Lecture notes: Introduction to Partial Differential Equations

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CHAPTER 1

Preliminaries

1. Euclidean spaces

1.1. Euclidean space as a vector space. Any element (or a *point*) of a Euclidean space $\mathbb{R}^n$ is an ordered n-tuple of real numbers and will be denoted by $\mathbf{x} = \langle x_1, x_2, ..., x_n\rangle$ or $\mathbf{r} = \langle x, y, z, ...\rangle$. Two points $\mathbf{x}$ and $\mathbf{y}$ are said to be *equal* if the corresponding *components* are equal $x_j = y_j$, $j = 1, 2, ..., n$.

A Euclidean space is a *vector space* over real numbers. This means the following. For any two elements $\mathbf{x}$ and $\mathbf{y}$ and a real number c, the addition of two elements and a multiplication of an element by a number are defined by

$$\mathbf{x} + \mathbf{y} = \langle x_1 + y_1, x_2 + y_2, ..., x_n + y_n\rangle$$
$$c\mathbf{x} = \langle cx_1, cx_2, ..., cx_n\rangle$$

By this definition, *any linear combination* $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$ of elements $\mathbf{x}_\alpha \in \mathbb{R}^n$, $\alpha = 1, 2, ..., k$, belongs to $\mathbb{R}^n$. In other words, $\mathbb{R}^n$ is *closed* with respect to the addition and multiplication by a number defined above. These operations are generalizations of the operations of vector algebra in plane or space. For this reason, elements of $\mathbb{R}^n$ are also called *n-dimensional vectors*.

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called *parallel* if they are proportional $\mathbf{x} = c\mathbf{y}$ for some $c \in \mathbb{R}$. The zero vector $\mathbf{0} = \langle 0, 0, ..., 0\rangle$ is parallel to any vector.

1.2. A distance in $\mathbb{R}^n$. The Euclidean distance between two points $\mathbf{x} = \langle x_1, x_2, ..., x_n\langle$ and $\mathbf{y} = \langle y_1, y_2, ..., y_n\rangle$ is defined by

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

It satisfies the distance axioms:

$$|\mathbf{x} - \mathbf{y}| \geq 0 \quad \text{and} \quad |\mathbf{x} - \mathbf{y}| = 0 \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{y}$$
$$|\mathbf{x} - \mathbf{y}| = |\mathbf{y} - \mathbf{x}|$$
$$|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|$$

that is, the distance is a non-negative symmetric function of two points that satisfies the triangle inequality and it vanishes if and only if the points are equal.

1.3. The inner product in $\mathbb{R}^n$. A number defined by the rule

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

for any two elements $\mathbf{x}$ and $\mathbf{y}$ is called *the dot or inner product* in $\mathbb{R}^n$. It follows from the definition that the dot product has the properties

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= \mathbf{y}\cdot\mathbf{x}\\ \mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) &= \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}\\ \mathbf{x}\cdot(c\mathbf{y}) &= (c\mathbf{x})\cdot\mathbf{y} = c(\mathbf{x}\cdot\mathbf{y})\end{aligned}$$

It also follows that

$$\mathbf{x}\cdot\mathbf{x}\geq 0 \quad\text{and}\quad \mathbf{x}\cdot\mathbf{x}=0 \quad\Leftrightarrow\quad \mathbf{x}=\mathbf{0}$$

The number

$$|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$$

is called the *norm* or *length* of $\mathbf{x}$. Evidently, the norm is the distance between $\mathbf{x}$ and the zero vector: $|\mathbf{x}| = |\mathbf{x}-\mathbf{0}|$.

Cauchy-Schwartz inequality. For any two vectors the following inequality holds

$$|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$$

and the equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are parallel.

PROOF. If $\mathbf{y}=\mathbf{0}$, then the inequality is true. Suppose $\mathbf{y}\neq\mathbf{0}$ and, hence, $|\mathbf{y}|\neq 0$. Consider a non-negative function defined by

$$f(t) = |\mathbf{x}+t\mathbf{y}|^2 \geq 0\,,\quad t\in\mathbb{R}$$

Using the relation between the distance and the dot product and the basic properties of the latter

$$\begin{aligned}f(t) &= (\mathbf{x}+t\mathbf{y})\cdot(\mathbf{x}+t\mathbf{y})\\ &= \mathbf{x}\cdot\mathbf{x} + t^2(\mathbf{y}\cdot\mathbf{y}) + 2t(\mathbf{x}\cdot\mathbf{y})\\ &= |\mathbf{x}|^2 + t^2|\mathbf{y}|^2 + 2t(\mathbf{x}\cdot\mathbf{y})\end{aligned}$$

So $f(t)$ is a quadratic polynomial. Since the coefficient at t^2 is positive, the polynomial attains its *minimal* value when $f'(t)=0$, that is, at

$$t = t^* = -\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|^2}$$

Since $f(t) \geq 0$ for any t,

$$\begin{aligned} & f(t^*) = |\mathbf{x}|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{y}|^2} \geq 0 \\ \Rightarrow \quad & (\mathbf{x}\cdot\mathbf{y})^2 \leq |\mathbf{x}|^2|\mathbf{y}|^2 \\ \Rightarrow \quad & |\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}| \end{aligned}$$

Note that the smallest possible value of $f(t)$ is zero. The condition $f(t) = 0$ holds if and only if $\mathbf{x} + t\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = -t\mathbf{y}$ which means that the vectors $\mathbf{x}$ and $\mathbf{y}$ are parallel. Thus, the equality in the Cauchy-Schwartz inequality is possible only if $\mathbf{x}$ and $\mathbf{y}$ are parallel.

Angles between vectors. It follows from the Cauchy-Schwartz inequality that for any two non-zero vectors

$$-1 \leq \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|\,|\mathbf{y}|} \leq 1$$

The angle θ between two non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as a root to the equation

$$\cos(\theta) = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|\,|\mathbf{y}|}\,, \quad \theta \in [0, \pi]$$

Note that the equation has a unique solution in the stated range because $\cos(\theta)$ is one-to-one on $[0, \pi]$. Since the equality in the Cauchy-Schwartz inequality is reached only if $\mathbf{x}$ and $\mathbf{y}$ are parallel or proportional, one infers that

$$\begin{aligned} \theta = 0 \quad &\Leftrightarrow \quad \mathbf{x} = c\mathbf{y}\,,\ c > 0 \\ \theta = \pi \quad &\Leftrightarrow \quad \mathbf{x} = c\mathbf{y}\,,\ c < 0 \end{aligned}$$

In the latter case, the vectors are sometimes called *anti-parallel.* Two vectors are called *orthogonal* (or perpendicular) if their dot product vanishes:

$$\mathbf{x} \perp \mathbf{y} \quad \Leftrightarrow \quad \mathbf{x}\cdot\mathbf{y} = 0 \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}$$

Note that the zero vector is parallel and orthogonal to any vector, and there is only one vector with such properties (there reader is advised to prove the latter assertion).

1.4. Bases and subspaces in $\mathbb{R}^n$. For any finite collection of vectors $\mathbf{x}_\alpha$, $\alpha = 1, 2, ..., k$, the collection of all linear combinations is called a span:

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k \in \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}$$

Vectors $\mathbf{x}_\alpha$, $\alpha = 1, 2, ..., k \leq n$ are called *linearly independent* if

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0} \quad \Leftrightarrow \quad c_1 = c_2 = \cdots = c_k = 0$$

In other words, none of the vectors $\mathbf{x}_\alpha$ can be expressed as a linear combination of the others.

Any vector can be expanded into the following linear combination

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

where all components of the vector $\mathbf{e}_k$ are equal to zero except the k-th components which is set to one:

$$(\mathbf{e}_k)_j = \delta_{kj} = \begin{cases} 1 \ , \ k = j \\ 0 \ , \ k \neq j \end{cases}$$

The vectors $\mathbf{e}_k$, $k = 1, 2, ..., n$, are linearly independent. Furthermore they are mutually orthogonal:

$$\mathbf{e}_k \cdot \mathbf{e}_m = \delta_{km}$$

In particular, the components of a vector $\mathbf{x}$ are uniquely defined by the dot product:

$$x_j = \mathbf{x} \cdot \mathbf{e}_j \ , \quad j = 1, 2, ..., n$$

so that the identity holds

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{x} \cdot \mathbf{e}_2)\mathbf{e}_2 + \cdots + (\mathbf{x} \cdot \mathbf{e}_n)\mathbf{e}_n$$

The collection of vectors $\mathbf{e}_k$, $k = 1, 2, ..., n$, is called the *standard basis* in $\mathbb{R}^n$.

A collection of vectors $\mathbf{u}_k$, $k = 1, 2, ..., n$, is called a *basis* in $\mathbb{R}^n$ if

$$\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\} = \mathbb{R}^n$$

In other words, any element of $\mathbb{R}^n$ is a linear combination of basis vectors. Clearly, basis vectors are linearly independent. If, in addition, the basis vectors are mutually orthogonal, then the basis is called *orthogonal*. An orthogonal basis is called *orthonormal* if $|\mathbf{u}_k| = 1$, $j = 1, 2, ..., n$. The standard basis is orthonormal. Any orthogonal basis can be converted into an orthonormal basis by the *renormalization* procedure

$$\mathbf{u}_k \quad \to \quad \mathbf{v}_k = |\mathbf{u}_k|^{-1}\,\mathbf{u}_k$$

The Gram-Schmidt process. The vector

$$\mathrm{Proj}_{\mathbf{x}}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{x}}{|\mathbf{x}|^2}\,\mathbf{x} = \frac{|\mathbf{y}|\cos(\theta)}{|\mathbf{x}|}\,\mathbf{x}$$

is called the *vector projection* of $\mathbf{y}$ onto $\mathbf{x} \neq \mathbf{0}$, where θ is the angle between $\mathbf{y}$ and $\mathbf{x}$. By construction, the vector projection of $\mathbf{y}$ is a vector parallel to $\mathbf{x}$ with the length $|\mathbf{y}||\cos(\theta)|$. The triangle with adjacent sides being $\mathbf{y}$ and $\mathrm{Proj}_{\mathbf{x}}\mathbf{y}$ is a right-angled triangle ($\mathbf{y}$ is its hypotenuse).

Any basis $\mathbf{u}_k$ in $\mathbb{R}^n$ can be converted into an orthogonal basis by the Gram-Schmidt process:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \mathrm{Proj}_{\mathbf{v}_1}\mathbf{u}_2 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \mathrm{Proj}_{\mathbf{v}_1}\mathbf{u}_3 - \mathrm{Proj}_{\mathbf{v}_2}\mathbf{u}_3 \\
&\dots \\
\mathbf{v}_k &= \mathbf{u}_k - \mathrm{Proj}_{\mathbf{v}_1}\mathbf{u}_k - \mathrm{Proj}_{\mathbf{v}_2}\mathbf{u}_k - \cdots - \mathrm{Proj}_{\mathbf{v}_{k-1}}\mathbf{u}_k
\end{aligned}$$

where $k = 2, 3, ..., n$. The reader is advised to verify that

$$\mathbf{v}_j \cdot \mathbf{v}_k = 0\,, \quad j \neq k\,, \quad j, k = 1, 2, ..., n$$

The collection $\{\mathbf{v}_k\}$ form an orthogonal basis which can be converted into an orthonormal basis by the renormalization procedure.

Subspaces in $\mathbb{R}^n$. A span of some vectors $\{\mathbf{x}_\alpha\}$ is closed with respect to vector addition of its elements and with respect to multiplication of its elements by a number because results of these operations are linear combinations of the vectors $\mathbf{x}_\alpha$. Therefore the span is also a linear space and it is called a *subspace* of $\mathbb{R}^n$. The greatest number of linearly independent vectors in a span is called the *dimension* of the span. In particular, the dimension of $\mathbb{R}^n$ is equal to n

$$\dim\,\mathbb{R}^n = n$$

because the standard basis has n linearly independent elements. The dimension of any subspace cannot exceed n and, hence,

$$\dim\,\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\} \leq \min\{k, n\}$$

and the equality is reached only if $\mathbf{x}_\alpha$ are linearly independent. In this case, the vectors $\mathbf{x}_\alpha$ form a basis in the span.

Lines and planes. A collection of points

$$\mathbf{x} = \mathbf{y} + t\mathbf{v}\,, \quad t \in \mathbb{R}$$

is called a *line* through the point $\mathbf{y}$ and parallel to a vector $\mathbf{v} \neq \mathbf{0}$. Clearly, position vectors of points of the line relative to a particular point $\mathbf{y}$, that is, $\mathbf{x} - \mathbf{y}$, form the span of $\mathbf{v}$ which is a one-dimensional subspace of $\mathbb{R}^n$.

Let $\mathbf{x}_\alpha$, $\alpha = 1, 2, ..., k < n$, be linearly independent. The collection of points

$$\mathbf{x} = \mathbf{y} + t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k\,, \quad t_\alpha \in \mathbb{R}$$

is called a *k-dimensional hyper-plane* or simply a *a k-dimensional plane* through a point $\mathbf{y}$. Position vectors of all points $\mathbf{x}$ of a hyper-plane relative to a particular point $\mathbf{y}$, that is, $\mathbf{x}-\mathbf{y}$, form the span of linearly independent vectors $\mathbf{x}_\alpha$, $\alpha = 1, 2, ..., k < n$, which has the dimension k (this is why it is called a k-th dimensional hyper-plane). The vectors $\mathbf{x}_\alpha$ form a basis for all vector in the hyper-plane.

If $n = 3$ and $k = 2$, then the above vector equation states that any vector in a plane is a linear combination of two non-parallel vectors in the plane. This condition can also be stated via the dot product: $(\mathbf{x}-\mathbf{y}) \cdot \mathbf{n} = 0$ where $\mathbf{n} = \mathbf{x}_1 \times \mathbf{x}_2$ is defined via the cross product in $\mathbb{R}^3$ of any two non-parallel vectors in the plane. Recall the cross product is orthogonal to both vectors in it.

2. Functions on $\mathbb{R}^n$

2.1. Sets in $\mathbb{R}^n$. A collection of all points whose distance is strictly less than $a > 0$ from a given point $\mathbf{y}$,

$$B_a(\mathbf{y}) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, |\mathbf{x} - \mathbf{y}| < a \right\}$$

is called an *open ball of radius a centered at* $\mathbf{y}$ or a *neighborhood* of the point $\mathbf{y}$. If $\mathbf{y} = \mathbf{0}$, that is, the ball is centered at the origin, then it is denoted as B_a.

A neighborhood of a set. Let Ω be a set of points in $\mathbb{R}^n$. A *neighborhood of a set* Ω is called the union of open balls centered at each point of Ω:

$$\Omega_a = \bigcup_{\mathbf{x} \in \Omega} B_a(\mathbf{x})$$

For example, a neighborhood of a ball B_R is a ball B_{R+a}.

Open sets. A set Ω is said to be *open* if for every point $\mathbf{x} \in \Omega$ there is a neighborhood $B_a(\mathbf{x})$ that is contained in Ω. For example $B_R(\mathbf{y})$ is open.

Limit points of a set. A point $\mathbf{y}$ is a *limit point* of a set Ω if the intersection $\Omega \cap B_a(\mathbf{y})$ for any neighborhood $B_a(\mathbf{y})$ contains at least one point of Ω different from the point $\mathbf{y}$. Note that $\mathbf{y}$ does not necessarily belong to Ω. For example, let $\Omega = B_1$ (all $\mathbf{x}$ such that $|\mathbf{x}| < 1$). Clearly, every point of the ball is a limit point because there is a ball of a sufficiently small radius centered at any point of Ω that is contained in Ω (the set B_1 is open as was noted above). Take any point $\mathbf{y}$ such that $|\mathbf{y}| = 1$ (at a distance 1 from the center of the ball). Clearly, any ball $B_a(\mathbf{y})$ has a non-empty intersection wit B_1. So all points on the *sphere* $|\mathbf{r}| = 1$ are limit points of B_1 but none of them belong to B_1.

Closed sets. The *closure* of a set Ω is the union of all limit points of Ω with Ω. It is denoted by $\bar{\Omega}$. A set Ω is called *closed* if it contains all its limit points so that $\bar{\Omega} = \Omega$. For example,

$$\overline{B_a(\mathbf{y})} = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, |\mathbf{x} - \mathbf{y}| \leq a \right\}$$

is a closed ball of radius a centered at $\mathbf{y}$. A point $\mathbf{y} \in \Omega$ is said to be *isolated* if there is a neighborhood $B_a(\mathbf{y})$ whose intersection with Ω contains only the point $\mathbf{y}$. Clearly, a finite collection of isolated points is a closed set (as it has no limit points). An infinite set of isolated point points $\{\mathbf{x}_k\}_0^\infty$ (a sequence in $\mathbb{R}^n$) may have a limit point that does not belong to the set. For example, if the sequence has the limit $\mathbf{x}$, that is,

$$\lim_{k\to\infty} |\mathbf{x}_k - \mathbf{x}| = 0$$

then $\mathbf{x}$ is a limit point of the set (any ball $B_a(\mathbf{x})$) contains $\mathbf{x}_k$ for all $k > N$ for some (large enough) N).

The interior of a set. The largest open set Ω_o that is contained in Ω is called the *interior* of Ω. So, for any Ω

$$\Omega_o \subseteq \Omega \subseteq \bar{\Omega}$$

One can prove that *the closure $\bar{\Omega}$ is the smallest closed set that contains* Ω. Note that the interior Ω_o can be empty. For example a set containing only finitely many points has an empty interior. The interior of an open set coincides with the set itself.

