CHAPTER 2

First-order PDEs

8. Characteristics of linear first-order PDEs

A first-order PDE is a relation

$$F(\mathbf{x}, u, Du) = 0\,, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n$$

If F is linear in Du, then the PDE is called *quasi-linear*:

$$\sum_{j=1}^{n} a_j(\mathbf{x}, u)\,\frac{\partial u(\mathbf{x})}{\partial x_j} = f(\mathbf{x}, u)$$

If F is linear in Du and u, then the PDE is called *linear*:

$$\sum_{j=1}^{n} a_j(\mathbf{x})\,\frac{\partial u(\mathbf{x})}{\partial x_j} + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x})$$

Solving a first order PDE in n variables is equivalent to solving an *autonomous* system of $n+1$ first order ordinary differential equations (ODEs). The method of reduction of a first order PDE to a system of ODEs is known as the *method of characteristics*. Here it is illustrated with linear and quasi linear first order PDEs in two variables. The method becomes technically complicated with increasing the number of variables because there is no general recipe for solving a general autonomous system of ODEs. In applications, this latter problem can be solved numerically by one of the standard subroutines such as, e.g., Runge-Kutta or Euler methods. In this sense, solving first order PDEs is still an ODE problem.

8.1. Linear first order PDE in two variables. Consider a general first order linear PDE in two variables (x, y)

$$a(x,y)u'_x(x,y) + b(x,y)u'_y(x,y) + c(x,y)u(x,y) = f(x,y) \qquad (8.1)$$

It is assumed that $u \in C^1$ and the coefficients a, b, c, and f are continuous functions (in an open region Ω in which the equation is to be solved), and a and b do not vanish simultaneously anywhere in Ω.

8.2. Basic idea for solving. Suppose that a and b are non-zero constants. In this case, is not difficult to construct a linear change of variables $(x, y) \to (\gamma, \eta)$ in which

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}$$

In this case, the PDE becomes an *ordinary differential equation* (ODE) in variable η, while the variables γ is a numerical parameter:

$$\frac{\partial u}{\partial \eta} + cu = f\,.$$

A linear first-order ODE can be solved by the standard method of variations of parameters. In particular, if $c = f = 0$, then

$$a\,\frac{\partial u}{\partial x} + b\,\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} = 0 \quad \Rightarrow \quad u(x, y) = g(\gamma)\,, \quad \gamma = \gamma(x, y)$$

where $g(\gamma)$ is any C^1 function of one real variable, and $\gamma = \gamma(x, y)$ is the new variable expressed in terms of the old ones.

Let us demonstrate the existence of such a change of variables. Put

$$\gamma = ay - bx\,, \quad \eta = \alpha y + \beta x$$

This transformation defines a change of variables if its Jacobian is not zero:

$$J = \det\begin{pmatrix} \eta'_x & \eta'_y \\ \gamma'_x & \gamma'_y \end{pmatrix} = \det\begin{pmatrix} \beta & \alpha \\ -b & a \end{pmatrix} = a\beta + b\alpha \neq 0$$

Then

$$\begin{aligned} a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} &= a\left(-b\,\frac{\partial}{\partial \gamma} + \beta\,\frac{\partial}{\partial \eta}\right) + b\left(a\,\frac{\partial}{\partial \gamma} + \alpha\,\frac{\partial}{\partial \eta}\right) \\ &= (a\beta + b\alpha)\,\frac{\partial}{\partial \eta} = J\,\frac{\partial}{\partial \eta} \end{aligned}$$

Since $J \neq 0$, any such a change of variables would do the job. In particular, one can set $\beta = \frac{1}{2a}$ and $\alpha = \frac{1}{2b}$ so that $J = 1$.

If a and b are not constant functions, then one can still attempt to find a *non-linear* change of variables with similar properties and reduce a first-order PDE to an ODE. This method is known as *the method of characteristics.*

8.3. Characteristics. If a and b are constants, then

$$d\gamma = a\,dy - b\,dx\,.$$

Therefore the curves (lines) on which the new variable γ takes constant values are solutions to the ordinary differential equation

$$a\,dy - b\,dx = 0 \quad \Rightarrow \quad \gamma(x,y) = ay - bx = p$$

Once the coordinate curves (lines) of γ are defined, one can find another function $\eta(x,y)$ whose level curves (lines) are intersecting level curves of $\gamma(x,y)$ at a non-zero angle, so that the transformation $(x,y) \to (\gamma, \eta)$ defines a change of variables (the Jacobian of the transformation does not vanish in this case). But the equation that defines level curves of the new variable γ makes perfect sense even if the coefficients a and b are functions of x and y. This offers us a possibility to find the desired change of variables to reduce a linear first-order PDE to an ordinary differential equation in one of the new variables, while the other variable remains a parameter.

An ordinary differential equation (ODE)

$$a(x,y)dy - b(x,y)dx = 0 \tag{8.2}$$

is called a *characteristic equation* for the PDE (**8**.**1**). Solutions to the characteristic equation are called *characteristics* of the PDE (**8**.**1**). A characteristic is a curve in the xy plane. If the curve is required to pass through a point (x_0, y_0), then the question arises if there is only one such curve. The theory of ODEs offers *sufficient conditions* for the existence and uniqueness of such curve.

Theorem **8**.**1**. *Suppose that functions $a(x,y)$ and $b(x,y)$ are from the class $C^1(\Omega)$ such that they do not vanish simultaneously anywhere. Then for every point $(x_0, y_0) \in \Omega$ there is a unique curve passing through it and satisfying Eq. (**8**.**2**) in some neighborhood of the point.*

A general solution to (**8**.**2**) can be written in the form

$$\gamma(x,y) = p$$

where p is an arbitrary constant and γ is a C^1 function. The characteristics are level sets of a C^1 function and labeled by a real parameter p. Since there is only one characteristic passing through each point of Ω, the characteristics for different p do not intersect in Ω. Consider a transformation

$$t = x\,, \quad p = \gamma(x,y)$$

Its Jacobian is

$$J = \det \begin{pmatrix} 1 & 0 \\ \gamma'_x & \gamma'_y \end{pmatrix} = \gamma'_y$$

Suppose that $\gamma'_y(x,y) \neq 0$ in Ω. The transformation is a change of variables in Ω and

$$a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} = a\Big(\frac{\partial}{\partial t} + \gamma'_x\,\frac{\partial}{\partial p}\Big) + b\,\gamma'_y\,\frac{\partial}{\partial p} = a\,\frac{\partial}{\partial t} + (a\gamma'_x + b\gamma'_y)\,\frac{\partial}{\partial p}$$

If $\gamma(x,y)$ is a solution to the original PDE

$$a\gamma'_x + b\gamma'_y = 0\,,$$

then the desired change of variables has been found because the original PDE becomes an ODE in the variable $t = x$

$$a\frac{\partial u}{\partial t} + cu = f$$

where a, c, and f must be expressed in the new variables. Let us find out if a function whose level curves are characteristics is indeed a solution to the PDE.

8.3.1. Implicit function theorem. The curve $\gamma(x,y) = \gamma(x_0,y_0)$ passes through the point (x_0,y_0). Near (x_0,y_0), the characteristic can always be represented as a graph $y = Y(x,p)$ or $x = X(y,p)$. Indeed, the equation $\gamma(x,y) = p$ defines implicitly y as a function of x or x as a function of y. The existence and smoothness of these functions is established by the implicit function theorem (studied in multivariable Calculus).

THEOREM 8.2. (Implicit function theorem)
Suppose that $\gamma(x,y)$ is from the class C^1 in a neighborhood of a point (x_0,y_0) and the partial derivatives $\gamma'_x(x_0,y_0)$ and $\gamma'_y(x_0,y_0)$ do not vanish simultaneously. If $\gamma'_y(x_0,y_0)) \neq 0$, then the equation $\gamma(x,y) = p$ can be solved for y in some interval $x_0 - \alpha < x < x_0 + \alpha$, $\alpha > 0$, and the solution is a differentiable function whose derivative is given by the following equation

$$\gamma(x,y) = p \quad \Rightarrow \quad y = Y(x,p)\,, \quad \frac{dy}{dx} = Y'_x(x,p) = -\frac{\gamma'_x}{\gamma'_y}\Big|_{y=Y}$$

If $\gamma'_x(x_0,y_0)) \neq 0$, then the equation $\gamma(x,y) = p$ can be solved for x in some interval $y_0 - \beta < y < y_0 + \beta$, $\beta > 0$, and the solution is a differentiable function whose derivative is given by the following equation

$$\gamma(x,y) = p \quad \Rightarrow \quad x = X(x,p)\,, \quad \frac{dx}{dy} = X'_y(y,p) = -\frac{\gamma'_y}{\gamma'_x}\Big|_{x=X}$$

The equations for the derivatives dy/dx and dx/dy are known as the *implicit differentiation equations*. Note also that by continuity of partial derivatives, the condition $\gamma'_y(x_0, y_0) \neq 0$ guarantees that $\gamma'_y(x, y) \neq 0$ in some neighborhood of (x_0, y_0). The radius of this neighborhood determines the interval $x \in (x_0 - \alpha, x_0 + \alpha)$ in which the implicit differentiation equation is valid.

It follows from the implicit differentiation equations that for each (fixed) value of the parameter p the function $y = Y(x, p)$ (or $x = X(x, p)$) is a solution to the differential equation

$$\gamma'_x(x, y)dx + \gamma'_y(x, y)dy = 0$$

Indeed, if $\gamma'_y \neq 0$, then the substitution of $dy = Y'_x(x, p)dx$ into the above equation reduces it to the implicit differentiation equation for dy/dx which is true for all points where $\gamma'_y \neq 0$ by the implicit function theorem. Since the graph $y = Y(x, p)$ is also a characteristic (a solution to (**8.2**)), the slope Y'_x must also be given by

$$\frac{dy}{dx} = Y'_x(x, p) = \frac{b}{a}\Big|_{y=Y} \tag{8.3}$$

Similarly, if the graph $x = X(y, p)$ is a characteristic, then

$$\frac{dx}{dy} = X'_y(y, p) = \frac{a}{b}\Big|_{x=X} \tag{8.4}$$

This equations show that for any point of a characteristic $\gamma(x, y) = p$, $a \neq 0$ implies that $\gamma'_y \neq 0$, and $b \neq 0$ implies that $\gamma'_x \neq 0$.

For example, a general solution to the following equation defines a family of concentric circles:

$$xdx + ydy = 0 \quad \Rightarrow \quad \gamma(x, y) = x^2 + y^2 = p^2 > 0$$

Note that $a = x$ and $b = -y$ vanish simultaneously at $(x, y) = (0, 0)$. So, the origin is excluded from a region in which this equation is considered. The equation can be solved for y:

$$y = Y(x, p) = \pm\sqrt{p^2 - x^2}\,, \quad -p < x < p$$

The positive solution describes the level curve near any point (x_0, y_0) with $x_0 \neq \pm p$ and $y_0 > 0$, while the negative one does so for $x_0 \neq \pm p$ and $y_0 < 0$. Neither of the solution can describe the level curve near points $(x_0, y_0) = (\pm p, 0)$. Note that $\gamma'_y(\pm p, 0) = 0$ and the implicit function theorem does not guarantee the existence of a solution. But $\gamma'_x(\pm p, 0) = \pm 2p \neq 0$. So, near those points there is a differentiable solution

$$x = X(x, p) = \pm\sqrt{p^2 - y^2}\,, \quad -p < y < p$$

It is not difficult to verify that

$$Y'_x(x,p) = \frac{b}{a}\Big|_{y=Y} = -\frac{\gamma'_x}{\gamma'_y}\Big|_{y=Y} = -\frac{x}{Y(x,p)}$$

$$X'_y(y,p) = \frac{a}{b}\Big|_{x=X} = -\frac{\gamma'_y}{\gamma'_x}\Big|_{x=X} = -\frac{y}{X(y,p)}$$

8.3.2. Properties of characteristics. The following theorem establishes the relation between characteristics and solutions to some PDEs.

THEOREM **8.3**. (Properties of characteristics)
Level sets of a function γ from the class C^1 with the non-vanishing gradient $(\gamma'_x, \gamma'_y) \neq (0,0)$ and satisfying the first order homogeneous linear PDE

$$a(x,y)\gamma'_x(x,y) + b(x,y)\gamma'_y(x,y) = 0 \tag{8.5}$$

*are solutions to the characteristic equation (**8.2**), where the functions a and b do not vanish simultaneously anywhere. Conversely, if $\gamma(x,y) = p$ is a general solution to (**8.2**), then the function $\gamma(x,y)$ is a solution to (**8.5**).*

PROOF. Let $\gamma(x,y)$ be a solution to (**8.5**) near a point (x_0,y_0). Since a and b do not vanish simultaneously anywhere, without loss of generality, $a(x_0,y_0) \neq 0$ so that by continuity of a, $a(x,y) \neq 0$ near (x_0,y_0). It follows from (**8.5**) that near (x_0,y_0)

$$\gamma'_x = -\frac{b}{a}\,\gamma'_y \tag{8.6}$$

Consider a level set $\gamma(x,y) = p$ containing the point (x_0,y_0), that is, $p = \gamma(x_0,y_0)$. The partial derivatives γ'_x and γ'_y do not vanish simultaneously. The relation (**8.6**) implies that $\gamma'_y(x_0,y_0) \neq 0$. By the implicit function theorem the equation $\gamma(x,y) = p$ can be solved for y near (x_0,y_0) so that the level set of γ containing the point (x_0,y_0) is the graph of a differentiable function $y = Y(x,p)$ passing through the point (x_0,y_0) and the derivative of Y with respect to x is given by the implicit differentiation equation:

$$\frac{dy}{dx} = Y'_x(x,p) = -\frac{\gamma'_x(x,y)}{\gamma'_y(x,y)}\Big|_{y=Y(x,p)}$$

In other words, the level curve $y = Y(x,p)$ is also a solution to ODE:

$$\gamma'_x(x,y)dx + \gamma'_y(x,y)dy = 0$$

The substitution of (**8**.6) into this equation yields the characteristic equation (**8**.2)

$$-\frac{b}{a}\,\gamma'_y\,dx + \gamma'_y dy = 0 \quad \Rightarrow \quad -\frac{\gamma'_y}{a}\Big(ady - bdx\Big) = 0$$
$$\Rightarrow \quad ady - bdx = 0$$

because $\gamma'_y \neq 0$. Thus, every level set of a solution to (**8**.5) is a characteristic.

Conversely, if $\gamma(x, y) = p$ is a characteristic curve passing through a point (x_0, y_0). Suppose that $a(x_0, y_0) \neq 0$. Then the slope of the characteristic at this point is

$$\frac{dy}{dx} = \frac{b(x_0, y_0)}{a(x_0, y_0)}$$

On the other hand, the slope is also given by the implicit differentiation equation:

$$\frac{dy}{dx} = -\frac{\gamma'_x(x_0, y_0)}{\gamma'_y(x_0, y_0)}$$

Equating the two expressions for the slope, one infers that at *any point* (x_0, y_0)

$$a\gamma'_x + b\gamma'_y = 0$$

Since the choice of (x_0, y_0) is arbitrary, it is concluded that $u(x, y) = \gamma(x, y)$ is a solution to (**8**.5). □

8.4. Solving PDEs with $c = f = 0$. Theorem **8**.3 asserts that solving PDE (**8**.1) with $c = f = 0$ is equivalent to solving ODE (**8**.2).

