37.6.2

Use the Fourier transform method to find a solution to the problem

$$(\nabla, A\nabla)G(x) = \delta(x)\,, \quad G \in \mathcal{S}'(\mathbb{R}^N)$$

where A is a strictly positive symmetric $N \times N$ matrix.

Solution: Consider a change of variables $y = Ux$ where U is an orthogonal matrix that diagonalizes the matrix A, that is, $A = U\Lambda U^T$ where $\Lambda_{ij} = \lambda_j \delta_{ij}$ is a diagonal matrix with $\lambda_j > 0$ being the eigenvalues of A. By the chain rule $\nabla_x = U^T \nabla_y$ and, hence, in the new variables, the equation reads

$$\sum_{j=1}^{N} \lambda_j \frac{\partial^2}{\partial y_j^2} G = \delta(U^T y) = \delta(y)$$

because $|\det U| = 1$. Next, let us scale the variables $y_j = \sqrt{\lambda_j} z_j$. Therefore

$$\Delta_z G = \delta(y) = \frac{1}{\sqrt{\det A}}\,\delta(z)$$

because $\lambda_1\lambda_2\cdots\lambda_N = \det A$. Thus, $G \in \mathcal{D}'$ is proportional to the Green's function of the Laplace operator that vanishes at infinity (modulo an additive harmonic function in $\mathbb{R}^N$) in the variables z:

$$G(x) = -\frac{1}{\sqrt{\det A}} \frac{1}{(N-2)\sigma_N |z|^{N-2}}\,, \quad N > 2\,,$$

where σ_N is the area of a unit sphere in $\mathbb{R}^N$. For $N = 2$, $G = (2\pi)^{-1}\ln(|z|)$. Inverting all the transformation that have been made,

$$z = \Lambda^{1/2} y = \Lambda^{-1/2} U x \quad \Rightarrow \quad |z|^2 = (z, z) = (x, U^T \Lambda^{-1} U x) = (x, A^{-1} x)$$

The final answer reads

$$G(x) = -\frac{1}{(N-2)\sigma_N} \frac{1}{\sqrt{\det A}(x, A^{-1}x)^{\frac{N-2}{2}}}\,, \quad N > 2\,,$$

and for $N = 2$

$$G(x) = \frac{1}{\pi\sqrt{\det A}}\,\ln(x, A^{-1}x)\,.$$

Alternative solution: By taking the Fourier transform of the equation, one infers that

$$-(k, Ak)\mathcal{F}[G](k) = 1$$

Since the matrix A is strictly positive, the equation $(k, Ak) = 0$ has only one solution $k = 0$. Therefore a general solution in $\mathcal{S}'$ has the form

$$\mathcal{F}[G](k) = -\mathcal{R}\,\frac{1}{(k, Ak)} + g(k)$$

where $g(k)$ is any temperate distribution with a point support. Any such distribution is a linear combination of $\delta(k)$ and its partial derivatives. Therefore $\mathcal{F}^{-1}[g](x)$ is a (harmonic) polynomial.

So, $G(x)$ is sought in $\mathcal{S}'$ modulo an additive harmonic polynomial. For $N > 2$, the reciprocal of (k, Ak) is locally integrable. Therefore no distributional regularization is needed. For $N = 2$, some kind of a principal value regularization can be invoked. However, an explicit form of the regularization is irrelevant for finding $G(x)$.

To calculate $G(x)$, it is convenient to use the property of the Fourier transform of a distribution under a linear change of variables. The idea is to note that there exists a transformation of the variable k so that $(k, Ak) = (q, q)$ because A is strictly positive matrix. Indeed, if $k = Bq$, then $B^T AB = I$. The latter equation is solved by the symmetric matrix $B = U\Lambda^{-1/2}U^T$ where U and Λ are defined above: $A = U\Lambda U^T$. Note that $B^2 = A^{-1}$ so B^{-1} is the square root of the matrix A. The inverse Fourier transform of $-\mathcal{R}\frac{1}{|q|^2}$ is the Green's function of the Laplace operator, $\mathcal{E}_{_N}(x)$ (satisfying suitable boundary conditions at infinity to be a unique fundamental solution). Then

$$\mathcal{F}[G(x)](k) = \mathcal{F}[\mathcal{E}_{_N}(x)](B^{-1}k)$$

because $|B^{-1}k|^2 = (k, Ak)$. It follows from the aforementioned property of the Fourier transform that

$$G(x) = |\det B|\mathcal{E}_{_N}(Bx) = \frac{1}{\sqrt{\det A}}\,\mathcal{E}_{_N}(Bx)$$

Since $|Bx|^2 = (x, A^{-1}x)$, the previous result is recovered from the explicit form of $\mathcal{E}_{_N}(x)$.