The boundary of a set. The set

$$\partial\Omega = \bar{\Omega} \setminus \Omega_o$$

is called the boundary of Ω. For example, the boundary of a ball (closed or open)

$$\partial B_a(\mathbf{y}) = \overline{B_a(\mathbf{y})} \setminus B_a(\mathbf{y}) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, |\mathbf{x} - \mathbf{y}| = a \right\}$$

is the sphere of radius a centered at $\mathbf{y}$. The whole space has no boundary. Indeed, the whole space $\Omega = \mathbb{R}^n$ is open and closed (because all its points are interior points and it contains all its limit points) so that its interior coincides with its closure and the boundary is empty.

2.2. Functions of several variables. Let u be a function of several variables, denoted x, y, z, etc. Unless stated otherwise, $\mathbf{r} = \langle x, y, z, ...\rangle$ or $\mathbf{x} = \langle x_1, x_2, ..., x_n\rangle$ is a point in a Euclidean space $\mathbb{R}^n$. A function u is a rule that assigns a unique number, denoted $u(\mathbf{r})$ to every point $\mathbf{r} \in \Omega \subset \mathbb{R}^n$:

$$u : \ \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

A function

$$u : \ \Omega \subset \mathbb{R}^n \to \mathbb{C}$$

is called complex-valued. In accord with the definition of complex numbers, it can always be written as a linear combination of two real-valued functions

$$u(\mathbf{r}) = v(\mathbf{r}) + iw(\mathbf{r})\,, \qquad v(\mathbf{r}) = \mathrm{Re}\, u(\mathbf{r})\,, \quad w(\mathbf{r}) = \mathrm{Im}\, u(\mathbf{r})$$

Partial derivatives. Let $u:\ \Omega \to \mathbb{R}$ and Ω be an open set. Suppose that the function u has partial derivatives up to some order. The first order partial derivatives are denoted as

$$\frac{\partial u}{\partial x} = u'_x\,, \quad \frac{\partial u}{\partial y} = u'_y\,, \quad \frac{\partial u}{\partial z} = u'_z\,, \quad \ldots$$

The second order partial derivatives are denoted as

$$\frac{\partial^2 u}{\partial x^2} = u''_{xx}\,, \quad \frac{\partial^2 u}{\partial y^2} = u''_{yy}\,, \quad \frac{\partial^2 u}{\partial z^2} = u''_{zz}\,, \quad \ldots$$

There are also mixed partial derivatives of the second order:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = u''_{xy}\,, \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = u''_{yx}$$

and similarly for any other pair of variables, that is, u''_{xz}, u''_{zx}, etc. Partial derivatives of the third order and higher are denoted similarly. For example

$$u'''_{xyy} = \frac{\partial}{\partial x}\frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}\,, \quad u'''_{xyz} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial u}{\partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z}\,.$$

Note that the order of variables in the subscript indicates the order in which partial derivatives are taken. In general, the value of partial derivatives of higher order depends on the order of differentiation. *If, however, the partial derivatives are continuous functions, then the order of differentiation is irrelevant*. This statement is known as *Clairaut's theorem*. In what follows, it will always be assumed that partial derivatives are continuous (up to the order of interest) and, hence, $u''_{xy} = u''_{yx}$, or $u'''_{xyz} = u'''_{xzy}$, etc.

Functions from the class C^p. A function on an open set Ω whose partial derivatives up to order p are continuous on Ω is called a function from the class $C^p(\Omega)$. All continuous functions on Ω form the class $C^0(\Omega)$. The functions whose partial derivatives of any order are continuous form the class $C^\infty(\Omega)$. If $\Omega = \mathbb{R}^n$, then $C^p(\mathbb{R}^n) = C^p$ for simplicity of notations. Functions from $C^p(\Omega)$ form a linear space because the functions $h(\mathbf{r}) = u(\mathbf{r}) + v(\mathbf{r})$ and $cu(\mathbf{r})$ are from $C^p(\Omega)$ if $v, u \in C^p(\Omega)$.

A collection of all partial derivatives of a function $u \in C^p(\Omega)$ of order $q \le p$ is denoted by $D^q u$. For example, for a function of two variables x and y

$$Du = \{u'_x\,,\ u'_y\}\,, \quad D^2 u = \{u''_{xx}\,,\ u''_{xy}\,,\ u''_{yy}\}$$

Similarly, for functions of n variables

$$Du = \left\{ \frac{\partial u}{\partial x_j} \;\Big|\; j = 1, 2, ..., n \right\},$$

$$D^2 u = \left\{ \frac{\partial^2 u}{\partial x_j \partial x_k} \;\Big|\; j \le k = 1, 2, ..., n \right\}.$$

2.3. The gradient. The vector in the domain of a function u whose components are partial derivatives are called the *gradient*:

$$\begin{aligned} \mathbb{R}^2: &\quad \boldsymbol{\nabla} u = (u'_x, u'_y)\,, \\ \mathbb{R}^3: &\quad \boldsymbol{\nabla} u = (u'_x, u'_y, u'_z)\,, \\ \mathbb{R}^n: &\quad \boldsymbol{\nabla} u = (u'_{x_1}, u'_{x_2}, ..., u'_{x_n})\,. \end{aligned}$$

For any unit vector $\mathbf{n}$, $|\mathbf{n}| = 1$, in the domain of a C^1 function u, the dot product

$$\mathbf{n} \cdot \boldsymbol{\nabla} u(\mathbf{x}) = \frac{\partial u}{\partial \mathbf{n}}$$

is called the directional derivative of u in the direction of $\mathbf{n}$ at a point $\mathbf{x}$. It defines the rate of change of u in the direction of $\mathbf{n}$ at a point $\mathbf{x}$. If θ is the angle between $\mathbf{n}$ and ∇u at a point $\mathbf{x}$, then

$$\frac{\partial u}{\partial \mathbf{n}} = |\nabla u| \cos(\theta)$$

This shows that u is increasing most rapidly at a point $\mathbf{x}$ in the direction parallel to the gradient ($\cos(\theta) = 1$ or $\theta = 0$) and the maximal rate is equal to $|\nabla u|$, whereas u is decreasing most rapidly at a point $\mathbf{x}$ in the direction anti-parallel to the gradient ($\cos(\theta) = -1$ or $\theta = \pi$) and the minimal rate is equal to $-|\nabla u|$.

Level sets. A collection of points at which a function u has the same value

$$u(\mathbf{x}) = K$$

is called a *level set* of u. For example, a level set of a constant function is either empty or coincides with$\mathbb{R}^n$. Level sets of the function

$$u(\mathbf{x}) = |\mathbf{x}|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

are concentric spheres $|\mathbf{x}| = R > 0$ and the level set $u(\mathbf{x}) = 0$ contains just one point $\mathbf{x} = \mathbf{0}$.

Smooth surfaces. Let $\mathbf{x} \in \mathbb{R}^{n-1}$. Consider a function $g(\mathbf{x})$. A collection of points $(g(\mathbf{x}), \mathbf{x}) \in \mathbb{R}^n$ is called a *graph* of g. For example, the curve $y = g(x)$ in the plane spanned by $(x, y) \in \mathbb{R}^2$ is a graph of a single-variable function g. Similarly, a surface $z = g(x, y)$ in the space $(x, y, z) \in \mathbb{R}^3$ is a graph of the two-variable function g, and so on.

A surface in $\mathbb{R}^n$ is defined as a set of points that looks like a graph of some function on $n-1$ variables in a neighborhood of each point. A surface in $\mathbb{R}^n$ is called *smooth* if in a neighborhood of each point it coincides with a graph of a C^1 function of $n-1$ variables. For example, a sphere $|\mathbf{x}| = R$ is a smooth surface because near its every point it is a graph of a C^∞ function. For example, near the point $x_j = R\delta_{jn}$ it coincides with the graph

$$x_n = \sqrt{R^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}$$

Near the point $x_j = -R\delta_{jn}$ it coincides with the graph

$$x_n = -\sqrt{R^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}$$

and near $x_j = R\delta_{j1}$ it coincides with the graph

$$x_1 = \sqrt{R^2 - x_2^2 - x_3^2 - \cdots - x_n^2}$$

Smooth boundary of an open set. Suppose that $u \in C^1$ and $\boldsymbol{\nabla} u \neq \mathbf{0}$, then a level set $u(\mathbf{x}) = K$ is a *smooth surface*. The assertion can be proved by means of *the implicit function theorem*. The equation $u(\mathbf{x}) = K$ can be viewed as a relation that defines one of the variables as an implicit function of the others. By the implicit function theorem, under stated conditions, the equation $u(\mathbf{x}) = K$ can be solved for one of the variables near any point in the level set. For example, if $u'_{x_n} \neq 0$ at that point, then there exists $x_n = g(x_1, x_2, ..., x_{n-1})$ and g is from the class C^1. Therefore near every point the level set is a graph of a function with continuous partial derivatives.

Consider a *parametric curve* in $\mathbb{R}^n$ which is a vector function of one variable $\mathbf{x} = \mathbf{x}(t)$. The derivative vector $\mathbf{x}'(t)$ is tangent to the curve (it is assumed that the derivative is non-zero and continuous). Suppose that the curve lies in the level surface $u(\mathbf{x}) = K$. Then the function $F(t) = u(\mathbf{x}(\mathbf{t})) = K$ has a constant value for all t and $F'(t) = 0$. On the other hand using the chain rule

$$\begin{aligned} 0 = F'(t) &= \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt} \\ &= \boldsymbol{\nabla} u \cdot \mathbf{x}'(t) \end{aligned}$$

It follows that the gradient at any point of a level set is orthogonal to a tangent vector to a curve in the level set passing through that point. If one takes all possible curves through a given point of the level set, then tangent vectors of the curves at this point would form a *tangent plane* to the level set at this point. Thus, *a level set of a C^1 function with non-vanishing gradient is a smooth surface and the gradient is normal to the level surface at any point.*

A boundary $\partial\Omega = \bar{\Omega}\setminus\Omega$ of an open set $\Omega \subset \mathbb{R}^n$ is called *smooth* if it is a level set of a function g from the class $C^1(\mathbb{R}^n)$ with non-vanishing gradient, $\nabla g \neq \mathbf{0}$. The vector

$$\mathbf{n} = |\nabla g|^{-1}\,\nabla g$$

is a *unit normal* to the boundary $\partial\Omega$. Note that the vector $-\mathbf{n}$ is also a unit normal. For example, a sphere $|\mathbf{x}| = R$ is the boundary of an open ball B_R. It is a level set of the C^1 function

$$g(\mathbf{x}) = |\mathbf{x}|^2$$

The gradient does not vanish

$$\boldsymbol{\nabla} g = 2\langle x_1, x_2, ..., x_n\rangle = 2\mathbf{x} \neq 0$$

on the level set $|\mathbf{x}|^2 = R^2$ (a sphere on a non-zero radius) so that $|\boldsymbol{\nabla} g| = 2|\mathbf{x}| = 2R$ on the level set $|\mathbf{x}|^2 = R^2$. Therefore the unit vector

$$\mathbf{n} = \frac{1}{R}\,\mathbf{x}$$

is a unit normal on the sphere $|\mathbf{x}| = R$.

2.4. An extension of a function. Let Ω be open and $u \in C^0(\Omega)$. Suppose that for every point $\mathbf{y} \in \partial\Omega$ the limit

$$\lim_{\mathbf{x}\to\mathbf{y}} u(\mathbf{x})$$

exists. Then u is said to be *continuously extendable* to $\bar{\Omega}$ so that

$$u(\mathbf{y}) = \lim_{\mathbf{x}\to\mathbf{y}} u(\mathbf{x})\,, \quad \mathbf{y} \in \bar{\Omega}$$

The class of all functions continuous functions on Ω that are continuously extendable to $\bar{\Omega}$ is denoted by $C^0(\bar{\Omega})$. The class $C^p(\bar{\Omega})$ consists of all functions from $C^p(\Omega)$ whose all partial derivatives have continuous extensions to $\bar{\Omega}$ so that

$$D^q u(\mathbf{y}) = \lim_{\mathbf{x}\to\mathbf{y}} D^q u(\mathbf{x})\,, \quad \mathbf{y} \in \bar{\Omega}\,, \quad q = 0, 1, ..., p$$

For example, multivariable polynomials on any open set Ω are continuously extendable together with all partial derivatives to the boundary of Ω. So, polynomials are from the class $C^\infty(\bar{\Omega})$ for any open Ω.

The function of two variables

$$g(x,y) = \sqrt{R^2 - x^2 - y^2}\,, \quad (x,y) \in \Omega = \{(x,y)|x^2 + y^2 < R^2\}$$

is from the class $C^\infty(\Omega)$. It has a continuous extension to the boundary circle $x^2 + y^2 = R^2$:

$$g(x,y) = 0\,, \quad x^2 + y^2 = R^2$$

Therefore $g \in C^0(\bar{\Omega})$. However, the partial derivatives

$$g'_x = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}\,, \quad g'_y = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

have no extension to the boundary circle and, hence, $g \notin C^1(\bar{\Omega})$.

3. Partial differential equations (PDEs)

Let u be a function on an open set $\Omega \subset \mathbb{R}^n$. *A partial differential equation* is a relation between the function u, its partial derivatives $D^q u$ up to some order $q \leq p$, and the argument $\mathbf{x}$ of u. The highest order of partial derivatives involved into the relation is called *the order of the partial differential equation*:

$$F(\mathbf{x}, u, Du, D^2u, ..., D^pu) = 0\,, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n\,. \tag{3.1}$$

It indicates that a function u is not an arbitrary function but such that it and its partial derivatives satisfy a certain relation F that must hold for every point $\mathbf{x}$ in an open set of interest $\Omega \subset \mathbb{R}^n$. The integer p is called *the order* of the partial differential equation.

3.1. A solution to a PDE. A function $u(\mathbf{x})$ is called *a solution* to a partial differential equation in an open set Ω if u and its partial derivatives satisfy the relation (**3**.**1**) for all $\mathbf{x} \in \Omega$.

EXAMPLE **3**.**1**. *Let $t > 0$ and $x \in \mathbb{R}$. Find a constant k such that the function*

$$u(t,x) = t^{-1/2} e^{-kx^2/t}$$

is a solution of the equation

$$u'_t = u''_{xx}\,, \quad t > 0\,, \quad x \in \mathbb{R}\,,$$

or show that no such constant exists.

SOLUTION: In this case, $\Omega = (0, \infty) \times (-\infty, \infty)$ is an open half-plane in $\mathbb{R}^2$. One has to calculate the partial derivatives u'_t and u''_{xx}, substitute them into the equation, and check whether there is a particular numerical value of k such that the relation between the partial derivatives is fulfilled for all positive t and all real x (the set Ω). The partial derivatives are

$$\begin{aligned} u'_t &= -\frac{1}{2} t^{-3/2} e^{-kx^2/t} + kx^2 t^{-5/2} e^{-kx^2/t} \\ u'_x &= -2kxt^{-3/2} e^{-kx^2/t} \\ u''_{xx} &= -2kt^{-3/2} e^{-kx^2/t} + 4k^2x^2t^{-5/2} e^{-kx^2/t} \end{aligned}$$

The substitution of the partial derivatives into the equation yields (after the cancellation of the positive factor $e^{-kx^2/t} > 0$)

$$-\frac{1}{2} t^{-3/2} + kx^2 t^{-5/2} = -2kt^{-3/2} + 4k^2x^2t^{-5/2}$$

The terms $t^{-3/2}$ and $x^2 t^{-5/2}$ are independent and, hence, the equality is possible for all values of $t > 0$ and $x \in \mathbb{R}$ if and only if the coefficients at these terms in the left and right side match, which gives a system of two equations for the unknown k:

$$\begin{cases} -\frac{1}{2} = -2k \\ k = 4k^2 \end{cases} \quad \Rightarrow \quad k = \frac{1}{4}$$

The first equation has the solution $k = 1/4$, while the second one has two solutions $k = 0$ and $k = 1/4$. The common solution is $k = 1/4$. Thus, the indicated function is a solution of the given partial differential equation only if $k = 1/4$. $\square$

EXAMPLE **3.2**. *If f and g are from the class $C^2(\mathbb{R})$, show that*

$$u(t,x) = f(x - ct) + g(x + ct)$$

is a solution to the wave equation in $\Omega = \mathbb{R}^2$:

$$u''_{tt} - c^2 u''_{xx} = 0\,.$$

SOLUTION: By the chain rule

$$\begin{aligned} u'_t &= -cf'(x - ct) + cg'(x + ct)\,, \\ u''_{tt} &= (-c)^2 f''(x - ct) + c^2 g''(x + ct) \\ u'_x &= f'(x - ct) + g'(x + ct)\,, \\ u''_{xx} &= f''(x - ct) + g''(x + ct)\,, \\ u''_{tt} - c^2 u''_{xx} &= c^2[f''(x - ct) + g''(x + ct)] \\ &\quad -c^2[f''(x - ct) + g''(x + ct)] \\ &= 0 \end{aligned}$$

for all $(t,x) \in \mathbb{R}^2$. $\square$

Note that a PDE may have infinitely many solutions. For example, a solution to the wave equation considered in Example **3.2** contains two arbitrary functions of a single variables.

3.2. Linear and non-linear PDEs. A partial differential equation is called *linear* if the relation F is a linear function in u and its partial derivatives. For example, the equation

$$u''_{xx} + f(x,y)u'_y + g(x,y)u = h(x,y)\,,$$

where f, g, and h are some functions of two variables x and y, is a linear second order partial differential equation for a function $u(x,y)$ of two variable. PDEs discussed in Examples **3.1** and **3.2** are linear equations. The equation

$$u''_{xx} + f(x,y)(u'_y)^2 + g(x,y)u^3 = h(x,y)$$

is also a second order partial differential equation, but it is not linear because it contains the square of the partial derivative u'_y and the cube of the function u.

