55.5.2. Differentiation operator on a half-line

Consider the differentiation operator in $\mathcal{L}_2(0,\infty)$:

$$
\begin{aligned}
A \quad &: \quad D_A \subset \mathcal{L}_2(0,\infty) \to \mathcal{L}_2(0,\infty)\,, \quad Au(x) = u'(x)\,,\\
D_A \quad &= \quad \Big\{ u \in C^1([0,\infty)) \cap \mathcal{L}_2(0,\infty) \,\Big|\, u(0)=0\,,\ \lim_{x\to\infty} u(x) = 0 \Big\}
\end{aligned}
$$

(i) Show that A is invertible.
(ii) Find the explicit form of $A^{-1}f$.
(iii) Show that A is not bounded away from zero.
(iv) Show that neither A nor its inverse A^{-1} is bounded.
(v) Show that the range of A is a proper subset of $C^0([0,\infty)) \cap \mathcal{L}_2(0,\infty)$. In particular, solve the equation $Au = f$, $u \in D_A$, where $f(x) = e^{-x} \in C^0([0,\infty)) \cap \mathcal{L}_2(0,\infty)$ or show that no solution exists.

SOLUTION: (i) The inverse A^{-1} exists if and only if the homogeneous problem $Au = 0$, $u \in D_A$, has only the trivial solution. A general solution to $u'(x) = 0$, $u \in C^1$, is a constant function, $u(x) = C$. The boundary condition $u(0) = 0$ demands that $C = 0$. Therefore the said problem has only the trivial solution and A is invertible.

(iii) One has to find a sequence $\{u_n\} \subset D_A$ such that $\|Au_n\|/\|u_n\| \to 0$ as $n \to \infty$. Let $u_n = xe^{-a_n x}$ where $a_n > 0$. Note that $u_n(0) = 0$. Then

$$
\begin{aligned}
\|u_n\|^2 &= \int_0^\infty x^2 e^{-2a_n x} dx = \frac{C_0}{a_n^3}\,, \quad C_0 = \int_0^\infty y^2 e^{-2y} dy\\
\|Au_n\|^2 &= \int_0^\infty |u_n'(x)|^2 dx = \int_0^\infty (1 - a_n x)^2 e^{-2a_n x} dx = \frac{C_A}{a_n}\,, \quad C_A = \int_0^\infty (1-y)^2 e^{-2y} dy\\
\frac{\|Au_n\|^2}{\|u_n\|^2} &= \frac{C_A}{C_0}\, a_n^2 \to 0
\end{aligned}
$$

as $n \to \infty$ if $a_n \to 0^+$ (e.g. $a_n = \frac{1}{n}$).

(iv) The operator A is invertible and not bounded away from zero. Therefore by Banach Theorem its inverse is not bounded, $\|A^{-1}\| = \infty$. To show that $\|A\| = \infty$, one has to find a sequence $\{u_n\} \subset D_A$ such that $\|Au_n\|/\|u_n\| \to \infty$ as $n \to \infty$. Take the same sequence as in Part (iii) but demand that $a_n \to \infty$ (e.g., $a_n = n$).

59.7.1. Spectrum of a projection operator

Let $\{v_n\}$ be an orthonormal set that is not complete in a Hilbert space $\mathcal{H}$. Define the operator:

$$P \;:\quad \mathcal{H} \to \mathcal{H}\,, \quad Pu = \sum_n \langle u, v_n\rangle v_n$$

(i) Show that

$$P^2 = P\,, \qquad P^* = P$$

(ii) Show that

$$\sigma_p(P) = \{1, 0\}$$

(iii) Determine the range of P. Show that for any $\lambda \notin \sigma_p(P)$, the resolvent is

$$\mathcal{R}_P(\lambda)f = (1-\lambda)^{-1}Pf - \lambda^{-1}(f - Pf)$$

(iv) Show that the resolvent is bounded and that

$$\sigma(P) = \sigma_p(P) = \{1, 0\}$$

Solution: Let us first establish the following properties of P:

$$\begin{array}{ll} (1) & \|P\| = 1 < \infty \\ (2) & P^2 = P \\ (3) & P^* = P \end{array}$$

By the Bessel inequality

$$\|Pu\|^2 = \sum_n |\langle u, v_n\rangle|^2 \le \|u\|^2$$

for any u in the Hilbert space because $D_P = \mathcal{H}$. It follows from this inequality that P is bounded and

$$\frac{\|Pu\|}{\|u\|} \le 1 \quad \Rightarrow \quad \|P\| = 1$$

because the equality can be reached when $u = v_n$. Therefore, by linearity, P is continuous on $\mathcal{H}$.

