CHAPTER 6

# Fourier method for 2D PDEs

## 35. Heat and wave equations on a circle

**35.1. The eigenvalue problem for a matrix.** Let $A$ be an $N \times N$ complex matrix. Consider the *eigenvalue problem* for $A$. This implies that one has to find all complex $\lambda$ for which the linear system

$$A\mathbf{x} = \lambda \mathbf{x}\,,$$

has a non-zero solution, and for each such $\lambda$ one has to find all linearly independent non-zero solutions. These values of $\lambda$ are called *eigenvalues* of $A$ and the corresponding solutions are called *eigenvectors* of $A$. Let $I$ denote the unit matrix, $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$.

Recall from the linear algebra that a homogeneous linear equation

$$B\mathbf{x} \equiv (A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a non-zero solution if and only if the determinant of $B$ vanishes. Therefore the eigenvalues of $A$ satisfies the equation

$$P_A(\lambda) = \det(A - \lambda I) = 0$$

The function $P_A(\lambda)$ is a polynomial of degree $N$. Any polynomial of degree $N$ has at most $N$ distinct complex roots:

$$\lambda = \lambda_k\,, \quad k = 1, 2, ..., n \leq N$$

If roots are counted with taking into account their multiplicity (e.g., a root with multiplicity 2 is counted twice), then the number of roots is exactly $N$. For each $\lambda = \lambda_k$, the linear system can be solved to obtain all eigenvectors corresponding to this eigenvalue. The dot product of two vectors with complex coefficients is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{N} x_j \overline{y_j}$$

A matrix $A^*$ is called the hermitian conjugate of $A$ if

$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^*\mathbf{y} \quad \text{for all vectors } \mathbf{x},\ \mathbf{y}$$

In a standard basis, the matrix elements of $A^*$ are obtained from the matrix elements of $A$ by transposition and complex conjugation:

$$A^*_{ij} = \overline{A_{ji}}$$

A matrix is called *hermitian or symmetric* if

$$A = A^* \quad \text{or} \quad A\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot A\mathbf{y} \quad \text{for all vectors } \mathbf{x},\ \mathbf{y}$$

It is proved in linear algebra that

- *the eigenvalues of a symmetric matrix are real*
- *among all eigenvectors of a symmetric matrix one can select $N$ mutually orthogonal unit vectors $\mathbf{e}_n$, $n = 1, 2, ..., N$ that form a basis, that is, any vector $\mathbf{x}$ can be expanded into a unique linear combination*

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_N\mathbf{e}_N\,, \quad x_n = \mathbf{x}\cdot\mathbf{e}_n$$

Consider the initial value problem for a vector function $\mathbf{x}(t)$:

$$\mathbf{x}'(t) = A\mathbf{x}(t)\,, \quad t > 0\,, \quad \mathbf{x}(0) = \mathbf{x}_0$$

Its solution defines a parametric curve in a Euclidean space. Suppose that the matrix $A$ is symmetric. Then there exist $N$ mutually orthogonal unit eigenvectors of $A$:

$$A\mathbf{e}_n = \lambda_n\mathbf{e}_n\,, \quad \mathbf{e}_n\cdot\mathbf{e}_m = \delta_{mn}$$

where $\delta_{mn} = 0$ if $n \neq m$ and $\delta_{nn} = 1$ (it is called the Kronecker symbol). Note that the eigenvalues $\lambda_n$ are not required to be distinct. If an eigenvalue $\lambda$ has a multiplicity $m$, then there are exactly $m$ linearly independent eigenvectors corresponding to $\lambda$. Using the Gramm-Schmidt process, these vectors can be chosen to be orthonormal. Eigenvectors corresponding to distinct eigenvalues are orthogonal. Any vector function can be expanded over the basis of eigenvectors of a symmetric matrix:

$$\mathbf{x}(t) = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + \cdots + x_N(t)\mathbf{e}_N$$

Then the initial value problem can be reduced to $N$ initial value problem for the components of the vector function:

$$\begin{aligned}
\mathbf{x}'(t) &= x_1'(t)\mathbf{e}_1 + x_2'(t)\mathbf{e}_2 + \cdots + x_N'(t)\mathbf{e}_N = A\mathbf{x}(t) \\
&= x_1(t)A\mathbf{e}_1 + x_2(t)A\mathbf{e}_2 + \cdots + x_N(t)A\mathbf{e}_N \\
&= \lambda_1 x_1(t)\mathbf{e}_1 + \lambda_2 x_2(t)\mathbf{e}_2 + \cdots + \lambda_n x_N(t)\mathbf{e}_N \\
\mathbf{x}(0) &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_N\mathbf{e}_N
\end{aligned}$$

Owing to orthogonality of the basis vectors, each components satisfies the initial value problem

$$x_n'(t) = \lambda_n x_n(t)\,, \quad x_n(0) = a_n \quad \Rightarrow \quad x_n(t) = a_n e^{\lambda_n t}$$

so that

$$\mathbf{x}(t) = \sum_{n=1}^{N} e^{\lambda_n t} a_n \mathbf{e}_n$$

In a similar fashion, one can solve the initial value problem for the second order equation

$$\mathbf{x}''(t) = A\mathbf{x}(t)\,, \quad t > 0\,, \quad \mathbf{x}(0) = \mathbf{a}\,, \quad \mathbf{x}'(0) = \mathbf{b}$$

if $A$ is a symmetric matrix, or the corresponding non-homogeneous problems

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)\,, \quad t > 0\,, \quad \mathbf{x}(0) = \mathbf{a}\,;$$
$$\mathbf{x}''(t) = A\mathbf{x}(t) + \mathbf{f}(t)\,, \quad t > 0\,, \quad \mathbf{x}(0) = \mathbf{a}\,, \quad \mathbf{x}'(0) = \mathbf{b}$$

where $\mathbf{f}(t)$ is a given vector function. Using the method of variation of parameters for ODEs one can show that for the solution to the first initial value problem reads

$$\mathbf{x}(t) = \sum_{n=1}^{N} \left( a_n e^{\lambda_n t} + \int_0^t e^{\lambda_n (t-\tau)} f_n(\tau) d\tau \right) \mathbf{e}_n$$

where $f_n = \mathbf{f} \cdot \mathbf{e}_n$. The answer to the second problem depends on the sign of eigenvalues of $A$. Suppose that $\lambda_n = -\omega_n^2 < 0$. In this case,

$$\mathbf{x}(t) = \sum_{n=1}^{N} \Bigg( a_n \cos(\omega_n t) + \frac{b_n}{\omega_n} \sin(\omega_n t) + \frac{1}{\omega_n} \int_0^t \sin[\omega_n (t-\tau)] f_n(\tau)\, d\tau \Bigg) \mathbf{e}_n$$

The technical details are for the reader as an exercise.

**35.2. The eigenvalue problem for a second-derivative operator on a circle.** Now recall the Cauchy problem for the parabolic or hyperbolic equations. They have a similar form

$$u_t'(x,t) = -c^2 Lu(x,t) \quad \text{or} \quad u_{tt}''(x,t) = -c^2 Lu(x,t)$$

where $c > 0$ is a constant, and $L$ is a *differential operator* acting with respect to the variable $x$:

$$Lu = -u_{xx}''$$

the wave equation. Is it possible to find a solution to a PDE problem as a series over *eigenfunctions* of differential operators, similarly to the initial value problems in a Euclidean space?

The question is not deprived from some sense. Indeed, the trigonometric functions

$$1\,,\quad \cos(nx)\,,\quad \sin(nx)\,,\quad n=1,2,...$$

form an orthogonal basis in the inner product space of continuous functions in an interval $[-\pi,\pi]$. Can these functions be *eigenfunctions* of $L$ so that the solution to the Cauchy problem can be expanded into a trigonometric Fourier coefficients? It is clear that not every continuous function can be differentiated twice. So, the *domain* of the operator $L$ must be defined. By definition, it consists of all continuous functions in $(-\pi,\pi)$ for which the *rule* of acting $L$ on the function make sense and some *additional conditions that prescribe the behavior of the function at the end points of the interval.* For example, let the domain $\mathcal{M}_L$ of the operator of second derivative consist of twice continuously differentiable functions whose first derivative can be extended to the end points and has equal values at the end points:

$$u\in\mathcal{M}_L \quad \text{if} \quad \begin{cases} u\in C^2(-\pi,\pi)\cap C^1[-\pi,\pi] \\ u(-\pi)=u(\pi) \\ u'(-\pi)=u'(\pi) \end{cases}$$

Thus, a differential operator is defined by the *rule* and its *domain.* Two operators having the same rules but different domains are *distinct* operators. For example, one could demand that the second derivative acts on functions with vanishing values at the endpoints $u(\pm\pi)=0$. It is a different operator from the one considered. Any function from $\mathcal{M}_L$ has a continuously differentiable $2\pi-$ periodic extension to the whole real axis:

$$u(x+2\pi)=u(x)\,,\quad u\in C^1(\mathbb{R})$$

For this reason, the operator $L$ will also be referred to as a *second-derivative operator on periodic functions or on a circle*, owing to that any continuous function on a circle is automatically periodic.

A differential operator $L$ is called *symmetric or Hermitian* if

$$\langle Lu,v\rangle=\langle u,Lv\rangle \quad \text{for all } u,v\in\mathcal{M}_L$$

Let us show that the second derivative operator with the stated domain $\mathcal{M}_L$ is symmetric. Omitting a real constant $c^2$ and using the inner product in the space of continuous functions on $(-\pi, \pi)$

$$\begin{aligned}\langle Lu, v\rangle &= -\int_{-\pi}^{\pi} u''(x)\overline{v(x)}\,dx \\ &= -u'(x)\overline{v(x)}\Big|_{-\pi}^{\pi} + u(x)\overline{v'(x)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u(x)\overline{v''(x)}\,dx \\ &= \langle u, Lv\rangle\,,\end{aligned}$$

where the integration by parts was carried out twice. The boundary terms vanish because $u$ and $v$ and their derivatives have equal values at the end points (note that belongs to the domain of the operator $L$).

Consider the *eigenvalue problem* for $L$:

$$Lu = \lambda u\,, \quad u \in \mathcal{M}_L$$

The problem is to find all non-zero (non-trivial) solutions to the differential equation in the *domain* of the differential operator, where $\lambda$ is a complex number. Just like in the finite dimensional case, not for every $\lambda$ a non-trivial solution exists. The values of $\lambda$ for which such solutions exist are called *eigenvalues* of $L$ and the corresponding solutions are called *eigenfunctions*.

Let us show that *eigenvalues of a Hermitian operator are real*. The following chain of equalities holds for any non-trivial solution to the equation $Lu = \lambda u$:

$$\lambda\langle u, u\rangle = \langle \lambda u, u\rangle = \langle Lu, u\rangle = \langle u, Lu\rangle = \langle u, \lambda u\rangle = \bar{\lambda}\langle u, u\rangle$$

Since $\langle u, u\rangle = \|u\|^2 > 0$, it is concluded that the eigenvalue is real: $\lambda = \bar{\lambda}$.

Let us find the eigenvalues and eigenfunctions for the second derivative operators in the stated domain. The following boundary value problem must be solved

$$Lu = -u''(x) = \lambda u(x)\,, \quad u(-\pi) = u(\pi)\,, \quad u'(-\pi) = u'(\pi)$$

First, note that $\lambda \geq 0$. It was already shown that $\lambda$ must be real. The non-negativity follows from

$$\lambda\langle u, u\rangle = \langle Lu, u\rangle = -\int_{-\pi}^{\pi} u''(x)\overline{u(x)}\,dx = \int_{-\pi}^{\pi} |u'(x)|^2 dx \geq 0$$

where the integration by parts was done, and the boundary term vanishes owing to the boundary conditions. Since $\langle u, u\rangle = \|u\|^2 > 0$, $\lambda \geq 0$ and $\lambda = 0$ if and only if the eigenfunction is constant. Note $u(x) = X_0(x) = 1$ satisfies the boundary conditions. Therefore $X_0(x) \in \mathcal{M}_L$.

