\frac{1}{\theta}}}\right)a^{\frac{\delta-\theta}{\theta}}\right] \tag{22}$$

$$\text{subject to:}\quad 2\,\alpha(c_3) \;\geq\; g(c_3) \quad\text{and}\quad \beta^{\theta}\,g(c_3) \;=\; a^{\delta-\theta}. \tag{23}$$

Letting $t \;=\; y - c$, (8) and (14) imply:

$$\alpha(c) \;=\; \int_{c}^{\infty}[y-c]^{\theta}\,\varphi(y)dy \;=\; \int_{0}^{\infty} t^{\theta}\,\varphi(t+c)dt,\quad\text{and} \tag{24}$$

$$g_4(c) \;\equiv\; \left[\frac{\delta}{2\theta}\right] g(c) \;=\; \int_c^{\infty} [y-c]^{\theta-1}\, y\, \varphi(y) dy \;=\; \int_0^{\infty} t^{\theta-1}[c+t]\, \varphi(t+c) dt. \qquad (25)$$

We will now prove: (i) $\alpha(0) = g_4(0)$; (ii) $\frac{\alpha(c)}{g_4(c)}$ is a decreasing function of c for all $p > 0$; and (iii) there exists a unique $\widehat{c}_3$ such that $2\,\alpha\left(\widehat{c}_3\right) = g\left(\widehat{c}_3\right)$.

To begin, define $s\,(t) \equiv \frac{e^{-t} t^{\theta-1}\,[t+c]^{p-2}}{\Gamma(p)}$ for $c > 0$ and $t \geq 0$. From (9) and (14):

$$\begin{aligned} \alpha(c) \;&=\; \int_c^{\infty} [y-c]^{\theta}\, \frac{e^{-y}\, y^{p-1}}{\Gamma\left(p\right)} dy \;=\; \int_0^{\infty} e^{-(t+c)}\, \frac{(t)^{\theta}\,[t+c]^{p-1}}{\Gamma\left(p\right)} dt \\ &=\; e^{-c} \int_0^{\infty} t\; [t+c]\, s\,(t)\, dt \end{aligned} \qquad (26)$$

$$\begin{aligned} \Rightarrow \quad \alpha'\,(c) \;&=\; -\,\alpha(c) + [p-1]\, e^{-c} \int_0^{\infty} \frac{e^{-t}\, t^{\theta}\,[t+c]^{p-2}}{\Gamma(p)} dt \\ &=\; -\,\alpha(c) + [p-1]\; e^{-c} \int_0^{\infty} t\, s\,(t)\, dt. \end{aligned} \qquad (27)$$

From (9) and (25):

$$\begin{aligned} g_4(c) \;&=\; \int_c^{\infty} \frac{[y-c]^{\theta-1}\, y^{p}\, e^{-y}}{\Gamma\left(p\right)} dy \;=\; \int_0^{\infty} \frac{(t)^{\theta-1}\,[t+c]^{p}\; e^{-(t+c)}}{\Gamma\left(p\right)} dt \\ &=\; e^{-c} \int_0^{\infty} [t+c]^2\; s\,(t)\, dt. \end{aligned} \qquad (28)$$

Therefore:

$$g_4'(c) \;=\; -\,g_4(c) \,+\, p\, e^{-c} \int_0^{\infty} [t+c]\; s\,(t)\, dt. \qquad (29)$$

(26) and (28) imply:

$$\alpha(0) \;=\; \int_0^{\infty} t^2\, s\,(t)\, dt \;=\; g_4(0). \qquad (30)$$

To show that $\frac{\alpha(c)}{g_4(c)}$ is a decreasing function of c, it suffices to show:

$$\alpha'\,(c)\; g_4(c) - g_4'(c)\, \alpha\,(c) \;<\; 0. \qquad (31)$$

Define: $\alpha_0 \;=\; \int_0^{\infty} s\,(t)\, dt$; $\quad \alpha_1 \;=\; \int_0^{\infty} t\, s\,(t)\, dt$; $\quad$ and $\quad \alpha_2 \;=\; \int_0^{\infty} t^2\, s\,(t)\, dt.$

(26) – (29) imply:

$$\alpha(c) \;=\; e^{-c}\left[c\,\alpha_1 + \alpha_2\right]; \qquad \alpha'\,(c) \;=\; -\,\alpha(c) + [p-1]\; e^{-c}\,\alpha_1;$$

$$g_4(c) \;=\; e^{-c}\left[c_0^2\alpha + 2\,c\,\alpha_1 + \alpha_2\right]; \quad \text{and} \quad g_4'(c) \;=\; -\,g_4(c) + p\, e^{-c}\left[c\,\alpha_0 + \alpha_1\right].$$

Therefore, the inequality in (31) holds if and only if:

$$\left[-\alpha(c) + (p-1)\, e^{-c}\alpha_1\right] g_4(c) \;<\; \left[-g_4(c) + pe^{-c}\left(c\,\alpha_0 + \alpha_1\right)\right] \alpha\,(c)$$

$$\Leftrightarrow\;\; [p-1]\, e^{-c}\alpha\, g_4(c) \;<\; p\, e^{-c}\left[c\,\alpha_0 + \alpha_1\right] \alpha\,(c)$$

$$\Leftrightarrow\;\; [p-1]\left[c_0^2\alpha_0 + 2c\alpha_1 + \alpha_2\right]\alpha_1 \;<\; p\left[c\,\alpha_1 + \alpha_2\right]\left[c\,\alpha_0 + \alpha_1\right]$$

$$\Leftrightarrow\;\; [p-1]\left[c_0^2\alpha_0\alpha_1 + 2\,c\,\alpha_1^2 + \alpha_1\alpha_2\right] \;<\; p\left[c^2\alpha_1\alpha_0 + c\,\alpha_1^2 + c\,\alpha_0\,\alpha_2 + \alpha_1\,\alpha_2\right]$$

$$\Leftrightarrow\;\; c^2\alpha_1\alpha_0 + c\,\alpha_1^2\,[2-p] + \alpha_1\alpha_2 + p\,c\,\alpha_0\,\alpha_2 \;>\; 0. \tag{32}$$

Since, α_0, α_1, and α_2 are strictly positive, (32) holds if $p \leq 2$.

Now suppose $p > 2$. The Cauchy - Schwartz inequality implies:

$$\begin{aligned}
\alpha_1 \;&=\; \int_0^\infty t\, s\,(t)\, dt \;=\; \int_0^\infty \left(t\sqrt{s\,(t)}\right)\left(\sqrt{s\,(t)}\right) dt \\
&\leq\; \left[\int_0^\infty \left(t\sqrt{s\,(t)}\right)^2 dt\right]^{\frac{1}{2}} \left[\int_0^\infty \left(\sqrt{s\,(t)}\right)^2 dt\right]^{\frac{1}{2}} \\
&=\; \left[\int_0^\infty t^2\, s\,(t)\, dt\right]^{\frac{1}{2}} \left[\int_0^\infty s\,(t)\, dt\right]^{\frac{1}{2}} \;=\; (\alpha_0)^{\frac{1}{2}}\,(\alpha_2)^{\frac{1}{2}}\,.
\end{aligned} \tag{33}$$

(33) implies:

$$\alpha_1^2 \;\leq\; \alpha_0\,\alpha_2 \;\;\Rightarrow\;\; c\,\alpha_1^2\,[2-p] \;\geq\; c\,[2-p]\,\alpha_0\,\alpha_2. \tag{34}$$

Using (34) in (32) provides:

$$\begin{aligned}
c^2\alpha_1\,\alpha_0 + c\,\alpha_1^2\,[2-p] + \alpha_1\,\alpha_2 + p\,c\,\alpha_0\alpha_2 \;&\geq\; c^2\alpha_1\,\alpha_0 + c\,[2-p]\,\alpha_0\,\alpha_2 + \alpha_1\,\alpha_2 + p\,c\,\alpha_0\,\alpha_2 \\
&=\; c^2\alpha_1\,\alpha_0 + 2\,c\,\alpha_0\,\alpha_2 + \alpha_1\,\alpha_2 \;>\; 0.
\end{aligned}$$

Therefore, (32) holds for $p > 2$ as well.

