Technical Appendix to Accompany

"On the Optimal Design of Demand Response Policies"

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This appendix presents conditions that ensure there exits a unique $\widetilde{\theta}_i$ such that $\Omega_i^D = (\widetilde{\theta}_i, \overline{\theta}\,]$ in the benchmark setting (analyzed in section 5 of the paper) where there are N_j consumers of type $j \in \{L, H\}$.

Recall that consumer i will provide demand response (DR) in state θ if:

$$\left.\frac{\partial V_i(x_i^u + x_i^o, \theta)}{\partial x_i^u}\right|_{x_i^u = \underline{x}_i} < \; r + m(\theta). \tag{1}$$

Let $\widetilde{\theta}_i$ denote a value of θ at which (1) holds when the inequality is replaced by an equality. $\widetilde{\theta}_i \in [\,\underline{\theta}, \overline{\theta}\,]$ will be unique and $\Omega_i^D = (\widetilde{\theta}_i, \overline{\theta}\,]$ (so individual i provides DR only for $\theta \in (\widetilde{\theta}_i, \overline{\theta}\,])$ if:

$$\left.\frac{\partial V_i(x_i^u + x_i^o, \theta)}{\partial x_i^u}\right|_{x_i^u = \underline{x}_i} > \; r + m(\theta) \quad \text{for all } \theta \in [\,\underline{\theta}, \widetilde{\theta}_i\,), \text{ and}$$

$$\left.\frac{\partial V_i(x_i^u + x_i^o, \theta)}{\partial x_i^u}\right|_{x_i^u = \underline{x}_i} < \; r + m(\theta) \quad \text{for all } \theta \in (\,\widetilde{\theta}_i, \overline{\theta}\,]. \tag{2}$$

(2) will hold if: (i) $\left.\frac{\partial V_i(x_i^u + x_i^o, \underline{\theta})}{\partial x_i^u}\right|_{x_i^u = \underline{x}_i} > \; r + m(\underline{\theta})$, so individual i does not provide DR for the smallest θ realization; (ii) $\left.\frac{\partial V_i(x_i^u + x_i^o, \overline{\theta})}{\partial x_i^u}\right|_{x_i^u = \underline{x}_i} < \; r + m(\overline{\theta})$, so individual i provides DR for the highest θ realization; and:

$$\text{(iii)} \quad \frac{\partial^2 V_i(x_i^u + x_i^o, \theta)}{\partial x_i^u \, \partial \theta} < \; m'(\theta) \quad \text{for all } \theta \in [\,\underline{\theta}, \overline{\theta}\,]. \tag{3}$$

(3) implies that consumer i's marginal valuation of electricity consumption increases with θ less rapidly than the compensation for DR increases with θ.

We begin by identifying sufficient conditions for (3) to hold. To do so, recall first that consumer i's marginal valuation of electricity consumption is:

$$\frac{\partial V_i(x_i^u + x_i^o, \theta)}{\partial x_i^u} \;=\; v_i\,\theta\,[\,x_i^u\,]^{\alpha_i}\;, \;\; \text{where } \alpha_i \;<\; 0\,. \tag{4}$$

Further recall that the amount of electricity consumer i purchases from the utility is:

$$x_i^u(\cdot) \;=\; \begin{cases} \left[\frac{r+m(\theta)}{v_i\,\theta}\right]^{\frac{1}{\alpha_i}} & \text{if } v_i\,\theta\,[\,\underline{x}_i\,]^{\alpha_i} \;<\; r+m(\theta) \\ \left[\frac{r}{v_i\,\theta}\right]^{\frac{1}{\alpha_i}} & \text{otherwise.} \end{cases} \tag{5}$$

In addition, under the optimal DR policy in this setting where $C(X) \;=\; F + a\,X + b\,X^2$:

$$m(\theta) \;=\; a + 2\,b \sum_{i=1}^{N} x_i^u(\cdot) - r\,. \tag{6}$$

(4) and (6) imply that (3) will hold if:

$$v_i\,[\,x_i^u\,]^{\alpha_i} + v_i\,\theta\,\alpha_i\,[\,x_i^u\,]^{\alpha_i - 1}\,\frac{\partial x_i^u(\cdot)}{\partial \theta} \;<\; 2\,b \sum_{i=1}^{N} \frac{\partial x_i^u(\cdot)}{\partial \theta}\,. \tag{7}$$