Non-linear waves. It is not difficult to verify that the first-order equation

$$u'_t(t, x) + cu'_x(t, x) = 0$$

admits solutions that looks like a traveling wave:

$$u(t, x) = f(x - ct)$$

where f is a differentiable function of a single variable. Consider the graph $y = f(x - ct)$ where t has a fixed value. Then with increasing t the solution $u(t, x)$ can be viewed as the shape $y = f(x)$ traveling with the rate c to the right (in the direction of increasing x). One can think of a *wave* of the shape $y = f(x)$ traveling with the *speed* c (e.g. like a wave traveling in a narrow channel; the variable x is the length counted along the channel from some point).

There are non-linear PDEs that admit solutions as traveling waves. One of them is the celebrated Korteweg-de Vries equation, or simple the KdV equation:

$$u'_t + 6u\, u'_x + u'''_{xxx} = 0$$

Note the second term is not linear in the function u. It describes waves in shallow waters (e.g., ocean waves on a shallow shore), and waves is a shallow channel. The graph of the function $u(t, x)$ defines the shape of the wave at a time t (x changes along the channel, or along a line perpendicular to the shore).

The simplest solution can be found in the form of a traveling wave:

$$u(t, x) = f(s)\,, \quad s = x - ct$$

where f should be three times differentiable ($f \in C^3(\mathbb{R})$). The KdV equation can be reduced to the following ordinary differential equation:

$$f''' - cf' + 6ff' = 0$$

and then to the first-order ordinary differential equation for f,

$$\frac{1}{2}\,(f')^2 + f^3 - \frac{c}{2}\,f^2 - Af = E$$

where A and E are some integration constants. Note that the 3rd order can be reduced to a second order equation by integration. The first-order equation is obtained by multiplying the second-order equation by f' and integrating it. The technical details are left to the reader as an exercise. A particularly simple solution corresponding to a specific choice of constants A and E in this equation has the form

$$u(t,x) = \frac{c}{2\cosh^2[\frac{\sqrt{c}}{2}(x-ct)]}$$

Recall that $\cosh(s) = \frac{1}{2}(e^s + e^{-s})$. It describes a solitary wave propagating with the seed c. The wave maximal height is equal to $\frac{c}{2}$. Since c is a parameter of the solution, it is concluded (by analyzing the graphs of $u(0,x)$ at different c) that the taller waves move faster and are more sharply peaked. You might want to check this conclusion on Florida beaches!

3.3. Superposition principle for linear homogeneous PDEs. Among all linear equations, consider *homogeneous linear PDEs*. The function F in (**3.1**) is linear in $D^q u$ and, in addition, has the property that the zero function $u(\mathbf{x}) = 0$ is a solution. For example, the following equation is linear

$$u''_{tt}(t,x) + c^2 u''_{xx}(t,x) = f(t,x)$$

but it is not homogeneous because $u = 0$ is not a solution ($f(x,t) \neq 0$). It becomes homogeneous when the *inhomogeneity* f vanishes. A homogeneous linear PDE has the form of a vanishing linear combination of u and its partial derivatives:

$$\sum_{q=0}^{p} a_q D^q u = a_0 u + a_1 Du + \cdots + a_p D^p u = 0\,,$$

where $a_q D^q u$ denotes a general linear combination of partial derivatives of order q (with coefficients a_q being functions of $\mathbf{x}$). Taking a partial derivative is a linear operation, meaning that a partial derivative of a linear combination of functions is a linear combination of the partial derivatives of the functions. For example,

$$\frac{\partial}{\partial x}\Big(c_a u_1(x,y) + c_2 u_2(x,y)\Big) = c_1\,\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x}$$

where c_1 and c_2 are constants. In general, a partial derivative of any order is also a linear operation:

$$D^q(c_1 u_1 + c_2 u_2) = c_1 D^q u_1 + c_2 D^q u_2\,,$$

For any two solutions, u_1 and u_2, to a linear homogeneous PDE, their linear combinations with constant coefficients

$$u(\mathbf{r}) = c_1 u_1(\mathbf{x}) + c_2 u_2(\mathbf{x})$$

is also a solution. Indeed,

$$\begin{aligned}\sum_{q=0}^{p} a_q D^q u &= \sum_{q=0}^{p} a_q D^q (c_1 u_1 + c_2 u_2)\\ &= \sum_{q=0}^{p} a_q (c_1 D^q u_1 + c_2 D^q u_2)\\ &= c_1 \sum_{q=0}^{p} a_q D^q u_1 + c_2 \sum_{q=0}^{p} a_q D^q u_2\\ &= 0 + 0 = 0\end{aligned}$$

because u_1 and u_2 are solutions This is called the *superposition principle* for linear PDEs.

In Example **3.2**, put $u_1(t,x) = f(x - ct)$ and $u_2(t,x) = g(x + ct)$. They are solutions to the homogeneous wave equation and, by the superposition principle, their sum is also a solution (their general linear combinations with constant coefficients would also be a solution). So, a solution can be interpreted as a superposition of two waves traveling in the opposite directions with the same speed.

3.4. Exercises.

1. Is the function $u(x,y) = \ln(e^x + e^y)$ a solution to the equation

$$u''_{xx}u''_{yy} - (u''_{xy})^2 = 0$$

in some open region in a plane?

2. If $u = u(x,y,z)$, find the most general solution to each of the following equations in $\mathbb{R}^3$:

$$\begin{aligned} u''_{xx} &= 0\\ u''_{xy} &= 0\\ u^{(n)}_{zz\cdots z} &= 0\\ u'''_{xyz} &= 0\\ u'_x &= yu^2\end{aligned}$$

3. Let f be a twice continuously differentiable function of a real variable. Under what conditions on the $n-$dimensional vector $\mathbf{k} = \langle k_1, k_2, ..., k_n\rangle$ is the function

$$u(t,\mathbf{x}) = f(ct + \mathbf{k}\cdot\mathbf{x})\,,\quad \mathbf{x}\in\mathbb{R}^n\,,$$

where $\mathbf{k}\cdot\mathbf{x} = k_1x_1 + k_2x_2 + \cdots + k_nx_n$ (the dot product in $\mathbb{R}^n$), a solution to the $n-$dimensional wave equation in $\mathbb{R}^n$:

$$c^{-2}u''_{tt} - u''_{x_1x_1} - u''_{x_2x_2} - \cdots - u''_{x_nx_n} = 0\,?$$

4. Let $\mathbf{x} = \langle x_1, x_2, ..., x_n\rangle \in \mathbb{R}^n$ and $\mathbf{a}\cdot\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ (the dot product in $\mathbb{R}^n$). Under what conditions on the vector $\mathbf{a}$ is the function $u(\mathbf{x}) = \exp(\mathbf{a}\cdot\mathbf{x})$ a solution to the equation

$$u''_{x_1x_1} + u''_{x_2x_2} + \cdots + u''_{x_nx_n} = u\,, \quad \mathbf{x}\in\mathbb{R}^n$$

5. Let $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. Find a value of the constant k, if any, such that the function

$$u(x,y,z) = \frac{\sin(kr)}{r}$$

is a solution of the (Helmholtz) equation

$$u''_{xx} + u''_{yy} + u''_{zz} + a^2u = 0\,, \quad |\mathbf{r}| \neq 0$$

where $a > 0$ is a given constant. Can the function u be extended to the origin $|\mathbf{r}| = 0$ so that it becomes a solution in the whole space $\mathbb{R}^3$? Hint: Consider the power series representation of the sine function.

6. Show that the function $u(x,y) = \ln(x^2+y^2)$ is a solution to the two-dimensional Laplace equation

$$u''_{xx} + u''_{yy} = 0\,, \qquad (x,y) \neq (0,0)$$

7. Let $r = |\mathbf{x}| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ (the distance from the origin in $\mathbb{R}^n$). Show that the function $u(\mathbf{x}) = r^{2-n}$, $n > 2$, is a solution to the $n-$dimensional Laplace equation

$$u''_{x_1x_1} + u''_{x_2x_2} + \cdots + u''_{x_nx_n} = 0\,, \qquad |\mathbf{x}| \neq 0$$

8. Find the most general solution to the 2D Laplace equation $u''_{xx} + u''_{yy} = 0$ that has the form

$$u(x,y) = f(s)\,, \quad s = x^2 + y^2$$

where f is a twice continuously differentiable function. Consider two cases:
(i) A solution is sought in an open region containing the origin;
(ii) A solution is sought in an open region that does not contain the origin.

9. Find the most general homogeneous polynomials in two variables of degree 2 and 3 that are solutions to the 2D Laplace equation, that is,

$$\begin{aligned} u(x,y) &= P_2(x,y) = Ax^2 + By^2 + Cxy\,, \\ u(x,y) &= P_3(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3 \end{aligned}$$

where A, B, C, and D are constants, and $u''_{xx} + u''_{yy} = 0$.

10. (i) Show that, if $u(t,x) = f(s)$, where $s = x - ct$, then the KdV equation is reduced to

$$f''(s) + 3f^2(s) - cf(s) = A$$

where A is a constant.
(ii) Multiply the above equation by $f'(s)$ and show that it can be reduced to the first order equation

$$\frac{1}{2}\,(f')^2 + f^3 - \frac{c}{2}\,f^2 - Af = E$$

where E is a constant.
(iii) Let $c > 0$. Show that the equation in Part (ii) can written in the form

$$\frac{1}{2}\,(f')^2 = -(f-\alpha)(f-\beta)^2\,, \quad \alpha > \beta$$

and, if $\beta = 0$, then $\alpha = \frac{c}{2}$. Solve the equation in this case and find the solitary wave solution given in the text above.

Answers.
1. Yes.
2. In the order of appearance,
$u = a(y,z)x + b(y,z)$;
$u = f(x,z) + g(y,z)$;
$u = a_0(x,y) + a_1(x,y)z + \cdots + a_{n-1}(x,y)z^{n-1}$;
$u = f(x,y) + g(y,z) + h(x,z)$;
$u = (C(y,z) - xy)^{-1}$.
3. $|\mathbf{k}| = 1$.
4. $|\mathbf{a}| = 1$.
5. $k = a$ and the extension is $u(0) = k$, the power series representation

$$u = k\sum_{n=0}^{\infty}(-1)^n\frac{(kr)^{2n}}{(2n-1)!} = k\sum_{n=0}^{\infty}(-1)^n\frac{k^{2n}(x^2+y^2+z^2)^n}{(2n-1)!}$$

has infinite radius of convergence and, hence, $u \in C^\infty(\mathbb{R}^3)$.
8. (i) $f(s) = A$; (ii) $f(s) = B\ln(s) + A$, where A and B are constants.
9. $P_2:\ B = -A$; $P_3:\ C = -3A\,,\ B = -3D$.

4. The method of separating variables

In contrast to ordinary differential equations, there are no general methods for solving PDEs. One can only talk about some particular methods applicable for solving rather limited classes of PDEs. One of such methods is known as *separation of variables*.

Let us illustrate its basic idea using PDEs in two variables x and y. A solution $u(x, y)$ is sought in a special form

$$u(x, y) = X(x)Y(y)$$

After the substitution of such a function into the PDE, one can attempt to reduce the PDE to a system of ordinary differential equations for the functions $X(x)$ and $Y(y)$. Of course, there is no guarantee that a solution of this form even exists. Even if solutions of this form do exist, they may not be all solutions of the PDE.

A multivariable extension of this idea is straightforward. A solution is sought as the product of single-variable functions:

$$u(x_1, x_2, ..., x_n) = X_1(x_1)X_2(x_2) \cdots X_n(x_n)$$

The main technical task is to reduce the PDE in question to a system of n ordinary differential equations for the functions X_j, $j = 1, 2, ..., n$ and solve the system.

4.1. Separation of variables in a 2D heat equation.

Example **4**.**1**. *Find all solutions to the PDE in Example* **3**.**1** *by the method of separation of variables.*

Solution: Put

$$u(t, x) = T(t)X(x)$$

Therefore

$$u'_t(t, x) = T'(t)X(x)\,, \quad u''_{xx}(t, x) = T(t)X''(x)$$

and, hence,

$$T'(t)X(x) = T(t)X''(x) \quad \Rightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

which must hold for all t and x in Ω. This is possible only if the functions in the left and right sides of this equation are constant:

$$\frac{T'(t)}{T(t)} = k \quad \Rightarrow \quad T'(t) = kT(t) \quad \Rightarrow \quad T(t) = Ce^{kt}$$

$$\frac{X''(x)}{X(x)} = k \quad \Rightarrow \quad X(x) = \begin{cases} Ax + B & , \ k = 0 \\ A\cos(mx) + B\sin(mx) & , \ k = -m^2 < 0 \\ Ae^{mx} + Be^{-mx} & , \ k = m^2 > 0 \end{cases}$$

where $m > 0$. Let $u(x,t;k)$ be a solution *obtainable* by separating variables for a particular value of the separation constant k. Since A, B, and C are arbitrary constants, the constant C can be included into A and B in the product $X(x)T(t)$ so that all solutions of this form read:

$$\begin{aligned} u(x,t;0) &= Ax + B\,, \\ u(x,t;m^2) &= \Big(A\cos(mx) + B\sin(mx)\Big)\, e^{-m^2 t}\,, \\ u(x,t;-m^2) &= \Big(Ae^{mx} + Be^{-mx}\Big)\, e^{m^2 t} \end{aligned}$$

where A, B, and $m > 0$ are arbitrary constants. □

This example shows that not all solutions of a PDE can be obtained by separating variables. In particular, the solution in Example **3.1** cannot be represented at the product of two single-variables functions.

4.2. Separation of variables in a 2D wave equation.

EXAMPLE **4.2**. *Find all solutions of the wave equation in Example* **3.2** *by separating variables.*

SOLUTION: Let $u(x,t) = X(x)T(t)$. It follows from the wave equation that

$$X(x)T''(t) - c^2X''(x)T(t) = 0 \quad \Rightarrow \quad \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}$$

The left and right sides of this equation must be constants because the equation holds for any (x,t). Therefore

$$X''(x) = kX(x)\,, \quad T''(t) = c^2kT(t)$$

where k is a constant of separation of variables.

Let $u(x,t;k)$ denote a solution obtainable by separating variables for a particular value of the separation constant. Then setting $k = 0$, $k = -m^2 < 0$, and $k = m^2 > 0$ as before (with $m > 0$), the solutions *obtainable* by separating variables are found:

$$\begin{aligned} u(x,t;0) &= (Ax + B)(Ct + D)\,, \\ u(x,t;-m^2) &= \Big(A\cos(mx) + \sin(mx)\Big)\Big(C\cos(cmt) + D\sin(cmt)\Big)\,, \\ u(x,t;m^2) &= \Big(Ae^{mx} + Be^{-mt}\Big)\Big(Ce^{cmt} + De^{-cmt}\Big)\,. \end{aligned}$$

where A, B, C, and D are arbitrary constants. □

Note that the solutions of the wave equation obtained by separating variables have the form given in Example **3.2**. Indeed, if $k = 0$, then

$$\begin{aligned} u(x,t) &= (Ax+B)(Ct+D) \\ &= A'\Big((x+ct)^2 - (x-ct)^2\Big) + B'(x+ct) + C'(x-ct) + D' \end{aligned}$$

where A', B', C', and D' can be expressed in terms of A, B, C, and D. Similarly, using some basic trigonometric identities for products of the sine and cosine functions and basic properties of the exponential function, a solution with $k \neq 0$ can be written in the form

$$\begin{aligned} u(x,t) &= A'\cos(x+ct) + B'\sin(x+ct) \\ &\quad + C'\cos(x-ct) + D'\sin(x-ct) \\ u(x,t) &= A'e^{x+ct} + B'e^{-x-ct} + C'e^{x-ct} + D'e^{-x+ct} \end{aligned}$$

The converse is obviously false. There are solutions to the wave equation that do not coincide with any of the above as was shown in Example **3.2**. Thus, the method of separation of variables does not produce all solutions to the 2D wave equation.

4.3. Limitations of the method. It is also noteworthy that the method does not generally works for non-homogeneous linear PDEs. For example, an attempt to separate variables in a non-homogeneous 2D wave equation

$$u''_{tt} - c^2 u''_{xx} = f(x,t)$$

fails for a generic inhomogeneity because

$$\begin{aligned} T''(t)X(x) - c^2T(t)X''(x) &= f(x,t) \\ \frac{T''(t)}{T(t)} - c^2\frac{X''(x)}{X(x)} &= \frac{f(x,t)}{T(t)X(x)} \end{aligned}$$

The separation of variables requires that the ratio in the right-hand site must be the sum of two single-variable functions

$$\frac{f(x,t)}{T(t)X(x)} = h(x) + g(t)$$

for some $h(x)$ and $g(t)$. This is possible if $f(x,t)$ is either proportional to $T(t)$ or $X(x)$, or more generally

$$f(x,t) = h(x)T(t) + g(t)X(x)$$

for some functions $h(x)$ and $g(t)$. This is not possible in general.

For example, put

$$f(x,t) = \sin(tx)\,.$$

If the above representation of f exists for all t and x, then for two particular values of x, e.g., $x = 1$ and $x = 2$, the functions $T(t)$ and $g(t)$ must satisfy the linear system

$$\begin{aligned} h_1 T(t) + X_1 g(t) &= \sin(t) \\ h_2 T(t) + X_2 g(t) &= \sin(2t) \end{aligned}$$

where $h_n = h(n)$ and $X_n = X(n)$, $n = 1, 2, \ldots$. The system has a unique solution if its determinant $h_1 X_2 - h_2 X_1$ is not equal to zero. In this case, the functions $T(t)$ and $g(t)$ are linear combinations of $\sin(t)$ and $\sin(2t)$. But for $x = 3$, their linear combination must be equal to $\sin(3t)$:

$$h_3 T(t) + X_3 g(t) = \sin(3t)$$

for all t, which is not possible because the functions $\sin(t)$, $\sin(2t)$, and $\sin(3t)$ are linearly independent. Thus, the determinant must be equal to zero.

