Technical Appendix to Accompany
"Extreme Screening Policies" (Bose et al., 2012)

by

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Section A of this Technical Appendix provides more detailed proofs of the conclusions in Bose et al. (2012). Section B provides an extension of the analysis in Bose et al. (2012).

A. Detailed Proofs of Conclusions in Bose et al. (2012).

Lemma 1. $x_L = \beta \, p_L V \left[\frac{1-q}{t_L} \right]$ *and* $x_H = \beta \, p_H V \left[\frac{q}{t_H} \right]$.

A detailed proof of Lemma 1 is provided in the text of Bose et al. (2012).

Lemma 2. *The sharing rate that maximizes the lender's profit when she adopts the SA policy and implements screening accuracy q is:*

$$\widetilde{\beta}(q) = \frac{1}{2} - \frac{I}{2\,V} \left[\frac{\phi_L \, p_L \, t_H \, (1-q)^2 + \phi_H \, p_H \, t_L \, q^2}{\phi_L \, p_L^2 \, t_H \, (1-q)^2 + \phi_H \, p_H^2 \, t_L \, q^2} \right] < \frac{1}{2}, \tag{1}$$

which is a strictly increasing function of q *for all* $q \in [\frac{1}{2}, 1)$.

<u>Proof</u>. From Lemma 1:

$$\pi(\beta, q) = \left[\frac{\beta \, V}{t_L} \right] \phi_L \, p_L \, [1-q]^2 \, [p_L \, (1-\beta) \, V - I] + \left[\frac{\beta \, V}{t_H} \right] \phi_H \, p_H \, q^2 \, [p_H \, (1-\beta) \, V - I] \, . \tag{2}$$

Differentiating equation (2) provides:

$$\frac{\partial \pi(\cdot)}{\partial \beta} = \frac{V^2}{t_L} \phi_L \, p_L^2 \, [1-q]^2 + \frac{V^2}{t_H} \phi_H \, p_H^2 \, q^2 - I \, V \left[\phi_L \, p_L \, (1-q)^2 \, \frac{1}{t_L} + \phi_H \, p_H \, q^2 \, \frac{1}{t_H} \right]$$
$$- \; 2 \, \beta \, V^2 \left[\phi_L \, p_L^2 \, (1-q)^2 \, \frac{1}{t_L} + \phi_H \, p_H^2 \, q^2 \, \frac{1}{t_H} \right]. \tag{3}$$

Since $q \in [\frac{1}{2}, 1]$, it is apparent from equation (3) that $\pi(\cdot)$ is a strictly concave function of β, that $\left. \frac{\partial \pi(\cdot)}{\partial \beta} \right|_{\beta = 1} < 0$, and that $\left. \frac{\partial \pi(\cdot)}{\partial \beta} \right|_{\beta = 0} > 0$ when Assumption 1 holds. Therefore, the expression for $\widetilde{\beta}(q)$ in equation (1) follows directly from equation (3).

Differentiating equation (1) provides:

$$
\begin{aligned}
\widetilde{\beta}'(q) \;&\overset{s}{=}\; -\left[\phi_L\, p_L^2\, t_H\, (1-q)^2 + \phi_H\, p_H^2\, t_L\, q^2\right] 2\left\{\phi_H\, p_H\, t_L\, q - \phi_L\, p_L\, t_H\, [1-q]\right\} \\
&\qquad + \left\{\phi_L\, p_L\, t_H\, [1-q]^2 + \phi_H\, p_H\, t_L\, q^2\right\} 2\left[\phi_H\, p_H^2\, t_L\, q - \phi_L\, p_L^2\, t_H\, (1-q)\right] \\
&= -\,2\, t_L\, t_H \{\, \phi_L\, p_L^2\, \phi_H\, p_H\, q\, [1-q]^2 - \phi_L\, p_L\, \phi_H\, p_H^2\, q^2\, [1-q] \\
&\qquad - \phi_L\, p_L\, \phi_H\, p_H^2\, q\, [1-q]^2 + \phi_L\, p_L^2\, \phi_H\, p_H\, q^2\, [1-q] \,\} \\
&\overset{s}{=} -\,\phi_L\, p_L\, \phi_H\, p_H\, q\, [1-q]\, [p_L\, (1-q) - p_H\, q - p_H\, (1-q) + p_L\, q] \;\overset{s}{=}\; p_H - p_L \;>\; 0. \quad \blacksquare
\end{aligned}
$$

Lemma 3. *The lender's revenue under the SA policy, $\widetilde{\pi}(q)$, is strictly increasing in q for all $q \in [\,\frac{1}{2}, \overline{q}\,]$. Furthermore, if $\phi_L \leq \widehat{\phi}_L$ and $G \geq \max_q K''(q)$, then the lender's profit under this policy, $\widetilde{\Pi}(q)$, is a strictly convex function of q for all $q \in [\,\frac{1}{2}, \overline{q}\,]$, for any $\overline{q} \in (\,\frac{1}{2}, 1\,]$.*

Proof. Substituting from Lemma 1 provides:

$$
\widetilde{\pi}(q) \;=\; \frac{[Z_1(q)]^2}{4\, t_L\, t_H\, Z_2(q)} \;>\; 0\,, \tag{4}
$$

where:

$$
Z_1(q) \;\equiv\; \phi_H\, p_H\, q^2\, t_L\, [p_H V - I] + \phi_L\, p_L\, [1-q]^2\, t_H\, [p_L V - I] \;>\; 0, \quad \text{and} \tag{5}
$$

$$
Z_2(q) \;\equiv\; \phi_H\, p_H^2\, q^2\, t_L + \phi_L\, p_L^2\, [1-q]^2\, t_H \;>\; 0\,. \tag{6}
$$

Differentiating (4) provides:

$$
\begin{aligned}
\widetilde{\pi}'(q) \;&=\; \frac{Z_1(q)}{4\, t_L\, t_H\, [Z_2(q)]^2} \{\, 4\, Z_2(q)\, [\phi_H\, p_H\, q\, t_L\, (p_H V - I) - \phi_L\, p_L\, (1-q)\, t_H\, (p_L V - I)] \\
&\qquad - 2\, Z_1(q)\, \left[\phi_H\, p_H^2\, q\, t_L - \phi_L\, p_L^2\, (1-q)\, t_H\right] \} \\
&= \frac{Z_1(q)}{2\, t_L\, t_H\, [Z_2(q)]^2} \{\, 2\left[\phi_H\, p_H^2\, q^2\, t_L + \phi_L\, p_L^2\, (1-q)^2\, t_H\right] \\
&\qquad \cdot [\phi_H\, p_H\, q\, t_L\, (p_H V - I) - \phi_L\, p_L\, (1-q)\, t_H\, (p_L V - I)] \\
&\qquad - \left[\phi_H\, p_H\, q^2\, t_L\, (p_H V - I) + \phi_L\, p_L\, (1-q)^2\, t_H\, (p_L V - I)\right] \\
&\qquad \cdot \left[\phi_H\, p_H^2\, q\, t_L - \phi_L\, p_L^2\, (1-q)\, t_H\right] \} \\
&= \frac{Z_1(q)}{2\, t_L\, t_H\, [Z_2(q)]^2} \{\, 2\, \phi_H^2\, p_H^3\, q^3\, t_L^2\, [p_H V - I] + 2\, \phi_L\, p_L^2\, \phi_H\, p_H\, q\, [1-q]^2\, t_L\, t_H\, [p_H V - I] \\
&\qquad - 2\, \phi_L\, p_L\, \phi_H\, p_H^2\, q^2\, [1-q]\, t_L\, t_H\, [p_L V - I] - 2\, \phi_L^2\, p_L^3\, [1-q]^3\; t_H^2\, [p_L V - I] \\
&\qquad - \phi_H^2\, p_H^3\, q^3\, t_L^2\, [p_H V - I] + \phi_L\, p_L^2\, \phi_H\, p_H\, q^2\, [1-q]\, t_L\, t_H\, [p_H V - I]
\end{aligned}
$$

$$- \phi_L\, p_L\, \phi_H\, p_H^2\, q\, [1-q]^2\, t_L\, t_H\, [p_L V - I] + \phi_L^2\, p_L^3\, [1-q]^3\; t_H^2\, [p_L V - I] \,\}$$

$$= \frac{Z_1(q)}{2\, t_L\, t_H\, [Z_2(q)]^2} \{\, \phi_H\, p_H\, q\, t_L\, [p_H V - I]\, \left[\phi_H\, p_H^2\, q^2\, t_L + \phi_L\, p_L^2\, (1-q)\,(2-q)\, t_H \right]$$

$$-\; \phi_L\, p_L\, [1-q]\, t_H\, [p_L V - I]\, \left[\phi_L\, p_L^2\, (1-q)^2\, t_H + \phi_H\, p_H^2\, q\,(1+q)\, t_L \right] \} \;>\; 0\,. \qquad (7)$$

The inequality in (7) holds because $p_H V - I \;>\; 0 \;>\; p_L V - I$.

We now show that $\widetilde{\pi}''(q) \;>\; 0$ if $\phi_L \;\leq\; \widehat{\phi}_L$. From (4):

$$\widetilde{\pi}(q) \;=\; \frac{\left[\phi_H\, p_H\, q^2\, t_L\, (p_H V - I) + \phi_L\, p_L\, (1-q)^2\, t_H\, (p_L V - I) \right]^2}{4\, t_H\, t_L\, \left[\phi_H\, p_H^2\, q^2\, t_L + \phi_L\, p_L^2\, (1-q)^2\, t_H \right]}$$

$$= \frac{[\phi_H\, p_H\, t_L\, (p_H V - I)]^2}{4\, t_H\, t_L\, \phi_H\, p_H^2\, t_L} \frac{\left[q^2 + \frac{\phi_L\, p_L\, t_H (p_L V - I)}{\phi_H\, p_H\, t_L (p_H V - I)}\, (1-q)^2 \right]^2}{\left[q^2 + \frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L}\, (1-q)^2 \right]} \;=\; M\, \frac{\left[q^2 + \delta_2\, (1-q)^2 \right]^2}{\left[q^2 + \delta_1\, (1-q)^2 \right]}\,, \qquad (8)$$

where:

$$\delta_1 \;=\; \frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L} \;>\; 0\,, \quad \delta_2 \;=\; \frac{\phi_L\, p_L\, t_H\, [p_L V - I]}{\phi_H\, p_H\, t_L\, [p_H V - I]} \;<\; 0\,, \quad \text{and} \qquad (9)$$

$$M \;=\; \frac{[\phi_H\, p_H\, t_L\, (p_H V - I)]^2}{4\, t_H\, t_L\, \phi_H\, p_H^2\, t_L}\,.$$

Define

$$\widehat{\pi} \;=\; \frac{\left[q^2 + \delta_2\, (1-q)^2 \right]^2}{q^2 + \delta_1\, [1-q]^2}\,. \qquad (10)$$

Since δ_1, δ_2 and M are independent of q, (8) and (10) provide:

Result A1. $\widetilde{\pi}$ is convex in q if and only if $\widehat{\pi}$ is convex in q.