37.6.3

Let $\rho(x)$ be a bounded function with bounded support. Use the Fourier transform method to find an integral representation of the most general distributional solution from to the problem

$$(\nabla, A\nabla)u(x) = \rho(x)\,, \quad |u(x)| \leq M\,, \quad x \in \mathbb{R}^3$$

where A is a strictly positive symmetric 3×3 matrix. Find a direction in which the solution is decreasing most rapidly with increasing $|x|$ in the asymptotic region $|x| \to \infty$.

SOLUTION: The solution is given by the convolution of the fundamental solution found in **37.6.2** for $N = 3$:

$$u(x) = (G * \rho)(x) = -\frac{1}{4\pi}\int_\Omega \frac{\rho(y)\, d^3y}{(x-y, A^{-1}(x-y)^{1/2}}$$

Since A is symmetric it has three orthonormal eigenvectors $A\hat{e}_j = \lambda_j \hat{e}_j$ $j = 1, 2, 3$. Let us change variables from y to η where $y = y_j \hat{e}_j$ and put $\xi_j = (x, \hat{e}_j)$. So, y and η are related by a rotation $y = U\eta$ where columns of U are $\hat{e}_j$ so that $d^3y = d^3\eta$

$$u(x) = -\frac{1}{4\pi}\int_{\Omega_U} \frac{\rho(U\eta)\, d^3\eta}{[\sum_j \lambda_j^{-1}(\xi_j - \eta_j)^2]^{1/2}}$$

if $|\xi| = |x| \to \infty$ then

$$u(x) = -\frac{1}{4\pi}\frac{Q}{[\sum_j \lambda_j^{-1}\xi_j^2]^{1/2}} + O\Big(\frac{1}{|x|^2}\Big)\,, \quad Q = \int_\Omega \rho(y)\, d^3y$$

because $\lambda_j > 0$. It follows from this representation, the Coulomb (or Newton) part of the potential falls off most rapidly in the direction of the eigenvector corresponding to the largest (smallest) eigenvalue of A^{-1} (of A) because $\sum_j \lambda_j^{-1}\xi_j^2 = (x, A^{-1}x)$ is maximal for a given $|x|$ when x is parallel to such an eigenvector.

38.6.2

(i) Find a distributional solution to the Helmholtz equation for $|x| \geq a > 0$ with a "quadrupole" point-like source:

$$(\Delta + k^2)v(x) = \sum_{i,j=1}^{3} p_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta(x) = (\nabla, p\nabla)\delta(x)\,, \quad x \in \mathbb{R}^3\,,$$

that satisfies the Sommerfeld radiation condition.
(ii) Show that the solution found for $x \neq 0$ is not locally integrable in the whole $\mathbb{R}^3$. So, a distributional solution in $\mathbb{R}^3$ is an extension of the classical solution to the singular point $x = 0$. Find this extension.

SOLUTION: (i) Put

$$G(x) = -\frac{e^{ik|x|}}{4\pi|x|}$$

This is the Green's function of the Helmholtz operator satisfying the Sommerfeld radiation condition. Then the solution in question is given by the convolution of G with the inhomogeneity, provided the convolution exists. The latter is the case as the inhomogeneity has a point support. Thus

$$v(x) = G * (\nabla, p\nabla)\delta(x) = (\nabla, p\nabla)(G * \delta)(x) = (\nabla, p\nabla)G(x)$$

where the derivatives are understood in the distributional sense.

(ii) The function $G(x)$ is locally integrable and from class C^∞ near any point but $x = 0$. So, in any open set that does not contain $x = 0$, the distributional and classical derivatives match. To find them, put $r = |x|$. Then G depends only on r and $\nabla r = x/r = \hat{x}$ is the unit vector $|\hat{x}| = 1$. Therefore for $x \neq 0$

$$\begin{aligned}
p\nabla G(x) &= \partial_r G(x) p\hat{x} \\
(\nabla, p\nabla)G(x) &= \partial_r^2 G(x)(\hat{x}, p\hat{x}) + \partial_r G(x)(\nabla, p\hat{x}) \\
&= \partial_r^2 G(x)(\hat{x}, p\hat{x}) + \partial_r G(x)\Big(\frac{\operatorname{tr} p}{r} - \frac{(\hat{x}, p\hat{x})}{r}\Big)
\end{aligned}$$

where $\operatorname{tr} p = \sum_i p_{ii}$ is the trace of the matrix p. The derivatives of G are

$$\begin{aligned}
\partial_r G(x) &= ikG(x) - \frac{G(x)}{r} \\
\partial_r^2 G(x) &= \Big(\frac{2}{r^2} - \frac{2ik}{r} - k^2\Big)G(x)
\end{aligned}$$

