Technical Appendix to Accompany "Innovation in Vertically Related Markets"
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Analysis of the Linear Bertrand Setting

Notation

Retail demand function:	$P(Q) = a - bQ$.
Upstream unit production cost:	$c^u = c_h - \alpha$ (= initial cost − cost reduction).
Cost of upstream cost reduction:	$K(\alpha) = \frac{k}{2}\alpha^2$.
Downstream unit cost of Di:	c_i^S under vertical separation, c_i^I under vertical integration.
Input price:	$w = c_h - \alpha + t\alpha = c_h - [1-t]\alpha$, where $t \in [0,1]$.

Vertical Separation in the Linear Bertrand Setting

The equilibrium price in this setting is $w + c_2^S$, since U sets L to ensure $c_2^S \geq c_1^S$ and Bertrand competition drives the retail price to the level of the highest unit production cost. Because D1 has the lowest unit production cost, it will produce the entire industry output:

$$Q(\cdot) \;=\; a - b[w + c_2^S]. \tag{1}$$

Recall:

$$w \;=\; c_h - [1-t]\alpha^S \quad \Rightarrow \quad w - c^u \;=\; c_h - [1-t]\alpha - \left[c_h - \alpha^S\right] \;=\; t\alpha^S. \tag{2}$$

(1) and (2) imply:

$$Q(\cdot) \;=\; a - b[c_h - (1-t)\,\alpha^S + c_2^S]. \tag{3}$$

(2) and (3) imply that U's objective is to:

$$Maximize\;\; \Pi_U^{SB} \;=\; [w - c^u]Q \;=\; t\alpha^S\left\{a - b\left[c_h - (1-t)\,\alpha^S + c_2^S\right]\right\} - \frac{k}{2}\left(\alpha^S\right)^2. \tag{4}$$

Differentiating (4) with respect to α provides:

$$t\alpha^S b[1-t] + t\left\{a - b\left[c_h - (1-t)\,\alpha^S + c_2^S\right]\right\} - k\alpha^S \;=\; 0$$

$$\Leftrightarrow \quad ta - tbc_h - tbc_2^S \;=\; k\alpha^S - \alpha^S bt[1-t] - \alpha^S bt[1-t] \quad \Leftrightarrow \quad \alpha^{SB} \;=\; \frac{t[a - bc_h - bc_2^S]}{k - 2bt[1-t]}. \tag{5}$$

(5) implies that U's R&D costs will be:

$$\frac{k}{2}\left(\alpha^{SB}\right)^2 \;=\; \frac{kt^2[a - bc_h - bc_2^S]^2}{2\left\{k - 2bt[1-t]\right\}^2}. \tag{6}$$

Substituting (5) into (3) provides:

$$Q^{SB}(\cdot) \;=\; a - bc_h - bc_2^S + \frac{b[1-t]t[a - bc_h - bc_2^S]}{k - 2bt[1-t]}$$

$$=\; \frac{\left[a - bc_h - bc_2^S\right]}{k - 2bt[1-t]}\left\{k - 2bt[1-t] + bt[1-t]\right\} \;=\; \frac{[k - bt(1-t)]\left[a - bc_h - bc_2^S\right]}{k - 2bt[1-t]}. \tag{7}$$

(4), (5), (6), and (7) imply that U's profit is:

$$\Pi_U^{SB} \;=\; t\alpha Q^{SB} - \frac{k}{2}\left(\alpha^{SB}\right)^2 \;=\; \left\{\frac{t^2[a - bc_h - bc_2^S]}{k - 2bt[1-t]}\right\}\frac{[k - bt(1-t)]\left[a - bc_h - bc_2^S\right]}{k - 2bt[1-t]} - \frac{kt^2[a - bc_h - bc_2^S]^2}{2\left\{k - 2bt[1-t]\right\}^2}$$

$$= \frac{2t^2[a - bc_h - bc_2^S]^2\,[k - bt(1-t)] - kt^2[a - bc_h - bc_2^S]^2}{2\,\{k - 2bt[1-t]\}^2}$$

$$= \frac{t^2\,[k - 2bt(1-t)]\,[a - bc_h - bc_2^S]^2}{2\,\{k - 2bt[1-t]\}^2} = \frac{t^2[a - bc_h - bc_2^S]^2}{2\,[k - 2bt\,(1-t)]}. \qquad (8)$$

From (7), the corresponding profit of D1 is:

$$\pi_1^{SB} = \left[w + c_2^S - (w + c_1^S)\right] Q^{SB} = \frac{\left[c_2^S - c_1^S\right]\,[k - bt(1-t)]\,\left[a - bc_h - bc_2^S\right]}{k - 2bt[1-t]}. \qquad (9)$$

From (7), consumer surplus in this setting is:

$$Z^{SB} = \frac{b}{2}[Q^{SB}]^2 = \frac{b\,[k - bt(1-t)]^2\,\left[a - bc_h - bc_2^S\right]^2}{2\,\{k - 2bt[1-t]\}^2}. \qquad (10)$$

Vertical Integration in the Linear Bertrand Setting

The equilibrium price in this setting is $w + c_2^I$ since U sets L to ensure $c_2^I > c_1^I$. Because D1 produces the entire industry output:

$$q_1^I = Q(w + c_2^I) = a - b[w + c_2^I]. \qquad (11)$$

The input price in this setting is:

$$w = c_h - [1-t]\alpha^I. \qquad (12)$$

Substituting (12) into (11) provides:

$$q_1 = a - bc_2^I - b\left[c_h - (1-t)\,\alpha^I\right] = a - bc_2^I - bc_h + b[1-t]\alpha^I. \qquad (13)$$

(12) and (13) imply that U-D1 chooses α in this setting to:

$$Maximize \;\; [p - (c_h - \alpha^I) - c_1^I]q_1 - \frac{k}{2}\alpha^2 = [w + c_2^I - c_1^I - c_h + \alpha^I]q_1 - \frac{k}{2}\alpha^2$$

$$\Rightarrow \;\; Maximize \;\; [t\alpha^I + c_2^I - c_1^I]\left[a - bc_2^I - bc_h + b(1-t)\alpha^I\right] - \frac{k}{2}\alpha^2. \qquad (14)$$

Differentiating (14) with respect to α provides:

$$b[1-t][t\alpha^I + c_2^I - c_1^I] + t\left[a - bc_2^I - bc_h + b\,(1-t)\,\alpha^I\right] - k\alpha^I = 0$$

$$\Rightarrow \;\; \alpha^I\,\{k - 2bt[1-t]\} = b[1-t][c_2^I - c_1^I] + t[a - bc_2^I - bc_h]$$

$$\Rightarrow \;\; \alpha^{IB} = \frac{t[a - bc_2^I - bc_h] + b[1-t][c_2^I - c_1^I]}{k - 2bt[1-t]}. \qquad (15)$$

(15) implies that U-D1's R&D costs are:

$$\frac{k}{2}\left(\alpha^{IB}\right)^2 = \frac{k\left\{t[a - bc_2^I - bc_h] + b[1-t][c_2^I - c_1^I]\right\}^2}{2\,\{k - 2bt[1-t]\}^2}. \qquad (16)$$

Substituting (15) into (13) provides:

$$q_1^{IB} = a - bc_2^I - bc_h + \frac{bt[1-t][a - bc_2^I - bc_h] + b^2[1-t]^2[c_2^I - c_1^I]}{k - 2bt[1-t]}$$

$$= \frac{1}{k-2bt[1-t]}\{[k-2bt\,(1-t)]\left[a-bc_2^I-bc_h\right]+bt[1-t][a-bc_2^I-bc_h]+b^2[1-t]^2[c_2^I-c_{1\,1}^I]\}$$

$$= \frac{[k-bt\,(1-t)]\left[a-bc_2^I-bc_h\right]+b^2[1-t]^2[c_2^I-c_1^I]}{k-2bt[1-t]}. \tag{17}$$