PROPOSITION **8**.1. *Let $\gamma(x, y) = p$ be characteristics of PDE (**8**.1) in an open region Ω and $g(t)$ be a C^1 function of a real variable t. Then the composition*

$$u(x, y) = g(\gamma(x, y))$$

*is the most general solution to (**8**.1) with $c = f = 0$ in Ω.*

Indeed, by the chain rule

$$\left.\begin{aligned} u'_x &= g'\gamma'_x \\ u'_y &= g'\gamma'_y \end{aligned}\right\} \quad \Rightarrow \quad au'_x + bu'_y = g'(a\gamma'_x + b\gamma'_y) = 0$$

because γ satisfies (**8**.5). This arbitrariness is related to that there are many functions which have common level sets. For example, level sets of any function of two variables that depends only on the combination $x^2 + y^2$ are concentric circles (the argument of such a function has a fixed value of a circle and, hence, the value of the function is constant on a circle). In other words, *the choice of* $\gamma(x,y)$ *whose level sets are characteristics is not unique.*

8.4.1. Some methods to find characteristics. Recall that a first order ODE

$$N(x,y)dx + M(x,y)dy = 0$$

is called *exact* if

$$\frac{\partial N}{\partial y} = \frac{\partial M}{\partial x}$$

and in this case there exists a function $\gamma(x,y)$ such that

$$\gamma'_x(x,y) = N(x,y)\,, \quad \gamma'_y(x,y) = M(x,y)$$

Therefore the exact equation can be cast in the form

$$d\gamma = \gamma'_x dx + \gamma'_y dy = N dx + M dy = 0$$

from which it follows that $\gamma(x,y) = p$ is a general solution. The characteristic equation (**8.2**) is *exact* if

$$\frac{\partial a}{\partial x} = -\frac{\partial b}{\partial y}$$

In this case,

$$\gamma'_x = -b\,, \quad \gamma'_y = a$$

If the characteristic equation is not exact, then it has to be solved by one of the standard methods for solving first-order ODEs (e.g., by separation of variables or by finding an integrating factor).

Example **8.1**. *Solve*

$$(x+2y)u'_x + (2x-y)u'_y = 0\,, \qquad (x,y) \in \mathbb{R}^2$$

Solution: Here $a = x+2y$ and $b = 2x-y$. The characteristic equation reads

$$ady - bdy = (x+2y)dy - (2x-y)dx = 0$$

It is an exact equation because

$$\frac{\partial a}{\partial x} = \frac{\partial}{\partial x}(x+2y) = 1\,, \qquad -\frac{\partial b}{\partial y} = -\frac{\partial}{\partial y}(2x-y) = 1$$

Therefore

$$\gamma'_x = y - 2x \quad \Rightarrow \quad \gamma(x,y) = xy - x^2 + h(y)$$

for some $h(y)$. It follows from $\gamma'_y = a$ that

$$x + h'(y) = x + 2y \quad \Rightarrow \quad h'(y) = 2y \quad \Rightarrow \quad h(y) = y^2$$

Thus, the function

$$u(x,y) = \gamma(x,y) = y^2 - xy + x^2$$

is a solution to the equation in equation and a general solution reads

$$u(x, y) = g(y^2 - xy + x^2)$$

where g is any differentiable function of one variable. □

EXAMPLE **8.2**. *Find a general solution to*

$$x^2 u'_x + (x^2 + y^2 + xy)u'_y = 0\,, \quad x > 0\,,\ y > 0$$

SOLUTION: Here $a = x^2$ and $b = x^2 + y^2 + xy$. The characteristic equation

$$x^2 dy - (x^2 + y^2 + xy)dx = 0$$

is not exact, but it is *homogeneous* because dy/dx depends only on the combination y/x. Indeed, since $a \neq 0$ for all $x > 0$

$$\frac{dy}{dx} = \frac{x^2 + y^2 + xy}{x^2} = \Big(\frac{y}{x}\Big)^2 + \frac{y}{x} + 1$$

A homogeneous equation can be solved by the substitution $v = y/x$ so that $y = xv$ and

$$\frac{dy}{dx} = v + x\,\frac{dv}{dx}$$

By means of this substitution the characteristic equation is reduced to a *separable* equation

$$v + x\,\frac{dv}{dx} = v^2 + v + 1 \quad \Rightarrow \quad \frac{dv}{v^2 + 1} = \frac{dx}{x}$$

Its general solution is

$$\int \frac{dv}{v^2 + 1} = \int \frac{dx}{x} \quad \Rightarrow \quad \arctan(v) = \ln(x) + p$$

where p is an integration constant. The characteristics are level sets $\gamma(x, y) = p$ of the function

$$\gamma(x, y) = \arctan(y/x) - \ln(x)$$

In this case, they can also be represented as graphs:

$$\gamma(x, y) = p \quad \Rightarrow \quad y = Y(x, p) = x \tan(\ln(x) + p)$$

A general solution to the PDE in question reads

$$u(x, y) = g(\gamma(x, y)) = g\Big(\arctan(y/x) - \ln(x)\Big)\,, \quad x > 0\,,\ y > 0\,,$$

where g is any differentiable function of one real variable. □

8.5. Exercises.

1-5. Find a general solution to each of the following linear PDEs:

1.
$$e^{-x}u'_x + (ye^{-x} - 3x^2)u'_y = 0\,, \quad (x,y) \in \mathbb{R}^2\,.$$

2.
$$xu'_x + y\Big(\ln(y) - \ln(x) + 1\Big)u'_y = 0\,, \qquad x > 0\,,\ y > 0\,.$$

3.
$$(2xy-1)u'_x + \Big(3x^3 - \frac{y}{x}\Big)u'_y = 0\,, \qquad x > 0$$

Hint: Find an integrating factor as a function of x to solve the characteristics equation.

4.
$$(y - x - 3)u'_x + (1 - x - y)u'_y = 0\,, \qquad x > -1\,.$$

Hint: Make a shift transformation $x = \alpha + \alpha_0$, $y = \beta + \beta_0$ so that $y - x - 3 = \beta - \alpha$ and $x + y - 1 = \alpha + \beta$ by choosing suitable constants α_0 and β_0.

5.
$$2u'_x - (y^3e^x + y)u'_y = 0\,, \qquad y > 0$$

Hint: The characteristic equation is a Bernoulli equation.

Selected answers. **1**. $u = g(\gamma)$, $g \in C^1$, $\gamma = ye^{-x} + x^3$.

2. $u = g(\gamma)$, $g \in C^1$, $\gamma = \frac{1}{x}\ln\Big(\frac{y}{x}\Big)$.

3. $u = g(\gamma)$, $g \in C^1$, $\gamma = y^2 - \frac{y}{x} - x^3$.

4. $u = g(\gamma)$, $g \in C^1$, $\gamma = \arctan\Big(\frac{\beta}{\alpha}\Big) - \ln\Big(\sqrt{\alpha^2 + \beta^2}\Big)$ where $\alpha = x + 1$ and $\beta = y - 2$.

5. $u = g(\gamma)$, $g \in C^1$, $\gamma = y^{-2}e^{-x} - x$.

9. The method of characteristics for linear first-order PDEs

9.1. Constant coefficients. If a, b, and c are constant in the equation

$$au'_x + bu'_y + cu = f(x, y)\,,$$

then a general solution is not difficult to find by changing variables. Without loss of generality, it is assumed that a and b do not vanish. If $a = 0$ or $b = 0$, then the equation is a linear first-order ordinary differential equation in either y or x. The characteristics are lines $\gamma(x, y) = ay - bx = p$. Consider the transformation

$$\gamma = ay - bx\,, \quad \eta = x$$

Its Jacobian $J = a$ is not zero. Hence, the transformation is a change of variables, and

$$a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial x} = a\,\frac{\partial}{\partial \eta}$$

so that

$$a\,\frac{\partial u}{\partial \eta} + cu = f$$

Since x and η are equal, the new variables are denoted as x and γ. The inverse transformation is given by

$$x = x\,, \quad y = Y(x, \gamma) = \frac{1}{a}\,(\gamma + bx)\,, \quad a \neq 0\,.$$

Let $u(x, y)$ be a solution to the original PDE, then the function

$$Z(x, \gamma) = u(x, Y(x, \gamma))$$

satisfies the ordinary differential equation

$$aZ'_x(x, \gamma) + cZ(x, \gamma) = f(x, Y(x, \gamma))\,,$$

This equation can be divided by a to obtain a standard form of a linear ODE with constant coefficients:

$$Z'_x(x, \gamma) + \frac{c}{a}\,Z(x, \gamma) = g(x, \gamma)$$

where

$$g(x, \gamma) = \frac{1}{a}\,f(x, Y(x, \gamma))$$

A particular solution can be found by the method of variation of parameter. Since a general solution to the associated homogeneous equation is $C_0 e^{-cx/a}$, where C_0 is a constant, a particular solution is sought in the form

$$Z(x, \gamma) = e^{-cx/a} V(x, \gamma)$$

A substitution to the equation yields the following equation for the unknown function V:

$$V'_x(x,\gamma) = e^{cx/a} g(x,\gamma)$$

By integrating this equation, one infers that

$$V(x,\gamma) = \int e^{cx/a} g(x,\gamma)\,dx$$

The integration constant can depend on the variable γ. Therefore a general solution reads

$$Z(x,\gamma) = Z_0(\gamma)e^{-cx/a} + e^{-cx/a}\int e^{cx/a} g(x,\gamma)\,dx$$

for some $Z_0(\gamma)$. Hence, a general solution to the original PDE is

$$u(x,y) = Z(x,\gamma)\Big|_{\gamma=ay-bx}.$$

It turns out that this procedure can be extended to the case when the coefficients are not constant. A technical complication is merely associated with finding the explicit form of the function $Y(x,\gamma)$.

9.2. Non-constant coefficients. Let $\gamma(x,y) = p$ be a general solution for the characteristic equation in some open Ω. By Theorem **8.1**, there is only characteristic passing through each point of Ω. Thus the characteristics for different values of p do not intersect. Yet, since the characteristics pass through every point of Ω, the union of all characteristics, labeled by real variable p, contains Ω. Suppose that $\gamma'_y \neq 0$ in Ω. Then, by the implicit function theorem $\gamma(x,y) = p$ can be solved for y and the graph $y = Y(x,p)$ is the level curve $\gamma(x,y) = p$.

Recall that if $\gamma'_y \neq 0$ (or $a \neq 0$), then the transformation

$$t = x\,, \quad p = \gamma(x,y)$$

is a change of variables and Eq. (**8.1**) becomes

$$a\frac{\partial u}{\partial t} + cu = f$$

Let us express a, c, and f in the new variables, and in what follows $t = x$ explicitly. Let $u(x,y)$ be a solution to (**8.1**) in Ω and $a(x,y) \neq 0$. Put

$$Z(x,p) = u(x,y)\Big|_{y=Y(x,p)} = u(x,Y(x,p))$$

For each (fixed) value of the parameter p, the function $Z(x,p)$ satisfies the equation

$$
\textbf{(9.1)}\qquad \begin{aligned} &Z'_x(x,p) + w(x,p)Z(x,p) = g(x,p)\,,\\ &w(x,p) = \frac{c(x,Y(x,p))}{a(x,Y(x,p))}\,,\quad g(x,p) = \frac{f(x,Y(x,p))}{a(x,Y(x,p))} \end{aligned}
$$

Let us prove this assertion. By the chain rule

$$
\begin{aligned}
Z'_x \; &= \; u'_x + u'_y Y'_x\Big|_{y=Y} = u'_x - u'_y\,\frac{\gamma'_x}{\gamma'_y}\Big|_{y=Y} = u'_x + u'_y\,\frac{b}{a}\Big|_{y=Y}\\
&= \; \frac{1}{a}\Big(au'_x + bu'_y\Big)\Big|_{y=Y} \; = \frac{1}{a}\Big(f - cu\Big)\Big|_{y=Y}\\
&\Rightarrow \quad Z'_x + \frac{c}{a}\Big|_{y=Y} Z = \frac{f}{a}\Big|_{y=Y}
\end{aligned}
$$

The second equality is obtained by the implicit differentiation, the third is nothing but (**8.3**), the fourth equality is obtained by factoring out $1/a$, and the final equality follows from that u is a solution to (**8.1**). This final equation coincides with (**9.1**).

Conversely, suppose that $Z(x,p)$ is a solution to (**9.1**) for a generic characteristic $y = Y(x,p)$ where p is viewed as an *arbitrary* parameter. Then the function

$$
\textbf{(9.2)}\qquad u(x,y) = Z(x,p)\Big|_{p=\gamma(x,y)} = Z(x,\gamma(x,y))
$$

is a solution to (**8.1**). Indeed, by the chain rule

$$
u'_x = Z'_x + Z'_p\gamma'_x\,,\quad u'_y = Z'_p\gamma'_y
$$

The substitution of the partial derivatives into the left side of (**8.1**) shows that $u(x,y)$ is a solution to (**8.1**):

$$
\begin{aligned}
au'_x + bu'_y + cu &= a(Z'_x + Z'_p\gamma'_x) + bZ'_p\gamma'_y + cZ\\
&= Z'_p\Big(a\gamma'_x + b\gamma'_y\Big) + a\Big(Z'_x + \frac{c}{a}\,Z\Big)\\
&= 0 + a\,\frac{f}{a} = f
\end{aligned}
$$

because γ is a solution to (**8.5**) and Z solves (**9.1**) for arbitrary value of p. The above arguments hold under the *assumption* that $Z(x,p)$ is a differentiable function in *both variables x and p* otherwise Z'_p may not exist or the chain rule does not hold. It is possible to show that for a sufficiently smooth c and f, the solution $Z(x,p)$ is from the class C^1 in (x,p) (see Section **9.4** below).

Thus, having found a family of characteristics $\gamma(x,y)=p$ labeled by a real variable p, any solution to a linear first-order PDE can be obtained by the substitution (**9.2**) where $Z(x,p)$ is a solution to the first-order ODE (**9.1**). *This comprises the method of characteristics for solving linear first-order PDEs.*

PROPOSITION **9.1**. (Method of characteristics for linear PDEs) *Let $\gamma(x,y)=p$ be characteristics of a linear first order PDE*

$$au'_x + bu'_y + cu = f\,,\quad (x,y)\in\Omega$$

where $a \neq 0$ in an open region Ω, and the functions a, b, c, and f are from the class $C^1(\Omega)$. Suppose that all characteristics are graphs $y = Y(x,p)$, that is, $\gamma(x, Y(x,p)) = p$ for all x. Let $Z(x,p)$ be a general solution to the linear ODE

$$Z'_x + \frac{c}{a}\Big|_{y=Y} Z = \frac{f}{a}\Big|_{y=Y}$$

for any value of the parameter p. Then the function

$$u(x,y) = Z(x,\gamma(x,y))$$

is a general solution to the PDE.

Note that the solution u must be from the class $C^1(\Omega)$. This is true if the solution $Z(x,p)$ has continuous partial derivatives with respect to x and the parameter p. The latter is shown to be true in Section 11.2 using an explicit form of $Z(x,p)$ if the functions a, b, c, and f have continuous partial derivatives.

9.3. A geometrical interpretation of the method. If $u(x,y)$ is a solution to (**8.1**) in some $\Omega\subset\mathbb{R}^2$, then the surface $z=u(x,y)$ in $\mathbb{R}^3$ is the graph of the solution. The parametric curve

$$\text{(9.3)}\qquad \begin{cases} x = X(t,p) = t \\ y = Y(t,p) \\ z = Z(t,p) = u(X,Y) = u(t,Y(t,p)) \end{cases}$$

lies in the graph $z=u(x,y)$ and its vertical projection onto the xy plane is the characteristic $\gamma(x,y)=p$ whose parametric equations are $x=t$, $y=Y(t,p)$. Since there is only one characteristic passing through each point of Ω, the characteristics do not intersect and their union covers Ω. In other words, the region Ω is *foliated* into a family of non-intersecting curves (characteristics) labeled by a parameter p. Every characteristic in Ω can be vertically lifted to the surface $z=u(x,y)$ by means of the rule $z=Z(t,p)$ established in (**9.1**). But Ω is the union of the characteristics and, hence, the entire surface $z = u(x, y)$ is obtained by this lifting procedure, that is, the union of parametric curves (**9.3**).