Put $\lambda = \nu^2$ where $\nu > 0$. In this case a general solution to the differential equation reads

$$u(x) = A\cos(\nu x) + B\sin(\nu x)\,.$$

The first boundary condition yields

$$A\cos(\pi\nu) - B\sin(\pi\nu) = A\cos(\pi\nu) + B\sin(\pi\nu)$$

which implies that either $B = 0$ or $\sin(\pi\nu) = 0$ or $\nu = n$, $n = 1, 2, ...$ The second boundary condition gives

$$A\nu\sin(\pi\nu) + B\nu\cos(\pi\nu) = -A\nu\sin(\pi\nu) + B\nu\cos(\pi\nu)$$

If $\sin(\pi\nu) = 0$, then the condition is identically satisfied. If $B = 0$, then either $A = 0$ or $\sin(\pi\nu) = 0$ again. It is therefore concluded that the eigenvalue $\lambda = \lambda_0 = 0$ has only one linearly independent eigenfunction, while the eigenvalues $\lambda = \lambda_n = n^2$ have two linearly independent eigenfunctions:

$$Lu = 0\,, \quad u \in \mathcal{M}_L \quad \Rightarrow \quad u = X_0(x) = 1\,,$$
$$Lu = \lambda u\,, \quad u \in \mathcal{M}_L \quad \Rightarrow \quad u = \begin{cases} X_n^c(x) = \cos(nx) \\ X_n^s(x) = \sin(nx) \end{cases}$$

where $\lambda = \lambda_n = n^2$, $n = 1, 2, ....$ Thus, the linearly independent eigenfunctions of the operator $L$ are trigonometric harmonics that form an orthogonal basis in the space of continuous functions in the interval $[-\pi, \pi]$.

**35.3. Formal solution to the 2D heat equation on a circle.** Is there any physical significance for the boundary conditions discussed? Consider the 2D heat equation for a circular rod whose transverse dimensions can be neglected as compared to the radius of the rod (like a wire). Then the arclength is $s = Rx$ where $x$ is an angle counted from a particular point of the rod. Therefore if $u(x, t)$ is the temperature along the circular rod at a time $t$, then the rates of change of the temperature along the rod are

$$\frac{\partial u}{\partial s} = \frac{1}{R}\frac{\partial u}{\partial x}\,, \quad \frac{\partial^2 u}{\partial^2 s} = \frac{1}{R^2}\frac{\partial^2 u}{\partial^2 x}$$

The temperature must be continuous along the circle. If $-\pi \le x \le \pi$, then one must demand that

$$u(t, -\pi) = u(t, \pi)\,, \quad t \ge 0$$

Furthermore, the rate $\partial u/\partial s$ defines the heat energy flow along the circular rod which is also continuous and, hence,

$$u'_x(t, -\pi) = u'_x(t, \pi)\,, \quad t \ge 0$$

Thus, a solution to the Cauchy problem

$$u'_t = -c^2 Lu\,, \quad u\Big|_{t=0} = v(x)\,; \quad u(x,t) \in \mathcal{M}_L\,,\ t \geq 0$$

describes the temperature evolution in a circular rod for a given initial temperature distribution $v$, where all the physical and geometric parameters (like the radius $R$) are included into the constant $c^2$. Let

$$v(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx)\Big)$$

be a trigonometric Fourier series of the initial data. By Fejér's theorem it converges to $v$ at each point because $v$ is continuous and periodic. A solution is sought in the form of a *formal Fourier series over basis eigenfunctions of* $L$:

$$\begin{aligned} u(x,t) &= A_0(t)X_0(x) + \sum_{n=1}^{\infty} \Big(A_n(t)X_n^c(x) + B_n(t)X_n^s(x)\Big) \\ &= A_0(t) + \sum_{n=1}^{\infty} \Big(A_n(t)\cos(nx) + B_n(t)\sin(nx)\Big) \end{aligned}$$

Substituting this series into the equation and carrying out *formal term-by-term differentiation* of the series, it is concluded that, if the said differentiation is justified, then the expansion coefficients satisfy the initial value problem:

$$\begin{aligned} A'_n(t) &= -c^2n^2A_n(t)\,, \quad A_n(0) = a_n \quad \Rightarrow \quad A_n(t) = a_n e^{-c^2n^2t}\,, \\ B'_n(t) &= -c^2n^2B_n(t)\,, \quad B_n(0) = b_n \quad \Rightarrow \quad B_n(t) = b_n e^{-c^2n^2t}\,, \end{aligned}$$

The obtained solution is a *formal solution* to the Cauchy problem

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx)\Big)e^{-c^2n^2t}$$

**35.4. Smoothness of the formal solution.** Tor all $t \geq \delta > 0$ (where $\delta$ is an arbitrary positive number), the terms of the formal solution are bounded by

$$\Big|a_n \cos(nx) + b_n \sin(nx)\Big| e^{-c^2n^2t} \leq (|a_n| + |b_n|)e^{-c^2n^2\delta}$$

If the initial data $v \in C^1$ is continuously differentiable and periodic, then by Theorem **32.3**,

$$\sum_n (|a_n| + |b_n|) < \infty$$

Therefore the formal solution exists for all $t \geq 0$ (one can set $\delta = 0$ in the above bound) and all $x$. Moreover, it is continuous in the region of interest including its boundaries:

$$u(x,t) \in C^0(t \geq 0)$$

In the region $t > 0$ (or $\delta > 0$), recall that the Fourier coefficients are bounded $|a_n| \leq M$ and $|b_n| \leq M$ for any continuous $v$. Any term-by-term differentiation of the formal solution bring an extra factor $n$ (for the $x$ derivative) or $n^2$ (for the time derivative) so that the terms of the series are bounded by

$$Kn^p e^{-c^2 n^2 \delta} \quad \Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{Kn^p e^{-c^2 n^2 \delta}} = 0 < 1$$

where $K$ is proportional to $M$ and some power of $c$. The series of these upper bounds converges by the root test for any $\delta > 0$ and any $p > 0$. Since all derivatives of the terms are continuous, it is concluded that

$$u \in C^\infty(t > 0)$$

A classical solution must be from the class $C^2(t > 0) \cap C^0(t \geq 0)$. Thus, *if $v \in C^1$, then the formal solution is the classical solution of the Cauchy problem.*

**35.5. A non-homogeneous heat equation on the circle.** Following the analogy with the linear system of ODEs considered above, the Fourier method can be used to solve the heat equation on a circle with an external heat source:

$$\begin{aligned} &u'_t = -c^2 Lu + f(x,t)\,, \quad t > 0\,,\\ &u\Big|_{t=0} = v(x)\,,\\ &u(x,t) \in \mathcal{M}_L\,, \quad t \geq 0 \end{aligned}$$

where the source $f$ is assumed to be continuous and periodic

$$f(x,t) = f(x + 2\pi, t)\,, \quad t \geq 0$$

Just like in the case of the Poisson equation, the inhomogeneity can be expanded into the Fourier series over the basis of eigenfunctions of the operator $L$:

$$f(x,t) = F_0(t) + \sum_{n=1}^{\infty} \Big( F_n^c(t)\cos(nx) + F_n^s(t)\sin(nx) \Big)$$

where the expansion coefficients are continuous functions of $t$. In this case, the expansion coefficients of a formal solution satisfy the initial value problem

$$\begin{aligned}
&A_n'(t) = -c^2n^2A_n(t) + F_n^c(t)\,, \quad A_n(0) = a_n \\
\Rightarrow\quad &A_n(t) = a_ne^{-c^2n^2t} + \int_0^t e^{-c^2n^2(t-\tau)}F_n^c(\tau)\,d\tau \\
&B_n'(t) = -c^2n^2B_n(t) + F_n^s(t)\,, \quad B_n(0) = b_n \\
\Rightarrow\quad &B_n(t) = b_ne^{-c^2n^2t} + \int_0^t e^{-c^2n^2(t-\tau)}F_n^s(\tau)\,d\tau
\end{aligned}$$

where the solution is obtained the method of variation of parameters. Note that $A_0(t)$ is obtained by setting $n = 0$ in $A_n(t)$.

**35.6. Formal solution to the 2D wave equation on a circle.** Consider a circular elastic string that can vibrate in a direction transverse to the plane in which the circle lies (the plane in which a circular string (or a circular rod) at equilibrium lies). If $u(x,t)$ denotes a vertical deviation of a point $x$ of the string from its equilibrium position at a time $t$, then $u(x+2\pi,t) = u(x,t)$ if an angle $x$ is used to label points of the string (just like in the case of the heat equation for a circular rod). The derivative $u_x'$ defines the density of the elastic energy of a vibrating string, which also must be continuous along the string. Therefore the Cauchy problem

$$\begin{aligned}
&u_{tt}'' = -c^2Lu + f(x,t)\,, \quad t > 0\,, \\
&u\Big|_{t=0} = v_1(x)\,, \quad u_t'\Big|_{t=0} = v_2(x)\,, \\
&u(x,t) \in \mathcal{M}_L\,, \quad t \geq 0
\end{aligned}$$

determines an evolution of a circular elastic string under the action of the external force $f(x,t)$ (applied at a point $x$ and a time $t$) if the initial shape was $v_1(x)$ and the initial distribution of vertical velocities was $v_2(x)$. Naturally, the initial data and inhomogeneity are assumed to be continuous and periodic in $x$.

If the expansion of the initial data and inhomogeneity over the basis of eigenfunctions are

$$\begin{aligned}
v_j(x) &= a_0^{(j)} + \sum_{n=1}^{\infty}\Big(a_n^{(j)}\cos(nx) + b_n^{(j)}\sin(nx)\Big) \\
f(x,t) &= F_0(t) + \sum_{n=1}^{\infty}\Big(F_n^c(t)\cos(nx) + F_n^s(t)\sin(nx)\Big)
\end{aligned}$$

where $j = 1, 2$, then a formal solution can be found in the form of the Fourier series

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} \Big( A_n(t)\cos(nx) + B_n(x)\sin(nx) \Big)$$

where the expansion coefficients satisfy the initial value problem

$$H''(t) + c^2\lambda_n H(t) = F(t)\,, \quad H(0) = h_1\,, \quad H'(0) = h_2$$

where $\lambda_n = n^2$ and the legend is

$$H = \begin{cases} A_n\,, \text{ if } & F = F_n^c\,,\ h_1 = a_n^{(1)}\,,\ h_2 = a_n^{(2)} \\ B_n\,, \text{ if } & F = F_n^s\,,\ h_1 = b_n^{(1)}\,,\ h_2 = b_n^{(2)} \end{cases}$$

In particular,

$$A_0(t) = a_0^{(1)} + a_0^{(2)} t + \int_0^t \int_0^\tau F_0(s)\,ds\,d\tau$$

$$H(t) = h_1\cos(cnt) + \frac{h_2}{cn}\sin(cnt) + \frac{1}{cn}\int_0^t \sin[cn(t-\tau)]F(\tau)\,d\tau$$

The coefficient $A_0(t)$ describes the motion of a circular string as a whole (a vertical motion of its *center of mass*). Note that the string has no fixed or attached ends and, hence, its center of mass can move freely. When no external force is applied, it moves with a constant vertical velocity $a_0^{(2)}$ as required by the first Newton's law.

**35.7. The existence of a formal solution.** The existence and smoothness of the formal solution depends on the behavior of the Fourier coefficients of $f$ and $v_{1,2}$. The more continuous derivatives they have the faster their Fourier coefficients decay to zero with increasing $n$ (see Theorem **32.2**). If $f = 0$ and the initial data are from the class $C^1$ so that

$$\sum_n (|a_n^{(j)}| + |b_n^{(j)}|) < \infty\,, \quad j = 1, 2$$

then the terms of the formal solution are bounded by

$$|A_n(t)\cos(nx) + B_n(t)\sin(nx)| \le |A_n(t)| + |B_n(t)|$$

$$|A_n(t)| \le |a_n^{(1)}| + \frac{|a_n^{(2)}|}{cn}$$

$$|B_n(t)| \le |b_n^{(1)}| + \frac{|b_n^{(2)}|}{cn}$$

Therefore the formal solution exists for all $t \ge 0$ and, by continuity of its terms, the formal solution is also continuous for all $t \le 0$ and all $x$.

Suppose that the external force $f$ and its derivative $f'_x$ are continuous and bounded,

$$|f'_x(x,t)| \le M\,.$$

Then its Fourier coefficients are bounded too. Indeed, using the integration by parts

$$F_n^c(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,t)\cos(nx)\,dx = -\frac{1}{\pi n}\int_{-\pi}^{\pi} f'_x(x,t)\sin(nx)\,dx$$

where the boundary term vanishes by periodicity of $f$. It follows from this equality that

$$\begin{aligned}|F_n^c(t)| &= \frac{1}{\pi n}\left|\int_{-\pi}^{\pi} f'_x(x,t)\sin(nx)\,dx\right| \le \frac{1}{\pi n}\int_{-\pi}^{\pi} |f'_x(x,t)||\sin(nx)|\,dx \\ &\le \frac{2M}{n}\end{aligned}$$

and similarly, $|F_n^s(t)| \le 2M/n$. The contribution of $f$ to the formal solution is independent of the initial data. So, the convergence of the formal series that depends on $f$ can be studied with $v_1 = v_2 = 0$. In this case

$$|H(t)| \le \frac{1}{cn}\int_0^t |\sin[cn(t-\tau)]||F(\tau)|\,d\tau \le \frac{2Mt}{c\pi n^2}$$

Therefore the formals series converges in any interval $0 \le t \le T$ and any $x$. Moreover, since the terms of the series are continuous for all $x$ and $t$, the sum of the series is continuous in any rectangle $(x,t) \in \mathbb{R}\times[0,T]$.