The argument can be repeated for any three distinct values of x. Owing to the linear independence of $\sin(x_1 t)$, $\sin(x_2 t)$, and $\sin(x_3 t)$, it is concluded that for any two x_1 and x_2

$$h(x_1)X(x_2) - h(x_2)X(x_1) = 0$$

Since X is not identically zero, the function $h(x)$ must be proportional to $X(x)$:

$$h(x) = AX(x)$$

for some constant A. Repeating the same argument for three values of t, it is inferred that the function $g(t)$ must be proportional to $T(t)$:

$$g(t) = BT(t)$$

for some constant B. Therefore

$$\sin(xt) = CT(t)X(x)$$

where $C = A + B$. The above relation must hold for all x and t. Put $t = 1$. Then $X(x)$ is proportional to $\sin(x)$. Put $x = 1$. Then $T(t)$ is proportional to $\sin(t)$. Then

$$\sin(tx) = k \sin(t) \sin(x)$$

for some constant k and all x and t. The latter cannot be true because it would mean that $\sin(2t)$ is proportional to $\sin(t)$ (if $x = 2$), which is false.

It should be emphasized that a solution to the equation do exist and its explicit form will be found later. The analysis shows that the solution cannot be obtained by the method of separating variables (it does not have the form $u(x,t) = X(x)T(t)$).

4.4. Separation of variables in higher dimensions. If a PDE has more than two variables, solutions obtained by separating variables may contain more than one separating constants. For example, let us obtain all solutions of the 3-dimensional Laplace equation:

$$\Delta u(x,y,z) = u''_{xx}(x,y,z) + u''_{yy}(x,y,z) + u''_{zz}(x,y,z) = 0$$

The *differential operator* Δ is called the *Laplacian.* Put

$$u(x,y,z) = X(x)Y(y)Z(z)$$

The substitution of this function into the Laplace equation yields

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

Since each term is a function of an independent variable, their sum can be equal to zero for all (x,y,z) if and only if each term is constant, that is,

$$\frac{X''(x)}{X(x)} = a\,, \quad \frac{Y''(y)}{Y(y)} = b\,, \quad \frac{Z''(z)}{Z(z)} = c\,, \quad a+b+c=0$$

Solutions of these ordinary differential equations are readily obtained (see Example **4.2**). Note that only of the three separation constants are independent. One can take a and b to be arbitrary, whereas $c = -a-b$. The separation constants a, b, and c cannot have the same sign. So, the analysis can be carried out for the four possible cases of signs of the independent parameters a and b. For example, if $a = -\alpha^2 < 0$ and $b = -\beta^2 < 0$, then $c = \alpha^2 + \beta^2 = \gamma^2 > 0$ and, in this case,

$$\begin{aligned} X(x) &= A\cos(\alpha x) + B\sin(\alpha x) \\ X(x) &= C\cos(\beta y) + D\sin(\beta y) \\ Z(z) &= Fe^{\gamma z} + He^{-\gamma z}\,, \quad \gamma = \sqrt{\alpha^2+\beta^2} \end{aligned}$$

The reader is advised to find an explicit form of all such solutions as an exercise.

4.5. The superposition principle and separation of variables. Suppose a homogeneous linear PDE admits separation of variables. Then a linear combination of such solutions with *different* separation constants is also a solution to the PDE. *This principle allows us to obtain new solutions by means of separating variables.*

Let k stand for a separation constant (or a collection of all separation constants) and $u(\mathbf{x};k)$ is a solution to a linear PDE obtained by separating variables corresponding to a particular value of k. Then the most general solution obtainable by separating variables is a linear combination of such solutions with distinct values of k:

$$\begin{aligned} u(\mathbf{x}) &= c_1 u(\mathbf{x}; k_1) + c_2 u(\mathbf{x}; k_2) + \cdots + c_n u(\mathbf{x}; k_n) \\ &= \sum_{j=1}^{n} c_j u(\mathbf{x}; k_j) \end{aligned}$$

where c_j, $j = 1, 2, ..., n$, are arbitrary constants.

For example, the function

$$u(x,t) = A\cos(mx)e^{-m^2 t} + Ce^{-px}e^{p^2 t}$$

is a solution to the heat equation in Example **3.1**. As shown in Example **4.1**, two terms in this solution are solutions obtained by separating of variables. The first term corresponds to the case of a negative separation constant $k = -m^2 < 0$ and the second one to a positive separation constant $k = p^2 > 0$. Similarly,

$$u(x,t) = \sum_{l=1}^{L} \Big(A_l \cos(m_l x) + B_l \sin(m_l x)\Big) e^{-m_l^2 t}$$

is also a solution to the heat equation as a linear combination of solutions corresponding to different negative separation constants $k = -m_l^2$, $l = 1, 2, ..., L$.

Similarly, applying the superposition principle to the 2D wave equation, one infers, for example, that a linear combination

$$\begin{aligned} u(x,t) &= \sum_{l=1}^{L} c_l u(x,t; -m_l^2) \\ &= \sum_{l=1}^{L} \cos(m_l ct)\Big(A_l \cos(m_l x) + B_l \sin(m_l x)\Big) \\ &\quad + \sum_{l=1}^{L} \sin(m_l ct)\Big(C_l \cos(m_l x) + D_l \sin(m_l x)\Big) \end{aligned}$$

is a solution to the equation, where A_l, B_l, C_l, D_l, and $m_l > 0$ are parameters of this solution.

4.6. Exercises.

1. Find all solutions to the 2D Laplace equation

$$u''_{xx}(x,y) + u''_{yy}(x,y) = 0\,, \quad u \in C^2(\mathbb{R}^2)$$

obtainable by separating variables.

2. Find all solutions to the 3D Laplace equation

$$u''_{xx}(x,y,z) + u''_{yy}(x,y,z) + u''_{zz}(x,y,z) = 0\,, \quad u \in C^2(\mathbb{R}^3)$$

obtainable by separating variables.

3. Find all solutions to the 2D Helmholtz equation

$$u''_{xx}(x,y) + u''_{yy}(x,y) + a^2 u(x,y) = 0\,, \quad u \in C^2(\mathbb{R}^2)$$

obtainable by separating variables.

4. Find all solutions to the wave equation

$$u''_{tt} - c^2(u''_{xx}(t,x,y) + u''_{yy}(t,x,y)) = 0\,, \quad u \in C^2(\mathbb{R}^3)$$

obtainable by separating variables that cannot grow unboundedly with increasing or decreasing time t, that is, if $u(t,x,y) = T(t)v(x,y)$, then

$$|T(t)| \le M\,, \quad t \in \mathbb{R}$$

5. Find all solutions to the first-order PDE

$$u'_x - cu'_y + xu = 0$$

obtainable by separating variables.

6. Find all solutions to the telegraph equation

$$u''_{tt} + 2\gamma u'_t - c^2 u''_{xx} = 0\,, \quad t > 0\,,\ x \in \mathbb{R}$$

where $\gamma > 0$ that are bounded in time $t > 0$, that is, $u(t,x) = T(t)X(x)$ and $|T(t)| \le M$ for all $t > 0$. The telegraph equation describes propagation of electrical signals in a power line, where the parameter γ models Ohmic losses.

Selected answers.

1. $u(x,y) = (Ax + B)(Cy + D)$;
$u(x,y) = (A\cos(mx) + B\sin(mx))(Ce^{my} + De^{-my})$;
$u(x,y) = (A\cos(my) + B\sin(my))(Ce^{mx} + De^{-mx})$
and any linear combinations of the above solutions with different values of the parameters.

3. If $u(x,y) = X(x)Y(y)$, then the following cases are possible
$X(x) = Ax + B$, $Y(y) = C\cos(ay) + D\sin(ay)$;
$X(x) = Ae^{mx} + Be^{-mx}$, $Y(y) = C\cos(\gamma y) + D\sin(\gamma y)$, $\gamma = \sqrt{m^2 + a^2}$;
$X(x) = A\cos(mx) + B\sin(mx)$, $Y(y) = Ce^{\gamma y} + De^{-\gamma y}$,
$\gamma = \sqrt{m^2 - a^2}$, $m > a$;

$X(x) = A\cos(mx) + B\sin(mx)$, $Y(y) = C\cos(\gamma y) + D\sin(\gamma y)$,
$\gamma = \sqrt{a^2 - m^2}$, $a > m > 0$;
$u(x, y) = X(y)Y(x)$ for any of the above pairs of functions X and Y; any linear combinations of the above solutions with different values of the parameters.
4. $(A\cos(act) + B\sin(act))v(x, y)$ where $v(x, y)$ is any solution from Problem 3.
5. $u(x, y) = Ae^{-x^2/2}e^{k(y+cx)}$

5. The method of changing variables

5.1. Curvilinear coordinates. Consider a transformation of a region $\Omega \subset \mathbb{R}^2$ to a new region $\Omega' \subset \mathbb{R}^2$

$$T: \ \Omega \to \Omega'$$

defined by equations

$$\xi = \xi(x, y)\,, \quad \eta = \eta(x, y)\,, \quad (x, y) \in \Omega$$

where the functions $\xi(x, y)$ and $\eta(x, y)$ are from the class $C^1(\Omega)$. If the *Jacobian* of this transformation

$$J(x, y) = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} \neq 0\,, \quad (x, y) \in \Omega$$

does not vanish in an open region Ω, then the transformation is called *a change of variables* and (ξ, η) are called *new coordinates* in Ω. One can show that the transformation that defines a change of variables is *one-to-one*. In other words, one can find the *inverse transformation*, $x = x(\xi, \eta)$, $y = y(\xi, \eta)$. The latter functions of ξ and η are such that

$$x\Big(\xi(x, y), \eta(x, y)\Big) = x\,, \quad y\Big(\xi(x, y), \eta(x, y)\Big) = y$$

for all $(x, y) \in \Omega$. If the transformations is not linear, the the new coordinates are also called *curvilinear*.

Coordinate grid. Point sets in Ω on which the new coordinates take a constant value are smooth curves:

$$\xi(x, y) = \xi_0\,, \quad \eta(x, y) = \eta_0$$

Indeed, they are level sets of functions that have continuous partial derivatives and whose gradient does not vanish in Ω because $J \neq 0$. By the properties of the gradient (recall Multivariable Calculus), the level sets of such functions are smooth curves and the gradients $\boldsymbol{\nabla}\xi$ and $\boldsymbol{\nabla}\eta$ are perpendicular to the corresponding level curves. They are called *coordinate curves*. At any point in Ω, the gradients are not parallel because the Jacobian is not zero (note that $J = 0$ implies that the vectors $\boldsymbol{\nabla}\xi$ and $\boldsymbol{\nabla}\eta$ are proportional and, hence, parallel). Thus, each point of Ω is a point of intersection of two coordinate curves. The coordinate curves form a *coordinate grid* in Ω, just like vertical and horizontal lines form the coordinate grid of rectangular coordinates in a plane.

Any PDE can be restated in the new variables using the chain rule for partial derivatives:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\,, \qquad \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}$$

Higher order derivatives are obtained by repetitive application of these rules. It can happen that with a suitable choice of new (curvilinear) coordinates, a PDE in question is reduced to a simpler equation that can be solved. The idea is illustrated below with the example of a 2D wave equation from Example **3.2**. It is also possible to seek a solution of a PDE by separating variables in the new curvilinear coordinates:

$$u(x, y) = \Phi(\xi)\Theta(\eta)$$

This concept is illustrated below by separating variables in the 2D Laplace equation in polar coordinates.

It is also worth noting that the concept of solving a PDE in new curvilinear coordinates can be extended to any number of variables. For example, spherical or cylindrical coordinates can be used to solve PDEs in $\mathbb{R}^3$ (e.g., by separating variables in these coordinates). This will be studied later.

5.2. General solution to the 2D wave equation. Let us define new variables via a linear transformation:

$$x_+ = x + ct\,, \quad x_- = x - ct$$

so that by the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial x_+}{\partial x}\frac{\partial}{\partial x_+} + \frac{\partial x_-}{\partial x}\frac{\partial}{\partial x_-} = \frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-}$$
$$\frac{\partial}{\partial t} = \frac{\partial x_+}{\partial t}\frac{\partial}{\partial x_+} + \frac{\partial x_-}{\partial t}\frac{\partial}{\partial x_-} = c\frac{\partial}{\partial x_+} - c\frac{\partial}{\partial x_-}$$

Hence, the second partial derivatives have the form

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-}\right)^2 = \frac{\partial^2}{\partial x_+^2} + 2\frac{\partial^2}{\partial x_+ \partial x_-} + \frac{\partial^2}{\partial x_-^2}$$
$$\frac{\partial^2}{\partial t^2} = \left(c\frac{\partial}{\partial x_+} - c\frac{\partial}{\partial x_-}\right)^2 = c^2\frac{\partial^2}{\partial x_+^2} - 2c^2\frac{\partial^2}{\partial x_+ \partial x_-} + c^2\frac{\partial^2}{\partial x_-^2}$$

Therefore the wave equation in the new variables assumes the form

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = -4c^2\frac{\partial^2 u}{\partial x_+ \partial x_-} = 0$$

The latter equation is not difficult to solve. Indeed,

$$\frac{\partial^2 u}{\partial x_+ \partial x_-} = \frac{\partial}{\partial x_+}\left(\frac{\partial u}{\partial x_-}\right) = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x_-} = h(x_-)$$

for some single-variable function h. Therefore

$$\begin{aligned} u(x,t) &= \int h(x_-)dx_- + f(x_+) = g(x_-) + f(x_+) \\ &= g(x-ct) + f(x+ct) \end{aligned} \tag{5.1}$$

where g is an antiderivative of h.

It is noteworthy that this method allows us to find the *general* solution to a 2D homogeneous wave equation. In other words, it was proved that *any solution to a 2D homogeneous wave equation can be written in the form (***5.1***)* with a suitable choice of f and g. Of course, the functions f and g must be twice differentiable in order for the equation to make sense for any pair (x,t).

Remark. One might wonder how to come up with new variables that would allow one to integrate a PDE? The studied 2D wave equation is a particular case of the so-called *hyperbolic equations*. A suitable change of variables to integrate a 2D hyperbolic equation can be found by *the method of characteristics*. This method and a classification of PDEs will be discussed below in detail.

5.3. Separating variables in polar coordinates. Polar coordinates in a plane are defined by the transformation:

$$x = r\cos(\varphi)\,, \quad y = r\sin(\varphi)$$

Any function $u(x,y)$ becomes *periodic* in the polar angle φ:

$$u(x,y) = u(r\cos(\varphi), r\sin(\varphi)) = U(r,\varphi) = U(r,\varphi+2\pi)\,.$$

A solution to a PDE in an open set of a plane can be sought in the form

$$u(x,y) = R(r)\Phi(\varphi)\,, \quad \Phi(\varphi+2\pi) = \Phi(\varphi)$$

The procedure is called a separation of variables in polar coordinates. The idea can obviously be extended to other coordinates in a plane or in any Euclidean space.

All partial derivatives in a PDE should be expressed in the new variables by means of the chain rule. For example, in polar coordinates

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} = \cos(\varphi)\frac{\partial}{\partial r} - \frac{\sin(\varphi)}{r}\frac{\partial}{\partial \varphi}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} = \sin(\varphi)\frac{\partial}{\partial r} + \frac{\cos(\varphi)}{r}\frac{\partial}{\partial \varphi}$$

where

$$\begin{aligned} r = \sqrt{x^2+y^2} \quad &\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos(\varphi) \\ &\Rightarrow \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin(\varphi) \\ \varphi = \arctan\left(\frac{y}{x}\right) \quad &\Rightarrow \quad \frac{\partial \varphi}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin(\varphi)}{r} \\ &\Rightarrow \quad \frac{\partial \varphi}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos(\varphi)}{r} \end{aligned}$$

In particular, the Laplace equation in two variables assumes the following form in polar coordinates

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \varphi} = 0\,, \quad r > 0\,.$$

Note that the equation does not make sense for $r = 0$. The point with $r = 0$ is the origin $(x, y) = (0, 0)$. Recall that the Jacobian of polar coordinates is equal to r. So, the singular point $r = 0$ in the Laplace operator is associated with the zero of the Jacobian. So, the transformation makes sense in any open region of the plane that does not include the origin.

This is not a coincidence, but rather a common feature for any curvilinear coordinates whose Jacobian vanishes at some points. In other words, partial derivatives are generally singular at the points where the Jacobian vanishes and a PDE has no meaning at these points. Solutions can be found only in regions that do not contain the singular points. Then one should investigate if *the obtained solution can be extended to singular points by taking the limits of the solution at the singular points*. If the latter is true, then the obtained solution can be expressed in the original (rectangular) variables and it would be from the required smoothness class.

If the 2D Laplace equation is to be solved in any open set Ω that contains the origin, then in polar coordinates, a solution is sought first in an open set $\Omega' = \Omega \setminus \{(0, 0)\}$ where the origin is excluded, meaning that $r > 0$. After solving the equation, one has to investigate which of the found solutions have suitable extensions to the origin. For example, if one seeks solutions to the Laplace equation, then any such solution must be from the class $C^2(\Omega)$. Clearly, not every function from the class $C^2(\Omega')$ would have a continuous extension to the origin together with its partial derivatives up to order 2. The procedure is illustrated by the following example.