The equations $x = t$ and $y = Y(t, p)$ define a transformation, $T : (t, p) \to (x, y)$, that maps a tp plane into the xy plane. This transformation has an inverse, $T^{-1} : (x, y) \in \Omega \to (t, p)$, defined by the equations $t = x$, $p = \gamma(x, y)$. Therefore the z coordinate of every point on the graph $z = u(x, y)$ can be obtained by the substitution of the inverse transformations into $Z(t, p)$, thus recovering $u(x, y) = Z(x, \gamma(x, y))$. The procedure is illustrated in Fig. **2.1**.

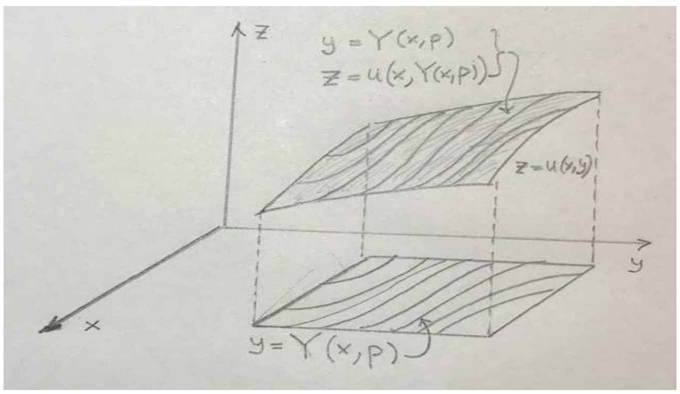

FIGURE **9.1**. A region Ω is foliated by characteristics $y = Y(x, p)$. Each point on any characteristic $(x, Y(x, p), 0)$ and, hence, each point of Ω can be lifted to the point $(x, Y(x, p), Z(x, p))$ that belongs to the graph $z = u(x, y)$ of a solution $u(x, y)$ to the PDE.

9.4. Structure of a general solution. The general solution to (**9.1**) is the sum of the general solution Z_{hm} to the associated homogeneous equation (with $g = 0$) and a particular solution Z_{pt}

$$Z(x, p) = Z_{\text{hm}}(x, p) + Z_{\text{pt}}(x, p) = Z_0 e^{-\int w(x,p)dx} + Z_{\text{pt}}(x, p)$$

where Z_0 is independent of x. A particular solution can be found by the method of undetermined coefficients

$$Z_{\text{pt}}(x, p) = V(x, p) e^{-\int w(x,p)dx}$$

where V is a new unknown function. A substitution of this relation into (**9.1**) yields

$$V'_x(x, p) = g(x, p) e^{\int w(x,p)dx} \quad \Rightarrow \quad V(x, p) = \int g(x, p) e^{\int w(x,p)dx}\, dx$$

It is important to note that the integration constant Z_0 can depend on the parameter p, that is,

$$Z_0 = Z_0(p)$$

Therefore the solution to (**8.1**) contains an arbitrary C^1 function $Z_0(p)$ of $p = \gamma(x, y)$:

$$(\mathbf{9.4})\qquad \begin{aligned} u(x,y) &= u_{\rm hm}(x,y) + u_{\rm pt}(x,y)\,, \\ u_{\rm hm}(x,y) &= Z_0(p)e^{-\int w(x,p)dx}\Big|_{p=\gamma(x,y)}\,, \\ u_{\rm pt}(x,y) &= e^{-\int w(x,p)dx}\int g(x,p)e^{\int w(x,p)dx}dx\Big|_{p=\gamma(x,y)} \end{aligned}$$

Note that by construction, the function $u_{\rm hm}(x, y)$ is a general solution to the associated homogeneous equation (Eq. (**8.1**) with $f(x, y) = 0$). Therefore

- *a general solution to a linear PDE (*$\mathbf{8.1}$*) is the sum of a general solution to the associated homogeneous equation and a particular solution.*

A completion of the proof of Proposition 9.1. The explicit form (**9.4**) of a solution to (**9.1**) allows us to investigate the question about differentiability of $Z(x, p)$ with respect both variables. Indeed, if $w(x, p)$ and $g(x, p)$ have continuous partial derivatives, then $Z(x, p)$ belongs to C^1 as one can readily see from (**9.4**). Partial derivatives of $w(x, p)$ and $g(x, p)$ are continuous if a, b, c, f, and $Y(x, p)$ have continuous partial derivatives. Therefore it is sufficient to require that all coefficients in a linear first-order PDE are from the class C^1. It remains to show that $Y(x, p)$ has continuous partial derivatives. Let us fix x and consider the equation $\gamma(x, y) = p$. Its solution $y = Y(x, p)$ defines y as a function of p for every x. Let us apply the implicit function theorem to the function $\mu(y, p) = \gamma(x, y) - p$ (x is fixed). Then $y = Y(x, p)$ is a solution to $\mu(y, p) = 0$ and by the implicit differentiation

$$Y'_p = -\frac{\mu'_p}{\mu'_y}\Big|_{y=Y} = \frac{1}{\gamma'_y}\Big|_{y=Y}$$

So, Y is a continuously differentiable function of two variables. Thus, it is concluded that, if a, b, c and f are from the class C^1, then a solution $Z(x, p)$ to (**9.1**) belongs to C^1.

9.5. Procedure to solve linear first order PDEs. Proposition **9.1** leads to the following procedure to solve a first order linear PDE where all the coefficients from the class C^1:

Step 1: Find the general solution to the characteristic equation (**8.2**) as level sets $\gamma(x,y) = p$ of a function;
Step 2: Solve $\gamma(x,y) = p$ for y to represent each characteristic as a graph $y = Y(x,p)$;
Step 3: Find the general solution to the linear first order ODE (**9.1**);
Step 4: Use the substitution (**9.2**) to obtain a solution to (**8.1**)

The procedure always gives a solution in a neighborhood of a point where the existence of solutions to Steps 1 and 2 is guaranteed by Theorem **8.1** and the implicit function Theorem **8.2**. If a solution is sought in a region Ω, then one has to show that every characteristic can be written as a graph $y = Y(x,p)$ in whole Ω, not only in a neighborhood of a point. In other words, an explicit form of the function $Y(x,p)$ should exist in whole range of x in Ω. For example, a circle $x^2 + y^2 = p^2$ in a plane is not a graph of a function, but near its each point can be represented as a graph of a function. This restriction will be eliminated in Section **11**. A trade-off is a technical complication of the method.

EXAMPLE **9.1**. *Find all solutions to*

$$u'_x + yu'_y + u = 2y\,, \qquad y > 0\,, \quad x \in \mathbb{R}$$

SOLUTION: Here $a = 1$, $b = y$, $c = 1$, and $f = 2y$.
Step 1. The characteristic equation is integrated by separating variables

$$dy - ydx = 0 \quad \Rightarrow \quad \int \frac{dy}{y} = \int dx \quad \Rightarrow \quad \gamma(x,y) = \ln(y) - x = p$$

where p is constant.
Step 2. The equation $\gamma(x,y) = p$ can be solved for y for any x and p:

$$y = e^p e^x = Y(x,p)$$

Step 3. In this case $w(x,p) = c/a = 1$ and

$$g(x,p) = \frac{f}{a}\Big|_{y=Y} = 2Y(x,p) = 2e^p e^x$$

Equation (**9.1**) reads

$$Z'_x(x,p) + Z(x,p) = 2e^p e^x$$

The associated homogeneous equation has the general solution

$$Z_{\rm hm}(x,p) = Z_0(p)e^{-x}$$

where $Z_0(p)$ is an arbitrary continuously differentiable function of a real variable p. A particular solution is sought in the form $Z_{\rm pt} = V(x,p)e^{-x}$ so that

$$V'_x = 2e^p e^{2x} \quad \Rightarrow \quad V(x,p) = e^p e^{2x}$$

Thus, the general solution to (**9.1**) is

$$Z(x,p) = Z_0(p)e^{-x} + e^p e^x$$

Step 4. The solution to the PDE in question reads

$$u(x,y) = Z_0(\ln(y) - x)e^{-x} + y = \mu(ye^{-x})e^{-x} + y$$

where $\mu(\tau)$ is a C^1 function of a real variable τ; here $Z_0(p) = \mu(e^p)$.
Checking the answer. The solution can be verified by calculating the partial derivatives:

$$\left.\begin{aligned} u'_x &= -\mu' ye^{-2x} - \mu e^{-x} \\ u'_y &= \mu' e^{-2x} + 1 \end{aligned}\right\} \quad \Rightarrow$$

$$\begin{aligned} u'_x + yu'_y + u &= -\mu' ye^{-2x} - \mu e^{-x} + y(\mu' e^{-2x} + 1) + \mu e^{-x} + y \\ &= 2y \end{aligned}$$

as required □

9.6. Exercises.

1-4. Find a general solution to each of the following linear PDEs:
1.

$$u'_x + b_0 u'_y + c_0 u = f_0\,, \qquad (x,y) \in \mathbb{R}^2$$

where b_0, $c_0 \neq 0$, and f_0 are constants.
2.

$$xu'_x + y\Big(\ln(y) - \ln(x) + 1\Big)u'_y + yu = 0\,, \qquad x > 0\,,\ y > 0\,.$$

Hint: Use the result of Problem 2 in Section 10.4.
3.

$$u'_x + 2u'_y + u = xy\,, \qquad (x,y) \in \mathbb{R}^2$$

4.

$$u'_x - 2xyu'_y = 2x^3 y\,, \qquad (x,y) \in \mathbb{R}^2$$

5. Define a change of variables $\tau = x$, $p = \gamma(x,y)$ where $\gamma(x,y) = p$ is a characteristic for each value of p so that the inverse is $\tau = x$, $y = Y(\tau,p)$. Show that

$$a(x,y)\,\frac{\partial}{\partial x} + b(x,y)\,\frac{\partial}{\partial y} = a\Big(\tau, Y(\tau,p)\Big)\,\frac{\partial}{\partial \tau}$$

Then show that the PDE (**8.1**) is reduced to ODE (**9.1**) by means of this change of variables in the case when $a \neq 0$ anywhere.

Selected answers.

1. $u = g(\gamma)e^{-c_0 x} + (f_0/c_0)$, $\gamma = y - b_0 x$ if $c_0 \neq 0$, and $u = f_0 x + g(\gamma)$ if $c_0 = 0$ where $g \in C^1$.
2. $u = g(\gamma)\exp\Big(-\frac{y}{\ln(y/x)}\Big)$, $g \in C^1$, $\gamma = \frac{1}{x}\ln\Big(\frac{y}{x}\Big)$
3. $u = g(\gamma)e^{-x} + xy - 2x - y + 4$, $\gamma = y - 2x$, $g \in C^1$.
4. $u = g(\gamma) - y(x^2+1)$, $\gamma = ye^{x^2}$, $g \in C^1$

10. The Cauchy problem for first order PDEs

The boundary value problem

$$\begin{cases} u'_t(\mathbf{x},t) = f(\mathbf{x},t,u,Du)\,, & t>0\,, \\ u\big|_{t=0} = u_0(\mathbf{x}) \end{cases} \tag{10.1}$$

is called the *Cauchy problem for first-order PDEs.* If t is time, then the PDE is the evolution law of a quantity $u(\mathbf{x},t)$ distributed in space, $\mathbf{x}\in\mathbb{R}^n$. A solution to the Cauchy problem defines the distribution of this quantity in space at a time $t>0$ if the initial distribution was u_0 (at the time $t=0$). The problem can be solved by the method of characteristics. The case of linear PDEs is discussed first.

10.1. Linear first order PDEs in two variables. Suppose f is a linear function in u and u'_x. Here x is real (from $\mathbb{R}$). Consider the Cauchy problem:

$$\begin{cases} u'_t + b(x,t)u'_x + c(x,t)u = f(x,t)\,, & t>0 \\ u\big|_{t=0} = u_0(x) \end{cases} \tag{10.2}$$

Let us use the method of characteristics for Eq. (**8.1**) to solve this problem. In this case $a=1$, and the other parameters b, c, and f are assumed to be functions from the class $C^1(t\geq 0)$ so any solution can be found by the method of characteristics (see Proposition **9.1**. The characteristics are solutions to Eq. (**8.2**) which has the following form in the case considered:

$$dx - b(x,t)dt = 0 \quad\Rightarrow\quad \gamma(x,t) = p$$

Consider a characteristic passing through the point $(x,t)=(p,0)$, that is, $\gamma(p,0)=p$. If such a characteristic exists, then it must be a solution to the initial value problem for ODE:

$$\frac{dx}{dt} = b(x,t)\,,\quad x\Big|_{t=0} = p \quad\Rightarrow\quad x = X(t,p) \tag{10.3}$$

Recall sufficient conditions for the existence and uniqueness of a solution to the initial value problem (**10.3**) from the theory of ODEs.

THEOREM **10.1**. *Suppose that $b(x,t)$ and $b'_x(x,t)$ are continuous in some neighborhood of $(p,0)$. Then the initial value problem (**10.3**) has a unique solution $X(t,p)$ in some open interval $-\delta<t<\delta$ where $\delta>0$.*

The graph $x=X(t,p)$ is also the level curve $\gamma(x,t)=p$. Therefore the function $X(t,p)$ must satisfy the identity

$$\gamma\big(X(t,p),t\big) = p$$

that holds for any t and p. In particular, letting $t = 0$ and using the initial condition in (**10**.**3**), it is concluded that the function γ has the following property

$$\gamma\big(X(0,p),0\big) = \gamma(p,0) = p\,, \tag{10.4}$$

which should hold for any p. The function $\gamma(x,t)$ can be obtained by solving $x = X(t,p)$ for p:

$$x = X(t,p) \quad \Rightarrow \quad p = \gamma(x,t)\,.$$

Next, all solutions to a linear PDE can be obtained from its characteristics and solutions $Z(t,p)$ to ODE (**9**.**1**):

$$\frac{dZ}{dt} + c(X,t)Z = f(X,t)\,, \quad t > 0\,, \tag{10.5}$$

where p is a numerical parameter. According to the theory of ODEs, the initial value problem for a linear ODE has a unique solution. Let $Z(t,p)$ be the solution to (**10**.**5**) that satisfies the initial condition:

$$Z(0,p) = u_0(p)\,.$$

According to the method of characteristics the function

$$u(x,t) = Z(t,p)\Big|_{p=\gamma(x,t)} = Z(t,\gamma(x,t))$$

satisfies PDE in (**10**.**2**) for any solution to the linear equation (**10**.**5**). Owing to the property (**10**.**4**), this solution also satisfies the initial condition:

$$u\Big|_{t=0} = Z(0,\gamma(x,0)) = Z(0,x) = u_0(x)$$

and, hence, it is a solution to the Cauchy problem.

Theorem **10**.**1** guarantees the existence and uniqueness of characteristics only in some open interval $t < \delta$, $\delta > 0$, not for all $t > 0$. So, if the initial value problem (**10**.**3**) happens to have a unique solution for any $p \in \mathbb{R}$ in the whole interval $t > 0$, then the solution to the stated Cauchy problem can be obtained by the method of characteristics.

10.2. Procedure to solve the Cauchy problem. The procedure for solving the Cauchy problem includes the same steps as given in Section **8.1**.

- Step 1: Solve the initial value problem (**10**.**3**) where p is a real parameter;
- Step 2: Represent the characteristics as level sets of a function $\gamma(x,t)$ by solving $x = X(t,p)$ for p;
- Step 3: Solve the initial value problem (**10**.**5**);
- Step 4: Obtain the solution to the Cauchy problem by the substitution $u(x,t) = Z(t,\gamma(x,t))$.

