CHAPTER 3

Classification of second-order PDEs

14. Characteristics of second-order PDEs

A classification of second-order PDEs is essential for questions about uniqueness of a solution. It is also important for choosing appropriate methods for solving second-order PDEs. The discussion will be limited to equations *linear* in second partial derivatives and begins with the simplest case of second-oder PDEs in two real variables. In particular, for two variables (x, y), A second-order PDE is *linear* relative to the second-order partial derivatives if it has the form

$$au''_{xx} + 2cu''_{xy} + bu''_{yy} = F(x, y, u, u'_x, u'_y)\,, \quad (x, y) \in \Omega \subset \mathbb{R}^2$$

where the coefficients a, b, and c are functions of x and y.

14.1. Basic idea of the method of characteristics. Recall that first-order PDEs linear in partial derivatives can be solved the method of characteristics. The characteristics of a differential equation define a change of variables $(x, y) \to (\tau, p)$ so that in the new variables the linear combination of partial derivatives is reduced to a singe derivative with respect to one of the new variables:

$$a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} = \frac{\partial}{\partial \tau}$$

and a PDE becomes an ODE the variable τ. In the case of second order PDEs, it is generally not possible to reduce the problem to an ODE by a change of variables:

$$\alpha = \alpha(x, y)\,, \quad \beta = \beta(x, y)\,.$$

However, it is possible to transform the linear combination of the second partial derivatives to one of the standard forms. For example, if a, b, and c are constant, then there exists a change of variables such that

$$au''_{xx} + 2cu''_{xy} + bu''_{yy} = 0 \quad \Rightarrow \quad \begin{array}{l} u''_{\alpha\alpha} - u''_{\beta\beta} = 0 \\ \text{or} \\ u''_{\beta\beta} = 0 \\ \text{or} \\ u''_{\alpha\alpha} + u''_{\beta\beta} = 0 \end{array}$$

Thus, the problem of solving the above PDE can be reduced to solving either the wave equation, or ODE, or the Laplace equation. General solutions for any of these problems have been already found. It is therefore important to develop a technique for finding such a change of variables. The method to find such a change of variables is known as *the method of characteristics for second order PDEs.*

Recall a classification of quadric curves, the curves defined by a general quadratic equation in variables x and y. It is possible to find a suitable rotation and translation of the coordinates (x, y) so that the equation describes one of the three possible conic sections: a hyperbola, or a parabola, or an ellipse:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,, \quad y = ax^2\,, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hyperbolic, parabolic, and elliptic PDEs. By the analogy with conic cross sections, a PDE in which the linear combination of second partials is reducible to that in the wave equation by a change of variables (with the minus sign between the derivatives) is called a *hyperbolic* equation. If this combination is reducible to that in the heat equation (where only one second partial with respect to one variable remains after the change of variables), then the PDE is called a *parabolic* equation. A PDE is called *elliptic* if the linear combination of second partials in it is reducible to that in the Laplace equation by a change of variables. It is clear that a correct classification of second order PDE is important for its solving.

14.2. Characteristics of PDEs with constant coefficients. Suppose that the coefficients a, b, and c are constant. Consider a linear transformation, that is, the functions $\alpha(x, y)$ and $\beta(x, y)$ are linear so that partial derivatives α'_x, α'_y, β'_x and β'_y are constant. The transformation can be written in the form

$$\alpha = \alpha'_x x + \beta'_y y\,, \quad \beta = \beta'_x x + \beta'_y y$$

The transformation is a change of variables if its Jacobian does not vanish

$$J = \det\begin{pmatrix} \alpha'_x & \alpha'_y \\ \beta'_x & \beta'_y \end{pmatrix} = \alpha'_x\beta'_y - \alpha'_y\beta'_x \neq 0$$

In this case, the linear equations can be solved for x and y to express the old variables as linear functions of the new ones. A geometrical significance of such a general linear transformation is that every rectangle in the xy plane is mapped onto a parallelogram in the $\alpha\beta$ plane.

Using the chain rules

$$\begin{aligned}
u'_x &= \alpha'_x u'_\alpha + \beta'_x u'_\beta\,,\\
u'_y &= \alpha'_y u'_\alpha + \beta'_y u'_\beta\,,\\
u''_{xx} &= \left(\alpha'_x \frac{\partial}{\partial\alpha} + \beta'_x \frac{\partial}{\partial\beta}\right) u'_x\\
&= (\alpha'_x)^2 u''_{\alpha\alpha} + 2\alpha'_x\beta'_x u''_{\alpha\beta} + (\beta'_x)^2 u''_{\beta\beta}\\
u''_{yy} &= \left(\alpha'_y \frac{\partial}{\partial\alpha} + \beta'_y \frac{\partial}{\partial\beta}\right) u'_y\\
&= (\alpha'_y)^2 u''_{\alpha\alpha} + 2\alpha'_y\beta'_y u''_{\alpha\beta} + (\beta'_y)^2 u''_{\beta\beta}\\
u''_{xy} &= \left(\alpha'_x \frac{\partial}{\partial\alpha} + \beta'_x \frac{\partial}{\partial\beta}\right) u'_y\\
&= \alpha'_x\alpha'_y u''_{\alpha\alpha} + (\alpha'_x\beta'_y + \alpha'_y\beta'_x) u''_{\alpha\beta} + \beta'_x\beta'_y u''_{\beta\beta}
\end{aligned}$$

Therefore the coefficients in the linear combination of the second partial derivatives in the new variables are

$$a u''_{xx} + 2c u''_{xy} + b u''_{yy} = A u''_{\alpha\alpha} + 2C u''_{\alpha\beta} + B u''_{\alpha\beta}$$

$$A = a(\alpha'_x)^2 + 2c\alpha'_x\alpha'_y + b(\alpha'_y)^2\,, \tag{14.1}$$

$$B = a(\beta'_x)^2 + 2c\beta'_x\beta'_y + b(\beta'_y)^2 \tag{14.2}$$

$$C = a\alpha'_x\beta'_x + c(\alpha'_x\beta'_y + \alpha'_y\beta'_x) + b\alpha'_y\beta'_y \tag{14.3}$$

Let us try to simplify this combination as much as possible by demanding $A = B = 0$. These equations are equivalent to solving a *first order PDE*:

$$a(\gamma'_x)^2 + 2c\gamma'_x\gamma'_y + b(\gamma'_y)^2 = 0 \tag{14.4}$$

If this PDE has two linearly independent solutions $\gamma = \alpha(x, y)$ and $\gamma = \beta(x, y)$, then these solutions can be used to construct a desired change of variables. Equation (**14.4**) is a first-order PDE, but it is not linear in partial derivatives and, hence, cannot be studied by the method of characteristics developed in Chapter 2. The method needs a generalization.

PROPOSITION **14.1**. *If a non-constant linear function $\gamma(x, y)$ is a solution to Eq. (**14.4**) with constant coefficients a, b, and c, then the level sets $\gamma(x, y) = \gamma_0$ (lines) are solutions to the first-order ordinary differential equation*

$$a(dy)^2 - 2cdxdy + b(dx)^2 = 0 \tag{14.5}$$

*Conversely, if level sets of a function $\gamma(x,y)$ are solutions to (**14.5**), then the function $\gamma(x,y)$ is a solution to (**14.4**).*

PROOF. Let a linear function $\gamma(x,y) = \gamma'_x x + \gamma'_y y$ be a solution to (**14.4**). Then on any level set $\gamma'_x x + \gamma'_y y = \gamma_0$ (a line)

$$\gamma'_x dx + \gamma'_y dy = 0 \tag{14.6}$$

The substitution of this relation into (**14.5**) shows that γ'_x and γ'_y must obey (**14.4**), which is true. Indeed, $\gamma(x,y)$ is not a constant function and, hence, γ'_x and γ'_y do not vanish simultaneously. Suppose $\gamma'_y \neq 0$. Then $dy = -(\gamma'_x/\gamma'_y)dx$ so that (**14.5**) yields

$$\left[a\left(\frac{\gamma'_x}{\gamma'_y}\right)^2 + 2c\,\frac{\gamma'_x}{\gamma'_y} + b\right](dx)^2 = 0$$

which coincides with (**14.4**) after multiplication by $(\gamma'_y)^2$.

Conversely, if level sets $\gamma(x,y) = \gamma_0$ define the general solution to (**14.5**), then (**14.6**) holds where γ'_x and γ'_y are such that $dy/dx = -\gamma'_x/\gamma'_y$ solves (**14.5**). After the substitution of dy/dx into the left side of (**14.5**), it becomes

$$0 = a\Big(\frac{dy}{dx}\Big)^2 - 2c\frac{dy}{dx} + b = a\left(\frac{\gamma'_x}{\gamma'_y}\right)^2 + 2c\,\frac{\gamma'_x}{\gamma'_y} + b$$

which is equivalent to (**14.4**) after multiplication by $(\gamma'_y)^2$. $\square$

14.3. Classification of second-order PDEs. Proposition **14.1** shows that the condition $A = 0$ or $B = 0$ is *equivalent* to solving ODE (**14.5**) which is called a *characteristic equation* for the studied second-order PDE and its solutions are called *characteristics*. If there are two characteristics (curves) $\alpha(x,y) = \alpha_0$ and $\beta(x,y) = \beta_0$ such that their Jacobian does not vanish, then they define a coordinate transformation $\alpha = \alpha(x,y)$, $\beta = \beta(x,y)$ after which $A = B = 0$, and in this case

$$au''_{xx} + 2cu''_{xy} + bu''_{yy} = 2Cu''_{\alpha\beta}$$

Let us find the characteristics.

If $a = b = 0$, then no coordinate transformation is needed. Suppose that $a \neq 0$. The characteristic equation has two (real or complex) roots

$$\frac{dy}{dx} = \frac{c \pm \sqrt{D}}{a}\,, \quad D = c^2 - ab \neq 0\,, \tag{14.7}$$

The sign of the discriminant D defines the type of a second-order PDE. If the discriminant vanishes, then the desired change of variables does not exist (after which $A = B = 0$). However, it will be shown that the only characteristic in the case $D = 0$ can still be used to simplify a second-order PDE so that $A = C = 0$ in the new variables.

DEFINITION **14.1**. (Classification of second-order PDEs)
The equation

$$au''_{xx} + 2cu''_{xy} + bu''_{yy} + F(x, y, u, u'_x, u'_y) = 0$$

with constant a, b, and c is called

- hyperbolic *if* $D = c^2 - ab > 0$
- elliptic *if* $D = c^2 - ab < 0$
- parabolic *if* $D = c^2 - ab = 0$

EXAMPLE **14.1**. *Determine the type of the equation*

$$u''_{xx} + 4u''_{xy} + 3u''_{yy} - x(u'_x)^2 + yxu = 0$$

SOLUTION: The discriminant of the characteristic equation with $a = 1$, $b = 3$, and $c = 2$

$$D = c^2 - ab = 4 - 9 = -5 < 0$$

is negative. Therefore the equation in question is elliptic. □

14.4. Exercises.

1. Classify each of the following PDEs:

$$\begin{aligned} &\text{(i)} && u''_{xx} - 6u''_{xy} = u'_x + xu\,, \\ &\text{(ii)} && u''_{xx} + 2u''_{xy} + 3u''_{yy} = yx\,, \\ &\text{(iii)} && u''_{xx} + 2u''_{xy} + 4u''_{yy} = yu \end{aligned}$$

15. Standard form of second-order PDEs

Let us use the characteristics to reduce a second-order PDE to the standard (most simple) form in each of the three cases, $D > 0$ (hyperbolic PDE), $D = 0$ (parabolic PDE), and $D < 0$ (elliptic PDE).