By Theorem **32.2**, it is clear that with the initial data and inhomogeneity that are smooth enough, the formal solution becomes a classical one

$$u \in C^2(t>0) \cap C^1(t \ge 0)$$

However, the method based on the convergence of the series of upper bounds of the terms of the formal solution and their derivatives is too restrictive, and weaker conditions on the initial data and inhomogeneity under which the formal solution becomes classical can be established (see the discussion in next Sections below).

### 35.8. Exercises.

**1**. Find a formal solution to the heat equation on a circle and investigate its existence

$$
\begin{aligned}
&u'_t(x,t) = u''_{xx} + |x|e^{-t}\,, \quad (x,t) \in (-\pi,\pi)\times(0,\infty)\\
&u\Big|_{t=0} = \begin{cases} x(\pi - x)\,, & 0 < x \le \pi \\ 0\,, & -\pi \le x \le 0 \end{cases}\\
&u(-\pi,t) = u(\pi,t)\,, \quad u'_x(-\pi,t) = u'_x(\pi,t)\,, \quad t \ge 0
\end{aligned}
$$

**2**. Find a formal solution to the wave equation on a circle and investigate its existence if $\omega \ne cn$, $n = 1, 2, ...,$

$$
\begin{aligned}
&u''_{tt}(x,t) = c^2 u''_{xx} + |x|\cos(\omega t)\,, \quad (x,t) \in (-\pi,\pi)\times(0,\infty)\\
&u\Big|_{t=0} = \begin{cases} x(\pi - x)\,, & 0 < x \le \pi \\ 0\,, & -\pi \le x \le 0 \end{cases}\\
&u'_t\Big|_{t=0} = 2\cos(2x) - 3\sin(x)\,,\\
&u(-\pi,t) = u(\pi,t)\,, \quad u'_x(-\pi,t) = u'_x(\pi,t)\,, \quad t \ge 0
\end{aligned}
$$

**3**. Solve Problem 2 if $\omega = c$. Show that the amplitude $|A_1^c(t)|$ can grow unboundedly with increasing $t$ and so does the solution $u(x,t)$. Show this happens whenever the frequency $\omega$ of the external force coincides with one of the *eigenfrequencies* of the vibrating string $\omega_n = cn$. In physics, this phenomenon is called a *resonance*.
*Hint*: Use the method of undetermined coefficients to solve the initial value problem for $A_1^c(t)$. Alternatively, one can take the limit $\omega \to c$ in the solution of Problem 2.

## 36. The Sturm-Liouville problem

**36.1. Eigenvalue problem for a differential operator.** In a space of continuous functions on an interval $[a, b]$, the inner product is defined by

$$\langle u, v\rangle = \int_a^b u(x)\overline{v(x)}\, dx\,.$$

Let $L$ be a linear differential operator and $\mathcal{M}_L$ be its domain. Functions in $\mathcal{M}_L$ are sufficiently smooth so that the rule $Lu$ makes sense and in addition they satisfy some boundary conditions at the endpoints of the interval $[a, b]$ In what follows, it is assumed that $Lu$ is a continuous function for any $u \in \mathcal{M}_L$:

$$L: \quad \mathcal{M}_L \to C^0[a, b]$$

The function $Lu$ must be It is further assumed that

$$\mathcal{M}_L = \text{a linear space}$$

that is, any linear combination of functions from $\mathcal{M}_L$ belongs to $\mathcal{M}_L$. The linearity of $L$ means that

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2$$

for any $u_{1,2} \in \mathcal{M}_L$ and complex numbers $c_{1,2}$. The inner product space of functions continuous on $[a, b]$ is linear. But $\mathcal{M}_L$ contains functions from this space that are required to satisfy additional conditions. These conditions must be linear so that if $u_{1,2}$ satisfy the conditions, their linear combination also satisfies these conditions. For example, a boundary condition

$$\alpha u(b) + \beta u(b) = 0$$

is linear for any numbers $\alpha$ and $\beta$. However, a boundary condition

$$u(a) = 1$$

is not linear because, if $u_1(a) = u_2(a) = 1$, then a linear combination $u(x) = c_1u_1(x) + c_2u_2(x)$ does not satisfy the boundary condition:

$$u(a) = c_1u_1(a) + c_2u_2(a) = c_1 + c_2 \neq 1$$

for all $c_1$ and $c_2$. The linearity of the domain $\mathcal{M}_L$ implies that *the zero or trivial function is always in* $\mathcal{M}_L$.

The problem

$$Lu = \lambda u\,, \quad u \in \mathcal{M}_L$$

is called an *eigenvalue problem* for an operator $L$. Any (complex) value of $\lambda$ at which this differential equation has a non-zero solution from the functional set $\mathcal{M}_L$ is called an *eigenvalue* of $L$, and a non-zero solution corresponding to such $\lambda$ is called an *eigenfunction* of $L$. Note that a non-zero solution to a linear differential equation always exists for any $\lambda$, those solutions may not be from the class $\mathcal{M}_L$ because, in addition, to some natural smoothness conditions required for $Lu$ to exist, $u$ must also fulfill some other conditions (e.g., boundary conditions).

**36.2. Hermitian or symmetric operator.** A differential operator $L$ is called *hermitian* or *symmetric* if

$$\langle Lu, v\rangle = \langle u, Lv\rangle\,, \quad u, v \in \mathcal{M}_L$$

for any functions from its domain.

THEOREM **36.1**. (Necessary and sufficient conditions for hermiticity) *In order for an operator L to be hermitian it is necessary and sufficient that the quadratic form $\langle Lu, u\rangle$ be real for all $u \in \mathcal{M}_L$.*

PROOF. Suppose that $L$ is hermitian. Then by the properties of the inner product

$$\langle Lu, u\rangle = \langle u, Lu\rangle = \overline{\langle Lu, u\rangle}$$

and, hence, the quadratic form is real.

Conversely, suppose that the quadratic form $\langle Lu, u\rangle$ is real. One has to show that $\langle Lu, v\rangle = \langle u, Lv\rangle$ for any $u$ and $v$ in $\mathcal{M}_L$. Put

$$\begin{aligned}\langle Lu, v\rangle &= \mathrm{Re}\,\langle Lu, v\rangle + i\mathrm{Im}\,\langle Lu, v\rangle \equiv A + iB\,,\\ \langle u, Lv\rangle &= \mathrm{Re}\,\langle u, Lv\rangle + i\mathrm{Im}\,\langle u, Lv\rangle \equiv C + iD\,.\end{aligned}$$

Then $\langle Lu, v\rangle = \langle u, Lv\rangle$ means that $A = C$ and $B = D$. By assumption, the following quadratic form is real:

$$\langle Lw, w\rangle - \langle Lu, u\rangle - \langle Lv, v\rangle = \text{real}$$

for any choice of $w \in \mathcal{M}_L$. Put $w = u + iv$. Then

$$\begin{aligned}\langle Lw, w\rangle &= \langle L(u + iv), u + iv\rangle\\ &= \langle Lu, u\rangle + \langle iLv, iv\rangle + \langle Lu, iv\rangle + \langle iLv, u\rangle\\ &= \langle Lu, u\rangle + \langle Lv, v\rangle - i\langle Lu, v\rangle + i\langle Lv, u\rangle\\ &= \langle Lu, u\rangle + \langle Lv, v\rangle - i(A + iB) + i(C - iD)\,,\end{aligned}$$

because $\langle Lv, u\rangle = \overline{\langle u, Lv\rangle} = B - iD$. It follows that

$$\begin{aligned}&\langle Lw, w\rangle - \langle Lu, u\rangle - \langle Lv, v\rangle = B + D - i(A - C) = \text{real}\\ \Rightarrow\quad &A = C\end{aligned}$$

Similarly, but putting $w = u + v$, it is shown that

$$\begin{aligned}&\langle Lw, w\rangle - \langle Lu, u\rangle - \langle Lv, v\rangle = A + C + i(B - D) = \text{real}\\ \Rightarrow\quad &B = D\end{aligned}$$

as required. □

**Positive operators.** A linear operator $L$ is called *positive semi-definite* (or simply *positive*) if

$$\langle Lu, u\rangle \geq 0\,, \quad u \in \mathcal{M}_L\,.$$

If the above inequality is strict $\langle Lu, u\rangle > 0$ in $\mathcal{M}_L$, then $L$ is said to be *positive definite* (or *strictly positive*). It follows from Theorem **36.1** that *a positive linear operator is hermitian*.

EXAMPLE **36.1**. *Show that the operator*

$$Lu(x) = -\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right]\,, \quad u \in C^2(-1,1)\cap C^1([-1,1]) = \mathcal{M}_L$$

*is hermitian in the inner product space of continuous functions on the interval* $[-1,1]$.

SOLUTION: One has to show that $\langle Lu, u\rangle$ is real for any $u \in \mathcal{M}_L$. Integrating by parts

$$\begin{aligned}\langle Lu, u\rangle &= -\int_{-1}^{1}[(1-x^2)u'(x)]'\,\overline{u(x)}\,dx \\ &= -(1-x^2)u'(x)\Big|_{-1}^{1} + \int_{-1}^{1}(1-x^2)u'(x)\overline{u'(x)}\,dx \\ &= \int_{-1}^{1}(1-x^2)|u'(x)|^2\,dx \geq 0\end{aligned}$$

where the boundary term vanishes because the derivatives $u'(\pm 1)$ exist (note $u \in C^1([-1,1])$); the desired inequality follows from the inequality $(1-x^2)|u'(x)|^2 \geq 0$ for all $x \in [-1,1]$ and any continuously differentiable $u$. Thus, the operator is positive. By Theorem **36.1**, the operator $L$ is hermitian. □

**Eigenvalue problem for a hermitian operator.**

THEOREM **36.2**. (Eigenvalue problem for a hermitian operator) *Let $L$ be a linear hermitian operator. Then all its eigenvalues are real and eigenfunctions corresponding to distinct eigenvalues are orthogonal. If, in addition, the operator is positive, then its eigenvalues are non-negative.*

PROOF. Let $\lambda_0$ be an eigenvalue of $L$ and $u_0$ be a normalized eigenfunction, $\|u_0\| = 1$, corresponding to $\lambda_0$. Consider the inner product of $Lu_0 = \lambda_0 u_0$ with $u_0$ and transform it as follows:

$$\langle Lu_0, u_0\rangle = \langle \lambda_0 u_0, u_0\rangle = \lambda_0\langle u_0, u_0\rangle = \lambda_0\|u_0\|^2 = \lambda_0\,.$$

The left side of this equality is a quadratic form of $L$ which must be real for a hermitian operator and so is $\lambda_0$. If, in addition, the operator is positive, then its quadratic form is non-negative so that $\lambda_0 \geq 0$.