Since $|G(x)| \sim \frac{1}{r}$ and $|(\hat{x}, p\hat{x})| \leq \|p\|$, where $\|p\|$ is the Frobenius norm of the matrix p, all terms in $v(x)$ are locally integrable in $\mathbb{R}^3$ except for $2(\hat{x}, p\hat{x})G(x)/r^2$ because it is proportional to r^{-3} which is not an integrable singularity in $\mathbb{R}^3$. So, $v(x) \neq \{(\nabla, p\nabla)G\}$ but rather $v(x)$ is a distributional extension of the classical derivative $\{(\nabla, p\nabla)G\}$ to the singular point $x = 0$:

$$v(x) = \mathcal{R}\{(\nabla, p\nabla)G\} \in \mathcal{D}'$$

To find it, let us compute v in the distributional sense. Note that $\nabla G = \{\nabla G\}$ because the classical gradient $\{\nabla G\}$ is locally integrable in $\mathbb{R}^3$. For any test function φ, one infers using continuity of the Lebesgue integral and integration by parts that

$$\begin{aligned}
(v,\varphi) &= -\Big((p\nabla G,\nabla\varphi)\Big) = -\int (p\nabla G,\nabla\varphi)\,d^3x \\
&= -\lim_{a\to 0^+}\int_{|x|>a}(p\nabla G,\nabla\varphi)\,d^3x \\
&= -\lim_{a\to 0^+}\left(\int_{|x|=a}(\hat{n},p\nabla G)\varphi\,dS - \int_{|x|>a}\{(\nabla,p\nabla)G\}\varphi\,d^3x\right) \\
&= -\lim_{a\to 0^+}\int_{|x|=a}(\hat{n},p\nabla G)\varphi\,dS + \Big(\mathcal{P}\,\{(\nabla,p\nabla)G\},\varphi\Big)\,.
\end{aligned}$$

where $\hat{n} = -x/a$. The volume integral can be viewed as the spherical principal value regularization $\mathcal{P}$ of the classical derivative of G that is calculated above for $x \neq 0$, provided the limit of the surface integral exists. In fact, the regularization is needed for the term

$$\mathcal{R}\frac{2(\hat{x},p\hat{x})G(x)}{|x|^2} = \mathcal{P}\frac{2(\hat{x},p\hat{x})G(x)}{|x|^2}$$

as the other terms are locally integrable. It follows from the estimate on the sphere $|x| = a$

$$|(\hat{n},p\nabla G)| \le \|p\||\hat{n}||\nabla G| \le \|p\|\Big(k|G(x)| + \frac{|G(x)|}{a}\Big) = \|p\|\Big(\frac{k}{4\pi a} + \frac{1}{4\pi a^2}\Big)$$

that only the second term in ∇G can contribute to the limit because $dS \sim a^2$ while the limit of the term proportional to k vanishes. Therefore

$$\begin{aligned}
-\lim_{a\to 0^+}\int_{|x|=a}(\hat{n},p\nabla G)\varphi\,dS &= \lim_{a\to 0^+}\frac{e^{ika}}{4\pi a^2}\int_{|x|=a}(\hat{x},p\hat{x})\varphi(x)\,dS \\
&= \frac{1}{4\pi}\lim_{a\to a^+}\int_{|z|=1}(z,pz)\varphi(az)dS_z \\
&= \frac{1}{4\pi}\int_{|z|=1}(z,pz)dS_z\,\varphi(0) \equiv C_p(\delta,\varphi)
\end{aligned}$$

by the Lebesgue dominated convergence theorem. Note that the integrand is bounded on the sphere $|(z,pz)\varphi(az)| \le \|p\|\sup|\varphi|$ and a constant is integrable on a sphere. Thus

$$v(x) = C_p\delta(x) + \mathcal{P}\,\{(\nabla,p\nabla)G\}$$

where C_p is the integral mean value of (z,pz) on the unit sphere, $|z| = 1$. The constant C_p does not depend on the choice of a basis in $\mathbb{R}^3$ because of the rotation invariance of the integral (the sphere and the integrand defined by the inner product are invariant under rotations about the origin).

Let λ_a be eigenvalues of p. They are real as p is symmetric. Let e_a be the corresponding eigenvectors chosen so that they form an orthonormal basis in $\mathbb{R}^3$. Such a choice of eigenvectors is always possible by the spectral theorem for symmetric matrices. So, the integral can be evaluated in the spherical coordinates in which the zenith angle ϕ is counted from e_3 and the polar angle θ is counted from e_1 in the plane spanned by e_1 and e_2. Then

$$\begin{aligned}
(z,pz) &= \lambda_1\sin^2(\phi)\cos^2(\theta) + \lambda_2\sin^2(\phi)\sin^2(\theta) + \lambda_3\cos^2(\phi) \\
C_p &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(z,pz)\,\sin(\phi)\,d\phi\,d\theta = \frac{1}{3}(\lambda_1+\lambda_2+\lambda_3) = \frac{\mathrm{tr}\,p}{3}
\end{aligned}$$