Consider the sequence $w_m = \sum_{n=1}^m \langle u, v_n\rangle v_n$ where $u \in \mathcal{H}$. Clearly $w_m \to Pu$ as $m \to \infty$. By continuity of P, $Pw_m \to P(Pu) = P^2 u$ for any $u \in \mathcal{H}$. On the other hand, P is linear and $Pv_n = v_n$ so that $Pw_m = w_m$ and, hence

$$P^2 u = \lim_{m\to\infty} Pw_m = \lim_{m\to\infty} w_m = Pu\,, \quad \forall u \in \mathcal{H}$$

which means that $P^2 = P$.

For any u and v in $\mathcal{H}$, put $w_m = \sum_{n=1}^m \langle u, v_n\rangle v_n$ and $\tilde{w}_m = \sum_{n=1}^m \langle v, v_n\rangle v_n$ so that $w_m \to Pu$ and $\tilde{w}_m \to Pv$ as $m \to \infty$. By continuity of the inner product

$$\begin{aligned} \langle Pu, v\rangle &= \lim_{m\to\infty} \langle w_m, v\rangle = \lim_{m\to\infty} \langle u, \tilde{w}_m\rangle = \langle u, Pv\rangle \\ \langle w_m, v\rangle &= \sum_{n=1}^m \langle u, v_n\rangle\langle v_n, v\rangle = \langle u, \sum_{n=1}^m \overline{\langle v_n, v\rangle} v_n\rangle = \langle u, \sum_{n=1}^m \langle v, v_n\rangle v_n\rangle = \langle u, \tilde{w}_m\rangle\,. \end{aligned}$$

So, $P = P^*$ (it is symmetric and its domain is the whole Hilbert space).

(i) Since P is self-adjoint, its spectrum is real and the residual spectrum $\sigma_r(P)$ is empty.

(ii) To find the point spectrum $\sigma_p(P)$, let M be the closure of span$\{v_n\}$ which is a Hilbert space (a subspace of $\mathcal{H}$) (recall the Riesz-Fisher theorem about orthogonal sets). Then by continuity of P, $Pu = u$ for any $u \in M$ and $Pv = 0$ for any v from the orthogonal complement of M. Furthermore M is closed and by the orthogonal projection theorem for any $w \in \mathcal{H}$, there exists a unique decomposition $w = u + v$ where $u \in M$ and $v \in M^\perp$. Therefore the equation

$$Pw = \lambda w$$

can have a nontrivial solution only if $\lambda = 0$ or $\lambda = 1$. This means that the point spectrum of P is has only two values: $\sigma_p(P) = \{0, 1\}$. Note also that the projection theorem could have been used to prove $P^2 = P$: $P(Pw) = Pu = Pw$ for any w.

(iii) By construction, the range of P is M. Consider the equation

$$Pw - \lambda w = f$$

where $\lambda \notin \sigma_p(P)$. By the orthogonal projection theorem, for any $w \in \mathcal{H}$ there is a unique decomposition $w = u + v$ where $u \in M$ and $v \in M^\perp$ where $u = Pw$ and $v = u - w = (I - P)w$. Since $P(Pw) = Pw$ and $P(I - P)w = Pw - Pw = 0$, the equation reads

$$(I - \lambda)Pw - \lambda(I - P)w = f = Pf + (I - P)f$$

Owing to the orthogonality of terms, the equation is equivalent to two equations (in M and in $M^\perp$, respectively):

$$\begin{aligned}
(I - \lambda)Pw = Pf \quad &\Rightarrow \quad Pw = (I - \lambda)^{-1}Pf \\
-\lambda(I - P)w = (I - P)f \quad &\Rightarrow \quad (I - P)w = -\lambda^{-1}(I - P)f \\
\mathcal{R}_P(\lambda)f = Pw + (I - P)w \quad &= \quad (1 - \lambda)^{-1}Pf - \lambda^{-1}(f - Pf)
\end{aligned}$$

for any $\lambda \notin \sigma_p(P)$.