From (10):

$$\left[\, q^2 + \delta_1\, (1-q)^2 \,\right] \widehat{\pi} \;=\; \left[q^2 + \delta_2\, (1-q)^2 \right]^2\,. \qquad (11)$$

Let:

$$g_1 \;\equiv\; q^2 + \delta_1\, [1-q]^2 \;>\; 0 \quad \text{and} \quad g_2 \;\equiv\; q^2 + \delta_2\, [1-q]^2\,. \qquad (12)$$

(11) and (12) provide:

$$g_1\, \widehat{\pi} \;=\; (g_2)^2\,. \qquad (13)$$

Differentiating (13) with respect to q provides:

$$g_1\, \widehat{\pi}' + g_1'\, \widehat{\pi} \;=\; 2\, g_2\, g_2'\,. \qquad (14)$$

Differentiating (14) with respect to q provides:

$$g_1'\, \widehat{\pi}' + g_1\, \widehat{\pi}'' \;+\; g_1''\, \widehat{\pi} + g_1'\, \widehat{\pi}' \;=\; 2\, (g_2')^2 + 2\, g_2\, g_2''$$

$$\Leftrightarrow \quad g_1\,\widehat{\pi}'' + \,g_1''\,\widehat{\pi} + 2\,g_1'\,\widehat{\pi}' \;=\; 2\left(g_2'\right)^2 + 2\,g_2\,g_2''$$

$$\Leftrightarrow \quad g_1\,\widehat{\pi}'' \;=\; 2\left(g_2'\right)^2 + 2\,g_2\,g_2'' - \,g_1''\,\widehat{\pi} - 2\,g_1'\,\widehat{\pi}' \,. \tag{15}$$

Since $g_1 > 0$, (15) implies that $\widehat{\pi}$ is convex in q if the expression to the right of the equality in (15) is positive.

From (12):

$$g_1' \;=\; 2\,q - 2\,\delta_1\,[1-q] \quad \Rightarrow \quad g_1'' \;=\; 2 + 2\,\delta_1 \;=\; 2\,[1+\delta_1]\,, \quad \text{and} \tag{16}$$

$$g_2' \;=\; 2\,q - 2\,\delta_2\,[1-q] \quad \Rightarrow \quad g_2'' \;=\; 2 + 2\,\delta_2 \;=\; 2\,[1+\delta_2]\,. \tag{17}$$

Using (16) and (17) in (15) provides:

$$g_1\,\widehat{\pi}'' \;=\; 2\left(g_2'\right)^2 + 4\,g_2\,[1+\delta_2] - 2\,[1+\delta_1]\,\widehat{\pi} - 2\,g_1'\,\widehat{\pi}'$$

$$\Leftrightarrow \quad \left[\frac{g_1}{2}\right]\widehat{\pi}'' \;=\; \left(g_2'\right)^2 + 2\,g_2\,[1+\delta_2] - [1+\delta_1]\,\widehat{\pi} - g_1'\,\widehat{\pi}' \,. \tag{18}$$

From (14):

$$\widehat{\pi}' \;=\; \frac{2\,g_2\,g_2' - g_1'\,\widehat{\pi}}{g_1} \,. \tag{19}$$

Relations (13), (18), and (19) provide:

$$\left[\frac{g_1}{2}\right]\widehat{\pi}'' \;=\; \left(g_2'\right)^2 + 2\,g_2\,[1+\delta_2] - [1+\delta_1]\,\widehat{\pi} - g_1'\left[\frac{2\,g_2\,g_2' - g_1'\,\widehat{\pi}}{g_1}\right] \quad \Leftrightarrow$$

$$\left[\frac{g_1^2}{2}\right]\widehat{\pi}'' \;=\; g_1\left(g_2'\right)^2 + 2\,g_1\,g_2\,[1+\delta_2] - g_1\,[1+\delta_1]\,\widehat{\pi} - g_1'\,g_2\,g_2' + \frac{g_1'\,g_2}{g_1}\,[g_1'\,g_2 - g_1\,g_2'] \,. \tag{20}$$

From (12), (16), and (17):

$$\begin{aligned} g_1'\,g_2 - g_1\,g_2' \;&=\; [2\,q - 2\,\delta_1\,(1-q)]\left[q^2 + \delta_2\,(1-q)^2\right] \\ &\quad - \left[q^2 + \delta_1\,(1-q)^2\right][2\,q - 2\,\delta_2\,(1-q)] \\ &=\; 2\,q^3 - 2\,\delta_1\,q^2\,[1-q] + 2\,q\,\delta_2\,[1-q]^2 - 2\,\delta_1\,\delta_2\,[1-q]^3 \\ &\quad - \left[\,2\,q^3 + 2\,q\,\delta_1\,(1-q)^2 - 2\,\delta_2\,q^2\,(1-q) - 2\,\delta_1\,\delta_2\,(1-q)^3\right] \\ &=\; -\,2\,\delta_1 q\,[1-q] + 2\,q\,\delta_2\,[1-q] \;=\; 2\,q\,[1-q]\,[\delta_2 - \delta_1]\,. \end{aligned} \tag{21}$$

Relation (21) implies:

$$g_1\left(g_2'\right)^2 - g_1'\,g_2\,g_2' \;=\; g_2'\,[g_1\,g_2' - g_1'\,g_2] \;=\; -\,2\,g_2'\,q\,[1-q]\,[\delta_2 - \delta_1]\,. \tag{22}$$

From (20), (21), and (22):

$$\left[\frac{g_1^2}{2}\right]\widehat{\pi}'' \;=\; g_1\left(g_2'\right)^2 - g_1'\,g_2\,g_2' + \frac{g_1'\,g_2}{g_1}\,[g_1'\,g_2 - g_1\,g_2'] + 2\,g_1\,g_2\,[1+\delta_2] - g_1\,[1+\delta_1]\,\widehat{\pi}$$

$$= \; - \, 2\, g_2'\, q\, [1-q]\, [\delta_2 - \delta_1] + \frac{g_1'\, g_2}{g_1}\, 2\, q\, [1-q]\, [\delta_2 - \delta_1] + 2\, g_1\, g_2\, [1+\delta_2] - g_1\, [1+\delta_1]\, \widehat{\pi}$$

$$= \; \frac{2\, q\, [1-q]\, [\delta_2 - \delta_1]}{g_1}\, [g_1'\, g_2 - g_1\, g_2'] + 2\, g_1\, g_2\, [1+\delta_2] - g_1\, [1+\delta_1]\, \widehat{\pi}$$

$$= \; \frac{4\, q^2\, [1-q]^2\, [\delta_2 - \delta_1]^2}{g_1} + 2\, g_1\, g_2\, [1+\delta_2] - g_1\, [1+\delta_1]\, \widehat{\pi}\,. \tag{23}$$

From (12) and (13):

$$2\, g_1\, g_2\, [1+\delta_2] - g_1\, [1+\delta_1]\, \widehat{\pi} \; = \; 2\, g_1\, g_2\, [1+\delta_2] - g_1\, [1+\delta_1]\, \frac{(g_2)^2}{g_1}$$

$$= \; 2\, g_1\, g_2\, [1+\delta_2] - [1+\delta_1]\, (g_2)^2 \; = \; g_2\, [2\, g_1\, (1+\delta_2) - g_2\, (1+\delta_1)]$$

$$= \; g_2 \left\{ 2 \left[q^2 + \delta_1\, (1-q)^2 \right] [1+\delta_2] - \left[q^2 + \delta_2\, (1-q)^2 \right] [1+\delta_1] \right\}$$

$$= \; g_2 \left\{ q^2\, [1 + 2\, \delta_2 - \delta_1] + [1-q]^2\, [2\, \delta_1 + \delta_1\, \delta_2 - \delta_2] \right\}\,. \tag{24}$$

Using (24) in (23) provides:

$$\left[\frac{g_1^2}{2} \right] \widehat{\pi}'' \; = \; \frac{4\, q^2\, [1-q]^2\, [\delta_2 - \delta_1]^2}{g_1} + g_2 \left\{ q^2\, [1 + 2\, \delta_2 - \delta_1] + [1-q]^2\, [2\, \delta_1 + \delta_1\, \delta_2 - \delta_2] \right\}$$

$$\Leftrightarrow \quad \left[\frac{g_1^3}{2} \right] \widehat{\pi}'' \; = \; 4\, q^2\, [1-q]^2\, [\delta_2 - \delta_1]^2 + g_1\, g_2\, \{ \, q^2\, [1 + 2\, \delta_2 - \delta_1] + [1-q]^2\, [2\, \delta_1 + \delta_1\, \delta_2 - \delta_2] \, \}\,. \tag{25}$$

Notice from (9) that $\delta_2 > -1$ since $\phi_H\, p_H\, t_L\, [p_H V - I] + \phi_L\, p_L\, t_H\, [p_L V - I] > 0$. Therefore, from (12):

$$g_2 \; = \; q^2 + \delta_2\, [1-q]^2 \; > \; q^2 - [1-q]^2 \; \geq \; 0, \; \text{ since } \; q \; \geq \; \frac{1}{2}\,. \tag{26}$$

Since $\delta_1 > 0$ and $\delta_2 \in (-1, 0)$:

$$2\, \delta_1 + \delta_1\, \delta_2 - \delta_2 \; = \; \delta_1\, [2 + \delta_2] - \delta_2 \; > \; 0\,. \tag{27}$$

Using (26) and (27) in (25) provides:

Result A2. $\widehat{\pi}'' > 0$ if $1 + 2\, \delta_2 - \delta_1 \geq 0$.

Now, suppose $\widehat{\pi}'' > 0$ for all $q \in \left[\frac{1}{2}, 1 \right]$. Then, $\widehat{\pi}'' \geq 0$ when $q = 1$. Using $q = 1$ in (25) provides:

$$\left[\frac{g_1^3}{2} \right] \widehat{\pi}'' \; = \; g_1\, g_2\, [1 + 2\, \delta_2 - \delta_1]\,. \tag{28}$$

Result A3 follows from (28).

Result A3. If $\widehat{\pi}'' \geq 0$ for all $q \in \left[\frac{1}{2}, 1\right]$, then $1 + 2\,\delta_2 - \delta_1 \geq 0$.

Result A4 follows from Results A1, A2, and A3.

Result A4. $\widetilde{\pi}'' \geq 0$ for all $q \in \left[\frac{1}{2}, 1\right]$ if and only if $1 + 2\,\delta_2 - \delta_1 \geq 0$.

To simplify the condition $1 + 2\,\delta_2 - \delta_1 \geq 0$, notice from (9) that:

$$\delta_2 \; = \; \frac{\phi_L\, p_L\, t_H\, [\, p_L V - I\,]}{\phi_H\, p_H\, t_L\, [\, p_H V - I\,]} \; = \; \delta_1 \, \frac{p_H}{p_L} \left[\frac{p_L V - I}{p_H V - I} \right]. \tag{29}$$

From (9) and (29):

$$\begin{aligned}
1 + 2\,\delta_2 - \delta_1 \; &= \; 1 + 2\,\delta_1 \frac{p_H}{p_L}\left[\frac{p_L V - I}{p_H V - I}\right] - \delta_1 \; = \; 1 + \delta_1\left[2\,\frac{p_H}{p_L}\left(\frac{p_L V - I}{p_H V - I}\right) - 1\right] \\
&= \; 1 + \delta_1 \left[\frac{2\, p_H\, (p_L V - I) - p_L\, (p_H V - I)}{p_L\, (p_H V - I)} \right] \; \geq \; 0
\end{aligned}$$

$$\Leftrightarrow \quad \delta_1 \left[2\, p_H\, (p_L V - I) - p_L\, (p_H V - I) \right] \; \geq \; -\, p_L\, [p_H V - I]$$

$$\Leftrightarrow \quad \delta_1 \; \leq \; \frac{p_L\, [p_H V - I]}{p_L\, [p_H V - I] - 2\, p_H\, [p_L V - I]} \tag{30}$$

$$\Leftrightarrow \quad \frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L} \; \leq \; \frac{p_L\, [p_H V - I]}{p_L\, [p_H V - I] - 2\, p_H\, [p_L V - I]}$$

$$\Leftrightarrow \quad \frac{\phi_H}{\phi_L} \; \geq \; \frac{p_L\, [p_H V - I] - 2\, p_H\, [p_L V - I]}{p_L\, [p_H V - I]} \left[\frac{p_L^2\, t_H}{p_H^2\, t_L} \right]$$

$$\Leftrightarrow \quad \frac{1 - \phi_L}{\phi_L} \; \geq \; \frac{p_L^2\, t_H\, [p_H V - I] - 2\, p_H\, p_L\, t_H\, [p_L V - I]}{p_H^2\, t_L\, [p_H V - I]}$$

$$\Leftrightarrow \quad \frac{1}{\phi_L} \; \geq \; \frac{[p_H V - I]\, [p_L^2\, t_H + p_H^2\, t_L] - 2\, p_H\, p_L\, t_H\, [p_L V - I]}{p_H^2\, t_L\, [p_H V - I]} \quad \Leftrightarrow \quad \phi_L \; \leq \; \widehat{\phi}_L\,. \tag{31}$$

Result A4 and (31) ensure that $\widetilde{\pi}$ is convex in q for all $q \in \left[\frac{1}{2}, 1\right]$ if $\phi_L \leq \widehat{\phi}_L$.