(14), (16), and (17) imply that U-D1's profit in this setting is:

$$\Pi_{U1}^{IB} = \left[\frac{t^2[a-bc_2^I-bc_h]+bt[1-t][c_2^I-c_1^I]}{k-2bt[1-t]}+c_2^I-c_1^I\right]\left[\frac{[k-bt\,(1-t)]\left[a-bc_2^I-bc_h\right]+b^2[1-t]^2[c_2^I-c_1^I]}{k-2bt[1-t]}\right]$$

$$-\frac{k\left\{t[a-bc_2^I-bc_h]+b[1-t][c_2^I-c_1^I]\right\}^2}{2\left\{k-2bt[1-t]\right\}^2}. \tag{18}$$

From (17), consumer surplus in this setting is:

$$Z^{IB} = \frac{b}{2}[q_1^{IB}]^2 = \frac{b\left\{[k-bt\,(1-t)]\left[a-bc_2^I-bc_h\right]+b^2[1-t]^2[c_2^I-c_1^I]\right\}^2}{2\left\{k-2bt[1-t]\right\}^2}. \tag{19}$$

Analysis of the Linear Cournot Setting

Vertical Separation (VS) in the Linear Cournot Setting

When it faces input price w_1^S under VS, D1 chooses q_1^{SC} to:

$$Maximize \quad \pi_1^{SC} = [a-b(q_1^{SC}+q_2^{SC})-w_1^S-c_1^S]q_1^{SC}. \tag{20}$$

Differentiating (20) with respect to q_1 provides:

$$\frac{\partial\pi_1^{SC}}{\partial q_1^S} = a-b\left[2q_1^{SC}+q_2^{SC}\right]-w_1^S-c_1^S = 0 \quad\Rightarrow\quad q_1^{SC} = \frac{1}{2b}[a-w_1^S-c_1^S-bq_2^{SC}]. \tag{21}$$

Similarly, D2's profit-maximizing choice of q_2^S, given input price w_2^S, is:

$$q_2^{SC} = \frac{1}{2b}[a-w_2^S-c_2^S-bq_1^{SC}]. \tag{22}$$

Substituting (21) into (22) provides:

$$q_2^{SC} = \frac{1}{2b}[a-w_2^S-c_2^S]-\frac{1}{4b}[a-w_1^S-c_1^S-bq_2^{SC}] \;\Rightarrow\; \frac{3}{4}q_2^{SC} = \frac{1}{4b}[2a-2w_2^S-2c_2^S-a+w_1^S+c_1^S]$$

$$\Rightarrow\quad q_2^{SC} = \frac{1}{3b}[a-2w_2^S+w_1^S-2c_2^S+c_1^S]. \tag{23}$$

Substituting (23) into (21) provides:

$$q_1^{SC} = \frac{1}{3b}[a-2w_1^S+w_2^S-2c_1^S+c_2^S]. \tag{24}$$

(23) and (24) provide:

$$Q^{SC} = q_1^{SC}+q_2^{SC} = \frac{1}{3b}[2a-w_1^S-w_2^S-c_1^S-c_2^S]. \tag{25}$$

(25) implies:

$$P(Q^{SC}) = a-bQ^{SC} = \frac{1}{3}[a+w_1^S+w_2^S+c_1^S+c_2^S]. \tag{26}$$

The input prices under VS are:

$$w^S \;=\; w_1^S \;=\; w_2^S =\; c_h - \alpha^S[1-t] \quad\Rightarrow\quad w^S - c^u \;=\; t\alpha. \tag{27}$$

Substituting (27) into (25) provides:

$$Q^{SC} \;=\; \frac{1}{3b}[2a - 2c_h + 2\alpha^S\,(1-t) - c_1^S - c_2^S]. \tag{28}$$

(27) and (28) imply that U chooses α to:

$$Maximize \quad [w^S - c^u]Q^{SC} - \frac{k}{2}\alpha^2 \;=\; \frac{t\alpha}{3b}[2a - 2c_h + 2\alpha^S\,(1-t) - c_1^S - c_2^S] - \frac{k}{2}\alpha^2. \tag{29}$$

Differentiating (29) with respect to α^S provides:

$$\frac{2t[1-t]\alpha^S}{3b} + \frac{t}{3b}[2a - 2c_h + 2\alpha^S\,(1-t) - c_1^S - c_2^S] - k\alpha^S \;=\; 0$$

$$\Rightarrow \quad 2t[1-t]\alpha^S + t[2a - 2c_h + 2\alpha^S\,(1-t) - c_1^S - c_2^S] \;=\; 3bk\alpha^S$$

$$\Rightarrow \quad \alpha^{SC} \;=\; \frac{t\left[2a - 2c_h - c_1^S - c_2^S\right]}{3bk - 4t[1-t]}. \tag{30}$$

Substituting (30) into (28) provides:

$$Q^{SC} \;=\; \frac{1}{3b}\left[2a - 2c_h - c_1^S - c_2^S\right] + \frac{2t[1-t]}{3b}\left[\frac{2a - 2c_h - c_1^S - c_2^S}{3bk - 4t[1-t]}\right]$$

$$=\; \frac{\left[2a - 2c_h - c_1^S - c_2^S\right][3bk - 4t\,(1-t)] + 2t[1-t]\left[2a - 2c_h - c_1^S - c_2^S\right]}{3b\,[3bk - 4t\,(1-t)]}$$

$$=\; \frac{\left[2a - 2c_h - c_1^S - c_2^S\right][3bk - 2t\,(1-t)]}{3b\,[3bk - 4t\,(1-t)]}. \tag{31}$$

(27) and (30) imply:

$$w^S - c^u \;=\; \frac{t^2\left[2a - 2c_h - c_1^S - c_2^S\right]}{3bk - 4t[1-t]}. \tag{32}$$

(29) – (32) imply that U's profit under VS is:

$$\Pi_U^{SC} \;=\; \frac{t^2\,[3bk - 2t\,(1-t)]\left[2a - 2c_h - c_1^S - c_2^S\right]^2}{3b\left\{3bk - 4t[1-t]\right\}^2} - \frac{kt^2\left[2a - 2c_h - c_1^S - c_2^S\right]^2}{2\left\{3bk - 4t[1-t]\right\}^2}. \tag{33}$$

(24), (27), and (30) imply:

$$q_1^{SC} \;=\; \frac{1}{3b}[a - c_h + \alpha^S[1-t] - 2c_1^S + c_2^S] \;=\; \frac{1}{3b}[a - c_h - 2c_1^S + c_2^S] + \frac{t[1-t]\left[2a - 2c_h - c_1^S - c_2^S\right]}{3b\,[3bk - 4t\,(1-t)]}$$

$$=\; \frac{[a - c_h - 2c_1^S + c_2^S]\,[3bk - 4t\,(1-t)] + t[1-t]\left[2a - 2c_h - c_1^S - c_2^S\right]}{3b\,[3bk - 4t\,(1-t)]}$$

$$=\; \frac{[a - c_h]\,[3bk - 2t\,(1-t)] - c_1^S\,[6bk - 7t\,(1-t)] + c_2^S\,[3bk - 5t\,(1-t)]}{3b\,[3bk - 4t\,(1-t)]}. \tag{34}$$

Similarly, (23), (27), and (30) imply:

$$q_2^{SC} \;=\; \frac{1}{3b}[a - c_h + \alpha^S[1-t] - 2c_2^S + c_1^S] \;=\; \frac{1}{3b}[a - c_h - 2c_2^S + c_1^S] + \frac{t[1-t]\left[2a - 2c_h - c_1^S - c_2^S\right]}{3b\left[3bk - 4t\left(1-t\right)\right]}$$

$$= \frac{\left[a - c_h\right]\left[3bk - 2t\left(1-t\right)\right] - c_2^S\left[6bk - 7t\left(1-t\right)\right] + c_1^S\left[3bk - 5t\left(1-t\right)\right]}{3b\left[3bk - 4t\left(1-t\right)\right]}. \quad (35)$$

(27) and (30) imply:

$$w^{SC} \;=\; c_h - \frac{t[1-t]\left[2a - 2c_h - c_1^S - c_2^S\right]}{3bk - 4t[1-t]}. \quad (36)$$

From (20):

$$\pi_1^{SC} \;=\; [a - bQ^{SC} - w^S - c_1^S]q_1^{SC} \quad \text{and} \quad \pi_2^{SC} \;=\; [a - bQ^{SC} - w^S - c_2^S]q_2^{SC}, \quad (37)$$

where Q^S is defined in (31), w^S is defined in (36), q_1^S is defined in (34), and q_2^S is defined in (35).