15.1. Standard form of hyperbolic equations. In the case of hyperbolic equations, two roots in (**14.7**) are distinct and real. The characteristic equation has two solutions:

$$\begin{aligned} a dy - (c + \sqrt{D})dx = 0 \quad &\Rightarrow \quad \gamma_+(x,y) = ay - (c + \sqrt{D})x = \alpha_0 \\ a dy - (c - \sqrt{D})dx = 0 \quad &\Rightarrow \quad \gamma_-(x,y) = ay - (c - \sqrt{D})x = \beta_0 \end{aligned}$$

for any constants α_0 and β_0. By Proposition **14.1**,

$$A = B = 0$$

if the new variables are defined by

$$\alpha = \gamma_+(x,y)\,, \qquad \beta = \gamma_-(x,y)\,.$$

Therefore

$$\begin{aligned} \alpha'_x &= -(c + \sqrt{D})\,, \quad \alpha'_y = a \\ \beta'_x &= -(c - \sqrt{D})\,, \quad \beta'_y = a\,. \end{aligned}$$

and the Jacobian of the transformation is

$$J = \det \begin{pmatrix} \alpha'_x & \alpha'_y \\ \beta'_x & \beta'_y \end{pmatrix} = -2a\sqrt{D} \neq 0$$

because $a \neq 0$ and $D \neq 0$. In the new variables, the hyperbolic equation has the form

$$u''_{\alpha\beta} = G_1(\alpha, \beta, u, u'_\alpha, u'_\beta) \tag{15.1}$$

after dividing it by the coefficient C, where $G_1 = -G/C$ and

$$C = -2aD \neq 0$$

because $a \neq 0$ and $D \neq 0$.

Equation (**15.1**) is called *the standard form of the hyperbolic equation* in two variables. In the case when in the original equation $a = 0$, the characteristics cannot have the form (**14.7**). However, $b \neq 0$ and the characteristics can still be found in a similar fashion so that (**15.1**) holds.

Remark. If $a = 0$, then the needed change of variables is obtained by solving (**14.5**):

$$-2cdxdy + b(dx)^2 = 0 \quad \Rightarrow \begin{cases} dx = 0 \\ bdx - 2cdy = 0 \end{cases} \Rightarrow \begin{cases} x = \alpha \\ bx - 2cy = \beta \end{cases}$$

EXAMPLE **15.1**. *Show that the equation*

$$2cu''_{xy} + bu''_{yy} = F(x, y, u, u'_x, u'_y)$$

is hyperbolic if $c \neq 0$ *and parabolic otherwise. Find a coordinate transformation after which the hyperbolic equation would have the standard form.*

SOLUTION: In this case, $a = 0$. Therefore $D = c^2$ and the equation is hyperbolic if $c \neq 0$ otherwise $D = 0$ and the equation is parabolic (if $b \neq 0$). The associated characteristic equation has the form

$$-2cdydx + b(dx)^2 = 0 \quad \Rightarrow \quad dx = 0 \quad \text{or} \quad -2cdy + bdx = 0$$

The characteristics $\gamma_{1,2}(x, y) = \text{const}$ are obtained by integrating these equations:

$$\gamma_1(x, y) = x\,, \quad \gamma_2(x, y) = bx - 2cy$$

The first characteristic $x = \alpha_0$ is a horizontal line, while the second one, $bx - 2cy = \beta_0$, is a line that intersects the first one at an angle for any choice of α_0 and β_0. The new variables are

$$\alpha = x\,, \qquad \beta = bx - 2cy$$

The transformation has the inverse

$$x = \alpha\,, \qquad y = \frac{1}{2c}(b\alpha - \beta)$$

By the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} + b\,\frac{\partial}{\partial \beta}\,, \qquad \frac{\partial}{\partial y} = -2c\,\frac{\partial}{\partial \beta}$$

$$u''_{xy} = -2c\,\frac{\partial}{\partial \beta}\left(\frac{\partial}{\partial \alpha} + b\,\frac{\partial}{\partial \beta}\right)u = -2cu''_{\alpha\beta} - 2cbu''_{\beta\beta}$$

$$u''_{yy} = 4c^2\left(\frac{\partial}{\partial \beta}\right)^2 u = 4c^2 u''_{\beta\beta}$$

$$2cu''_{xy} + bu''_{yy} = -4c^2u''\alpha\beta - 4c^2bu''_{\beta\beta} + 4c^2bu''_{\beta\beta} = -4c^2u''_{\alpha\beta}$$

Therefore in the new variables the hyperbolic equation has the form

$$2u''_{xy} + u''_{yy} = F \quad \Leftrightarrow \quad u''_{\alpha\beta} = -\frac{1}{4c^2}F$$

□

An alternative standard form of a hyperbolic equation. Let us make yet another change of variables in Eq. (**15.1**)

$$\alpha = \xi + \eta\,, \qquad \beta = \xi - \eta$$

or

$$\xi = \frac{1}{2}(\alpha + \beta)\,, \qquad \eta = \frac{1}{2}(\alpha - \beta)$$

By the chain rule

$$\frac{\partial}{\partial \alpha} = \frac{1}{2}\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial}{\partial \eta}\,, \qquad \frac{\partial}{\partial \beta} = \frac{1}{2}\frac{\partial}{\partial \xi} - \frac{1}{2}\frac{\partial}{\partial \eta}$$

so that

$$u''_{\alpha\beta} = \frac{1}{4}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) u = \frac{1}{4}(u''_{\xi\xi} - u''_{\eta\eta})$$

Therefore Eq. (**15.1**) becomes

$$u''_{\xi\xi} - u''_{\eta\eta} = 4G_1 \tag{15.2}$$

Equation (**15.2**) is also called the *standard form* of a hyperbolic equation in two variables.

15.2. Standard form of parabolic equations. If $D = 0$, then one $ab = c^2 > 0$ and one can always choose $a > 0$ and $b > 0$. Two characteristics (**14.7**) coincide, that is, there is just one independent solution to the characteristic equation. If $c > 0$ so that $c = \sqrt{ab}$, then the characteristic is

$$\begin{aligned} &a(dy)^2 - 2cdydx + b(dx)^2 = (\sqrt{a}dy - \sqrt{b}dx)^2 = 0 \\ \Rightarrow\quad &\gamma(x,y) = \sqrt{a}\,y - \sqrt{b}\,x = \gamma_0 \end{aligned}$$

In this case, put

$$\begin{cases} \alpha = \gamma(x,y) = \sqrt{a}\,y - \sqrt{b}\,x \\ \beta = \varphi(x,y) = \sqrt{a}\,y + \sqrt{b}\,x \end{cases}, \qquad c = \sqrt{ab} > 0$$

If $c < 0$, that is, $c = -\sqrt{ab}$, then

$$\begin{cases} \alpha = \gamma(x,y) = \sqrt{a}\,y + \sqrt{b}\,x \\ \beta = \varphi(x,y) = \sqrt{a}\,y - \sqrt{b}\,x \end{cases}, \qquad c = -\sqrt{ab} < 0\,.$$

With this choice of α,

$$A = 0$$

by Proposition **14.1**. For $c > 0$, the other coefficients are

$$\begin{aligned} B &= a(\beta'_x)^2 + 2c\beta'_x\beta'_y + b(\beta'_y)^2 \\ &= ab + 2c\sqrt{a}\sqrt{b} + ba \\ &= 4c^2 \\ C &= a\alpha'_x\beta'_x + c(\alpha'_x\beta'_y + \alpha'_y\beta'_x) + b\alpha'_y\beta'_y \\ &= -a\sqrt{b}\beta'_x + c(-\sqrt{b}\beta'_y + \sqrt{a}\beta'_x) + b\sqrt{a}\beta'_y \\ &= \sqrt{a}(c - \sqrt{ab})\beta'_x + \sqrt{b}(c - \sqrt{ab})\beta'_y \\ &= 0 \end{aligned}$$

The same result holds if $c < 0$. Note that the latter equality, $C = 0$, holds *regardless* of the choice $\beta = \varphi(x, y)$. So, in the case when $D = 0$, the second variable can be chosen in any convenient way (of course, the Jacobian of the transformation should not vanish, $J \neq 0$).

The equation

$$u''_{\beta\beta} = -\frac{1}{4c^2} G(\alpha, \beta, u, u'_\alpha, u'_\beta) \tag{15.3}$$

is called the *standard form of a second-order parabolic PDE in two variables*. If the right side G is independent of u'_α, then the parabolic equation is reduced to an *ordinary differential equation* in which the variable α is a parameter.

EXAMPLE **15.2**. *Show that the equation*

$$u''_{xx} - 2u''_{xy} + u''_{yy} = -2(x + y)^2(u'_x - u'_y)$$

is parabolic and find its general solution by reducing it to the standard form.

SOLUTION: In this equation $a = b = 1$ and $c = -1$. Therefore $D = c^2 - ab = 0$ and, hence, the equation is parabolic. The characteristic equation

$$a(dy)^2 - 2cdydx + b(dx)^2 = 0 \quad \Rightarrow \quad (dy + dx)^2 = 0$$

has only one solution

$$\gamma(x, y) = y + x = \gamma_0$$

To bring the PDE to a standard form, put

$$\alpha = \gamma(x, y) = y + x\,, \qquad \beta = y - x$$

The Jacobian of this transformation does not vanish ($J = 2 \neq 0$). So, the choice of β defines a change of variables. By the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}$$
$$u'_x - u'_y = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = -2\,\frac{\partial u}{\partial \beta}$$
$$u''_{xx} - 2u''_{xy} + u''_{yy} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u = 4\,\frac{\partial^2 u}{\partial \beta^2}$$

Thus, in the new variables the equation assumes the standard form

$$u''_{\beta\beta} = -\alpha^2\, u'_\beta$$

The equation is an ordinary differential equation because it does not contain u'_α. Let $v = u'_\beta$. Then for every α,

$$\frac{dv}{d\beta} = -\alpha^2 v\,,$$
$$\Rightarrow \quad v = C_1 e^{-\alpha^2\beta}$$
$$\Rightarrow \quad u = \int v\, d\beta = -\frac{C_1}{\alpha^2}\, e^{-\alpha^2\beta} + C_2$$

where the integration constants C_1 and C_2 are arbitrary functions of α. So without loss of generality

$$u(x,y) = f(\alpha)e^{-\alpha^2\beta} + g(\alpha)\,, \quad \alpha = y + x\,, \quad \beta = y - x$$

for some twice continuously differentiable functions f and g. $\square$

15.3. Standard form of elliptic equations. Note that $D < 0$ implies that $c^2 < ab$ and, hence, $a \neq 0$. So, for elliptic equations, the characteristics (**14.7**) are complex

$$\gamma(x,y) = ay - \left(c + i\sqrt{-D}\right)x = \gamma_0\,,$$
$$\bar\gamma(x,y) = ay - \left(c - i\sqrt{-D}\right)x = \bar\gamma_0$$

where $\bar\gamma$ denotes the complex conjugated function γ. The proof of Proposition **14.1** does not require that γ is real. Therefore in the new complex variables

$$\alpha = \gamma(x,y) = ay - \left(c + i\sqrt{-D}\right)x\,,$$
$$\beta = \bar\gamma(x,y) = ay - \left(c - i\sqrt{-D}\right)x$$

the coefficients A and B vanish by Lemma **14.1**

$$A = B = 0$$

as in the case of hyperbolic equations so that the equation assumes the form (**15.1**) where α and $\beta = \bar{\alpha}$ are complex variables. To avoid the use of complex variables, let us introduce two real variables

$$\alpha = \xi + i\eta\,, \quad \beta = \xi - i\eta$$

They are related to the original variables as

$$\xi = \frac{1}{2}(\alpha + \beta) = ay - cx\,, \quad \eta = \frac{1}{2i}(\alpha - \beta) = -\sqrt{-D}x$$