Finally, to demonstrate the convexity of $\widetilde{\Pi}\,(q)$, differentiating (2) provides:

$$\frac{\partial^2 \widetilde{\Pi}\,(q)}{\partial q^2} \; = \; m\,(\beta) - K''\,(q) \tag{32}$$

where:

$$m\,(\beta) \; \equiv \; \frac{2\,\beta\, V}{t_H}\, \phi_H\, p_H\, [\, p_H V\, (1 - \beta) - I\,] + \frac{2\,\beta\, V}{t_L}\, \phi_L\, p_L\, [\, p_L V\, (1 - \beta) - I\,]\,. \tag{33}$$

Observe that $m\,(\beta)$ is a concave function of β because:

$$m'\,(\beta) \; = \; \frac{2\,\phi_H\, p_H V\, [\, p_H V\, (1 - 2\,\beta) - I\,]}{t_H} + \frac{2\,\phi_L\, p_L V\, [\, p_L V\, (1 - 2\,\beta) - I\,]}{t_L} \tag{34}$$

$$\Rightarrow \quad m''(\beta) \;=\; -\frac{4\,\phi_H\,p_H^2\,V^2}{t_H} - \frac{4\,\phi_L\,p_L^2\,V^2}{t_L} \;<\; 0\,. \tag{35}$$

Recall from Lemma 2 that β is an increasing function of q. Furthermore, from (1):

$$\beta\big|_{q=\frac{1}{2}} \;=\; \beta^{\min} \;=\; \frac{\phi_L\,p_L\,t_H\,[\,p_L V - I\,] + \phi_H\,p_H\,t_L\,[\,p_H V - I\,]}{2\,V\,[\phi_L\,p_L^2\,t_H + \phi_H\,p_H^2\,t_L]}\,, \quad \text{and} \tag{36}$$

$$\beta\big|_{q=1} \;=\; \beta^{\max} \;=\; \frac{\phi_H\,p_H\,t_L\,[\,p_H V - I\,]}{2\,V\,\phi_H\,p_H^2\,t_L} \;=\; \frac{p_H V - I}{2\,p_H V}\,. \tag{37}$$

Straightforward calculations employing (36) reveal:

$$\frac{2\,\phi_H\,p_H V\left[\,p_H V\left(1 - 2\,\beta^{\min}\right) - I\,\right]}{t_H} \;=\; \frac{2\,\phi_L\,p_L\,\phi_H\,p_H\,I\,V\,[\,p_H - p_L\,]}{\phi_L\,p_L^2\,t_H + \phi_H\,p_H^2\,t_L}\,, \quad \text{and} \tag{38}$$

$$\frac{2\,\phi_L\,p_L V\left[\,p_L V\left(1 - 2\,\beta^{\min}\right) - I\,\right]}{t_L} \;=\; -\,\frac{2\,\phi_L\,p_L\,\phi_H\,p_H\,I\,V\,[\,p_H - p_L\,]}{\phi_L\,p_L^2\,t_H + \phi_H\,p_H^2\,t_L}\,. \tag{39}$$

(34), (38), and (39) provide:

$$m'(\beta)\big|_{\beta=\beta^{\min}} \;=\; 0\,. \tag{40}$$

(35) and (40) imply that $m'(\beta) < 0$ for all $\beta \in (\beta^{\min}, \beta^{\max})$ and so $m(\cdot)$ attains its minimum at $\beta = \beta^{\max}$. From (33) and (37):

$$\begin{aligned}
m(\beta^{\max}) \;&=\; \frac{2\,\phi_H\,p_H V}{t_H}\left[\frac{p_H V - I}{2\,p_H V}\right]\left[p_H V\left(\frac{2\,p_H V - (p_H V - I)}{2\,p_H V}\right) - I\right] \\
&\quad + \frac{2\,\phi_L\,p_L V}{t_L}\left[\frac{p_H V - I}{2\,p_H V}\right]\left[p_L V\left(\frac{2\,p_H V - (p_H V - I)}{2\,p_H V}\right) - I\right] \\
&= \frac{2\,\phi_H\,p_H V}{t_H}\left[\frac{p_H V - I}{2\,p_H V}\right]\left[\frac{p_H V + I}{2} - I\right] + \frac{2\,\phi_L\,p_L V}{t_L}\left[\frac{p_H V - I}{2\,p_H V}\right]\left[\frac{p_L}{p_H}\left(\frac{p_H V + I}{2}\right) - I\right] \\
&= \frac{\phi_H}{t_H}\,[p_H V - I]\left[\frac{p_H V - I}{2}\right] + \frac{\phi_L\,p_L}{p_H\,t_L}\,[p_H V - I]\left[\frac{p_L\,p_H\,V - (2\,p_H - p_L)\,I}{2\,p_H}\right] \;=\; G\,.
\end{aligned} \tag{41}$$

(32) and (41) imply:

$$\frac{\partial^2 \widetilde{\Pi}\,(q)}{\partial\,(q)^2} \;\geq\; G - K''\,(q)\;. \tag{42}$$

(42) implies that $\widetilde{\Pi}\,(\cdot)$ is a strictly convex function of q when $G \geq \max_q K''\,(q)$ if $\frac{d^2\Pi(q)}{dq^2} \geq \frac{\partial^2\Pi(q)}{\partial q^2}$. The Second Order Envelope Theorem (Cornes, 1992, pp. 24 - 26) ensures that this

inequality holds. ■

Proposition 1. *Suppose the conditions of Lemma 3 hold and* $\widetilde{\pi}'(\frac{1}{2}) > K'(\frac{1}{2})$*. Then the lender maximizes her profit by implementing the AA policy when* $\overline{q} < q^c$ *and by implementing the SA policy and setting* $q = \overline{q}$ *when* $\overline{q} > q^c$*, where* $q^c \in (\frac{1}{2}, 1)$ *is defined by* $\widetilde{\pi}(q^c) = \pi_A$*.*

Proof. We first prove that:

$$\pi_A = \frac{[\,\phi_L\, p_L\, t_H\,(p_L V - I) + \phi_H\, p_H\, t_L\,(p_H V - I)\,]^2}{4\, t_L\, t_H\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]}\,. \tag{43}$$

To do so, observe that when the lender approves every request for funding, the i entrepreneur located farthest from the origin that applies for financing is located at $x_i^A = \beta\, p_i V / t_i$ for $i \in \{L, H\}$.[1] Therefore, the lender's expected profit under the AA policy is:

$$\varphi_A(\beta) = \phi_L\,[\,p_L\,(1-\beta)\,V - I\,]\left[\frac{\beta\, p_L V}{t_L}\right] + \phi_H\,[\,p_H\,(1-\beta)\,V - I\,]\left[\frac{\beta\, p_H V}{t_H}\right]. \tag{44}$$

Differentiating (44) provides:

$$\varphi_A'(\beta) = \frac{\phi_L\, p_L V}{t_L}\,[\,p_L\,(1-2\,\beta)\,V - I\,] + \frac{\phi_H\, p_H V}{t_H}\,[\,p_H\,(1-2\,\beta)\,V - I\,]\,. \tag{45}$$

It is apparent that $\varphi_A''(\beta) < 0$. Furthermore, Assumption 1 ensures $\varphi_A'(\beta)|_{\beta=0} > 0$. Therefore, (45) implies that the lender's preferred sharing rate under the AA policy is given by:

$$\varphi_A'(\beta) = 0 \quad \Leftrightarrow \quad \beta_A = \frac{\phi_L\, p_L\, t_H\,[\,p_L V - I\,] + \phi_H\, p_H\, t_L\,[\,p_H V - I\,]}{2\,V\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]} \tag{46}$$

$$\Rightarrow \quad 1 - \beta_A = \frac{1}{2\,V\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]}\{\,2\,V\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]$$

$$- \;\phi_L\, p_L\,[p_L V - I]\, t_H - \phi_H\, p_H\,[p_H V - I]\, t_L\,\}$$

$$= \frac{V\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L] + I\,[\phi_L\, p_L\, t_H + \phi_H\, p_H\, t_L]}{2\,V\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]}$$

$$\Rightarrow \quad [1-\beta_A]\,V\left[\phi_L\, p_L^2\,\frac{1}{t_L} + \phi_H\, p_H^2\,\frac{1}{t_H}\right]$$

$$= \frac{[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]\,\{V\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L] + I\,[\phi_L\, p_L\, t_H + \phi_H\, p_H\, t_L]\}}{2\, t_L\, t_H\,[\phi_L\, p_L^2\, t_H + \phi_H\, p_H^2\, t_L]}\,. \tag{47}$$

(44) and (47) imply:

$$\varphi_A(\beta_A) = \beta_A\, V\left\{\frac{\phi_L\, p_L}{t_L}\,[\,p_L\,(1-\beta)\,V - I\,] + \frac{\phi_H\, p_H}{t_H}\,[\,p_H\,(1-\beta)\,V - I\,]\right\}$$

[1] t_L and t_H are assumed to be sufficiently large relative to V that $x_i^A < 1$ for $i = L, H$.

$$= \beta_A V \left\{ \frac{[\phi_L p_L^2 t_H + \phi_H p_H^2 t_L] \{V [\phi_L p_L^2 t_H + \phi_H p_H^2 t_L] + I [\phi_L p_L t_H + \phi_H p_H t_L]\}}{2 t_L t_H [\phi_L p_L^2 t_H + \phi_H p_H^2 t_L]} - I \left[\frac{\phi_L p_L}{t_L} + \frac{\phi_H p_H}{t_H} \right] \right\}$$

$$= \beta_A V \frac{1}{2 t_L t_H} \{V [\phi_L p_L^2 t_H + \phi_H p_H^2 t_L] + I [\phi_L p_L t_H + \phi_H p_H t_L]\} - \beta V \frac{2 I [\phi_L p_L t_H + \phi_H p_H t_L]}{2 t_L t_H}$$

$$= \frac{\beta_A V}{2 t_L t_H} \{\phi_L p_L t_H [p_L V - I] + \phi_H p_H t_L [p_H V - I]\} . \tag{48}$$

(46) and (48) imply that the lender's profit under the AA policy is as specified in (43).

We now prove that $\widetilde{\pi}(\frac{1}{2}) < \pi_A < \widetilde{\pi}(1)$. To do so, observe from (4) and (43) that:

$$\widetilde{\pi}(\frac{1}{2}) = \frac{[\, \phi_L p_L t_H (p_L V - I) + \phi_H p_H t_L (p_H V - I) \,]^2}{16 \, t_L t_H [\, \phi_L p_L^2 t_H + \phi_H p_H^2 t_L \,]} = \frac{1}{4} \pi_A .$$

Also, from (43):

$$\pi_A < \frac{[\, \phi_H p_H t_L (p_H V - I) \,]^2}{4 \, t_L t_H \phi_H p_H^2 t_L} = \frac{\phi_H [\, p_H V - I \,]^2}{4 \, t_H} = \widetilde{\pi}(1) . \tag{49}$$

The inequality in (49) holds because:

$$0 < \phi_L p_L t_H [\, p_L V - I \,] + \phi_H p_H t_L [\, p_H V - I \,] < \phi_H p_H t_L [\, p_H V - I \,] . \tag{50}$$

The first inequality in (50) reflects Assumption 1. The second inequality holds because $p_L V - I < 0$.