EXAMPLE **10.1**. *Solve the Cauchy problem*

$$\begin{cases} u'_t + xu'_x - u = x\cos(t)\,, & t > 0 \\ u\big|_{t=0} = \frac{1}{1+x^2} \end{cases}$$

SOLUTION: Here $b = x$, $c = -1$, and $g = x\cos(t)$.
Step 1. The initial value problem (**10.3**)

$$\frac{dX}{dt} = X\,, \quad X(0) = p$$

is integrated by separating variables

$$\int \frac{dX}{X} = \int dt \quad \Rightarrow \quad \ln|X| = t + t_0 \quad \Rightarrow \quad X(t) = Ce^t$$

where $C = \pm e^{t_0}$ and t_0 are integration constants. Using the initial condition,

$$X(t,p) = pe^t\,, \quad p \in \mathbb{R}\,,\ t \geq 0$$

So, the solution exists and is unique for all p and $t > 0$.
Step 2. The characteristics are level curves of the following function:

$$x = pe^t \quad \Rightarrow \quad p = \gamma(x,t) = xe^{-t}\,.$$

Step 3. The solution to the initial value problem (**10.5**)

$$\frac{dZ}{dt} - Z = pe^t\cos(t)\,, \quad Z(0) = Z_0 = \frac{1}{1+p^2}$$

can be found by the method of undermined coefficients. The function e^t is a solution to the associated homogeneous equation, $Z' - Z = 0$. A particular solution should have the form

$$Z(t) = V(t)e^t$$

where the function V is a solution to

$$\frac{dV}{dt}\,e^t = pe^t\cos(t) \quad \Rightarrow \quad V(t,p) = p\sin(t)$$

so that the general solution reads

$$Z(t,p) = Ce^t + p\sin(t)e^t \quad \Rightarrow \quad Z(t,p) = \frac{e^t}{1+p^2} + +p\sin(t)e^t\,.$$

The value of the integration constant C was found from the initial condition.
Step 4. The solution to the Cauchy problem is

$$u(x,t) = Z(t,p)\Big|_{p=\gamma(x,t)} = \frac{e^t}{1+x^2e^{-2t}} + x\sin(t)\,.$$

Checking the solution. One has

$$
\begin{aligned}
u\Big|_{t=0} &= u(x,0) = \frac{1}{1+x^2}\,, \\
u'_t &= \frac{e^t(1+x^2e^{-2t})+2x^2e^{-t}}{(1+x^2e^{-2t})^2} + x\cos(t)\,, \\
u'_x &= -\frac{2xe^{-t}}{(1+x^2e^{-2t})^2} + \sin(t)
\end{aligned}
$$

The substitution of the partial derivatives into the equation shows that

$$
\begin{aligned}
u'_t + xu'_x - u &= \frac{e^t(1+x^2e^{-2t})+2x^2e^{-t}-2x^2e^{-t}}{(1+x^2e^{-2t})^2} \\
&\quad + x\cos(t) + x\sin(t) - \frac{e^{-t}}{1+x^2e^{-2t}} - x\sin(t) \\
&= x\cos(t)
\end{aligned}
$$

as required. □

10.3. Structure of the solution. The solution to the Cauchy problem (**10.2**) can always be written as the sum

$$
u(x,t) = u_{\rm hm}(x,t) + u_{\rm pt}(x,t)\,,
$$

where $u_{\rm hm}(x,t)$ is the solution to the Cauchy problem for the associated homogeneous equation (the problem (**10.2**) with $f(x,t)=0$), whereas the function $u_{\rm pt}(x,t)$ is the solution to the Cauchy problem (10.**2**) with zero initial condition ($u_0 = 0$):

$$
\begin{cases} u'_t + bu'_x + cu = 0 \\ u\Big|_{t=0} = u_0 \end{cases} \quad\Rightarrow\quad u = u_{\rm hm}\,,
$$

$$
\begin{cases} u'_t + bu'_x + cu = f \\ u\Big|_{t=0} = 0 \end{cases} \quad\Rightarrow\quad u = u_{\rm pt}\,.
$$

Indeed, it was shown a general solution to the linear problem is always the sum of a general solution to the associated homogeneous equation and a particular solution. Therefore $u(x,t)$ is a solution to the equation in (**10.2**), but it also satisfies the initial condition:

$$
u\Big|_{t=0} = u_{\rm hm}\Big|_{t=0} + u_{\rm pt}\Big|_{t=0} = u_0(x) + 0 = u_0(x)
$$

as required.

Using the general solution (**9.4**) it is not difficult to obtain an explicit form of the solution of two above Cauchy problems. Put

$$
w(t,p) = c\big(X(t,p),t\big)\,, \qquad g(t,p) = f\big(X(t,p),t\big)
$$

Then

$$u_{\rm hm}(x,t) = u_0(p)e^{-\int_0^t w(\tau,p)d\tau}\Big|_{p=\gamma(t,x)},$$

$$u_{\rm pt}(x,t) = e^{-\int_0^t w(\tau,p)d\tau}\int_0^t g(\tau,p)e^{\int_0^\tau w(\tau',p)d\tau'}d\tau\Big|_{p=\gamma(t,x)}$$

This approach is illustrated with an example of the Cauchy problem with constant coefficients.

10.4. The Cauchy problem with constant coefficients. Consider the Cauchy problem for a general linear first order PDE with constant coefficients

$$\textbf{(10.6)}\qquad \begin{cases} u'_t + bu'_x + cu = f(x,t)\,, & t>0\,,\\ u\big|_{t=0} = u_0(x) \end{cases}$$

where b and c are constants. The problem is solved in two steps. First, let us solve the Cauchy problem for the associated homogeneous equation:

$$\textbf{(10.7)}\qquad \begin{cases} u'_t + bu'_x + cu = 0\,, & t>0\,,\\ u\big|_{t=0} = u_0(x) \end{cases}$$

The characteristics satisfying (**10.3**) are

$$dx - bdt = 0 \quad\Rightarrow\quad x - bt = p \quad\Rightarrow\quad x = X(t,p) = p + bt$$

Note that $X(0,p) = p$. The general solution to the equation in (**10.7**) is found by solving the linear ODE problem:

$$\begin{aligned} Z'(t) + cZ(t) = 0 \quad&\Rightarrow\quad Z(t) = Z_0(p)e^{-ct}\\ &\Rightarrow\quad u(x,t) = Z_0(x-bt)e^{-ct} \end{aligned}$$

where Z_0 is any function from C^1. It is chosen to satisfy the initial condition in (**10.7**)

$$u(x,0) = Z_0(x) = u_0(x)$$

Thus, the solution to the homogeneous Cauchy problem (**10.7**) reads

$$\textbf{(10.8)}\qquad u(x,t) = u_0(x-bt)e^{-ct}$$

Note that the initial condition u_0 must be from class C^1 in order for $u(t,x)$ be also from C^1 as required for any solution.

Next, let us solve the Cauchy problem (**10.6**) with vanishing initial condition:

$$\textbf{(10.9)}\qquad \begin{cases} u'_t + bu'_x + cu = f(x,t)\,, & t>0\,,\\ u\big|_{t=0} = 0 \end{cases}$$

Its solution is found by solving the initial value for linear ODE with zero boundary condition:

$$Z' + cZ = f(t, X(t,p)), \quad Z(0) = 0$$

where $X(t,p) = x - bt$. Its solution is found by the substitution $Z(t) = V(t)e^{-ct}$ where $V(0) = 0$ so that

$$\begin{cases} V'(t) = f(p + bt, t)e^{ct} \\ V(0) = 0 \end{cases} \quad \Rightarrow \quad V(t) = \int_0^t f(p + b\tau, \tau)e^{c\tau}\, d\tau$$

and, hence,

$$u(x,t) = e^{-ct} \int_0^t f\Big(x - b(t-\tau), \tau\Big)e^{c\tau}\, d\tau \tag{10.10}$$

Therefore the solution to the Cauchy problem (**10.6**) is the sum of the functions (**10.8**) and (**10.10**)

$$u(t,x) = u_0(x - bt)e^{-ct} + e^{-ct}\int_0^t f\Big(x - b(t-\tau), \tau\Big)e^{c\tau}\, d\tau$$

Thus, the linear Cauchy problem with constant coefficients can be solved in quadratures for generic initial conditions and inhomogeneities.

If $c = 0$ and $f = 0$, the solution describes propagation of the initial shape $u_0(x)$ without distortion with a constant speed b along the x axis. If $b > 0$, the shape moves in the direction of increasing x and in the opposite direction if $b < 0$. If $c > 0$, then the amplitude of the propagating shape is exponentially attenuating with increasing time t.

10.5. Transfer equation. The result can be extended to a multivariable case. The function

$$u(t,\mathbf{x}) = u_0(\mathbf{x} - \mathbf{v}t)e^{-ct} + e^{-ct}\int_0^t f\Big(\mathbf{x} - \mathbf{v}(t-\tau), \tau\Big)e^{c\tau} d\tau \tag{10.11}$$

is the solution to the Cauchy problem

$$\begin{cases} u'_t + \mathbf{v}\cdot\nabla_x u + cu = f(\mathbf{x}, t), \quad t > 0,\ \mathbf{x} \in \mathbb{R}^n \\ u\big|_{t=0} = u_0(\mathbf{x}) \end{cases}$$

where u_0 and f are C^1 functions, $\nabla_x u$ is the gradient of u with respect to $\mathbf{x}$, c is a constant, and $\mathbf{v}$ is a constant vector. The above equation is also known as a *transfer equation*. It describes the energy transfer by radiation or by a flow of particles (e.g., neutrons in a nuclear reactor) in a homogeneous media.

10.6. Exercises.

1-**3**. Solve each of the following Cauchy problems:

1.

$$u'_t + 2u'_x = x\sin(t)\,, \quad t > 0\,, \quad u\Big|_{t=0} = \cos(x)\,.$$

2.

$$u'_t - 2u'_x + u = xe^{-t}\,, \quad t > 0\,, \quad u\Big|_{t=0} = e^{-x^2}\,.$$

3.

$$u'_t + 2tu'_x + xu = 0\,, \quad t > 0\,, \quad u\Big|_{t=0} = e^{x}$$

4. Show that the function (**10**.**11**) is the solution to the Cauchy problem for the transfer equation.

5. Solve the Cauchy problem

$$u'_t + u'_x - 3u'_y + 2u = \cos(t - 2x + y)\,, \quad t > 0\,, \quad u\Bigg|_{t=0} = u_0(x, y)$$

Express the answer in terms of the function $u_0(x, y)$.

Selected answers.

1. $u = \cos(x - 2t) + x(1 - \cos(t)) + 2\sin(t) - 2t$.
2. $u = e^{-(x+2t)^2}e^{-t} + (xt + t^2)e^{-t}$.
3. $u = e^{v}$, $v = \frac{2}{3}t^3 - t^2 - tx + x$.

11. The method of parametric characteristics

In the method of characteristics discussed so far, finding a solution $u(x, y)$ to a PDE it was necessary to represent characteristics $\gamma(x, y) = p$ as as graphs $y = Y(x, p)$. This is possible if $\gamma'_y \neq 0$ everywhere. The function $Y(x, p)$ is then used to find $u(x, y)$, and the method was also based on the assumption that $a(x, y)$ does not vanish in the region where the solution was sought. If these stated conditions are not fulfilled, then an alternative would be to solve $\gamma(x, y) = p$ with respect to x to obtain graphs $x = X(y, p)$ (which is possible only if $\gamma'_x \neq 0$ everywhere). The functions $X(y, p)$ can then be used to find the solution $u(x, y)$ by a similar method (by solving a first order ODE with respect to y), provided $b(x, y) \neq 0$ everywhere. In other words, the restrictions are essentially the same.

11.1. Autonomous systems of ODEs. The necessity to solve the equation $\gamma(x, y) = p$ for either of the variables can be eliminated by using general *parametric equations for characteristics.* A smooth planar curve always admits a parameterization

$$\gamma(x, y) = p \quad \Leftrightarrow \quad \begin{cases} x = X(\tau, p) \\ y = Y(\tau, p) \end{cases}$$

where τ is a parameter on the curve, with a continuous non-vanishing tangent vector

$$\mathbf{T} = \langle X'_\tau, Y'_\tau \rangle$$

Each curve defined as a level curve of a function has many parameterization. Note that the parameter τ labels points on the curve. Clearly, points of a given curve (as a point set in a plane) can be labeled in many ways. But any parameterization satisfy the condition:

$$\gamma\Big(X(\tau, p), Y(\tau, p)\Big) = p$$

for all values of the parameter τ.

For example, a circle admits the following parameterization

$$x^2 + y^2 = p^2 \quad \Leftrightarrow \quad \begin{cases} x = p\cos(\tau) \\ y = p\sin(\tau) \end{cases}$$

because $\cos^2(\tau) + \sin^2(\tau) = 1$ for all τ. The tangent vector

$$\mathbf{T} = \langle -p\sin(\tau), p\cos(\tau) \rangle$$

does not vanish for any τ. Note that a circle is neither a graph $y = Y(x, p)$ nor $x = X(\tau, p)$.

A parameterization of a characteristic curve can always be chosen so that

$$\mathbf{T} = \Big\langle a(x,y), b(x,y)\Big\rangle .$$

at any point (x,y) of the curve. Indeed, for each fixed value of p, $(dx,dy) = (X'_\tau d\tau, Y'_\tau d\tau) = \mathbf{T}d\tau$, and it follows from the characteristic equation that

$$a(X,Y)Y'_\tau - b(X,Y)X'_\tau = 0$$

for all τ (or for any point of the curve). This equation means that $\mathbf{T}$ must be parallel to the vector (a,b) or *proportional* to it:

$$\textbf{(11.1)} \qquad \mathbf{T} = \langle X'_\tau, Y'_\tau\rangle = \lambda(\tau)\Big\langle a(X,Y), b(X,Y)\Big\rangle$$

for some continuous function $\lambda(\tau) \neq 0$ that does not vanish anywhere. But $\lambda(\tau)$ can always be set to 1 by a *reparameterization* of the curve (or by *relabeling* points of the curve). Let a new parameter be

$$\tilde{\tau} = \int \lambda(\tau)d\tau \quad \Rightarrow \quad d\tilde{\tau} = \lambda(\tau)d\tau$$

Then by the chain rule

$$\frac{d}{d\tilde{\tau}}X = \frac{1}{\lambda}\frac{d}{d\tau}X = \frac{1}{\lambda}\lambda a = a\,,$$
$$\frac{d}{d\tilde{\tau}}Y = \frac{1}{\lambda}\frac{d}{d\tau}Y = \frac{1}{\lambda}\lambda b = b$$

In other words, a reparameterization scales the tangent vector by a non-zero factor and, hence, and this freedom can be used to set $\lambda(\tau) = 1$. The following fact has been established.

PROPOSITION **11.1**. *All characteristics can be found by solving the following autonomous system of ODEs*

$$\textbf{(11.2)} \qquad \begin{cases} X'_\tau = a(X,Y) \\ Y'_\tau = b(X,Y) \end{cases}$$

If $X = X(\tau)$ and $Y = Y(\tau)$ are solutions to (**11.2**), then the *shifted functions*

$$\textbf{(11.3)} \qquad X = X(\tau+\tau_0)\,, \quad Y = Y(\tau+\tau_0)$$

are also solutions for any τ_0 because a and b are independent of a parameter τ. This is a general property of all autonomous systems. Solutions to (**11.2**) are curves in the xy plane and the collection of all such curves is called a *phase portrait* of the autonomous system.