41.9.1

Let $x \in \mathbb{R}^3$. Suppose that there is no source in the heat equation. Then the solution gives the temperature distribution $u(x,t)$ for time $t > 0$ if the initial temperature distribution was $u(x,0) = u_0(x)$. The energy needed to rise the temperature by ΔT of a substance of mass m is equal to $mC\Delta T$, where C is the so called specific heat capacity of the substance. If the substance is uniform then $m = \rho V$ where ρ is the mass density and V is the volume. Therefore if one uses the same amount of energy to rise the temperature from 0 to T for a volume V, then VT is constant. Let T_0 be the temperature of a ball of radius R. Suppose that the energy needed to rise the temperature of the ball from 0 to T_0 is used to heat a spherical layer $R < |x| < R + \Delta R$. Then its temperature is $T_1 = T_0V_0/V_1$ where V_0 is the volume of the ball and V_1 is the volume of the layer. Similarly, if the energy needed to rise the temperature of the ball from 0 to T_0 is used to heat a circular torus of radius R with a circular cross section of radius $\Delta R < R$, the temperature of the torus is $T_2 = T_0V_0/V_2$ where V_2 is the volume of the torus. Suppose the initial temperature distribution is given by

$$u_0(x) = u_a(x, \Delta R) = T_a\chi_a(x, \Delta R)$$

where $a = 1, 2$ and χ_1 and χ_2 are the characteristic functions of the spherical layer and torus, respectively.

(i) Show that

$$u_1(x, \Delta R) \to \frac{T_0R}{3}\,\delta_{S_R}(x)\,, \quad u_2(x, \Delta R) \to \frac{2T_0R^2}{3}\,\delta_{C_R}(x)$$

in $\mathcal{D}'$ as $\Delta R \to 0^+$, where δ_{S_R} and δ_{C_R} are spherical and circular delta-functions supported, respectively, on the sphere $|x| = R$ and on a circle of of radius R. Find an integral representation for the solution of the heat equation when the initial data are given by the above limit distributions. Justify your answer by an appropriate analysis based on the distribution theory. The solutions describe temperature distributions when the same amount of heat energy was initially distributed in a thin spherical layer of radius R and in thin torus of the same radius.

(ii) Calculate the temperature at the origin, $u(0,t)$, as a function of time t for the both initial distributions. Sketch their graphs. Compare them with the case when $u_0(x) = T_0V_0\delta(x - x_0)$ where $|x_0| = R$. What is the physical significance of the latter case? For the case of the torus, calculate $u(x,t)$ where the point x is on the line through the center of the circle C_R that is perpendicular to the plane containing C_R, and $|x| = z$.

Solution: For any test function φ

$$\begin{aligned}
(u_1, \varphi) &= T_1 \int_{R<|x|R+\Delta R} \varphi(x)\, d^3x = \frac{T_0R^3}{(R+\Delta R)^3 - R^3}\int_R^{R+\Delta R} \phi(r)\, r^2 dr\,, \\
\phi(r) &= \int_{|y|=1} \varphi(ry)\, dS_y
\end{aligned}$$

The function $\phi(r)$ is continuous because $\varphi(ry)$ is continuous in the parameter r, has a bound $|\varphi(x)| \le M$ independent of r that is integrable on a unit sphere. By the integral mean value theorem, there exists $R \le r_* \le R + \Delta R$ such that

$$(u_1, \varphi) = \frac{T_0R^3\Delta R}{(R+\Delta R)^3 - R^3}\,\phi(r_*)r_*^2$$

In the limit $\Delta R \to 0$, $r_* \to R$ and, hence,

$$\lim_{\Delta R\to 0}(u_1,\varphi) = \frac{T_0 R^3}{3}\,\phi(R) = \frac{T_0 R}{3}\int_{|x|=R}\varphi(x)\,dS = \frac{T_0 R}{3}(\delta_{S_R},\varphi)$$

To carry out a similar analysis for u_2, let us make the following representation for an integral over the torus. Put $x = R\hat{e}(s) + y$ where $0 \le s \le 2\pi R$ is the arclength over the central circle of the torus, $\hat{e}(s)$ is the unit vector such that $R\hat{e}(s)$ is the position vector of a point on the central circle for a given s, and y is orthogonal to $\hat{e}(s)$ and spans the disk $|y - R\hat{e}(s)| < \Delta R$ which is denoted by D_s. The volume of the torus is $2\pi R\Delta A$ where ΔA is the area of the disk D_s. In this representation

$$\begin{aligned}(u_2,\varphi) &= T_2\int_0^{2\pi R}\int_{D_s}\varphi(R\hat{e}(s)+y)d^2y ds = \frac{2T_0R^2}{3\Delta A}\int_0^{2\pi R}\phi(s,\Delta R)\,ds\\ \phi(s,\Delta R) &= \int_{D_s}\varphi(R\hat{e}(s)+y)d^2y = \Delta A\varphi(R\hat{e}(s)+y_s)\end{aligned}$$

for some $|y_s| \le \Delta R$ in D_s by the integral mean value theorem. In the limit $\Delta R \to 0$, $y_s \to 0$- the center of the disk D_s. The limit of $\phi(s,\Delta R)$ is calculated by interchanging the order with integration. This is legitimate because $\varphi(R\hat{e}(s)+y_s)$ is continuous in y_s and its bound $|\varphi(x)| \le M$ is independent of y_s and integrable over $[0, 2\pi R]$. Therefore

$$\lim_{\Delta R\to 0}(u_2,\varphi) = \frac{2T_0R^2}{3}\int_0^{2\pi R}\int_{D_s}\varphi(R\hat{e}(s))\,ds = \frac{2T_0R^2}{3}(\delta_{C_R},\varphi)$$