(iv) Next, it follows from the triangle inequality that

$$\begin{aligned}
\|\mathcal{R}_P(\lambda)f\| \quad &\leq \quad \frac{\|Pf\|}{|\lambda - 1|} + \frac{\|f - Pf\|}{|\lambda|} \\
&\leq \quad \frac{\|f\|}{|\lambda - 1|} + \frac{\|f\|}{|\lambda|}
\end{aligned}$$

Note that $I - P$ is a projection operator onto $M^\perp$ and, hence, $\|I - P\| \leq 1$ just like for the projection operator P. Since the above inequality hold for any $f \in \mathcal{H}$,

$$\|\mathcal{R}_P(\lambda)\| \leq \frac{1}{|\lambda - 1|} + \frac{1}{|\lambda|}$$

which shows that the resolvent exists and is bounded for any complex λ that is not equal to 1 or 0. Therefore all such λ form the resolvent set, or the continuum spectrum of P is empty. Thus, the spectrum consists only of the point spectrum, $\sigma(P) = \{0, 1\}$.

59.7.2. The derivative operator on a circle

Define the operator

$$A\ :\quad D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = u(1)\,\}\,,\quad Au(x) = -iu'(x)$$

(i) Show that

$$\sigma_p(A) = \{2\pi n\}_{-\infty}^{\infty}$$

(ii) Construct the adjoint A^*. Show that A has a self-adjoint extension and

$$\sigma_r(A) = \emptyset$$

(iii) Show that the resolvent is

$$\begin{aligned} \mathcal{R}_A f(x) &= C_f e^{i\lambda x} + i\int_0^x e^{i\lambda(x-y)} f(y)\, dy \\ C_f &= \frac{ie^{i\lambda}}{1-e^{i\lambda}}\int_0^1 e^{-i\lambda y} f(y)\, dy \end{aligned}$$

for any complex $\lambda \notin \sigma_p(A)$.
(iv) Prove that $\|\mathcal{R}_A\| < \infty$ and find the spectrum of the operator.

Solution: (i) The operator is symmetric. Its domain is dense in the Hilbert space and

$$\langle Au, v\rangle = -i\int_0^1 u'\bar{v}\, dx = -iu\bar{v}\Big|_0^1 + i\int_0^1 u\bar{v}' dx = \langle u, Av\rangle$$

because the boundary term vanishes for any u and v from D_A. Therefore its approximate spectrum is real. The point spectrum is given by all real λ for which the following boundary value problem has a non-trivial solution:

$$\left\{\begin{array}{l} -iu'(x) = u(x) \\ u(0) = u(1) \end{array}\right. \quad\Rightarrow\quad \left\{\begin{array}{l} u(x) = Ce^{i\lambda x} \\ u(0) = u(1) \end{array}\right. \quad\Rightarrow\quad e^{i\lambda} = 1 \quad\Rightarrow\quad \sigma_p(A) = \{2\pi n\}_{-\infty}^{\infty}$$

(ii) If $v \in D_{A^*}$, then there exists $g \in \mathcal{L}_2(0,1)$ such that $\langle Au, v\rangle = \langle u, g\rangle$ for any $u \in D_A$. Recall that for any $g \in \mathcal{L}_2(0,1)$ there exists an absolutely continuous v such that $v' \sim g$. Therefore the domain $D_{A*} \subset AC^0[0,1]$. Let v be absolutely continuous. Then by integration by parts

$$\langle Au, v\rangle = -iu(x)\overline{v(x)}\Big|_0^1 + i\langle u, v'\rangle = -iu(1)[\overline{v(1)} - \overline{v(1)}] + \langle u, g\rangle$$

This shows that such g exists if the boundary term vanish and, in this case, $g(x) = -iv'(x) \in \mathcal{L}_2(0,1)$.