Let $u_1$ and $u_2$ be eigenfunctions of $L$ corresponding to eigenvalues $\lambda_1$ and $\lambda_2$, respectively, and $\lambda_1 \neq \lambda_2$. One has to show that $\langle u_1, u_2 \rangle = 0$. Since $Lu_1 = \lambda_1 u_2$, $Lu_2 = \lambda_2 u_2$, $\lambda_{1,2} = \overline{\lambda_{1,2}}$ are real, and $L$ is hermitian the following equalities hold:

$$\begin{aligned}\lambda_1 \langle u_1, u_2 \rangle &= \langle \lambda_1 u_1, u_2 \rangle = \langle Lu_1, u_2 \rangle = \langle u_1, Lu_2 \rangle = \langle u_1, \lambda_2 u_2 \rangle \\ &= \lambda_2 \langle u_1, u_2 \rangle\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, this equality implies $\langle u_1, u_2 \rangle = 0$. $\square$

**Example: Legendre polynomials.** Linearly independent eigenfunctions of the operator in Example **36**.**1** are *Legendre polynomials*:

$$\begin{aligned}&Lu = -[(1-x^2)u']' = \lambda u\,, \quad u \in C^2(0,1) \cap C^1[0,1] = \mathcal{M}_L \\ &\Rightarrow \quad \lambda = n(n+1)\,, \quad u(x) = P_n(x)\,, \quad n = 0, 1, 2, \ldots\,.\end{aligned}$$

This result can be established by *the Frobenius theory*. The differential equation in the eigenvalue problem has *two singular points* $x = \pm 1$. It is known as the *Legendre differential equation*. By the Frobenius theory, two linearly independent solutions of the Legendre equation are sought in the form of power series:

$$u_1(x) = \sum_{k=0}^{\infty} a_k x^{2k}\,, \quad u_2(x) = \sum_{n=0} b_k x^{2k+1}$$

So, a general solution for any $\lambda$ is given by a power series

$$u(x) = \sum_{n=0}^{\infty} c_n x^n\,, \quad |x| < 1$$

Recall that by the Frobenius theory the series converge at least for $|x| < 1$ (1 is the distance to the nearest singular point $x = \pm 1$). By substituting these series into the equation and comparing the coefficients at the same powers of $x$, the recurrence relation for the coefficients $c_n$ is found:

$$c_{n+2} = \left( \frac{n(n+1) - \lambda}{(n+1)(n+2)} \right) c_n$$

The coefficient $c_0$ remains arbitrary and the above recurrence relation defines all even coefficients $c_{2k} = c_0 a_k$. Similarly, $c_1$ defines all odd coefficients $c_{2k-1} = c_1 b_k$. The solution is a linear combinations of two linearly independent solutions, $c_0 u_1 + c_1 u_2$, as expected. The technicalities are left to the reader as an exercise.

Next, one *demands* that the solution $u$ must have from the domain of the operator. This implies in particular that $u(\pm 1)$ and $u'(\pm 1)$ exist because $u \in C^1[-1,1]$. This implies that the series must also converge at $x = \pm 1$, which turns out to be possible if and only if $\lambda = n(n+1)$, $n = 0, 1, 2, ...$, and, under this condition, the series become a *finite sum* or a *polynomial* because, by the recurrence relation, $c_k$ vanish for all $k > n$. For $\lambda = n(n+1)$, $u_1$ and $u_2$ are proportional to even and odd Legendre polynomials, respectively.

It follows from Theorem **36.2** that Legendre polynomials form an orthogonal set in the space of continuous functions in $[-1,1]$.

**36.3. Sturm-Liouville operator.** The Sturm-Liouville operator in an interval $(a,b)$ is defined by the following rule

$$Lu(x) = -\Big(p(x)u'(x)\Big)' + q(x)u(x)\,, \quad a < x < b\,, \tag{36.1}$$

where the function $p(x)$ and $q(x)$ have the properties

$$p \in C^1[a,b]\,, \quad q \in C^0[a,b]\,, \quad p(x) > 0\,, \quad q(x) \geq 0\,. \tag{36.2}$$

The domain of the Sturm-Liouville operator is

$$u \in \mathcal{M}_L : \begin{cases} u \in C^2(a,b) \cap C^1[a,b]\,, \\ \alpha_a u(a) - \beta_a u'(a) = 0\,, \quad \alpha_b u(b) + \beta_b u'(b) = 0 \\ \alpha_j \geq 0\,, \quad \beta_j \geq 0\,, \quad \alpha_j + \beta_j > 0\,, \quad j = a, b \end{cases} \tag{36.3}$$

The first condition ensures that $Lu(x)$ exists and is continuous in $[a,b]$. It also guarantees that $u$ and $u'$ have values at the endpoints of the interval so that the stated boundary conditions make sense. The third condition states that $\alpha_j$ and $\beta_j$, $j = a, b$, are non-negative but cannot be zero simultaneously. Its significance will become clear when the hermiticity of $L$ is studied. The eigenvalue problem

$$Lu = \lambda u\,, \quad u \in \mathcal{M}_L \tag{36.4}$$

is called the *regular Sturm-Liouville problem*. It plays a fundamental role in separating variables in many PDEs that are used in engineering and physics. If $p(a) = 0$ or $p(b) = 0$ or both, then the Sturm-Liouville problem is called *singular*. An example is provided by the Legendre operator discussed above. For singular problem, boundary conditions require that a function is just *regular* at boundary points where $p$ vanishes.

### 36.4. Properties of the Sturm-Liouville operator.

PROPOSITION **36.1**. *The Sturm-Liouville operator is hermitian.*

PROOF. It is sufficient to show that the quadratic form $\langle Lu, u\rangle$ is real for any $u \in \mathcal{M}_L$. One has

$$\langle Lu, u\rangle = -\int_a^b \Big(p(x)u'(x)\Big)'\,\overline{u(x)}\,dx + \int_a^b q(x)|u(x)|^2 dx$$

The last term is real and non-negative because $q(x) \geq 0$ and $|u(x)|^2 > 0$. So, the first term must be proved to be real. Integration by parts in the first term yields

$$-\int_a^b \Big(p(x)u'(x)\Big)'\,\overline{u(x)}\,dx = -p(x)u'(x)\overline{u(x)}\,\Big|_a^b + \int_a^b p(x)|u'(x)|^2 dx\,.$$

Note that the existence of the boundary term (or the validity of the integration by parts in this case) is guaranteed by that $u(x)$ has continuous derivatives at the boundary points and $p(x)$ is continuous on the closed interval $[a, b]$. The boundary term is transformed using the boundary conditions in (**36.3**) from which $u'(a)$ and $u'(b)$ are expressed in terms of $u(a)$ and $u(b)$, respectively:

$$\textbf{(36.5)}\qquad -p(x)u'(x)\overline{u(x)}\,\Big|_a^b = p(a)u'(a)\overline{u(a)} - p(b)u'(b)\overline{u(b)} = \frac{\alpha_a}{\beta_a}p(a)|u(a)|^2 + \frac{\alpha_b}{\beta_b}p(b)|u(b)|^2$$

if $\beta_a \neq 0$ and $\beta_b \neq 0$, and if $\beta_b = 0$ or $\beta_b = 0$, then the term containing $\beta_a$ or $\beta_b$ in the right side must be omitted (for $\beta_a = \beta_b = 0$ the boundary term is not present at all). Thus,

$$\langle Lu, u\rangle = \int_a^b \Big(p|u'|^2 + q|u|^2\Big)dx + \frac{\alpha_a}{\beta_a}\,p(a)|u(a)|^2 + \frac{\alpha_b}{\beta_b}\,p(b)|u(b)|^2$$

$$\langle Lu, u\rangle \geq 0$$

because of $p(x) > 0$ (the properties (**36.2**)) and the third property in (**36.3**). So, the Sturm-Liouville operator is real and, in fact, positive semi-definite and, hence, hermitian. □

THEOREM **36.3**. (Zero eigenvalue of the Sturm-Liouville operator) *In order for $\lambda = 0$ to be an eigenvalue of the Sturm-Liouville operator, it is necessary and sufficient that $q = 0$ and $\alpha_a = \alpha_b = 0$, and in this case $\lambda = 0$ is a simple eigenvalue and the corresponding eigenfunction is a constant function.*

PROOF. *Necessity.* Let $\lambda = 0$ be an eigenvalue of the Sturm-Liouville $L$ and $u$ be the corresponding eigenfunction so that $Lu = 0$, $u \in \mathcal{M}_L$. It has been shown earlier by integration by parts that

$$\langle Lu, u\rangle = \int_a^b \Big(p|u'|^2 + q|u|^2\Big)dx + \frac{\alpha_a}{\beta_a}\,p(a)|u(a)|^2 + \frac{\alpha_b}{\beta_b}\,p(b)|u(b)|^2$$

If $Lu = 0$, then $\langle Lu, u\rangle = 0$. The first term in the above integral vanishes if $p(x)|u'(x)|^2 = 0$ and $q(x)|u(x)|^2 = 0$. Since $p(x) > 0$ and $q(x) \geq 0$ in $[a, b]$, it is concluded that $u'(x) = 0$ or $u(x) = u_0$ is a non-zero constant function in $(a, b)$. The latter implies that $q(x) = 0$. Furthermore, if $u(x)$ is a constant function from $\mathcal{M}_L$, then it must also satisfy the boundary condition which yields $\alpha_a u(a) = 0$ and $\alpha_b u(b) = 0$ or $\alpha_a = \alpha_b = 0$ because $u(x) \neq 0$ and $u'(x) = 0$. The given line of arguments also shows that $u(x) = 1$ is the only (linearly independent) eigenfunction corresponding the zero eigenvalue, that is, if $\lambda = 0$ is eigenvalue, then it is a simple eigenvalue.

*Sufficiency.* Let $q(x) = 0$ and $\alpha_a = \alpha_b = 0$. Since $\alpha_a + \beta_a > 0$ and $\alpha_b + \beta_b > 0$ for any function in $\mathcal{M}_L$, it is concluded that $\beta_a > 0$ and $\beta_b > 0$. In this case, the eigenvalue problem for $\lambda = 0$ reads

$$Lu = -\Big(p(x)u'(x)\Big)' = 0\,, \quad u'(a) = u'(b) = 0\,.$$

A constant function $u(x) = \text{const}$ is a solution to this problem and, hence, an eigenfunction of $L$ corresponding to the eigenvalue $\lambda = 0$. □

The following theorem about the properties of eigenvalues and eigenfunctions of the Sturm-Liouville operator can be proved (a proof is omitted).

THEOREM **36**.4. (The Sturm-Liouville problem)
*Let $L$ be the Sturm-Liouville operator. Then:*

(i) *Eigenvalues of $L$ are non-negative;*
(ii) *The set $\{\lambda_k\}_1^\infty$ of eigenvalues of $L$ is countable and has no limit points;*
(iii) *Each $\lambda_k$ is simple;*
(iv) *The set of eigenfunctions $\{X_k\}_1^\infty$ can be chosen real and orthonormal, and furthermore $X_k \in C^2([a, b])$;*
(v) *The set $\{X_k\}_1^\infty$ is complete in the inner product space of continuous functions on $[a, b]$, that is, it is an orthogonal basis in this space.*

A proof of Parts (ii) and (v) goes beyond the scope of this course as it requires a theory of compact operators (or, at least, a theory of Hilbert-Schmidt operators).

Part (i) follows from Theorem **36.2**.

Part (iii). As any ordinary differential equation of second order, Eq. (**36.4**) has two linearly independent solutions for any real $\lambda$ (the boundary conditions (**36.3**) are ignored). Two solutions $\phi$ and $\psi$ of (**36.4**) are linearly independent in an interval $[a, b]$ if their Wronskian does not vanish in the interval:

$$W(x) = \det\begin{pmatrix} \phi & \psi \\ \phi' & \psi' \end{pmatrix} = \phi\psi' - \phi'\psi \neq 0\,, \quad a \leq x \leq b\,.$$

Suppose that $\phi_k$ and $\psi_k$ are two eigenfunctions corresponding to an eigenvalue $\lambda_k$, that is, $L\phi_k = \lambda_k\phi_k$ and $L\psi_k = \lambda_k\psi_k$. Since they are from $\mathcal{M}_L$, they must satisfy the boundary conditions (**36.3**). It follows from the first boundary condition that

$$\alpha_a\phi_k(a) - \beta_a\phi_k'(a) = 0\,, \quad \alpha_a\psi_k(a) - \beta_a\psi_k'(a) = 0\,.$$

These conditions can be viewed as a system of linear equations for $\alpha_a$ and $\beta_a$. Since $\alpha_a + \beta_a > 0$, the parameters $\alpha_a$ and $\beta_a$ cannot be zero simultaneously, which is possible only if the determinant of this system vanishes:

$$\det\begin{pmatrix} \phi_k(a) & -\phi_k'(a) \\ \psi_k(a) & -\psi_k'(a) \end{pmatrix} = \psi_k(a)\phi_k'(a) - \psi_k'(a)\phi_k(a) = W(a) = 0\,.$$

By the Liouville-Ostrogradsky theorem,

$$p(x)W(x) = p(a)W(a)\,, \quad a \leq x \leq b\,. \tag{36.6}$$

Since $p(x) > 0$ in $[a, b]$, it is concluded that $W(x) = 0$ in $[a, b]$. This implies that the function $\phi_k$ and $\psi_k$ are not linearly independent in $[a, b]$ and there is a constant $C$ such that $\psi_k(x) = C\phi_k(x)$ for all $x \in [a, b]$. So, each eigenvalue is indeed simple.

Part (iv). The equation is real. Therefore real and imaginary parts of an eigenfunction are also eigenfunctions because eigenvalues are real. Since all eigenvalues are simple, real and imaginary parts are linearly dependent, and one can take a real part of an eigenfunction as a linearly independent solution. Eigenfunctions corresponding to different eigenvalues of a hermitian operator are orthogonal. By scaling eigenfunctions so that their norm is unit, an orthonormal set is obtained.