Since: (i) $\widetilde{\pi}(\frac{1}{2}) < \pi_A < \widetilde{\pi}(1)$; (ii) π_A does not vary with q (from (43)); and (iii) $\widetilde{\pi}(q)$ is a strictly increasing function of q (from (7)), it follows that there exists a $q^c \in (\frac{1}{2}, 1)$ for which $\widetilde{\pi}(q^c) = \pi_A$. Consequently, if $\widetilde{\Pi}(q)$ is a strictly convex function of q for all $q \in [\, \frac{1}{2}, \overline{q} \,]$, and if $\widetilde{\Pi}'(\frac{1}{2}) > 0$ (so that $\widetilde{\Pi}'(q) > 0$ for all $q \in [\, \frac{1}{2}, \overline{q} \,]$), then $\pi_A > \widetilde{\Pi}(q)$ for all $q \in [\frac{1}{2}, \overline{q}\,]$ when $\overline{q} < q^c$ and $\widetilde{\Pi}(\overline{q}) > \pi_A$ when $\overline{q} > q^c$. ■

Corollary 1. *Suppose* $\overline{q} = 1$, $K(1) < \widetilde{\pi}(1) - \pi_A$ *when* $\phi_H = \widehat{\phi}_H = 1 - \widehat{\phi}_L$, *and the conditions of Proposition 1 hold. Then the lender will adopt the SA policy and set* $q = 1$ *if* $\phi_H < \phi_H^c \equiv \frac{2 \gamma_1}{2 \gamma_1 + \gamma_2 + \sqrt{(\gamma_2)^2 - 4 \gamma_1 \gamma_3}}$. *The lender will adopt the AA policy if* $\phi_H > \phi_H^c$. *Consequently, as* ϕ_H *declines from just above* ϕ_H^c *to just below* ϕ_H^c, *the welfare of* L *entrepreneurs declines from* $\frac{\phi_L}{2 t_L} [\, p_L \beta_A V \,]^2$ *to* 0 *and the welfare of* H *entrepreneurs increases* *from* $\frac{\phi_H}{2\,t_H}\,[\,p_H\,\beta_A V\,]^2$ *to* $\frac{\phi_H}{2\,t_H}\left[\,p_H\,\widetilde{\beta}(1)\,V\,\right]$.[2]

Proof. (4) and (43) imply that the difference between the lender's profit under the SA policy with $q = 1$ and her profit under the AA policy is:

$$\Omega\left(\phi_H\right) \;\equiv\; \frac{\phi_H\,[\,p_H V - I\,]^2}{4\,t_H} - \;K(1) - \frac{[\,\phi_L\,p_L\,t_H\,(p_L V - I) + \phi_H\,p_H\,t_L\,(p_H V - I)\,]^2}{4\,t_L\,t_H\,[\phi_L\,p_L^2\,t_H + \phi_H\,p_H^2\,t_L]}\,. \tag{51}$$

Therefore, the Corollary holds if there exists a unique ϕ_H^c such that $\Omega\left(\phi_H\right) \lesseqgtr 0$ as $\phi_H \gtreqless \phi_H^c$. Observe that $\Omega\left(1\right) < 0$ because the AA policy and the SA policy generate the same revenue for the lender when $\phi_H = 1$ but the SA policy is more costly, since $K(1) > 0$. By assumption, $\Omega(\widehat{\phi}_H) = \{\,\widetilde{\pi}(\cdot) - K(1) - \pi_A\}_{\phi_H = \widehat{\phi}_H} > 0$.

It is readily shown that $\Omega''\left(\phi_H\right) = -\frac{I^2\,p_H^2\,p_L^2\,[p_H - p_L]^2 t_H\,t_L}{2\left[\phi_L\,p_L^2\,t_H + \phi_H\,p_H^2\,t_L\right]^3} < 0$, so $\Omega\left(\phi_H\right)$ is a concave function of ϕ_H. Therefore, since $\Omega(\widehat{\phi}_H) > 0$ and $\Omega\left(1\right) < 0$, there exists a unique $\phi_H^c \in (\widehat{\phi}_H, 1)$ such that $\Omega(\phi_H^c) = 0$.

Because $\Omega\left(\phi_H\right)$ is a quadratic function of ϕ_H, the equation $\Omega\left(\phi_H\right) = 0$ may have two real roots. Since $\Omega\left(\phi_H\right)$ is a concave function of ϕ_H, $\Omega(\widehat{\phi}_H) > 0$, and $\Omega\left(1\right) < 0$, the larger of the two roots lies between $\widehat{\phi}_H$ and 1.

Let $y \equiv \frac{\phi_L}{\phi_H}$. Then:

$$y \;=\; \frac{1 - \phi_H}{\phi_H} \;=\; \frac{1}{\phi_H} - 1 \;\;\Rightarrow\;\; \frac{1}{\phi_H} \;=\; 1 + y \;\;\Rightarrow\;\; \phi_H \;=\; \frac{1}{1 + y}. \tag{52}$$

From (51) and (52), $\Omega\left(\phi_H\right) = 0$ if and only if:

$$\frac{\phi_H\,[\,p_H V - I\,]^2}{4\,t_H} - \frac{\phi_H\left[\,\frac{\phi_L}{\phi_H}\,p_L\,t_H\,(p_L V - I) + \,p_H\,t_L\,(p_H V - I)\,\right]^2}{4\,t_L\,t_H\left[\frac{\phi_L}{\phi_H}\,p_L^2\,t_H + \,p_H^2\,t_L\right]} - K(1) \;=\; 0$$

$$\Leftrightarrow\;\; \frac{[\,p_H V - I\,]^2}{4\,t_H} - \frac{\left[\,\frac{\phi_L}{\phi_H}\,p_L\,t_H\,(p_L V - I) + \,p_H\,t_L\,(p_H V - I)\,\right]^2}{4\,t_L\,t_H\left[\frac{\phi_L}{\phi_H}\,p_L^2\,t_H + \,p_H^2\,t_L\right]} \;=\; \frac{K(1)}{\phi_H}$$

$$\Leftrightarrow\;\; \frac{[\,p_H V - I\,]^2}{4\,t_H} - \frac{[\,y\,p_L\,t_H\,(p_L V - I) + \,p_H\,t_L\,(p_H V - I)\,]^2}{4\,t_L\,t_H\,[y\,p_L^2\,t_H + \,p_H^2\,t_L]} \;=\; K(1)\,[1 + y]$$

$$\Leftrightarrow\;\; t_L\,\left[y\,p_L^2\,t_H + \,p_H^2\,t_L\right][\,p_H V - I\,]^2 - [\,y\,p_L\,t_H\,(p_L V - I) + \,p_H\,t_L\,(p_H V - I)\,]^2$$
$$=\; 4\,t_L\,t_H\left[\,y\,p_L^2\,t_H + \,p_H^2\,t_L\right]K(1)\,[\,1 + y\,]\,. \tag{53}$$

[2]Recall that $\widetilde{\beta}(1) = \frac{p_H V - I}{2\,p_H V}$, from equation (1).

Let $A \equiv p_L V - I$ and $B \equiv p_H V - I$. Then, (53) can be written as:

$$t_L \left[y\, p_L^2\, t_H + p_H^2\, t_L \right] [\, B\,]^2 - [\, y\, p_L\, t_H\, A + p_H\, t_L\, B\,]^2 \;=\; 4\, t_L\, t_H \left[y\, p_L^2\, t_H + p_H^2\, t_L \right] K(1)\, [1 + y]$$

$$\Leftrightarrow \;\; y\, p_L^2\, t_H\, t_L\, B^2 + p_H^2\, t_L^2\, B^2 - y^2\, p_L^2\, t_H^2\, A^2 - p_H^2\, t_L^2\, B^2 - 2\, y\, p_L\, t_H\, A\, p_H\, t_L\, B$$
$$= \; 4\, t_L\, t_H\, K(1) \left[y\, p_L^2\, t_H + p_H^2\, t_L + y^2\, p_L^2\, t_H + y\, p_H^2\, t_L \right]$$

$$\Leftrightarrow \;\; y^2 \left[p_L^2\, t_H\, A^2 + 4\, t_L\, K(1)\, p_L^2\, t_H \right] + 4\, t_L^2\, p_H^2\, K(1)$$
$$-\; y \left[p_L^2\, t_L\, B^2 - 2\, p_L\, p_H\, t_L\, A\, B - 4\, t_L\, K(1) \left(p_L^2\, t_H + p_H^2\, t_L \right) \right] \;=\; 0$$

$$\Leftrightarrow \;\; \gamma_1\, y^2 - \gamma_2\, y + \gamma_3 \;=\; 0 \tag{54}$$

where γ_1, γ_2, and γ_3 are defined in the text. From (52), the smaller value of y corresponds to the larger value of ϕ_H. Therefore, (54) implies that the critical value of y corresponding to $\widetilde{\phi}_H$ is $\widetilde{y} = \frac{\gamma_2 - \sqrt{(\gamma_2)^2 - 4\,\gamma_1\,\gamma_3}}{2\,\gamma_1}$, and so:

$$\phi_H^c \;=\; \frac{1}{1 + \widetilde{y}} \;=\; \frac{1}{1 + \frac{\gamma_2 - \sqrt{(\gamma_2)^2 - 4\,\gamma_1\,\gamma_3}}{2\,\gamma_1}} \;=\; \frac{2\,\gamma_1}{2\,\gamma_1 + \gamma_2 - \sqrt{(\gamma_2)^2 - 4\,\gamma_1\,\gamma_3}}. \qquad \blacksquare$$

Lemma 4. $\alpha_H^*(q) \;=\; 1$ *for all* $q \in (\frac{1}{2}, \overline{q}\,]$.

Proof. Let x_i^m denote the location of the i entrepreneur located farthest from the origin that applies for financing when the lender pursues the MA policy. It is readily verified that x_L^m and x_H^m are:

$$x_L^m \;=\; \frac{p_L V \beta}{t_L} [\, q\, \alpha_L + (1 - q)\, \alpha_H] \quad \text{and} \quad x_H^m \;=\; \frac{p_H V \beta}{t_H} [\, q\, \alpha_H + (1 - q)\, \alpha_L]\,. \tag{55}$$

Let $\pi\,(q, \alpha_L, \alpha_H, \beta)$ denote the lender's expected revenue when she chooses screening accuracy q, finances a project with probability α_i upon seeing signal s_i, and sets sharing rate β for a financed project. From (55):

$$\pi\,(q, \alpha_L, \alpha_H, \beta) \;=\; \phi_L\, x_L^m \,\{\, q\, \alpha_L\, [p_L V\, (1 - \beta) - I] + [1 - q]\, \alpha_H\, [p_L V\, (1 - \beta) - I\,]\}$$
$$+\;\; \phi_H\, x_H^m \,\{[1 - q]\;\, \alpha_L\, [p_H V\, (1 - \beta) - I] + q\, \alpha_H\, [p_H V\, (1 - \beta) - I\,]\}$$

$$= \; \frac{\phi_L\, p_L V \beta}{t_L} [\, q\, \alpha_L + (1 - q)\, \alpha_H] \,\{\, q\, \alpha_L\, [p_L V\, (1 - \beta) - I\,] + [1 - q]\, \alpha_H\, [p_L V\, (1 - \beta) - I\,]\}$$
$$+\; \frac{\phi_H\, p_H V \beta}{t_H} [\, q\, \alpha_H + (1 - q)\, \alpha_L] \,\{[1 - q]\;\, \alpha_L\, [p_H V\, (1 - \beta) - I\,] + q\, \alpha_H\, [p_H V\, (1 - \beta) - I\,]\}$$

$$= \; \frac{\phi_L\, p_L V \beta}{t_L} [\, q\, \alpha_L + (1 - q)\, \alpha_H]^2\, [p_L V\, (1 - \beta) - I\,]$$

$$+ \quad \frac{\phi_H \, p_H V \beta}{t_H} \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \left[p_H V \, (1-\beta) - I \, \right]. \tag{56}$$

For any $q \in \left(\frac{1}{2}, 1\right]$, the lender chooses $\beta \in [0,1]$ and $\alpha_i \in [0,1]$ to maximize $\pi(\cdot)$. We will denote elements of the revenue-maximizing lending policy by *'s.