Finally, note that $\frac{2\,\alpha(0)}{g(0)} = \frac{\delta}{\theta} > 1$ and that $\frac{2\,\alpha(c)}{g(c)}$ is a decreasing function of c for all $p > 0$. Furthermore, from (8), $\frac{2\alpha(c)}{g(c)} \to 0$ as $p \to \infty$. Hence, there exists a unique $\widehat{c}_3$ such that $2\,\alpha\,(\widehat{c}_3) = g\,(\widehat{c}_3)\,.$

These conclusions facilitate a restatement of [P-L]. When (23) holds:

$$\frac{a^\delta}{2\,a^\theta\alpha\,(c_3)} \;\leq\; \frac{a^{\delta-\theta}}{g\,(c_3)} \;\;\Rightarrow\;\; 2\,\alpha\,(c_3) \;\geq\; g\,(c_3) \;\;\Rightarrow\;\; c_3 \;\leq\; \widehat{c}_3. \tag{35}$$

(22), (23), and (35) imply that [P-L] can be written as:

$$\underset{a,\ c_3 \le \widehat{c}_3}{\text{Maximize}} \quad \widehat{L} \;=\; a \left[p \;-\; \frac{g_2\,(c_3)}{[g\,(c_3)]^{\frac{1}{\theta}}}\; a^{\frac{\delta-\theta}{\theta}} \right]. \tag{36}$$

Because $\widehat{L}$ is concave in a, the unconstrained optimum for a occurs where:

$$\frac{\partial \widehat{L}}{\partial a} \;=\; p - \frac{g_2\,(c_3)}{[g\,(c_3)]^{\frac{1}{\theta}}}\, a^{\frac{\delta-\theta}{\theta}} \;-\; a\, \frac{g_2\,(c_3)}{[g\,(c_3)]^{\frac{1}{\theta}}} \left[\frac{\delta-\theta}{\theta} \right] a^{\frac{\delta-2\theta}{\theta}} \;=\; 0$$

$$\Rightarrow \quad a^{\frac{\delta-\theta}{\theta}}\, \frac{g_2\,(c_3)}{[g\,(c_3)]^{\frac{1}{\theta}}} \left[\frac{\delta}{\theta} \right] \;=\; p \quad \Rightarrow \quad a \;=\; \left[\frac{p\,\theta}{\delta}\, \frac{[g\,(c_3)]^{\frac{1}{\theta}}}{g_2\,(c_3)} \right]^{\frac{\theta}{\delta-\theta}}. \tag{37}$$

(36) and (37) imply that the solution to [P-L] is given by:

$$a \;=\; \left[\frac{p\,\theta}{\delta}\, \frac{[g\,(c_3^*)]^{\frac{1}{\theta}}}{g_2\,(c_3^*)} \right]^{\frac{\theta}{\delta-\theta}} \quad \text{and} \quad \pi^L \;=\; a \left[p - \frac{p\theta}{\delta} \right] \;=\; a\,p \left[1 - \frac{\theta}{\delta} \right], \tag{38}$$

where c_3^* is the point at which $\rho(c_3) \equiv \left[\frac{\delta\, g_2(c_3)}{p\ \theta\, [g(c_3)]^{\frac{1}{\theta}}} \right]^{\frac{\delta\theta}{\delta-\theta}}$ attains its minimum value in the range $[0,\ \widehat{c}_3]$.

We next show that $\widehat{c}_3$ is the value of c that solves (10). To demonstrate this conclusion, recall that by definition, $2\,\alpha\,(c) = g\,(c)$ at $\hat{c}_3$. Therefore, from (8) and (26):

$$2\,e^{-c} \int_0^\infty e^{-t}\, \frac{(t)^\theta\,[t+c]^{p-1}}{\Gamma\,(p)} dt \;=\; \left[\frac{2\,\theta}{\delta} \right] e^{-c} \int_0^\infty \frac{(t)^{\theta-1}\,[t+c]^p\, e^{-t}}{\Gamma\,(p)} dt. \tag{39}$$

Let $l(t) = (t)^{\theta-1}\,[t+c]^{p-1}\, e^{-t}$. Then (39) implies:

$$\delta \int_0^\infty t\, l(t) dt \;=\; \theta \int_0^\infty [t+c]\ l(t) dt \quad \Leftrightarrow \quad [\delta-\theta] \int_0^\infty t\, l(t) dt \;=\; c\,\theta \int_0^\infty l(t) dt. \tag{40}$$

Substituting for $l(t)$ in (40) provides (10).

Finally, notice that $\beta = \left[\frac{a^{\delta-\theta}}{g\left(c_3^*\right)} \right]^{\frac{1}{\theta}}$ and $x_0 = a c_3^*$, from (14) and (20). ■

Proposition A1. Suppose $p = 1$ and $\delta > 1$. Then at the solution to [P-L]:

$$\pi^L \;=\; a \left[1 - \frac{\theta}{\delta} \right] \quad \text{where} \quad a \;=\; \left(\frac{\theta}{\delta}\, \frac{\left(\left[\frac{2\theta}{\delta} \right] e^{-c_3^*}\, \Gamma(\theta)\ [\theta + c_3^*] \right)^{\frac{1}{\theta}}}{e^{-c_3^*}} \right)^{\frac{\theta}{\delta-\theta}} \tag{41}$$

$$\text{and} \quad c_3^* \;=\; \begin{cases} \frac{1-\theta+\theta^2}{1-\theta} & if \quad \delta \;\geq\; \frac{1}{1-\theta} \\ \delta - \theta & if \quad \delta \;\leq\; \frac{1}{1-\theta}. \end{cases} \tag{42}$$

Proof. The proof follows from substituting into (38) the expressions for c_3^* identified in Observation A3 below. The proof of Observation A3 employs the conclusions recorded as Observations A1 and A2.

Observation A1. $\widehat{c}_3 \;=\; \delta - \theta$ when $p = 1$.

Proof. When $p = 1$:

$$[\delta - \theta] \int_0^\infty e^{-t}\, t^\theta\, [t+c]^{p-1}\, dt \;=\; [\delta - \theta]\; \Gamma\,(\theta + 1)$$

$$\text{and} \quad c\,\theta \int_0^\infty e^{-t}\, t^{\theta-1}\, [t+c]^{p-1}\, dt \;=\; c\,\theta\,\Gamma\,(\theta)\,. \tag{43}$$

Since $\theta\,\Gamma\,(\theta) \;=\; \Gamma\,(\theta+1)$, (9) and (43) imply:

$$[\delta - \theta]\; \Gamma\,(\theta+1) \;=\; c\,\Gamma\,(\theta+1) \quad \Rightarrow \quad \widehat{c}_3 \;=\; \delta - \theta. \quad \square$$

Observation A2. Suppose $p = 1$. Then $\rho(c_3) \equiv \left[\frac{\delta\, g_2(c_3)}{p\;\theta\,[g(c_3)]^{\frac{1}{\theta}}}\right]^{\frac{\delta\theta}{\delta-\theta}}$ attains its global minimum at $c_3 \;=\; \frac{1-\theta+\theta^2}{1-\theta}$.

Proof. From (8) and (9):

$$g_2(c) \;=\; \int_c^\infty [y-c]\,\varphi\,(y)\; dy \;=\; \int_c^\infty [y-c]\,\frac{e^{-y}\, y^{p-1}}{\Gamma\,(p)} dy. \tag{44}$$

Substituting $y - c \;=\; t$ into (44) provides:

$$g_2(c) \;=\; \int_0^\infty t\, e^{-(t+c)}\,\frac{[t+c]^{p-1}}{\Gamma(p)} dt \;=\; \frac{e^{-c}}{\Gamma(p)} \int_0^\infty t\, e^{-t}\, [t+c]^{p-1}\, dt. \tag{45}$$

(45) implies that if $p = 1$, then $g_2(c) \;=\; e^{-c}$.

From (25), $g_4(c) = \int_0^\infty t^{\theta-1}[c+t]^p\, \frac{e^{-(t+c)}}{\Gamma(p)} dt$. Hence, if $p = 1$, then:

$$g_4(c) \;=\; e^{-c} \int_0^\infty t^{\theta-1}\,[c+t]\, e^{-t} dt \;=\; e^{-c} \left[c \int_0^\infty t^{\theta-1}\, e^{-t} dt + \int_0^\infty t^\theta\, e^{-t} dt \right]$$

$$\;=\; e^{-c}\,[c\,\Gamma(\theta) + \Gamma\,(\theta+1)] \;=\; e^{-c}\,\Gamma(\theta)\,[\theta + c]\,. \tag{46}$$

(25) and (46) imply:

$$g(c) \;=\; \left[\frac{2\,\theta}{\delta}\right] g_4(c) \;=\; \left[\frac{2\,\theta}{\delta}\right] e^{-c}\,\Gamma(\theta)\;[\theta + c]$$

$$\Rightarrow\;\; \frac{\delta\, g_2\,(c_3)}{p\,\theta\,[g\,(c_3)]^{\frac{1}{\theta}}} \;=\; \frac{\delta\, e^{-c}}{\theta\left[\left(\frac{2\theta}{\delta}\right)\; e^{-c}\,\Gamma(\theta)\;(\theta + c_3)\right]^{\frac{1}{\theta}}} \;=\; k_0 \left[\frac{e^{-c}}{\left[e^{-c}\,(\theta + c_3)\right]^{\frac{1}{\theta}}}\right] \qquad (47)$$

$$\text{where} \quad k_0 \;=\; \frac{\delta}{\theta\left[\frac{2\theta}{\delta}\,\Gamma(\theta)\right]^{\frac{1}{\theta}}}.$$

Note that:

$$\ln\left(\frac{\delta\, g_2\,(c_3)}{p\,\theta[g\,(c_3)]^{\frac{1}{\theta}}}\right) \;=\; \ln k_0 - c_3 + \frac{c_3}{\theta} - \frac{\ln\,(c_3 + \theta)}{\theta}.$$

Let $v(c_3) = \ln\left\{\frac{\delta\, g_2(c_3)}{p\,\theta[g(c_3)]^{\frac{1}{\theta}}}\right\}$. Then:

$$\frac{\partial v(c_3)}{\partial c_3} \;=\; -\,1 + \frac{1}{\theta} - \frac{1}{\theta\,[c_3 + \theta]} \quad \text{and} \quad \frac{\partial^2 v(c_3)}{\partial\,(c_3)^2} \;=\; \frac{1}{\theta\,[c_3 + \theta]^2} \;>\; 0. \qquad (48)$$

(48) implies that $\left.\frac{\partial v(c_3)}{\partial c_3}\right|_{c_3=0} < 0$, $\left.\frac{\partial v(c_3)}{\partial c_3}\right|_{c_3\to\infty} > 0$, and $v(c_3)$ is convex. Therefore, $v(c_3)$ reaches its minimum at $\widetilde{c}_3$, where:

$$-1 + \frac{1}{\theta} - \frac{1}{\theta\,[\widetilde{c}_3 + \theta]} \;=\; 0 \quad\Rightarrow\quad \widetilde{c}_3 \;=\; \frac{1 - \theta + \theta^2}{1 - \theta}. \quad \square$$

Observation A3. Suppose $p = 1$ and $\theta \leq \frac{1}{2}$. Then:

(i) $c_3^* = \frac{1-\theta+\theta^2}{1-\theta}$ when $\delta \geq \frac{1}{1-\theta}$, and $c_3^* = \widehat{c}_3$ if $\theta = \frac{1}{2}$ and $\delta = 2$.