From (31), consumer surplus in this setting is:

$$Z^{SC} \;=\; \frac{b}{2}(Q^{SC})^2 \;=\; \frac{\left[2a - 2c_h - c_1^S - c_2^S\right]^2\left[3bk - 2t\left(1-t\right)\right]^2}{18b\left[3bk - 4t\left(1-t\right)\right]^2}. \quad (38)$$

Vertical Integration (VI) in the Linear Cournot Setting

From (22), D2's profit-maximizing output given input price w^I is:

$$q_2^{IC} \;=\; \frac{1}{2b}[a - w^I - c_2^I - bq_1^{IC}]. \quad (39)$$

U-D1 chooses q_1^{IC} to:

$$Maximize \quad \Pi_{U1}^{IC} \;=\; [w^I - c^u]q_2^{IC} + [a - b(q_1^{IC} + q_2^{IC}) - c^u - c_1^I]q_1^{IC}. \quad (40)$$

(40) implies:

$$\frac{\partial \Pi_{U1}^{IC}}{\partial q_1^I} \;=\; a - 2bq_1^{IC} - bq_2^{IC} - c^u - c_1^I \;=\; 0 \quad \Rightarrow \quad q_1^{IC} \;=\; \frac{1}{2b}[a - c^u - c_1^I - bq_2^{IC}]. \quad (41)$$

Substituting (41) into (39) provides:

$$q_2^{IC} \;=\; \frac{1}{2b}[a - w_2 - c_2^{IC}] - \frac{1}{4b}[a - c^u - c_1^I - bq_2^{IC}] \quad \Rightarrow \quad \frac{3}{4}q_2^{IC} \;=\; \frac{1}{4b}\{2a - 2w^I - 2c_2^I - a + c^u + c_1^I\}$$

$$\Rightarrow \quad q_2^{IC} \;=\; \frac{1}{3b}\left[a - 2w^I + c^u - 2c_2^I + c_1^I\right]. \quad (42)$$

Substituting (42) into (41) provides:

$$q_1^{IC} \;=\; \frac{1}{2b}[a - c^u - c_1] - \frac{1}{6b}\left[a - 2w^I + c^u - 2c_2^I + c_1^I\right] \;=\; \frac{1}{6b}\left[3a - 3c^u - 3c_1^I - a + 2w^I - c^u + 2c_2^I - c_1^I\right]$$

$$\Rightarrow \quad q_1^{IC} \;=\; \frac{1}{3b}[a - 2c^u + w^I - 2c_1^I + c_2^I]. \quad (43)$$

(42) and (43) imply:

$$Q^{IC} \;=\; q_1^{IC} + q_2^{IC} \;=\; \frac{1}{3b}[2a - c^u - w^I - c_1^I - c_2^I]. \quad (44)$$

Recall from (27) that for given upstream cost reduction α:

$$w^I \;=\; c_h - [1-t]\alpha. \quad (45)$$

Substituting (45) into (42) and (43) provides:

$$q_1^{IC} \ = \ \frac{1}{3b}[a - 2(c_h - \alpha) + c_h - (1-t)\alpha - 2c_1^I + c_2^I] = \ \frac{1}{3b}[a - c_h - 2c_1^I + c_2^I + (1+t)\alpha]. \tag{46}$$

$$q_2^{IC} \ = \ \frac{1}{3b}\left[a - 2(c_h - [1-t]\alpha) + c_h - \alpha - 2c_2^I + c_1^I\right] \ = \ \frac{1}{3b}\left[a - c_h - 2c_2^I + c_1^I + (1-2t)\,\alpha\right]. \tag{47}$$

(46) and (47) imply:

$$Q^{IC} \ = \ \frac{1}{3b}[2a - 2c_h - c_1^I - c_2^I + (2-t)\,\alpha]. \tag{48}$$

(48) implies:

$$P^{IC} \ \equiv \ P(Q^{IC}) \ = \ a - bQ^{IC} \ = \ \frac{1}{3}[a + 2c_h + c_1^I + c_2^I - (2-t)\,\alpha]. \tag{49}$$

(49) implies:

$$P^{IC} - c^u - c_1^I \ = \ \frac{1}{3}[a + 2c_h + c_1^I + c_2^I - (2-t)\,\alpha] - c_h + \alpha - c_1^I = \ \frac{1}{3}[a - c_h - 2c_1^I + c_2^I + (1+t)\,\alpha]. \tag{50}$$

Substituting (45) – (47) and (50) into (40) implies that U-D1 chooses α to:

$$Maximize \ \ \Pi_{U1}^{IC}(\alpha) = \frac{t\alpha}{3b}\left[a - c_h - 2c_2^I + c_1^I + (1-2t)\,\alpha\right] \ + \ \frac{1}{9b}[a - c_h - 2c_1^I + c_2^I + (1+t)\,\alpha]^2 - \frac{k}{2}\alpha^2. \tag{51}$$

(51) implies:

$$\Pi_{U1}^{IC\,\prime}(\alpha) \ = \ \frac{t\alpha[1-2t]}{3b} + \frac{t}{3b}\left[a - c_h - 2c_2^I + c_1^I + (1-2t)\,\alpha\right]$$

$$+\frac{2[1+t]}{9b}[a - c_h - 2c_1^I + c_2^I + (1+t)\,\alpha] - k\alpha \ = \ 0$$

$$\Rightarrow \quad 3t\alpha[1-2t] + 3t\left[a - c_h - 2c_2^I + c_1^I + (1-2t)\,\alpha\right] + 2[1+t][a - c_h - 2c_1^I + c_2^I + (1+t)\,\alpha] \ = \ 9bk\alpha$$

$$\Rightarrow \quad 3t\left[a - c_h - 2c_2^I + c_1^I\right] + 2[1+t][a - c_h - 2c_1^I + c_2^I] \ = \ \alpha\left\{9bk - 6t[1-2t] - 2[t^2 + 2t + 1]\right\}$$

$$\Rightarrow \quad \alpha^{IC} \ = \ \frac{3t\left[a - c_h - 2c_2^I + c_1^I\right] + 2[1+t][a - c_h - 2c_1^I + c_2^I]}{9bk - 2[1 + 5t - 5t^2]}, \tag{52}$$

since

$$6t[1-2t] + 2[t^2 + 2t + 1] \ = \ -10t^2 + 10t + 2 \ = \ 2[1 + 5t - 5t^2].$$

From (48):

$$Q^{IC} \ = \ \frac{1}{3b}[2a - 2c_h - c_1^I - c_2^I + (2-t)\,\alpha^{IC}], \tag{53}$$

where α^{IC} is as defined in (52).