**36.5. Fourier series over Sturm-Liouville eigenfunctions.** Suppose that the set of eigenfunctions is normalized so that $\|X_k\| = 1$. Then

$$\langle X_k, X_n\rangle = \int_a^b X_k(x)X_n(x)\,dx = \delta_{kn}$$

For any continuous function $u$ one can define a formal Fourier series

$$u(x) \sim \sum_{n=1}^{\infty} a_n X_n(x)\,, \quad a_n = \langle u, X_n\rangle = \int_a^b u(x) X_n(x)\, dx$$

where $a_n$ are called the *Fourier coefficients of* $u$ *over the basis* $\{X_n\}_1^\infty$. The completeness of the set of eigenfunctions implies that the Parseval-Steklov equality holds:

$$\|u\|^2 = \int_a^b |u(x)|^2 dx = \sum_{n=1}^{\infty} |a_n|^2$$

and the Fourier series converges in the mean:

$$\lim_{n\to\infty} \Big\| u - \sum_{k=1}^{n} a_k X_k \Big\| = 0$$

The convergence in the mean implies that the sum of the series can differ from $u(x)$ on sets whose total length is less any positive number (e.g., a finite collections of points), or one says that the series converges to $u$ *almost everywhere*:

$$u(x) = \sum_{n=1}^{\infty} a_n X_n(x) \quad a.e.$$

What is an analog of Fejér's theorem for the Sturm-Liouville basis? The following two theorems are due to V.A. Steklov. They answer the question.

THEOREM **36.5**. (First Steklov theorem)
*Let $L$ be the Sturm-Liouville operator and $\{X_k\}_1^\infty$ be an orthonormal set of its eigenfunctions. Then For any $u \in \mathcal{M}_L$ the Fourier series*

$$u(x) \sim \sum_{k=1}^{\infty} \langle u, X_k\rangle\, X_k(x)$$

*converges to $u(x)$ for all $x \in [a,b]$. Moreover, the Fourier series can be differentiated term-by-term and the obtained series converges to the derivative $u'(x)$ in the mean:*

$$u'(x) = \sum_{k=1}^{\infty} \langle u, X_k\rangle\, X_k'(x) \quad a.e.$$

THEOREM **36.6**. (Second Steklov theorem)
*For any continuously differentiable function $u$ on a closed interval $[a,b]$ that vanishes at the endpoints,*

$$u \in C^1([a,b])\,, \quad u(a) = u(b) = 0\,,$$

*its Fourier series over the set of eigenfunctions of the Sturm-Liouville operator converges to $u(x)$ for all $x \in [a, b]$.*

Note that the second Steklov theorem is stronger (compare $C^1([a,b])$ with $\mathcal{M}_L$): The existence and continuity of the second derivative is not necessary for the convergence of the Fourier series.

**Remark.** If the eigenfunctions are not normalized, then the Fourier coefficients are defined by

$$a_k = \frac{1}{\|X_k\|^2}\int_a^b u(x)X_k(x)\,dx\,, \quad \|X_k\|^2 = \int_a^b |X_k(x)|^2\,dx$$

Recall that $X_m^c(x) = \cos(mx)$ and $X_m^s(x) = \sin(mx)$ are not normalized:

$$\|X_m^c\|^2 = \int_{-\pi}^{\pi} \cos^2(mx)\,dx = \pi\,, \quad \|X_m^s\|^2 = \pi$$

This explains the factor $\frac{1}{\pi}$ in the corresponding trigonometric Fourier coefficients.

**36.6. Solving for eigenvalues and eigenfunctions.** The basic theory of ordinary differential equations asserts that:

• *The initial value problem*

$$u''(x) + g(x)u'(x) + h(x)u(x) = 0\,, \quad x > 0\,, \quad u(0) = u_0\,,\ u'(0) = u_1$$

*has a unique solution for given continuous functions $g$ and $h$;*

• *The general solution to the second order differential equation is a linear combination of two linearly independent solutions;*

• *The initial data determine uniquely the coefficients in the linear combination to obtain the unique solution to the initial value problem.*

The linear differential equation (**36**.**4**) can be written in the form

$$u''(x) + \frac{p'(x)}{p(x)}\,u'(x) + \frac{\lambda - q(x)}{p(x)}\,u(x) = 0$$

because $p(x) > 0$. The functions $g(x) = p'(x)/p(x)$ and $h(x) = (\lambda - q(x))/p(x)$ are continuous on $[a,b]$ by definition of the Sturm-Liouville operator. Therefore its general solution is a linear combination of two linearly independent solutions. Since the solution of the initial value problem is unique, two linearly independent solutions can be found by solving the initial value problem with two suitable sets of initial data.

Let $u_1(x;\lambda)$ and $u_2(x;\lambda)$ be the solutions to (**36.4**) that satisfy the conditions

$$\begin{array}{ll} u_1(a;\lambda) = 1\,, & u_1'(a;\lambda) = 0\,; \\ u_2(a;\lambda) = 0\,, & u_2'(a;\lambda) = 1 \end{array}\,. \tag{36.7}$$

As noted, they are unique for any $\lambda \in \mathbb{R}$ and they are also linearly independent. Note that the linear dependence $u_1(x;\lambda) = Cu_2(x;\lambda)$ for some number $C \neq 0$ contradicts the chosen initial conditions. Then the function

$$u(x;\lambda) = \beta_a u_1(x;\lambda) + \alpha_a u_2(x;\lambda) \tag{36.8}$$

satisfies (**36.4**) and the first boundary condition in (**36.3**) because $u(a;\lambda) = \beta_a$ and $u'(a;\lambda) = \alpha_a$ so that for any $\lambda$

$$\alpha_a u(a;\lambda) - \beta_a u'(a;\lambda) = \alpha_a\beta_a - \beta_a\alpha_a = 0\,.$$

In order to satisfy the second boundary condition in (**36.3**), one has to demand that

$$\alpha_b u(b;\lambda) + \beta_b u'(b;\lambda) = 0\,. \tag{36.9}$$

The roots $\lambda = \lambda_k$ of this equation define all the eigenvalues of the Sturm-Liouville operator. Theorem **36.4** also implies that this equation must have countably many simple roots that do not have any limit point. The corresponding eigenfunctions are given by

$$X_k(x) = u(x;\lambda_k)\,.$$

They are not normalized, but they form an orthogonal basis and can be used to expand any continuous function into the Fourier series that converges in the mean (almost everywhere).

EXAMPLE **36.2**. *Solve the eigenvalue problem*

$$-u''(x) = \lambda u(x)\,, \quad u(0) = 0\,, \quad u'(l) + \alpha u(l) = 0\,, \quad \alpha \geq 0$$

*Find an asymptotic expression for large eigenvalues.*

SOLUTION: Here $p = 1$, $q = 0$, and $[a,b] = [0,l]$. In addition, $\beta_a = 0$, $\alpha_a = 1$, $\beta_b = 1$, and $\alpha_b = \alpha$, then the above eigenvalue problem is an eigenvalue problem for the Sturm-Liouville operator. Thus, all eigenvalues are non-negative. To find them consider, the general solution of the equation

$$-u'' = \lambda u \quad \Rightarrow \quad u(x) = \begin{cases} C_1\cos(\sqrt{\lambda}\,x) + C_2\sin(\sqrt{\lambda}\,x)\,, & \lambda > 0 \\ C_1 + C_2 x\,, & \lambda = 0 \end{cases}$$

But $\lambda = 0$ cannot be an eigenvalue because $\alpha_0 = 1 \neq 0$ by Theorem **36.3**. Alternatively, if $\lambda = 0$ is an eigenvalue, then the corresponding eigenfunction is constant, $u(x) = C_1$. However, the condition $u(0) = 0$ implies that $u(x) = 0$ for any constant function.

Following the above general procedure, the solutions (**36**.**7**) are found by a suitable choice of $C_1$ and $C_2$ for $\lambda > 0$:

$$u_1(x;\lambda) = \cos(\sqrt{\lambda}\, x)\,, \quad u_2(x;\lambda) = \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\, x)\,.$$

Therefore the solution (**36**.**8**) satisfying the first boundary condition reads

$$u(x;\lambda) = u_2(x;\lambda) = \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\, x)$$

The eigenvalues are the roots of the equation (**36**.**9**)

$$u'(l;\lambda) + \alpha u(l;\lambda) = 0 \quad \Rightarrow \quad \cos(l\sqrt{\lambda}) + \frac{\alpha}{\sqrt{\lambda}}\sin(l\sqrt{\lambda}) = 0$$

This shows that $\lambda = 0$ is not an eigenvalue. The positive eigenvalues are roots of the transcendental equation

$$-\frac{\sqrt{\lambda}}{\alpha} = \tan(l\sqrt{\lambda}) \quad \Rightarrow \quad -\frac{z}{l\alpha} = \tan(z)\,, \quad z = l\sqrt{\lambda}\,.$$

The roots $z = z_k$, $k = 1, 2, ...$, are easy to analyze by graphing the functions $y = -z/(l\alpha)$ and $y = \tan(z)$ for $z > 0$ in the $zy$ plane. The graph $y = \tan(z)$ has vertical asymptotes $z = \pi k - \frac{\pi}{2}$, $k = 1, 2, ...$, such that $\tan(z) \to \mp\infty$ if $z \to (\pi k - \frac{\pi}{2})^{\pm}$ (the limits from the left and right). Therefore the straight line with negative slope $y = -z/(l\alpha)$ intersects the graph $y = \tan(z)$ one time in each interval $\pi k - \pi/2 < z < \pi k$, $k = 1, 2, ....$ Thus,

$$\frac{\pi^2(k-\frac{1}{2})^2}{l^2} < \lambda_k < \frac{\pi^2 k^2}{l^2}\,, \quad k = 1, 2, \ldots\,.$$

Furthermore, $\lambda_k$ monotonically approaches the left end of the interval with increasing $k$. If $\epsilon_k > 0$ denotes a deviation of $\lambda_k$ from the left end of the interval, then

$$\lambda_k = \frac{\pi^2(k-\frac{1}{2})^2}{l^2} + \epsilon_k\,, \quad \epsilon_k \to 0\,,\ k \to \infty\,;$$

the sequence $\epsilon_k$ is monotonically decreasing. In full accord with Theorem **36**.**4**, the eigenvalues are all simple and their sequence does not have any limit point (a point whose any neighborhood contains infinitely mane eigenvalues). Orthogonal (not orthonormal) eigenfunctions

$$X_k(x) = \sin\left(\sqrt{\lambda_k}\, x\right)\,, \quad k = 1, 2, ....$$

are real (as stated in Theorem **36.4**). The following is worth noting. By Theorem **36.4**,

$$\langle X_k, X_n\rangle = \int_0^l \sin\left(\sqrt{\lambda_k}\, x\right)\sin\left(\sqrt{\lambda_n}\, x\right)\, dx = 0\,, \quad k \neq n\,,$$

where $\lambda_k$ satisfy the above transcendental equation. The reader is advised to evaluate this integral and show that this is indeed so by using the equation for $\lambda_k$. This *rather technical* exercise illustrates the power of Theorem **36.4** (the orthogonality is guaranteed by the theorem, which otherwise is not obvious at all). □

EXAMPLE **36.3**. *Find an orthogonal set of eigenfunctions of the following Sturm-Liouville operator in an interval* $(0,1)$*, expand the function* $f(x) = x^2(1-x)^2$ *over the set into a formal Fourier series, and investigate its convergence and term-by-term differentiation:*

$$Lu = -u''(x) = \lambda u\,, \quad u(0) = u(1) = 0$$

SOLUTION: Zero is not an eigenvalue because a non-zero constant function does not satisfy the boundary conditions. Put $\lambda = \nu^2$, $\nu > 0$. A general solution reads

$$u(x) = C_1\cos(\nu x) + C_2\sin(\nu x)$$

The zero boundary condition at $x = 0$ is satisfied by

$$u(x;\nu) = \sin(\nu x)$$

Therefore the eigenvalues are roots of the equation:

$$u(1;\nu) = 0 \quad\Rightarrow\quad \sin(\nu) = 0 \quad\Rightarrow\quad \nu = \pi n\,,\ n = 1, 2, ...$$

An orthogonal basis in the space of continuous function on $[0,1]$ is

$$X_n(x) = u(x;\pi n) = \sin(\pi n x)\,, \quad \|X_n\|^2 = \int_0^1 \sin^2(\pi n x)\, dx = \frac{1}{2}$$