EXAMPLE **5.1**. *Find all solutions to the Laplace equation in an open set Ω of a plane by separating variables in polar coordinates. Discuss the cases when Ω includes the origin and when Ω does not contain the origin.*

SOLUTION: A solution is sought in the form $u(x,y) = R(r)\Phi(\varphi)$:

$$\frac{1}{r}\Big(rR'(r)\Big)'\Phi(\varphi) + \frac{1}{r^2}R(r)\Phi''(\varphi) = 0\,, \quad r > 0$$

Multiplying this equation by r^2 and dividing by Φ, one infers that

$$\frac{r(rR'(r))'}{R(r)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0$$

The two terms in the left side of this equation must be constant because they are function of different variables:

$$\frac{r(rR'(r))'}{R(r)} = k\,, \quad \frac{\Phi''(\varphi)}{\Phi(\varphi)} = -k$$

where k is the separation constant. The general form of Φ is easy to find:

$$\Phi(\varphi) = \begin{cases} A\cos(m\varphi) + B\sin(m\varphi) & ,\ k = m^2\,,\ m > 0\,, \\ Ae^{m\varphi} + Be^{-m\varphi} & ,\ k = -m^2\,,\ m > 0 \\ A + B\varphi & ,\ k = 0 \end{cases}$$

Not all solutions are periodic, $\Phi(\varphi + 2\pi) = \Phi(\varphi)$. If $k = m^2 > 0$, then m must be an integer. In this case, it is sufficient to take only positive integers, $m = 1, 2, ...$, because A and B are arbitrary constants and no new (linearly independent) solutions correspond to negative integers, $m = -1, -2, ...$ If $k = -m^2 < 0$, then the periodicity condition cannot be satisfied for any real m. If $k = 0$, then one has to set $B = 0$ to obtain a periodic solution (any constant function is periodic).

Thus, it follows from the periodicity $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ that

$$\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi)\,, \quad k = m^2\,,\ m = 0, 1, 2, \dots\,,$$

The equation for the function $R(r)$ reads

$$\frac{r(rR')'}{R} = m^2 \quad \Rightarrow \quad r^2R'' + rR' - m^2R = 0\,, \quad r > 0$$

This is an *equidimensional equation*. Its solutions are sought is the form $R = r^\alpha$. Taking the derivatives of the power function, the *characteristic equation* for α is found:

$$\alpha(\alpha - 1) + \alpha - m^2 = 0 \quad \Rightarrow \quad \alpha^2 - m^2 = 0$$

It has two distinct real roots $\alpha = \pm m$ if $m \neq 0$ and the real root $\alpha = 0$ of multiplicity 2 if $m = 0$. According to the theory of equidimensional equations, this means that a general solution has the form

$$R(r) = \begin{cases} Cr^m + Dr^{-m} & , \ m = 1, 2, ... \\ C + D\ln(r) & , \ m = 0 \end{cases}$$

where C and D are arbitrary constants. Since the Laplace equation is linear, its most general solution that is obtainable by separating variables in polar coordinates is a linear combination of solutions

$$\begin{aligned} u_m(x, y) &= \Big(A\cos(m\varphi) + B\sin(m\varphi)\Big)\Big(Cr^m + Dr^{-m}\Big) \\ u_0(x, y) &= A\Big(C + D\ln(r)\Big)\,, \end{aligned}$$

where A, B, C, and D are arbitrary constants.

The Laplace equation is linear and homogeneous. Therefore by the superposition principle, a general solution obtainable by separating variables in polar coordinates is a linear combination of the above solutions:

$$\begin{aligned} u(x, y) = A_0 &+ \sum_{m=1}^{M} \Big(A_m\cos(m\varphi) + B_m\sin(m\varphi)\Big)r^m \\ &+ C_0\ln(r) + \sum_{m=1}^{M} \Big(C_m\cos(m\varphi) + D_m\sin(m\varphi)\Big)r^{-m}\,, \end{aligned}$$

where A_m, B_m, C_m, and D_m are arbitrary constants.

The origin excluded. It is important to observe that the limit of u as $r \to 0^+$ does not exist if the coefficients C_m and D_m are not equal to zero because $\ln(r)$ and r^{-m} become infinite in this limit. Therefore the solution has no continuous extension to the origin $r = 0$, unless $C_m = D_m = 0$ for all m, and can only be used in any open set Ω which *does not* contain the origin $(x, y) = (0, 0)$. In this case, the function $u(x, y)$ has continuous partial derivatives of any order in Ω:

$$(0,0) \notin \Omega \quad \Rightarrow \quad u \in C^\infty(\Omega) \subset C^2(\Omega)$$

The origin included. If the origin belongs to Ω, then in order for $u(x, y)$ to have a continuous extension to the origin and, hence, to be continuous on Ω, it is necessary and sufficient that $C_m = D_m = 0$ for all m. In this case, the function $u(x, y)$ also have continuous partial derivatives of any order in Ω:

$$u(x,y) = A_0 + \sum_{m=1}^{M} \Big(A_m \cos(m\varphi) + B_m \sin(m\varphi)\Big) r^m \,, \quad (0,0) \in \Omega \,.$$

$\square$

It is worth noting that the solution found in the above example for an open set that includes the origin are polynomials in x and y. They are known as *harmonic polynomials in two variables.* This can be seen by means of trigonometric identities. For example,

$$\begin{aligned} \sin(2\varphi) r^2 &= 2\sin(\varphi)\cos(\varphi) r^2 = 2xy \\ \cos(3\varphi) r^3 &= \Big(\cos^3(\varphi) - 3\sin^2(\varphi)\cos(\varphi)\Big) r^3 = x^3 - 3xy^2 \end{aligned}$$

and similarly for any terms $\cos(m\varphi) r^m$ and $\sin(m\varphi) r^m$. The corresponding trigonometric identities can be deduced by means of the Euler formula

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

and binomial expansion:

$$\begin{aligned} \cos(m\varphi) &= \operatorname{Re} e^{im\varphi} = \operatorname{Re}\Big(\cos(\varphi) + i\sin(\varphi)\Big)^m , \\ \sin(m\varphi) &= \operatorname{Im} e^{im\varphi} = \operatorname{Im}\Big(\cos(\varphi) + i\sin(\varphi)\Big)^m , \end{aligned}$$

Harmonic functions. Any function that satisfies the Laplace equation in a region Ω is also called a *harmonic function* in Ω. In particular, the found polynomials are harmonic functions in the whole plane $\mathbb{R}^2$. If C_m and D_m are not equal to zero, then the found solution is a harmonic function in any open region that does not contain the origin. Harmonic functions will be studied later in detail.

5.4. Exercises.

1. (i) Use a linear change of variables to solve the equation

$$au'_x(x,y) + bu'_y(x,y) = 0$$

(ii) Separate variables in the equation and find the most general solution obtainable by this method. Compare the solution with the solution from Part (i). Are they the same?
Hint: find a suitable linear transformation

$$\xi = \alpha_1 x + \alpha_2 y\,, \quad \eta = \beta_1 x + \beta_2 y$$

where $\alpha_{1,2}$ and $\beta_{1,2}$ are parameters, such that the equation is reduced to $u'_\xi = 0$. Recall that a linear transformation is change of variables if its Jacobian does not vanish.

2. Use the method of changing variables to find the general solution to the equation

$$u'_x(x,y) + 3u'_y(x,y) = (x+y)^{p-1}\,, \quad p > 1\,, \quad \Omega:\ x+y > 0$$

Can this equation be solved by separating variables x and y? Support your answer by reasonings.
Hint: Use the hint for the previous problem.

3. Find the general solution to the 2D Laplace equation in the complement Ω of the disk of radius a centered at the origin, $\Omega:\ x^2+y^2 > a^2$, that is obtainable by separating variables in polar coordinates and
(i) that is bounded in Ω, that is, $|u(x,y)| \le M$ for some constant M and all $(x,y) \in \Omega$;
(ii) that vanishes at infinity, that is $u \to 0$ as $(x,y) \to \infty$.

4. Find the general solution to the 2D Laplace equation in the complement Ω of the disk of radius a centered at the origin, $\Omega:\ x^2+y^2 > a^2$, that is obtainable by separating variables in polar coordinates and
(i) whose gradient is bounded in Ω, that is, $|\nabla u(x,y)| \le M$ for some constant M and all $(x,y) \in \Omega$.
(ii) whose gradient vanishes at infinity, that is $|\nabla u| \to 0$ as $(x,y) \to \infty$.
Hint: Use the relation between partial derivatives to show that the squared length of the gradient vector has the following form in polar coordinates:

$$|\nabla u|^2 = (u'_x)^2 + (u'_y)^2 = (u'_r)^2 + \frac{(u'_\varphi)^2}{r^2}$$

Conclude that the gradient is bounded if and only if u'_r and u'_φ/r are bounded in Ω. Use the explicit form of the general solution to calculate the derivatives and analyze conditions under which they are bounded.

5. Use the change of variables $\eta = x+y$, $\gamma = x-y$, to find a general solution to the following first-order PDE:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (x-y)u^2$$

Answers.
1. $u(x,y) = f(s)$, where $s = bx - ay$ and f is a differentiable function.
2. $u(x,y) = (x+y)^p/(4p) + f(s)$, where $s = 3x - y$ and f is a differentiable function. If p is not an integer, then the solution cannot be written in the form $X(x)Y(y)$ or as a linear combinations of such products.
3. (i)

$$u(x,y) = A_0 + \sum_{m=1}^{M} \Big(C_m \cos(m\varphi) + D_m \sin(m\varphi)\Big) r^{-m} \,,$$

where A_0, C_m, and D_m are arbitrary constants.
(ii) $A_0 = 0$ in the above solution.
4. (i)

$$\begin{aligned} u(x,y) = {} & A_0 + r\Big(A_1\cos(\varphi) + B_1 \sin(\varphi)\Big) + C_0 \ln(r) \\ & + \sum_{m=1}^{M} \Big(C_m \cos(m\varphi) + D_m \sin(m\varphi)\Big) r^{-m} \,, \end{aligned}$$

where $A_{0,1}$, B_1, C_m, and D_m are arbitrary constants.
(ii) $A_1 = B_1 = 0$ in the above solution.
5. $u(x,y) = f(x-y) + \frac{1}{2}(y^2 - x^2)$ where f is any continuously differentiable function of one real variable.

6. Complex-valued solutions to linear PDEs

6.1. Functions of complex variables. Any point of a plane can be views as an ordered pair of real numbers $(x, y) \in \mathbb{R}^2$. The elements of $\mathbb{R}^2$ can be added and multiplied by a real number. Note that elements of a Euclidean space cannot be multiplied so that the product of two elements is an element of the Euclidean space and the multiplication is associative and commutative just as for real numbers. It turns out, an associative and commutative multiplication exists only in $\mathbb{R}^2$ and does not exist in higher dimensional Euclidean spaces. If in addition to addition of elements of $\mathbb{R}^2$, a (commutative and associative) multiplication is defined, then the space is called a *complex plane*. With every $(x, y) \in \mathbb{R}^2$, one can associated a *complex number* $z = x + iy$ where the symbol i is postulated to have the following algebraic property $i^2 = -1$ relative to multiplication. The numbers $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are called the *real* and *imaginary* parts of the *complex number* $z = x + iy$. Two complex numbers are added according to the usual vector rule

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

that is, by adding the corresponding components or real and imaginary parts. Two complex numbers are said to be equal $z_1 = z_2$ if $x_1 = x_2$ and $y_1 = y_2$ (the corresponding components of two vectors are equal), that is, their real and imaginary parts are equal. The product of two complex numbers is defined by the rule

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

One can check that it is commutative, $z_1 z_2 = z_2 z_1$ and associative $(z_1 z_2) z_3 = z_1 (z_2 z_3)$. If uses the ordinary rules of algebra for real numbers with an additional agreement that $i^2 = -1$, then the right side of the above equation follows from its left side. If $\operatorname{Im} z_1 = y_1 = 0$, or z_1 is real, then the above rule coincides with the rule of multiplication of a 2D vector (x_2, y_2) by a real number x_1.

The complex number $\bar{z} = x - iy$ is called the *complex conjugate* of z. The *magnitude* of z is

$$|z| = |\bar{z}| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \geq 0$$

and $|z| = 0$ if and only if $z = 0$. A complex number z^{-1} or $\frac{1}{z}$ is called a *reciprocal* of z if $z^{-1} z = 1$. It is not difficult to verify that

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}$$

Division of a complex number by a non-zero complex number is defined by

$$\frac{z_1}{z_2} = z_2^{-1} z_1 = \frac{\bar{z}_1 z_2}{|z_1|^2}$$

A complex plane is denoted by $\mathbb{C}$. Since there is a one-to-one correspondence between points of $\mathbb{R}^2$ and complex numbers, one says that a plane is spanned by a complex variable z (this is indicated as $\mathbb{R}^2 \sim \mathbb{C}$). Note that the magnitude $|z|$ is nothing by the distance of the point $(x, y) \in \mathbb{R}^2$ from the origin, and $|z_1 - z_2|$ is the distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$. A complex variable z is said to approach z_0, or $z \to z_0$, if the distance $|z - z_0| \to 0$. The condition $|z - z_0| < \delta$ defines an open disk or radius δ centered at z_0. So, open and closed sets in a complex plane are defined in the same way as in $\mathbb{R}^2$. Any collection of complex numbers $\Omega \subset \mathbb{C}$ can always be interpreted as a collection of two dimensional vectors in $\mathbb{R}^2$. In what follows no difference between sets of $\mathbb{C}$ and $\mathbb{R}^2$ will be made. One any set $\Omega \subset \mathbb{C}$ one can defined a function:

$$f: \ \Omega \subset \mathbb{C} \to \mathbb{C}$$

which is *a rule that assigns a unique number $f(z)$ to every complex number $z \in \Omega$*. For example, the rule $f(z) = z^n$, where n is an integer, defines a power function on the whole $\mathbb{C}$ if $n \geq 0$, while for $n < 0$, the power function is not defined at $z = 0$ (the origin). One can define a function of two or more complex variables, e.g., $f(z, \xi)$ as a rule that assign a unique complex number to any pair of complex numbers z and ξ.

Let $z = x + iy$. Then the complex number $f(z)$ also has real and imaginary parts that depend on two real variables (x, y):

$$f(z) = u(x, y) + iv(x, y)$$

The function $f(z)$ has a limit at $z = z_0 = a + ib$ if the functions u and v have the limits as $(x, y) \to (a, b)$. The function $f(z)$ is *continuous at* z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)\,.$$

One says that $f \in C^0(\Omega)$, $\Omega \subset \mathbb{C}$, if f is continuous at every point of Ω. The derivative at a point $z_0 \in \mathbb{C}$ is defined in the same way as for functions of real variable:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the limit exists. In particular, the derivative of a power function reads

$$(z^n)' = nz^{n-1}$$

for any z in its domain.

Functions defined by power series. A sequence of complex numbers z_n, $n = 1, 2, ...,$ is said to converge to a complex number z if

$$\lim_{n\to\infty} |z_n - z| = 0$$

and in this case one write $\lim_{n\to\infty} z_n = z$ or $z_n \to z$ as $n \to \infty$. A sequence of functions $f_n(z)$, $n = 1, 2, ...,$ is said to converge to a function $f(z)$ pointwise on $\Omega \subset \mathbb{C}$ if

$$\lim_{n\to\infty} f_n(z) = f(z)$$

for every $z \in \Omega$. In particular, one can prove that the power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k = \lim_{n\to\infty} \sum_{k=0}^{n} c_k z^k$$

where c_k are complex numbers, converges for every point in the disk

$$|z| < R\,, \quad R = \frac{1}{\alpha}\,, \quad \alpha = \lim_{n\to\infty} \sqrt[n]{|c_n|}$$

provided the limit exists. The number R is called the radius of convergence of the power series. If $\alpha = 0$, then $R = \infty$, which means that the power series converges for all complex z. If $\alpha = \infty$, then $R = 0$, which means that the power series converges only for $z = 0$.

A function f is called *analytic* in an open region $\Omega \subset \mathbb{C}$ if $f(z)$ near any point $z_0 \in \Omega$ its values are given by a power series with a non-zero convergence radius:

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k\,, \quad |z - z_0| < \delta$$

for some $\delta > 0$. Let $f(x)$ be represented by a power series in real variable x:

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

which converges in an interval $-R < x < R$ (R can also be infinite). For example,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,, \quad -\infty < x < \infty$$
$$\sin(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k-1}}{(2k-1)!}\,, \quad -\infty < x < \infty$$
$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k}x^{2k}}{(2k)!}\,, \quad -\infty < x < \infty$$
$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k\,, \quad -1 < x < 1$$

It turns out that any such function can be *extended* to a complex plane by the same power series:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k\,, \quad |z| < R$$

In other words, the series is proved to converge for any z in the disk of radius R in the complex plane. Recall that any function of a real variable represented by a power series is from the class C^∞ in the interval $(-R, R)$ and its derivatives also have power series representations that are obtained term-by-term differentiation of the power series:

$$f'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}\,, \quad -R < x < R$$

and these power series have the same radius of convergence. The Taylor theorem asserts that

$$c_k = \frac{f^{(k)}(0)}{k!}$$

This relation can readily be established by setting $x = 0$ in power series representations for the derivatives $f^{(k)}(x)$. For example $f(0) = c_0$, $f'(0) = c_1$, $f''(0) = 2c_2$, etc. It follows from the power series representation of the derivatives of f that a complex continuation of f to the disk $|z| < R$ is a differentiable function in the disk and

$$f'(z) = \sum_{k=1}^{\infty} kc_k z^{k-1}\,, \quad |z| < R$$

In fact, $f \in C^\infty$.

Holomorphic functions. In complex analysis, a function of a complex variable that is differentiable at every point of its domain is called *holomorphic*. In particular, any analytic function is holomorphic because a function represented by a power series is differentiable. In the theory of functions of complex variables, the converse is proved: *holomorphic functions are analytic*. So, a differentiable function of a complex variable is differentiable infinitely many times.