If $\frac{\partial x_i^u(\cdot)}{\partial \theta} > 0$ for all $i = 1, 2, ..., N$, then because $\alpha_i < 0$, (7) holds if:

$$b \;>\; \frac{v_i\,[\,x_i^u\,]^{\alpha_i}}{2\sum_{i=1}^{N} \frac{\partial x_i^u(\cdot)}{\partial \theta}} \quad \text{for all } \; i \;=\; 1, 2, ..., N\,. \tag{8}$$

We now derive conditions that ensure $\frac{\partial x_i^u(\cdot)}{\partial \theta} > 0$ for all $i = 1, 2, ..., N$.

Case 1. No Consumer Ever Provides Demand Response

From (5), for $i = 1, 2, ..., N$:

$$\frac{\partial x_i^u(\cdot)}{\partial \theta} \;=\; -\frac{1}{\alpha_i}\left[\frac{r}{v_i}\right]^{\frac{1}{\alpha_i}} \theta^{-\frac{1+\alpha_i}{\alpha_i}} \;>\; 0 \;\text{ for all } \theta \,\in\, \Omega_i^{-D}\,. \tag{9}$$

Case 2. Both Types of Consumers Provide Some Demand Response

(5) and (6) imply that for $\theta \,\in\, \Omega_i^D$:

$$\frac{\partial x_i^u(\cdot)}{\partial \theta} \;=\; \frac{1}{\alpha_i}\left[\frac{r+m(\theta)}{v_i\,\theta}\right]^{\frac{1-\alpha_i}{\alpha_i}} \left[\frac{2\,b}{v_i\,\theta}\sum_{j=1}^{N}\frac{\partial x_j^u(\cdot)}{\partial \theta} - \frac{r+m(\theta)}{v_i\,\theta^2}\right]. \tag{10}$$

(10) implies that if $\theta \,\in\, \Omega_L^D$ and $\theta \,\in\, \Omega_H^D$, then:

$$\frac{\partial x_L^u(\cdot)}{\partial \theta} \;=\; \frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}} \left[\frac{2\,b}{v_L\,\theta}\left(N_L\,\frac{\partial x_L^u(\cdot)}{\partial \theta} + N_H\,\frac{\partial x_H^u(\cdot)}{\partial \theta}\right) - \frac{r+m(\theta)}{v_L\,\theta^2}\right]; \;\text{ and}$$

$$\frac{\partial x_H^u(\cdot)}{\partial\theta} \;=\; \frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\left[\frac{2\,b}{v_H\,\theta}\left(N_L\frac{\partial x_L^u(\cdot)}{\partial\theta}+N_H\frac{\partial x_H^u(\cdot)}{\partial\theta}\right)-\frac{r+m(\theta)}{v_H\,\theta^2}\right]$$

$$\Rightarrow\quad \frac{\partial x_L^u(\cdot)}{\partial\theta} \;=\; \frac{\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\left[\frac{2\,b\,N_H}{v_L\,\theta}\,\frac{\partial x_H^u(\cdot)}{\partial\theta}-\frac{r+m(\theta)}{v_L\,\theta^2}\right]}{1-\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_L}{v_L\,\theta}};\ \text{and} \qquad (11)$$

$$\frac{\partial x_H^u(\cdot)}{\partial\theta} \;=\; \frac{\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\left[\frac{2\,b\,N_L}{v_H\,\theta}\,\frac{\partial x_L^u(\cdot)}{\partial\theta}-\frac{r+m(\theta)}{v_H\,\theta^2}\right]}{1-\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\frac{2\,b\,N_H}{v_H\,\theta}}. \qquad (12)$$