Sufficient conditions for the existence and uniqueness of the initial value problem for autonomous systems is stated in the following theorem.

THEOREM **11.1**. *Suppose that $a(x, y)$ and $b(x, y)$ are from $C^1(\Omega)$ for some open Ω and they do not vanish simultaneously anywhere in Ω. Then for any $(x_0, y_0) \in \Omega$ the autonomous system (***11.2***) has a unique solution $X(\tau)$ and $Y(\tau)$ in some interval $t_0 - \delta < \tau < \tau_0 + \delta$, $\delta > 0$, satisfying the initial conditions*

$$X(\tau_0) = x_0\,, \quad Y(\tau_0) = y_0$$

The theorem asserts that there is only one characteristic passing through each point of Ω. Therefore any two points in Ω either lie on the same characteristics or on two non-intersecting characteristics. This implies that the region Ω is the union of non-intersecting characteristics, or Ω is said to be *foliated* by the characteristics. The shifted solution (**11.3**) describes the same curve in the foliation of Ω.

11.2. Hamiltonian mechanics as an autonomous system. Hamiltonian mechanics for one degree of freedom (like a particle on a line) is defined by a Hamiltonian which is a C^2 function $H(q, p)$ of two variables q, called a coordinate, and p, called a momentum. The plane spanned by ordered pairs (q, p) is called a *phase space*. A Hamiltonian of a dynamical system is the energy as a function on the phase space. A Hamiltonian system evolves in time according to the Hamiltonian equations of motion

$$q' = H'_p(q, p)\,, \quad p' = -H'_q(q, p)$$

For example, the energy of a particle of a mass m moving under the action of a conservative force $F(q) = -V'(q)$ is

$$H(p, q) = \frac{p^2}{2m} + V(q)$$

where the function $V(q)$ is called a *potential energy*. The Hamiltonian equations of motion are equivalent to Newton's equations:

$$q' = \frac{p}{m}\,, \quad p' = -V'(q) \quad \Rightarrow \quad mq'' = -V'(q) = F(q)$$

Recall that any ODE of order n can be reduced to a system of first order ODEs. In this way, Newton's equations for conservative forces can be reduced to Hamiltonian ones.

A solution to Hamiltonian equations, $q = q(t)$ and $p = p(t)$, is a trajectory of the system in the phase space. The value of the Hamiltonian is constant along each trajectory. In other words, the energy is conserved in due course of evolution. Indeed, using the chain rule and Hamiltonian equations of motion

$$\frac{d}{dt} H\big(q(t), p(t)\big) = H'_q q'(t) + H'_p p'(t) = -p'(t)q'(t) + q'(t)p'(t) = 0$$

The trajectories of a Hamiltonian system are level sets of the Hamiltonian

$$H(q,p) = E$$

The parameter E is called the energy of the system. For example, a harmonic oscillator is a Hamiltonian system with

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2}\,q^2$$

where $k > 0$ is a parameter. It describes vibrations of a particle of mass m attached to a spring (with the spring constant k). The phase space trajectories (or the *phase portrait*) are ellipses with the p and q semi-axes equal to $(2mE)^{1/2}$ and $(2E/k)^{1/2}$, respectively.

The phase portrait defines the shapes of phase-space trajectories, but does not say anything how the system moves along them. A solution to the Hamiltonian equation of motion gives a *particular parameterization* of these trajectories where the parameter is a physical time. Suppose that the Hamiltonian H is such that only one level curve passes through each point (q_0, p_0) (see Theorem **11.1**). Then level curves do not intersect and each level curve corresponds to a unique value of the energy E. In this case, all solutions to the Hamiltonian equations of motion are uniquely labeled by values of E:

$$q = q(t,E)\,, \quad p = p(t,E)\,,$$

and there exists an *initial data curve* $q = q_0(E)$, $p = p_0(E)$, where E is a parameter along the curve, that intersects each trajectory only once at a point $q_0(E)$, $p_0(E)$ so that

$$q(0,E) = q_0(E)\,, \quad p(0,E) = p_0(E)\,.$$

In the example of a harmonic oscillator, any ray from the origin can chosen as an initial value curve because it intersects each ellipse only once. For instance,

$$p_0(E) = \sqrt{2mE}\,, \quad q_0(E) = 0\,, \quad E > 0$$

The solutions to the initial value problem for the autonomous system

$$q' = H'_p = \frac{p}{m}\,, \quad p' = -H'_q = -kq$$

reads

$$q(t,E) = \sqrt{\frac{2E}{k}}\,\sin(\omega t)\,, \quad p(t,E) = \sqrt{2mE}\,\cos(\omega t)$$

where $\omega = \sqrt{k/m}$ is the *frequency* (it defines the period of vibrations $T = 2\pi/\omega$). Each curve in the phase portrait is labeled by a parameter (energy) $E > 0$, and points of the curve for a given value of E are labeled by the parameter (time) t.

11.3. Initial data curve. Suppose that hypotheses of Theorem **11.1** hold for the system (**11.2**) for each point in an open region Ω. Therefore all characteristics do not intersect in Ω and their union is Ω. Consider a smooth parametric curve C, $x = \nu(p)$, $y = \mu(p)$, where p is a parameter, that is not a characteristic in Ω (not a solution to (**11.2**) with $\tau = p$). Therefore this curve is intersecting characteristics in Ω. Suppose that this curve can be chosen so that it intersects each characteristic in Ω at a single point. Let us choose the initial data for the autonomous system (**11.2**) to be points of this curve

$$X(\tau_0) = \nu(p) = x_0\,, \quad Y(\tau_0) = \mu(p) = y_0\,. \tag{11.4}$$

For every value of the parameter p, there is a unique solution

$$X = X(\tau, p)\,, \quad Y = Y(\tau, p)\,,$$

so that all characteristics are labeled by values of p.

If there exists one initial data curve with the said properties, then there are infinitely many other such curves. Indeed, owing to the shift property (**11.3**), the functions $X(\tau + s, p)$ and $Y(\tau + s, p)$ with a shifted argument are also solutions to the autonomous system for any s, and

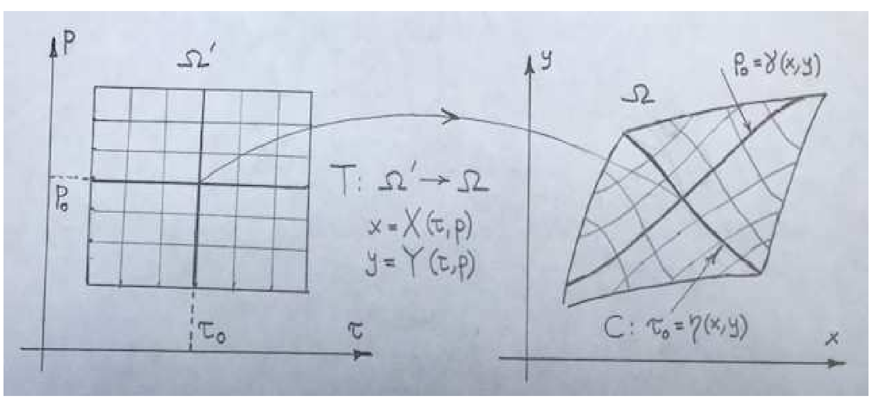

FIGURE **11.1**. Transformation T of an open region Ω' in the τp plane to an open region Ω in the xy plane defined by all characteristics in Ω passing through an initial data curve C: $\tau_0 = \eta(x, y)$. The transformation (**11.5**) is a change of variables in Ω. The characteristics and initial data curves are the coordinate curves of the new variables τ and p.

the parametric curve $x = X(\tau_0 + s, p)$, $y = Y(\tau_0 + s, p)$, with p being the parameter, also intersects all characteristics because the points $(X(\tau_0, p), Y(\tau_0, p)) = (\nu(p), \mu(p))$ and $(X(\tau_0 + s, p), Y(\tau_0 + s, p))$ lie on the same characteristic corresponding to a given p. Therefore every point $(x, y) \in \Omega$ can be viewed as the point of intersection of a characteristic corresponding to a unique value of p and an initial data curve that corresponds to a unique value of the parameter τ. This implies that all solutions to (**11.2**) define a transformation of a part Ω' of the τp plane into Ω in the xy plane which has the inverse. In other words, this transformation is a *change of variables* on Ω. The coordinate curves of the variables τ and p in the xy plane are intersecting only once, forming a curvilinear grid in Ω. The inverse transformation is obtained by solving the parametric equations for τ and p:

$$\begin{matrix} x = X(\tau, p) \\ y = Y(\tau, p) \end{matrix} \quad \Rightarrow \quad \begin{matrix} \tau = \eta(x, y) \\ p = \gamma(x, y) \end{matrix} \tag{11.5}$$

The level curves of the functions η and γ are new coordinate curves. The constructed change of variables is illustrated in Fig. **13.1**.

If at least one initial data curve exists in Ω, then all solutions to (**8.1**) can be found as follows using the change of variables defined by the corresponding characteristics.

PROPOSITION **11.2**. (Method of parametric characteristics) *Suppose that there exists a smooth curve $x = \nu(p)$, $y = \mu(p)$ in an open region Ω such that every characteristic of the linear equation*

$$au'_x + bu'_y + cu = f\,, \quad (x, y) \in \Omega$$

where the functions a, b, c, and f are from the class $C^1(\Omega)$, is a unique solution to the initial value problem

$$\begin{aligned} &X'_\tau = a(X, Y)\,, \quad Y'_\tau = b(X, Y)\,, \\ &X\Big|_{\tau=0} = \nu(p)\,, \quad Y\Big|_{\tau=0} = \mu(p) \end{aligned} \tag{11.6}$$

If $\tau = \eta(x, y)$ and $p = \gamma(x, y)$ is the inverse transformation on Ω of the transformation defined by characteristics $x = X(\tau, p)$ and $y = Y(\tau, p)$, then a general solution to the PDE is

$$u(x, y) = Z\Big(\eta(x, y), \gamma(x, y)\Big) \tag{11.7}$$

where the function $Z(\tau, p)$ is a general solution to the linear ODE:

$$Z'_\tau + c(X, Y)Z = f(X, Y) \tag{11.8}$$

for each value of the parameter p.

A proof is similar to the proof of Proposition **9.1**. However, it is technically more involved because one has to deal with a general change of variables defined by parametric characteristics. It is given at the and of this section. Here it is only emphasized that the constructed change of variables is a realization of the basic idea for solving linear first order PDEs that was introduced in Section **10.1**. It will be shown in Section **13.6** that in the new variables

$$a(x,y)\,\frac{\partial}{\partial x}+b(x,y)\,\frac{\partial}{\partial y}=\frac{\partial}{\partial \tau}$$

by the chain rule so that the PDE in question is reduced to the ODE (**11.8**).

Remark. If $a(x,y)\neq 0$ everywhere, then Eq. (**8.1**) is equivalent to

$$u'_x+\frac{b}{a}\,u'_y+wu=g\,,\quad w=\frac{c}{a}\,,\quad g=\frac{f}{a}$$

In this case, the system (**11.2**) has the form

$$X'_\tau=1\,,\quad Y'_\tau=\frac{b(X,Y)}{a(X,Y)}$$

Therefore $x=X(\tau)=\tau$ and $y=Y(\tau,p)$ and, hence, the characteristics are graphs $y=Y(x,p)$. The inverse transformation is obtained by solving the equation $y=Y(x,p)$ for p (to obtain $p=\gamma(x,y)$) because $\tau=\eta(x,y)=x$.

11.4. Geometrical significance of the method. The established result shows that for each p the parametric curve

$$x=X(t,p)\,,\quad y=Y(t,p)\,,\quad z=Z(t,p)\,, \tag{11.9}$$

obtained by solving the autonomous system (**11.6**) and the linear problem (**11.8**), lies in the graph $z=u(x,y)$ of a solution to (**8.1**) and the whole graph is the union of the non-intersecting space curves (**11.9**). This procedure is illustrated in Fig. **13.2**. Since the transformation (**11.5**) is a change of variables in Ω, the graph of any solution to a linear first order PDE is obtained by the substitution (**11.7**).

11.5. A procedure to solve linear 1st order PDEs. Proposition **11.2** gives the following procedure for solving any linear first order PDE in two variables.

Step 1: Find the phase portrait of characteristics in Ω by solving either $ady-bdx=0$ or the system (**11.2**).

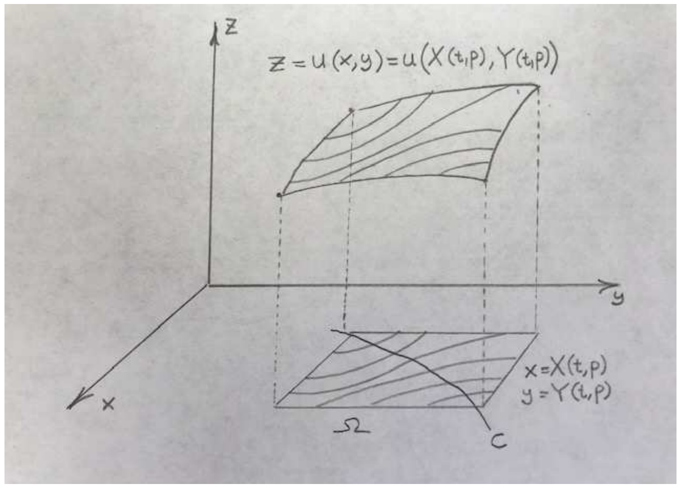

FIGURE **11.2**. Lifting the foliation of Ω by parametric characteristics to obtain the graph $z = u(x, y)$ of a solution to a first order PDE. A foliation of Ω is obtained by solving (**11.2**) with initial data along some smooth curve C and the lifting rule by solving (**11.8**).

Step 2: Use the phase portrait to find an initial data curve so that there is only one characteristic passes through each point of the initial data curve, and a region Ω in which a solution to the PDE is sought is the union of all such characteristics;
Step 3: Solve the initial value problem (**11.6**) to determine the change of variables in Ω;
Step 4: Find a general solution to the linear ODE (**11.8**);
Step 5: Find the inverse transformation of the change of variables defined by the characteristics, and construct a general solution to the PDE by the substitution (**11.7**).

EXAMPLE **11.1**. *Find all characteristics as parametric curves and use them to solve the equation:*

$$-yu'_x + xu'_y + 4xyu = 0\,, \quad \Omega:\ x^2 + y^2 > 0\,.$$

Investigate if any solution can be extended to the origin. Compare the set of solutions with all solutions when $\Omega = \mathbb{R}^2$

SOLUTION: Step 1: The characteristics are

$$-ydy - xdx = 0 \quad \Rightarrow \quad x^2 + y^2 = p^2\,, \quad p > 0\,.$$

So, the characteristics are concentric circles with the center at the origin. This is the phase portrait of characteristics in Ω.

Step 1. Alternative method: The autonomous system (**11.2**) is

$$X' = -Y\,, \quad Y' = X$$

Substituting the first equation $Y = -X'$ into the second one, one infers that

$$X'' + X = 0 \quad \Rightarrow \quad X = A\cos(\tau) - B\sin(\tau)$$

and from the first equation it follows that

$$Y = -X' = A\sin(\tau) + B\cos(\tau)$$

where A and B are integration constants. These relations are nothing but relations between components of a planar vector obtained from another planar vector by a rotation through an angle τ. It can be written in a matrix form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

The 2×2 matrix in this relation is an orthogonal matrix. A rotation preserves the length of a vector:

$$x^2 + y^2 = A^2 + B^2\,.$$

Therefore all characteristics are concentric circles.