(51) implies:

$$\Pi_{U1}^{IC} \ = \ \frac{t\alpha^{IC}}{3b}\left[a - c_h - 2c_2^I + c_1^I + (1-2t)\,\alpha^{IC}\right] + \ \frac{1}{9b}[a - c_h - 2c_1^I + c_2^I + (1+t)\,\alpha^{IC}]^2 - \frac{k}{2}\left(\alpha^{IC}\right)^2, \tag{54}$$

where α^I is as defined in (52).

Consumer surplus in this setting is:

$$Z^{IC} = \frac{b}{2}(Q^{IC})^2, \tag{55}$$

where Q^I is as defined in (53).

Analysis of the Linear Demand, Quadratic Cost Setting with Product Differentiation

Consider the setting in which the welfare of a representative consumer that consumes q_1 units of D1's retail product and q_2 units of D2's retail product is:

$$U(q_1,\, q_2) \;=\; \widetilde{\alpha}[q_1 + q_2] \;-\; \frac{1}{2}\beta\left[(q_1)^2 + (q_2)^2\right] \;-\; \delta q_1 q_2, \tag{56}$$

where $\beta > \delta$. Also assume that U's cost of reducing upstream unit cost by α units is:

$$K(\alpha) \;=\; \frac{k}{2}\alpha^2. \tag{57}$$

The regulator sets the access price:

$$w \;=\; c_h - \alpha + t\alpha \;=\; c_h - [1-t]\alpha. \tag{58}$$

Differentiating $U(\cdot)$ provides consumer demand functions:

$$p_i \;=\; \frac{\partial U(\cdot)}{\partial q_i} \;=\; \widetilde{\alpha} - \beta q_i - \delta q_j \quad\Rightarrow\quad q_i \;=\; \frac{1}{\beta}\left[\widetilde{\alpha} - p_i - \delta q_j\right]. \tag{59}$$

(59) implies:

$$q_i \;=\; \frac{1}{\beta}[\widetilde{\alpha} - p_i] - \left[\frac{\delta}{\beta}\right]\frac{1}{\beta}\left[\widetilde{\alpha} - p_j - \delta q_i\right] \quad\Rightarrow\quad q_i\left[1 - \frac{\delta^2}{\beta^2}\right] \;=\; \frac{1}{\beta^2}\left\{\beta[\widetilde{\alpha} - p_i] - \delta\left[\widetilde{\alpha} - p_j\right]\right\}$$

$$\Rightarrow\quad q_i \;=\; \left[\frac{1}{\beta^2 - \delta^2}\right]\left[(\beta - \delta)\widetilde{\alpha} - \beta p_i + \delta p_j\right] \;=\; a - bp_i + dp_j, \tag{60}$$

where:

$$a \;\equiv\; \frac{[\beta - \delta]\,\widetilde{\alpha}}{\beta^2 - \delta^2}; \quad b \;\equiv\; \frac{\beta}{\beta^2 - \delta^2}; \quad \text{and} \quad d \;\equiv\; \frac{\delta}{\beta^2 - \delta^2}. \tag{61}$$

The downstream unit cost of production for Di is $c_i(L)$. U chooses L at the same time it chooses α. The downstream cost structure is:

$$c_1(L) \;=\; \underline{c}_1 + cL \quad\text{and}\quad c_2(L) \;=\; \underline{c}_2 + c[1-L], \quad \text{where } \underline{c}_1 \;=\; \underline{c}_2 = \underline{c}. \tag{62}$$

w_i^r will denote the effective access charge faced by downstream producer Di under regime r in this "LQD setting" for $i \in \{1, 2\}$ and $r \in \{S, I\}$.

Vertical Separation (VS) in the LQD Setting

From (60), downstream competition under VS can be modeled as firm Di choosing p_i to maximize:

$$\pi_i^S(p_i,\, p_j) \;=\; [p_i - w_i^S - c_i^S]q_i(p_i,\, p_j) \;=\; [p_i - w_i^S - c_i^S]\left[a - bp_i + dp_j\right]. \tag{63}$$

(60) and (63) imply that firm Di's profit-maximizing choice of p_i is determined by:

$$\frac{\partial \pi_i^S(\cdot)}{\partial p_i} \;=\; [p_i - w_i^S - c_i^S][-b] + a - bp_i + dp_j \;=\; 0 \quad\Rightarrow\quad p_i \;=\; \frac{1}{2b}\left[a + bw_i^S + bc_i^S + dp_j\right]. \tag{64}$$

(64) implies that the equilibrium retail price of Di under VS (given w_i and w_j) is:

$$p_i^S \;=\; \frac{1}{2b}\left[a + bw_i^S + bc_i^S\right] + \left[\frac{d}{2b}\right]\frac{1}{2b}\left[a + bw_j^S + bc_j^S + dp_i^S\right]$$

$$\Rightarrow\quad p_i^S\left[1 - \frac{d^2}{4b^2}\right] \;=\; \frac{1}{4b^2}\left\{2b\left[a + bw_i^S + bc_i^S\right] + d\left[a + bw_j^S + bc_j^S\right]\right\}$$

$$\Rightarrow\quad p_i^S \;=\; \left[\frac{1}{4b^2 - d^2}\right]\left\{[2b + d]\,a + 2b^2[w_i^S + c_i^S] + bd[w_j^S + c_j^S]\right\}. \tag{65}$$

Substituting (65) into (60) provides:

$$
\begin{aligned}
q_i^S &= a - \left[\frac{b}{4b^2-d^2}\right]\left\{[2b+d]\,a + 2b^2[w_i^S + c_i^S] + bd[w_j^S + c_j^S]\right\} \\
&\qquad + \left[\frac{d}{4b^2-d^2}\right]\left\{[2b+d]\,a + 2b^2[w_j^S + c_j^S] + bd[w_i^S + c_i^S]\right\} \\
&= \left[\frac{1}{4b^2-d^2}\right]\left\{\left[4b^2 - d^2 - b\,(2b+d) + d\,(2b+d)\right]a - [2b^3 - bd^2][w_i^S + c_i^S] - [b^2 d - 2b^2 d][w_j^S + c_j^S]\right\} \\
&= \left[\frac{1}{4b^2-d^2}\right]\left\{[2b^2 + bd]a - b[2b^2 - d^2][w_i^S + c_i^S] + b^2 d[w_j^S + c_j^S]\right\} \\
&= \left[\frac{b}{4b^2-d^2}\right]\left\{[2b+d]a - [2b^2 - d^2][w_i^S + c_i^S] + bd[w_j^S + c_j^S]\right\}. \qquad (66)
\end{aligned}
$$

(66) implies:

$$
\begin{aligned}
Q^S &= q_1^S + q_2^S = \left[\frac{b}{4b^2-d^2}\right]\left\{2[2b+d]a - [2b^2 - bd - d^2][w_i^S + c_i^S + w_j^S + c_j^S]\right\} \\
&= \left[\frac{b[2b+d]}{4b^2-d^2}\right]\left\{2a - [b-d][w_i^S + c_i^S + w_j^S + c_j^S]\right\} = \left[\frac{b}{2b-d}\right]\left\{2a - [b-d][w_i^S + c_i^S + w_j^S + c_j^S]\right\}. \qquad (67)
\end{aligned}
$$

(67) implies that as long as $\frac{\partial w_1^S}{\partial L} = \frac{\partial w_2^S}{\partial L} = 0$:

$$
\frac{dQ^S}{dL} = \frac{\partial Q^S}{\partial c_i}\left[\frac{\partial c_i^S}{\partial L}\right] + \frac{\partial Q^S}{\partial c_j^S}\left[\frac{\partial c_j^S}{\partial L}\right] = -\left[\frac{b(b-d)}{2b-d}\right]\left\{\frac{\partial c_i^S}{\partial L} + \frac{\partial c_j^S}{\partial L}\right\} = 0. \qquad (68)
$$

The last equality in (68) holds because $\frac{\partial\{c_1(L)+c_2(L)\}}{\partial L} = 0$, by assumption.