(11) and (12) imply:

$$\frac{\partial x_L^u(\cdot)}{\partial\theta}\left[1-\frac{1}{\alpha_L}\left(\frac{r+m(\theta)}{v_L\,\theta}\right)^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_L}{v_L\,\theta}\right]$$

$$= -\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{r+m(\theta)}{v_L\,\theta^2}+\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_H}{v_L\,\theta}\,\frac{\partial x_H^u(\cdot)}{\partial\theta}$$

$$= -\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{r+m(\theta)}{v_L\,\theta^2}$$

$$+\;\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_H}{v_L\,\theta}\left[\frac{\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\left[\frac{2\,b\,N_L}{v_H\,\theta}\,\frac{\partial x_L^u(\cdot)}{\partial\theta}-\frac{r+m(\theta)}{v_H\,\theta^2}\right]}{1-\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\frac{2\,b\,N_H}{v_H\,\theta}}\right]$$

$$\Leftrightarrow\quad \frac{\partial x_L^u(\cdot)}{\partial\theta}\left\{1-\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_L}{v_L\,\theta}\right.$$

$$\left.-\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_H}{v_L\,\theta}\left[\frac{\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\frac{2\,b\,N_L}{v_H\,\theta}}{1-\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\frac{2\,b\,N_H}{v_H\,\theta}}\right]\right\}$$

$$= -\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{r+m(\theta)}{v_L\,\theta^2}$$

$$
-\;\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\;\frac{2\,b\,N_H}{v_L\,\theta}\left[\frac{\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{r+m(\theta)}{v_H\,\theta^2}}{1-\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{2\,b\,N_H}{v_H\,\theta}}\right]
$$

$$
\Leftrightarrow\;\frac{\partial x_L^u(\cdot)}{\partial\theta}\;=\;-\,\frac{B}{A} \tag{13}
$$

where:

$$
A\;\equiv\;1-\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\;\frac{2\,b\,N_L}{v_L\,\theta}\left[1+\frac{\frac{1}{\alpha_H}\left(\frac{r+m(\theta)}{v_H\,\theta}\right)^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{2\,b\,N_H}{v_H\,\theta}}{1-\frac{1}{\alpha_H}\left(\frac{r+m(\theta)}{v_H\,\theta}\right)^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{2\,b\,N_H}{v_H\,\theta}}\right];\quad\text{and}
$$

$$
B\;\equiv\;\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\;\frac{r+m(\theta)}{v_L\,\theta^2}
$$

$$
\cdot\left[1+\frac{2\,b\,N_H}{v_L\,\theta}\left(\frac{\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{v_L}{v_H}}{1-\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{2\,b\,N_H}{v_H\,\theta}}\right)\right]. \tag{14}
$$

Define $D\;\equiv\;\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{2\,b\,N_H}{v_H\,\theta}\;<\;0$. Since $\alpha_L\;<\;0$, (14) implies:

$$
A\;>\;0\;\;\Leftrightarrow\;\;1+\frac{D}{1-D}\;>\;0\;\;\Leftrightarrow\;\;\frac{1}{1-D}\;>\;0.
$$

This inequality holds because $D\;<\;0$.

Since $\alpha_L<0$ and $r+m(\theta)\;=\;C'(\cdot)>0$, (14) implies:

$$
B\;<\;0\;\;\Leftrightarrow\;\;1+\frac{2\,b\,N_H}{v_L\,\theta}\left[\frac{\frac{1}{\alpha_H}\left(\frac{r+m(\theta)}{v_H\,\theta}\right)^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{v_L}{v_H}}{1-D}\right]\;>\;0
$$

$$
\Leftrightarrow\;\;\frac{1-D+\frac{2\,b\,N_H}{v_L\,\theta}\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\;\frac{v_L}{v_H}}{1-D}\;>\;0\;\;\Leftrightarrow\;\;\frac{1-D+D}{1-D}\;>\;0\;\;\Leftrightarrow\;\;\frac{1}{1-D}\;>\;0\,.
$$

This inequality holds because $D<0$. Since $A>0$ and $B<0$, (13) implies $\frac{\partial x_L^u(\cdot)}{\partial\theta}>0$.

Since $\alpha_H\;<\;0$, $A>\;0$, and $\frac{\partial x_L^u(\cdot)}{\partial\theta}\;>\;0$, (12), (13), and (14) imply:

$$\frac{\partial x_H^u(\cdot)}{\partial \theta} \;>\; 0 \;\;\Leftrightarrow\;\; \frac{2\,b\,N_L}{v_H\,\theta}\left[\frac{\partial x_L^u(\cdot)}{\partial \theta}\right] - \frac{r+m(\theta)}{v_H\,\theta^2} \;<\; 0$$

$$\Leftrightarrow\;\; -\,\frac{2\,b\,N_L}{v_H\,\theta}\left[\frac{B}{A}\right] - \frac{r+m(\theta)}{v_H\,\theta^2} \;<\; 0 \;\;\Leftrightarrow\;\; 2\,b\,N_L\,B + \left[\frac{r+m(\theta)}{\theta}\right]A \;>\; 0$$