Since $u(1)$ is arbitrary for $u \in D_A$, one has to demand that $v(0) = v(1)$. Thus,

$$A^*v(x) = g = -iv'(x)\,,\quad D_A \subset D_{A^*} = \{AC^0[0,1] | v(0) = v(1)\}$$

Let us calculate the double adjoint, A^{**}. If $u \in D_{A^{**}}$, then there exists $g \in \mathcal{L}_2(0,1)$ such that $\langle A^*v, u\rangle = \langle v, g\rangle$ for all $v \in D_{A^*}$. Since $D_{A^{**}} \subset AC^0[0,1]$, let u be absolutely continuous. Then by integration by parts

$$\langle A^*v, u\rangle = -iv(x)\overline{u(x)}\Big|_0^1 + i\langle v, u'\rangle = -iv(1)[\overline{u(1)} - \overline{u(1)}] + \langle v, g\rangle$$

This shows that $u(1) = u(0)$ and $A^{**}u(x) = -u'(x)$. Hence $A^{**} = A^*$ and A^* is the self-adjoint extension of A (the operator A is essentially self-adjoint, which could also be established by showing that $A^*v = \pm iv$, $v \in D_{A^*}$, has only the trivial solution). Thus, its residual spectrum is empty $\sigma_r(A) = \emptyset$.

(iii) For any complex λ that is not in the point spectrum $\sigma_p(A)$, consider the problem

$$\begin{cases} Au = f \\ u \in D_A \end{cases} \Rightarrow \begin{cases} -iu'(x) = f(x) \\ u(0) = u(1) \end{cases} \Rightarrow \begin{cases} u(x) = C_f e^{i\lambda x} + i \int_0^x e^{i\lambda(x-y)} f(y)\, dy \equiv \mathcal{R}_A(\lambda) f(x) \\ u(0) = u(1) \end{cases}$$

where the constant C_f is obtained from the boundary condition

$$C_f = C_f e^{i\lambda} + ie^{i\lambda} \int_0^1 e^{-i\lambda y} f(y)\, dy \quad \Rightarrow \quad C_f = \frac{ie^{i\lambda}}{1 - e^{i\lambda}} \int_0^1 e^{-i\lambda y} f(y)\, dy$$

as required. Note that if A is extended to A^* (a self-adjoint extension), then the resolvent is defined on the whole $\mathcal{L}_2(0,1)$ if $\lambda \notin \sigma_p(A)$. Without the extension, its domain is $C^0([0,1])$ which is dense in $\mathcal{L}_2(0,1)$.

(iv). Since the residual spectrum is empty, any non-real λ lies in the resolvent set and hence $\|\mathcal{R}_A(\lambda)\| < \infty$ in this case. Let λ be real but not an integer multiple of 2π. Then

$$|C_f| \le \frac{1}{2\sin(\lambda/2)} \int_0^1 |f(y)|\, dy \le \frac{\|f\|}{2\sin(\lambda/2)}$$

Therefore

$$\begin{aligned} |\mathcal{R}_A(\lambda) f(x)| &\le |C_f| + \int_0^1 |f(x)|\, dx \le \frac{\|f\|}{2\sin(\lambda/2)} + \|f\| \equiv M(\lambda)\|f\| \\ \|\mathcal{R}_A(\lambda) f\| &= \Big(\int_0^1 |\mathcal{R}_A(\lambda) f(x)|^2 dx\Big)^{1/2} \le M(\lambda)\|f\| \end{aligned}$$

Since $M(\lambda) < \infty$ for any $\lambda \notin \sigma_p(A)$, the continuum spectrum is empty, $\sigma_c(A) = \emptyset$. Thus, $\sigma(A) = \sigma_p(A) = \{2\pi n\}_{-\infty}^{\infty}$.

59.7.4.

Repeat the analysis of Problem 59.7.2 for the operator

$$A \; : \quad D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = zu(1) \,\}\,, \quad Au(x) = -iu'(x)$$

where z is a complex number. Find the spectrum $\sigma(A)$. Note that the operator is not longer symmetric if $|z| \neq 1$.