Under VS, U chooses α^S to maximize:

$$
\Pi_U^S(\alpha^S) \equiv [w - (c_h - \alpha^S)][q_1^S + q_2^S] - K(\alpha^S) = t\alpha^S Q^S - \frac{k}{2}\left(\alpha^S\right)^2. \qquad (69)
$$

Furthermore, under VS:

$$
w_1^S = w_2^S = c_h - [1-t]\alpha^S \quad \Rightarrow \quad \frac{dw_i^S}{d\alpha^S} = -[1-t] \quad \text{for } i = 1, 2. \qquad (70)
$$

(70) implies that when U chooses L and α^S simultaneously, $\frac{\partial w_1^S}{\partial L} = \frac{\partial w_2^S}{\partial L} = 0$. Therefore, (68) and (69) imply that U is indifferent among all $L \in [0, 1]$ under VS. We will assume that, when indifferent among all $L \in [0, 1]$, U sets $L = \frac{1}{2}$ when downstream costs are as specified in (62) (perhaps to respect laws or customs against discrimination).

Differentiating (69) provides:

$$
\Pi_U^{S\prime}(\alpha^S) = tQ^S + t\alpha^S\frac{dQ^S}{d\alpha} - k\alpha^S. \qquad (71)
$$

(67) and (70) imply:

$$
\frac{dQ^S}{d\alpha^S} = -\frac{b[b-d]}{2b-d}\frac{d\{w_1^S + w_2^S\}}{d\alpha^S} = \frac{2b[b-d][1-t]}{2b-d}. \qquad (72)
$$

Substituting (67) and (72) into (71) provides:

$$
\Pi_U^{S\prime}(\alpha^S) = \left[\frac{bt}{2b-d}\right]\left\{2a - [b-d][w_1^S + c_1^S + w_2^S + c_2^S]\right\} + \frac{2bt\alpha[b-d][1-t]}{2b-d} - k\alpha^S = 0. \qquad (73)
$$

Solving (73), using (70), provides:

$$2abt - bt[b-d][w_1^S + c_1^S + w_2^S + c_2^S] + 2bt\alpha^S[b-d][1-t] \;=\; k\alpha^S\,[2b-d]$$

$$\Rightarrow \quad 2bta - bt[b-d][c_1^S + c_2^S] \;=\; 2bt[b-d][c_h - (1-t)\,\alpha^S] - 2bt[1-t]\alpha^S[b-d] + k\alpha^S\,[2b-d]$$

$$\Rightarrow \quad 2bta - 2bt[b-d]c_h - bt[b-d][c_1^S + c_2^S] \;=\; \alpha^S\,\{-2bt[1-t][b-d] + k\,[2b-d] - 2bt[1-t][b-d]\}$$

$$\Rightarrow \quad \alpha^S \;=\; \frac{bt\left\{2a - [b-d][2c_h + c_1^S + c_2^S]\right\}}{k\,[2b-d] - 4bt[1-t][b-d]}. \tag{74}$$

<u>Vertical Integration (VI) in the LQD Setting</u>

Under VI, the effective input prices faced by D1 and D2, respectively, are:

$$w_1^I \;=\; c_h - \alpha^I \quad \text{and} \quad w_2^I \;=\; c_h - [1-t]\alpha^I. \tag{75}$$

From (64), the profit-maximizing price for D2 given D1's price is:

$$p_2^I \;=\; \frac{1}{2b}\left[a + bw_2^I + bc_2^I + dp_1^I\right]. \tag{76}$$

Using (60), the integrated firm, U-D1, chooses p_1 to maximize:

$$\begin{aligned}\Pi_{U1}^I \;&=\; [w_2^I - w_1^I]q_2^I(p_2^I,\, p_1^I) + [p_1^I - w_1^I - c_1^I]q_1(p_1^I, p_2^I)\\ &=\; t\alpha^I\left[a - bp_2^I + dp_1^I\right] + [p_1^I - w_1^I - c_1^I]\left[a - bp_1^I + dp_2^I\right].\end{aligned} \tag{77}$$

Differentiating (77) with respect to p_1 provides:

$$t\alpha^I d - b[p_1^I - w_1^I - c_1^I] + a - bp_1^I + dp_2^I \;=\; 0 \quad\Leftrightarrow\quad p_1^I \;=\; \frac{1}{2b}\left[a + bw_1^I + bc_1^I + t\alpha^I d + dp_2^I\right]. \tag{78}$$

Substituting (76) into (78) provides:

$$p_1^I \;=\; \frac{1}{2b}\left[a + bw_1^I + bc_1^I + t\alpha^I d\right] + \left[\frac{d}{2b}\right]\frac{1}{2b}\left[a + bw_2^I + bc_2^I + dp_1^I\right]$$

$$\Rightarrow \quad p_1^I\left[1 - \frac{d^2}{4b^2}\right] \;=\; \frac{1}{4b^2}\left\{2b\left[a + bw_1^I + bc_1^I + t\alpha^I d\right] + d\left[a + bw_2^I + bc_2^I\right]\right\}$$

$$\Rightarrow \quad p_1^I \;=\; \left[\frac{1}{4b^2 - d^2}\right]\left\{[2b+d]a + 2bt\alpha^I d + 2b^2[w_1^I + c_1^I] + bd\left[w_2^I + c_2^I\right]\right\}. \tag{79}$$

Substituting (79) into (76) provides:

$$p_2^I \;=\; \frac{1}{2b}\left[a + bw_2 + bc_2^I\right] + \frac{d}{2b}\left[\frac{1}{4b^2 - d^2}\right]\left\{[2b+d]a + 2bt\alpha^I d + 2b^2[w_1^I + c_1^I] + db\left[w_2^I + c_2^I\right]\right\}$$

$$= \frac{1}{2b}\left[\frac{1}{4b^2 - d^2}\right]\left\{\left[4b^2 - d^2\right]\left[a + bw_2^I + bc_2^I\right] + d\left([2b+d]a + 2bt\alpha^I d + 2b^2[w_1^I + c_1^I] + db\left[w_2^I + c_2^I\right]\right)\right\}$$

$$= \frac{1}{2b}\left[\frac{1}{4b^2 - d^2}\right]\left\{[4b^2 + 2bd]a + 2bt\alpha^I d^2 + 2b^2 d[w_1^I + c_1^I] + \left(\left[4b^2 - d^2\right]b + bd^2\right)\left[w_2^I + c_2^I\right]\right\}$$

$$= \left[\frac{1}{4b^2 - d^2}\right]\left\{[2b+d]a + t\alpha^I d^2 + bd[w_1^I + c_1^I] + 2b^2\left[w_2^I + c_2^I\right]\right\}. \tag{80}$$

(60), (79), and (80) imply:

$$q_1^I \;=\; a - bp_1^I + dp_2^I \;=\; \left[\frac{1}{4b^2-d^2}\right]\{\left[4b^2-d^2\right]a - b\left([2b+d]a + 2bt\alpha^I d + 2b^2[w_1^I+c_1^I] + bd\left[w_2^I+c_2^I\right]\right)$$

$$+d\left([2b+d]a + t\alpha^I d^2 + bd[w_1^I+c_1^I] + 2b^2\left[w_2^I+c_2^I\right]\right)\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{\left[4b^2-d^2-2b^2-bd+2bd+d^2\right]a - t\alpha^I d\left[2b^2-d^2\right] - b[w_1^I+c_1^I]\left[2b^2-d^2\right] + b^2 d\left[w_2^I+c_2^I\right]\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]\,ba - t\alpha^I d\left[2b^2-d^2\right] - b\left[2b^2-d^2\right]\left[w_1^I+c_1^I\right] + b^2 d\left[w_2^I+c_2^I\right]\}. \tag{81}$$