(ii) If $G(x,t)$ is the causal Green's function for the heat operator, then the solution for any initial compactly supported temperature distribution $u_0(x)$ is given by the convolution $G(x,t)*u_0(x)$. Let u_0 be a surface delta function Δ_S supported on a smooth M-surface S (here $M = 1$ (curve in space) or $M = 2$ (surface in space)). If $x = x(\xi)$ is a parameterization of S so that

$$(\delta_S,\varphi) = \int_S \varphi(x(\xi))\,d\mu(\xi)\,,\quad \varphi\in\mathcal{D}\,,$$

where $d\mu(\xi)$ is the Lebesgue measure on S, and the total Lebesgue measure of S is demanded to be finite, $\int_S d\mu(\xi) < \infty$. Let us calculate the convolution $G(x,t)*\delta_S(x)$. For any test function,

$$(G*\delta_S,\varphi) \;=\; \Big(G(y,t),\big(\delta_S(x),\varphi(x+y)\big)\Big) = \int G(y,t)\int_S\varphi(y+x(\xi))\,d\mu(\xi)\,d^3y$$

Recall that $G(x,t)$ is from class $C^\infty(t>0)$. By Fubini's theorem the order of integration can be changed because $|G(y,t)\varphi(y+x(\xi))| \le MG(y,t) \in \mathcal{L}(\mathbb{R}^3\times S)$ for $t > 0$. Therefore by changing integration variables $z = y + x(\xi)$, $d^z = d^3y$, one has

$$(G*\delta_S,\varphi) = \int\varphi(z)\int_S G(z-x(\xi),t)\,d\mu(\xi)\,d^3z$$

Thus,

$$u(z,t) = G(z,t)*\delta_S(z) = \int_S G(z-x(\xi),t)\,d\mu(\xi) \quad\Rightarrow\quad u(0,t) = \int_S G(x(\xi),t)\,d\mu(\xi)$$

If S is the sphere $|z| = R$, then

$$u(0,t) = 4\pi R^2 G(R,t)$$

because $G(x,t)$ depends only on $|x| = R$. Similarly, if S is a circle of radius R centered at the origin, then

$$u(0,t) = 2\pi R G(R,t)$$

The answer is obtained by scaling the above expressions with appropriate factors in the limit distributions $u_{1,2}$.

43.7.4. Implosion waves

Consider the regular distribution in $\mathbb{R}^3$

$$v(x, \Delta R) = \frac{P_0}{\Delta R}\chi(x, \Delta R)$$

where χ is the characteristic function of a spherical layer, $R < |x| < R + \Delta R$. As $\Delta R \to 0^+$, it converges to $P_0\delta_{S_R}(x)$ in the distributional sense, where δ_{S_R} is the spherical delta-function supported on the sphere $|x| = R$.

(i) Consider the surface wave potentials for $u_0 = v$ and $u_1 = v$. Sketch the supports of the surface wave potentials in the limit $\Delta R \to 0^+$ for two moments of time $R/c > t_2 > t_1 > 0$, indicate the wave fronts of propagating (sound) waves described by these potentials. If the wave equation is used to model sound waves, then the surface wave potentials can be interpreted as propagating perturbations of the air pressure relative to the atmospheric pressure. The initial pressure perturbation is created in a thin spherical layer (e.g., by an explosion). Owing to the spherical symmetry, there will be waves collapsing or focusing to a point (so called imploding waves).

(ii) Calculate the wave potentials from Part (i) at $x = 0$ as functions of time $t > 0$. Sketch their graphs indicating how the shape is changing as $\Delta R \to 0^+$. Can the wave potentials at $x = 0$ be considered as distributions of time t? If so, find the distributional limit in $\mathcal{D}'_+$ of the wave potentials at $x = 0$ when $\Delta R \to 0^+$.

Solution: (i) The wave fronts are obtained by the Huygens' principle. Draw spheres of radius ct centered at every point of the sphere $|x| = R$. The intersection of these spheres is the support of the surface wave potentials $V_3^{(0,1)}(x, t)$ if initially they are supported on the sphere $|x| = R$ (in the limit $\Delta R \to 0$). The support is a spherical layer $R - ct \le |x| \le R + ct$, $t < R/c$.