The Fourier coefficients are calculated by integration by parts 4 times:

$$\begin{aligned} a_n = \frac{\langle f, X_n\rangle}{\|X_n\|^2} &= 2\int_0^1 x^2(1-x)^2\sin(\pi n x)\, dx \\ &= -\frac{2}{\pi n}\int_0^1 x^2(1-x)^2 d\cos(\pi n x) \\ &= \frac{2}{\pi n}\int_0^1 (2x - 6x^2 + 4x^3)\cos(\pi n x) dx = \cdots \\ &= \frac{4}{\pi^3 n^3}\left(\cos(\pi n) - 1 + \frac{12}{\pi^2 n^2}\int_0^1 \sin(\pi n x)\, dx\right) \end{aligned}$$

It follows that

$$a_{2k} = 0\,, \quad a_{2k-1} = \frac{8}{\pi^3 n^3}\left(\frac{12}{\pi^2 n^2} - 1\right)$$

and

$$f(x) = x^2(1-x)^2 \sim \frac{8}{\pi^3}\sum_{k=1}^{\infty}\left(\frac{12}{\pi^2(2k-1)^2} - 1\right)\frac{\sin[(2k-1)\pi x]}{(2k-1)^3}$$

The domain of the operator in question consists of twice continuously differentiable functions in $(0,1)$ that have zero values at the endpoints. The function $f(x)$ is therefore from the domain of the operator. By the first Steklov theorem, the series converges to $f(x)$ for all $x \in [0,1]$ and the derivative $f'(x)$ can be obtained by the term-by-term differentiation of the series for $f$:

$$f'(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{12}{\pi^2(2k-1)^2} - 1\right)\frac{\cos[(2k-1)\pi x]}{(2k-1)^2} \quad a.e.$$

However it is possible to conclude that the differentiated Fourier series converges to $f'(x)$ for all $x \in [0,1]$ because the terms of the series are majorated by

$$\left|\left(\frac{12}{\pi^2(2k-1)^2} - 1\right)\frac{\cos[(2k-1)\pi x]}{(2k-1)^2}\right| \le \frac{3}{(2k-1)^2}$$

and the series of the upper bounds converges $\sum_k \frac{1}{(2k-1)^2} < \infty$. Since the terms of the series are continuous in $[0,1]$ so is the sum by Theorem **34.3**. But the sum coincides with a *continuous* $f'(x)$ almost everywhere and, hence, $f'(x)$ and the sum must be equal for all $x$.

Alternatively, one can use the second Steklov theorem to establish the convergence of the Fourier series to $f(x)$ everywhere. The term-by-tern differentiation is shown to hold by Theorem **34.3**. □

**36.7. Exercises.**

1. (Liouville-Ostrogradsky theorem)
Suppose that $\phi(x)$ and $\psi(x)$ are non-zero solutions of (**36.4**) for some $\lambda$ and $W(x)$ is their Wronskian. Use (**36.4**) to show that

$$W'(x) = -\frac{p'(x)}{p(x)}\,W(x)\,.$$

Use this equation to prove the Liouville-Ostrogradsky theorem (**36.6**). Solve the equation with the initial condition $W(x_0) = W_0$ for some $a \le x_0 \le b$. Use the solution to prove that if $W(x) = 0$ for some

$a \le x_0 \le b$, then $W(x) = 0$ in $[a, b]$. Use the latter fact to prove that if the Wronskian vanishes at some point in $[a, b]$, then $\phi(x) = C\psi(x)$ in $[a, b]$ for some constant $C$, that is, the solutions are linearly dependent.

**2**. Consider the eigenvalue problem

$$Lu \equiv -(e^x u')' = \lambda u\,, \quad 0 < x < l\,,$$
$$u \in C^2(0, l) \cap C^0[0, l]\,, \quad u(0) = u(l) = 0.$$

Is $\lambda = 0$ is an eigenvalue of $L$?

**3**. Solve the eigenvalue problem and use the Sturm-Liouville theory to show that linearly independent eigenfunctions form an orthogonal basis in the space of continuous functions on $[0, l]$:

$$Lu(x) = -u''(x) = \lambda u(x)\,, \quad 0 < x < l\,,$$
$$u \in C^2(0, l) \cap C^0[0, l]\,, \quad u(0) = u(l) = 0.$$

**4**. Solve the eigenvalue problem and use the Sturm-Liouville theory to show that linearly independent eigenfunctions form an orthogonal basis in the space of continuous functions on $[0, l]$:

$$Lu(x) = -u''(x) = \lambda u(x)\,, \quad 0 < x < 1\,,$$
$$u \in C^2(0, 1) \cap C^1[0, 1]\,, \quad u'(0) = u'(1) = 0.$$

Show that eigenfunctions can be chosen real and orthonormal by giving their explicit form. Expand the function $f(x) = x^2(1-x)^2$ into a formal Fourier series over these eigenfunctions and investigate its convergence and term-by-term differentiation.

**5**. Solve the eigenvalue problem and use the Sturm-Liouville theory to show that linearly independent eigenfunctions form an orthogonal basis in the space of continuous functions on $[-a, a]$:

$$Lu(x) = -u''(x) = \lambda u(x)\,, \quad -a < x < a\,,$$
$$u \in C^2(-a, a) \cap C^0[-a, a]\,, \quad u(-a) = u(a) = 0.$$

**6**. Solve the eigenvalue problem and use the Sturm-Liouville theory to show that linearly independent eigenfunctions form an orthogonal basis in the space of continuous functions on $[-a, a]$:

$$Lu(x) = -u''(x) = \lambda u(x)\,, \quad -a < x < a\,,$$
$$u \in C^2(-a, a) \cap C^1[-a, a]\,, \quad u'(-a) = u'(a) = 0.$$

**7**. Use the method of Section **36.6** to solve the Sturm-Liouville problem:

$$Lu(x) = -u''(x) = \lambda u(x)\,, \quad 0 < x < l\,,$$
$$u \in C^2(0,l) \cap C^1[0,l]\,, \quad -u'(0) + \alpha u(0) = 0\,, \quad u'(l) + \alpha u(l) = 0$$

where $\alpha \geq 0$. Use a graphical method to analyze eigenvalues by analogy with Example **36.2**.

**8**. Is there a complex number $z$ for which the operator

$$L_z u(x) = -iu'(x)\,, \quad 0 < x < 2\pi$$
$$u \in C^1(0,2\pi) \cap C^0[0,2\pi]\,, \quad u(2\pi) = zu(0)\,,$$

hermitian? If so, solve the eigenvalue problem for $L_z$ in this case. Do linearly independent eigenfunctions form an orthogonal basis?

**9**. Consider $Lu(x) = iu'(x)$ where $u \in \mathcal{M}_L = C^1(\mathbb{R}) \cap C_2^0(\mathbb{R})$ where

$$u \in C_2^0(\mathbb{R}) \quad \text{if} \quad \int_{-\infty}^{\infty} |u(x)|^2\, dx < \infty$$

Show that $\langle Lf, g\rangle = \langle f, Lg\rangle$ for any $f$ and $g$ from $\mathcal{M}_L$.

**10**. Let $Lu(x) = -u''(x)$ where $u \in \mathcal{M}_L = C^2(\mathbb{R}) \cap C_2^0(\mathbb{R})$, where

$$u \in C_2^0(\mathbb{R}) \quad \text{if} \quad \int_{-\infty}^{\infty} |u(x)|^2\, dx < \infty$$

Show that $\langle Lu, u\rangle \geq 0$ for all $u \in \mathcal{M}_L$.

**11**. Let $Lu(x) = iu'(x)$ where $u \in C^1(0,a) \cap C^0[0,a]$. Find the largest subset $\mathcal{M}_L \subset C^1(0,a) \cap C^0[0,a]$ such that $\langle Lf, g\rangle = \langle f, Lg\rangle$ for any $f$ and $g$ from $\mathcal{M}_L$. Hint: Integrate by parts in the inner product and find the most general boundary conditions under which the boundary term in the integration by parts vanishes.

**12**. Let $Lu(x) = -u''(x)$ where $u \in C^2(0,a) \cap C^1[0,a]$. Find the largest subset $\mathcal{M}_L \subset C^2(0,a) \cap C^1[0,a]$ by imposing boundary conditions on $u(x)$ and $u'(x)$ at $x = 0$ and $x = a$ so that $\langle Lu, u\rangle \geq 0$ for all $u \in \mathcal{M}_L$.
Hint: Put $\alpha_1 u(0) + \beta_1 u'(0) = 0$ and $\alpha_2 u(a) + \beta_2 u'(a) = 0$. Integrate by parts in $\langle Lu, u\rangle$ and use the boundary conditions to find conditions on the numbers $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ under which $\langle Lu, u\rangle \geq 0$.

**Selected answers.**
**2**. No.

**3**. $\lambda = \lambda_k = \left(\frac{\pi k}{l}\right)^2$, $X_k(x) = (2/l)^{1/2}\sin(\pi kx/l)$, $k = 1, 2, ...,$ $\langle X_k, X_n\rangle = \delta_{kn}$. Since $L$ is a Sturm-Liouville operator, the completeness follows from Theorem **36**.**4**.

**4**. $\lambda = \lambda_k = \left(\frac{\pi k}{l}\right)^2$, $X_0(x) = l^{-1/2}$, $X_k(x) = (2/l)^{1/2}\cos(\pi kx/l)$, $k = 1, 2, ...,$ $\langle X_k, X_n\rangle = \delta_{kn}$. Since $L$ is a Sturm-Liouville operator, the completeness follows from Theorem **36**.**4**.

**5**. $\lambda = \lambda_k = \left(\frac{\pi k}{2a}\right)^2$, $X_k(x) = a^{-1/2}\sin(\sqrt{\lambda_k}(x+a))$, $k = 1, 2, ...,$ $\langle X_k, X_n\rangle = \delta_{kn}$. Since $L$ is a Sturm-Liouville operator, the completeness follows from Theorem **36**.**4**

**6**. $\lambda = \lambda_k = \left(\frac{\pi k}{2a}\right)^2$, $X_0(x) = (2a)^{-1/2}$, $X_k(x) = a^{-1/2}\cos(\sqrt{\lambda_k}(x+a))$, $k = 1, 2, ...,$ $\langle X_k, X_n\rangle = \delta_{kn}$. Since $L$ is a Sturm-Liouville operator, the completeness follows from Theorem **36**.**4**.

**7**. Let $z = \sqrt{\lambda}l$. Then

$$\cot(z) = \frac{1}{2\alpha}\left(\frac{\alpha^2 l}{z} - \frac{z}{l}\right)$$

**8**. $z = e^{i\varphi}$ or $|z| = 1$. For $z = 1$, an orthogonal basis is given by complex trigonometric Fourier harmonics $X_n(x) = e^{inx}$, where $n = 0, \pm 1, \pm 2, ....$

## 37. The Cauchy problem for 2D parabolic PDEs

**37.1. Formulation of the problem.** Let $L$ be a Sturm-Liouville operator on an interval $[a, b]$. Consider the initial and boundary value problem in an open rectangle

$$\Pi_\infty = (a, b) \times (0, \infty)$$

for a *parabolic* equation:

$$u_t'(x,t) = -L_x u(x,t) + f(x,t)\,, \quad (x,t) \in \Pi_\infty\,, \tag{37.1}$$

$$u(x,0) = v(x)\,, \quad x \in [a,b]\,, \tag{37.2}$$

$$\begin{cases} \alpha_a u(a,t) - \beta_a u_x'(a,t) = 0 \\ \alpha_b u(b,t) + \beta_b u_x'(b,t) \ = 0 \end{cases}, \quad t \ge 0\,. \tag{37.3}$$

The condition (**37.2**) is the initial condition. Since the equation (**37.1**) is of the first order with respect to the evolution variable $t$, only one initial condition is need. The problem is to fund a function that has continuous second partial derivatives in the rectangle $\Pi_\infty$, satisfies Eq. (**37.1**), the initial condition (**37.2**), and the boundary condition (**37.3**). The boundary condition requires the existence of partial derivatives if $\beta_{a,b} \neq 0$ so the classical solution must be from the class

$$u \in C^2(\Pi_\infty) \cap C^0(\overline{\Pi}_\infty)\,, \quad u_x' \in C^0(\overline{\Pi}_\infty)\,, \quad \overline{\Pi}_\infty = [a,b] \times [0,\infty)\,.$$

In addition, it will also be assumed that the inhomogeneity $f$ is a continuous function in $\Pi_\infty$ and its boundary.

$$f \in C^0(\,\overline{\Pi_\infty}\,)\,.$$

The initial data should also be sufficiently smooth in order for a classical solution to exist. The boundary conditions need to be compatible with the initial conditions. By setting $t = 0$ in the boundary conditions and using the initial condition one gets

$$\left(\alpha_a v - \beta_a \frac{dv}{dx}\right)\bigg|_{x=a} = \left(\alpha_b v + \beta_b \frac{dv}{dx}\right)\bigg|_{x=b} = 0$$

Therefore, if the parameters $\beta_{a,b}$ do not vanish simultaneously, the initial data $v$ must be differentiable the interval $[a,b]$ (including the endpoints) in order for this condition to make sense:

$$v \in C^1[a,b]$$

It is possible to prove that the classical exists and is unique, small variations of the parameters (the inhomogeneity and initial data) lead to small variations of the classical solution.