(60), (79), and (80) also imply:

$$q_2^I \;=\; a - bp_2^I + dp_1^I \;=\; \left[\frac{1}{4b^2-d^2}\right]\{\left[4b^2-d^2\right]a - \; b\{[2b+d]a + t\alpha d^2 + bd[w_1^I+c_1^I] + 2b^2\left[w_2^I+c_2^I\right]\}$$

$$+\; d\left\{[2b+d]a + 2bt\alpha^I d + 2b^2[w_1^I+c_1^I] + bd\left[w_2^I+c_2^I\right]\right\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{\left[4b^2-d^2-2b^2-bd+2bd+d^2\right]a - t\alpha^I bd^2[1-2] - b^2 d[1-2][w_1^I+c_1^I] - b[2b^2-d^2]\left[w_2^I+c_2^I\right]\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]\,ba + t\alpha^I bd^2 + b^2 d[w_1^I+c_1^I] - b[2b^2-d^2]\left[w_2^I+c_2^I\right]\}. \tag{82}$$

(81) and (82) imply:

$$Q^I \;=\; q_1^I + q_2^I \;=\; \left[\frac{1}{4b^2-d^2}\right]\{2\,[2b+d]\,ba - t\alpha^I d\left[2b^2-bd-d^2\right]$$

$$-\left[2b^2-bd-d^2\right]b[w_1^I+c_1^I] - \left[2b^2-bd-d^2\right]b\left[w_2^I+c_2^I\right]\}$$

$$= \left[\frac{1}{2b-d}\right]\{2ba - [b-d][t\alpha^I d + b\left(w_1^I+c_1^I\right) + b\left(w_2^I+c_2^I\right)]\}. \tag{83}$$

(75) and (83) imply:

$$\frac{dQ^I}{d\alpha^I} \;=\; -\left[\frac{b-d}{2b-d}\right]\left[td + b\frac{dw_1^I}{d\alpha} + b\frac{dw_2^I}{d\alpha}\right] \;=\; -\frac{[b-d]}{2b-d}\left[td - b - b(1-t)\right] \;=\; \frac{[b-d][2b-t(b+d)]}{2b-d}. \tag{84}$$

From (77), U-D1 chooses α to maximize:

$$\Pi^I(\alpha^I) \;=\; t\alpha^I q_2^I + [p_1^I - c_h + \alpha^I - c_1^I]q_1^I - \frac{1}{2}k\left(\alpha^I\right)^2. \tag{85}$$

Differentiating (85) with respect to α provides:

$$\Pi^{I\prime}(\alpha^I) \;=\; tq_2^I + t\alpha^I\frac{dq_2^I}{d\alpha} + [p_1^I - c_h + \alpha^I - c_1^I]\frac{dq_1^I}{d\alpha^I} + q_1^I\left[\frac{dp_1^I}{d\alpha^I} + 1\right] - k\alpha^I. \tag{86}$$

(81) and (75) imply:

$$\frac{dq_1^I}{d\alpha^I} \;=\; \left[\frac{1}{4b^2-d^2}\right]\left\{-td\left[2b^2-d^2\right] - b\left[2b^2-d^2\right]\frac{dw_1^I}{d\alpha^I} + b^2 d\frac{dw_2^I}{d\alpha^I}\right\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{-td\left[2b^2-d^2\right] + b\left[2b^2-d^2\right] - b^2 d[1-t]\} \;=\; \left[\frac{1}{4b^2-d^2}\right]\{-td\left[b^2-d^2\right] + b\left[2b^2-d^2\right] - b^2 d\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{-td\left[b^2-d^2\right]+b[2b+d][b-d]\} = \left[\frac{b-d}{4b^2-d^2}\right]\{b[2b+d]-d\,[b+d]\,t\}. \tag{87}$$

(82) and (75) imply:

$$\frac{dq_2^I}{d\alpha^I} = \left[\frac{1}{4b^2-d^2}\right]\left\{bd^2t+b^2d\frac{dw_1^I}{d\alpha^I}-b[2b^2-d^2]\frac{dw_2^I}{d\alpha^I}\right\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\left\{bd^2t-b^2d+b[2b^2-d^2][1-t]\right\} = \left[\frac{1}{4b^2-d^2}\right]\left\{-2bt[b^2-d^2]+b[2b^2-bd-d^2]\right\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\left\{-2bt[b^2-d^2]+b[2b+d][b-d]\right\} = \left[\frac{b-d}{4b^2-d^2}\right]\{b[2b+d]-2b\,[b+d]\,t\}. \tag{88}$$

(79) and (75) imply:

$$\frac{dp_1^I}{d\alpha^I} = \left[\frac{1}{4b^2-d^2}\right]\left\{2btd+2b^2\left[\frac{dw_1^I}{d\alpha^I}\right]+bd\left[\frac{dw_2^I}{d\alpha^I}\right]\right\}$$

$$= \left[\frac{b}{4b^2-d^2}\right]\{2td-2b-d[1-t]\} = \frac{b[3dt-2b-d]}{4b^2-d^2}. \tag{89}$$

(89) implies:

$$\frac{dp_1^I}{d\alpha^I}+1 = \frac{b[3dt-2b-d]+4b^2-d^2}{4b^2-d^2} = \frac{2b^2-bd-d^2+3bdt}{4b^2-d^2} = \frac{[2b+d][b-d]+3bdt}{4b^2-d^2}. \tag{90}$$

(75) and (79) imply:

$$p_1^I-c_h+\alpha^I-c_1^I = \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]a+2bt\alpha^Id+2b^2[c_h-\alpha^I+c_1^I]$$

$$+\,bd\left[c_h-\alpha^I(1-t)+c_2\right]-\left[4b^2-d^2\right]\left[c_h-\alpha^I+c_1^I\right]\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]a+2bt\alpha^Id-2b^2\alpha^I-\alpha bd\,[1-t]+\alpha^I\left[4b^2-d^2\right]$$

$$+c_h\left(2b^2+bd-\left[4b^2-d^2\right]\right)-\left[4b^2-2b^2-d^2\right]c_1^I+bdc_2^I\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]a+\alpha^I\left[3btd-bd-2b^2+4b^2-d^2\right]-c_h\left[2b^2-bd-d^2\right]-\left[2b^2-d^2\right]c_1^I+bdc_2^I\}$$

$$= \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]a+\alpha^I\left[(2b+d)(b-d)+3btd\right]-c_h\left[2b^2-bd-d^2\right]-\left[2b^2-d^2\right]c_1^I+bdc_2^I\}. \tag{91}$$