(ii) $c_3^* = \widehat{c}_3 = \delta - \theta$ when $\delta \leq \frac{1}{1-\theta}$.

Proof. From Observations A1 and A2:

$$c_3^* \;\leq\; \widehat{c}_3 \;\Leftrightarrow\; \frac{1 - \theta + \theta^2}{1 - \theta} \;\leq\; \delta - \theta \;\;\Leftrightarrow\;\; 1 - \theta + \theta^2 \;\leq\; [1 - \theta]\,[\delta - \theta]$$

$$\Leftrightarrow\;\; 1 - \theta + \theta^2 \;\leq\; \delta - \theta\delta - \theta + \theta^2 \;\;\Leftrightarrow\;\; 1 \;\leq\; \delta\,[1 - \theta]\,. \qquad (49)$$

The result follows from (49), since $v\,(c_3)$ is a convex function, from (48). $\square$

Finally, notice that if $p = 1$, then (47) implies:

$$\frac{p\,\theta}{\delta}\frac{[g\,(c_3^*)]^{\frac{1}{\theta}}}{g_2\,(c_3^*)} \;=\; \frac{\theta}{\delta}\frac{\left[\left(\frac{2\theta}{\delta}\right) e^{-c_3^*}\,\Gamma(\theta)\;(\theta + c_3^*)\right]^{\frac{1}{\theta}}}{e^{-c_3^*}}$$

$$\Rightarrow\ a_0 \ = \ \left(\frac{\theta}{\delta}\frac{\left[\left(\frac{2\theta}{\delta}\right)e^{-c_3^*}\,\Gamma(\theta)\,(\theta+c_3^*)\right]^{\frac{1}{\theta}}}{e^{-c_3^*}}\right)^{\frac{\theta}{\delta-\theta}}. \quad \blacksquare$$

Proposition A2. Suppose $p = 1$ and $\theta = \frac{1}{2}$. Then for all $\delta \in (1,2)$:

$$\frac{\pi^L}{\pi} \ \geq \ \left(\frac{\left[\delta^2+1\right]\pi}{4}\, e^{-\left(\delta-\frac{1}{2}\right)}\right)^{\frac{1}{2\delta-1}}. \tag{50}$$

Proof. Since $\theta = \frac{1}{2}$ and $\delta \in (1,2)$, $\delta < \frac{1}{1-\theta}$. Hence, from Proposition A1:

$$a_0 \ = \ \left(\frac{\theta}{\delta}\frac{\left(\left[\frac{2\theta}{\delta}\right]e^{-c_3^*}\Gamma(\theta)\ \left[\theta+c_3^*\right]\right)^{\frac{1}{\theta}}}{e^{-c_3^*}}\right)^{\frac{\theta}{\delta-\theta}} \quad \text{and} \quad c_3^* \ = \ \delta-\theta. \tag{51}$$

Substituting for θ and c_3^* in (51) provides:

$$a_0 \ = \ \left(\frac{1}{2\delta}\frac{\left(e^{-\left(\delta-\frac{1}{2}\right)}\Gamma(\frac{1}{2})\right)^2}{e^{-\left(\delta-\frac{1}{2}\right)}}\right)^{\frac{1}{2\delta-1}} \ = \ \left(\frac{1}{2\delta}e^{-\left(\delta-\frac{1}{2}\right)}\left(\Gamma(\frac{1}{2})\right)^2\right)^{\frac{1}{2\delta-1}} \ = \ \left(\frac{\pi}{2\delta}\, e^{-\left(\delta-\frac{1}{2}\right)}\right)^{\frac{1}{2\delta-1}}.$$

Therefore from Finding 3:

$$\pi^L \ = \ \left[1-\frac{1}{2\delta}\right]\left(\frac{\pi}{2\delta}\, e^{-\left(\delta-\frac{1}{2}\right)}\right)^{\frac{1}{2\delta-1}}. \tag{52}$$

From Finding 2:

$$\pi \ \leq \ \left[1-\frac{1}{2\delta}\right]\left(\frac{2}{\delta\left[1+\delta^2\right]}\right)^{\frac{1}{2\delta-1}}. \tag{53}$$

(52) and (53) imply that the inequality in (50) holds. $\blacksquare$

Proposition A3. $\frac{\pi^L}{\pi} \geq 0.9495$ for all $\delta > 1$ when $p = 1$ and $\theta = \frac{1}{2}$.

Proof. From Proposition 2, $\frac{\pi^L}{\pi} \geq 0.9495$ for all $\delta \geq 2$ when $\theta = \frac{1}{2}$. Therefore, from Proposition A2, it will suffice to prove that for all $\delta \in (1,2)$, the minimum value of $\left(\frac{\left[\delta^2+1\right]\pi}{4}\, e^{-\left(\delta-\frac{1}{2}\right)}\right)^{\frac{1}{2\delta-1}}$ is greater than or equal to 0.9495. Let $B(\delta) \equiv \left(\frac{\left[\delta^2+1\right]\pi}{4} e^{-\left(\delta-\frac{1}{2}\right)}\right)^{\frac{1}{2\delta-1}}$. Then:

$$\ln\left(B\left(\delta\right)\right) \ = \ \frac{1}{2\delta-1}\left[\ln\left(\delta^2+1\right)+\ln\left(\pi\right)-\delta+\frac{1}{2}-2\ln\left(2\right)\right]$$

$$\Rightarrow \quad \frac{\partial \ln (B(\delta))}{\partial \delta} = \frac{1}{2\delta - 1}\left[\frac{2\delta}{\delta^2+1} - 1\right] - \frac{2}{[2\delta-1]^2}\left[\ln\left(\delta^2+1\right) + \ln(\pi) - \delta + \frac{1}{2} - 2\ln(2)\right]$$

$$\Rightarrow \quad \left.\frac{\partial \ln (B(\delta))}{\partial \delta}\right|_{\delta=1} = -2\left[-\ln(2) + \ln(\pi) - \frac{1}{2}\right] > 0; \text{ and} \tag{54}$$

$$\left.\frac{\partial \ln (B(\delta))}{\partial \delta}\right|_{\delta=2} = \frac{1}{3}\left[\frac{4}{5} - 1\right] - \frac{2}{9}\left[\ln(5) + \ln(\pi) - 2 + \frac{1}{2} - 2\ln(2)\right] < 0. \tag{55}$$

We will now show that there exists a unique $\delta \in [1, 2]$ such that $\frac{\partial \ln(B(\delta))}{\partial \delta} = 0$. This fact, (54), and (55) imply that $B(\delta)$ is minimized either at $\delta = 1$ or $\delta = 2$. To show that there exists a unique δ such that $\frac{\partial \ln(B(\delta))}{\partial \delta} = 0$, note that $\frac{\partial \ln(B(\delta))}{\partial \delta} = 0$ if and only if:

$$\left[\frac{1}{2\delta-1}\right]\left[\frac{2\delta}{\delta^2+1} - 1\right] - \frac{2}{[2\delta-1]^2}\left[\ln\left(\delta^2+1\right) + \ln(\pi) - \delta + \frac{1}{2} - 2\ln(2)\right] = 0$$

$$\Leftrightarrow \quad M(\delta) \equiv 2\delta^2 - \delta - \left[1+\delta^2\right]\left[\ln\left(\delta^2+1\right) + \ln(\pi) - 2\ln(2)\right] = 0. \tag{56}$$

(56) implies:

$$M(\delta = 1) = 2 - 1 - 2\left[\ln(2) + \ln(\pi) - 2\ln(2)\right] > 0;$$

$$M(\delta = 2) = 8 - 2 - 5\left[\ln(5) + \ln(\pi) - 2\ln(2)\right] < 0; \text{ and}$$

$$M''(\delta) = 1.484 - 2\ln\left(1+\delta^2\right) - \frac{4\delta^2}{1+\delta^2}.$$

Since $1 + \delta^2$ and $\frac{\delta^2}{1+\delta^2}$ are both increasing functions of δ, it follows that $M''(\delta) \leq 1.484 - 2\ln(2) - 2 < 0$. Therefore, there exists a unique δ such that $M(\delta) = 0$.