Step 2: All circles in the phase portrait intersects the positive x axis, $(x, y) = (p, 0)$, $p > 0$. So, by setting the initial data to be

$$X(0) = \nu(p) = p\,, \quad Y(0) = \mu(p) = 0\,, \quad p > 0$$

all characteristics are uniquely labeled by the parameter $p > 0$. The choice of initial data to label all characteristics is not unique, but it is important that it exists. For example, one can take any strait line through the origin: $x = p$, $y = mp$, $p > 0$ for some slope m. The choice of an initial data curve is not relevant for a final form of the solution $u(x, y)$.

Step 3: The initial value problem for the autonomous system (**11.6**) reads

$$X' = -Y\,, \quad Y' = X\,, \quad X\Big|_{\tau=0} = p\,, \quad Y\Big|_{\tau=0} = 0$$

Using the general solution found above and the initial data, the integration constants are determined: $A = p$ and $B = 0$. The change of variables reads

$$\begin{cases} x = X(\tau, p) = p\cos(\tau) \\ y = Y(\tau, p) = p\sin(\tau) \end{cases} \qquad p > 0\,, \quad \tau \in [0, 2\pi]$$

The new variables are polar coordinates in Ω with p and τ being the radial variable and polar angle, respectively. The initial data can also be set at any point $\tau = \tau_0$. This would only change the range of

variables τ (in polar coordinates the polar angle can be counted from any line through the origin).
Step 4: Next, let us find a general solution to Eq. (**11**.**8**):

$$Z'_\tau + 4XYZ = 0 \quad \Rightarrow \quad Z'_\tau + 2p^2\sin(2\tau)Z = 0\,,$$

where the double angle formula was used, $2\sin(\tau)\cos(\tau) = \sin(2\tau)$. Separating the variables, one infers that

$$Z(\tau,p) = Z_0(p)e^{-2p^2\int\sin(2\tau)d\tau} = Z_0(p)e^{p^2\cos(2\tau)}$$

for some $Z_0(p)$.
Step 4: The inverse transformation is

$$\tau = \arctan\Big(\frac{y}{x}\Big) = \eta(x,y)\,, \quad p = \sqrt{x^2+y^2} = \gamma(x,y)\,.$$

with appropriately chosen branches of the arctangent function, just like in polar coordinates. Therefore a general solution to the PDE in question reads

$$u(x,y) = Z\big(\eta(x,y),\gamma(x,y)\big) = Z_0\Big(\sqrt{x^2+y^2}\Big)\,e^{x^2-y^2}$$

where Z_0 is any C^1 function of a single *positive* variable. Here the trigonometric identity $\cos(2\tau) = \cos^2(\tau) - \sin^2(\tau)$ was used so that $p^2\cos(2\tau) = x^2 - y^2$.

Including the origin. If the origin is included into Ω, then $u(x,y)$ must have continuous partial derivatives at the origin. This implies that $Z'_0(0)$ must exist. Therefore not every solution would have a smooth extension to the origin. For example, solutions with $Z_0(p) = \ln(p)$ or $Z(p) = p^{-m}$, $m > 0$, are not solutions to the PDE in the whole plane, but they are solutions in the plane with the origin removed. This is similar to solutions obtained by separating variables in the 2D Laplace equation in polar coordinates. Note well that the PDE considered is reduced to ODE by means of the chain rule in polar coordinates

$$-y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} = \frac{\partial}{\partial \tau}$$

that makes sense everywhere except the origin where the Jacobian vanishes (it is a singular point of the change of variables). □

11.5.1. On the choice of parameterization of characteristics. The change of variables is obtained by solving the initial value problem (**11**.**6**). If all characteristics are determined as level curves $\gamma(x,y) = p$, then there many ways to parameterize them. Can any of these parameterizations be used? For example, the concentric circles in the above example can be parameterized as

$$x = p\cos(\tau^3)\,, \quad y = p\sin(\tau^3) \quad \Rightarrow \quad x^2 + y^2 = p^2$$

These functions do not satisfy (**11.6**) but they describe the same curves and, hence, both parameterization are related via a reparameterization: $\tau^3 \to \tau$, in this case. It was noted that the tangent vector $\mathbf{T} = \langle X'_\tau, Y'_\tau\rangle$ is parallel to the vector $\langle a, b\rangle$. Therefore for a generic parameterization $X'_\tau = \lambda a$ and $Y'_\tau = \lambda b$. It is not difficult to see that $\lambda = 3\tau^2$ in the example considered (with $a = -y$ and $b = x$). The reader is asked to prove that the change of variables defined by a generic parameterization of characteristics has the property

$$a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} = \frac{1}{\lambda}\,\frac{\partial}{\partial \tau}$$

which changes ODE (**11.8**) because the autonomous equations (**11.2**), where $\lambda = 1$, were used to derived it. So, if a parameterization is found from $\gamma(x, y) = p$, then either (**11.6**) have to be checked, and, if needed, a reparameterization must be carried out so that $\lambda = 1$, or (**11.8**) is modified as stated above (Z'_τ is changed to $(1/\lambda)Z'_\tau$).

11.5.2. Singular points. The procedure is based on the assumption that the initial value problem for the autonomous system has a unique solution at any point of Ω. In this case, the characteristics define a change of variables. If there are points in Ω where this assumption is false, and there are more than one characteristic passing through such a point (see Theorem **11.1**), then the change of variables is not defined, and the method cannot be used to construct a solution (or show that it exists) at such points. Points where a and b vanish simultaneously are singular points of the change of variables defined by characteristics because $\gamma'_x = \gamma'_y = 0$ and the Jacobian vanishes at these points:

$$J = \det\begin{pmatrix} \eta'_x & \eta'_y \\ \gamma'_x & \gamma'_y \end{pmatrix} = 0$$

Indeed,

$$a\gamma'_x + b\gamma'_y = 0 \quad \Rightarrow \quad \boldsymbol{\nabla}\gamma = h\langle -b, a\rangle$$

for some function h. If the gradient $\boldsymbol{\nabla}\gamma$ exists everywhere, then it must vanish at any point where $a = b = 0$. The method of parametric characteristic can still be used in a region from which these points are removed. However not all solutions obtained *near* these points can be *extended* to singular points of the change of variables defined by the characteristics. For instance, in Example **11.1**, the characteristics are level curves of $\gamma(x, y) = x^2 + y^2$. Its gradient $\boldsymbol{\nabla}\gamma = 2\langle x, y\rangle$ vanishes at the origin, where $a = b = 0$. It was shown that not all solutions obtained by the method of parametric characteristics near the origin are smoothly extendable to the origin. If a solution is sought in a region which contains singular points, then one can find all solutions with the singular points excluded and select only those solutions that have a smooth extension to the singular points.

11.6. A proof of Proposition 11.2. Let $u(x, y)$ be a solution to (**8.1**). Put

$$Z(\tau, p) = u(X(\tau, p), Y(\tau, p))$$

Let us show that it satisfies (**11.8**). By the chain rule

$$\begin{aligned} Z'_\tau + c(X,Y)Z &= u'_x(X,Y)X'_\tau + u'_y(X,Y)Y'_\tau + c(X,Y)Z \\ &= u'_x(X,Y)a(X,Y) + u'_y(X,Y)b(X,Y) + c(X,Y)Z \\ &= f(X,Y) \end{aligned}$$

where the second equality follows from (**11.2**) and the last equality follows from (**8.1**).

Conversely, let $Z(\tau, p)$ be solutions to (**11.8**) in some neighborhood of (τ_0, p_0) so that $X(\tau_0, p_0) = x_0$ and $Y(\tau_0, p_0) = y_0$. Let the functions $\tau = \eta(x, y)$ and $p = \gamma(x, y)$ define the inverse transformation (**11.5**) near (x_0, y_0) so that $\tau_0 = \eta(x_0, y_0)$ and $p_0 = \gamma(x_0, y_0)$. Let us show that the function

$$u(x, y) = Z(\tau(x, y), \gamma(x, y))$$

is a solution to the original PDE (**8.1**) near (x_0, y_0).

In order to do so, one has to establish some properties of the transformation (**11.5**). Suppose first that $a(x_0, y_0) \neq 0$ (and, hence, $\gamma'_y(x_0, y_0) \neq 0$ by the properties of the characteristics). For all (x, y) near (x_0, y_0), the following identity holds:

$$x = X\big(\eta(x, y), \gamma(x, y)\big)$$

Differentiating this identity with respect to x and y, the following two relations are obtained by the chain rule

$$\begin{aligned} 1 &= X'_\tau \eta'_x + X'_p \gamma'_x \\ 0 &= X'_\tau \eta'_y + X'_p \gamma'_y \end{aligned}$$

The second equation is solved for X'_p and the latter is substituted into the first one to obtain the following relation that holds for any (τ, p)

near (τ_0, p_0):

$$1 = X'_\tau\left(\eta'_x - \eta'_y\,\frac{\gamma'_x}{\gamma'_y}\right) = X'_\tau\left(\eta'_x + \eta'_y\,\frac{b}{a}\right)$$

$$= a\eta'_x + b\eta'_y \tag{11.10}$$

where the second equality is justified by (**8**.**6**) and the last one by $X'_\tau = a(X, Y)$ in (**11**.**2**). Relation (**11**.**10**) also holds if $a(x_0, y_0) = 0$. In this case, $b(x_0, y_0) \neq 0$ (and $\gamma'_x(x_0, y_0) \neq 0$). Relation (**11**.**10**) can be inferred from the identity

$$y = Y\big(\eta(x, y), \gamma(x, y)\big)$$

by differentiating it with respect to y and x and combining the obtained equations in a fashion similar to the previous case. The technicalities are left to the reader as an exercise.

In a neighborhood of (x_0, y_0), the chain rule holds

$$\frac{\partial}{\partial x} = \frac{\partial\tau}{\partial x}\frac{\partial}{\partial\tau} + \frac{\partial p}{\partial x}\frac{\partial}{\partial p} = \eta'_x\,\frac{\partial}{\partial\tau} + \gamma'_x\,\frac{\partial}{\partial p}$$

$$\frac{\partial}{\partial y} = \frac{\partial\tau}{\partial y}\frac{\partial}{\partial\tau} + \frac{\partial p}{\partial y}\frac{\partial}{\partial p} = \eta'_y\,\frac{\partial}{\partial\tau} + \gamma'_y\,\frac{\partial}{\partial p}$$

$$a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} = \left(a\eta'_x + b\eta'_y\right)\frac{\partial}{\partial\tau} + \left(a\gamma'_x + b\gamma'_y\right)\frac{\partial}{\partial p} = \frac{\partial}{\partial\tau}$$

where the identity (**11**.**10**) and the property of characteristics (**8**.**5**) were used. Using this rule the left side of Eq. (**8**.**1**) for the function (**11**.**7**) at (x_0, y_0) is found to be

$$au'_x + bu'_x + cu = Z'_\tau + cZ = f\,;$$

where the last equality follows from (**11**.**8**). Thus, the equation is fulfilled at (x_0, y_0). But the choice of (x_0, y_0) is arbitrary. Therefore the function (**11**.**7**) satisfies Eq. (**8**.**1**) at any point.

11.7. Exercises.

1. Sketch the phase portrait for the Hamiltonian system with the following Hamiltonian

$$H(p, q) = \frac{1}{2}\,p^2 + V(q)\,, \quad V(q) = V_0(a^2 - q^2)^2\,, \quad V_0 > 0\,.$$

2. Show that (**11**.**10**) is valid in a neighborhood of any point (x_0, y_0) where $a(x_0, y_0) = 0$.

3. Solve

$$yu'_x - 4xu'_y + (4x^2 + y^2)u = -xy\,, \qquad y > 0$$

by using a parametric representation of characteristics. Show that the characteristics are ellipses and each characteristic intersects the positive y axis only once and, hence, the positive y axis can be chosen as an initial data curve. Follow the procedure of Example **11.1** to finish the problem.

4. Suppose that parametric characteristics $x = X(\tau, p)$, $y = Y(\tau, p)$ are obtained directly from $\gamma(x, y) = p$ and so that (**11.1**) holds for some $\lambda(\tau) \neq 0$. Show that in this case,

$$a(x, y)\,\frac{\partial}{\partial x} + b(x, y)\,\frac{\partial}{\partial y} = \frac{1}{\lambda(\tau)}\frac{\partial}{\partial \tau}$$

Show that $x = p\cos(\alpha\tau)$, $y = 2p\sin(\alpha\tau)$, where $\alpha \neq 0$, describe all characteristics in Problem 3 and they satisfy (**11.1**) with some $\lambda \neq 1$. Use this change of variables and the above chain rule to solve the PDE in Problem 3.

Selected answers.

3.

$$u = g(s)e^{-\frac{s}{2}\arctan(\frac{2x}{y})} + \frac{y^2 - 4x^2 - sxy}{16 + s^2}\,, \quad s = 4x^2 + y^2\,,$$

where g is any function from the class $C^1((0, \infty))$.

12. Quasi-linear first order PDEs

12.1. Characteristics. Consider the equation

$$a(x, y, u)u'_x + b(x, y, u)u'_y = f(x, y, u)\,, \quad (x, y) \in \Omega \tag{12.1}$$

and the following autonomous system of ODEs associated with it

$$\begin{cases} X' = a(X, Y, Z) \\ Y' = b(X, Y, Z) \\ Z' = f(X, Y, Z) \end{cases} \tag{12.2}$$

A solution to this system defines a parametric curve in space

$$x = X(\tau)\,, \quad y = Y(\tau)\,, \quad z = Z(\tau)\,.$$

It is called a *characteristic* of the quasi-linear equation (**12.1**).

Let V_Ω be a cylinder in space with the horizontal cross section Ω, $V_\Omega = \Omega \times (z_1, z_2)$. Consider the initial value problem for this system so that

$$X(\tau_0) = x_0\,, \quad Y(\tau_0) = y_0\,, \quad Z(\tau_0) = z_0$$

Suppose $a(x, y, z)$, $b(x, y, z)$ and $f(x, y, z)$ are from the class $C^1(V_\Omega)$ and do not vanish simultaneously at any point in V_Ω. Then the initial value problem has a unique solution for any $(x_0, y_0, z_0) \in V_\Omega$ in some open interval containing τ_0, and the solution defines a smooth curve in V_Ω. By the uniqueness of the solution, the characteristics do not intersect anywhere in V_Ω and, hence, V_Ω is the union of all characteristics. The characteristics fill out V_Ω like spaghetti in a sauce pan.

Suppose that (**12.1**) has a solution. Then there exists a particular collection of characteristics in V_Ω whose union forms the graph $z = u(x, y)$ of the solution, much like in the case of linear PDEs.

Proposition **12.1**. *Let $u(x, y)$ be a solution to (**12.1**). Any characteristic through a point (x_0, y_0, z_0), where $z_0 = u(x_0, y_0)$, lies in the graph $z = u(x, y)$ of the solution.*

Proof. Let $x = X(\tau)$, $y = Y(\tau)$, $z = Z(\tau)$ be a characteristic through a point (x_0, y_0, z_0) of the graph $z = u(x, y)$. Consider the parametric curve

$$x = X(\tau)\,, \quad y = Y(\tau)\,, \quad z = u(X(\tau), Y(\tau)) \equiv \tilde{Z}(\tau)$$

By construction, this curve lies in the graph and passes through the same point in the graph as the characteristic:

$$X(\tau_0) = x_0\,, \quad Y(\tau_0) = y_0\,, \quad \tilde{Z}(\tau_0) = u(x_0, y_0) = z_0$$

Therefore, if this curve is proved to be a solution to the initial value problem (**12.2**), then by the uniqueness of the solution $Z(t) = \tilde{Z}(t)$, which implies that the characteristic lies in the graph. By the chain rules

$$\begin{aligned}\frac{d\tilde{Z}}{d\tau} &= u'_x(X,Y)\frac{dX}{d\tau} + u'_y(X,Y)\frac{dY}{d\tau}\\ &= u'_x(X,Y)a(X,Y,Z) + u'_y(X,Y)b(X,Y,Z)\\ &= f(X,Y,Z)\\ &= \frac{dZ}{d\tau}\end{aligned}$$

where the second equality follows from (**12.1**), the third is obtained from that u is a solution to (**12.1**), and the last one is again a consequence of (**12.1**). Therefore the functions Z and $\tilde{Z}$ can differs by an additive constant. This constant must be zero because $Z(\tau_0) = z_0 = \tilde{Z}(\tau_0)$. □

Since there is only one characteristic passing through each point of the graph $z = u(x,y)$, the graph is the union of non-intersecting characteristics. So, a solution to (**12.1**) can be found if one identifies all characteristics whose union would form the graph of the solution. In the case of linear PDEs, this task was accomplished by analyzing the phase portrait of the autonomous system and finding an initial data curve.