SOLUTION: Following the procedure for finding the spectrum of an operator (densely defined), let us first find the point spectrum:

$$-iu' = \lambda u\,, \quad u(0) = zu(1) \quad \Rightarrow \quad 1 = ze^{i\lambda}$$

If $z = 0$, then the point spectrum is empty. Let $z \neq 0$. Put $z = |z|e^{i\theta}$, $0 \leq \theta < 2\pi$. Then the point spectrum is

$$\sigma_p(A) = \{-\theta + 2\pi n + i\ln(|z|)\}_{-\infty}^{\infty}\,.$$

Next, let us construct the adjoint. If $v \in D_{A^*}$, then there exists $g \in \mathcal{L}_2(0,1)$ such that $\langle Au, v\rangle = \langle u, g\rangle$ for all $u \in D_A$. Since $D_{A^*} \subset AC^0[0,1]$, let v be absolutely continuous. By integration by parts

$$\langle Au, v\rangle = -iu(1)[z\overline{v(0)} - \overline{v(1)}] + i\langle u, v'\rangle = \langle u, g\rangle$$

Thus $A^*v = g = -iv'$ and

$$D_{A^*} = \{AC^0[0,1]|\bar{z}v(0) = v(1)\}$$

in order for the boundary term to vanish because $u(1)$ is arbitrary.

The next step is to find the compression spectrum of A. To accomplish this task, one has to find the point spectrum of A^* and take its complex conjugation: $\sigma_{\rm com}(A) = \overline{\sigma_p(A^*)}$. If μ is an eigenvalue of A^*, then

$$\mu = \mu_n = -\theta + 2\pi n - i\ln(|z|)$$

because the corresponding boundary value problem is identical to that for A with the only difference that $z \to (\bar{z})^{-1}$ or $|z| \to 1/|z|$. Therefore

$$\sigma_{\rm com}(A) = \{\bar{\mu}_n\}_{-\infty}^{\infty} = \{-\theta + 2\pi n + i\ln(|z|)\}_{-\infty}^{\infty} = \sigma_p(A)$$

Hence, the residual spectrum is empty $\sigma_r(A) = \emptyset$ if $z \neq 0$. If $z = 0$, then $\bar{z} = 0$ in D_{A^*}, and the point spectrum of A^* is empty. So, $\sigma_r(A) = \emptyset$ in this case, as well.

The resolvent is obtained in the same way as in Problem 59.7.2 with only difference that the constant C_f should be fixed by $u(0) = zu(1)$. Thus, for any $\lambda \notin \sigma_p(A)$

$$\begin{aligned} \mathcal{R}_A(\lambda)f(x) &= C_f(z)e^{i\lambda x} + i\int_0^x e^{i\lambda(x-y)}f(y)\,dy\,, \quad f \in \mathcal{L}_2(0,1) \\ C_f(z) &= \frac{ize^{i\lambda}}{1 - ze^{i\lambda}}\int_0^1 e^{-i\lambda y}f(y)\,dy \end{aligned}$$

Using the estimates $|e^{\pm ix\lambda}| \leq e^{|{\rm Im}\lambda|}$ for all $x \in [0,1]$ and that $|1 - ze^{i\lambda}| \geq \delta > 0$ for any $\lambda \notin \sigma_p(A)$, it is straightforward to see that

$$|C_f(z)| \leq \frac{|z|e^{|{\rm Im}\lambda|}}{\delta}\|f\| \quad \Rightarrow \quad \|\mathcal{R}_A(\lambda)\| \leq \frac{|z|e^{2|{\rm Im}\lambda|}}{\delta}\|f\| + e^{|{\rm Im}\lambda|}\|f\| \equiv M\|f\|$$

Thus, the continuum spectrum is empty, $\sigma_c(A) = \emptyset$.

REMARK: If $|z| = 1$, the operator is essentially self-adjoint because $A^{**} = A^*$ (A^* is a self-adjoint extension of A by the same argument as in Problem 59.7.2).