Hence, when $p = 1$, $\theta = \frac{1}{2}$, and $1 < \delta \leq 2$, the lower bound of $\frac{\pi^L}{\pi}$ is minimized either at $\delta = 2$ or as $\delta \to 0$. From the proof of Proposition 2, the lower bound of $\frac{\pi^L}{\pi}$ is minimized at $\delta = 2.55899$ when $p = 1$, $\theta = \frac{1}{2}$, and $\delta \geq 2$. Therefore, it will suffice to compare the lower bounds of $\left.\frac{\pi^L}{\pi}\right|_{\delta=2.55899}$ and $\left.\frac{\pi^L}{\pi}\right|_{\delta\to 1}$. The lower bound of $\left.\frac{\pi^L}{\pi}\right|_{\delta\to 1} = \frac{\pi}{2}e^{-\frac{1}{2}} = 0.95225 > 0.94955$ = the lower bound of $\left.\frac{\pi^L}{\pi}\right|_{\delta=2.55899}$. Therefore, $\frac{\pi^L}{\pi} \geq 0.94955$ for all $\delta > 1$ when $p = 1$ and $\theta = \frac{1}{2}$. ■

Appendix B. Proofs of Conclusions in the Text

This appendix provides detailed proofs of the formal conclusions in the paper. The formal conclusions are the following:

Finding 1. $\pi \leq \widetilde{a}\left[1-\frac{\theta}{\delta}\right]$, *where* $\widetilde{a} = \left[\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\frac{\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}{1+\delta}\right]^{\frac{1}{\delta-\theta}}$.

Finding 2. $\pi \leq \widehat{a}\left[1-\frac{1}{2\delta}\right]$, *where* $\widehat{a} = \left(\frac{2}{\delta\left[1+\delta^2\right]}\right)^{\frac{1}{2\delta-1}}$ *when* $\theta = \frac{1}{2}$.

Finding 3. *At the solution to [P-L]:*

$$\beta = \left(\frac{\theta}{\delta}\right)e^{\frac{1-\theta+\theta^2}{1-\theta}}; \quad x_0 = \left[\frac{1-\theta+\theta^2}{1-\theta}\right]a; \quad \textit{and} \quad \pi^L = \left[\frac{\delta-\theta}{\delta}\right]a; \tag{57}$$

$$\textit{where} \quad a = \left[\left(\frac{2\,\Gamma(\theta)}{1-\theta}\right)\left(\frac{\theta}{\delta}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)}\right]^{\frac{1}{\delta-\theta}}. \tag{58}$$

Corollary 1. $a < 1$ *at the solution to [P-L].*

Corollary 2. $\frac{d\beta}{d\delta} < 0$ *and* $\frac{d\pi^L}{d\delta} > 0$ *at the solution to [P-L]. Furthermore,* $\frac{da}{d\delta} > 0$ *and* $\frac{dx_0}{d\delta} > 0$ *when* $\delta \geq 4e^{\frac{1}{4}}$.

Corollary 3. $\frac{d\beta}{d\theta} > 0$ *at the solution to [P-L].*

Proposition 1. $\frac{\pi^L}{\pi} \geq \left[\left(\frac{\theta}{\delta}\right)\frac{[1+\delta]\,e^{-\left(1-\theta+\theta^2\right)}\Gamma(\theta)}{[1-\theta]\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}\right]^{\frac{1}{\delta-\theta}} \geq .743$ *for all* $\theta \in (0, \frac{1}{2}]$ *and* $\delta \geq 2$.

Proposition 2. $\frac{\pi^L}{\pi} \geq \left(\frac{\left[\delta^2+1\right]\pi\, e^{-\frac{3}{2}}}{\delta^2}\right)^{\frac{1}{2\delta-1}} \geq 0.9495$ *for all* $\delta \geq 2$ *when* $\theta = \frac{1}{2}$.

Proof of Finding 1.

The principal's problem $[P]$ is:

$$\underset{w(x),\, a}{Maximize} \quad L \;=\; \int_0^\infty [x - w(x)]\, f(x|a)\, dx \tag{59}$$

$$\text{subject to:} \quad \int_0^\infty 2\,(w(x))^\theta\, f(x|a)\, dx - a^\delta \;\geq\; 0, \;\text{ and} \tag{60}$$

$$\int_0^\infty 2\,(w(x))^\theta\, f_a(x|a)\, dx - \delta a^{\delta-1} \;=\; 0. \tag{61}$$

Let X be a random variable denoting output, and let x denote a specific value of X. The density function for X is:

$$f(x|a) \;=\; \frac{1}{a^p\,\Gamma(p)}\, x^{p-1}\, e^{-\frac{x}{a}} \qquad \text{for } x \;\geq\; 0.$$

Define the random variable $Y = \frac{X}{a}$, and let y denote a specific value of Y. It can be shown that $Y \sim \varphi(y)$, where:

$$\varphi(y) \;=\; \frac{1}{\Gamma(p)}\, y^{p-1}\, e^{-y} \qquad \text{for } y \;\geq\; 0.$$

Letting $E(\cdot)$ denote "expectation," (59) can be written as:

$$L \;=\; \int_0^\infty x f(x|a)\, dx - \int_0^\infty w(x)\, f(x|a)\, dx \;=\; ap - \int_0^\infty w(x)\, f(x|a)\, dx$$

$$= \; ap - \int_0^\infty w(ay)\, \varphi(y)\, dy \;=\; ap - E(w(aY)). \tag{62}$$

Similarly, (60) can be rewritten as:

$$\int_0^\infty 2\,(w(ay))^\theta\, \varphi(y)\, dy - a^\delta \;\geq\; 0 \quad \Leftrightarrow \quad 2E\left((w(aY))^\theta\right) \;\geq\; a^\delta. \tag{63}$$

Furthermore, (61) can be rewritten as:

$$\int_0^\infty 2\,(w(ay))^\theta \left(\frac{ay - ap}{a^2}\right) \varphi(y)\, dy \;=\; \delta a^{\delta-1}$$

$$\Leftrightarrow \quad \int_0^\infty 2\,(w(ay))^\theta\, y\varphi(y)\, dy - \int_0^\infty 2\,(w(ay))^\theta\, p\varphi(y)\, dy \;=\; \delta a^\delta$$

$$\Leftrightarrow \quad 2E\left((w(aY))^\theta\, Y\right) - 2pE\left((w(aY))^\theta\right) \;=\; \delta a^\delta. \tag{64}$$

Notice that:

$$2E\left((w(aY))^{\theta}Y\right) - pa^{\delta} \;\geq\; \delta a^{\delta} \;\;\Leftrightarrow\;\; E\left((w(aY))^{\theta}Y\right) \;\geq\; \frac{[p+\delta]\,a^{\delta}}{2}. \qquad (65)$$

From Holder's inequality, if X and Y are two non-negative functions, then:

$$E(XY) \;\leq\; [E(X^{p})]^{\frac{1}{p}}\,[E(X^{q})]^{\frac{1}{q}} \quad \text{for all } p>1, \;\text{ where } \frac{1}{p}+\frac{1}{q} \;=\; 1.$$

Consequently:

$$E\left((w(aY))^{\theta}Y\right) \;\leq\; \left[E\left(\left((w(aY))^{\theta}\right)^{\frac{1}{\theta}}\right)\right]^{\theta}\left[E\left((Y)^{\frac{1}{1-\theta}}\right)\right]^{1-\theta}$$

$$\Rightarrow \quad E\left((w(aY))^{\theta}Y\right) \;\leq\; [E((w(aY)))]^{\theta}\left[E\left((Y)^{\frac{1}{1-\theta}}\right)\right]^{1-\theta}. \qquad (66)$$

Notice that:

$$[E((w(aY)))]^{\theta}\left[E\left((Y)^{\frac{1}{1-\theta}}\right)\right]^{1-\theta} \;\geq\; \frac{[p+\delta]\,a^{\delta}}{2}$$

$$\Leftrightarrow \quad [E((w(aY)))]^{\theta} \;\geq\; \frac{[p+\delta]\,a^{\delta}}{2\left[E\left((Y)^{\frac{1}{1-\theta}}\right)\right]^{1-\theta}}$$

$$\Leftrightarrow \quad E((w(aY))) \;\geq\; \left[\frac{[p+\delta]\,a^{\delta}}{2\left[E\left((Y)^{\frac{1}{1-\theta}}\right)\right]^{1-\theta}}\right]^{\frac{1}{\theta}} \;=\; \left[\frac{[p+\delta]\,a^{\delta}\,(\Gamma(p))^{1-\theta}}{2\left(\Gamma\left(p+\frac{1}{1-\theta}\right)\right)^{1-\theta}}\right]^{\frac{1}{\theta}} \qquad (67)$$

$$\Leftrightarrow \quad E((w(aY))) \;\geq\; ka^{\frac{\delta}{\theta}}, \qquad \text{where } k \;=\; \left[\frac{[p+\delta]\,(\Gamma(p))^{1-\theta}}{2\left(\Gamma\left(p+\frac{1}{1-\theta}\right)\right)^{1-\theta}}\right]^{\frac{1}{\theta}}. \qquad (68)$$

The equality in (67) holds because, since $Y \sim \Gamma(p)$:

$$E\left((Y)^{\frac{1}{1-\theta}}\right) \;=\; \int_{0}^{\infty}(y)^{\frac{1}{1-\theta}}\,\frac{1}{\Gamma(p)}\,(y)^{p-1}\,e^{-y}dy$$

$$= \frac{1}{\Gamma(p)}\int_{0}^{\infty}(y)^{p-1+\frac{1}{1-\theta}}\,e^{-y}dy \;=\; \frac{\Gamma\left(p+\frac{1}{1-\theta}\right)}{\Gamma(p)}. \qquad (69)$$

(67) and (69) imply:

$$E((w(aY))) \;\geq\; \left[\frac{(p+\delta)\,a^{\delta}\,(\Gamma(p))^{1-\theta}}{2\left(\Gamma\left(p+\frac{1}{1-\theta}\right)\right)^{1-\theta}}\right]^{\frac{1}{\theta}}.$$

(62) and (68) imply:

$$L \;\leq\; ap - ka^{\frac{\delta}{\theta}} \;=\; L_u\left(a\right). \tag{70}$$

We will now maximize $L_u(a)$ to derive an upper bound, L^*, for the maximum value of L.