First observe that $\alpha_L^* = 0$ and $\alpha_H^* = 1$ if $q = 1$. To see why, observe from (56) that when $q = 1$:

$$\begin{aligned} \frac{\partial \pi(\cdot)}{\partial \alpha_L} &= \frac{2 \, \phi_L \, p_L V \beta^*}{t_L} \alpha_L \left[\, p_L V \, (1-\beta^*) - I \, \right] \; < \; 0 \qquad \Rightarrow \qquad \alpha_L^* \; = \; 0 \, ; \; \text{ and} \\ \frac{\partial \pi(\cdot)}{\partial \alpha_H} &= \frac{2 \, \phi_H \, p_H V \beta^*}{t_H} \alpha_H \left[\, p_H V \, (1-\beta^*) - I \, \right] \; > \; 0 \qquad \Rightarrow \qquad \alpha_H^* \; = \; 1 \, . \end{aligned} \tag{57}$$

The inequality in (57) reflects the fact that the lender will never implement a sharing rate that generates negative revenue for the lender on both low quality and high quality projects.

We now prove that if $q \in (\frac{1}{2}, 1)$, then:

$$\begin{aligned} \beta^* \left(q, \alpha_L, \alpha_H \right) \; &= \; \frac{1}{2} \\ &- \; \frac{I \left\{ \, \phi_L \, p_L \, t_H \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \phi_H \, p_H \, t_L \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\}}{2 \, V \left\{ \, \phi_L \, p_L^2 \, t_H \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \phi_H \, p_H^2 \, t_L \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\}} \, . \end{aligned} \tag{58}$$

To do so, observe from (56) that:

$$\begin{aligned} \frac{\partial \pi(\cdot)}{\partial \beta} \; &= \; \frac{\phi_L \, p_L V}{t_L} \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 \left[p_L V \, (1 - 2 \, \beta) - I \, \right] \\ &\qquad + \; \frac{\phi_H \, p_H V}{t_H} \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \left[p_H V \, (1 - 2 \, \beta) - I \, \right]. \end{aligned} \tag{59}$$

It is apparent from (59) that $\frac{\partial^2 \pi(\cdot)}{\partial \beta^2} < 0$ and that $\left. \frac{\partial \pi(\cdot)}{\partial \beta} \right|_{\beta = 1} < 0$. Therefore, $\beta^* \left(\cdot \right) = 0$ or $\beta^* \left(\cdot \right)$ is determined by:

$$\begin{aligned} & \left[1 - 2 \, \beta \right] \left\{ \, \frac{\phi_L \, p_L^2 \, V^2}{t_L} \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \frac{\phi_H \, p_H^2 \, V^2}{t_H} \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\} \\ & \qquad = \; I \left\{ \, \frac{\phi_L \, p_L V}{t_L} \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \frac{\phi_H \, p_H V}{t_H} \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\} \\ & \Leftrightarrow \quad 1 - 2 \, \beta \; = \; \frac{I \left\{ \, \phi_L \, p_L \, t_H \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \phi_H \, p_H \, t_L \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\}}{V \left\{ \, \phi_L \, p_L^2 \, t_H \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \phi_H \, p_H^2 \, t_L \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\}} \\ & \Leftrightarrow \quad \beta \; = \; \frac{1}{2} - \frac{I \left\{ \, \phi_L \, p_L \, t_H \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \phi_H \, p_H \, t_L \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\}}{2V \left\{ \, \phi_L \, p_L^2 \, t_H \left[\, q \, \alpha_L + (1-q) \, \alpha_H \right]^2 + \phi_H \, p_H^2 \, t_L \left[\, q \, \alpha_H + (1-q) \, \alpha_L \right]^2 \right\}} \, . \end{aligned}$$

It remains to verify that $\beta^* \left(\cdot \right) \neq 0$. If $\beta^* \left(\cdot \right) = 0$, then the lender's revenue is zero. In contrast, as demonstrated in the proof of Lemma **??**, the lender can secure strictly positive revenue by setting $\alpha_L = 0$, $\alpha_H = 1$, and $\beta^*(q, 0, 1) > 0$ for all $q \in (\frac{1}{2}, 1]$. Therefore, since

$$\pi\left(q, \alpha_L^*, \alpha_H^*, \beta^*\left(q, \alpha_L^*, \alpha_H^*\right)\right) \;\geq\; \pi\left(q, 0, 1, \beta^*\left(q, 0, 1\right)\right),$$

it follows that $\beta^*\left(q, \alpha_L^*, \alpha_H^*\right) > 0$.

Conclusion L4.1. If $\alpha_L^* \in (0, 1)$, then the lender's expected revenue from funding a project after observing the unfavorable signal is 0.

Proof. Given β, the lender's expected revenue from funding a project after observing the unfavorable signal is:

$$\begin{aligned}
\pi_L^m &\equiv q\,\phi_L\,x_L^m\left[p_L V\left(1-\beta\right) - I\right] + \left[1-q\right]\phi_H\,x_H^m\left[p_H V\left(1-\beta\right) - I\right] \\
&= q\,\phi_L\,\frac{p_L V \beta}{t_L}\left[q\,\alpha_L + (1-q)\,\alpha_H\right]\left[p_L V\left(1-\beta\right) - I\right] \\
&\quad + \left[1-q\right]\phi_H\,\frac{p_H V \beta}{t_H}\left[q\,\alpha_H + (1-q)\,\alpha_L\right]\left[p_H V\left(1-\beta\right) - I\right]. \qquad (60)
\end{aligned}$$

The equality in (60) reflects (55).

Straightforward differentiation of (56) reveals that:

$$\frac{\partial \pi\left(q, \alpha_L, \alpha_H, \beta\right)}{\partial \alpha_L} \;=\; 2\,\pi_L^m. \qquad (61)$$

(61) implies that if $\alpha_L^* \in (0, 1)$, then it must be the case that $\pi_L^m = 0$. □

Conclusion L4.2. If $\alpha_H^* \in (0, 1)$, then the lender's expected revenue from funding a project after observing the favorable signal is 0.

Proof. Given β, the lender's expected revenue from funding a project after observing the favorable signal is:

$$\begin{aligned}
\pi_H^m &\equiv \left[1-q\right]\phi_L\,x_L^m\left[p_L V\left(1-\beta\right) - I\right] + q\,\phi_H\,x_H^m\left[p_H V\left(1-\beta\right) - I\right] \\
&= \left[1-q\right]\phi_L\,\frac{p_L V \beta}{t_L}\left[q\,\alpha_L + (1-q)\,\alpha_H\right]\left[p_L V\left(1-\beta\right) - I\right] \\
&\quad + q\,\phi_H\,\frac{p_H V \beta}{t_H}\left[q\,\alpha_H + (1-q)\,\alpha_L\right]\left[p_H V\left(1-\beta\right) - I\right]. \qquad (62)
\end{aligned}$$

The equality in (62) reflects Lemma 1.

Straightforward differentiation of (56) reveals that:

$$\frac{\partial \pi\left(q, \alpha_L, \alpha_H, \beta\right)}{\partial \alpha_H} \;=\; 2\,\pi_H^m. \qquad (63)$$

(63) implies that if $\alpha_H^* \in (0, 1)$, then it must be the case that $\pi_H^m = 0$. □

We now show that $\pi_H^m > \pi_L^m$. To do so, suppose that $\pi_L^m \geq \pi_H^m$. Then, from (60) and (62):

$$\begin{aligned} & q\,\phi_L\,x_L\,[p_L V\,(1-\beta) - I\,] + [1-q]\,\phi_H\,x_H\,[p_H V\,(1-\beta) - I\,] \\ & \qquad \geq \ [1-q]\,\phi_L\,x_L\,[p_L V\,(1-\beta) - I\,] + q\,\phi_H\,x_H\,[p_H V\,(1-\beta) - I\,] \\ \Rightarrow & \ \ [\,2\,q-1]\,\{\phi_L\,x_L\,[p_L V\,(1-\beta) - I\,] - \phi_H\,x_H\,[p_H V\,(1-\beta) - I\,]\} \ \geq \ 0 \\ \Rightarrow & \ \ \phi_L\,x_L\,[p_L V\,(1-\beta) - I\,] \ \geq \ \phi_H\,x_H\,[p_H V\,(1-\beta) - I\,] \qquad (64) \\ \Rightarrow & \ \ p_H V\,[1-\beta] - I \ < \ 0\,. \qquad (65) \end{aligned}$$

(64) holds because $2\,q > 1$. (65) holds because $p_L V\,[1-\beta] - I \leq p_L V - I < 0$. If the inequality in (65) holds, then the lender incurs negative expected revenue whenever she finances a project. But this cannot constitute an optimal policy for the lender because Assumption 1 ensures that she can secure strictly positive revenue. Hence, by contradiction, $\pi_H^m > \pi_L^m$.

From (61) and (63):

$$\frac{\partial \pi(q, \alpha_L, \alpha_H, \beta)}{\partial \alpha_H} \ = \ 2\,\pi_H^m \ > \ 2\,\pi_L^m \ = \ \frac{\partial \pi(q, \alpha_L, \alpha_H, \beta)}{\partial \alpha_L}\,. \qquad (66)$$

We can now prove that $\alpha_H^* = 1$. To do so, first suppose $\alpha_L^* = 0$. Then because Assumption 1 ensures that the lender can secure strictly positive revenue, it must be the case that $\alpha_H^* > 0$. If $\alpha_H^* < 1$, then $\pi_H^m = 0$, from Conclusion L4.2. But then the lender's expected revenue is zero, and so this policy cannot be optimal.

Now suppose $\alpha_L^* > 0$. Then (66) implies $\frac{\partial \pi(q^*, \alpha_L^*, \alpha_H^*, \beta^*)}{\partial \alpha_H} > \frac{\partial \pi(q^*, \alpha_L^*, \alpha_H^*, \beta^*)}{\partial \alpha_H} \geq 0$, and so $\alpha_H^* = 1$. ■

Lemma 5. *Suppose Assumption 2 holds. Then* $\alpha_L^* = 0$ *if* $q = 1$, $\alpha_L^* \in (0, 1)$ *if* $q \in (q_1, 1)$, *and* $\alpha_L^* = 1$ *if* $q \in (\frac{1}{2}, q_1)$.

Proof. From (60) and (61):

$$\begin{aligned} \frac{\partial^2 \pi(q, \alpha_L, \alpha_H, \beta)}{\partial \alpha_L^2} \ &= \ 2 \left[\frac{\partial \pi_L^m}{\partial \alpha_L} \right] \\ &= \ 2\,\{\,q^2\,\phi_L\,\frac{p_L V \beta}{t_L}\,[\,p_L V\,(1-\beta) - I\,] + [1-q]^2\,\phi_H\,\frac{p_H\,V\beta}{t_H}\,[\,p_H V\,(1-\beta) - I\,]\,\} \\ &\overset{s}{=} \ [1-q]^2\,\phi_H\,p_H\,t_L\,[\,p_H V\,(1-\beta) - I\,] + q^2\,\phi_L\,p_L\,t_H\,[\,p_L V\,(1-\beta) - I\,] \ \equiv \ h(q)\,. \qquad (67) \end{aligned}$$

Let $\widehat{q} \in [\frac{1}{2}, 1]$ denote the value of q at which $h(q) = 0$. $\widehat{q}$ exists and is unique because: (i) $h(1) < 0$ (since $p_L V - I < 0$); (ii) $h(\frac{1}{2}) > 0$ (from Assumption 2 and the fact that $\beta < \frac{1}{2}$, from (58)); and (iii) $h'(q) < 0$ for $q \in (\frac{1}{2}, 1)$. (67) implies:

$$\frac{\partial^2 \pi(q,\alpha_L,\alpha_H,\beta)}{\partial \alpha_L^2} \gtreqless 0 \quad \Leftrightarrow \quad q \lesseqgtr \widehat{q}\,. \tag{68}$$

(68) implies that $\pi(q,\alpha_L,\alpha_H,\beta)$ is a concave function of α_L when $q > \widehat{q}$.