$$\Leftrightarrow\;\; 2\,b\,N_L\,\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}} \frac{r+m(\theta)}{v_L\,\theta^2}\left[1+\frac{2\,b\,N_H}{v_L\,\theta}\left(\frac{\frac{1}{\alpha_H}\left[\frac{r+m(\theta)}{v_H\,\theta}\right]^{\frac{1-\alpha_H}{\alpha_H}}\frac{v_L}{v_H}}{1-D}\right)\right]$$

$$+\;\; \frac{r+m(\theta)}{\theta}\left[1-\frac{1}{\alpha_L}\left(\frac{r+m(\theta)}{v_L\,\theta}\right)^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_L}{v_L\,\theta}\left(1+\frac{D}{1-D}\right)\right] \;>\; 0$$

$$\Leftrightarrow\;\; 2\,b\,N_L\,\frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}} \frac{r+m(\theta)}{v_L\,\theta^2}\left[1+\frac{D}{1-D}\right]$$

$$+\;\; \frac{r+m(\theta)}{\theta}\left[1-\frac{1}{\alpha_L}\left(\frac{r+m(\theta)}{v_L\,\theta}\right)^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_L}{v_L\,\theta}\left(1+\frac{D}{1-D}\right)\right] \;>\; 0$$

$$\Leftrightarrow\;\; \frac{1}{\alpha_L}\left[\frac{r+m(\theta)}{v_L\,\theta}\right]^{\frac{1-\alpha_L}{\alpha_L}}\frac{2\,b\,N_L}{v_L\,\theta}\left[\frac{r+m(\theta)}{\theta}\right]\left[\frac{1-D+D}{1-D}-\frac{1-D+D}{1-D}\right]$$

$$+\;\; \frac{r+m(\theta)}{\theta} \;>\; 0$$

$$\Leftrightarrow\;\; \frac{r+m(\theta)}{\theta} \;>\; 0 \qquad (15)$$

Because $r + m(\theta) = C'(\cdot) > 0$, the inequality in (15) holds.

Case 3. Only One Consumer Type Provides Some Demand Response

From (5) and (9):

$$\frac{\partial x_k^u(\cdot)}{\partial \theta} \;=\; -\,\frac{1}{\alpha_k}\left[\frac{r}{v_k}\right]^{\frac{1}{\alpha_k}}\theta^{-\frac{1+\alpha_k}{\alpha_k}} \;>\; 0 \;\text{ for all }\; \theta \in \Omega_k^{-D}\,; \;\text{ and} \qquad (16)$$

$$\frac{\partial x_i^u(\cdot)}{\partial \theta} \;=\; \frac{1}{\alpha_i}\left[\frac{r+m(\theta)}{v_i\,\theta}\right]^{\frac{1-\alpha_i}{\alpha_i}}\left[\frac{2\,b}{v_i\,\theta}\left(N_i\,\frac{\partial x_i^u(\cdot)}{\partial \theta}+N_k\,\frac{\partial x_k^u(\cdot)}{\partial \theta}\right)-\frac{r+m(\theta)}{v_i\,\theta^2}\right]$$

$$\Leftrightarrow \quad \frac{\partial x_i^u(\cdot)}{\partial \theta} \left[1 - \frac{1}{\alpha_i} \left(\frac{r + m(\theta)}{v_i\, \theta} \right)^{\frac{1-\alpha_i}{\alpha_i}} \frac{2\, b\, N_i}{v_i\, \theta} \right]$$

$$= \frac{1}{\alpha_i} \left[\frac{r + m(\theta)}{v_i\, \theta} \right]^{\frac{1-\alpha_i}{\alpha_i}} \left[\frac{2\, b\, N_k}{v_i\, \theta} \frac{\partial x_k^u(\cdot)}{\partial \theta} - \frac{r + m(\theta)}{v_i\, \theta^2} \right] \text{ for } \theta \in \Omega_i^D. \tag{17}$$