THEOREM **37.1**. (Continuity of a classical solution)
*A classical solution of the problem (**37.1**)–(**37.3**) is unique, if it exists, and depends continuously on the initial data $v$ and $f$ in the following sense. If $u(x,t)$ and $\tilde{u}(x,t)$ are classical solutions corresponding to two sets $v$, $f$ and $\tilde{v}$, $\tilde{f}$ such that*

$$\max_{\overline{\Pi}_T} |f(x,t) - \tilde{f}(x,t)| \le \epsilon\,, \quad \overline{\Pi}_T = [0,l] \times [0,T]\,,$$
$$\max_{[a,b]} |v(x) - \tilde{v}(x)| \le \epsilon_0\,,$$

*for any $T > 0$, then*

$$\max_{\overline{\Pi}_T} |u(x,t) - \tilde{u}(x,t)| \le \epsilon_0 + T\epsilon\,.$$

In this theorem a small number $\epsilon_0$ limits the maximal difference of two initial data sets $v$ and $\tilde{v}$, and a small number $\epsilon$ limits the maximal difference of two inhomogeneities $f$ and $\tilde{f}$. Then the theorem asserts that the maximal difference of two corresponding classical solutions over any finite interval $0 \le t \le T$ is limited by $\epsilon_0 + T\epsilon$. This implies that small variations of the parameters lead to small variations of the solution, or *the solution depends continuously on the parameters.* The continuity of a solution is a must-have feature in any well-posed PDE problem that describes a real world phenomenon. Parameters are always known with some accuracy, and a solution is not expected to fluctuate wildly under small fluctuations of the parameters.

**37.2. A physical significance of a general parabolic equation.** In general the problem (**37.1**)–(**37.3**) can be interpreted as a heat equation for a non-homogeneous rod, where $u(x,t)$ is the temperature of the rod at a position $x$ and a time $t$. The function $p(x)$ plays the role of a non-uniform heat conductance coefficient. The boundary conditions (**37.3**) for $\beta_{a,b} \neq 0$ mean that the temperature at the end point of the rod is not held constant, but rather there is heat energy flow through the endpoints. In general, a flow of the thermal energy is proportional to the gradient $\nabla T$ of the temperature $T$. Consequently, the flow of the thermal energy across a unit surface is proportional to the normal component of the temperature gradient $\mathbf{n} \cdot \nabla T$, where $\mathbf{n}$ is the unit normal vector to the surface. According to Newton's law of cooling, the thermal energy flow across a surface separating two bodies at different temperatures is proportional to the temperature difference of the bodies. Suppose that the temperature of the rod is non-negative, $u(x,t) \ge 0$ (one can always count it from the absolute zero).

Since $\alpha$ and $\beta$ are non-negative, the conditions

$$u'_x(b,t) = -\frac{\alpha_b}{\beta_b} u(b,t) \le 0\,, \quad u'_x(a,t) = \frac{\alpha_a}{\beta} u(a,t) \ge 0$$

imply that the rod is loosing the thermal energy through its ends. Indeed, the derivative $u'_x(b,t)$ is positive and, hence, the thermal energy is flowing in the direction of increasing $x$, that is, from the rod through its endpoint $x = b$. The derivative $u'_x(a,t)$ is negative and, hence, the thermal energy is flowing in the direction *opposite* to the direction in which $x$ is increasing. So, the following physical interpretation may be given to the boundary conditions (**37**.**3**)

- If $\beta = 0$, then the corresponding endpoint of a cooling rod is kept at a fixed temperature;
- If $\alpha = 0$, the there is a constant thermal energy flow from the rod through the corresponding endpoint;
- If $\alpha$ and $\beta$ are both not zero, then there is a thermal energy flow from the rod through the corresponding endpoint of the rod attached to a large body of a constant temperature.

In the limit $t \to \infty$, the temperature $u(x,t)$ does not approach 0 if and only if the Sturm-Liouville operator has the zero eigenvalue and, in this case, $u(x,t) \to \text{const} > 0$ as $t \to \infty$. The fist eigenvalue $\lambda_1$ can vanish if and only if $\alpha_a = \alpha_b = 0$. This means that there is no thermal energy flow from the rod. A uniform temperature distribution $u(x,t) = \text{const}$ (an eigenfunction corresponding to the zero eigenvalue) implies that there is no thermal energy flow in the rod. The thermal energy becomes uniformly distributed. For any other boundary condition with $\alpha > 0$ and $\beta \ge 0$, the rod is going to lose all its thermal energy through its endpoints as $\lambda_k > 0$.

**Remark.** If $\alpha$ and $\beta$ are allowed to take negative values, then the corresponding Sturm-Liouville operator is no longer positive, although it is still hermitian. This implies that some of its eigenvalues can be negative so that the solution (**37**.**5**) can grow exponentially with increasing time $t$ even if $f = 0$. Physically, this describe a situation when there is a thermal energy flow *into* the rod through its endpoints, that is, this describes a *heating* process rather than a cooling process.

The parabolic problem (**37**.**1**)–(**37**.**3**) also describes a diffusion process. In this case, $u(x,t)$ is interpreted as a concentration of particles. For example, if atoms of coper are non-uniformly placed in iron, then the concentration of coper atoms is a function of position. Yet, due to a diffusion process it will also change with time. In this case, the function $p(x)$ is a so-called diffusion coefficient of coper atoms in an iron rod. The derivative $u'_x$ defines a flow of coper atoms across the cross section of the rod. The boundary conditions (**37.3**) have the following physical interpretation:

- If $\beta = 0$, then the concentration of diffusive particle vanishes at the corresponding endpoint;
- If $\alpha = 0$, then there is no flow of diffusive particles across the corresponding endpoint of the rod;
- If $\alpha \neq 0$ and $\beta \neq 0$, then there is a flow of diffusive particles from the rod through the corresponding endpoint of the rod that is proportional to the difference of concentrations in the rod and an outer body to which the rod is attached (known as *Fick's law of diffusion*)

Just like in the heat equation case, all diffusive particles will eventually leave the rod through its endpoints if at least one of the $\alpha$'s does not vanish and the $\alpha$'s and $\beta$'s are assumed to be non-negative. No diffusive particle will leave the rod if $\alpha_a = \alpha_b = 0$ (no diffusive flow through the endpoints). The solution approaches a stationary constant concentration of diffusive particles.

The note qualitative behavior of the solution can easily be verified by the Fourier method.

**37.3. Formal solution.** The conditions (**37.3**) are the most general boundary conditions under which the Sturm-Liouville operator is hermitian. The latter property allows one to construct a formal solution of the problem by separation of variables and the Fourier method, just as in the case of the 2D heat equation in a circle, where the trigonometric harmonics were used as an orthogonal basis. The existence of the formal solution requires an investigation of convergence of the series in $\Pi_\infty$. Whether or not the formal solution is a classical one is to yet to be determined.

Let $X_k$ and $\lambda_k$, $k = 1, 2, \ldots$, be eigenfunctions and the corresponding eigenvalues of the operator $L$:

$$L_x X(x) = \lambda X(x) \quad \Rightarrow \quad \lambda = \lambda_k\,, \quad X(x) = X_k(x)\,, \quad k = 1, 2, \ldots,$$

where the eigenfunction satisfy the boundary condition (**37.3**). The eigenfunctions form a real orthonormal basis in the space of continuous functions on $[a, b]$

$$\langle X_k, X_j \rangle = \int_a^b X_k(x) X_j(x)\, dx = \delta_{kj}\,.$$

Consider the sequences of partial sums of the Fourier series for the initial data $v$ and the function $f$:

$$v_n(x) = \sum_{k=1}^{n} a_k X_k(x)\,, \quad a_k = \langle v, X_k\rangle = \int_a^b u_0(x) X_j(x)\,dx\,,$$

$$f_n(x,t) = \sum_{k=1}^{n} F_k(t) X_k(x)\,, \quad F_k(t) = \langle f, X_k\rangle = \int_a^b f(x,t) X_j(x)\,dx\,.$$

Consider the problem (**37.1**)–(**37.3**) where $v$ is replaced by $v_n$ and $f$ by $f_n$. Then the solution is sought in the form

$$u_n(x,t) = \sum_{k=1}^{n} V_k(t) X_k(x)\,.$$

The substitution of this relation into (**37.1**) and (**37.2**) yields

$$\sum_{k=1}^{n} V_k'(t) X_k(x) = \sum_{k=0}^{n} V_k(t) L_x X_k(x) + \sum_{k=1}^{n} F_k(t) X_k(x)\,.$$

or using $L_x X_k = \lambda_k X_k$

$$\sum_{k=1}^{n} \Big[V_k'(t) - \lambda_k V_k(t)\Big] X_k(x) = \sum_{k=1}^{n} F_k(t) X_k(x)\,.$$

Since $X_k$ are linearly independent, the expansion coefficients $T_k$ must satisfy the initial value problem:

$$V_k'(t) + \lambda_k V_k(t) = F_k(t)\,, \quad V_k(0) = a_k\,,$$

whose solution can be obtained by the method of variation of parameters or by the Laplace transform. It reads

(**37.4**) $$V_k(t) = a_k e^{-\lambda_k t} + \int_0^t e^{-\lambda_k (t-\tau)} F_k(\tau) d\tau\,.$$

The series

(**37.5**) $$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \sum_{k=1}^{\infty} V_k(t) X_k(x)\,,$$

where the coefficients are given by (**37.4**), is called a *formal solution* to the problem (**37.1**)–(**37.3**).

**37.4. Classical and formal solutions.** The convergence (or the existence) of the formal solution and its smoothness can be studied with the help of basic theorems discussed earlier (recall the use of Theorems **34.2** and **34.3**, when studying the heat equation in a circle). It is also possible to state *sufficient conditions* on the smoothness of the initial data in order for the formal solution to be the classical one.

THEOREM **37.2**. (Classical and formal solutions)
*If $v \in \mathcal{M}_L$ and $f = 0$, then a formal solution to the problem (**37.1**)–(**37.3**) given by the Fourier series (**37.5**) is the classical solution. The solution is also infinitely many times differentiable with respect to $t$ for $t > 0$ and $a \le x \le b$.*

Note well that the theorem provides *sufficient* conditions for a formal solution to be the classical ones. Even if the hypotheses do not hold, the formal solution can still be smooth enough to be a classical solution.

**37.5. An example: Cooling of a homogeneous rod.** Consider a heat conducting homogeneous rod of length $l$. Suppose that its ends are kept at a fixed temperature $T_c$. At the initial moment of time $t = 0$, the temperature as function of position on the rod is $T_0(x)$, $0 \le x \le l$, so that $T_0(0) = T_0(l) = T_c$. The problem is to find the temperature of the rod as a function of position and time, $T(x,t)$. Let

$$u(x,t) = T(x,t) - T_c$$

so that

$$u(x,0) = v(x) = T_0(x) - T_c\,.$$

Then by basic laws of heat conductance, one can show that $u(x,t)$ is a (classical) solution to the following problem

$$u'_t = \alpha^2 u''_{xx}\,, \quad u\Big|_{t=0} = v(x)\,, \quad u\Big|_{x=0} = u\Big|_{x=l} = 0\,, \tag{37.6}$$

where $\alpha$ is called a *heat conductance constant* for the material of the rod.