Substituting (81), (87), (88), (90), and (91) into (86) provides:

$$\Pi_{U1}^I{}'(\alpha^I) = \left[\frac{t}{4b^2-d^2}\right]\{[2b+d]\,ba+t\alpha^Ibd^2+b^2d[w_1^I+c_1^I]-b[2b^2-d^2]\left[w_2^I+c_2^I\right]\}$$

$$+\,t\alpha^I\left[\frac{b-d}{4b^2-d^2}\right]\{b[2b+d]-2b\,[b+d]\,t\} + \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]a+\alpha^I\left[(2b+d)(b-d)+3btd\right]$$

$$-\,c_h\left[2b^2-bd-d^2\right]-\left[2b^2-d^2\right]c_1+bdc_2\}\left[\frac{b-d}{4b^2-d^2}\right]\{b[2b+d]-td\,[b+d]\}$$

$$+\ \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]\,ba - t\alpha^I d\left[2b^2-d^2\right] - b\left[2b^2-d^2\right][w_1^I + c_1^I]$$
$$+b^2 d\left[w_2^I + c_2^I\right]\}\left[\frac{(2b+d)(b-d)+3bdt}{4b^2-d^2}\right] - k\alpha^I \ = \ 0 \tag{92}$$

$$\Leftrightarrow\ \ t\left[4b^2-d^2\right]\{[2b+d]\,ba + t\alpha^I bd^2 + b^2 d[w_1^I + c_1^I] - b[2b^2-d^2]\left[w_2^I + c_2^I\right]\}$$
$$+\ t\left[4b^2-d^2\right][b-d]\{b[2b+d] - 2b\,[b+d]\,t\}\alpha^I$$
$$+\,[b-d]\{b[2b+d] - td\,[b+d]\}\{[2b+d]a + \alpha^I\,[(2b+d)(b-d)+3btd]$$
$$-\ c_h\left[2b^2-bd-d^2\right] - \left[2b^2-d^2\right]c_1 + bdc_2\}$$
$$+\{[2b+d]\,ba - t\alpha^I d\left[2b^2-d^2\right] - b\left[2b^2-d^2\right][w_1^I + c_1^I]$$
$$+\ b^2 d\left[w_2^I + c_2^I\right]\}\,[(2b+d)(b-d)+3bdt] - k\alpha^I\left[4b^2-d^2\right]^2 \ = \ 0 \tag{93}$$

$$\Leftrightarrow\ \ t\left[4b^2-d^2\right]\{[2b+d]\,ba + t\alpha^I bd^2 + b^2 d[c_h - \alpha^I + c_1^I] - b[2b^2-d^2]\left[c_h - \alpha^I(1-t) + c_2^I\right]\}$$
$$+\ t\left[4b^2-d^2\right][b-d]\{b[2b+d] - 2b\,[b+d]\,t\}\alpha^I + [b-d]\{b[2b+d] - td\,[b+d]\}\{[2b+d]a$$
$$-c_h\left[2b^2-bd-d^2\right] - \left[2b^2-d^2\right]c_1^I + bdc_2\} + [(2b+d)(b-d)+3btd]\,[b-d]\{b[2b+d] - td\,[b+d]\}\alpha^I$$
$$+\ \{[2b+d]\,ba - t\alpha^I d\left[2b^2-d^2\right] - b\left[2b^2-d^2\right][c_h - \alpha^I + c_1^I]$$
$$+\ b^2 d\left[c_h - \alpha^I(1-t) + c_2^I\right]\}\,[(2b+d)(b-d)+3bdt] - k\left[4b^2-d^2\right]^2\alpha^I \ = \ 0 \tag{94}$$

$$\Leftrightarrow\ \ t\left[4b^2-d^2\right]\{[2b+d]\,ba + b^2 d[c_h + c_1^I] - b[2b^2-d^2]\left[c_h + c_2^I\right]\}$$
$$+\,[b-d]\{b[2b+d] - td\,[b+d]\}\{[2b+d]a - c_h\left[2b^2-bd-d^2\right] - \left[2b^2-d^2\right]c_1 + bdc_2^I\}$$
$$+\{[2b+d]\,ba - b\left[2b^2-d^2\right][c_h + c_1^I] + b^2 d\left[c_h + c_2^I\right]\}\,[(2b+d)(b-d)+3bdt]$$
$$+\ t\left[4b^2-d^2\right]\{tbd^2 - b^2 d + b[2b^2-d^2][1-t]\}\alpha^I +\ t\left[4b^2-d^2\right][b-d]\{b[2b+d] - 2b\,[b+d]\,t\}\alpha^I$$
$$+\ \ [(2b+d)(b-d)+3btd]\,[b-d]\{b[2b+d] - td\,[b+d]\}\alpha^I$$
$$+\ \ [(2b+d)(b-d)+3bdt]\{b\left[2b^2-d^2\right] - td\left[2b^2-d^2\right] - b^2 d[1-t]\}\alpha^I - k\left[4b^2-d^2\right]^2\alpha^I \ = \ 0 \tag{95}$$

$$\Leftrightarrow\ \ \alpha^I = \ -\frac{G}{H},\quad \text{where} \tag{96}$$

$$G \ \equiv\ t\left[4b^2-d^2\right]\{[2b+d]\,ba - b\,[2b+d]\,[b-d]\,c_h + b^2 dc_1 - b[2b^2-d^2]c_2^I\}$$
$$+\ \ [b-d]\{b[2b+d] - td\,[b+d]\}\{[2b+d]a - c_h\,[2b+d][b-d] - \left[2b^2-d^2\right]c_1 + bdc_2^I\}$$
$$+\ \{[2b+d]\,ba - b\left[2b^2-d^2\right][c_h + c_1^I] + b^2 d\left[c_h + c_2^I\right]\}\,[(2b+d)(b-d)+3bdt]\,, \tag{97}$$

and:

$$H \ \equiv\ t\left[4b^2-d^2\right]\{tbd^2 - b^2 d + b[2b^2-d^2][1-t]\} - k\left[4b^2-d^2\right]^2$$
$$+\ t\left[4b^2-d^2\right][b-d]\{b[2b+d] - 2b\,[b+d]\,t\} +\ \ [(2b+d)(b-d)+3btd]\,[b-d]\{b[2b+d] - td\,[b+d]\}$$
$$+\ \ [(2b+d)(b-d)+3bdt]\{b\left[2b^2-d^2\right] - td\left[2b^2-d^2\right] - b^2 d[1-t]\}. \tag{98}$$

Notice that:

$$tbd^2 - b^2d + b[2b^2 - d^2][1-t] + [b-d]\{b[2b+d] - 2b[b+d]t\}$$

$$= tbd^2 - b^2d + 2b^3 - bd^2 - 2b^3t + bd^2t + [b-d][2b^2 + bd - 2b^2t - 2bdt]$$

$$= 2b^3 - b^2d - bd^2 - 2b^3t + bd^2t + bd^2t + 2b^3 + b^2d - 2b^2d - bd^2 - 2b^3t - 2b^2dt + 2b^2dt + 2bd^2t$$

$$= 4b^3 - 2b^2d - 2bd^2 - 4b^3t + 4bd^2t = 2b[2b^2 - bd - d^2] - 4bt[b^2 - d^2] = 2b[b-d]\{2b + d - 2t[b+d]\}. \quad (99)$$

Further notice that:

$$[b-d]\{b[2b+d] - td[b+d]\} + b\left[2b^2 - d^2\right] - td\left[2b^2 - d^2\right] - b^2d[1-t]$$

$$= [b-d][2b^2 + bd - bdt - d^2t] + 2b^3 - bd^2 - b^2d - 2b^2dt + b^2dt + d^3t$$

$$= 2b^3 + b^2d - 2b^2d - bd^2 - b^2dt - bd^2t + bd^2t + d^3t + 2b^3 - b^2d - bd^2 - 2b^2dt + b^2dt + d^3t$$

$$= 4b^3 - 2b^2d - 2bd^2 + t[-2b^2d - 2d^3] = 2b[2b^2 - bd - d^2] - 2dt[b^2 - d^2]$$

$$= 2b[2b^2 - bd - d^2] - 2dt[b^2 - d^2] = 2[b-d]\{b[2b+d] - dt[b+d]\}. \quad (100)$$

(98), (99), and (100) imply:

$$H = \left[4b^2 - d^2\right]\{t2b[b-d][2b + d - 2t(b+d)] - k\left[4b^2 - d^2\right]\}$$
$$+ 2[b-d]\left[(2b+d)(b-d) + 3btd\right]\{b[2b+d] - dt[b+d]\}. \quad (101)$$