Using (16), (17) can be written as:

$$\frac{\partial x_i^u(\cdot)}{\partial \theta} = - \frac{ \frac{1}{\alpha_i} \left[\frac{r+m(\theta)}{v_i\, \theta} \right]^{\frac{1-\alpha_i}{\alpha_i}} \left[\frac{2\, b\, N_k}{v_i\, \theta} \frac{1}{\alpha_k} \left[\frac{r}{v_k} \right]^{\frac{1}{\alpha_k}} \theta^{-\frac{1+\alpha_k}{\alpha_k}} + \frac{r+m(\theta)}{v_i\, \theta^2} \right] }{ 1 - \frac{1}{\alpha_i} \left[\frac{r+m(\theta)}{v_i\, \theta} \right]^{\frac{1-\alpha_i}{\alpha_i}} \frac{2\, b\, N_i}{v_i\, \theta} } . \tag{18}$$

Since $\alpha_i < 0$, (18) implies that $\frac{\partial x_i^u(\cdot)}{\partial \theta} > 0$ in this θ region if and only if:

$$\frac{2\, b\, N_k}{v_i\, \theta} \frac{1}{\alpha_k} \left[\frac{r}{v_k} \right]^{\frac{1}{\alpha_k}} \theta^{-\frac{1+\alpha_k}{\alpha_k}} + \frac{r + m(\theta)}{v_i\, \theta^2} > 0$$

$$\Leftrightarrow \quad \frac{2\, b\, N_k}{v_i\, \theta^2} \frac{1}{\alpha_k} \left[\frac{r}{v_k} \right]^{\frac{1}{\alpha_k}} \theta^{-\frac{1}{\alpha_k}} + \frac{r + m(\theta)}{v_i\, \theta^2} > 0$$

$$\Leftrightarrow \quad \frac{2\, b\, N_k}{\alpha_k} \left[\frac{r}{v_k\, \theta} \right]^{\frac{1}{\alpha_k}} + r + m(\theta) > 0\, . \tag{19}$$

From (6):

$$r + m(\theta) = C'(\cdot) = a + 2\, b \sum_{j=1}^{N} x_j^u(\cdot)$$

$$= a + 2\, b \left[N_k \left(\frac{r}{v_k\, \theta} \right)^{\frac{1}{\alpha_k}} + N_i \left(\frac{r + m(\theta)}{v_i\, \theta} \right)^{\frac{1}{\alpha_i}} \right]. \tag{20}$$

(19) and (20) imply that $\frac{\partial x_i^u(\cdot)}{\partial \theta} > 0$ in this setting if:

$$\frac{2\, b\, N_k}{\alpha_k} \left[\frac{r}{v_k\, \theta} \right]^{\frac{1}{\alpha_k}} + a + 2\, b \left[N_k \left(\frac{r}{v_k\, \theta} \right)^{\frac{1}{\alpha_k}} + N_i \left(\frac{r + m(\theta)}{v_i\, \theta} \right)^{\frac{1}{\alpha_i}} \right] > 0$$

$$\Leftrightarrow \quad \frac{N_k}{\alpha_k} \left[\frac{r}{v_k\, \theta} \right]^{\frac{1}{\alpha_k}} + N_k \left[\frac{r}{v_k\, \theta} \right]^{\frac{1}{\alpha_k}} + N_i \left[\frac{r + m(\theta)}{v_i\, \theta} \right]^{\frac{1}{\alpha_i}} + \frac{a}{2\, b} > 0$$

$$\Leftrightarrow \quad N_k \left[\frac{r}{v_k\, \theta} \right]^{\frac{1}{\alpha_k}} \left[1 + \frac{1}{\alpha_k} \right] + N_i \left[\frac{r + m(\theta)}{v_i\, \theta} \right]^{\frac{1}{\alpha_i}} + \frac{a}{2\, b} > 0$$

$$\Leftrightarrow \quad N_k \left[\frac{r}{v_k\,\theta} \right]^{\frac{1}{\alpha_k}} \left[1 - \frac{1}{|\,\alpha_k\,|} \right] + N_i \left[\frac{r + m(\theta)}{v_i\,\theta} \right]^{\frac{1}{\alpha_i}} + \frac{a}{2\,b} \;>\; 0\,. \tag{21}$$

Because $r + m(\theta) \;=\; C'(\cdot) \;>\; 0$, the inequality in (21) holds if $1 \;>\; \frac{1}{|\,\alpha_k\,|}$. This inequality always holds for the parameter values considered in the numerical solutions analyzed in Section 5 of the paper.