By the chain rule

$$\frac{\partial}{\partial \alpha} = \frac{1}{2}\frac{\partial}{\partial \xi} + \frac{1}{2i}\frac{\partial}{\partial \eta}\,, \quad \frac{\partial}{\partial \beta} = \frac{1}{2}\frac{\partial}{\partial \xi} - \frac{1}{2i}\frac{\partial}{\partial \eta}\,,$$

$$u''_{\alpha\beta} = \frac{1}{4}\left(\frac{\partial}{\partial \xi} + \frac{1}{i}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} - \frac{1}{i}\frac{\partial}{\partial \eta}\right) u = \frac{1}{4}\Big(u''_{\xi\xi} + u''_{\eta\eta}\Big)$$

The substitution of the latter relation to (**15.1**) shows that an elliptic equation can always be transformed to the standard form

$$u''_{\xi\xi} + u''_{\eta\eta} = 4G_1 \tag{15.4}$$

which is to be compared to the second standard form of a hyperbolic equation (**15.2**).

Example **15.3**. *Show that the equation*

$$u''_{xx} - 2u''_{xy} + 5u''_{yy} = xu'_y u$$

is an elliptic equation and find the standard form of the equation by making a suitable change of variables.

Solution: In this equation $a = 1$, $b = 5$, and $c = -1$. Therefore $D = c^2 - ab = -4 < 0$ and, hence, the equation is elliptic. It has two complex characteristics

$$\gamma(x, y) = y + (1 - 2i)x = \gamma_0\,, \quad \bar{\gamma}(x, y) = y + (1 + 2i)x = \gamma_0$$

The real and imaginary parts of the characteristic function γ define new variables in which the equation should have the standard form

$$\xi = y + x\,, \quad \eta = -2x$$

Using the chain rules

$$\begin{aligned}
\frac{\partial}{\partial x} &= \xi'_x \frac{\partial}{\partial \xi} + \eta'_x \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}\,,\\
\frac{\partial}{\partial y} &= \xi'_y \frac{\partial}{\partial \xi} + \eta'_y \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi}\,,\\
u''_{xx} &= \left(\frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}\right)^2 u = u''_{\xi\xi} - 4u''_{\xi\eta} + 4u''_{\eta\eta}\,,\\
u''_{xy} &= \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}\right) = u''_{\xi\xi} - 2u''_{\xi\eta}\,,\\
u''_{yy} &= \left(\frac{\partial}{\partial \xi}\right)^2 u = u''_{\xi\xi}
\end{aligned}$$

The substitution of the above relations into the equation in question yields

$$4u''_{\xi\xi} + 4u''_{\eta\eta} = -\frac{1}{2}\,\eta u'_{\xi} u \quad \Rightarrow \quad u''_{\xi\xi} + u''_{\eta\eta} = -\frac{1}{8}\,\eta u'_{\xi} u$$

□

15.4. Exercises.

1. For each of the following PDEs, find the characteristics and reduce the equation to the standard form:

$$\begin{aligned}
&\text{(i)} && u''_{xx} - 6u''_{xy} = u'_x + xu\,,\\
&\text{(ii)} && u''_{xx} + 2u''_{xy} + 3u''_{yy} = yx\,,\\
&\text{(iii)} && u''_{xx} + 2u''_{xy} + 4u''_{yy} = yu
\end{aligned}$$

2. Find a general solution by reducing it the following equation to a standard form

$$u''_{yy} - 6u''_{xy} = 0$$

3. Find all polynomial solutions to the following equation by reducing it to the standard form

$$u''_{xx} + 2u''_{xy} + 5u''_{yy} = 0$$

Hint: Recall harmonic polynomials

4. Find a general solution to the following equation by reducing it to the standard form

$$u''_{xx} + 4u''_{xy} + 4u''_{yy} = -4u$$

Selected answers.

2. $u(x,y) = f(x) + g(x+6y)$, f and g are any functions from C^2

3.

$$u(x,y) = A_0 + \sum_{n=1}^{N} \Big(A_n \operatorname{Re} z^n + B_n \operatorname{Im} z^n \Big), \quad z = y - x + 2ix$$

where A_n and B_n are constants.

4.

$$u(x,y) = f(y-2x)\cos\left(x + \frac{1}{2}\,y\right) + g(y-2x)\sin\left(x + \frac{1}{2}\,y\right)$$

where f and g are any functions from C^2

16. Linear PDE with constant coefficients

16.1. Standard form of linear PDEs with constant coefficients. Consider the most general linear equation of second order with constant coefficients

$$au''_{xx} + bu''_{yy} + 2cu''_{xy} + pu'_x + qu'_y + mu = f(x,y)$$

Here a, b, c, p, q, and m are constants. Using a change of variables associated with the characteristics of this equation, the latter can be reduced to one of the following forms

$$\begin{aligned} \text{elliptic}: &\quad u''_{\alpha\alpha} + u''_{\beta\beta} + Pu'_\alpha + Qu'_\beta + mu = f\,, \\ \text{hyperbolic}: &\quad u''_{\alpha\alpha} - u''_{\beta\beta} + Pu'_\alpha + Qu'_\beta + mu = f\,, \\ \text{parabolic}: &\quad u''_{\beta\beta} + Pu'_\alpha + Qu'_\beta + mu = f\,. \end{aligned}$$

For a further simplification, let us make a substitution

$$u = e^{k\alpha + n\beta} v$$

where v is a new function and k and n are constants. It follows that

$$\begin{aligned} u'_\alpha &= e^{k\alpha+n\beta}(v'_\alpha + kv)\,, \\ u'_\beta &= e^{k\alpha+n\beta}(v'_\beta + nv)\,, \\ u''_{\alpha\alpha} &= e^{k\alpha+n\beta}(v''_{\alpha\alpha} + 2kv'_\alpha + k^2 v)\,, \\ u''_{\beta\beta} &= e^{k\alpha+n\beta}(v''_{\beta\beta} + 2nv'_\beta + n^2 v)\,. \end{aligned}$$

By a suitable choice of parameters k and n, one can always reduce *the elliptic, hyperbolic, and parabolic equations with constant coefficients to the following standard forms*:

$$\begin{aligned} \text{elliptic}: &\quad v''_{\alpha\alpha} + v''_{\beta\beta} + \mu v = g\,, \\ \text{hyperbolic}: &\quad v''_{\alpha\alpha} - v''_{\beta\beta} + \mu v = g\,, \\ \text{parabolic}: &\quad v''_{\beta\beta} + \mu v'_\alpha = g\,. \end{aligned} \tag{16.1}$$

For example, in the case of the elliptic equation, the aforementioned substitution yields (after multiplying each side of the equation by $e^{-k\alpha-n\beta}$)

$$\begin{aligned} &v''_{\alpha\alpha} + v''_{\beta\beta} + (P+2k)v'_\alpha + (Q+2n)v'_\beta \\ &+(k^2+n^2+Pk+Qn+m)v = fe^{-k\alpha-n\beta} \end{aligned}$$

Set

$$k = -\frac{P}{2}\,, \quad n = -\frac{Q}{2}$$

to make the coefficients at the first partial derivatives vanish. Thus, in the case of the elliptic equation

$$\mu = \frac{1}{2}(P^2+Q^2) + m\,, \quad g = fe^{-k\alpha-n\beta}$$

EXAMPLE **16.1**. *Determine the type of the equation and reduce it to one of the standard form:*

$$u''_{xx} + 2u''_{xy} + u''_{yy} + 4u'_x - 2u'_y + \tfrac{1}{4}\, u = 0$$

SOLUTION: In this equation $a = b = c = 1$. Therefore $D = c^2 - ab = 0$ and, hence, this is a parabolic equation. The characteristic equation has the form

$$(dy)^2 - 2dxdy + (dx)^2 = 0 \quad \Rightarrow \quad (dy - dx)^2 = 0$$

It has one solution

$$\gamma(x, y) = y - x = \gamma_0$$

The change of variables needed to reduce the equation to the standard form can be taken as

$$\alpha = \gamma(x, y) = y - x\,, \qquad \beta = y + x$$

The choice of the second variable β is arbitrary in the parabolic case, safe for one condition that the Jacobian of the transformation is not zero, which is the case with the above choice of β. By the chain rule

$$\begin{aligned}
\frac{\partial}{\partial x} &= -\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\,,\\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\,,\\
u''_{xx} &= \left(-\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)^2 u = u''_{\alpha\alpha} - 2u''_{\alpha\beta} + u''_{\beta\beta}\,,\\
u''_{yy} &= \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)^2 u = u''_{\alpha\alpha} + 2u''_{\alpha\beta} + u''_{\beta\beta}\,,\\
u''_{xy} &= \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)\left(-\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right) u = -u''_{\alpha\alpha} + u''_{\beta\beta}\,.
\end{aligned}$$

Substituting the partial derivatives in the new variables into the given equation, the latter is reduced to the following form

$$4u''_{\beta\beta} - 6u'_\alpha + 2u'_\beta + \tfrac{1}{4}\, u = 0$$

Next, let us make a substitution

$$u = e^{a\alpha + b\beta} v$$

where v is a new function, and the constants a and b are chosen to simplify the linear combination of the unknown function and its first partial derivatives as much as possible

$$4(v''_{\beta\beta} + 2bv'_\beta + b^2 v) - 6(v'_\alpha + av) + 2(v'_\beta + bv) + \tfrac{1}{4}v = 0\,,$$
$$4v''_{\beta\beta} + (8b+2)v'_\beta - 6v'_\alpha + (4b^2 - 6a + 2b + \tfrac{1}{4})v = 0$$

By choosing

$$b = -\frac{1}{4}$$

the coefficient at v'_β can be reduced to zero. The coefficient at v'_α cannot be modified, while the coefficient at v can be reduced to zero by choosing

$$a = \frac{1}{6}(4b^2 + 2b + \tfrac{1}{4}) = \frac{1}{6}$$

Thus, the standard form of the equation reads

$$v''_{\beta\beta} - \tfrac{3}{2}v'_\alpha = 0\,,$$

where

$$\alpha = y - x\,, \quad \beta = y + x\,, \quad u = e^{\frac{1}{6}\alpha - \frac{1}{4}\beta} v\,.$$

$\square$

16.2. Exercises.

1-3. Reduce each of the following equations to the standard form (**16.1**) by means of the substitution $u = e^{kx+my}v$ with a suitable choice of constants k and m:

1.

$$u''_{xx} + u''_{yy} = au'_x + bu'_y + cu$$

2.

$$u''_{xx} - u''_{yy} = au'_x + bu'_y + cu$$

3.

$$u''_{xx} = au'_x + bu'_y + cu$$

4-7. Determine the type of each of the following equations with constant coefficients and use a suitable change of variables and, if necessary, a substitution to reduce the equations to the standard form (**16.1**):

4.

$$u''_{xx} - 2u''_{xy} + u''_{yy} + au'_x + bu'_y + cu = 0$$

5.

$$u''_{xx} - 2u''_{xy} - 3u''_{yy} + au'_x + bu'_y + cu = 0$$

6.

$$u''_{xx} - 2u''_{xy} + 5u''_{yy} + au'_x + bu'_y + cu = 0$$

7.

$$u''_{xy} = au'_x + bu'_y + cu$$

Selected answers.
1. $v''_{xx} + v''_{yy} = Cv$, $C = c + \frac{1}{4}a^2 + \frac{1}{4}b^2$
2. $v''_{xx} - v''_{yy} = Cv$, $C = c + \frac{1}{4}a^2 - \frac{1}{4}b^2$
3. $v''_{xx} = bv'_y$ if $b \neq 0$, and $v''_{xx} = Cv$ if $b = 0$ where $C = c + \frac{1}{4}a^2$
4. $v''_{\beta\beta} + Av'_\alpha = 0$ if $a \neq -b$ where $A = \frac{1}{4}(a+b)$, and $v''_{\beta\beta} + Cv = 0$ if $a = -b$ where $C = \frac{1}{4}(c - \frac{1}{4}a^2)$; $\alpha = x + y$, $\beta = x - y$.
5. $v''_{\alpha\beta} - Cv = 0$, $C = \frac{1}{16}[\frac{1}{16}(3a+b)(b-a) + c]$, $\alpha = y - x$, $\beta = y + 3x$.
6. $v''_{\alpha\alpha} + v''_{\beta\beta} + Cv = 0$, $C = \frac{1}{4}c - \frac{1}{64}(a+b)^2 - \frac{1}{16}a^2$, $\alpha = x + y$, $\beta = \pm 2x$.
7. $v''_{xy} = (c + ab)v$.

17. Non-constants coefficients

It turns out that the analysis of the case with constant coefficients at the second partial derivatives is easily extended to a general case when the coefficients are general C^1 functions:

$$a = a(x,y)\,, \quad b = b(x,y)\,, \quad c = c(x,y)$$

First, note that under a general change of variables

$$\alpha = \alpha(x,y)\,, \quad \beta = \beta(x,y)$$

where $\alpha(x,y)$ and $\beta(x,y)$ are from the class C^2, the equation

$$au''_{xx} + 2cu''_{xy} + bu''_{yy} = F(x,y,u,u'_x,u'_y)$$

becomes

$$Au''_{\alpha} + 2Cu''_{\alpha\beta} + Bu''_{\beta\beta} = G(\alpha,\beta,u,u'_\alpha,u'_\beta) \tag{17.1}$$

where the coefficients A, B, and C are given by Eqs. (**14.1**)–(**14.3**). The terms containing second partial derivatives of α and β depend only on u'_α and u'_β and, hence, are included into G. Indeed, if α'_x, α'_y, β'_x, and β'_y are not constants, then the chain rules for u'_x and u'_y remains the same, while the chain rule for the second partials yields additional terms. For example,

$$\begin{aligned} u''_{xx} &= \frac{\partial}{\partial x}\, u'_x = \frac{\partial}{\partial x}\,(\alpha'_x u'_\alpha + \beta'_x u'_\beta) \\ &= \alpha'_x \frac{\partial}{\partial x}\, u'_\alpha + \beta'_x \frac{\partial}{\partial x}\, u'_\beta + \alpha''_{xx} u'_\alpha + \beta''_{xx} u'_\beta \\ &= \alpha'_x \left(\alpha'_x \frac{\partial}{\partial \alpha} + \beta'_x \frac{\partial}{\partial \beta}\right) u'_\alpha + \beta'_x \left(\alpha'_x \frac{\partial}{\partial \alpha} + \beta'_x \frac{\partial}{\partial \beta}\right) u'_\beta \\ &\quad + \alpha''_{xx} u'_\alpha + \beta''_{xx} u'_\beta \\ &= (\alpha'_x)^2 u''_{\alpha\alpha} + 2\alpha'_x\beta'_x u''_{\alpha\beta} + (\beta'_x)^2 u''_{\beta\beta} + \alpha''_{xx} u'_\alpha + \beta''_{xx} u'_\beta \end{aligned}$$

Note that the coefficients at the second partials with respect to the new variables have the same form as in the case of a linear change of variables. The difference is that there are additional linear combinations of u'_α and u'_β in the second partials u''_{xx}, u''_{xy}, and u''_{yy}. These additional linear combinations are included into a general function G, while the expressions (**14.1**), (**14.2**), and (**14.3**) for the coefficients A, B, and C, respectively, remains true for a general change of variables. This implies that in order to reduce A or B or both to zero, one has to solve the first order PDE (**14.4**).

17.1. The method of characteristics. The first order PDE (**14.4**) can be solved by the method of characteristics that has a natural extension to the case of non-constant coefficients a, b, and c. First, Proposition **14.1** can be extended to the case of a general change of variables.

THEOREM **17.1**. *Let a, b, and c be functions from the class C^1 such that a and b do not vanish simultaneously anywhere. If a function $\gamma(x,y) \in C^1$ is a solution to*

$$a(x,y)(\gamma_x')^2 + 2c(x,y)\gamma_x'\gamma_y' + b(x,y)(\gamma_y')^2 = 0 \tag{17.2}$$

such that γ_x' and γ_y' do not vanish simultaneously anywhere, then level curves $\gamma(x,y) = \gamma_0$ are solutions to the ordinary differential equation

$$a(x,y)(dy)^2 - 2c(x,y)dxdy + b(x,y)(dx)^2 = 0\,. \tag{17.3}$$

*Conversely, if level curves of a function $\gamma(x,y) \in C^1$ are solutions to (**17.3**), then the function $\gamma(x,y)$ is a solution to (**17.2**).*

PROOF. Let $\gamma(x,y)$ be a solution to (**17.2**). Let (x_0,y_0) be a point in a level set of the function γ, that is, $\gamma(x_0,y_0) = \gamma_0$. Suppose that $\gamma_y'(x_0,y_0) \neq 0$. By the implicit function theorem (Theorem **8.2**), the equation $\gamma(x,y) = \gamma_0$ has a unique solution $y = y(x)$ in a neighborhood of a point (x_0,y_0) such that $y_0 = y(x_0)$ and the function $y(x)$ is differentiable and

$$\frac{dy}{dx} = -\frac{\gamma_x'}{\gamma_y'} \tag{17.4}$$

$$\Rightarrow \quad \gamma_x'(x,y)dx + \gamma_y'(x,y)dy = 0 \tag{17.5}$$

If $\gamma_y'(x_0,y_0) = 0$, then $\gamma_x'(x_0,y_0) \neq 0$ and, by the implicit function theorem, the equation $\gamma(x,y) = \gamma_0$ can be solved for x so that the solution $x = x(y)$ is a differentiable function of y and $dx/dy = -\gamma_y'/\gamma_x'$. The latter leads to (**17.5**) again.

Next, it is shown that any curve satisfying (**17.5**) also satisfies ODE (**17.3**). Without loss of generality it is assumed that $\gamma_y'(x_0,y_0) \neq 0$, then near (x_0,y_0) the slope of the level curve is given by (**17.4**). The substitution of (**17.4**) into (**17.3**) yields

$$\begin{aligned} a\left(\frac{dy}{dx}\right)^2 - 2c\,\frac{dy}{dx} + b &= a\left(-\frac{\gamma_x'}{\gamma_y'}\right)^2 - 2c\left(-\frac{\gamma_x'}{\gamma_y'}\right) + b \\ &= \frac{1}{(\gamma_y')^2}\Big(a\,(\gamma_x')^2 + 2c\,\gamma_x'\gamma_y' + b\,(\gamma_y')^2\Big) \\ &= 0 \end{aligned}$$

because γ is a solution to (**17.2**). So, the level curve of γ passing through (x_0, y_0) is a solution to ODE (**17.3**). Since the choice of (x_0, y_0) is arbitrary, the conclusion holds everywhere.

Conversely, suppose $a(x_0, y_0) \neq 0$ so that $a(x, y) \neq 0$ near (x_0, y_0) by continuity of a. Then near (x_0, y_0) Eq. (**17.3**) has two solutions

$$\frac{dy}{dx} = \frac{c(x,y) \pm \sqrt{D(x,y)}}{a(x,y)},$$

where $D(x, y) = c^2(x, y) - a(x, y)b(x, y)$. By Theorem **10.1**, the initial value problem $y(x_0) = y_0$ for any of these equations has a unique solution because a, b, and c are from the class C^1. The solution is a smooth curve through (x_0, y_0) which can be represented as a level curve of a function $\gamma(x, y) \in C^1$ so that

$$\frac{dy}{dx} = \frac{c(x,y) \pm \sqrt{D(x,y)}}{a(x,y)} = -\frac{\gamma'_x(x,y)}{\gamma'_y(x,y)}$$

by the implicit function theorem. The latter relation shows that $-\gamma'_x/\gamma'_y$ is a root of the quadratic equation for the slope of the characteristic and, hence, γ satisfies (**17.2**):

$$\begin{aligned} a\left(\frac{\gamma'_x}{\gamma'_y}\right)^2 + 2c\,\frac{\gamma'_x}{\gamma'_y} + b &= a\left(\frac{c \pm \sqrt{D}}{a}\right)^2 - 2c\,\frac{c \pm \sqrt{D}}{a} + b \\ &= 0 \end{aligned}$$

If $a(x_0, y_0) = 0$, then $b(x_0, y_0) \neq 0$. Equation (**17.3**) can be solved for the slope dx/dy near such point. The obtained ODE has a unique solution that defines a smooth curve passing through the point (x_0, y_0). This curve can always be viewed as a level curve of some function $\gamma(x, y)$ such that $\gamma'_x(x_0, y_0) \neq 0$. By the implicit function theorem, the slope of this curve is $dx/dy = -\gamma'_y/\gamma'_x$ and, hence, is also a root of the quadratic equation (**17.3**). The latter implies that γ must be a solution to (**17.2**). Since the choice of (x_0, y_0) is arbitrary, the conclusion holds everywhere. □

17.2. Classification of second order PDEs. The discriminant $D = D(x, y)$ is a function of x and y. Therefore its sign depends on (x, y) and so does the type of PDE.