Also from (60) and (61):

$$\begin{aligned}
\left.\frac{\partial \pi(q,\alpha_L,\alpha_H,\beta)}{\partial \alpha_L}\right|_{\alpha_L=0} &\overset{s}{=} \left.\pi_L^m\right|_{\alpha_L=0} \\
&= q\,[1-q]\,\phi_H \frac{p_H V \beta}{t_H}\,[\,p_H V\,(1-\beta)-I\,] + q\,[1-q]\,\phi_L \frac{p_L V \beta}{t_L}\,[\,p_L V\,(1-\beta)-I\,] \\
&\overset{s}{=} \frac{\phi_H\, p_H}{t_H}\,[\,p_H V\,(1-\beta)-I\,] + \frac{\phi_L\, p_L}{t_L}\,[\,p_L V\,(1-\beta)-I\,] \\
&\equiv z_1(\beta) \;>\; 0 \;\text{ for all }\; q \in [\,\frac{1}{2},1\,]\,.
\end{aligned} \tag{69}$$

The inequality in (69) holds because: (i) $\beta \leq \frac{1}{2}$, from (58); (ii) $z_1(\frac{1}{2}) > 0$, from Assumption 2; and (iii) $z_1'(\beta) < 0$.

In addition, (60) and (61) imply:

$$\begin{aligned}
\left.\frac{\partial \pi(q,\alpha_L,\alpha_H,\beta)}{\partial \alpha_L}\right|_{\alpha_L=1} &\overset{s}{=} \left.\pi_L^m\right|_{\alpha_L=1} \;=\; [1-q]\,\phi_H \frac{p_H V \beta}{t_H}\,[\,p_H V\,(1-\beta)-I\,] \\
&+\; q\,\phi_L \frac{p_L V \beta}{t_L}\,[\,p_L V\,(1-\beta)-I\,] \;\equiv\; z_2(q) \gtreqless 0 \;\text{ as }\; q \lesseqgtr q_1\,.
\end{aligned} \tag{70}$$

(70) holds because: (i) $z_2'(q) < 0$; and (ii) $z_2(q_1) = 0$, from the definition of q_1.

Observe from their definitions that $\widehat{q} < q_1$. Therefore, (68) and (70) imply:

$$\alpha_L^* \;=\; 1 \;\text{ if }\; q \;<\; q_1 \quad\text{and}\quad \alpha_L^* \;<\; 1 \;\text{ if }\; q \;>\; q_1\,. \tag{71}$$

Consequently, from (57) and (71), $\alpha_L^* = 1$ if $q \in (\widehat{q}, q_1)$, $\alpha_L^* \in (0,1)$ if $q \in (q_1, 1)$, and $\alpha_L^* = 0$ if $q = 1$.

If $q < \widehat{q}$, then $\pi(\cdot)$ is a strictly convex function of α_L, from (68). Furthermore, $\left.\frac{\partial \pi(q,\alpha_L,\alpha_H,\beta)}{\partial \alpha_L}\right|_{\alpha_L=0} > 0$, from (69). Therefore, $\alpha_L^* = 1$. ■

Proposition 2. *Suppose* $\phi_L \leq \widehat{\phi}_L$ *and Assumption 2 holds. Then the lender's revenue under the MA policy is a strictly convex function of* q.

<u>Proof</u>. From the Second Order Envelope Theorem:

$$\frac{d^2 \pi(q,\alpha_L^*(q),\alpha_H^*(q),\beta^*(q))}{dq^2} \;\geq\; \frac{\partial^2 \pi(q,\alpha_L^*(q),\alpha_H^*(q),\beta^*(q))}{\partial q^2}\,. \tag{72}$$

Differentiating (56) provides:

$$\frac{\partial \pi(q, \alpha_L^*(q), \alpha_H^*(q), \beta^*(q))}{\partial q} =$$

$$\frac{2\,\beta^*(q)V}{t_L\,t_H}\{\,\phi_L\,p_L\,t_H\,[\,p_L V\,(1-\beta^*(q))-I\,]\,[\,\alpha_L^*(q)-\alpha_H^*(q)\,]\,[\,q\,\alpha_L^*(q)+(1-q)\,\alpha_H^*(q)\,]$$
$$+\;\phi_H\,p_H\,t_L\,[\,p_H V\,(1-\beta^*(q))-I\,]\,[\,\alpha_H^*(q)-\alpha_L^*(q)\,]\,[\,q\,\alpha_H^*(q)+(1-q)\,\alpha_L^*(q)\,]\,\}$$

$$\Rightarrow \quad \frac{\partial^2 \pi(q, \alpha_L^*(q), \alpha_H^*(q), \beta^*(q))}{\partial q^2} = \frac{2\,[\,\alpha_H^*(q)-\alpha_L^*(q)\,]^2\,\beta^*(q)\,V\,\widehat{G}}{t_L\,t_H}, \tag{73}$$

$$\text{where } \widehat{G} \equiv \phi_H\,p_H\,t_L\,[\,p_H V\,(1-\beta^*(q))-I\,]+\phi_L\,p_L\,t_H\,[\,p_L V\,(1-\beta^*(q))-I\,]\,. \tag{74}$$

(72), (73), (74), and Lemma 4 imply that $\pi(\cdot)$ is a convex function of q if $\widehat{G} > 0$ for all $q \in \left[\frac{1}{2}, \overline{q}\right]$.

It is apparent from (74) that $\frac{\partial \widehat{G}}{\partial \beta^*} < 0$. Also, from (58), $\beta^*(q) \leq \frac{1}{2}$ for all $q \in \left[\frac{1}{2}, \overline{q}\right]$. Consequently, if $\widehat{G} > 0$ at $\beta^* = \frac{1}{2}$, then $\widehat{G} > 0$ and so $\frac{\partial^2 \pi(\cdot)}{\partial q^2} > 0$ for all $q \in \left[\frac{1}{2}, \overline{q}\right]$. From (74) and Assumption 2:

$$\widehat{G}\Big|_{\beta^* = \frac{1}{2}} = \phi_H\,p_H\,t_L\left[\frac{p_H V}{2}-I\right]+\;\phi_L\,p_L\,t_H\left[\frac{p_L V}{2}-I\right] > 0. \tag{75}$$

The proposition follows from (72), (73), and (75). ■

B. Extension: The Setting with Variable Screening Costs.

The analysis in Bose et al. (2012) can be extended to allow the lender's cost of securing any desired level of screening accuracy to vary with the number of projects she screens. Suppose the lender's cost of screening n applicants with screening accuracy q is $\widetilde{K}(q,n) = F(q) + c(q)\, n$, where $F(\cdot)$ represents a fixed cost of screening.

$\pi^v(\beta, q)$ will denote the lender's variable profit in this setting with variable screening costs when she offers sharing rate β and implements screening accuracy q. This variable profit is the difference between the lender's revenue and her variable screening costs. Formally:

$$\begin{aligned}\pi^v(\beta, q) \;=\; & \phi_L\, x_L\, [1-q]\, [p_L V\,(1-\beta) - I] + \phi_H\, x_H\, q\, [p_H V\,(1-\beta) - I] \\ & - \; c\,(q)\, [\phi_L\, x_L + \phi_H\, x_H\,]\,. \end{aligned} \tag{76}$$

Lemma A1 identifies the lender's profit-maximizing sharing rate, $\beta(q)$, when she adopts the SA policy in this setting. This rate is derived by substituting the values of x_L and x_H identified in Lemma 1 in the text into equation (76) and maximizing the resulting expression with respect to β. Lemma A1 refers to Condition 1, which is the natural counterpart to Assumption 1. When Condition 1 holds for all $q \in [\frac{1}{2}, 1]$, the lender will optimally implement a strictly positive sharing rate in the present setting with variable screening costs. Consequently, the condition precludes the trivial outcome in which the lender finances no projects.

Condition 1.
$$\begin{aligned}& \phi_H\, p_H\, t_L\, [p_H V - I]\, q^2 + \phi_L\, p_L\, t_H\, [p_L V - I]\, [1-q]^2 \\ & \quad > \; c\,(q)\, [\phi_L\, p_L\, t_H\, (1-q) + \phi_H\, p_H\, t_L\, q]\,.\end{aligned}$$

Lemma A1. Suppose $\widetilde{K}(q,n) = F(q) + c(q)\, n$ and Condition 1 holds for all $q \in [\frac{1}{2}, 1]$. Then the sharing rate that maximizes the lender's profit when she adopts the SA policy and implements screening accuracy q is:

$$\begin{aligned}\beta(q) \;=\; & \frac{1}{2\, V\, \left[\phi_L\, p_L^2\, t_H\, (1-q)^2 + \phi_H\, p_H^2\, t_L\, q^2\right]} \{\phi_L\, p_L\, t_H\, (1-q)^2\, [p_L V - I] \\ & + \; \phi_H\, p_H\, t_L\, q^2\, [p_H V - I] - c\,(q)\, [\phi_L\, p_L\, t_H\, (1-q) + \phi_H\, p_H\, t_L\, q]\}.\end{aligned} \tag{77}$$

Proof. From (76) and Lemma 1 in the text:

$$\begin{aligned}\pi^v(\beta, q) \;=\; & \phi_L\, [1-q]\, [p_L V\,(1-\beta) - I] \left[\frac{p_L V\,(1-q)\,\beta}{t_L}\right] + \phi_H\, q\, [p_H V\,(1-\beta) - I] \left[\frac{p_H V\, q\, \beta}{t_H}\right] \\ & - \; F\,(q) - c\,(q) \left[\frac{\phi_L\, p_L V\,(1-q)\,\beta}{t_L} + \frac{\phi_H\, p_H\, V\, q\, \beta}{t_H}\right]\end{aligned} \tag{78}$$

$$\Rightarrow \quad \frac{\partial \pi^v(\cdot)}{\partial \beta} \;=\; \frac{\phi_L\, p_L V\,(1-q)^2\, [p_L V\,(1-2\,\beta) - I]}{t_L} + \frac{\phi_H\, p_H V\, q^2\, [p_H V\,(1-2\beta) - I]}{t_H}$$

$$- c\,(q) \left[\frac{\phi_L\, p_L V\,(1-q)}{t_L} + \frac{\phi_H\, p_H V\, q}{t_H} \right] \;=\; 0$$

$$\Leftrightarrow \;\; \phi_L\, p_L\, t_H\, [1-q]^2\, [p_L V\,(1-2\,\beta) - I]$$

$$+\;\; \phi_H\, p_H\, t_L\, q^2\, [p_H V\,(1-2\beta) - I] - c\,(q)\, [\phi_L\, p_L\, t_H\,(1-q) + \phi_H\, p_H\, t_L\, q] \;=\; 0$$

$$\Leftrightarrow \;\; \phi_L\, p_L\, t_H\,(1-q)^2\, [p_L V - I] + \phi_H\, p_H\, t_L\, q^2\, [p_H V - I]$$

$$-\; 2\,\beta \left[\phi_L\, p_L^2 V\, t_H\,(1-q)^2 + \phi_H\, p_H^2 V\, t_L\, q^2\right] -\; c\,(q)\, [\phi_L\, p_L\, t_H\,(1-q) + \phi_H\, p_H\, t_L\, q] \;=\; 0$$

$$\Rightarrow \;\; \beta \;=\; \frac{1}{2\,V\left[\phi_L\, p_L^2\, t_H\,(1-q)^2 + \phi_H\, p_H^2\, t_L\, q^2\right]} \{\phi_L\, p_L\, t_H\,(1-q)^2\, [p_L V - I]$$

$$+\;\; \phi_H\, p_H\, t_L\, q^2\, [p_H V - I] - c\,(q)\, [\phi_L\, p_L\, t_H\,(1-q) + \phi_H\, p_H\, t_L\, q]\}. \quad (79)$$

It is readily verified that $\pi^v(\cdot)$ is a strictly concave function of β, that $\left.\frac{\partial^v \pi(\cdot)}{\partial \beta}\right|_{\beta=1} < 0$, and that $\left.\frac{\partial \pi^v(\cdot)}{\partial \beta}\right|_{\beta=0} > 0$ when Condition 1 holds. Therefore, when Condition 1 holds, (79) identifies the sharing rate that maximizes the lender's profit given screening accuracy q. ■

Substituting the expression for $\beta(q)$ in equation (77) into the expression for $\pi^v(\beta, q)$ in equation (76) provides an expression for $\pi^v(q) = \max\limits_{\beta} \pi^v(\beta, q)$, the lender's maximum variable profit in this setting when she implements screening accuracy q.