$$\frac{\partial L_u\left(a\right)}{\partial a} \;=\; p - k\left(\frac{\delta}{\theta}\right)a^{\frac{\delta}{\theta}-1} \;=\; 0 \quad\Rightarrow\quad \widetilde{a} \;=\; \left[\frac{\theta p}{\delta k}\right]^{\frac{\theta}{\delta-\theta}}. \tag{71}$$

Using (71) in (70) provides:

$$L^* \;\leq\; \widetilde{a}\left[p - k\,\widetilde{a}^{\frac{\delta}{\theta}-1}\right] \;=\; \widetilde{a}\left[p - \frac{p\,\theta}{\delta}\right] \;=\; p\left[1-\frac{\theta}{\delta}\right]\left[\frac{\theta p}{\delta k}\right]^{\frac{\theta}{\delta-\theta}}. \tag{72}$$

Substituting for k from (68) into (72) provides:

$$\pi \;\leq\; L^* \;\leq\; p\left[1-\frac{\theta}{\delta}\right]\left[\frac{p\theta}{\delta}\left(\frac{(2)^{\frac{1}{\theta}}\left(\Gamma\left(p+\frac{1}{1-\theta}\right)\right)^{\frac{1-\theta}{\theta}}}{\left[p+\delta\right]^{\frac{1}{\theta}}\left(\Gamma\left(p\right)\right)^{\frac{1-\theta}{\theta}}}\right)\right]^{\frac{\theta}{\delta-\theta}}$$

$$=\; p\left[1-\frac{\theta}{\delta}\right]\left[p\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\frac{\left(\Gamma\left(p+\frac{1}{1-\theta}\right)\right)^{1-\theta}}{\left[p+\delta\right]\left(\Gamma\left(p\right)\right)^{1-\theta}}\right]^{\frac{1}{\delta-\theta}}. \tag{73}$$

From (73), when $p = 1$:

$$\pi \;\leq\; \left[1-\frac{\theta}{\delta}\right]\left[\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\frac{\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}{1+\delta}\right]^{\frac{1}{\delta-\theta}}. \quad \blacksquare \tag{74}$$

Proof of Finding 2.

Suppose $\theta = \frac{1}{2}$ and $f(x|a)$ is as specified in (1). Lemma A1 in Appendix A implies that if $w(x)$ is not constrained to be non-negative for all realizations of x, then the solution to the principal's problem is determined by:

$$w\left(x\right) \;=\; \left(\lambda + \mu\left[\frac{f_a\left(x,a\right)}{f\left(x,a\right)}\right]\right)^2; \quad \lambda \;=\; \frac{a^\delta}{2}; \quad \mu \;=\; \frac{\delta a^{\delta+1}}{2p}; \tag{75}$$

$$\frac{f_a\left(x,a\right)}{f\left(x,a\right)} \;=\; \frac{x-ap}{a^2}; \quad \text{and} \quad \delta^3\,a^{2\delta-1} + 2\,p\,\lambda\,\delta\,a^{\delta-1} - 2\,p^2 \;=\; 0. \tag{76}$$

(75) and (76) imply that an upper bound (π^u) for π is:

$$
\begin{aligned}
\pi^u &= E\left\{x - w(x)\right\} \;=\; ap - \int_0^\infty \left[\lambda + \mu\left(\frac{x - pa}{a^2}\right)\right]^2 f\left(x|a\right)dx \\
&= ap - \int_0^\infty \left[\lambda^2 + 2\lambda\mu\left(\frac{x-pa}{a^2}\right) + \mu^2\left(\frac{(x-pa)^2}{a^4}\right)\right] f\left(x|a\right)dx \\
&= ap - \left[\lambda^2 + \frac{\mu^2 p}{a^2}\right] \;=\; ap - \left[\lambda^2 + \frac{p}{a^2}\left(\frac{\delta a^{\delta+1}}{2p}\right)^2\right] \\
&= ap - \left[\lambda^2 + \frac{p}{a^2}\frac{1}{4p^2}\delta^2 a^{2\delta+2}\right] = ap - \left[\frac{\delta^2 a^{2\delta}}{4p} + \frac{a^{2\delta}}{4}\right] = ap - \frac{1}{4p}\left[\delta^2 a^{2\delta} + p\,a^{2\delta}\right] \\
&= ap - \frac{1}{4p}\left[a\,\delta^2\,a^{2\delta-1} + p\,a^{2\delta}\right] \;=\; ap - \frac{1}{4p}\left[a\frac{\left[2p^2 - 2p\lambda\delta a^{\delta-1}\right]}{\delta} + pa^{2\delta}\right] \\
&= ap - \frac{1}{4p}\left[\frac{2ap^2}{\delta} - 2p\lambda a^{\delta} + pa^{2\delta}\right] \;=\; ap - \frac{1}{4p}\left[\frac{2ap^2}{\delta} - pa^{\delta}a^{\delta} + pa^{2\delta}\right] \\
&= ap - \frac{1}{4p}\left[\frac{2ap^2}{\delta}\right] = ap - \frac{ap}{2\delta} \;=\; \frac{ap\left[2\delta - 1\right]}{2\delta}. \qquad (77)
\end{aligned}
$$

(76) and (77) imply that when $p = 1$:

$$
\pi \;\leq\; a\left[1 - \frac{1}{2\delta}\right], \quad \text{where } a \;=\; \left(\frac{2}{\delta\left[1+\delta^2\right]}\right)^{\frac{1}{2\delta-1}}. \quad \blacksquare
$$

Proof of Finding 3.

The proof follows directly from the proof of Proposition A1 in Appendix A. ■

Proof of Corollary 1.

From Finding 3, $a < 1$ at the solution to [P-L] if:

$$
\left[\frac{2\,\Gamma(\theta)}{1-\theta}\right]\left(\frac{\theta}{\delta}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)} \;<\; 1. \qquad (78)
$$

Since $\delta \geq 2$:

$$
\left[\frac{2\,\Gamma(\theta)}{1-\theta}\right]\left(\frac{\theta}{\delta}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)} \;\leq\; \left[\frac{2\,\Gamma(\theta)}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)}
$$

$$= \left[\frac{\theta\Gamma(\theta)}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)} = \left[\frac{\Gamma(\theta+1)}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)}. \tag{79}$$

Notice that:

$$\Gamma(\theta+1) = \int_0^{\infty} e^{-t}\,(t)^{\theta+1-1}\,dt = \int_0^{\infty} e^{-t}t^{\theta}dt = E\left\{T^{\theta}\right\}, \tag{80}$$

where the density function for the random variable T is exponential with mean 1. (80) and Holder's inequality imply:

$$\Gamma(\theta+1) = E\left\{T^{\theta}\right\} \le E\left\{\left(T^{\theta}\right)^{\frac{1}{\theta}}\right\}^{\theta} = E\left\{T\right\} = 1 \tag{81}$$

$$\Rightarrow \left[\frac{\Gamma(\theta+1)}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)} \le \left[\frac{1}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)}. \tag{82}$$

(79) and (82) imply:

$$\left[\frac{2\,\Gamma(\theta)}{1-\theta}\right]\left(\frac{\theta}{\delta}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)} \le \left[\frac{1}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)}. \tag{83}$$

$$\text{Let} \quad z\left(\theta\right) = \ln\left(\left[\frac{1}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)}\right)$$

$$= -\ln(1-\theta) + \theta\ln\left(\theta\right) - \theta\ln(2) - \left(1-\theta+\theta^2\right) \tag{84}$$

$$\Rightarrow z'\left(\theta\right) = \frac{1}{1-\theta} + 1 + \ln\left(\theta\right) - \ln(2) + 1 - 2\theta = \frac{1}{1-\theta} + 2\left[1-\theta\right] + \ln\left(\theta\right) - \ln(2) \tag{85}$$

$$\Rightarrow z''\left(\theta\right) = \frac{1}{\left[1-\theta\right]^2} - 2 + \frac{1}{\theta} \ge \frac{1}{\left[1-\theta\right]^2}. \tag{86}$$

The inequality in (86) holds because $\frac{1}{\theta} \ge 2$, since $\theta \le \frac{1}{2}$. (86) implies that $z\left(\theta\right)$ is convex in θ for all $\theta \in \left[0, \frac{1}{2}\right]$. Furthermore, from (84) and (85):

$$z\left(\theta=0\right) = -1 < 0; \quad z'\left(\theta=0\right) = -\infty; \quad z\left(\theta=\frac{1}{2}\right) = -\frac{3}{4} < 0; \quad \text{and} \tag{87}$$

$$z'\left(\theta=\frac{1}{2}\right) = 2 + 1 + \ln\left(\frac{1}{2}\right) - \ln(2) = 3 + \ln\left(\frac{1}{4}\right) > 0. \tag{88}$$

(87) and (88) imply that $z\left(\theta\right) < 0$ for all $\theta \in \left[0, \frac{1}{2}\right]$. Therefore, from (84):

$$\ln\left(\left[\frac{1}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)}\right) < 0 \text{ for all } \theta \in \left[0, \frac{1}{2}\right]$$

$$\Rightarrow \quad \left[\frac{1}{1-\theta}\right]\left(\frac{\theta}{2}\right)^{\theta} e^{-\left(1-\theta+\theta^2\right)} \; < \; 1 \;\text{ for all } \theta \in \left[0, \frac{1}{2}\right]. \tag{89}$$

(83) and (89) imply that the inequality in (78) holds, so $a < 1$ for all $\theta \in \left[0, \frac{1}{2}\right]$. ■

Proof of Corollary 2.