(ii) From the Notes

$$V_3^{(0)}(0, t) = \frac{t}{4\pi}\int_{|z|=1} v(ctz, \Delta R)\, dS_z$$

where $v(ctz, \Delta R) = P_0/\Delta R$ if $R/c < t < (R + \Delta R)/c$ and $v = 0$ otherwise. So, $V_3^{(0)}(0, t) = 0$ for $t < R/c$ and for $t > (R + \Delta R)/c$ and

$$V_3^{(0)}(0, t) = \frac{P_0 t}{\Delta R}, \quad R/c < t < (R + \Delta R)/c$$

For any test function $\varphi(t)$,

$$\left(V_3^{(0)}(0, t), \varphi(t)\right) = \frac{P_0}{\Delta R}\int_{R/c}^{(R+\Delta R)/c} t\varphi(t)\, dt = P_0\, t^*\varphi(t^*)\,, \quad R/c \le t^* \le (R + \Delta R)/c$$

by the integral mean value theorem for some t^*. In the limit, $\Delta R \to 0$, $T^* \to R/c$ so that

$$\lim_{\Delta R \to 0}\left(V_3^{(0)}(0, t), \varphi(t)\right) = \frac{P_0 R}{c}\varphi(R/c)$$

This shows that $V_3^{(0)}(0,t) \to \frac{P_0 R}{c}\delta(t-\frac{R}{c})$. So, the pressure at the center of the exploding sphere becomes a delta function in time. It "blows up" at $t = R/c$. Similarly,

$$V_3^{(1)}(x,t) = \frac{d}{dt}\frac{t}{4\pi}\int_{|z|=1} v(ctz, \Delta R)\, dS_z = \frac{d}{dt}V_3^{(0)}(t) \to \frac{P_0 R}{c}\delta'\Big(t - \frac{R}{c}\Big)$$

as $\Delta R \to 0$ by continuity of differentiation on $\mathcal{D}'$.

44.9.3. Radiation of a magnetic dipole

(i) Formulate the generalized Cauchy problem for a point-like magnetic dipole.

(ii) Solve the Cauchy problem and calculate the explicit form of electromagnetic fields as vector-valued distributions in the same fashion as for an electric dipole in the Notes.

(iii) Find the far fields, calculate their Poynting vector, and find its outward flux across a sphere of an arbitrary large radius. Put $\boldsymbol{\mu}(t) = \theta(t)\boldsymbol{\mu}_0 \cos(\omega t)$ (a monochromatic magnetic dipole) and caculate the average rate per one cycle $T = 2\pi/\omega$ at which the dipole emits electromagnetic energy.

SOLUTION: (i) The electric current density is given by $\mathbf{J} = \boldsymbol{\mu}(t) \times \boldsymbol{\nabla}\delta(x)$, where $\boldsymbol{\mu}(t)$ is a sufficiently smooth vector function of time t (from class C^2 by analogy with the case of an electric dipole), and the electric charge density vanishes. The electric charge conservation law holds because

$$(\boldsymbol{\nabla}, \mathbf{J}) = -(\boldsymbol{\mu}, \boldsymbol{\nabla} \times \boldsymbol{\nabla}\delta) = (\boldsymbol{\mu}, \mathbf{0}) = 0$$

Therefore the electric and magnetic fields are distributional solutions to Maxwell's equations that vanish in the half-space $t < 0$:

$$\begin{aligned}
&\frac{\partial \mathbf{E}}{\partial t} - c\boldsymbol{\nabla} \times \mathbf{B} = -4\pi \mathbf{J}\,,\\
&\frac{\partial \mathbf{B}}{\partial t} + c\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}\,,\\
&(\boldsymbol{\nabla}, \mathbf{E}) = (\boldsymbol{\nabla}, \mathbf{B}) = 0\,,\\
&\mathbf{E}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}, t) = \mathbf{0}\,, \quad t < 0\,.
\end{aligned}$$

if, in addition, the electric and magnetic fields are assumed to be zero at $t = 0^+$.

(ii) By (??) and (??), the solution is given by

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,,$$

where the vector valued distribution $\mathbf{A}$ is given by the convolution

$$\begin{aligned}
\mathbf{A}(\mathbf{x}, t) &= 4\pi c(G_3 * \mathbf{J})(\mathbf{x}, t) = -4\pi c\boldsymbol{\nabla} \times \Big(G_3(\mathbf{x}, t) * \boldsymbol{\mu}(t) \cdot \delta(\mathbf{x})\Big)\\
&= -\boldsymbol{\nabla} \times \frac{\boldsymbol{\mu}(t_r)}{|\mathbf{x}|}\,, \quad t_r = t - \frac{|\mathbf{x}|}{c}
\end{aligned}$$