DEFINITION **17.1**. (Classification of second-order PDEs)
The equation

$$a(x,y)u''_{xx} + 2c(x,y)u''_{xy} + b(x,y)u''_{yy} = F(x,y,u,u'_x,u'_y)$$

is called

- hyperbolic in Ω *if* $D(x,y>0\,,\quad (x,y)\in\Omega$
- elliptic in Ω *if* $D(x,y)<0\,,\quad (x,y)\in\Omega$
- parabolic in Ω *if* $D(x,y)=0\,,\quad (x,y)\in\Omega$

where Ω *is an open region and*

$$D(x,y)=c^2(x,y)-a(x,y)b(x,y)$$

Recall that a PDE is defined in an open set Ω. The type of PDE determines a change of variables in Ω that is needed to reduce the PDE to one of the standard forms. For example, the equation

$$u''_{xx}+xu''_{yy}=0 \quad \text{is} \quad \begin{cases} \text{elliptic} & \text{in } \Omega_+=\{(x,y)|x>0\} \\ \text{hyperbolic} & \text{in } \Omega_-=\{(x,y)|x<0\} \end{cases}$$

The set where $D=0$ is the line $x=0$. It has no interior points at all as a set in $\mathbb{R}^2$ and, hence, it has no open subset in which the equation may be viewed as a parabolic equation.

Remark. In general case, integration of the characteristic equation (**17**.**3**) is a difficult task. If, however, if two independent integrals (characteristics) are known, a PDE can be reduced to the corresponding standard form in the same way as in the case with constant coefficients a, b, and c.

17.3. Hyperbolic equations. If there is an open set Ω in which $D(x,y)>0$, then the characteristic equation has two real independent integrals (curves) satisfying two ordinary differential equations

$$\frac{dy}{dx}=\frac{c(x,y)+\sqrt{D(x,y)}}{a(x,y)} \quad \Rightarrow \quad \alpha(x,y)=\alpha_0$$
$$\frac{dy}{dx}=\frac{c(x,y)-\sqrt{D(x,y)}}{a(x,y)} \quad \Rightarrow \quad \beta(x,y)=\beta_0$$

where α_0 and β_0 are constants. Let us show that the transformation

$$\alpha=\alpha(x,y)\,,\quad \beta=\beta(x,y)$$

defines a change of variables in Ω. Since a and b do not vanish simultaneously, let $a\neq 0$ in a neighborhood of some point. Then in this neighborhood, $\alpha'_y\neq 0$ and $\beta'_y\neq 0$ because $\alpha(x,y)$ and $\beta(x,y)$ are solutions to are solutions to the first-order PDE (**17**.**2**) (see (**17**.**4**)).

Then

$$\frac{\alpha'_x}{\alpha'_y} = -\frac{c(x,y) + \sqrt{D(x,y)}}{a(x,y)}\,,$$
$$\frac{\beta'_x}{\beta'_y} = -\frac{c(x,y) - \sqrt{D(x,y)}}{a(x,y)}$$

Therefore the Jacobian of the transformation is

$$J = \alpha'_x\beta'_y - \alpha'_y\beta'_x == \alpha'_y\beta'_y\left(\frac{\alpha'_x}{\alpha'_y} - \frac{\beta'_x}{\beta'_y}\right) = \frac{2\alpha'_y\beta'_y}{a}\sqrt{D} \neq 0$$

because $D > 0$. If $a(x,y) = 0$, then one of the characteristics is an integral of the equation $dx = 0$ so that $\beta(x,y) = x$, whereas the function $\alpha(x,y)$ solves (**17.2**)

$$\frac{\alpha'_x}{\alpha'_y} = -\frac{2c(x,y)}{b(x,y)}\,, \qquad \alpha'_y(x,y) \neq 0\,.$$

Therefore $J = -\alpha'_y \neq 0$. The points where the gradients of α or β vanish are singular points of the change of variables. Just like in the case of first order PDE, if such a point happens to be in Ω, then any solution found near this point has to have a smooth extension to this point otherwise it is not a solution to the original PDE.

In the new variables associated with the two independent characteristics, $A = B = 0$ in (**17.1**) and a hyperbolic PDE is reduced to (**15.1**) or to (**15.2**) if, in addition,

$$\xi = \frac{1}{2}(\alpha + \beta)\,, \quad \eta = \frac{1}{2}(\alpha - \beta)\,.$$

17.4. An operator method to transform partial derivatives. Although the coefficients A and B are proved to vanish in the new variables, all second partial derivatives also contains terms linear in the first-order partial derivatives that have to be calculated in order to reduce the equation to its standard form. It is convenient to use the following technical trick. Let ∂_x denotes an *operator* of taking a partial derivative with respect to x, that is, the operator *acts* on a function u producing u'_x, which is written as

$$\partial_x u = u'_x$$

The second partial derivative u''_{xx} is the result of a repeated action of ∂_x on u. For example,

$$\partial_x^2 u = \partial_x(\partial_x u) = \partial_x u'_x = u''_{xx}\,, \quad \partial_x\partial_y u = u''_{xy}\,, \quad \partial_y^2 u = u''_{yy}$$

One can consider operators obtained by multiplication of a function and a differentiation operators. For example,

$$D_x = u\partial_x\,, \qquad D_x v = u\partial_x v = uv'_x\,,$$

for any function v. The operators of differentiation satisfy the product rule. For example,

$$\partial_x(uv) = (\partial_x u)v + u(\partial_x v)\,, \quad \partial_x D_y = \partial_x(u\partial_y) = u'_x\partial_y + u\partial_x\partial_y\,.$$

There is a relation between the operators of differentiation in the old and new variables:

$$\partial_x = \alpha'_x\partial_\alpha + \beta'_x\partial_\beta\,, \qquad \partial_y = \alpha'_y\partial_\alpha + \beta'_y\partial_\beta\,.$$

established by the chain rules. Then

$$\begin{aligned}
\partial_x^2 &= \partial_x(\alpha'_x\partial_\alpha + \beta'_x\partial_\beta) = (\partial_x\alpha'_x)\partial_\alpha + (\partial_x\beta'_x)\partial_\beta + \alpha'_x\partial_x\partial_\alpha + \beta'_x\partial_x\partial_\beta \\
&= \alpha''_{xx}\partial_\alpha + \beta''_{xx}\partial_\beta + \alpha'_x(\alpha'_x\partial_\alpha + \beta'_x\partial_\beta)\partial_\alpha + \beta'_x(\alpha'_y\partial_\alpha + \beta'_y\partial_\beta)\partial_\beta \\
&= \alpha''_{xx}\partial_\alpha + \beta''_{xx}\partial_\beta + (\alpha'_x)^2\partial_\alpha^2 + 2\alpha'_x\beta'_x\partial_\alpha\partial_\beta + (\beta'_x)^2\partial_\beta^2 \\
u''_{xx} &= \alpha''_{xx}u'_\alpha + \beta''_{xx}u'_\beta + (\alpha'_x)^2u''_{\alpha\alpha} + 2\alpha'_x\beta'_xu''_{\alpha\beta} + (\beta'_x)^2u''_{\beta\beta}
\end{aligned}$$

Other second partial derivatives can be computed similarly.

EXAMPLE **17.1**. *Reduce the equation*

$$u''_{xx} - xyu''_{yy} = 0\,, \quad (x,y) \in \Omega = \{(x,y)|x > 0\,,\ y > 0\}$$

to the standard form.

SOLUTION: Here $a = 1$, $c = 0$, and $b = xy$. Therefore $D(x,y) = xy > 0$ in Ω. The characteristic equation reads

$$(dy)^2 - xy(dx)^2 = 0 \quad \Rightarrow \quad dy = \pm\sqrt{xy}\,dx$$

which are easily integrated by separating variables

$$dy = \sqrt{xy}\,dx \quad \Rightarrow \quad \int \frac{dy}{\sqrt{y}} = \int \sqrt{x}dx \quad \Rightarrow \quad 2y^{1/2} - \frac{2}{3}x^{3/2} = \alpha_0$$

$$dy = -\sqrt{xy}\,dx \quad \Rightarrow \quad \int \frac{dy}{\sqrt{y}} = -\int \sqrt{x}dx \quad \Rightarrow \quad 2y^{1/2} + \frac{2}{3}x^{3/2} = \beta_0$$

Put

$$\alpha = 2y^{1/2} - \frac{2}{3}x^{3/2}\,, \quad \beta = 2y^{1/2} + \frac{2}{3}x^{3/2}$$

and

$$\xi = \frac{1}{2}(\alpha + \beta) = 2y^{1/2}\,, \qquad \eta = \frac{1}{2}(\beta - \alpha) = \frac{2}{3}x^{3/2}$$

Using the chain rules

$$\frac{\partial}{\partial x} = x^{1/2}\,\frac{\partial}{\partial \eta}\,, \qquad \frac{\partial}{\partial y} = y^{-1/2}\,\frac{\partial}{\partial \xi}\,,$$

$$u''_{xx} = \frac{\partial}{\partial x}\left(x^{1/2}u'_\eta\right) = x^{1/2}\frac{\partial}{\partial x}\,u'_\eta + \frac{1}{2}\,x^{-1/2}u'_\eta = xu''_{\eta\eta} + \frac{1}{2}\,x^{-1/2}u'_\eta$$

$$u''_{yy} = \frac{\partial}{\partial y}\left(y^{-1/2}u'_\xi\right) = y^{-1/2}\frac{\partial}{\partial y}\,u'_\xi - \frac{1}{2}\,y^{-3/2}u'_\xi = y^{-1}u''_{\xi\xi} - \frac{1}{2}\,y^{-3/2}u'_\xi$$

The substitution of the second partials into the original equation with a subsequent replacement of x and y by the corresponding functions of ξ and η yield:

$$\begin{aligned} u''_{xx} - xyu''_{yy} &= xu''_{\eta\eta} + \frac{1}{2}\,x^{-1/2}u'_\eta - xu''_{\xi\xi} + \frac{1}{2}\,xy^{-1/2}u'_\xi \\ &= x(u''_{\eta\eta} - u''_{\xi\xi}) + \frac{x}{2}\left(x^{-3/2}u'_\eta + y^{-1/2}u'_\xi\right) \\ &= 0 \\ u''_{\eta\eta} - u''_{\xi\xi} &= -\frac{1}{3\eta}\,u'_\eta - \frac{1}{4\xi}\,u'_\xi \end{aligned}$$

□

17.5. Parabolic equations. In the case of parabolic equations $D(x, y) = 0$ in an open region Ω and two characteristics coincide. A change of variables that reduces the equation to the standard form (**15.3**) is obtained in the same way as in the case with constant coefficients, namely, one of the new variables is determined by the characteristic (a solution to (**17.2**) or (**17.3**)), while the other is arbitrary subject to the only condition that the Jacobian of the transformation does not vanish in Ω:

$$\frac{dy}{dx} = \frac{c(x,y)}{a(x,y)} \quad \Rightarrow \quad \gamma(x,y) = \alpha_0\,,$$
$$\alpha = \gamma(x,y)\,, \quad \beta = \varphi(x,y)$$

where φ is a C^2 function such that

$$J = \gamma'_x\varphi'_y - \gamma'_y\varphi'_x \neq 0\,, \quad (x,y) \in \Omega\,.$$

If the right side of Eq. (**15.3**) is independent of u'_α, then the equation becomes an ordinary differential equation in which the coordinate α plays the role of a parameter.

EXAMPLE **17.2**. *Determine the type of the equation and use a suitable change of variable to reduce the equation to the standard form:*

$$xu''_{xx} + 2\sqrt{xy}u''_{xy} + yu''_{yy} + \tfrac{1}{2}\,u'_y = 0\,, \quad x > 0\,,\ y > 0$$

Specify the new region in which the standard equation is to be solved.