12.2. An initial data curve. Consider a smooth parametric curve C in V_Ω:

$$x = \nu(p)\,, \quad y = \mu(p)\,, \quad z = \lambda(p)$$

Then the initial value problem with

$$X(\tau_0) = \nu(p)\,, \quad Y(\tau_0) = \mu(p)\,, \quad Z(\tau_0) = \lambda(p)$$

for the system (**12.2**) has a unique solution for every p. and this solution defines a *parametric surface*

$$x = X(\tau,p) \quad y = Y(\tau,p)\,, \quad z = Z(\tau,p)\,, \quad (\tau,p)\in\Omega' \tag{12.3}$$

This surface is the union of the characteristics labeled by a parameter p as depicted in Figure **14.1**. Of course, one has to assume that either C does not intersect a characteristic in V_Ω or it does so only at one point.

In general, a parametric surface is not a graph of some function. For example, a unit sphere is a parametric surface

$$\begin{cases} x = X(\tau,p) = \sin(p)\cos(\tau)\\ y = Y(\tau,p) = \sin(p)\sin(\tau)\\ z = Z(\tau,p) = \cos(p)\end{cases} \quad\Rightarrow\quad x^2+y^2+z^2 = 1$$

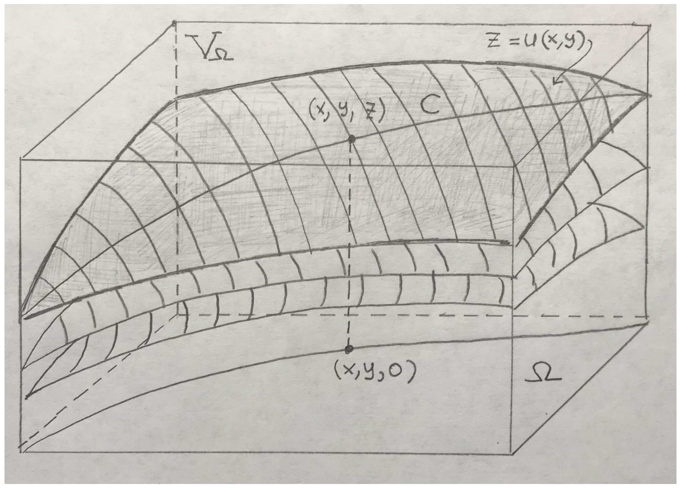

FIGURE **12.1**. Non-intersecting characteristics filling out the open cylinder $V_\Omega = \Omega \times (z_1, z_2)$. The curve C is an initial data curve. All characteristics intersecting C form the graph $z = u(x, y)$ of a solution to PDE. Different choices of C correspond different solutions to PDE

Here the parameters τ and p are the polar and zenith angles, respectively, in the spherical coordinates. For each value of the zenith angle p, the above parametric curve is a circle of radius $\sin(p)$ that is the intersection of the sphere with the horizontal plane $z = \cos(p)$ so that the sphere is the union of all such circles. However, the sphere is not graph because for each (x, y) there are two points on the sphere $z = \pm\sqrt{1 - x^2 - y^2}$.

Proposition **12.1** asserts that if the initial data curve lies in the graph of a solution to (**12.1**), that is,

$$\lambda(p) = u(\nu(p), \mu(p))$$

for any p, then the parametric surface (**12.3**) coincides with the graph $z = u(x, y)$ for some range of parameters (τ, p). As in the case of linear equations, there are many solutions to (**12.1**). How about the converse? Under what conditions on the initial data curve do the corresponding characteristics form the graph of a solution?

PROPOSITION **12.2**. *Suppose that an initial data curve is such that the two first relations in (***12.3***) define a change of variables in Ω. Let $\tau = \eta(x, y)$ and $p = \gamma(x, y)$ be the inverse transformation. Then the* *function*

$$u(x,y) = Z\Big(\eta(x,y), \gamma(x,y)\Big) \tag{12.4}$$

*is a solution to (**12.1**).*

Proof. A proof is done by a direct verification of the equation. Since the transformation $(\tau, p) \to (x, y)$ is a change of variables, the following identities hold

$$p = \gamma\Big(X(\tau,p), Y(\tau,p)\Big)\,, \quad \tau = \eta\Big(X(\tau,p), Y(\tau,p)\Big)$$

for all (τ, p). Taking the partial derivative with respect to τ of both sides of these identities and using the chain rule, it is concluded that

$$0 = \gamma'_x X'_\tau + \gamma'_y Y'_\tau\,, \quad 1 = \eta'_x X'_\tau + \eta'_y Y'_\tau$$

Substituting the first two equations in (**12.2**) into these relations, the following identities are proved to hold at any point of the parametric surface defined by (**12.3**):

$$a\gamma'_x + b\gamma'_y = 0\,, \quad a\eta'_x + b\eta'_y = 1 \tag{12.5}$$

Let us substitute the function (**12.4**) into the left side of (**12.1**). By the chain rule

$$\begin{aligned} a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} &= (a\eta'_x + b\eta'_y)\frac{\partial}{\partial \tau} + (a\gamma'_x + b\gamma'_y)\frac{\partial}{\partial p} \\ &= \frac{\partial}{\partial \tau} \end{aligned}$$

where the functions a, b, and the partial derivatives of γ and η are taken on the parametric surface (**12.3**) and, hence, they satisfy the identities (**12.5**). Therefore on the parametric surface

$$au'_x + bu'_y = Z'_\tau = f$$

Hence, the function u is a solution to (**12.1**) □

12.3. Method of characteristics for quasi-linear PDEs. For each initial data curve for which the first two equations in (**12.3**) define a change of variables, a solution of (**12.1**) is given by (**12.4**). It is clear from the geometrical interpretation of the method that the are many initial data curves that can lead to the same solution (all such curves lie in the same surface $z = u(x, y)$). However, in general, for different choices of the initial data curve one can get different solutions. The method allows us to construct a solution in a neighborhood of a point that is a point of intersection of a characteristic and an initial data curve. But the curvilinear coordinate grid cannot always be extended from this neighborhood to the whole Ω. Finding an initial data curve that leads to a change of variables in a given Ω might be a challenging problem that may not even have a solution. In this case, the PDE has no solution in Ω, but it can have solutions in subsets of Ω. Another significant technical difficulty of the method is that the inversion of the change of variables can lead to transcendental equations, and, hence, cannot be carried out analytically.

A summary of the method of characteristics:

- Step 1: Solve the initial value problem for the autonomous system (**12.2**) with arbitrary initial data (x_0, y_0, z_0) where $(x_0, y_0) \in \Omega$;
- Step 2: Let the coordinates of the initial point be continuously differentiable functions of a parameter p, $x_0 = x_0(p)$, $y_0 = y_0(p)$, and $z_0 = z_0(p)$. Analyze conditions on these functions under which the transformation $x = X(\tau, p)$, $y = Y(\tau, p)$ defines a change of variables in Ω. In particular, the Jacobian of the transformation cannot vanish in Ω;
- Step 3: Find the inverse transformation and the solution (**12.4**) to the PDE.

Example **12.1**. *Find a general solution to the equation*

$$u'_x + uu'_y = 0\,,$$

in some open region Ω. Give an explicit form of two particular solutions that contain the line $x = 0$, $z = y$ and the parabola $x = 0$, $y = z^2/4$. Indicate Ω in which the solutions exist. Sketch the coordinate grids associated with the change of variables in Ω generated by these initial data curves.

Solution: Step 1: Consider a characteristic in a cylinder V_Ω passing through a point (x_0, y_0, z_0) at $\tau = 0$. In this case, $a = 1$, $b = z$, and $f = 0$ so that

$$\begin{aligned} &X' = 1\,, \quad Y' = Z\,, \quad Z' = 0\,, \\ \Rightarrow \quad &X = \tau + x_0\,, \quad Y = z_0\tau + y_0\,, \quad Z = z_0 \end{aligned}$$

Step 2: Let $x = x_0(p)$, $y = y_0(p)$, and $z = z_0(p)$ be an initial data curve in V_Ω. Since the shift of the parameter τ does not change the characteristic (as a point set in space), without loss of generality, any collection of characteristics, whose union is the graph of a solution to the PDE, has the form

$$x = X(\tau, p) = \tau\,, \quad y = Y(\tau, p) = z_0(p)\tau + y_0(p)\,, \quad z = Z(\tau, p) = z_0(p)$$

where $y_0(p)$ and $z_0(p)$ are C^1 functions of a parameter p. They have to be chosen so that the first two relations define a change of variables on Ω and can be inverted. They define a one-to-one transformation of Ω' (spanned by (τ, p)) to Ω. In particular, the Jacobian of the transformation cannot vanish:

$$J = \det \begin{pmatrix} X'_\tau & X'_p \\ Y'_\tau & Y'_p \end{pmatrix} = z'_0(p)\tau + y'_0(p) \neq 0\,, \quad (\tau, p) \in \Omega'\,.$$

The inverse transform is $\tau = \eta(x, y) = x$ and $p = \gamma(x, y)$ where γ is defined implicitly by the equation

$$F(x, y, p) = z_0(p)x - y + y_0(p) = 0 \quad \Rightarrow \quad p = \gamma(x, y)$$

Note that this equation cannot always be solved analytically. For example, if $y_0(p) = p$ and $z_0(p) = e^p$, then the equation becomes transcendental and cannot be solved explicitly for p. However the existence of a solution is guaranteed in a neighborhood of any zero of F by the implicit function theorem if $F'_p \neq 0$. The latter is indeed the case because

$$F'_p = z'_0(p)x + y'_0(p) = J \neq 0$$

Furthermore, the theorem also guarantees that the solution γ is from the class C^1 and

$$\gamma'_x = -\frac{F'_x}{F'_p} = -\frac{z_0}{F'_p}\Big|_{p=\gamma}, \quad \gamma'_y = -\frac{F'_y}{F'_p} = \frac{1}{F'_p}\Big|_{p=\gamma}$$

Note that $F'_p = J \neq 0$.
Step 3: Thus, the function

$$u(x, y) = z_0(\gamma(x, y))$$

is a general solution in any open Ω in which the equation $F = 0$ has a solution. It is not difficult to check that u is indeed a solution. By the chain rule

$$\begin{aligned} u'_x + uu'_y &= z'_0(\gamma)\gamma'_x + z_0(\gamma)z'_0(\gamma)\gamma'_y \\ &= -z'_0(\gamma)\,\frac{z_0(\gamma)}{F'_p} + z_0(\gamma)z'_0(\gamma)\,\frac{1}{F'_p} = 0 \end{aligned}$$

If a solution is to be found in a given Ω, then for any point $(\alpha, \beta) \in \Omega$, it is not difficult to find $y_0(p)$ and $z_0(p)$ such that $F(\alpha, \beta, p) = 0$ is satisfied for some $p = p_0$. So, the function $\gamma(x, y)$ always exists in a neighborhood of any $(\alpha, \beta) \in \Omega$ for any choice of $y_0(p)$ and $z_0(p)$ that generates a change of variables in this neighborhood and $F(\alpha, \beta, p_0) = 0$. A difficult question is: whether this neighborhood is large enough to cover the whole Ω.

Example 1: Let us find a solution whose graph contains the line $x = 0$, $y = z$, that is, $z_0(p) = y_0(p) = p$. In this case, $F(x, y, p) = p(x+1) - y$

$$p = \gamma(x, y) = \frac{y}{x+1} \quad \Rightarrow u(x, y) = z_0|_{p=\gamma} = \frac{y}{x+1}$$

This solution exists in a half-plane, either for $x > -1$ or $x < -1$. Note that $J = F'_p = x + 1 = 0$ if $x = -1$. All points with $x = -1$ are singular of the constructed change of variables. The curves of constant $\tau = \tau_0$ are vertical lines $x = \tau_0$. A curve of constant $p = p_0$ is the line intersecting the x axis at $x = -1$ and having slopes p_0, that is, $y = p_0(x+1)$. Any point in the half-plane $x > -1$ or $x < -1$ is the point of intersection of two coordinates lines. The coordinate grid cannot be extended beyond the line $x = -1$ from either of the half-planes.

It is also interesting to note that the surface $z = \frac{y}{x+1}$ (the graph of the solution) can be viewed as the union of horizontal straight lines $y = p(x+1)$, $z = p$, where either $x > -1$ or $x < -1$ (which are the characteristics). With increasing p, the line moves up along the z axis and at the same time rotates about this axis (because the slope p in the xy plane is increasing), thus sweeping the surface of the graph.

Example 2: Let us find a solution that contains the parabola $x = 0$, $y = z^2/4$, that is, $z_0(p) = 2p$ and $y_0(p) = p^2$. Then

$$F(x, y, p) = 2px + p^2 - y \quad \Rightarrow \quad J = F'_p = 2x + 2p$$

There are two such solutions:

$$p = \gamma_\pm(x, y) = -x \pm \sqrt{x^2 + y} \quad \Rightarrow \quad u(x, y) = -2x \pm 2\sqrt{x^2 + y}$$

These solutions exist in the part of the plane where $x^2 + y > 0$. Note that the points of the parabola $y = -x^2$ are singular points of the change of variables because $J = F'_p = \pm 4\sqrt{x^2 + y} = 0$ on the parabola. The new coordinate grid cannot be extended beyond the parabola.

Curves of constant $\tau = \tau_0$ are vertical half-line $x = \tau_0$, $y > -\tau^2$, originated from points $(\tau_0, -\tau_0^2)$ on the parabola $y = -x^2$. Curves of constant $p = p_0$ looks like lines $y = 2p_0 + p_0^2$. Each line is tangent to the parabola $y = -x^2$ at the point $(-p_0, -p_0^2)$. However, the whole cannot be a coordinate line of constant p. Note that in a new coordinate grid, each point of Ω (the region where $x^2 + y > 0$) must be an intersection of two coordinates lines, that is, an intersection of said vertical and tangents lines. It is not difficult to see that there are two lines through any point $(x_0, y_0$ in Ω that are tangent to the boundary parabola $y = -x^2$, and these lines correspond to *distinct* values of $p = -x_0 \pm \sqrt{x_0^2 + y_0} = \gamma_\pm(x_0, y_0)$. To have only one coordinate line of constant p defining each point in Ω, let us take only a *half of the*

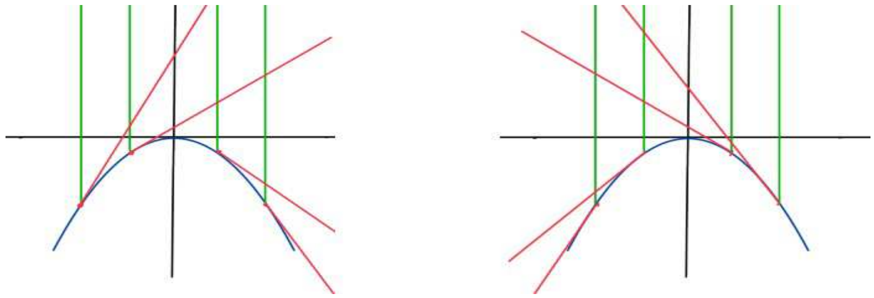

FIGURE **12.2**. Left panel: The coordinate grid of new variables $\tau = x$ and $p = \gamma_+(x, y)$. Vertical lines originating from the points of parabola $y = -x^2$ are curves of constant τ. The half-lines tangent to the parabola are curves of constant p. Right panel: The coordinate grid of new variables $\tau = x$ and $p = \gamma_-(x, y)$. The same legend as in the left panel.

tangent line bounded by the point at which the line is tangent to the parabola, that is,

$$y = 2p_0 x + p_0^2\,, \quad x > -p_0 \quad \Rightarrow \quad p_0 = -x + \sqrt{x^2 + y} = \gamma_+(x, y)$$

In this case, the new coordinate grid is formed by these half-lines and the vertical half-lines. It is associated with the solution generated by γ_+. Similarly, one can also take the other half of each tangent line as curves of constant p:

$$y = 2p_0 x + p_0^2\,, \quad x < -p_0 \quad \Rightarrow \quad p_0 = -x - \sqrt{x^2 + y} = \gamma_-(x, y)$$

Together with the vertical lines, they form yet another coordinate grid which generates the solution with $\gamma_-(x, y)$. The coordinate grids of the changes of variables are shown in Figure **14.2**.