The exponential and trigonometric functions can be extended to the whole complex plane using their power series. The extensions are differentiable and

$$\Big(e^z\Big)' = e^z\,, \quad \Big(\cos(z)\Big)' = -\sin(z)\,, \quad \Big(\sin(z)\Big)' = \cos(z)\,.$$

that is, they are holomorphic in the whole complex plane $\mathbb{C}$. The extensions have the same algebraic properties. For example,

$$e^{z_1}e^{z_2} = e^{z_1+z_2}\,, \quad \sin(2z) = 2\sin(z)\cos(z)\,, \quad \text{etc.}$$

In particular, it follows from $i^{2k} = (-1)^k$ and $i^{2k+1} = (-1)^k i$ and the power series representation of the exponential and trigonometric functions that

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)\,, \quad \varphi \in \mathbb{R}$$

This relation is known as *the Euler formula*. By taking the complex conjugation of the Euler equation one infers that

$$e^{-i\varphi} = \cos(\varphi) - i\sin(\varphi)$$

and hence

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2}\,, \quad \sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}\,,$$

Using the power series representations for the exponential and trigonometric functions the latter relations can be extended to any complex number z:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}\,, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}\,, \quad z \in \mathbb{C}$$

If $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$, the polar coordinates in a plane, then

$$z = x + iy = r\Big(\cos(\varphi) + i\sin(\varphi)\Big) = re^{i\varphi}\,, \quad r = |z|$$

The angle φ is called a *phase* of a complex number z. This representation of complex numbers is used to define any power of a complex number z:

$$z^p = \Big(re^{i\varphi}\Big)^p = r^p e^{ip\varphi}\,.$$

In particular, the real or imaginary part of any polynomial in a complex variable $z = x + iy = r^{i\varphi}$ has a general form in polar coordinates

$$P_n(x, y) = A_0 + \sum_{m=1}^{n} r^m \Big(A_m \cos(m\varphi) + B_m \sin(m\varphi) \Big)$$

Indeed, for any monomial $Cz^m = Cr^m e^{im\varphi}$ and, hence, if $C = A - iB$,

$$\begin{aligned} \operatorname{Re} Cz^m &= \operatorname{Re}(A - iB) r^m \Big(\cos(m\varphi) + i \sin(m\varphi) \Big) \\ &= r^m \Big(A \cos(m\varphi) + B \sin(m\varphi) \Big) \end{aligned}$$

and similarly for the imaginary part. On the other hand,

$$\operatorname{Re} C(x + iy)^m$$

is a polynomial in two real variables, x and y. For example

$$\begin{aligned} \operatorname{Re}(A - iB)(x + iy)^2 &= \operatorname{Re}(A - iB)(x^2 - y^2 + 2ixy) \\ &= A(x^2 - y^2) + 2Bxy \end{aligned}$$

These polynomials satisfy the Laplace equation. It was shown that such solutions to the Laplace equation can be obtained by separating variables in polar coordinates. Therefore *the real or imaginary parts of any polynomial in a complex variable is a harmonic polynomial in a plane*. Their properties are discussed in detail below.

6.2. Complex-valued functions on $\mathbb{R}^n$. A rule that assigns a unique complex number to any point of a Euclidean space is called a *complex-valued function of several variables*. If $\mathbf{x} \in \mathbb{R}^n$, then $f(\mathbf{x}) \in \mathbb{C}$ is the value of the function f at $\mathbf{x}$. For example, let $\mathbf{k} = (a, b, c) \in \mathbb{R}^3$ be a constant vector in space, then put $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and

$$u(x, y, z) = e^{i(\mathbf{k} \cdot \mathbf{x})} = e^{i(ax+by+cz)}$$

The above rule assigns a unique complex number to each ordered triple of real numbers (x, y, z) and, hence, u is a complex-valued function of three real variables. Any complex-valued functions is uniquely defined by two real-valued functions that are its real and imaginary parts:

$$f(\mathbf{x}) = v(\mathbf{x}) + iw(\mathbf{x}), \quad v(\mathbf{x}) = \operatorname{Re} f(\mathbf{x}), \quad w(\mathbf{x}) = \operatorname{Im} f(\mathbf{x})$$

For example, by the Euler formula

$$e^{i(ax+by+cz)} = \cos(ax + by + cz) + i \sin(ax + by + cz)$$

Partial derivatives of complex-valued functions are defined by the rule

$$Df(\mathbf{x}) = Dv(\mathbf{x}) + iDw(\mathbf{x})$$

For example,

$$\begin{aligned}\frac{\partial}{\partial x} e^{i(ax+by+cz)} &= -a\sin(ax+by+cz) + ia\cos(ax+by+cz)\\ &= iae^{i(ax+by+cz)}\end{aligned}$$

6.3. Complex-valued solutions to PDEs. A complex-valued function u on an open $\Omega \subset \mathbb{R}^n$ is called a solution to a PDE

$$F(\mathbf{x}, u, Du, D^2u, ..., D^pu) = 0\,, \quad \mathbf{x} \in \Omega$$

if the above relation holds for any $\mathbf{x}$. Note that in order for the equation to make sense, the function F must be defined as a function of several complex variables $D^q u$, $q = 0, 1, ..., p$. The latter is always true for linear PDEs:

$$Lu = D^p u + a_{p-1} D^{p-1} u + \cdots + a_1 Du + a_0 u = f$$

where the coefficients a_q and f can be real- or complex-valued functions of $\mathbf{x}$. The linear equation is called *homogeneous* if $f = 0$ and *non-homogeneous* otherwise.

Suppose that the coefficients a_q, $q = 0, 1, ..., p-1$, are real and $f = 0$. Then the real and imaginary parts of any complex-valued solution $u(\mathbf{x})$ are also real-valued solutions of this PDE. This follows from the rule of differentiation of a complex-valued functions

$$D^q u = D^q(v + iw) = D^q v + iD^q w$$

Since the coefficients in the differential operator L are real,

$$Lu = 0 \quad \Leftrightarrow \quad Lv + iLw = 0 \quad \Leftrightarrow \quad \begin{cases} Lv = 0 \\ Lw = 0 \end{cases}$$

because a complex number is zero if and only if its real and imaginary parts vanish. For example, the real and imaginary parts of the monomial $z^m = (x+iy)^m = v(x,y) + iw(x,y)$ are solutions to the Laplace equation: $\Delta v(x,y) = 0$ and $\Delta w(x,y) = 0$, where $L = \Delta$ is the Laplace operator.

Example **6.1**. (Plane waves)
Find a condition on a vector $\mathbf{k} = (k_x, k_y, k_z) \in \mathbb{R}^3$ and a constant ω under which the function

$$u(t, x, y, z) = e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\,,$$

where $\mathbf{x} = (x, y, z)$ is a solution to the wave equation in four-dimensional spacetime

$$u''_{tt} - c^2(u''_{xx} + u''_{yy} + u''_{zz}) = 0\,,$$

where c is a real constant. Find the corresponding real-valued solutions.

Solution: By taking the partial derivatives

$$\begin{aligned} u''_{tt} &= (i\omega)^2 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} = -\omega^2 u \\ u''_{xx} &= (-ik_x)^2 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} = -k_x^2 u \\ u''_{yy} &= (-ik_y)^2 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} = -k_y^2 u \\ u''_{zz} &= (-ik_z)^2 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} = -k_z^2 u \end{aligned}$$

Therefore

$$u''_{tt} - c^2(u''_{xx} + u''_{yy} + u''_{zz}) = \Big(-\omega^2 + c^2(k_x^2 + k_y^2 + k_z^2)\Big)u = 0$$

The latter must hold for any (t, x, y, z). Since $u(t, x, y, z) \neq 0$, this implies that u is a solution if

$$\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) \quad \text{or} \quad \omega = \pm c|\mathbf{k}|$$

The corresponding real valued solutions are

$$\begin{aligned} \operatorname{Re} u &= \cos(\omega t - \mathbf{k}\cdot\mathbf{x}) \\ \operatorname{Im} u &= \sin(\omega t - \mathbf{k}\cdot\mathbf{x}) \end{aligned}$$

as one infers from the Euler equation. Since the wave equation is linear, a linear combination of these solution is also a real solution

$$v = A\cos(\omega t - \mathbf{k}\cdot\mathbf{x}) + B\sin(\omega t - \mathbf{k}\cdot\mathbf{x})$$

by the superposition principle. Put $A = C\cos(\varphi)$ and $B = C\sin(\varphi)$, where $C = (A^2 + B^2)^{1/2}$. Then the solution can also be written in the form

$$v(t, \mathbf{x}) = C\cos(\omega t - \mathbf{k}\cdot\mathbf{x} - \varphi)$$

where C and φ are arbitrary constants. $\square$

Plane waves. The above solution to the 4D wave equation is called a *plane wave*. Note that the amplitude of the wave has a constant value $v = C\cos(\varphi)$ in a plane $\mathbf{k}\cdot\mathbf{x} = \omega t$ for each given moment of time t. The plane is perpendicular to the vector $\mathbf{k}$. With increasing time t, the plane moves in the direction of $\mathbf{k}$ if $\omega = c|\mathbf{k}|$ or in the direction opposite to $\mathbf{k}$ if $\omega = -c|\mathbf{k}|$ with the rate $\omega/|\mathbf{k}| = c$. Recall that the distance between two parallel planes $\mathbf{k}\cdot\mathbf{x} = d_1$ and $\mathbf{k}\cdot\mathbf{x} = d_2$ is $|d_1 - d_2|/|\mathbf{k}| = c|t_1 - t_2|$ because $d_{1,2} = \omega t_{1,2}$. For this reason, the constant c is called the *wave speed* (it can be a speed of sound or speed of light, depending on the physical nature of the function u). The vector $\mathbf{k}$ is called the *wave vector*. It determines the direction in which the plane wave propagates. Its magnitude determines the *wave frequency* $\omega = c|\mathbf{k}|$. A relation between the wave vector and frequency is called the *wave dispersion relation*. The constant φ in the plane wave solution is often called a phase of the wave.

If $v(t, \mathbf{x})$ describes a plane sound wave, then it gives a deviation of the pressure from the background (atmospheric) pressure. Suppose one measures the pressure at a particular point $\mathbf{x}$ in space. Then the observed signal $v = C\cos(\omega t + \varphi')$, where the phase φ' depends on the point of observation, is periodic in time with the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{c|\mathbf{k}|}$$

This is why the constant ω is called the frequency. For each particular moment of time t, the spatial variations of the pressure are also periodic. Indeed, consider any line parallel to the wave vector. If s is the distance along the line counted from a particular point, then the position vector of any point of the line relative to the particular point is $\mathbf{x} = s\hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ is the unit vector parallel to $\mathbf{k}$ so that $\mathbf{k}\cdot\mathbf{x} = |\mathbf{k}||\mathbf{x}| = s\mathbf{k}|$ Then the pressure along the line at a give moment of time is $v = C\cos(s|\mathbf{k} + \varphi'')$ where the phase φ'' depends on time. So, the observed spatial distribution of the signal is periodic with the period

$$\lambda = \frac{2\pi}{|\mathbf{k}|}$$

The parameter λ is called the *wave length*. Thus, the frequency determines periodicity in time, whereas the wave length determines periodicity in space. The frequency and wave length are not independent and related by the wave dispersion relation:

$$\lambda = cT = \frac{2\pi c}{\omega}$$

that is, the wave length is the distance a plane wave travels in one period of its temporal oscillations.

6.4. Harmonic functions in a plane. It turns out that the relation between polynomials of a complex variable and harmonic polynomials is a consequence of a more general relation between holomorphic functions and harmonic functions of two variables.

Consider the Laplace equation in a plane spanned by two real variables (x, y):

$$\textbf{(6.1)} \qquad \Delta u(x,y) = u''_{xx}(x,y) + u''_{yy}(x,y) = 0\,, \quad (x,y)\in\mathbb{R}^2$$

A solution $u(x,y)$ to (**6.1**) is a *harmonic function* in a plane. If

$$u(x,y) = v(x,y) + iw(x,y)$$

is a complex-valued harmonic function, then its real and imaginary parts, $\operatorname{Re} u(x,y) = v(x,y)$ and $\operatorname{Im} u(x,y) = w(x,y)$, are also harmonic functions because of the linearity of the Laplace equation:

$$\Delta(v + iw) = 0 \quad \Leftrightarrow \quad \begin{cases} \Delta v = 0 \\ \Delta w = 0 \end{cases}$$

Let $z = x + iy$ be the complex variable associated with the pair (x, y) of rectangular coordinates. The the complex conjugated number reads and $\bar{z} = x - iy$. One can find the inverse transformation from the pair $(z, \bar{z})$ to (x, y):

$$z = x + iy\,, \;\; \bar{z} = x - iy\,; \qquad x = \frac{1}{2}(z + \bar{z})\,, \;\; y = \frac{1}{2i}(z - \bar{z})$$

A function $u(x, y)$ in a plane can be viewed as a function of new variables z and $\bar{z}$:

$$u(x, y) = u\left(\tfrac{1}{2}(z + \bar{z}), \tfrac{1}{2}(z + \bar{z})\right) = U(z, \bar{z})$$

Similarly, the Laplace operator can written in terms of the new variables z and $\bar{z}$. Using the chain rule for differentiation with respect to a complex variable

$$\frac{df}{dz} = \frac{df}{dw}\,\frac{dw}{dz}\,, \quad w = w(z)\,,$$

the Laplace operator Δ can be written in terms of partial derivatives with respect to z and $\bar{z}$:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial z}\,\frac{\partial z}{\partial x} + \frac{\partial u}{\partial \bar{z}}\,\frac{\partial \bar{z}}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}}\right) = \frac{\partial^2 u}{\partial z^2} + 2\,\frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{\partial^2 u}{\partial \bar{z}^2} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial z}\,\frac{\partial z}{\partial y} + \frac{\partial u}{\partial \bar{z}}\,\frac{\partial \bar{z}}{\partial y} = i\,\frac{\partial u}{\partial z} - i\,\frac{\partial u}{\partial \bar{z}} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial y}\left(i\,\frac{\partial u}{\partial z} - i\,\frac{\partial u}{\partial \bar{z}}\right) = i^2\,\frac{\partial^2 u}{\partial z^2} + 2i(-i)\,\frac{\partial^2 u}{\partial z \partial \bar{z}} + (-i)^2\,\frac{\partial^2 u}{\partial \bar{z}^2} \\
&= -\frac{\partial^2 u}{\partial z^2} + 2\,\frac{\partial^2 u}{\partial z \partial \bar{z}} - \frac{\partial^2 u}{\partial \bar{z}^2}\,.
\end{aligned}$$

Therefore a harmonic function is a solution to the equation

$$\Delta u = 4\,\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0 \tag{6.2}$$

Equation (**6.2**) is easy to solve. Since u has continuous second partial derivatives, the order of partial differentiation does not matter

(Clairaut's theorem) and, hence,

$$\text{either}\quad \frac{\partial u}{\partial z} = F(z) \quad\text{or}\quad \frac{\partial u}{\partial \bar z} = G(\bar z)$$

for some functions F and G of a *single* complex variable. Since the equation is linear, the general solution is given by

$$u(x,y) = f(z) + g(\bar z)\,,\quad z = x+iy\,,$$

where f and g are *differentiable* functions of a single complex variable ($f'(z) = F(z)$ and $g'(\bar z) = G(\bar z)$. In other words, *any solution to the 2D Laplace equation can be obtained from a holomorphic function by taking its real or imaginary part.*

THEOREM **6.1**. (Harmonic functions on a plane)
A general (complex valued) solution to the Laplace equation on a plane

$$\Delta u(x,y) = 0\,,\qquad (x,y)\in\mathbb{R}^2$$

is given by the sum

$$u(x,y) = f(z) + g(\bar z)\,,\qquad z = x+iy$$

where f and g are holomorphic functions on the complex plane.

The Laplace equation is linear. Therefore the real and imaginary parts of a complex valued solution are also solutions (real valued harmonic functions).

COROLLARY **6.1**. *The real and imaginary parts of a holomorphic function $f(z)$, $z = x+iy$, in the complex plane are harmonic functions in the plane:*

$$\begin{array}{ll} v(x,y) = \operatorname{Re} f(z)\,, & w(x,y) = \operatorname{Im} f(z) \\ \Delta v(x,y) = 0\,, & \Delta w(x,y) = 0 \end{array}$$

For example, the functions e^z, $\sin(z)$, and $\cos(z)$ are analytic functions in the complex plane because they are defined by power series with infinite radius of convergence. As a consequence, the real and imaginary parts of these functions are real harmonic functions:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$
$$\Rightarrow\quad \begin{cases} v(x,y) = \operatorname{Re} e^z = e^x\cos(y)\,, & \Delta v = 0 \\ w(x,y) = \operatorname{Im} e^z = e^x\sin(y)\,, & \Delta w = 0 \end{cases}$$

Similarly,

$$v(x,y) = \text{Re}\,\cos(z) = \frac{1}{2}\text{Re}\left(e^{iz}+e^{-iz}\right) = \cosh(y)\cos(x)$$
$$w(x,y) = \text{Im}\,\cos(z) = \frac{1}{2}\text{Im}\left(e^{iz}+e^{-iz}\right) = -\sinh(y)\sin(x)$$

It is worth noting that these solutions to the 2D Laplace equation can also be obtained by separating variables in the rectangular coordinates, $u(x, y) = X(x)Y(y)$.

EXAMPLE **6.2**. *Show that the functions*

$$\begin{aligned} &e^{x^2-y^2}\cos(2xy)\,,\ e^{x^2-y^2}\sin(2xy)\,,\\ &e^{-2xy}\cos(x^2-y^2)\,,\ e^{-2xy}\sin(x^2-y^2) \end{aligned}$$

are harmonic.