(75) and (77) imply that when L and α are chosen simultaneously (so $\frac{\partial w_i}{\partial L} = 0$):

$$\frac{\partial \Pi_{U1}^I}{\partial L} = t\alpha^I\left[\frac{\partial q_2^I}{\partial L}\right] + [p_1^I - w_1^I - c_1^I]\left[\frac{\partial q_1^I}{\partial L}\right] + q_1^I\left[\frac{\partial p_1^I}{\partial L}\right]. \quad (102)$$

From (81), since $\frac{\partial c_1}{\partial L} = -\frac{\partial c_2}{\partial L}$ by assumption:

$$\frac{\partial q_1^I}{\partial L} = \left[\frac{1}{4b^2 - d^2}\right]\left\{b^2d\frac{\partial c_2^I}{\partial L} - b[2b^2 - d^2]\frac{\partial c_1^I}{\partial L}\right\} = -b\left[\frac{2b^2 + bd - d^2}{4b^2 - d^2}\right]\frac{\partial c_1^I}{\partial L} = -b\left[\frac{b+d}{2b+d}\right]\frac{\partial c_1^I}{\partial L}. \quad (103)$$

From (82):

$$\frac{\partial q_2^I}{\partial L} = \left[\frac{1}{4b^2 - d^2}\right]\left\{b^2d\frac{\partial c_1^I}{\partial L} - b[2b^2 - d^2]\frac{\partial c_2^I}{\partial L}\right\} = b\left[\frac{2b^2 + bd - d^2}{4b^2 - d^2}\right]\frac{\partial c_1^I}{\partial L} = b\left[\frac{b+d}{2b+d}\right]\frac{\partial c_1^I}{\partial L}. \quad (104)$$

From (79):

$$\frac{\partial p_1^I}{\partial L} = \left[\frac{1}{4b^2 - d^2}\right]\left\{2b^2\left[\frac{\partial c_1^I}{\partial L}\right] + bd\left[\frac{\partial c_2^I}{\partial L}\right]\right\} = b\left[\frac{2b - d}{4b^2 - d^2}\right]\frac{\partial c_1^I}{\partial L} = \left[\frac{b}{2b+d}\right]\frac{\partial c_1^I}{\partial L}. \quad (105)$$

From (91):

$$p_1^I - w_1^I - c_1^I = \left[\frac{1}{4b^2 - d^2}\right]\{[2b+d]a + \alpha\left[(2b+d)(b-d) + 3btd\right] - c_h\left[2b^2 - bd - d^2\right] - \left[2b^2 - d^2\right]c_1^I + bdc_2^I\}.$$

From (75) and (81):

$$q_1^I = \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]\,ba - t\alpha^I d\left[2b^2-d^2\right] - b\left[2b^2-d^2\right][c_h - \alpha + c_1^I] + b^2 d\left[c_h - (1-t)\,\alpha^I + c_2^I\right]\}. \quad (106)$$

(102) – (106) imply:

$$\frac{\partial \Pi_{U1}^I}{\partial L} = \left[\frac{\partial c_1^I}{\partial L}\right]\left[\frac{1}{2b+d}\right]\left\{t\alpha^I b[b+d] - \frac{b[b+d]}{4b^2-d^2}Z_1 + \left[\frac{b}{4b^2-d^2}\right]Z_2\right\}, \quad (107)$$

where:

$$Z_1 = [2b+d]a + \alpha\left[(2b+d)(b-d)+3btd\right] - c_h\left[2b^2-bd-d^2\right] - \left[2b^2-d^2\right]c_1^I(L) + bdc_2^I(L), \text{ and} \quad (108)$$

$$Z_2 = [2b+d]\,ba - t\alpha d\left[2b^2-d^2\right] - b\left[2b^2-d^2\right][c_h - \alpha + c_1^I(L)] + b^2 d\left[c_h - (1-t)\,\alpha + c_2^I(L)\right]. \quad (109)$$

From (108) and (109):

$$\frac{\partial Z_1}{\partial L} = -\left[2b^2-d^2\right]\left[\frac{\partial c_1^I}{\partial L}\right] \quad \text{and} \quad \frac{\partial Z_2}{\partial L} = -\,b\left[2b^2-d^2\right]\left[\frac{\partial c_1^I}{\partial L}\right] - b^2 d\left[\frac{\partial c_1^I}{\partial L}\right]. \quad (110)$$

(107) – (110) imply that when (62) holds (so $\frac{\partial^2 c_1}{\partial L^2} = 0$):

$$\frac{\partial^2 \Pi_{U1}^I}{\partial L^2} = \left[\frac{\partial c_1^I}{\partial L}\right]^2\left[\frac{1}{2b+d}\right]\left\{\frac{b[b+d]}{4b^2-d^2}\left[2b^2-d^2\right] - \left[\frac{b}{4b^2-d^2}\right]\left[b\left(2b^2-d^2\right)+b^2 d\right]\right\}$$

$$\stackrel{s}{=} b[b+d]\left[2b^2-d^2\right] - b^2[2b^2-d^2+bd] = bd\left[2b^2-d^2\right] - b^3 d = bd[b^2-d^2] > 0. \quad (111)$$

(111) implies that when (62) holds, Π_{U1}^I is a convex function of L. Therefore, U will either set $L = 0$ or $L = 1$.

From (97), define:

$$G(L) \equiv t\left[4b^2-d^2\right]\{[2b+d]\,ba - b\,[2b+d]\,[b-d]\,c_h + b^2 dc_1^I(L) - b[2b^2-d^2]c_2^I(L)\}$$

$$+ [b-d]\{b[2b+d] - td\,[b+d]\}\{[2b+d]a - c_h\,[2b+d][b-d] - \left[2b^2-d^2\right]c_1^I(L) + bdc_2^I(L)\}$$

$$+ \{[2b+d]\,ba - b\left[2b^2-d^2\right][c_h + c_1^I(L)] + b^2 d\left[c_h + c_2^I(L)\right]\}\left[(2b+d)(b-d)+3bdt\right]. \quad (112)$$

Also, using (96), (101), and (112), define:

$$\alpha^I(L) = -\,\frac{G(L)}{H}. \quad (113)$$

Using (75), (81), and (113), define:

$$q_1^I(L) = \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]\,ba - td\alpha^I(L)\left[2b^2-d^2\right]$$

$$-\, b\left[2b^2-d^2\right][c_h - \alpha^I(L) + c_1^I(L)] + b^2 d\left[c_h - (1-t)\alpha^I(L) + c_2^I(L)\right]\}. \quad (114)$$

Similarly, from (75), (82), and (113), define:

$$q_2^I(L) = \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]\,ba + tbd^2\alpha^I(L) + b^2 d[c_h - \alpha^I(L) + c_1^I(L)]$$

$$-b[2b^2-d^2]\left[c_h - (1-t)\alpha^I(L) + c_2^I(L)\right]\}. \quad (115)$$

Also, from (75), (91), and (113), define:

$$p_1^I(L) - c_h + \alpha^I(L) - c_1^I(L) = \left[\frac{1}{4b^2-d^2}\right]\{[2b+d]a + \alpha^I(L)\left[(2b+d)(b-d)+3btd\right]$$

$$- c_h \left[2b^2 - bd - d^2\right] - \left[2b^2 - d^2\right] c_1^I(L) + bdc_2^I(L)\}. \tag{116}$$

Using (77) and (113) – (116), define:

$$\Pi_{U1}^I(L) \;=\; t\alpha^I(L)q_2^I(L) + \left[p_1^I(L) - c_h + \alpha^I(L) - c_1^I(L)\right] q_1^I(L) - \frac{k}{2}\left[\alpha^I(L)\right]^2. \tag{117}$$

(111) and (117) imply that U's profit-maximizing choice of L is:

$$L^I \;=\; \underset{L\,\in\,\{0,1\}}{\arg\max}\;\; \Pi_{U1}^I(L). \tag{118}$$