The last equality is analogous to the scalar potential Φ for an electric dipole (the divergence becomes the curl in the present case). For $\mathbf{x} \neq 0$, the vector potential is from class C^∞. Therefore the distributional and classical derivative match:

$$\mathbf{A}(\mathbf{x}, t) = \frac{\hat{\mathbf{x}} \times \dot{\boldsymbol{\mu}}(t_r)}{c|\mathbf{x}|} + \frac{\hat{\mathbf{x}} \times \boldsymbol{\mu}(t_r)}{|\mathbf{x}|^2}\,, \quad \mathbf{x} \neq \mathbf{0}$$

where $\boldsymbol{\nabla}|\mathbf{x}| = \mathbf{x}/|\mathbf{x}| = \hat{\mathbf{x}}$ and the over-dot denote the time derivative as in the case of an electric dipole. The singularity at $\mathbf{x} = \mathbf{0}$ is locally integrable in $\mathbb{R}^3$. Therefore, the above equation holds in $\mathcal{D}'$.

For $\mathbf{x} \neq \mathbf{0}$ the distributional and classical derivatives match and, hence,

$$
\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= -\frac{\hat{\mathbf{x}} \times \ddot{\boldsymbol{\mu}}(t_r)}{c^2|\mathbf{x}|} - \frac{\hat{\mathbf{x}} \times \dot{\boldsymbol{\mu}}(t_r)}{c|\mathbf{x}|^2} \\
\mathbf{B}(\mathbf{x},t) &= \{\boldsymbol{\nabla} \times \mathbf{A}\} = \frac{\ddot{\boldsymbol{\mu}}(t_r) - \hat{\mathbf{x}}(\hat{\mathbf{x}}, \ddot{\boldsymbol{\mu}}(t_r))}{c^2|\mathbf{x}|} + \frac{\dot{\boldsymbol{\mu}}(t_r) - 3\hat{\mathbf{x}}(\hat{\mathbf{x}}, \dot{\boldsymbol{\mu}}(t_r))}{c|\mathbf{x}|^2} \\
&\quad + \frac{4\boldsymbol{\mu}(t_r) - 3\hat{\mathbf{x}}(\hat{\mathbf{x}}, \boldsymbol{\mu}(t_r))}{|\mathbf{x}|^3}, \quad \mathbf{x} \neq \mathbf{0}
\end{aligned}
$$

The electric field is locally integrable and, hence, the above expression defines the electric field as a regular distribution. The last term in the expression for the magnetic field is not locally integrable in $\mathbb{R}^3$ because its magnitude is proportional to $|\mathbf{x}|^{-3}$. So, the distributional curl of $\mathbf{A}$ is not equal to the classical one. Since $\mathbf{A}$ is locally integrable, for any test function

$$
\begin{aligned}
(\mathbf{B},\varphi) &= \big(\boldsymbol{\nabla} \times \mathbf{A}, \varphi\big) = \big(\mathbf{A}, \times \boldsymbol{\nabla}\varphi\big) = \int\int_{\mathbb{R}^3} \mathbf{A} \times \boldsymbol{\nabla}\varphi \, d^3x dt \\
&= \int \lim_{a\to 0^+} \int_{|x|>a} \mathbf{A} \times \boldsymbol{\nabla}\varphi \, d^3x dt \\
&= \int \lim_{a\to 0^+} \left(\int_{|x|=a} \mathbf{A} \times \hat{\mathbf{n}}\, \varphi \, dS + \int_{|x|>a} \{\boldsymbol{\nabla} \times \mathbf{A}\}\, \varphi \, d^3x \right) dt
\end{aligned}
$$

by continuity of the Lebesgue integral. The second term is the spherical principal value regularization of the classical curl of $\mathbf{A}$ (in fact, only the term proportional to $|\mathbf{x}|^{-3}$ requires this regularization). The limit of the surface integral is a contribution of a distribution supported at $\mathbf{x} = \mathbf{0}$.

Since the normal $\hat{\mathbf{n}} = -\mathbf{x}/a$, one has the following estimate of the integrand on the sphere $|\mathbf{x}| = a$

$$
|\mathbf{A} \times \hat{\mathbf{n}}| \leq \frac{|\dot{\boldsymbol{\mu}}(t - \frac{a}{c})|}{ca} + \frac{|\boldsymbol{\mu}(t - \frac{a}{c})|}{a^2}
$$