Lemma A2. Suppose $K(q,n) = F(q) + c(q)\,n$ and Condition 1 holds for all $q \in [\frac{1}{2}, 1]$. Then the lender's maximum variable profit when she adopts the SA policy and implements screening accuracy q is:

$$\pi^v(q) \;=\; \frac{1}{4\,t_L\, t_H\left[\phi_L\, p_L^2\, t_H\,(1-q)^2 + \phi_H\, p_H^2\, t_L\, q^2\right]} \{\phi_L\, [1-q]^2\, p_L\, t_H\, [p_L V - I]$$

$$+\;\; \phi_H\, q^2 p_H\, t_L\, [p_H V - I] - c\,(q)\, [\phi_L\, p_L\, t_H\,(1-q) + \phi_H\, p_H\, t_L\, q]\}^2. \quad (80)$$

<u>Proof</u>. Substituting from (79) into (78) provides:

$$\Pi^v(q) \;=\; \pi^v - F\,(q)\,, \qquad \text{where}$$

$$\pi^v \;=\; \phi_L\, [1-q]\, [p_L V\,(1-\beta) - I] \left[\frac{p_L V\,(1-q)\,\beta}{t_L}\right] + \phi_H\, q\, [p_H V\,(1-\beta) - I] \left[\frac{p_H V\, q\,\beta}{t_H}\right]$$

$$-\; c\,(q) \left[\frac{\phi_L\, p_L V\,(1-q)\,\beta}{t_L} + \frac{\phi_H\, p_H V\, q\,\beta}{t_H}\right]$$

$$=\; \frac{V\beta}{t_L\, t_H} \left\{\phi_L\,(1-q)^2\, p_L\, t_H\, [p_L V\,(1-\beta) - I] + \phi_H\, q^2\, p_H\, t_L\, [p_H V\,(1-\beta) - I]\right.$$

$$\left. -\, c\,(q)\, [\phi_L\, p_L\, t_H\,(1-q) + \phi_H\, p_H\, t_L\, q]\right\}$$

$$
\begin{aligned}
&= \frac{V\beta}{t_L\,t_H}\left\{\phi_L\,(1-q)^2\,p_L\,t_H\,[p_LV-I]+\phi_H\,q^2\,p_H\,t_L\,[p_HV-I]\right.\\
&\qquad \left. -\;c\,(q)\,[\phi_L\,p_L\,t_H\,(1-q)+\phi_H\,p_H\,t_L\,q]-\beta V\left[\phi_L\,(1-q)^2\,p_L^2\,t_H+\phi_H\,q^2\,p_H^2\,t_L\right]\right\}\\
&= \frac{V\beta}{t_L\,t_H}\Big[\{\phi_L\,[1-q]^2\,p_L\,t_H\,[p_LV-I]+\phi_H\,q^2p_H\,t_L\,[p_HV-I]\\
&\qquad -\;c\,(q)\,[\phi_L\,p_L\,t_L\,(1-q)+\phi_H\,p_H\,t_H\,q]\}\\
&\qquad -\frac{1}{2}\{\phi_L\,(1-q)^2\,p_L\,t_H\,[p_LV-I]+\phi_H\,q^2p_H\,t_L\,[p_HV-I]\\
&\qquad -c\,(q)\,[\phi_L\,p_L\,t_H\,(1-q)+\phi_H\,p_H\,t_L\,q]\}\Big]\\
&= \frac{V\beta}{2\,t_L\,t_H}\{\phi_L\,[1-q]^2\,p_L\,t_H\,[p_LV-I]+\phi_H\,q^2p_H\,t_L\,[p_HV-I]\\
&\qquad -\;c\,(q)\,[\phi_L\,p_L\,t_H\,(1-q)+\phi_H\,p_H\,t_L\,q]\}\,. \qquad (81)
\end{aligned}
$$

Substituting $V\beta$ from (79) into (81) provides:

$$
\begin{aligned}
\pi^v(q) \;=\; & \frac{1}{4\,t_L\,t_H\left[\phi_L\,p_L^2\,t_H\,(1-q)^2+\phi_H\,p_H^2\,t_L\,q^2\right]}\{\phi_L\,[1-q]^2\,p_L\,t_H\,[p_LV-I]\\
&+\;\phi_H\,q^2p_H\,t_L\,[p_HV-I]-c\,(q)\,[\phi_L\,p_L\,t_H\,(1-q)+\phi_H\,p_H\,t_L\,q]\}^2.\;\blacksquare \qquad (82)
\end{aligned}
$$

Proposition A1 now explains when $\pi^v(q)$ will be a convex function of q. The proposition refers to:

Condition 2.
$$
\begin{aligned}
&\left[\phi_H\,p_H^2\,t_L\,q^2+\phi_L\,p_L^2\,t_H\,(1-q)^2\right]\{2\,[\phi_L\,p_L\,t_H\,(p_LV-I)+\phi_H\,p_H\,t_L\,(p_HV-I)]\\
&\quad -\;2\,c'\,(q)\,[\phi_H\,p_H\,t_L-\phi_L\,p_L\,t_H]-c''\,(q)\,[\phi_L\,p_L\,t_H\,(1-q)+\phi_H\,p_H\,t_L\,q]\}\\
&> \;\left[\phi_H\,p_H^2\,t_L\;+\phi_L\,p_L^2\,t_H\right]\{\phi_H\,p_H\,t_L\,q^2\,[p_HV-I]+\phi_L\,p_L\,t_H\,[1-q]^2\,[p_LV-I]\\
&\quad -\;c\,(q)\,[\phi_L\,p_L\,t_H\,(1-q)+\phi_H\,p_H\,t_L\,q]\}.
\end{aligned}
$$

Proposition A1. Suppose $K(q,n)=F(q)+c(q)\,n$. Also suppose Conditions 1 and 2 hold for all $q\in[\frac{1}{2},1]$. Then the lender's maximum variable profit is a convex function of q, i.e., $\pi^{v\,\prime\prime}(q)>0$ for all $q\in[\,\frac{1}{2},\,\overline{q}\,]$.[3]

Proof. To determine whether $\pi^v(\cdot)$ is a convex function of q, let $p\;=\;1-q$. Then from (80):

$$
\pi^v(q) \;=\; \frac{\{\phi_L\,p^2\,p_L\,t_H\,[p_LV-I]+\phi_H\,q^2p_H\,t_L\,[p_HV-I]-c\,(q)\,[\phi_L\,p_L\,t_H\,p+\phi_H\,p_H\,t_L\,q]\}^2}{4\,t_L\,t_H\,[\phi_L\,p_L^2\,t_H\,p^2+\phi_H\,p_H^2\,t_L\,q^2]}
$$

[3]Furthermore, it can be shown that $\pi^{v\,\prime}(q)>0$ for all $q\in[\frac{1}{2},\overline{q}\,]$ if $c'(\frac{1}{2})<[A_1+A_2]/B$, where $A_1=3\,\phi_L\,p_L\,\phi_H\,p_H\,t_L\,t_H\,I\,[p_H-p_L]$, $A_2=\phi_H^2\,p_H^3\,t_L^2\,[p_HV-I]-\phi_L^2\,p_L^3\,t_H^2\,[p_LV-I]$, and $B=[\phi_L\;p_L\,t_H+\phi_H\,p_H\,t_L]\,[\phi_L\,p_L^2\,t_H+\phi_H\,p_H^2\,t_L]$.

$$= \frac{\{\phi_L\, p^2\, p_L\, t_H\, [p_L V - I] + \phi_H\, q^2\, p_H\, t_L\, [p_H V - I] - c\,(q)\, [\phi_L\, p_L\, t_H\, p + \phi_H\, p_H\, t_L\, q]\}^2}{4\, t_L\, t_H\, \phi_H\, p_H^2\, t_L\, \left[\frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L} p^2 + q^2\right]}$$

$$= \frac{[\phi_H\, p_H\, t_L\, (p_H V - I)]^2 \left[\frac{\phi_L\, p_L\, t_H [p_L V - I] p^2}{\phi_H\, p_H\, t_L [p_H V - I]} + q^2 - \frac{c(q)}{\phi_H\, p_H\, t_L [p_H V - I]} [\phi_L\, p_L\, t_L\, p + \phi_H\, p_H\, t_H\, q]\right]^2}{4\, t_L\, t_H\, \phi_H\, p_H^2\, t_L\, \left[\frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L} p^2 + q^2\right]}$$

$$= \frac{[\phi_H\, p_H\, t_L\, (p_H V - I)]^2}{4\, t_L\, t_H\, \phi_H\, p_H^2\, t_L} \cdot \left[\frac{\left[\frac{\phi_L\, p_L\, t_H [p_L V - I] p^2}{\phi_H\, p_H\, t_L [p_H V - I]} + q^2 - \frac{c(q)}{\phi_H\, p_H\, t_L [p_H V - I]} [\phi_L\, p_L\, t_H\, p + \phi_H\, p_H\, t_L\, q]\right]^2}{\left[\frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L} p^2 + q^2\right]} \right]. \tag{83}$$

Define:

$$\xi_1 = \frac{\phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L} > 0; \quad \xi_2 = \frac{\phi_L\, p_L\, t_H\, [p_L V - I]}{\phi_H\, p_H\, t_L\, [p_H V - I]} < 0; \quad \text{and}$$

$$\xi_3 = -\frac{c\,(q)}{\phi_H\, p_H\, t_L\, [p_H V - I]} < 0\,. \tag{84}$$

(83) and (84) provide:

$$\pi^v(q) = \frac{[\phi_H\, p_H\, t_L\, (p_H V - I)]^2}{4\, t_L\, t_H\, \phi_H\, p_H^2\, t_L}\, P \tag{85}$$

where:

$$P = \frac{[q^2 + \xi_2\, p^2 + \xi_3\, (\phi_L\, p_L\, t_H\, p + \phi_H\, p_H\, t_L\, q)]^2}{q^2 + \xi_1\, p^2}\,. \tag{86}$$

(85) and (86) imply that $\pi^v(\cdot)$ is convex in q if and only if P is convex in q. To determine whether P is convex in q, define:

$$h_1 = q^2 + \xi_1\, p^2; \qquad h_2 = q^2 + \xi_2\, p^2;$$

$$h_3 = \xi_3\, [\phi_L\, p_L\, t_H\, p + \phi_H\, p_H\, t_L\, q]\,; \quad \text{and} \quad H = h_2 + h_3. \tag{87}$$