SOLUTION: In this equation $a = x$, $b = y$, and $c = \sqrt{xy}$. Therefore

$$D(x,y) = c^2 - ab = xy - xy = 0\,, \quad x > 0\,,\ y > 0$$

and the equation is parabolic in the first quadrant. The characteristic equation (**17.3**) is easily integrated by separating variables

$$\frac{dy}{dx} = \frac{c}{a} = \frac{\sqrt{y}}{\sqrt{x}} \quad \Rightarrow \quad \int \frac{dy}{\sqrt{y}} = \int \frac{dx}{\sqrt{x}} \quad \Rightarrow \quad \sqrt{y} - \sqrt{x} = C$$

Therefore the new variables can be taken in the form

$$\alpha = \sqrt{y} - \sqrt{x}\,, \quad \beta = y$$

because the Jacobian of the transformation does not vanish in the first quadrant:

$$J = \alpha'_x \beta'_y - \alpha'_y \beta'_x = -\frac{1}{2\sqrt{x}} \neq 0\,, \quad x > 0\,,\ y > 0\,.$$

Using the chain rules

$$\begin{aligned}
\frac{\partial}{\partial x} &= -\frac{1}{2\sqrt{x}}\,\frac{\partial}{\partial \alpha}\,, \quad \frac{\partial}{\partial y} = \frac{1}{2\sqrt{y}}\,\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\,,\\
u''_{xx} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2\sqrt{x}}\,u'_\alpha\right) = -\frac{1}{2\sqrt{x}}\,\frac{\partial}{\partial x}\,u'_\alpha + \frac{1}{4}\,x^{-3/2}u'_\alpha\\
&= \frac{1}{4x}\,u''_{\alpha\alpha} + \frac{1}{4}x^{-3/2}\,u'_\alpha\\
u''_{yy} &= \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2\sqrt{y}}\,u'_\alpha + u'_\beta\right)\\
&= \frac{1}{2\sqrt{y}}\,\frac{\partial}{\partial y}\,u'_\alpha + \frac{\partial}{\partial y}\,u'_\beta - \frac{1}{4}\,y^{-3/2}\,u'_\alpha\\
&= \frac{1}{4y}\,u''_{\alpha\alpha} + \frac{1}{\sqrt{y}}\,u''_{\alpha\beta} + u''_{\beta\beta} - \frac{1}{4}\,y^{-3/2}\,u'_\alpha\\
u''_{xy} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{y}}\,u'_\alpha + u'_\beta\right) = \frac{1}{2\sqrt{y}}\frac{\partial}{\partial x}u'_\alpha + \frac{\partial}{\partial x}u'_\beta\\
&= -\frac{1}{4}\,(xy)^{-1/2}u''_{\alpha\alpha} - \frac{1}{2}\,x^{-1/2}u''_{\alpha\beta}
\end{aligned}$$

The substitution of the second partials into the equation yields

$$
\begin{aligned}
xu''_{xx} + 2\sqrt{xy}u''_{xy} + yu''_{yy} &= x\Big(\tfrac14 x^{-1}\, u''_{\alpha\alpha} + \tfrac14 x^{-3/2}\, u'_\alpha\Big)\\
&\quad + 2(xy)^{1/2}\Big(-\tfrac14 (xy)^{-1/2}u''_{\alpha\alpha} - \tfrac12 x^{-1/2}u''_{\alpha\beta}\Big)\\
&\quad + y\Big(\tfrac14 y^{-1}\, u''_{\alpha\alpha} + y^{-1/2}\, u''_{\alpha\beta} + u''_{\beta\beta} - \tfrac14\, y^{-3/2}\, u'_\alpha\Big)\\
&= yu''_{\beta\beta} - \tfrac14 (y^{-1/2} - x^{-1/2})\, u'_\alpha
\end{aligned}
$$

A subsequent substitution of the latter relation and $y = \beta$ and $\sqrt{x} = \sqrt{\beta} - \alpha$ reduces the equation to the standard form:

$$
\begin{aligned}
&yu''_{\beta\beta} - \tfrac14 (y^{-1/2} - x^{-1/2})\, u'_\alpha + \tfrac12\Big(\tfrac12 y^{-1/2}u'_\alpha + u'_\beta\Big) = 0\\
&u''_{\beta\beta} = -\frac{1}{4\beta(\sqrt{\beta}-\alpha)}\, u'_\alpha - \frac{1}{2\beta}\, u'_\beta\,, \qquad (\alpha,\beta)\in\Omega'
\end{aligned}
$$

To find Ω' in the $\alpha\beta$ plane, note that the boundary $\partial\Omega$ of Ω should be mapped onto the boundary of Ω' by the transformation considered. The region Ω is bounded by the coordinate axes so that the line $y = 0$ (the x axis) is mapped to $\beta = 0$ (the α axis) and the line $x = 0$ (the y axis) is mapped to the parabola $\alpha = \sqrt{\beta}$. Therefore the equation is to be solved in the open region

$$
\Omega' = \Big\{(\alpha,\beta)\,|\,\alpha < \sqrt{\beta}\,,\ \beta > 0\Big\}
$$

□

Peculiarities of parabolic equations. There are a few features in the standard form of a parabolic PDE that are independent of the choice of the second variable β. For example, in Example **17.2** one can generally set

$$
\alpha = \sqrt{y} - \sqrt{x}\,, \qquad \beta = \beta(x,y)
$$

where $\beta(x,y)$ is a C^2 function such that the Jacobian does not vanish in Ω (the first quadrant):

$$
J = -\frac{\beta'_y}{2\sqrt{x}} - \frac{\beta'_x}{2\sqrt{y}} = -\frac{1}{2\sqrt{xy}}\Big(\sqrt{x}\,\beta'_x + \sqrt{y}\,\beta'_y\Big) \neq 0\,, \quad x>0\,,\ y>0
$$

By the chain rules similar to those obtained in Example **17.2**:

$$
\begin{aligned}
\frac{\partial}{\partial x} &= -\frac{1}{2\sqrt{x}}\,\frac{\partial}{\partial \alpha} + \beta'_x\,\frac{\partial}{\partial \beta}\,, \qquad \frac{\partial}{\partial y} = \frac{1}{2\sqrt{y}}\,\frac{\partial}{\partial \alpha} + \beta'_y\,\frac{\partial}{\partial \beta}\,,\\
u''_{xx} &= \frac{1}{4x}\,u''_{\alpha\alpha} - \frac{\beta'_x}{\sqrt{x}}\,u''_{\alpha\beta} + (\beta'_x)^2 u''_{\beta\beta} + \frac{1}{4x\sqrt{x}}\,u'_\alpha + \beta''_{xx}\,u'_\beta\,,\\
u''_{yy} &= \frac{1}{4y}\,u''_{\alpha\alpha} + \frac{\beta'_y}{\sqrt{y}}\,u''_{\alpha\beta} + (\beta'_y)^2\,u''_{\beta\beta} - \frac{1}{4y\sqrt{y}}\,u'_\alpha + \beta''_{yy}\,u'_\beta\,,\\
u''_{xy} &= -\frac{1}{4\sqrt{xy}}\,u''_{\alpha\alpha} + \left(\frac{\beta'_x}{2\sqrt{y}} - \frac{\beta'_y}{2\sqrt{x}}\right) u''_{\alpha\beta} + \beta'_x\beta'_y u''_{\beta\beta} + \beta''_{xy}\,u'_\beta
\end{aligned}
$$

Therefore the equation is reduced to the standard form

$$
\begin{aligned}
0 &= x\,u''_{xx} + y\,u''_{yy} + 2\sqrt{xy}\,u''_{xy} + \tfrac{1}{2}\,u'_y\\
&= u''_{\alpha\alpha}\left(\tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{2}\right) + u''_{\alpha\beta}\left(-\sqrt{x}\beta'_x + \sqrt{y}\beta'_y + \sqrt{x}\beta'_x - \sqrt{y}\beta'_y\right)\\
&\quad + u''_{\beta\beta}\left(x(\beta'_x)^2 + y(\beta'_y)^2 + 2\sqrt{xy}\beta'_x\beta'_y\right)\\
&\quad + u'_\alpha\left(\frac{1}{4\sqrt{x}} - \frac{1}{4\sqrt{y}} + \frac{1}{4\sqrt{y}}\right)\\
&\quad + u'_\beta\left(x\beta''_{xx} + y\beta''_{yy} + 2\sqrt{xy}\beta''_{xy} + \frac{\beta'_y}{2}\right)\\
&= 4xyJ^2 u''_{\beta\beta} + \frac{1}{4\sqrt{x}}\,u'_\alpha + u'_\beta\left(x\beta''_{xx} + y\beta''_{yy} + 2\sqrt{xy}\beta''_{xy} + \frac{\beta'_y}{2}\right)
\end{aligned}
$$

The equation can be divided by $4xyJ^2$ because $J(x,y) \neq 0$ in Ω so that it assumes the desired form (**15.3**).

Can the freedom in choosing $\beta(x,y)$ be used to further simplify the parabolic equation? First, *the choice of $\beta(x,y)$ cannot be used to make the coefficient at u'_α to be zero and thereby to reduce the equation to an ordinary differential equation in which the variable α is a parameter.* This is a general feature, not specific to the example considered. That this coefficient vanishes or does not vanish is determined *only* by the characteristic $\alpha(x,y)$ of a parabolic PDE and, of course, by coefficients a, b, and c.

Second, *the choice of $\beta(x,y)$ may only be used to simplify the dependence of the equation on u'_β.* For instance, if

$$
\beta(x,y) = \sqrt{y} + \sqrt{x} \quad \Rightarrow \quad J = -\frac{1}{2\sqrt{xy}}
$$

then the standard form of the parabolic equation in Example **17.2** reads

$$u''_{\beta\beta} = \frac{1}{2(\beta-\alpha)}\Big(u'_\beta - u'_\alpha\Big)\,, \qquad \beta > |\alpha| > 0$$

With the choice $\beta = x + \sqrt{y}$, the coefficient at u'_β vanishes in the standard form, but the coefficient at u'_α has a more complicated dependence on α and β. Yet, the new region in which the equation is to be solved has a more complicated boundary: $\beta > \alpha > 0$ and $\beta^2 > -\alpha > 0$. Technical details are left to the reader as an exercise.

17.6. Elliptic equations. For an elliptic PDE, $D(x,y) < 0$ in an open region Ω. In this case, the characteristic equation has two complex independent integrals

$$\frac{dy}{dx} = \frac{c(x,y) - i\sqrt{-D(x,y)}}{a(x,y)} \quad \Rightarrow \quad \gamma(x,y) = C$$

$$\frac{dy}{dx} = \frac{c(x,y) + i\sqrt{-D(x,y)}}{a(x,y)} \quad \Rightarrow \quad \bar{\gamma}(x,y) = \bar{C}$$

where $\bar{\gamma}$ is the complex conjugation of γ. The equation can be reduced to the standard form (**15.4**) by means of the change of variables

$$\xi = \operatorname{Re}\gamma(x,y) = \frac{1}{2}\Big(\gamma(x,y) + \bar{\gamma}(x,y)\Big)\,,$$

$$\eta = \operatorname{Im}\gamma(x,y) = \frac{1}{2i}\Big(\gamma(x,y) - \bar{\gamma}(x,y)\Big)\,.$$

One can show that the Jacobian of this transformation is not equal to zero. A proof of this assertion is similar to the case of hyperbolic equations, and technicalities are left to the reader as an exercise.