This shows that the term containing $\dot{\boldsymbol{\mu}}$ in $\mathbf{A}$ does not contribute to the limit because $dS \sim a^2$. Therefore putting $\mathbf{x} = a\mathbf{z}$, one infers that

$$
\begin{aligned}
\lim_{a\to 0^+} \int_{|x|=a} \mathbf{A} \times \hat{\mathbf{n}}\varphi \, dS &= \lim_{a\to 0^+} \int_{|z|=1} \mathbf{z} \times \Big(\mathbf{z} \times \boldsymbol{\mu}(t - \tfrac{a}{c})\Big)\varphi(a\mathbf{z}, t)\, dS \\
&= \int_{|z|=1} \mathbf{z} \times \Big(\mathbf{z} \times \boldsymbol{\mu}(t)\Big)\, dS\, \varphi(\mathbf{0}, t)
\end{aligned}
$$

by the Lebesgue dominated convergence theorem. Note that the magnitude of the integrand is bounded for all a by $|\boldsymbol{\mu}(t)| \sup|\varphi|$. So, by continuity of $\boldsymbol{\mu}$ and the test function the limit can be moved into the integral. Put

$$
\mathbf{C}(t) = \int_{|z|=1} \mathbf{z} \times \Big(\mathbf{z} \times \boldsymbol{\mu}(t)\Big)\, dS \; = \; \int_{|z|=1} \Big[\mathbf{z}\Big(\mathbf{z}, \boldsymbol{\mu}(t)\Big) - \boldsymbol{\mu}(t)\Big]\, dS
$$

The latter integral is convenient to calculate in spherical coordinates in which the zenith angle ϕ is counted from the vector $\boldsymbol{\mu}$ so that $(\mathbf{z}, \boldsymbol{\mu}) = |\boldsymbol{\mu}|\cos(\phi)$. If $\mathbf{e}_{1,2}$ are orthonormal vectors in the plane perpendicular to $\boldsymbol{\mu}$, then

$$
\mathbf{z} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + \cos(\phi)\hat{\boldsymbol{\mu}}(t)
$$

where $z_1 = \sin(\phi)\cos(\theta)$, $z_2 = \sin(\phi)\sin(\theta)$, and $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}/|\boldsymbol{\mu}|$ is the unit vector parallel to $\boldsymbol{\mu}$. Since $dS = \sin(\phi)\,d\phi\,d\theta$ and the integration over θ is taken over the interval $[0, 2\pi]$, the integral of $z_{1,2}$ vanishes. Therefore

$$\mathbf{C}(t) = 2\pi\boldsymbol{\mu}(t)\int_0^\pi \Big(\cos^2(\phi) - 1\Big)\sin(\phi)\,d\phi = -\frac{8\pi}{3}\,\boldsymbol{\mu}(t)$$

and

$$\begin{aligned}
(\mathbf{B},\varphi) &= -\frac{8\pi}{3}\int \boldsymbol{\mu}(t)\varphi(\mathbf{0},t)\,dt + \Big(\mathcal{P}\{\boldsymbol{\nabla}\times\mathbf{A}\},\varphi\Big)\,, \\
\mathbf{B}(\mathbf{x},t) &= -\frac{8\pi}{3}\boldsymbol{\mu}(t)\cdot\delta(\mathbf{x}) + \mathcal{P}\{\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\}
\end{aligned}$$

(iii) The flux of the electromagnetic energy in the asymptotic region $|\mathbf{x}| \to \infty$ is determined by fields falling off as $|\mathbf{x}|^{-1}$ so that the Poynting vector reads

$$\begin{aligned}
\mathbf{E} &= -\frac{\hat{\mathbf{x}}\times\ddot{\boldsymbol{\mu}}(t_r)}{c^2|\mathbf{x}|} + O(|\mathbf{x}|^{-2})\,, \\
\mathbf{B} &= \frac{\ddot{\boldsymbol{\mu}}(t_r) - \hat{\mathbf{x}}(\hat{\mathbf{x}},\ddot{\boldsymbol{\mu}}(t_r))}{c^2|\mathbf{x}|} + +O(|\mathbf{x}|^{-2})\,, \\
\mathbf{S} &= \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{|\ddot{\boldsymbol{\mu}}(t_r)|^2 - (\hat{\mathbf{x}},\ddot{\boldsymbol{\mu}}(t_r))^2}{4\pi c^3|\mathbf{x}|^2} + +O(|\mathbf{x}|^{-3})
\end{aligned}$$

The latter expression is identical to that of the asymptotic Poynting vector for a radiating electric dipole. Using the results from the Notes, where the electric dipole vector $\mathbf{p}(t)$ is to be replaced by the magnetic dipole vector $\boldsymbol{\mu}(t)$, one infers that

$$\begin{aligned}
\int_{|\mathbf{x}|=R}(\mathbf{S},d\boldsymbol{\Sigma}) &= \frac{2|\ddot{\boldsymbol{\mu}}(t-\frac{R}{c})|^2}{3c^3} \\
\Big\langle \int_{|\mathbf{x}|=R}(\mathbf{S},d\boldsymbol{\Sigma})\Big\rangle_T &= = \frac{1}{T}\int_0^T\int_{|\mathbf{x}|=R}(\mathbf{S},d\boldsymbol{\Sigma})\,dt = \frac{\omega^2}{3c^3}\,|\boldsymbol{\mu}_0|^2
\end{aligned}$$

if $\boldsymbol{\mu}(t) = \boldsymbol{\mu}_0\cos(\omega t)$ where $T = 2\pi/\omega$ is the period of oscillations of the magnetic moment.