Recall from Finding 3 that at the solution to [P-L]:

$$\beta = \left[\frac{\theta}{\delta}\right] e^{\frac{1-\theta+\theta^2}{1-\theta}} \text{ and } \pi^L = a\left[\frac{\delta-\theta}{\delta}\right], \text{ where } a = \left[\left(\frac{2\,\Gamma(\theta)}{1-\theta}\right)\left(\frac{\theta}{\delta}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)}\right]^{\frac{1}{\delta-\theta}}. \tag{90}$$

It is apparent from (90) that $\frac{d\beta}{d\delta} \; < \; 0$.

To show $\frac{d\pi^L}{d\delta} \; > \; 0$, let:

$$t_0 \; = \; \left[\frac{2\,\Gamma(\theta)}{1-\theta}\right]\left(\frac{\theta}{\delta}\right)^{\theta+1} e^{-\left(1-\theta+\theta^2\right)}. \tag{91}$$

(90) and (91) imply:

$$\ln\left(\pi^L\right) \; = \; \ln(\delta-\theta) - \ln(\delta) + \left[\frac{1}{\delta-\theta}\right]\ln(t_0)$$

$$\Rightarrow \quad \frac{\partial \ln\left(\pi^L\right)}{\partial\delta} \; = \; \frac{1}{\delta-\theta} - \frac{1}{\delta} - \frac{\ln(t_0)}{[\delta-\theta]^2} - \left[\frac{1}{\delta-\theta}\right]\left[\frac{1+\theta}{\delta}\right]$$

$$\Rightarrow \quad \left[\delta(\delta-\theta)^2\right]\frac{\partial \ln\left(\pi^L\right)}{\partial\delta} \; = \; \delta[\delta-\theta] - (\delta-\theta)^2 - \delta\ln(t_0) - [\delta-\theta][1+\theta]$$

$$= \; [\delta-\theta][\delta-(\delta-\theta)-(1+\theta)] - \delta\ln(t_0) \; = \; -[\delta-\theta] - \delta\ln(t_0). \tag{92}$$

(92) implies:

$$\frac{\partial \ln\left(\pi^L\right)}{\partial\delta} \; \geq \; 0 \quad \Leftrightarrow \quad \delta-\theta+\delta\ln(t_0) \; \leq \; 0. \tag{93}$$

(81) and (91) imply:

$$t_0 \; = \; \left[\frac{2\,\Gamma(\theta+1)}{1-\theta}\right]\frac{(\theta)^{\theta}}{(\delta)^{\theta+1}} e^{-\left(1-\theta+\theta^2\right)} \; \leq \; \left[\frac{2}{1-\theta}\right]\frac{(\theta)^{\theta}}{(\delta)^{\theta+1}} e^{-\left(1-\theta+\theta^2\right)} \tag{94}$$

$$\Rightarrow \quad \ln(t_0) \; \leq \; \ln(2) - \ln(1-\theta) + \theta\ln(\theta) - [\theta+1]\ln(\delta) - \left(1-\theta+\theta^2\right). \tag{95}$$

(93) and (95) imply:

$$\frac{\partial \ln\left(\pi^L\right)}{\partial\delta} \; \geq \; 0 \;\text{ if }\; b(\theta) \; \leq \; 0, \tag{96}$$

where $b(\theta) = \; \delta-\theta+\delta\left[\ln(2) - \ln(1-\theta) + \theta\ln(\theta) - [\theta+1]\ln(\delta) - \left(1-\theta+\theta^2\right)\right]. \quad (97)$

From (97):

$$b\,(0) \;=\; \delta + \delta\,[\ln(2) - \ln(1) - \ln(\delta) - 1] \;=\; \delta\,[\ln(2) - \ln(\delta)]\,. \tag{98}$$

Since $\delta \geq 2$, (98) implies:

$$b\,(0) \;\leq\; 0 \;\text{ for all } \delta \geq 2. \tag{99}$$

From (97):

$$b\left(\frac{1}{2}\right) \;=\; \delta - \frac{1}{2} + \delta\left[\ln(2) - \ln\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{3}{2}\ln(\delta) - \frac{3}{4}\right] \;\equiv\; h(\delta). \tag{100}$$

(100) implies:

$$h(2) \;=\; 2 - \frac{1}{2} + 2\left[0.28972 - \left(\frac{3}{2}\right)\ln(2)\right] \;=\; 0. \tag{101}$$

Differentiating (100) provides:

$$h'(\delta) \;=\; 0.28972 - \left(\frac{3}{2}\right)\ln(\delta) - \frac{3}{2} \;<\; 0 \;\text{ for all } \delta \;\geq\; 1. \tag{102}$$

(100) – (102) imply:

$$b\left(\frac{1}{2}\right) \;\leq\; 0 \;\text{ for all } \delta \;\geq\; 2. \tag{103}$$

Also, differentiating (97) provides:

$$b'(\theta) \;=\; -1 + \delta\left[\frac{1}{1-\theta} + \ln(\theta) + 1 - \ln(\delta) + 1 - 2\theta\right]$$

$$\Rightarrow \quad b''(\theta) \;=\; \delta\left[\frac{1}{(1-\theta)^2} + \frac{1}{\theta} - 2\right] \geq 0, \;\text{ since } \theta \in \left[0, \frac{1}{2}\right]. \tag{104}$$

(99), (103), and (104) imply:

$$b(\theta) \;\leq\; 0 \;\text{ for all } \theta \in \left[0, \frac{1}{2}\right] \;\text{ and } \delta \;\geq\; 2. \tag{105}$$

(96) and (105) imply:

$$\frac{\partial \ln\left(\pi^L\right)}{\partial \delta} \;\geq\; 0 \;\text{ for all } \theta \in \left[0, \frac{1}{2}\right] \;\text{ and }\; \delta \;\geq\; 2.$$

It remains to prove that $\frac{da}{d\delta} > 0$ and $\frac{dx_0}{d\delta} > 0$ when $\delta \geq 4e^{\frac{1}{4}}$. We will first show that if there exists a $\widetilde{\delta}$ such that $\frac{\partial a}{\partial \delta}\big|_{\delta\,=\,\widetilde{\delta}} > 0$, then $\frac{\partial a}{\partial \delta} > 0$ for all $\delta > \widetilde{\delta}$. From (90):

$$\ln(a) \;=\; \left[\frac{1}{\delta - \theta}\right]\left[\ln(2\,\Gamma(\theta)) - \ln(1-\theta) + [\theta+1]\ln(\theta) - [\theta+1]\ln(\delta) - \left(1 - \theta + \theta^2\right)\right]$$

$$= \left[\frac{1}{\delta-\theta}\right]\left[k\left(\theta\right)-\left(\theta+1\right)\ln\left(\delta\right)\right], \tag{106}$$

$$\text{where } k\left(\theta\right) = \ln\left(2\,\Gamma(\theta)\right)-\ln\left(1-\theta\right)+\left[\theta+1\right]\ln\left(\theta\right)-\left(1-\theta+\theta^2\right). \tag{107}$$

Differentiating (106) provides:

$$\frac{1}{a}\left[\frac{\partial a}{\partial\delta}\right] = -\left(\frac{1}{\left[\delta-\theta\right]^2}\right)\left[k\left(\theta\right)-\left[\theta+1\right]\ln\left(\delta\right)\right]-\left[\frac{1}{\delta-\theta}\right]\left[\frac{\theta+1}{\delta}\right]$$

$$\Rightarrow \left[\frac{\delta-\theta}{a}\right]\left[\frac{\partial a}{\partial\delta}\right] = -\frac{k\left(\theta\right)-\left(\theta+1\right)\ln\left(\delta\right)}{\delta-\theta}-\frac{\theta+1}{\delta}. \tag{108}$$

(108) implies that if $\frac{\partial a}{\partial\delta}\big|_{\delta\,=\,\widetilde{\delta}} > 0$, then:

$$-\frac{k\left(\theta\right)-\left[\theta+1\right]\ln\left(\widetilde{\delta}\right)}{\widetilde{\delta}-\theta}-\frac{\theta+1}{\widetilde{\delta}} > 0$$

$$\Leftrightarrow -\widetilde{\delta}\left[k\left(\theta\right)-\left(\theta+1\right)\ln\left(\widetilde{\delta}\right)\right]-\left[\widetilde{\delta}-\theta\right]\left[\theta+1\right] > 0$$

$$\Leftrightarrow -\widetilde{\delta}\left[\frac{k\left(\theta\right)}{\theta+1}-\ln\left(\widetilde{\delta}\right)\right]-\left(\widetilde{\delta}-\theta\right) > 0$$

$$\Leftrightarrow -\left[\frac{k\left(\theta\right)}{\theta+1}-\ln\left(\widetilde{\delta}\right)\right]-\left(1-\frac{\theta}{\widetilde{\delta}}\right) > 0 \Rightarrow \ln\left(\widetilde{\delta}\right)+\frac{\theta}{\widetilde{\delta}} > \frac{k\left(\theta\right)}{\theta+1}+1. \tag{109}$$

Notice that:

$$\frac{\partial}{\partial\delta}\left(\ln\left(\delta\right)+\frac{\theta}{\delta}\right) = \frac{1}{\delta}-\frac{\theta}{\delta^2} = \frac{1}{\delta}\left[1-\frac{\theta}{\delta}\right] > 0.$$

Since $\ln\left(\widetilde{\delta}\right)+\frac{\theta}{\widetilde{\delta}}$ is increasing in δ while $\frac{k(\theta)}{\theta+1}+1$ is independent of δ, it follows from (109) that $\frac{\partial a}{\partial\delta} > 0$ for all $\delta > \widetilde{\delta}$. Therefore, to show that $\frac{\partial a}{\partial\delta} > 0$ for all $\delta > 4e^{\frac{1}{4}}$ for all $\theta \in (0, \frac{1}{2}]$, it suffices to show that $\frac{\partial a}{\partial\delta}\big|_{\delta\,=\,4e^{\frac{1}{4}}} > 0$ for all $\theta \in (0, \frac{1}{2}]$.