EXAMPLE **17.3**. *Reduce the equation to the standard form by a suitable change of variables*

$$y^2 u''_{xx} + e^{2x} u''_{yy} + \frac{1}{2}\, y\, u'_y = 0\,, \quad y > 0\,,\ x \in \mathbb{R}$$

Find the new region in which the standard equation is to be solved.

SOLUTION: In the PDE in question $a = y^2$, $c = 0$, and $b = e^{2x}$. Therefore $D = c^2 - ab = -y^2 e^{2x} < 0$ in the upper half of the xy plane and, hence, this an elliptic PDE The complex characteristic is determined by a general integral of the equation that is solved by separation of variables

$$\frac{dy}{dx} = -\frac{ie^x}{y} \quad \Rightarrow \quad \int y\,dy = i\int e^x dx \quad \Rightarrow \quad \gamma(x,y) = \frac{1}{2}\,y^2 + ie^x = C$$

Therefore the sought-after change of variables reads

$$\xi = \operatorname{Re}\gamma(x,y) = \frac{1}{2}y^2\,, \quad \eta = \operatorname{Im}\gamma(x,y) = e^x$$

The Jacobian reads

$$J = \xi'_x\eta'_y - \xi'_y\eta'_x = ye^x \neq 0\,, \quad y > 0\,,\ x \in \mathbb{R}\,.$$

By the chain rules

$$\begin{aligned}
&\frac{\partial}{\partial x} = e^x\,\frac{\partial}{\partial \eta}\,, \quad \frac{\partial}{\partial y} = y\,\frac{\partial}{\partial \xi}\,,\\
&u''_{xx} = \frac{\partial}{\partial x}\left(e^x u'_\eta\right) = e^x\frac{\partial}{\partial x}u'_\eta + e^x u'_\eta = e^{2x}u''_{\eta\eta} + e^x u'_\eta\,,\\
&u''_{yy} = \frac{\partial}{\partial y}\left(yu'_\xi\right) = y\,\frac{\partial}{\partial y}\,u'_\xi + u'_\xi = y^2u''_{\xi\xi} + u'_\xi
\end{aligned}$$

The substitution of the partials in the new variables into the equation yields

$$\begin{aligned}
y^2\Big(e^{2x}u''_{\eta\eta} + e^x u'_\eta\Big) + e^{2x}\Big(y^2u''_{\xi\xi} + u'_\xi\Big) + \frac{y^2}{2}\,u'_\xi &= 0\,,\\
u''_{\eta\eta} + u''_{\xi\xi} + \frac{e^{2x} + \frac{1}{2}y^2}{y^2e^{2x}}\,u'_\xi + e^{-x}\,u'_\eta &= 0\,,\\
u''_{\eta\eta} + u''_{\xi\xi} + \frac{\eta^2 + \xi}{2\xi\eta^2}\,u'_\xi + \frac{1}{\eta}\,u'_\eta &= 0
\end{aligned}$$

The latter equation is to be solved in the open region that is the image of the upper half-plane under the transformation $(x,y) \to (\xi,\eta)$:

$$(\xi,\eta) \in \Omega' = \Big\{(\xi,\eta)\,|\,\xi > 0\,,\ \eta > 0\Big\}\,.$$

□

17.7. Exercises.

1-5. Reduce each of the following equations to the standard form in the specified open region by means of a suitable change of variables and determine the region spanned by the new variables in which the reduced equation is to be solved:

1.

$$u''_{xx} + xyu''_{yy} = 0\,, \qquad x > 0\,,\ y > 0$$

2.

$$e^{2x}u''_{xx} + 2e^{x+y}u''_{xy} + e^{2y}u''_{yy} = 0\,, \qquad (x,y) \in \mathbb{R}^2$$

3.

$$u''_{xx} - 2xyu''_{xy} - 3(xy)^2u''_{yy} = 0\,, \qquad x > 0\,,\ y > 0$$

4. Here $b(y) > 1$, $b(y) \to 1$ as $|y| \to \infty$, and $b \in C^1$

$$u''_{xx} - b^2(y)u''_{yy} = 0\,, \qquad (x,y) \in \mathbb{R}^2\,.$$

5. Here $b(y) > 1$, $b(y) \to 1$ as $|y| \to \infty$, and $b \in C^1$

$$u''_{xx} + b^2(y)u''_{yy} = 0\,, \qquad (x,y) \in \mathbb{R}^2\,.$$

6. Is there any choice of functions $f(x,y)$ and $g(x,y)$ for which the following equation can be reduced to an ordinary differential equation?

$$e^{2x}u''_{xx} + 2e^{x+y}u''_{xy} + e^{2y}u''_{yy} = fu'_x + gu'_y\,, \qquad (x,y) \in \mathbb{R}^2$$

Selected answers.

1. $u''_{\alpha\alpha} + u''_{\beta\beta} = \frac{1}{\beta}\,u'_\beta - \frac{1}{3\alpha}\,u'_\alpha$, $\alpha = \frac{2}{3}x^{3/2}$, $\beta = 2y^{1/2}$.
2. $u''_{\beta\beta} + \frac{\beta}{\beta^2-\alpha^2}\,u'_\alpha - \frac{\alpha}{\beta^2-\alpha^2}\,u'_\beta = 0$, $\alpha = e^{-y} - e^{-x}$, $\beta = e^{-y} + e^{-x}$.
3. $u''_{\alpha\beta} = \frac{3}{16}\frac{2+\beta-\alpha}{\beta-\alpha}\,u'_\beta + \frac{1}{16}\frac{3\beta-3\alpha-2}{\beta-\alpha}\,u'_\alpha$, $\alpha = \ln(y) - \frac{1}{2}x^2$, $\beta = \ln(y) + \frac{3}{2}x^2$.
4. $u''_{\alpha\beta} = \frac{1}{4}b'(y)(u'_\alpha + u'_\beta)$ where y is the root of the equation $F(y) = \frac{1}{2}(\alpha+\beta)$ and $F(y) = \int_0^y \frac{dz}{b(z)}$ (there exists only one root. Prove it!)
5. $u''_{\alpha\alpha} + u''_{\beta\beta} = b'(y)u'_\beta$ where y is the root of the equation $F(y) = \beta$ and $F(y) = \int_0^y \frac{dz}{b(z)}$ (there exists only one root. Prove it!)
6. Yes. $e^{-x}f + e^{-x} = e^{-y}g + e^{y}$.

18. Multivariable second-order PDEs

Let $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. A general second-order PDE in n variables has the form

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}(\mathbf{x})\,\frac{\partial^2 u}{\partial x_j \partial x_k} = F(\mathbf{x}, u, \nabla u)\,, \quad \mathbf{x} \in \Omega \tag{18.1}$$

where F is a function of $\mathbf{x}$, u, and its gradient ∇u, and Ω is an open region in $\mathbb{R}^n$. Consider a change of variables in Ω

$$\alpha_j = \alpha_j(\mathbf{x})\,, \quad j = 1, 2, ..., n$$

This implies that the Jacobian of this transformation does not vanish in Ω:

$$J_{jk} = \frac{\partial \alpha_k}{\partial x_j}\,, \quad J(\mathbf{x}) = \det(J_{jk}) \neq 0\,, \quad \mathbf{x} \in \Omega$$

By the chain rule

$$\begin{aligned}
\frac{\partial}{\partial x_j} &= \sum_{m=1}^{n} \frac{\partial \alpha_m}{\partial x_j}\,\frac{\partial}{\partial \alpha_m} = \sum_{m=1}^{n} J_{jm}\,\frac{\partial}{\partial \alpha_m}\,,\\
\frac{\partial^2 u}{\partial x_j \partial x_k} &= \frac{\partial}{\partial x_k}\left(\sum_{m=1}^{n} J_{jm}\,\frac{\partial u}{\partial \alpha_m}\right) = \sum_{m=1}^{n} J_{jm}\,\frac{\partial}{\partial x_k}\frac{\partial u}{\partial \alpha_m} + \sum_{m=1}^{n} \frac{\partial J_{jm}}{\partial x_k}\,\frac{\partial u}{\partial \alpha_m}\\
&= \sum_{m=1}^{n}\sum_{l=1}^{n} J_{jm}J_{kl}\frac{\partial^2 u}{\partial \alpha_l \partial \alpha_m} + \sum_{m=1}^{n} \frac{\partial J_{jm}}{\partial x_k}\,\frac{\partial u}{\partial \alpha_m}
\end{aligned}$$

Therefore in the new variables the equation assumes the form

$$\sum_{m=1}^{n}\sum_{l=1}^{n} A_{ml}\frac{\partial^2 u}{\partial \alpha_l \partial \alpha_m} = G\,, \tag{18.2}$$

$$G = F - \sum_{m=1}^{n} \frac{\partial J_{jm}}{\partial x_k}\,\frac{\partial u}{\partial \alpha_m}\,,$$

$$A_{ml} = \sum_{j=1}^{n}\sum_{k=1}^{n} J_{jm}J_{kl}a_{jk}$$

The relation between coefficients A_{ml} and a_{jk} can be cast in the matrix form

$$\boldsymbol{A} = \boldsymbol{J}^T \boldsymbol{a} \boldsymbol{J}$$

where $\boldsymbol{A}$ and $\boldsymbol{a}$ are $n \times n$ symmetric matrices with elements A_{ml} and a_{jk}, respectively, and $\boldsymbol{J}$ is the *Jacobian matrix* with elements J_{jk} and $\boldsymbol{J}^T$ is the transposition of $\boldsymbol{J}$.

Suppose that the coefficients a_{jk} are constant. Consider a *quadratic form* associated with a symmetric matrix $\boldsymbol{a} = \boldsymbol{a}^T$:

$$\mathbf{y}^T \boldsymbol{a} \mathbf{y} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} y_j y_k \,, \qquad \mathbf{y} \in \mathbb{R}^n$$

Take a *linear* change of variables

$$y_k = \sum_{j=1}^{n} J_{kj} \, \beta_j \quad \text{or} \quad \mathbf{y} = \boldsymbol{J\beta}$$

Then the quadratic form in the new variables reads

$$\mathbf{y}^T \boldsymbol{a} \mathbf{y} = (\boldsymbol{J\beta})^T \boldsymbol{aJ\beta} = \boldsymbol{\beta}^T \boldsymbol{J}^T \boldsymbol{aJ\beta} = \boldsymbol{\beta}^T \boldsymbol{A\beta}$$

The new matrix in the quadratic form is also symmetric

$$\boldsymbol{A}^T = (\boldsymbol{J}^T \boldsymbol{aJ})^T = \boldsymbol{J}^T \boldsymbol{a}^T (\boldsymbol{J})^{TT} = \boldsymbol{J}^T \boldsymbol{aJ} = \boldsymbol{A}$$

It is known from the linear algebra that for every symmetric matrix $\boldsymbol{a} = \boldsymbol{a}^T$ there exists a linear transformation defined by a non-singular matrix $\boldsymbol{J}$ (that is, $\det \boldsymbol{J} \neq 0$) such the transformed matrix $\boldsymbol{A}$ is diagonal and its diagonal elements are either ± 1 or 0:

$$A_{ml} = 0 \,, \quad m \neq l \,, \quad A_{ll} \in \{1, -1, 0\}$$

Furthermore, *Sylvester's law of inertia for quadratic forms* asserts that the number of diagonal elements of each kind is an invariant of $\boldsymbol{a}$, that is, it does not depend on the matrix $\boldsymbol{J}$ used to reduce the quadratic form to its standard (diagonal) form.