SOLUTION: Consider the function $f(z) = e^{z^2}$. It is analytic everywhere in the complex plane because it is represented by a power series with infinite radius of convergence (the power series for e^s with $s = z^2$). Therefore it is holomorphic in the complex plane and its real and imaginary parts are solutions to the Laplace equation:

$$e^{z^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2}\Big(\cos(2xy) + i\sin(2xy)\Big)$$

so that the real and imaginary parts coincide with the first pair of functions in question. The second pair is obtained from the first one by swapping $x^2 - y^2$ and $2xy$ which are real and imaginary parts of z^2. The real and imaginary parts of a complex number are "swapped" (modulo a sign change of the real part) after multiplication by i. The function $f(z) = e^{iz^2}$ is also analytic (by the same argument as before). Therefore its real and imaginary parts solve the Laplace equation:

$$\begin{aligned} iz^2 &= i(x^2-y^2+2ixy) = -2xy + i(x^2-y^2)\\ e^{iz^2} &= e^{i(x^2-y^2)-2xy} = e^{-2xy}\Big(\cos(x^2-y^2) + i\sin(x^2-y^2)\Big) \end{aligned}$$

Of course, the problem can also be solved by taking partial derivatives of the given functions to show that they are indeed solutions to the Laplace equation. □

Remark. It is interesting to note that if in the Laplace equation a formal change of variables is made

$$y = -it\,,$$

then the Laplace equation turns into the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

Their general solutions are related by the same transformation

$$u = f(z) + g(\bar z) = f(x+iy) + g(x-iy) = f(x-t) + g(x+t)$$

as long as the transformation makes sense. For example, the functions f and g in the solution to the wave equation are from $C^2(\mathbb{R})$. If they happen to have an analytic extension to a complex plane, then the stated transformation does make sense. For example, if $f(x)$ is given by a power series in x with infinite radius of convergence, by replacing real x by complex z in the power series an analytic extension of f is obtained. The series for $f(z)$ converges in the whole complex plane.

6.5. Parabolic coordinates. It is clear that the solutions to the 2D Laplace equation in Example **6.2** cannot be obtained by separating variables either in rectangular or polar coordinates. However, they can be obtained by separating variables in *parabolic coordinates*:

$$x = \xi\eta\,, \quad y = \frac{1}{2}\,(\xi^2 - \eta^2)$$

The curves of constant ξ form confocal parabolas concave downward:

$$2y = -\frac{x^2}{\xi^2} + \xi^2$$

whereas the curves of constant η form confocal parabolas concave upward:

$$2y = \frac{x^2}{\eta^2} - \eta^2$$

The parabolas are conic sections with foci being at the origin.

It is left to the reader as an exercise to show that the Jacobian of the parabolic coordinates is

$$J = \xi^2 + \eta^2$$

and the Laplace equation has the form

$$\Delta u = \frac{1}{J}\Big(u''_{\xi\xi} + u''_{\eta\eta}\Big) = 0\,, \quad (\xi,\eta) \neq (0,0).$$

It is now straightforward to see that this equation admits solutions of the form $u(x,y) = \Phi(\xi)\Theta(\eta)$. Any such solution that admits a continuous extension to the singular point $(\xi,\eta) = (0,0)$ is a harmonic function in a plane.

6.6. Harmonic polynomials. Consider the monomials of one complex variables

$$f_l(z) = z^l\,, \qquad l = 0, 1, 2, \ldots$$

They are analytic functions on the complex plane. Then the real and imaginary parts of f_l

$$v_l(x,y) = \mathrm{Re}\,(x+iy)^l\,, \qquad w_l(x,y) = \mathrm{Im}\,(x+iy)^l$$

are polynomial solutions to the Laplace equation on a plane. For every given l, these solutions are *homogeneous* polynomials of degree l, meaning that

$$v_l(sx, sy) = s^l v_l(x, y)\,, \quad w_l(sx, sy) = s^l w_l(x, y)$$

for any real s. Polynomial solutions of the Laplace equation on a plane are called *harmonic polynomials*. Any harmonic polynomial is a linear combination of $v_l(x, y)$ and $w_l(x, y)$ which form a linearly independent set in the space of harmonic polynomials. Note that taking the real and imaginary parts of $\bar{z}^l$ does not produce any new linearly independent homogeneous polynomials because

$$\mathrm{Re}\,\bar{z}^l = v_l(x, -y) = v_l(x, y)\,, \qquad \mathrm{Im}\,\bar{z}^l = w_l(x, -y) = -w_l(x, y)\,.$$

The latter is easy to see by noting that $(iy)^n$ is real if n is even and imaginary if n is odd. So, it follows from the binomial expansion that v_l must be even function of y, while w_l must be odd.

Let us obtain linearly independent harmonic polynomials for a few l using the binomial expansion:

$$\begin{array}{lll}
l = 0: & v_0(x, y) = 1\,, & \\
l = 1: & v_1(x, y) = x\,, & w_1(x, y) = y\,, \\
l = 2: & v_2(x, y) = x^2 - y^2\,, & w_2(x, y) = 2xy\,, \\
l = 3: & v_3(x, y) = x^3 - 3xy^2\,, & w_3(x, y) = 3yx^2 - y^3\,, \\
l = 4: & v_4(x, y) = x^4 - 6x^2y^2 + y^4\,, & w_4(x, y) = 4x^3y - 4xy^3
\end{array}$$

For example, a general harmonic polynomial of degree 2 reads:

$$P_2(x, y) = a_0 + a_1 x + b_1 y + a_2(x^2 - y^2) + b_2 xy\,,$$

where a_0, a_1, a_2, b_1, and b_2 are arbitrary real constants. The harmonic polynomials satisfy the recurrence relation:

$$P_{l+1}(x, y) = P_l(x, y) + a_{l+1} v_{l+1}(x, y) + b_{l+1} w_{l+1}(x, y)\,.$$

Harmonic polynomials in polar coordinates. Consider a homogeneous harmonic polynomial of degree l:

$$p_l(x, y) = A_l \mathrm{Re}\, z^l + B_l \mathrm{Im}\, z^l\,, \quad l \geq 1\,,$$

where A_l and B_l are some constants. In polar coordinates

$$x = r\cos(\theta)\,, \quad y = r\sin(\theta)\,, \quad z = x + iy = re^{i\theta}\,,$$

the polynomial has the form

$$p_l = r^l \Big(A_l \cos(l\theta) + B_l \sin(l\theta) \Big)$$

A harmonic polynomial is a solution to the Laplace equation obtainable by separating variables in polar coordinates as a linear combination of

p_l. On any circle $r = a$, a harmonic polynomial is a linear combination of the trigonometric functions $\cos(l\varphi)$ and $\sin(l\varphi)$. They are called *trigonometric polynomials*, owing to that $\cos(l\varphi)$ and $\sin(l\varphi)$ can expressed as polynomials of $\cos(\varphi)$ and $\sin(\varphi)$ for any integer l.

6.7. Exercises.

1. Calculate the Jacobian and Laplace operator in the parabolic coordinates.

2. Find all solutions to the 2D Laplace equation obtainable by separating variables in the parabolic coordinates.

3. Put $f(z) = e^{z^p}$ where p is a positive integer. Find an explicit form of real and imaginary parts of f. They are solutions to the 2D Laplace equation. Can these solutions be obtained by separating variables in either rectangular, or polar, or parabolic coordinates?

4. Find the most general harmonic polynomial that has a constant value on a circle of radius a centered at the origin.
Hint: Recall that the functions 1, $\cos(m\varphi)$, and $\sin(m\varphi)$, $m = 1, 2, ...$, are linearly independent.

5. Find all non-zero complex-valued solutions to the 2D Helmholtz equation

$$u''_{xx} + u''_{yy} + a^2 u = 0$$

that have the form $u(x, y) = F(z)G(\bar{z})$, where $z = x + iy$, and F and G are holomorphic in the complex plane. Do real and imaginary parts of the solution satisfy the equation?

6. Let $u(x, y)$ and $v(x, y)$ be real-valued functions from the class C^2 on the whole plane and satisfy the conditions:

$$u'_x = v'_y\,, \quad u'_y = -v'_x$$

Show that:
(i) u and v are harmonic functions.
(ii) there exists a holomorphic function f such that $f(x+iy) = u(x, y) + iv(x, y)$. *Hint*: Note that $\partial f(z)/\partial \bar{z} = 0$. State the latter relation in terms of (x, y).

7. Find all polynomial solutions to the 2D equation

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 u}{\partial y^2} = 0$$

Selected answers.
2. $u(x,y) = (A\xi + B)(C\eta + D)$;
$u(x,y) = (A\cos(m\xi) + B\sin(m\xi))(Ce^{m\eta} + De^{-m\eta})$;
$u(x,y) = (A\cos(m\eta) + B\sin(m\eta))(Ce^{m\xi} + De^{-m\xi})$
and any linear combinations of the above solutions with different values of the parameters A, B, C, D, and m.
3. $f = e^{v_p}(\cos(w_p) + i\sin(w_p))$, where v_p and w_p are linearly independent homogeneous harmonic polynomials of degree p that are defined in Section 6.5. The harmonic functions $\operatorname{Re} f$ and $\operatorname{Im} f$ cannot be obtained by any of the said separations of variables if $p \geq 3$.
4. A constant polynomial, $p_0(x,y) = C$.
5. $u(x,y) = A\exp\left(k\bar{z} - \frac{a^2}{4k}z\right)$ where k and A are non-zero complex numbers. Its real and imaginary parts are also solutions to the Helmholtz equation.

7. Boundary conditions

A partial differential equation may have infinitely many solutions. In order for a partial differential equation to have a unique solution, some additional conditions must be imposed on the solutions. Suppose a family of solutions is found for a PDE in an open region Ω. Among all solutions, one seeks a particular one that has some specific properties on the boundary $\partial\Omega$. For example, a solution that has a specified form on $\partial\Omega$ or specified values of partial derivatives on $\partial\Omega$. These additional conditions are called *boundary conditions*. One of the main questions in an analysis of a PDE is:

- *Under what boundary conditions a PDE has a unique solution?*

7.1. An extension of a solution to the boundary. In order to formulate boundary conditions, one has to *extend* a solution and its partial derivatives given in an open set Ω to the boundary $\partial\Omega$ of Ω. Suppose $u(\mathbf{x})$ is a solution to a PDE (**3**.**1**) in an open set Ω. If the solution is required to have a specified value $u_0(\mathbf{y})$ at each point on the boundary $\partial\Omega$, then the function $u(\mathbf{x})$ must have the limit at each $\mathbf{y} \in \partial\Omega$, and this limit must have a specified value:

$$\lim_{\mathbf{x}\to\mathbf{y}} u(\mathbf{x}) = u_0(\mathbf{y})$$

For brevity in what follows, the stated boundary condition will be written in the form

$$u\Big|_{\partial\Omega} = u_0$$

Note that the boundary of Ω does not belong to Ω because Ω is open. Therefore it is not clear if the limit actually exists. If a solution to a PDE of order p has the limit at each point of the boundary, then the solution is said to have a *continuous extension* to the boundary (as defined Sec.**2.4**), and in this case, it is said to be from the class

$$u \in C^p(\Omega) \cap C^0(\bar{\Omega})$$

where $\bar{\Omega}$ is the union of Ω and its boundary. This class consists of all functions that have continuous partial derivatives up to order p in an open Ω, and every function has a continuous extension to the boundary of Ω. The boundary condition makes sense only for solutions from this class. Not every function from the class $C^p(\Omega)$ can be continuously extended to the boundary of Ω. Consequently, not every solution to a PDE can have a continuous extension to the boundary of an open region in which the solution is found.

For example, the function

$$u(x,t) = t^{-1/2}e^{-x^2/(4t)}\,, \quad t>0\,, \quad x\in\mathbb{R}$$

is from the class $C^\infty(t>0)$ (it has continuous partial derivatives of any order in the (open) half plane $t>0$). It is also a solution to the heat equation discussed in Example **3.1**. The boundary of the half-plane $t>0$ is the $x-$axis whose equation is $t=0$. The solution does not have a continuous extension to the boundary. Indeed, using the substitution $z=1/t$:

$$\lim_{t\to 0^+} u(x,t) = \lim_{z\to\infty}\sqrt{z}e^{-zx^2/4} = \begin{cases} 0\,, & x\neq 0 \\ \infty\,, & x=0\end{cases}$$

So, the limit has no numerical value at one point of the boundary and, hence, u cannot be continuously extended to the boundary of the half-plane.

If boundary condition involve partial derivatives of the solution, then one should demand that these partial derivatives have continuous extensions to the boundary. If all partial derivatives of a solution up order $q<p$ can be extended to the boundary, then one writes

$$u\in C^p(\Omega)\cap C^q(\bar{\Omega})$$

Any function from this class has continuous partial derivatives up order p in an open Ω, and the limit of all partial derivatives If continuous extension of u and its partial derivatives $D^q u$, $q=0,1,...$, are required to have specified values on the boundary, then corresponding boundary conditions mean that

$$\lim_{\mathbf{x}\to\mathbf{y}} D^q u(\mathbf{x}) = v_q(\mathbf{y})$$

for any point on the boundary $\mathbf{y}\in\partial\Omega$, where v_q are given functions on the boundary. For brevity, this boundary condition will also be written as

$$D^q u\Big|_{\partial\Omega} = v_q$$

7.2. The 2D wave equation in an interval. Consider the 2D wave equation where the position variable x is restricted to an interval

$$u''_{tt} - c^2 u''_{xx} = 0\,, \quad \Omega: 0<x<L\,,\ -\infty<t<\infty$$

Let us *find all solutions to the equation that are obtainable by separating variables and satisfy the zero boundary conditions*

$$u\Big|_{x=0} = u\Big|_{x=L} = 0\,, \quad -\infty<t<\infty$$

A physical significance of this problem can be understood if one recalls that one of the interpretations of the 2D wave equation is that its solutions describe evolution of the shape of an elastic string. The zero boundary conditions mean that the string has a finite length L and its endpoints are fixed and cannot move, like a guitar string.

If $u(t,x) = T(t)X(x)$, then the functions $T(t)$ and $X(x)$ satisfy the second-order differential equations whose solutions are continuous in the entire real line. Therefore

$$\begin{aligned} u\Big|_{x=0} &= \lim_{x\to 0^+} u(t,x) = T(t)X(0)\,, \\ u\Big|_{x=L} &= \lim_{x\to L^-} u(t,x) = T(t)X(L) \end{aligned}$$

by continuity of $X(x)$. Therefore the boundary conditions imply that

$$X(0) = X(L) = 0$$

where the function $X(x)$ is a solution to the equation

$$X''(x) + kX(x) = 0$$

with k being a separation constant. There are 3 types of solutions.

Case $\mathbf{k = 0}$. If $k = 0$, then $X = A + Bx$. The linear function can satisfy the boundary conditions only if $A = B = 0$:

$$\begin{cases} X(0) = A = 0 \\ X(L) = A + BL = 0 \end{cases} \quad \Rightarrow \quad A = B = 0$$

So, no non-trivial solution exists in this case.

Case $\mathbf{k < 0}$. If $k = -m^2 < 0$, then $X(x) = Ae^{mx} + Be^{-mx}$ and the boundary conditions are satisfied only if $A = B = 0$:

$$\begin{cases} X(0) = A + B = 0 \\ X(L) = Ae^{mL} + Be^{-mL} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} B = -A \\ 2A\sinh(mL) = 0 \end{cases}$$

$$\Rightarrow \quad A = B = 0$$

because $\sinh(mL) \neq 0$ if $m > 0$. No non-trivial solution exists in this case, either.

Case $\mathbf{k > 0}$. Finally, if $k = m^2 > 0$, then $X(x) = A\cos(mx) + B\sin(mx)$ and the boundary conditions are reduced to

$$\begin{cases} X(0) = A = 0 \\ X(L) = A\cos(mL) + B\sin(mL) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} A = 0 \\ B\sin(mL) = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} A = 0 \\ m = \frac{\pi n}{L}\,, \quad n = 1, 2, \ldots \end{cases}$$

because one has to have $B \neq 0$ in order to get a non-trivial solution. Thus,

$$\begin{aligned} u(x,t) &= u_n(x,t) = X_n(x)T_n(t) \\ &= \sin\left(\frac{\pi n x}{L}\right)\left[A_n \cos\left(\frac{\pi c n t}{L}\right) + B_n \sin\left(\frac{\pi c n t}{L}\right)\right] \end{aligned}$$

where $n = 1, 2, ...$, and A_n and B_n are arbitrary constants. Since the wave equation is linear and homogeneous, one can use the superposition principle to get the most general solution obtainable by separating variables:

$$u(x,t) = \sum_{n=1}^{N} \sin\left(\frac{\pi n x}{L}\right)\left[A_n \cos\left(\frac{\pi c n t}{L}\right) + B_n \sin\left(\frac{\pi c n t}{L}\right)\right]$$

The solution is not unique. From the physical point of view this conclusion should be anticipated because the evolution of the shape of a guitar string depends on its initial shape. It is not difficult to see that

$$\begin{aligned} u\Big|_{t=0} &= \sum_{n=1}^{N} A_n \sin\left(\frac{\pi n x}{L}\right), \\ u_t'\Big|_{t=0} &= \sum_{n=1}^{N} \frac{\pi c n B_n}{L} \sin\left(\frac{\pi n x}{L}\right) \end{aligned}$$

If one demands that initially, that is, at $t = 0$ the string shape is defined by a *given* function $u_0(x)$ and the distribution of initial velocities is given by a function $u_1(x)$, then the *initial conditions*

$$u\Big|_{t=0} = u_0(x)\,, \quad u_t'\Big|_{t=0} = u_1(x)$$

can be used to determine the remaining parameters A_n and B_n in the solution. Since the functions $\sin(\pi n x/L)$ are linearly independent, there is a unique choice of A_n and B_n to satisfy the initial conditions, provided the initial data are given by linear combinations of $\sin(\pi n x/L)$:

$$u_0(x) = \sum_{n=1}^{N} \alpha_n \sin\left(\frac{\pi n x}{L}\right), \quad u_1(x) = \sum_{n=1}^{M} \beta_n \sin\left(\frac{\pi n x}{L}\right)$$

Indeed, the initial conditions are satisfied if

$$A_n = \alpha_n\,, \quad B_n = \frac{L\beta_n}{\pi c n}$$

A natural question arises: Can one use arbitrary functions u_0 and u_1 as initial data, not just linear combinations of $\sin(\pi n x/L)$, to obtain a unique solution to the wave equation? The answer is affirmative. It will be discussed in detail later.