Using (87) in (86) and differentiating provides:

$$P = \frac{H^2}{h_1} \;\Rightarrow\; P\, h_1 = H^2 \;\Rightarrow\; P\, h_1' + P'\, h_1 = 2\, H\, [H'] \tag{88}$$

$$\Rightarrow\; P\, h_1'' + P'\, h_1' + P''\, h_1 + P'\, h_1' = 2\,(H')^2 + 2\, H\, [H'']$$

$$\Leftrightarrow\; P\, h_1'' + 2\, P'\, h_1' + P''\, h_1 = 2\,(H')^2 + 2\, H\, [H'']$$

$$\Leftrightarrow\; P''\, h_1 = 2\,(H')^2 + 2\, H\, [H''] - P\, h_1'' - 2\, P'\, h_1'\,. \tag{89}$$

From (87):

$$h_1 = q^2 + \xi_1\, p^2 = q^2 + \xi_1\, [1 - q]^2$$

$$\Rightarrow \quad h_1' \;=\; 2\,q - 2\,\xi_1\,[1-q] \quad \text{and} \quad h_1'' \;=\; 2\,[1+\xi_1]\,. \tag{90}$$

$$h_2 \;=\; q^2 + \xi_2\,p^2 \;=\; q^2 + \xi_2\,[1-q]^2$$
$$\Rightarrow \quad h_2' \;=\; 2\,q - 2\,\xi_2\,[1-q] \quad \text{and} \quad h_2'' \;=\; 2\,[1+\xi_2]\;. \tag{91}$$

$$H \;=\; h_2 + h_3$$
$$\Rightarrow \quad H' \;=\; h_2' + h_3' \quad \text{and} \quad H'' \;=\; h_2'' + h_3''\,. \tag{92}$$

(91) and (92) provides:

$$H'' \;=\; 2\,[1+\xi_2] + h_3''\,. \tag{93}$$

Using (90) and (93) in (89) provides:

$$P''h_1 \;=\; 2\,(H')^2 + 2\,H\,[h_2'' + h_3''] - P\,[2\,(1+\xi_1)] - 2\,P'\,h_1'$$
$$\Rightarrow \quad \frac{P''h_1}{2} \;=\; (H')^2 + H\,[h_2'' + h_3''] - P\,[1+\xi_1] - P'\,h_1'\,. \tag{94}$$

Also, from (88):

$$P\,h_1' + P'\,h_1 \;=\; 2\,H\,[H'] \quad \Rightarrow \quad P' \;=\; \frac{2\,H\,[H'] - P\,h_1'}{h_1}\,. \tag{95}$$

Using (88) and (95) in (94) provides:

$$\frac{P''\,h_1}{2} \;=\; (H')^2 + H\,[h_2'' + h_3''] - \left[\frac{H^2}{h_1}\right][1+\xi_1] - \left[\frac{2\,H\,(H') - P\,h_1'}{h_1}\right] h_1'$$

$$\Rightarrow \quad \frac{P''\,(h_1)^2}{2} \;=\; (H')^2\,h_1 + H\,[h_2'' + h_3'']\,h_1 - H^2\,[1+\xi_1] - 2\,H\,(H')\,h_1' + P\;(h_1')^2\;. \tag{96}$$

Using (88) and (91) in (96) provides:

$$\frac{P''\,(h_1)^2}{2} \;=\; (H')^2\,h_1 + H\,[2\,(1+\xi_2) + h_3'']\,h_1 - H^2\,[1+\xi_1] - 2\,H\,[H']\,h_1' + \left[\frac{H^2}{h_1}\right](h_1')^2$$

$$=\; (H')^2\,h_1 - H\,[H']\,h_1' + \left[\frac{H^2}{h_1}\right](h_1')^2 - H\,[H']\,h_1' + H\,[2\,(1+\xi_2) + h_3'']\,h_1 - H^2\,[1+\xi_1]$$

$$=\; H'\,[H'\,h_1 - H\,h_1'] + \frac{H\,h_1'}{h_1}\,[H\,h_1' - H'h_1] + H\,[2\,(1+\xi_2) + h_3'']\,h_1 - H^2\,[1+\xi_1]$$

$$=\; H'\,[H'h_1 - Hh_1'] - \frac{H\,h_1'}{h_1}\,[H'h_1 - Hh_1'] + H\,[2\,(1+\xi_2) + h_3'']\,h_1 - H^2\,[1+\xi_1]$$

$$=\; [H'h_1 - H\,h_1']\left[H' - \frac{Hh_1'}{h_1}\right] + H\,[2\,(1+\xi_2) + h_3'']\,h_1 - H^2\,[1+\xi_1]$$

$$=\; \frac{1}{h_1}\,[H'h_1 - H\,h_1']^2 + H\,[(2\,(1+\xi_2) + h_3'')\,h_1 - H\,(1+\xi_1)]\;. \tag{97}$$

Since $h_1 = q^2 + \xi_1 [1-q]^2 > 0$, the first term on the right hand side of (97) is non-negative. Therefore, a sufficient condition for $P'' > 0$ is

$$H\,[(2\,(1+\xi_2) + h_3'')\,h_1 - H\,(1+\xi_1)] \;>\; 0\,. \tag{98}$$

The inequality in (98) will hold if:

$$H \;>\; 0 \quad \text{and} \quad [2\,(1+\xi_2) + h_3'']\,h_1 - H\,[1+\xi_1] \;>\; 0\,. \tag{99}$$

From (84) and (87):

$$\begin{aligned} H \;&=\; h_2 + h_3 \;=\; q^2 + \xi_2\,[1-q]^2 + \xi_3\,[\phi_L\,p_L\,t_H\,p + \phi_H\,p_H\,t_L\,q] \\ &=\; q^2 + \left[\frac{\phi_L\,p_L\,t_H\,(p_L V - I)}{\phi_H\,p_H\,t_L\,(p_H V - I)}\right][1-q]^2 - \frac{c\,(q)\,[\phi_L\,p_L\,t_H\,p + \phi_H\,p_H\,t_L\,q]}{\phi_H\,p_H\,t_L\,[p_H V - I]} \\ &=\; \frac{1}{\phi_H\,p_H\,t_L\,[p_H V - I]}\{\,\phi_H\,p_H\,t_L\,[p_H V - I]\,q^2 \\ &\qquad +\; \phi_L\,p_L\,t_H\,[p_L V - I]\,[1-q]^2 - c\,(q)\,[\phi_L\,p_L\,t_H\,p + \phi_H\,p_H\,t_L\,q]\,\}\,. \end{aligned} \tag{100}$$

(100) implies:

$$\begin{aligned} H \;>\; 0 \;\;\Leftrightarrow\;\; &\phi_H\,p_H\,t_L\,[\,p_H V - I\,]\,q^2 + \phi_L\,p_L\,t_H\,[\,p_L V - I\,]\,[1-q]^2 \\ &-\; c\,(q)\,[\,\phi_L\,p_L\,t_H\,(1-q) + \phi_H\,p_H\,t_L\,q\,] \;>\; 0\,. \end{aligned} \tag{101}$$

Notice from (79) that the inequality in (101), which is Condition 1, ensures $\beta > 0$.

To analyze the other component of the sufficient condition in (99), notice from (84) and (87) that:

$$h_3 \;=\; \xi_3\,[\,\phi_L\,p_L\,t_H\,p + \phi_H\,p_H\,t_L\,q\,] \;=\; -\,\frac{c\,(q)\,[\phi_L\,p_L\,t_H\,(1-q) + \phi_H\,p_H\,t_L\,q]}{\phi_H\,p_H\,t_L\,[p_H V - I]} \tag{102}$$

$$\begin{aligned} \Rightarrow\;\; h_3' \;&=\; -\,\frac{c'\,(q)\,[\phi_L\,p_L\,t_H\,(1-q) + \phi_H\,p_H\,t_L\,q]}{\phi_H\,p_H\,t_L\,[p_H V - I]} - \frac{c\,(q)\,[-\,\phi_L\,p_L\,t_H + \phi_H\,p_H\,t_L]}{\phi_H\,p_H\,t_L\,[p_H V - I]} \\ \Rightarrow\;\; h_3'' \;&=\; -\,\frac{c''\,(q)\,[\phi_L\,p_L\,t_H\,(1-q) + \phi_H\,p_H\,t_L\,q]}{\phi_H\,p_H\,t_L\,[p_H V - I]} - \frac{2\,c'\,(q)\,[-\phi_L\,p_L\,t_H + \phi_H\,p_H\,t_L]}{\phi_H\,p_H\,t_L\,[p_H V - I]}\,. \end{aligned} \tag{103}$$

Also:

$$h_1 \;=\; q^2 + \xi_1\,[1-q]^2 \;=\; q^2 + \frac{\phi_L\,p_L^2\,t_H}{\phi_H p_H^2 t_L}\,[1-q]^2 = \frac{\phi_H\,p_H^2\,t_L\,q^2 + \phi_L\,p_L^2\,t_H\,[1-q]^2}{\phi_H\,p_H^2\,t_L}\,. \tag{104}$$

(84) and (103) imply:

$$\begin{aligned} 2\,[1+\xi_2] + h_3'' \;=\; &\frac{1}{\phi_H\,p_H\,t_L\,[p_H V - I]}\{2\,[\phi_L\,p_L\,t_H\,(p_L V - I) + \phi_H\,p_H\,t_L\,(p_H V - I)] \\ &-\; 2\,c'\,(q)\,[\phi_H\,p_H\,t_L - \phi_L\,p_L\,t_H] - c''\,(q)\,[\phi_L\,p_L\,t_H\,(1-q) + \phi_H\,p_H\,t_L\,q]\}\,. \end{aligned} \tag{105}$$

From (84) and (87):

$$1 + \xi_1 \;=\; \frac{\phi_H\, p_H^2\, t_L \,+ \phi_L\, p_L^2\, t_H}{\phi_H\, p_H^2\, t_L}\,. \tag{106}$$

(100), (104), (105), and (106) imply that the second inequality in (99) holds if and only if Condition 2 holds, i.e.:

$$\begin{aligned}
&\left[\phi_H\, p_H^2\, t_L\, q^2 + \phi_L\, p_L^2\, t_H\, (1-q)^2\right] \{2\, [\phi_L\, p_L\, t_H\, (p_L V - I) + \phi_H\, p_H\, t_L\, (p_H V - I)] \\
&\qquad -\; 2\, c'(q)\, [\phi_H\, p_H\, t_L - \phi_L\, p_L\, t_H] - c''(q)\, [\phi_L\, p_L\, t_H\, (1-q) + \phi_H\, p_H\, t_L\, q]\} \\
&> \;\left[\phi_H\, p_H^2\, t_L \,+ \phi_L\, p_L^2\, t_H\right] \{\, \phi_H\, p_H\, t_L\, q^2\, [p_H V - I] + \phi_L\, p_L\, t_H\, [1-q]^2\, [p_L V - I] \\
&\qquad -\; c(q)\, [\phi_L\, p_L\, t_H\, (1-q) + \phi_H\, p_H\, t_L\, q]\,\}\,. \quad \blacksquare
\end{aligned} \tag{107}$$

Condition 2 indicates that $\pi^v(\cdot)$ is more likely to be a convex function of q if the lender's marginal cost of screening does not increase too rapidly with q (so $c''(q)$ is small), *ceteris paribus*. Condition 2 also indicates that $\pi^v(q)$ is more likely to be convex if the lender's marginal cost declines with q (so $c'(q) < 0$) and ϕ_H / t_H is large relative to ϕ_L / t_L, *ceteris paribus*. Under these conditions, an increase in q reduces the marginal cost of screening, induces more entrepreneurs to seek funding for any given sharing rate, and increases the fraction of H entrepreneurs that apply for funding.