Returning to Eq. (**18.1**), suppose that a_{jk} are continuous functions on an open set Ω. For every point $\mathbf{x}_0 \in \Omega$, there is a linear non-singular transformation (a change of variables) such that

$$A_{ml}(\mathbf{x}_0) = 0 \,, \quad m \neq l \,; \qquad A_{ll}(\mathbf{x}_0) \in \{1, -1, 0\}$$

Note that the transformation depends on $\mathbf{x}_0$. Owing to Sylvester's law of inertia, the following classification of second-order higher dimensional PDEs is adopted.

DEFINITION **18.1**. *Equation (***18.1***) at a point $\mathbf{x}_0$ is called*

- *an* elliptic *equation if all n coefficients $A_{ll}(\mathbf{x}_0)$ have the same sign;*
- *a* normal hyperbolic *equation if $n-1$ coefficients $A_{ll}(\mathbf{x}_0)$ have the same sign, while one coefficient has an opposite sign;*
- *a* hyperbolic *equation if $n > m > 1$ coefficients $A_{ll}(\mathbf{x}_0)$ have the same sign, while the other $n-m$ coefficients have the opposite sign;*

- *a* parabolic *equation if at least one of the coefficients* $A_{ll}(\mathbf{x}_0)$ *is equal to zero.*

18.1. The case of constant coefficients. Suppose that a_{jk} are constants. Then the change of variables needed to reduce a second order PDE to a standard form is linear:

$$\alpha_k = \sum_{j=1}^{n} J_{jk} x_j \qquad \text{or} \qquad \boldsymbol{\alpha} = \mathbf{J}^T \mathbf{x}\,.$$

The matrix $\mathbf{J}$ can be found as follows. The matrix $\mathbf{a}$ is symmetric, $\mathbf{a}^T = \mathbf{a}$. It is known from the linear algebra that its eigenvalues are real, they are roots of the characteristic polynomial of degree n:

$$P_n(\lambda) = \det(\mathbf{a} - \lambda \mathbf{I}) = 0$$

where $\mathbf{I}$ is the identity matrix, $I_{jk} = \delta_{jk}$, and there exists an orthogonal transformation $\mathbf{O}$, $\mathbf{O}^T = \mathbf{O}^{-1}$ such that the matrix

$$\mathbf{O}^T \mathbf{a} \mathbf{O} = \boldsymbol{\lambda}\,, \qquad \boldsymbol{\lambda} = \mathrm{diag}\,(\lambda_1, \lambda_2, ..., \lambda_n)$$

is diagonal with diagonal elements being the eigenvalues of $\mathbf{a}$ (if an eigenvalue has a multiplicity m, then it is appears m times in the diagonal of $\boldsymbol{\lambda}$). Define a diagonal matrix $\boldsymbol{\Lambda}$ with matrix elements

$$\Lambda_{jk} = \Lambda_j \delta_{jk}\,, \quad \Lambda_j = \begin{cases} |\lambda_j|^{-1/2}\,, & \lambda_j \neq 0 \\ 1\,, & \lambda_j = 0 \end{cases}$$

and put

$$\mathbf{J} = \mathbf{O}\boldsymbol{\Lambda}$$

Then it follows from

$$\mathrm{sign}(\lambda_j) = \frac{\lambda_j}{|\lambda_j|}$$

that

$$\mathbf{J}^T \mathbf{a} \mathbf{J} = \boldsymbol{\Lambda}^T \mathbf{O}^T \mathbf{a} \mathbf{O} \boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T \boldsymbol{\lambda} \boldsymbol{\Lambda} = \mathbf{A}$$
$$A_{lm} = A_l \delta_{lm}\,, \quad A_l = \begin{cases} \mathrm{sign}(\lambda_l)\,, & \lambda_l \neq 0 \\ 0\,, & \lambda_l = 0 \end{cases}$$

Thus, $\mathbf{A}$ is a diagonal matrix with diagonal elements being ± 1 and 0 as is required for the classification of second order PDEs.

The constructed linear transformation is a change of variables because its Jacobian

$$\det(\mathbf{J}) = \det(\mathbf{O}) \det(\boldsymbol{\Lambda}) = \det(\boldsymbol{\Lambda}) \neq 0$$

because $\boldsymbol{\Lambda}$ is a diagonal matrix with non-zero diagonal elements. The Sylvester's law of inertia implies that the only freedom left in construction of $\mathbf{J}$ is permutations of eigenvalues λ_j. The corresponding coordinate transformations are swapping coordinates. For example, a permutation of λ_1 and λ_2 corresponds to a coordinate transformation $(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n) \to (\alpha_2, \alpha_1, \alpha_3, ..., \alpha_n)$.

It also follows from the above analysis that the eigenvalues of the matrix $\mathbf{a}$ can be used to classify a second order PDE

- *If all eigenvalues of* $\mathbf{a}$ *do not vanish and have the same sign, Eq. (*$\mathbf{18.1}$*) is* elliptic*;*
- *If all eigenvalues of* $\mathbf{a}$ *do not vanish and have different signs, Eq. (*$\mathbf{18.1}$*) is* hyperbolic*;*
- *If all eigenvalues of* $\mathbf{a}$ *do not vanish and only one of them has a different sign, Eq. (*$\mathbf{18.1}$*) is* normal hyperbolic*;*
- *If at least one of the eigenvalues of* $\mathbf{a}$ *vanishes, Eq. (*$\mathbf{18.1}$*) is* parabolic*.*

By the constructed linear change of variables Eq. ($\mathbf{18.1}$) can be reduced to one the following *standard forms*

$$
\begin{aligned}
\text{elliptic}: \quad & \frac{\partial^2 u}{\partial \alpha_1^2} + \frac{\partial^2 u}{\partial \alpha_2^2} + \cdots + \frac{\partial^2 u}{\partial \alpha_n^2} = G\,,\\
\text{normal hyperbolic}: \quad & \frac{\partial^2 u}{\partial \alpha_1^2} = \frac{\partial^2 u}{\partial \alpha_2^2} + \frac{\partial^2 u}{\partial \alpha_3^2} + \cdots + \frac{\partial^2 u}{\partial \alpha_n^2} + G\,,\\
\text{hyperbolic}: \quad & \frac{\partial^2 u}{\partial \alpha_1^2} + \frac{\partial^2 u}{\partial \alpha_2^2} + \cdots + \frac{\partial^2 u}{\partial \alpha_m^2}\\
& = \frac{\partial^2 u}{\partial \alpha_{m+1}^2} + \frac{\partial^2 u}{\partial \alpha_{m+2}^2} + \cdots + \frac{\partial^2 u}{\partial \alpha_n^2} + G\,,\\
\text{parabolic}: \quad & s_1\frac{\partial^2 u}{\partial \alpha_1^2} + s_2\frac{\partial^2 u}{\partial \alpha_2^2} + \cdots + s_m\frac{\partial^2 u}{\partial \alpha_m^2} = G\,,\\
& s_j = \pm 1\,, \quad j = 1, 2, ..., m < n
\end{aligned}
$$

18.2. The existence of the standard form in a general case. A non-constant matrix $\mathbf{a}(\mathbf{x})$ can be reduced to a diagonal matrix by an orthogonal transformation at any particular point $\mathbf{x}_0$. So, just like in the previous case, the eigenvalues of $\mathbf{a}$ at any particular point allows us to classify a second order PDE at that point. So, it is always possible to find a change of variables such that the coefficients A_{lm} take the standard values at a particular point $\mathbf{x_0}$:

$$
A_{lm}(\mathbf{x}_0) = s_{lm} = \begin{cases} 0\ , \ l \neq m \\ \pm 1, 0\ , \ l = m \end{cases} \tag{18.3}
$$

where the number of ± 1 and 0 in values of s_{ll} is determined by the numbers of positive, negative, and vanishing eigenvalues of the matrix $\mathbf{a}$ at a point $\mathbf{x}_0$.

The next question: Is it possible to find a change variables such that $A_{lm}(\mathbf{x})$ has the standard form in a neighborhood of $\mathbf{x}_0$ (or in an open set containing $\mathbf{x}_0$)? Since Eq. (**18.2**) can always be divided by a non-vanishing function, it is sufficient that

$$A_{lm}(\mathbf{x}) = B(\mathbf{x}) s_{lm}$$

for some $B \neq 0$ in order for (**18.1**) to have the standard form in a neighborhood of $\mathbf{x}_0$. As in the two-variable case, this question amounts to solving a system of the first-order PDEs:

$$\sum_{k=1}^{n} \sum_{j=1}^{n} a_{jk}(\mathbf{x}) \frac{\partial \alpha_m}{\partial x_j} \frac{\partial \alpha_l}{\partial x_k} = B(\mathbf{x}) s_{lm}$$

for all pairs (l, m), $l \geq m$ (because $s_{lm} = s_{lm}$) where $B \neq 0$ The system has n unknown functions α_j.

Let us count the number of conditions on α_j. For $l \neq m$, the system contains $n(n-1)/2$ equations, and for $l = m$ there is $n-1$ independent condition (because B is an arbitrary parameter). If $n = 2$, there are two independent equations for two unknowns. In this case, the reduction to the standard form is possible in an open set (as was shown above).

If $n = 3$, one can always choose three functions α_j to reduce the three non-diagonal elements A_{lm}, $l > m$, to zero. So, it is possible to achieve $A_{lm} = 0$, $l \neq m$, in a open set, but there is no freedom left to reduce the diagonal elements A_{ll} to the standard form. The absolute values of the three functions A_{ll} must be equal in order for Eq. (**18.1**) to be in the standard form, which comprises two more conditions on α_j.

For $n > 3$, it is not even possible to eliminate all mixed derivatives (corresponding to the off-diagonal elements of $\mathbf{A}$) by a change of variables because the number of equations is greater than the number of unknowns. Note that $\frac{1}{2}n(n-1) > n$ implies $n > 3$ and vice versa. Therefore, if $n > 3$, there is no change of variables under which Eq. (**18.1**) can be reduced to one of the standard forms for a general choice of $a_{jk}(\mathbf{x})$.

It is then concluded that for $n \geq 3$, it is not generally possible to reduce a second order PDE to a standard form in an open set by a change of variables. Only in the case $n = 2$, there always exists a change of variables under which (**18.1**) is reduced to one of the standard forms as was shown above.

18.3. Exercises.

1. Find conditions on the constants a, b, and c under which the equation

$$u''_{tt} + au''_{xx} + 2cu''_{xy} + bu''_{yy} = F(t, x, y, u, u'_t, u'_x, u'_y)$$

is either elliptic, or hyperbolic, or parabolic.

Selected answers.

1. Let $D = c^2 - ab$. The equation is: elliptic if $D > 0$ and $a > 0$ (or $b > 0$); hyperbolic if $D < 0$ or $D > 0$ and $a < 0$ (or $b < 0$); parabolic if $D = 0$.