EXAMPLE **7.1**. *Find all solutions to the 2D wave equation*

$$u''_{tt} - u''_{xx} = 0\,, \quad t > 0\,, \quad 0 < x < 1$$

by separating variables that satisfies the zero boundary conditions

$$u\Big|_{x=0} = u\Big|_{x=1}$$

and the following initial conditions

$$u\Big|_{t=0} = \sin(\pi x) - 3\sin(2\pi x)\,, \quad u'_t\Big|_{t=0} = 2\sin(3\pi x)$$

SOLUTION: Setting $c = 1$ and $L = 1$ in the general solution obtained above, it is concluded that any solution obtainable by separating variables that satisfies the boundary conditions reads

$$u(x,t) = \sum_{n=1}^{N} \sin(\pi n x)\Big(A_n \cos(\pi n t) + B_n \sin(\pi n t)\Big)$$

for some N. It follows from the initial conditions that

$$\begin{aligned}
u\Big|_{t=0} &= \sum_{n=1}^{N} A_n \sin(\pi n x) = \sin(\pi x) - 3\sin(2\pi x) \\
\Rightarrow \quad & A_1 = 1\,,\ A_2 = -3\,,\ A_n = 0\,,\ n > 2 \\
u'_t\Big|_{t=0} &= \sum_{n=1}^{N} \pi n B_n \sin(\pi n x) = 2\sin(3\pi x) \\
\Rightarrow \quad & B_3 = \frac{2}{3\pi}\,,\ B_n = 0\,,\ n \neq 3
\end{aligned}$$

so that the solution reads

$$u(x,t) = \sin(\pi x)\cos(\pi t) - 3\sin(2\pi x)\cos(2\pi t) + \frac{2}{3\pi}\sin(3\pi x)\sin(3\pi t)$$

□

7.3. The 2D heat equation in an interval. The above analysis can be repeated for the 2D heat equation in an interval with zero boundary conditions

$$\begin{aligned}
u'_t &= a^2 u''_{xx}\,, \quad 0 < x < L\,, \quad -\infty < t < \infty \\
u\Big|_{x=0} &= u\Big|_{x=L} = 0\,, \quad -\infty < t < \infty
\end{aligned}$$

The physical significance of this problem is easy to understand if one recalls that the heat equation describes the temperature distribution in a heat conducting rod, $u(x,t)$ is the temperature at point x of the rod and at a time t. The boundary conditions describe the situation in which the temperature of the end points is kept fixed. Indeed, let $T(x,t)$ be the physical temperature that satisfies the heat equation. If one demands that

$$T(0,t) = T_0\,, \quad T(L,t) = T_L$$

for any time t, then

$$T(x,t) = T_0 + \frac{T_L - T_0}{L}\,x + u(x,t)$$

where $u(x,t)$ satisfies the heat equation with zero boundary conditions at the endpoints because $T'_t = u'_t$ and $T''_{xx} = u''_{xx}$.

The most general solution obtainable by separating variables reads

$$u(x,t) = \sum_{n=1}^{N} A_n \sin\left(\frac{\pi n x}{L}\right) e^{-(a\pi n/L)^2 t}$$

where A_n are arbitrary constants. The technicalities are left for the reader as an exercise. The non-uniqueness of the solution may be anticipated from an intuitive idea that the evolution of the temperature distribution depends on its initial distribution (assuming, of course, that the heat equation offer a correct mathematical model of the heat energy transfer). Indeed, let us supplement the problem by an *initial condition*

$$u|_{t=0} = u_0(x)$$

where $u_0(x)$ is a given function. Since the solution at $t = 0$ is a linear combinations of linearly independent functions $\sin(\pi n x/L)$, the parameters A_n are uniquely determined, *provided the initial data is a linear combination of* $\sin(\pi n x)$. If

$$u_0(x) = \sum_{n=1}^{N} \alpha_n \sin(\pi n x)\,,$$

then the initial condition yields

$$A_n = \alpha_n\,.$$

It is interesting to note that any initial temperature distribution eventually becomes a linear (equilibrium) distribution:

$$\lim_{t\to\infty} T(x,t) = T_0 + \frac{T_L - T_0}{L}\,x$$

EXAMPLE **7.2**. *Solve the initial and boundary value problem for the heat equation by separating variables:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \quad t > 0\,, \quad 0 < x < 2\,,$$
$$u\Big|_{x=0} = u\Big|_{x=2} = 0\,, \quad t \geq 0\,,$$
$$u\Big|_{t=0} = 4\sin(\pi x/2)\cos(\pi x/2) - 3\sin(2\pi x)\,, \quad 0 \leq x \leq 2\,.$$

or show that no solution can be obtained by this method.

SOLUTION: Let $u(x,t) = X(x)T(t)$. Then by separating variables in the heat equation

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Since the variables t and x are independent, the left and right sides of this equation must be equal to a constant λ. So, the function X is a solution to the boundary value problem

$$X''(x) - \lambda X(x) = 0\,, \quad X(0) = X(2) = 0\,.$$

It was shown that a solution exists only if

$$\lambda = \lambda_n = -\left(\frac{\pi n}{2}\right)^2, \quad X(x) = X_n(x) = \sin\left(\frac{\pi n x}{2}\right), \quad n = 1, 2, \ldots$$

The associated function $T(t) = T_n(t)$ solves the equation

$$T_n'(t) = \lambda_n T_n(t) \quad \Rightarrow \quad T_n(t) = A_n e^{\lambda_n t}$$

where A_n is an integration constant. So, the most general solution that vanishes at $x = 0$ and $x = 2$ (and can be obtained by separating variables) reads

$$u(x,t) = \sum_{n=1}^{N} A_n \sin\left(\frac{\pi n x}{2}\right) e^{-\frac{\pi^2 n^2 t}{4}}$$

for some integer $N \geq 1$. Therefore, the initial condition at $t = 0$ is fulfilled only if

$$\begin{aligned}\sum_{n=1}^{N} A_n \sin\left(\frac{\pi n x}{2}\right) &= 4\sin(\pi x/2)\cos(\pi x/2) - 3\sin(2\pi x)\\ &= 2\sin(\pi x) - 3\sin(2\pi x)\end{aligned}$$

where the double angle trigonometric identity was used. The functions $X_n(x)$ are linearly independent. Therefore the above equality can hold for all x in the said interval only is the coefficients in the linear combinations of X_n match:

$$A_2 = 2\,, \quad A_4 = -3\,, \quad A_1 = A_3 = 0\,, \quad A_n = 0\,,\ n > 4\,.$$

Thus, the solution reads

$$u(x,t) = 2\sin(\pi x)e^{-\pi^2 t} - 3\sin(2\pi x)e^{-4\pi^2 n^2 t}\,.$$

□

7.4. Asymptotic boundary conditions. If the region in which a PDE is solved is not bounded, then *asymptotic* conditions can be imposed on a solution that prescribe how the solution should behave as one or more variables tends to infinity. For example, consider the 2D heat equation in the positive quadrant

$$u'_t = a^2 u''_{xx}\,, \quad x > 0\,, \quad t > 0\,.$$

Let us attempt to find all solutions that are bounded

$$|u(x,t)| \le M < \infty\,, \quad x \ge 0\,,\ t \ge 0\,,$$

and can be obtained by separating variables: $u(x,t) = T(t)X(x)$. Even if a solution can be continuously extended to the boundary of the positive quadrant, it can still grow arbitrary large as x or t or both become arbitrary large. The condition requires that no unbounded growth of the solution is permitted.

If λ is a separation constant, then

$$T'(t) = a^2\lambda T(t)\,, \quad X''(x) - \lambda X(x) = 0\,.$$

The first equation has bounded solutions only if $\lambda \le 0$. If $\lambda = 0$, then only a constant solution is bounded ($X(x) = A + Bx$ is bounded only if $B = 0$). Put $\lambda = -\omega^2 < 0$, so that

$$u(x,t) = e^{-\omega^2 a^2 t}\Big(A_\omega \cos(\omega x) + B_\omega \sin(\omega x)\Big)\,.$$

where A_ω and B_ω are arbitrary constants. This solution also contains a constant solution if $\omega = 0$. So, there is a family of bounded solutions labeled by a non-negative parameter $\omega \ge 0$. Since the heat equation is linear, the sum of any number of such solutions (corresponding to distinct values of ω) is also a bounded solution.

7.5. The 2D Laplace equation in an interval. Similarly, the following boundary value problem for the 2D Laplace equation

$$u''_{xx} + u''_{yy} = 0\,, \quad 0 < x < L\,, \quad -\infty < y < \infty$$
$$u\Big|_{x=0} = u\Big|_{x=L} = 0\,, \quad -\infty < y < \infty$$

has the following most general solution

$$u(x,y) = \sum_{n=1}^{N} \sin\left(\frac{\pi n x}{L}\right)\left[A_n e^{(\pi n/L)y} + B_n e^{-(\pi n/L)y}\right]$$

that can be obtained by separating variables. The technicalities are left again to the reader as an exercise.

Asymptotic conditions. The region is not bounded and, hence, one can impose asymptotic conditions on the solution when $|y| \to \infty$. For example, if one demands that

$$\lim_{y\to+\infty} u(x,y) = 0$$

then $A_n = 0$ in the found solution. If one demands that the solution must be bounded

$$|u(x,y)| \le M < \infty\,, \quad 0 \le x \le L\,, \quad -\infty < y < \infty\,.$$

then $A_n = B_n = 0$ and only the trivial solution, $u(x,y) = 0$, is permitted.

Of course, asymptotic and boundary conditions are determined by a physical process or phenomenon modeled by a PDE. For instance, in electrostatics, the electric field is conservative, that is, it is the gradient of a function called an electric potential, and the potential is shown to satisfy the Laplace equation in any region where no electric charges are present. The zero boundary condition physically correspond to a metal (conducting) surface that is grounded. So, the gradient ∇u of the above solution can be interpreted as an electric field between two infinite parallel conducting planes that are grounded. However, such an interpretation is not sound from the physical point of view because the potential $u(x,y)$ becomes infinite as $y \to \pm\infty$ so that the electric field becomes infinite too. So, solutions with unbounded electric field at spatial infinite are not physically acceptable. If one imposes the boundedness condition

$$|\nabla u| \le M \quad \text{or} \quad |u'_x| \le M_1\,,\ |u'_y| \le M_2$$

Then only the trivial solution $u(x,y) = 0$ would satisfy the zero boundary condition at $x = 0$ and $x = L$ and the boundedness condition. This agrees with the physical picture. There exists no electrical field in the described system.

7.6. Laplace equation in a disk. Let us investigate solutions of the Laplace equation in a disk that has a prescribed values on the boundary of the disk:

$$\Delta u = 0\,, \quad x^2 + y^2 < R^2\,, \quad u\Big|_{x^2+y^2=R^2} = v$$

The boundary condition is convenient to formulate in polar coordinates because any function on a circle is a $2\pi-$periodic function of the polar angle

$$v(\theta + 2\pi) = v(\theta)$$

Let us attempt to solve the problem by separating variables in polar coordinates. Since the origin is included into the disk in which a solution is sought,

$$u(x,y) = A_0 + \sum_{m=1}^{M} r^m \Big(A_m \cos(m\theta) + B_m \sin(m\theta)\Big)$$

Therefore the initial condition yields conditions on the parameters A_m and B_m

$$u\Big|_{r=R} = A_0 + \sum_{m=1}^{M} R^m \Big(A_m \cos(m\theta) + B_m \sin(m\theta)\Big) = \varphi(\theta)$$

Since the trigonometric harmonics, $\sin(m\theta)$ and $\cos(m\theta)$, are linearly independent, the coefficients A_m and B_m are uniquely determined by the boundary conditions, *provided the boundary data v is a linear combination of the trigonometric harmonics*:

$$v(\theta) = a_0 + \sum_{m=1}^{M} \Big(a_m \cos(mx) + b_m \sin(mx)\Big)$$

and in this case

$$A_0 = a_0\,, \quad A_n = \frac{a_n}{R^n}\,, \quad B_n = \frac{b_n}{R^n}$$

by comparing the coefficients at the same harmonics so that

$$u(x,y) = a_0 + \sum_{m=1}^{M} \Big(\frac{r}{R}\Big)^m \Big(a_m \cos(mx) + b_m \sin(mx)\Big)$$

Example **7.3**. *Solve the boundary value problem for the Laplace equation in a disk by separating variables in polar coordinates:*

$$\Delta u = 0\,, \quad x^2+y^2<1\,, \quad u\Big|_{x^2+y^2=1} = 4y^3\Big|_{x^2+y^2=1}$$

Express the solution in rectangular coordinates.

Solution: Let us first check if the boundary data is a linear combination of trigonometric harmonics:

$$\begin{aligned} v(\theta) &= 4\sin^3(\theta) = -\frac{1}{2i}(e^{i\theta}-e^{-i\theta})^3 \\ &= -\frac{1}{2i}(e^{3i\theta}-e^{-3i\theta}-3e^{i\theta}+3e^{-i\theta}) \\ &= 3\sin(\theta)-\sin(3\theta) \end{aligned}$$

The boundary condition yields

$$\begin{aligned} u\Big|_{r=1} &= A_0+\sum_{m=1}^{M}\Big(A_m\cos(m\theta)+B_m\sin(m\theta)\Big) = 3\sin(\theta)-\sin(3\theta) \\ &\Rightarrow \quad A_n=0\,,\ B_1=3\,,\ B_3=-1\,,\ B_2=B_n=0\,,\ n>3\,, \\ u(x,y) &= 3r\sin(\theta)-r^3\sin(3\theta) \\ &= 3y-r^3(3\sin(\theta)-4\sin^3(\theta)) \\ &= 3y-3y(x^2+y^2)+4y^3 = 3y+y^3-3yx^2 \end{aligned}$$

□

7.7. Exercises.

1. Derive the most general solution to the 2D heat equations on an interval with the zero boundary conditions as stated in Section 7.3 by separating variables.

2. Derive the most general solution to the 2D Laplace equation on a strip with the zero boundary conditions as stated in Section 7.4 by separating variables.

3. Find the most general solution to the 2D wave equation satisfying the following boundary conditions

$$\begin{aligned} &u''_{tt}-u''_{xx}=0\,, \quad -1<x<1\,, \quad t\in\mathbb{R} \\ &u\Big|_{x=-1}=u\Big|_{x=1}=0\,, \quad t\in\mathbb{R} \end{aligned}$$

by separating variables t and x.

4. Find the most general solution to the 2D heat equation satisfying periodic boundary conditions:

$$\begin{aligned} &u'_t = u''_{xx}\,, \quad 0 < x < L\,, \quad t \in \mathbb{R} \\ &u(x+L,t) = u(x,t)\,, \quad u'_x(x+L,t) = u'_x(x,t)\,, \end{aligned}$$

by separating variables t and x.

5. Find the most general solution to the 2D Laplace equation that is periodic in both variables x and y:

$$u(x+a,y) = u(x,y) = u(x,y+b)$$

by separating variables x and y.

6. Let Ω be a planar region bounded by two parabolas $2y = x^2 - 1$ and $2y = \frac{1}{4}x^2 - 4$. Use separation of variables in parabolic coordinates (see Section 6.4) to solve the following boundary value problem:

$$u''_{xx} + u''_{yy} = 0\,, \quad (x,y) \in \Omega\,, \qquad u\Big|_{\partial\Omega} = 0$$

7. Find all solutions to the boundary value problem by separating variables in polar coordinates

$$\Delta u = 0\,, \quad x^2 + y^2 < 4\,, \quad u\Big|_{x^2+y^2=4} = xy^3\Big|_{x^2+y^2=4}$$

Express the solution in rectangular coordinates.

8. Solve the boundary value problem by separating variables in polar coordinates with a suitable center

$$\Delta u = 0\,, \quad x^2 + y^2 < 4y\,, \quad u\Big|_{x^2+y^2=4y} = y^3\Big|_{x^2+y^2=4y}$$

9. Find all bounded solutions to the 3D Laplace equation

$$u''_{xx} + u''_{yy} + u''_{zz} = 0$$

by separating variables $u(x,y,z) = X(x)Y(y)Z(z)$.

Selected answers.

3.

$$u(x,t) = \sum_{n=1}^{N} \sin\left(\frac{\pi n}{2}(x+1)\right)\left[A_n \cos\left(\frac{\pi n}{2}\,t\right) + B_n \sin\left(\frac{\pi n}{2}\,t\right)\right]$$

4.

$$u(x,t) = A_0 + \sum_{n=1}^{N} e^{-(2\pi n/L)^2 t} \left[A_n \cos\left(\frac{2\pi n}{L}\, x\right) + B_n \sin\left(\frac{2\pi n}{L}\, x\right)\right]$$

5. $u(x,y) = A$

6. Put $x = \xi\eta$ and $y = \frac{1}{2}(\xi^2 - \eta^2)$. Then

$$u(x,y) = \sum_{n=1}^{N} \sin(\pi n \eta) \left[A_n e^{\pi n \xi} + B_n e^{-\pi n \xi}\right]$$

7. $u(x,y) = 2xy - \frac{1}{2}(yx^3 - xy^3)$.