From (107):

$$k\left(\theta\right) = \ln\left(2\,\frac{\theta\Gamma(\theta)}{\theta}\right)-\ln\left(1-\theta\right)+\left[\theta+1\right]\ln\left(\theta\right)-\left(1-\theta+\theta^2\right)$$

$$= \ln\left(2\,\frac{\Gamma(\theta+1)}{\theta}\right)-\ln\left(1-\theta\right)+\left[\theta+1\right]\ln\left(\theta\right)-\left(1-\theta+\theta^2\right)$$

$$= \ln\left(2\right)+\ln\left(\Gamma(\theta+1)\right)-\ln\left(1-\theta\right)+\theta\ln\left(\theta\right)-\left(1-\theta+\theta^2\right). \tag{110}$$

Since $\Gamma(\theta+1) \leq 1$ and $\ln(\theta) < 0$, (110) implies:

$$k(\theta) \leq \ln(2) - \ln(1-\theta) - \left(1-\theta+\theta^2\right) \leq \ln(2) - \ln\left(\frac{1}{2}\right) - \left(1-\theta+\theta^2\right)$$
$$= 2\ln(2) - \left(1-\theta+\theta^2\right) \leq 2\ln(2) - \frac{3}{4} \Rightarrow \quad -k(\theta) \geq -2\ln(2) + \frac{3}{4}. \qquad (111)$$

(108) and (111) imply:

$$\left[\frac{\delta(\delta-\theta)^2}{a}\right]\left[\frac{\partial a}{\partial \delta}\right] = \delta\left[-k(\theta) + (\theta+1)\ln(\delta) - \theta - 1\right] + \theta[\theta+1]$$
$$\geq \delta\left[-2\ln(2) + \frac{3}{4} + (\theta+1)\ln(\delta) - \theta - 1\right] + \theta[\theta+1]$$
$$= \delta\left[-2\ln(2) + \frac{3}{4}\right] + [\theta+1][\delta(\ln(\delta) - 1) + \theta]. \qquad (112)$$

The expression in (112) is an increasing function of θ. Therefore, it will suffice to find $\widetilde{\delta}$ such that:

$$\left[\frac{\delta(\delta-\theta)^2}{a}\right]\left.\frac{\partial a}{\partial \delta}\right|_{\delta = \widetilde{\delta}} \geq \delta\left[-2\ln(2) - \frac{1}{4} + \ln(\delta)\right] \geq 0. \qquad (113)$$

$\delta[-2\ln(2) - \frac{1}{4} + \ln(\delta)]$ is an increasing function of δ. Also $-2\ln(2) - \frac{1}{4} + \ln(\delta) \geq 0 \Leftrightarrow \delta \geq 4e^{\frac{1}{4}}$. Therefore, (113) implies that $\left[\frac{\delta(\delta-\theta)^2}{a}\right] \left.\frac{\partial a}{\partial \delta}\right|_{\delta = 4e^{\frac{1}{4}}} \geq 0$, and so $\left.\frac{\partial a}{\partial \delta}\right|_{\delta = 4e^{\frac{1}{4}}} > 0$ for all $\theta \in (0, \frac{1}{2}]$.

Finally, recall from Finding 3 that $x_0 = \left[\frac{1-\theta+\theta^2}{1-\theta}\right] a$ at the solution to [P-L]. Therefore, $\frac{\partial x_0}{\partial \delta} > 0$ for all $\delta \geq 4e^{\frac{1}{4}}$ and $\theta \in (0, \frac{1}{2}]$ since $\frac{\partial a}{\partial \delta} > 0$ for all $\delta \geq 4e^{\frac{1}{4}}$ and $\theta \in (0, \frac{1}{2}]$. ■

Proof of Corollary 3.

From (90):

$$\ln(\beta) = \ln(\theta) - \ln(\delta) + \frac{1-\theta+\theta^2}{1-\theta}$$
$$\Rightarrow \frac{\partial \ln(\beta)}{\partial \theta} = \frac{1}{\theta} + \frac{[1-\theta][-1+2\theta] + 1 - \theta + \theta^2}{[1-\theta]^2} = \frac{1}{\theta} + \frac{\theta[2-\theta]}{[1-\theta]^2} > 0. \ ■$$

Proof of Proposition 1.

(73) implies that when $p = 1$:

$$\pi \;\leq\; \left[1-\frac{\theta}{\delta}\right]\left[\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\frac{\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}{1+\delta}\right]^{\frac{1}{\delta-\theta}}. \tag{114}$$

Finding 3 implies that when $p = 1$:

$$\pi^L \;=\; \left[1-\frac{\theta}{\delta}\right]\left(\left(\frac{\theta}{\delta}\right)\frac{\left(\left[\frac{2\theta}{\delta}\right]e^{-c_3^*}\Gamma(\theta)\;[\theta+c_3^*]\right)^{\frac{1}{\theta}}}{e^{-c_3^*}}\right)^{\frac{\theta}{\delta-\theta}}, \text{ where } c_3^* \;=\; \frac{1-\theta+\theta^2}{1-\theta}.$$

$$\Rightarrow\;\; \pi^L \;=\; \left[1-\frac{\theta}{\delta}\right]\left[\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\frac{\left(\left(\frac{\theta}{\delta}\right)e^{-c_3^*}\Gamma(\theta)\;[\theta+c_3^*]\right)^{\frac{1}{\theta}}}{e^{-c_3^*}}\right]^{\frac{\theta}{\delta-\theta}}$$

$$=\; \left[1-\frac{\theta}{\delta}\right]\left[\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\left(\frac{\theta}{\delta}\right)e^{-(1-\theta)c_3^*}\,\Gamma(\theta)\;[\theta+c_3^*]\right]^{\frac{1}{\delta-\theta}}$$

$$=\; \left[1-\frac{\theta}{\delta}\right]\left[\left(\frac{\theta}{\delta}\right)(2)^{\frac{1}{\theta}}\right]^{\frac{\theta}{\delta-\theta}}\left[\left(\frac{\theta}{\delta}\right)e^{-\left(1-\theta+\theta^2\right)}\,\Gamma(\theta)\;\left(\frac{1}{1-\theta}\right)\right]^{\frac{1}{\delta-\theta}}. \tag{115}$$

(114) and (115) imply that when $p = 1$:

$$\frac{\pi^L}{\pi} \;\geq\; \left[\frac{\left(\frac{\theta}{\delta}\right)e^{-\left(1-\theta+\theta^2\right)}\Gamma(\theta)\;\left(\frac{1}{1-\theta}\right)}{\frac{\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}{1+\delta}}\right]^{\frac{1}{\delta-\theta}} \;=\; \left[\left(\frac{\theta}{\delta}\right)\frac{[1+\delta]\,e^{-\left(1-\theta+\theta^2\right)}\Gamma(\theta)}{[1-\theta]\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}\right]^{\frac{1}{\delta-\theta}}. \tag{116}$$

$$\text{Define}\quad d(\delta) \;\equiv\; \left(\frac{\theta}{\delta}\right)\frac{[1+\delta]\,e^{-\left(1-\theta+\theta^2\right)}\Gamma(\theta)}{[1-\theta]\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}},\ \text{and note that:} \tag{117}$$

$$d(\delta) \;=\; \left[\frac{1+\delta}{\delta}\right]\left[\frac{\theta\,e^{-\left(1-\theta+\theta^2\right)}\Gamma(\theta)}{(1-\theta)\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}\right] \;=\; \left[1+\frac{1}{\delta}\right]r(\theta), \tag{118}$$

$$\text{where}\quad r(\theta) \;\equiv\; \frac{\theta\,e^{-\left(1-\theta+\theta^2\right)}\Gamma(\theta)}{[1-\theta]\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}}.$$

Notice from (116) that $[d(\delta)]^{\frac{1}{\delta-\theta}} \leq 1$ since $\frac{\pi^L}{\pi} \leq 1$. Therefore, $d(\delta) \leq 1$. Consequently, since $1 + \frac{1}{\delta} > 1$, (118) implies that $r(\theta) < 1$.

$$\text{Let}\quad G(\delta) \;\equiv\; \ln[d(\delta)]^{\frac{1}{\delta-\theta}} \quad\Rightarrow\quad G(\delta) \;=\; \left[\frac{1}{\delta-\theta}\right]\ln[d(\delta)] \tag{119}$$

$$\Rightarrow\;\; G'(\delta) \;=\; -\frac{1}{[\delta-\theta]^2}\ln[d(\delta)] + \left[\frac{1}{\delta-\theta}\right]\left[\frac{d'(\delta)}{d(\delta)}\right]$$

$$\Rightarrow \quad [\delta-\theta]^2 G'(\delta) \;=\; -\ln[d(\delta)] + [\delta-\theta]\left[\frac{d'(\delta)}{d(\delta)}\right]. \tag{120}$$