The graphs of two solutions lie in a quadric surface

$$z = u(x, y) \quad \Rightarrow \quad (z + 2x)^2 = 4(x^2 + y) \quad \Rightarrow \quad z^2 + 4xz = 4y$$

This is a hyperbolic paraboloid (also known as a "saddle"). To see this, note that the intersection with the plane $z - x = 0$ is an upward parabola $y = 5x^2/4$, while the intersection with the perpendicular plane $z + x = 0$ is a downward parabola $y = -3x^2/4$. Of course, the equation can be brought to the standard form $y = (z/a)^2 - (x/b)^2$ by a rotation in the xz plane (recall a classification of quadric surfaces in multivariable calculus). The graphs of the two solutions are the parts of this saddle surface that lie above and below Ω in the xy plane. The parametric equations of this surface obtained by the method of characteristics are similar, in this sense, to the parametric equations of a sphere that is the union of two graphs (the upper and lower hemispheres). □

12.4. Multivariable case. The method of parametric characteristics can be extended to quasi-linear PDEs in n variables:

$$\sum_{j=1}^{n} a_j(\mathbf{x},u)\,\frac{\partial u}{\partial x_j} = f(\mathbf{x},u)\,, \quad \mathbf{x}\in\Omega\subset\mathbb{R}^n\,.$$

The curve (characteristic) in $\mathbb{R}^{n+1}$ that is a solution to the initial value problem for the system of $n+1$ ODEs

$$\frac{dX_j}{dt} = a_j(\mathbf{X},Z)\,, \quad \mathbf{X}(t_0)=\mathbf{x}_0\,, \quad j=1,2,...,n\,,$$
$$\frac{dZ}{dt} = f(\mathbf{X},Z)\,, \quad Z(t_0)=z_0$$

lies in the $n-$dimensional surface $z=u(\mathbf{x})$ in $\mathbb{R}^{n+1}$ where u is a solution to the PDE so that $z_0=u(\mathbf{x}_0)$. Conversely, if an initial data parametric surface $\mathbf{x}_0=\mathbf{x}_0(\mathbf{p})$, $z_0=z_0(\mathbf{p})$, $\mathbf{p}\in\mathbb{R}^{n-1}$, is such that the first n equations of parametric characteristics

$$\mathbf{x}=\mathbf{X}(t,\mathbf{p})\,, \quad z=Z(t,\mathbf{p})$$

define a change of variables in Ω so that the inverse transformation reads

$$p_\beta=\gamma_\beta(\mathbf{x})\,, \quad t=\eta(\mathbf{x})\,, \quad \beta=1,2,...,n-1$$

then the function

$$u(\mathbf{x}) = Z\Big(\eta(\mathbf{x}),\boldsymbol{\gamma}(\mathbf{x})\Big)$$

is a solution to the PDE. The technical complexity of the described procedure increases with the dimension n of the problem. For this reason the method is used mostly in numerical algorithms for solving first order PDEs.

12.5. Exercises.

1. Find all solutions to

$$e^y u'_x + u'_y = \cos^2(u)$$

2. Find all solutions to

$$u'_x + 2xu'_y = u^2+1$$

3. Find all solutions, if any, to

$$yu'_x + u^n u'_y = 0\,, \quad n>0$$

whose graphs, $z = u(x, y)$, consist of characteristics passing through the vertical line $x = x_0(p) = \alpha$, $y = y_0(p) = \beta$, $z = z_0(p) = p$, where α and β are constants, and p is a parameter on the line. Indicate the region in which such solutions exist.

Selected answers.

1. $u(x, y) = \arctan\Big(y + \tan(g(p))\Big)$, where $g(p)$ is any C^1 function, and $p = x - e^y$.

2. $u(x, y) = \tan\Big(x + \arctan(g(p))\Big)$, where $g(p)$ is any C^1 function, and $p = y - x^2$.

3. $u(x, y) = \left(\frac{y^2-\beta^2}{2(x-\alpha)}\right)^{1/n}$, and Ω is a natural domain of the function u.

13. Cauchy problem for quasi linear PDEs

The method of parametric characteristics is greatly simplified if the function a (or b) does not vanish anywhere. In this case, PDE (**12.1**) can be divided by a, thus obtaining an *equivalent* PDE in which the coefficient at u'_x is equal to 1. For this PDE the first equation in the system (**12.2**) is trivial to integrate. In particular, this type of PDE appear in the Cauchy problem, which is analyzed below.

Consider the Cauchy problem

$$\textbf{(13.1)} \qquad \begin{cases} u'_t + b(x,t,u)u'_x = f(x,t,u)\,, & t > 0\,, \\ u\big|_{t=0} = u_0(x) \end{cases}$$

In this case, the autonomous system (**12.2**)

$$T'(\tau) = 1\,, \quad X'(\tau) = b(T,X,Z)\,, \quad Z'(\tau) = f(T,X,Z)$$

where τ is a parameter, is reduced to a system of only two equations because $T(\tau) = \tau$ and one can choose the variable $\tau = t$ to be a parameter of any characteristic curve in the graph $z = u(x,t)$. Then the other two equations are

$$\frac{dX}{dt} = b(t,X,Z)$$
$$\frac{dZ}{dt} = f(t,X,Z)$$

Consider the initial value problem for this system

$$\textbf{(13.2)} \qquad X(0) = p\,, \quad Z(0) = u_0(p)$$

for a real p. Then, if b and f are C^1 functions that do not vanish simultaneously along the initial data curve, the solution exists and is unique

$$X = X(t,p)\,, \quad Z = Z(t,p)$$

at least in some interval $0 < t < t_0$, and the characteristics are uniquely labeled by a parameter p. By construction, for each p the parametric curve

$$t = T(\tau,p) = \tau\,, \quad x = X(\tau,p)\,, \quad z = Z(\tau,p)$$

passes through the point $(t,x,z) = (0,p,u_0(p))$. By the initial condition $u(x,0) = u_0(x)$, this point lies on the graph $z = u(x,t)$ of the solution to the Cauchy problem. According to the previous section, the graph $z = u(x,t)$ is the union of all such characteristics labeled by

p and the solution $u(x,t)$ is obtained by the substitution:

$$x = X(t,p) \quad \Rightarrow \quad p = \gamma(x,t)$$
$$u(x,t) = Z(t,p)\Big|_{p=\gamma} = Z(t,\gamma(x,t))$$

Yet, this solution also satisfies the initial condition

$$u\Big|_{t=0} = Z(0,\gamma(x,0)) = u_0(x)$$

because

$$X(0,p) = p \quad \Rightarrow \quad \gamma(p,0) = p$$

The solution $u(x,t)$ can be viewed as if each point of the curve $z = u(x,0) = u_0(x)$ in the graph $z = u(x,t)$ is moving along the corresponding characteristics thus sweeping the surface of the entire graph $z = u(x,t)$ (see Fig. **14.2**)

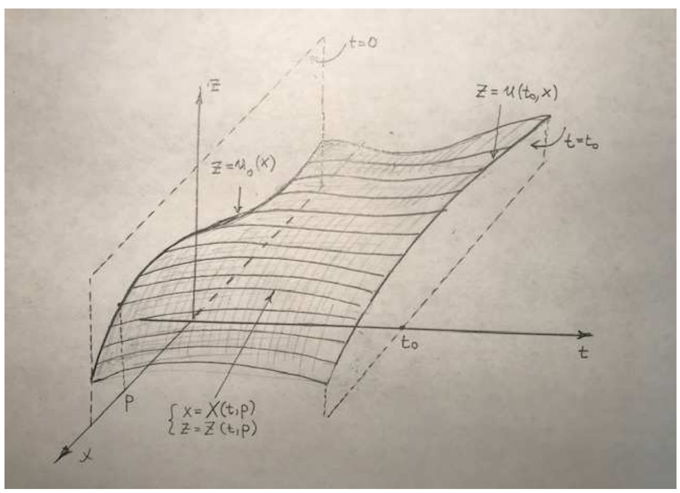

FIGURE **13.1**. The graph $z = u(x,t)$ of the solution $u(x,t)$ to the Cauchy problem is the union of non-intersecting characteristics originating from each point of the initial curve $z = u(x,0) = u_0(x)$ in the coordinate plane $t = 0$.

13.1. Procedure for solving the Cauchy problem.

Step 1: Solve the initial value problem (**13.2**) to obtain characteristic curves $x = X(t,p)$, $z = Z(t,p)$.
Step 2: Solve the equation $x = X(t,p)$ for p so that $p = \gamma(x,t)$;
Step 3: Obtain the solution to the Cauchy problem by the substitution $u(x,t) = Z(t,\gamma(x,t))$.

In order for the procedure to work, the coefficients b and f and the initial data u_0 must be functions from the class C^1 to guarantee that the solution is from C^1. In addition, the solution to the initial value problem should exist for all $t > 0$ as well as the function $\gamma(x,t)$. Even if the coefficients and initial data are smooth enough, the solution may exist only in an interval $0 \le t < t_0$ as is illustrated in an example below (the solution may "blow up" as t approaches some t_0). Yet, only for a limited choice of b and f this procedure can be carried out explicitly because it involves solving a general system of two first order ODE and no universal analytic algorithm exists for this. Even if this latter task has been accomplished, one still has to solve a general equation $X(t,p) = x$ for p and the latter equation often happens to be transcendental.

EXAMPLE **13.1**. *Solve the Cauchy problem*

$$u'_t + 3uu'_x = 2t\,, \quad t > 0\,; \quad u\Big|_{t=0} = x$$

SOLUTION: Here $b = 3u$ and $f = 2t$.
Step 1: The associated initial value problem is

$$\begin{cases} X' = 3Z \\ Z' = 2t \end{cases}, \qquad X(0) = p\,, \quad Z(0) = p$$

The solution to the second equation reads

$$Z(t,p) = t^2 + p$$

Therefore

$$X(t,p) = p + 3\int_0^t Z(\tau)\,d\tau = p(3t+1) + t^3$$

Step 2:

$$x = X(t,p) \quad \Rightarrow \quad p = \frac{x - t^3}{3t+1} = \gamma(x,t)$$

Step 3: The solution to the Cauchy problem reads

$$u(x,t) = Z(t,p)\Big|_{p=\gamma} = t^2 + \frac{x-t^3}{3t+1}$$

Checking the answer: The initial condition is obviously satisfied, $u(x,0) = x$. The substitution of the partial derivatives

$$u'_t = 2t - \frac{3t^2}{(3t+1)} + \frac{3t^3 - 3x}{(3t+1)^2}\,,$$
$$u'_x = \frac{1}{3t+1}$$

into the left side of the equation yields

$$\begin{aligned} u'_t + 3uu'_x &= 2t - \frac{3t^2}{(3t+1)} + \frac{3t^3 - 3x}{(3t+1)^2} + 3\left(t^2 + \frac{x - t^3}{3t+1}\right)\frac{1}{3t+1} \\ &= 2t \end{aligned}$$

as required. □

EXAMPLE **13.2**. *Solve the Cauchy problem*

$$u'_t - uu'_x = x\,, \quad 0 < t < t_0\,; \qquad u\Big|_{t=0} = x$$

for some $t_0 > 0$.

SOLUTION: In this case $b = -u$ and $f = x$.
Step 1: The characteristics are solutions to the initial value problem

$$X' = -Z\,, \quad Z' = X\,, \quad X(0) = p\,, \quad Z(0) = p$$

Differentiating the second equation and substituting the first one into the resulting equation, an equation for Z is obtained

$$Z'' + Z = 0$$

Its general solution is

$$Z(t) = A\cos(t) + B\sin(t) \quad \Rightarrow \quad X(t) = Z'(t) = B\cos(t) - A\sin(t)$$

The initial conditions require that $A = B = p$. Thus, the characteristics are

$$\begin{aligned} x &= X(t,p) = p\Big(\cos(t) - \sin(t)\Big) \\ z &= Z(t,p) = p\Big(\cos(t) + \sin(t)\Big)\,. \end{aligned}$$

Step 2: Expressing p in terms of t and x:

$$p = \frac{x}{\cos(t) - \sin(t)} = \gamma(x,t)\,, \quad t < t_0 = \frac{\pi}{4}$$

because $\gamma(x,t)$ does not exist if $t = \pi/4$.
Step 3: The solution to the Cauchy problem in the interval $0 < t < \pi/4$ has the form

$$u(x,t) = Z(t,p)\Big|_{p=\gamma} = x\,\frac{\cos(t) + \sin(t)}{\cos(t) - \sin(t)} = x\tan\Big(t + \tfrac{\pi}{4}\Big)$$

As $t \to \frac{\pi}{4}^-$, the solution "blows up", $u \to +\infty$.
Checking the solution: The solution satisfies the initial condition because $\tan(\pi/4) = 1$. Then

$$\begin{aligned} u'_t &= \frac{x}{\cos^2(t+\frac{\pi}{4})}\,, \quad u'_x = \tan\Big(t+\frac{\pi}{4}\Big) \\ u'_t - uu'_x &= \frac{x}{\cos^2(t+\frac{\pi}{4})} - x\tan^2(t+\tfrac{\pi}{4}) \\ &= x\,\frac{1-\sin^2(t+\frac{\pi}{4})}{\cos^2(t+\frac{\pi}{4})} = x \end{aligned}$$

as required. □

13.2. Exercises.

1. Solve the Cauchy problem

$$u'_t + uu'_x = x\,, \quad t > 0\,; \quad u\Big|_{t=0} = x$$

2. Solve the Cauchy problem

$$u'_t + u'_x + cu^n = 0\,, \quad u\Big|_{t=0} = u_0(x)\,,$$

Express the solution in terms of the function u_0.

3. Solve the Cauchy problem

$$u'_t + bu^n u'_x + u = 0\,, \quad u\Big|_{t=0} = x^{1/n}\,,\ x > 0$$

where $b > 0$ and $n > 0$ are constants.

Selected answers.

1. $u = x$.
2. $u = [u_0^\nu(x-t) - \nu ct]^{1/\nu}$, $\nu = 1 - n$ if $n \neq 1$, and $u = u_0(x-t)e^{-ct}$ if $n = 1$.
3. $u = (x/\beta)^{1/n}e^{-t}$, $\beta = 1 + \frac{b}{n}(1 - e^{-nt})$.