From (118):

$$d'(\delta) \;=\; -\left[\frac{1}{\delta^2}\right] r(\theta) \quad \Rightarrow \quad \frac{d'(\delta)}{d(\delta)} \;=\; \frac{-1/\delta^2}{[1+\delta]/\delta} \;=\; -\,\frac{1}{\delta\,[1+\delta]}. \tag{121}$$

Substituting from (121) into (120) and using (117) provides:

$$\begin{aligned}
[\delta-\theta]^2 G'(\delta) \;&=\; -\ln[d(\delta)] + [\delta-\theta]\left[-\frac{1}{\delta(1+\delta)}\right] \\
&=\; -\left[\ln(1+\delta) - \ln(\delta) + \ln r(\theta)\right] - \frac{\delta-\theta}{\delta\,[1+\delta]} \\
&=\; -\ln(1+\delta) + \ln(\delta) - \frac{\delta-\theta}{\delta\,[1+\delta]} - \ln r(\theta). \qquad (122)
\end{aligned}$$

(122) implies:

$$\begin{aligned}
G'(\delta) \gtreqless 0 \quad &\Leftrightarrow \quad -\ln(1+\delta) + \ln(\delta) - \frac{\delta-\theta}{\delta\,[1+\delta]} - \ln r(\theta) \;\gtreqless\; 0 \\
&\Leftrightarrow \quad \ln(1+\delta) - \ln(\delta) + \frac{\delta-\theta}{\delta\,[1+\delta]} \;\lesseqgtr\; -\ln r(\theta) \\
&\Leftrightarrow \quad R(\delta) \;\lesseqgtr\; -\ln r(\theta), \qquad (123)
\end{aligned}$$

$$\text{where} \quad R(\delta) \;\equiv\; \ln(1+\delta) - \ln(\delta) + \frac{\delta-\theta}{\delta\,[1+\delta]}. \tag{124}$$

From (124):

$$R(\delta) \to 0 \quad \text{as} \quad \delta \to \infty. \tag{125}$$

Differentiating (124) provides:

$$R'(\delta) \;=\; \frac{1}{1+\delta} - \frac{1}{\delta} + \frac{\delta\,[1+\delta] - [\delta-\theta]\,[1+2\delta]}{\delta^2\,[1+\delta]^2} \;=\; -\,\frac{[\delta-\theta]\,[1+2\delta]}{\delta^2\,[1+\delta]^2} \;<\; 0. \tag{126}$$

Furthermore, $-\ln r(\theta) > 0$ since $r(\theta) < 1$. Therefore, (123) – (126) imply that for a given θ, there exists a $\delta_0(\theta)$ such that $G(\delta)$ is decreasing for all $\delta < \delta_0(\theta)$ and $G(\delta)$ is increasing for all $\delta > \delta_0(\theta)$. If $\delta_0(\theta) \leq 2$ for a given θ, then the lower bound of $\frac{\pi^L}{\pi}$ is reached at $\delta = 2$ since $\delta \geq 2$ by assumption. Alternatively, if $\delta_0(\theta) > 2$ for a given θ, then the lower bound of $\frac{\pi^L}{\pi}$ is reached at $\delta = \delta_0(\theta)$, where $\delta_0(\theta)$ is found by solving $s(\delta) = -\ln r(\theta)$.

To determine a lower bound for $\frac{\pi^L}{\pi}$ for all $\theta \in (0, \frac{1}{2}]$ and $\delta \geq 2$, define:

$$H(\theta) \;\equiv\; \left[\left(\frac{\theta}{\delta}\right) \frac{[1+\delta]\, e^{-\left(1-\theta+\theta^2\right)} \Gamma(\theta)}{[1-\theta]\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}} \right]^{\frac{1}{\delta-\theta}} \;=\; \left[\frac{[1+\delta]\, e^{-\left(1-\theta+\theta^2\right)} \Gamma(\theta+1)}{\delta\,[1-\theta]\left(\Gamma\left(\frac{2-\theta}{1-\theta}\right)\right)^{1-\theta}} \right]^{\frac{1}{\delta-\theta}} . \quad (127)$$

It can be shown that $H'(\theta) > 0$ and so the minimum value of $H(\theta)$ is reached as $\theta \to 0$ for any fixed $\delta \in [2, \infty)$. To determine the limiting value of $H(\theta)$ as $\theta \to 0$, note that:

$$H(\theta) \;\to\; \left[\left(\frac{1+\delta}{\delta}\right)\left(\frac{1}{e}\right) \right]^{\frac{1}{\delta}} \quad \text{as} \quad \theta \to 0. \quad (128)$$

Since $\left[\left(\frac{1+\delta}{\delta}\right)\left(\frac{1}{e}\right)\right]^{\frac{1}{\delta}}$ is an increasing function of δ, $\left[\left(\frac{1+\delta}{\delta}\right)\left(\frac{1}{e}\right)\right]^{\frac{1}{\delta}}$ attains its minimum value when $\delta = 2$. From (128), $H(\theta) \to \left(\frac{3}{2}\left(\frac{1}{e}\right)\right)^{\frac{1}{2}} = 0.74285$ as $\theta \to 0$ when $\delta = 2$. Consequently, from (116), $\frac{\pi^L}{\pi} \geq .743$ for all $\theta \in (0, \frac{1}{2}]$ and $\delta \geq 2$. ■

Proof of Proposition 2.

Finding 2 and Proposition 1 imply that when $\theta = \frac{1}{2}$:

$$\frac{\pi^L}{\pi} \;\geq\; \frac{\left[1 - \frac{1}{2\delta}\right]\left(\frac{2}{\delta^3}\left[\frac{22}{7}\right] e^{-\frac{3}{2}}\right)^{\frac{1}{2\delta-1}}}{\left[1 - \frac{1}{2\delta}\right]\left(\frac{2}{\delta\left[\delta^2+1\right]}\right)^{\frac{1}{2\delta-1}}} \;=\; \left(\frac{\left[\delta^2+1\right] 22\, e^{-\frac{3}{2}}}{7\,\delta^2} \right)^{\frac{1}{2\delta-1}} . \quad (129)$$

Let $\widetilde{B}(\delta) \;\equiv\; \left(\frac{\left[\delta^2+1\right]22\, e^{-\frac{3}{2}}}{7\,\delta^2}\right)^{\frac{1}{2\delta-1}}$. Then:

$$\ln(\widetilde{B}(\delta)) \;=\; \left[\frac{1}{2\delta-1}\right] \ln\left(\left[1+\frac{1}{\delta^2}\right]\frac{22}{7}\, e^{-\frac{3}{2}}\right) \;=\; \left[\frac{1}{2\delta-1}\right]\left[\ln\left(1+\frac{1}{\delta^2}\right) + \ln\left(\frac{22}{7}\, e^{-\frac{3}{2}}\right)\right]$$

$$\Rightarrow \quad \frac{\partial}{\partial\delta}\ln(\widetilde{B}(\delta)) \;=\; \left[\frac{1}{2\delta-1}\right]\frac{\left(-2\,\delta^{-3}\right)}{\left[1+\frac{1}{\delta^2}\right]} - 2\left[\ln\left(1+\frac{1}{\delta^2}\right) + \ln\left(\frac{22}{7}\, e^{-\frac{3}{2}}\right)\right]\frac{1}{[2\delta-1]^2}$$

$$= \;-\;\frac{2}{[2\delta-1]^2}\left[\frac{2\delta-1}{\delta^3+\delta} + \ln\left(1+\frac{1}{\delta^2}\right) + \ln\left(\frac{22}{7}\, e^{-\frac{3}{2}}\right)\right]. \quad (130)$$

(130) implies:

$$\frac{\partial}{\partial\delta}\ln(\widetilde{B}(\delta)) \;=\; 0 \;\text{ when }\; \widetilde{M}(\delta) \;\equiv\; \frac{2\delta-1}{\delta^3+\delta} + \ln\left(1+\frac{1}{\delta^2}\right) + \ln\left(\frac{22}{7}\, e^{-\frac{3}{2}}\right) \;=\; 0. \quad (131)$$

Note that:

$$\frac{\partial}{\partial\delta}\left(\frac{2\delta-1}{\delta^3+\delta}\right) \;\stackrel{s}{=}\; 2\left[\delta^3+\delta\right] - [2\delta-1]\left[3\delta^2+1\right] \;=\; 2\delta^3 + 2\delta - 6\delta^3 - 2\delta + 3\delta^2 + 1$$

$$= \; -4\delta^3 + 3\delta^2 + 1 \; < \; -\delta^3 + 1 \; < \; 0. \tag{132}$$

(131) and (132) imply that $\widetilde{M}'(\delta) < 0$ for all $\delta \geq 2$. It can be verified that $\widetilde{M}(2.55899) = 0$. Hence, $\widetilde{M}(\delta) > 0$ if $\delta < 2.55899$, and $\widetilde{M}(\delta) < 0$ if $\delta > 2.55899$. Therefore, $\widetilde{B}'(\delta) < 0$ if $\delta < 2.55899$, $\widetilde{B}'(\delta) < 0$ if $\delta > 2.55899$, and $\widetilde{B}'(\delta) = 0$ if $\delta = 2.55899$. Hence, (130) and (131) imply that for all $\delta \geq 2$:

$$\widetilde{B}(\delta) \; \geq \; \widetilde{B}(2.55899) \; \approx \; \left(\frac{[7.5485]\,[3.14159]}{[6.5485][4.4817]} \right)^{\frac{1}{4.118}} \; \approx \; (.808026)^{.24284} \; \approx \; .94955. \; \blacksquare$$