Let us find the formal solution of the problem.
The associated Sturm-Liouville problem:

$$-X''(x) = \lambda X(x)\,, \quad X(0) = X(l) = 0\,,$$

First, $\lambda = 0$ is not an eigenvalue. It were an eigenvalue, then the corresponding eigenfunction must be a constant function (according to the general analysis of the Sturm-Liouville problem), but a non-zero constant function does not satisfy the boundary conditions. So, put

$\lambda = \nu^2$, $\nu > 0$. Following a general procedure for solving the Sturm-Liouville problem, A solution that satisfies the zero boundary condition at the left endpoint is

$$X(x;\nu) = \sin(\nu x)$$

The eigenvalues are found from the boundary condition at the right endpoint:

$$X(l;\nu) = 0 \quad \Rightarrow \quad \sin(\nu l) = 0 \quad \Rightarrow \quad \nu = \nu_k = \frac{\pi k}{l}\,, \quad k = 1, 2, ...$$

The corresponding orthogonal eigenfunctions are

$$X(x;\nu_k) = \sin(\nu_k x)$$

They are not normalized. Since

$$\int_0^l \sin^2(\nu_k x)\, dx = \frac{l}{2}$$

the functions

$$X_k(x) = \sqrt{\frac{2}{l}}\sin(\nu_k x)\,, \quad k = 1, 2, ...$$

form an orthonormal basis:

$$\langle X_k, X_n\rangle = \int_0^l X_k(x)X_n(x)\, dx = \delta_{kn}$$

Fourier coefficients of the initial data: Given the initial data $v(x)$, its Fourier coefficients over the found Sturm-Liouville basis are given by

$$a_k = \langle v, X_k\rangle = \sqrt{\frac{2}{l}}\int_0^l v(x)\sin(\nu_k x)\, dx$$

The formal solution: The formal solution is given by the Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} V_k(t)X_k(x)$$

where the coefficients satisfy the initial value problem:

$$V_k'(t) = -\alpha^2\nu_k^2 V_k(t)\,, \quad V_k(0) = a_k$$

whose solution reads

$$V_k(t) = a_k e^{-\alpha^2\nu_k^2 t}\,.$$

Thus, the formal solution reads

$$u(x,t) = \sqrt{\frac{2}{l}}\sum_{k=1}^{\infty} a_k e^{-\alpha^2\nu_k^2 t}\sin(\nu_k x)\,. \tag{37.7}$$

Note that the solution vanishes in the limit $t \to \infty$ because all $\nu_k^2 > 0$. To illustrate how the Fourier coefficients of the initial data are computed, consider the following example.

EXAMPLE **37.1**. *Find the formal solution (***37.7***) if*

$$v(x) = \frac{4T_0}{l^2}\, x(l-x)\,,$$

*and show that $u(x,t)$ has continuous partial derivatives of any order for $(x,t) \in [0,l] \times (0,\infty)$. Is the formal solution also a classical solution?*

SOLUTION: The initial temperature distribution $v(x)$ has the maximal value $T_0$ at the midpoint $x = l/2$. The Fourier coefficients of the initial data are obtained by integration by parts

$$\begin{aligned}
a_k &= \frac{4T_0}{l^2}\sqrt{\frac{2}{l}}\int_0^l x(l-x)\sin(\nu_k x)\,dx = -\frac{4T_0}{l^2\nu_k}\sqrt{\frac{2}{l}}\int_0^l x(l-x)\,d\cos(\nu_k x) \\
&= -\frac{4T_0}{l^2\nu_k}\sqrt{\frac{2}{l}}x(l-x)\,\cos(\nu_k x)\Big|_0^l + \frac{4T_0}{l^2\nu_k}\sqrt{\frac{2}{l}}\int_0^l (l-2x)\cos(\nu_k x)\,dx \\
&= 0 + \frac{4T_0}{l^2\nu_k^2}\sqrt{\frac{2}{l}}\int_0^l (l-2x)\,d\sin(\nu_k x) \\
&= \frac{4T_0}{l^2\nu_k^2}\sqrt{\frac{2}{l}}(l-2x)\,\sin(\nu_k x)\Big|_0^l + \frac{8T_0}{l^2\nu_k^2}\sqrt{\frac{2}{l}}\int_0^l \sin(\nu_k x)\,dx \\
&= 0 - \frac{8T_0}{l^2\nu_k^3}\sqrt{\frac{2}{l}}\cos(\nu_k x)\Big|_0^l \\
&= \frac{8T_0(1-(-1)^k)}{l^2\nu_k^3}\sqrt{\frac{2}{l}}
\end{aligned}$$

Thus, the formal solution reads

$$u(x,t) = 16T_0 \sum_{k=1}^{\infty} \frac{1-(-1)^k}{l^3\nu_k^3} e^{-\alpha^2\nu_k^2 t}\sin(\nu_k x)$$

Smoothness of the formal solution: There are several ways to analyze the existence and smoothness of the formal solution. The simplest is based on Theorem **37.2**. Although its hypotheses are too restrictive, it works in this particular example. Indeed, the domain of the Sturm-Liouville operator consists of twice continuously differentiable functions in the interval $(0,l)$ and these functions are continuous at the endpoints and vanish at these points. Note that the derivatives are not involved into the boundary conditions and the existence of the derivatives at the boundary is not required. The initial data $v(x)$ is twice continuously differentiable as a polynomial and vanishes at the endpoints of the interval $[0,l]$. Therefore $v \in \mathcal{M}_L$ as required by the hypotheses of Theorem **37.2**. Thus, the formal solution exists and is the classical one.

Another, a more general, approach is based on the use of Theorems **34.2** and **34.3** which require to find upper bounds of terms of the formal solution and investigate the convergence of the series of the upper bounds. Since

$$\left|\frac{1-(-1)^k}{l^3\nu_k^3}e^{-\alpha^2\nu_k^2 t}\sin(\nu_k x)\right| \le \frac{2}{\pi^3 k^3} \quad \text{and} \quad \sum \frac{1}{k^3} < \infty$$

The Fourier series is majorated by a convergent numerical series for all $(x,t) \in \overline{\Pi}_\infty$ and, hence, the formal solution exists and is a continuous function on the closed rectangle $\overline{\Pi}_\infty$.

To investigate its differentiability, the same procedure has to be carried out for the series obtained by term-by-term differentiation, but it is sufficient to do so only for $t > 0$ (the boundary $t = 0$ is not included). If one takes the term-by-term partial derivative with respect to $t$ of oder $p$, then each term gets a factor of $\nu_k^{2p}$. Taking a partial derivative of order $q$ with respect to $x$ yields a factor $\nu_k^q$. In other words, term-by-term differentiation produces factors $\nu_k^m \sim k^m$ for some $m > 0$. Therefore the upper bounds of partial derivatives of any order decay *exponentially* for any $t \ge \varepsilon > 0$:

$$|V_k^{(p)}(t)||X_k^{(q)}(x)| \le C\nu_k^m e^{-\alpha^2\nu_k^2 t} \le \nu_k^m e^{-\alpha^2\nu_k^2\epsilon}$$

where $C$ is independent of $k$ and includes all numerical factors (like $T_0$, $l$, and $\alpha$). The numerical series the upper bounds converges for any $m$ because

$$\sum_{k=1}^{\infty} k^m e^{-ck^2} < \infty$$

for any $m$ and any $c > 0$. For example by the root test

$$\begin{aligned}\lim_{k\to\infty}\sqrt[k]{k^m e^{-ck^2}} &= \lim_{k\to\infty}\left(\sqrt[k]{k}\right)^m e^{-ck}\\ &= \left(\lim_{k\to\infty}\sqrt[k]{k}\right)^m \lim_{k\to\infty} e^{-ck}\\ &= 1\cdot 0 = 0\end{aligned}$$

Thus, by Theorem **34.3**, the sum of the Fourier series is infinitely many times differentiable for $t > 0$ because $\varepsilon > 0$ is arbitrary. Thus, the formal solution is the classical solution. □

Finally, it should be noted that in the limit $t \to \infty$, the solution (**37**.7) approaches 0, which means that the temperature of the rod approaches a constant value equal to the temperature of its ends. Moreover, for $t$ large enough but finite, the first term in the sum gives a dominant contribution (as the others terms are exponentially small in comparison to it):

$$u(x,t) \approx a_1 \Big(\frac{2}{l}\Big)^{1/2} e^{-\alpha^2\pi^2 t/l^2} \sin\Big(\frac{\pi}{l}\,x\Big)\,, \quad t \to \infty\,.$$

Since the temperature $T$ differs from $u$ by a constant $T_0$, which is the temperature of the endpoints, for after a large enough time, the rod temperature exponentially approaches a constant value $T_0$.

EXAMPLE **37**.2. *Find the formal solution, if it exists,*

$$u'_t = u''_{xx} + f\,, \quad t > 0\,,\ -1 < x < 1\,, \quad f(x,t) = (x^2-1)e^{-t}$$
$$u\Big|_{t=0} = (x+1)^2(x-1)\,, \quad u'_x\Big|_{x=-1} = u\Big|_{x=1} = 0\,.$$

*Investigate if the formal solution is classical or not in the case when* $f = 0$.

SOLUTION: It is convenient to shift the variable $y = x+1$ so that the problem can be cast in the interval $[0,2]$:

$$u'_t = u''_{yy} + f\,, \quad t > 0\,,\ 0 < y < 2\,, \quad f(y,t) = y(y-2)e^{-t}$$
$$u\Big|_{t=0} = y^2(y-2)\,, \quad u'_y\Big|_{y=0} = u\Big|_{y=2} = 0\,.$$

The associated Sturm Liouville problem is

$$-X''(y) = \lambda X(y)\,, \quad X'(0) = X(0) = 0$$

Since all eigenvalues are strictly positive for these boundary conditions, put $\lambda = \nu^2$, $\nu > 0$. The solution that satisfies the boundary condition on the left endpoint of the interval is

$$X(y;\nu) = \cos(\nu y)\,.$$

Therefore the eigenvalues are obtained from the boundary condition at the right endpoint of the interval

$$X(2;\nu) = \cos(2\nu) = 0 \quad \Rightarrow \quad \nu = \nu_k = -\frac{\pi}{4} + \frac{\pi k}{2}\,,\ k = 1,2,...$$

The orthonormal set of eigenfunctions is

$$X_k(y) = X(y;\nu_k) = \cos(\nu_k y)\,, \quad \|X_k\|^2 = \int_0^2 \cos^2(\nu_k y)dy = 1$$

The integral is evaluated by using the double angle equation and

$$\sin(4\nu_k) = 0\,, \quad \cos(2\nu_k) = 0\,, \quad \sin(2\nu_k) = (-1)^{k+1}$$

The Fourier coefficients of the initial data are evaluated using integration by parts three times:

$$\begin{aligned}
a_k &= \int_0^2 u_0(y)X_k(y)\,dy = \frac{1}{\nu_k}\int_0^2 y^2(y-2)d\sin(\nu_k y)\\
&= -\frac{1}{\nu_k}\int_0^2 \Big(2y(y-2)+y^2\Big)\sin(\nu_k y)dy\\
&= \frac{1}{\nu_k^2}\int_0^2 \Big(2y(y-2)+y^2\Big)d\cos(\nu_k y)\\
&= -\frac{1}{\nu_k^2}\int_0^2 (6y-4)\cos(\nu_k y)dy\\
&= -\frac{1}{\nu_k^3}\int_0^2 (6y-4)d\sin(\nu_k y)\\
&= \frac{8(-1)^k}{\nu_k^3} + \frac{6}{\nu_k^3}\int_0^2 \sin(\nu_k y)dy\\
&= \frac{2}{\nu_k^3}\Big(4(-1)^k + \frac{3}{\nu_k}\Big)
\end{aligned}$$

The boundary terms vanish in the first two integration by parts because $\cos(2\nu_k) = 0$. In the third integration by parts, the equations $\sin(2\nu_k) = (-1)^{k+1}$ and $\cos(2\nu_k) = 0$ were used.

The Fourier coefficients of the inhomogeneity: It is sufficient to calculate the Fourier coefficients of the function $y(y-2)$ and multiply the result by $e^{-t}$. Using the integration by part twice

$$\begin{aligned}
b_k &= \int_0^2 y(y-2)\cos(\nu_k y)dy = \frac{1}{\nu_k}\int_0^2 y(y-2)d\sin(\nu_k x)\\
&= -\frac{1}{\nu_k}\int_0^2 (2y-2)\sin(\nu_k y)dy\\
&= \frac{1}{\nu_k^2}\int_0^2 (2y-2)d\cos(\nu_k y) = -\frac{2}{\nu_k^2} - \frac{12}{\nu_k^2}\int_0^2 \cos(\nu_k y)dy\\
&= -\frac{2}{\nu_k^2}\left(1 - \frac{6(-1)^k}{\nu_k}\right)
\end{aligned}$$

Therefore

$$F_k(t) = b_k e^{-t}$$

The formal solution: One has to solve the initial value problem

$$V_k'(t) = -\nu_k^2 V_k(t) + F_k(t)\,, \quad V_k(0) = a_k$$

Its solution has the form

$$\begin{aligned} V_k(t) &= a_k e^{-\nu_k^2 t} + b_k e^{-\nu_k^2 t}\int_0^t e^{\nu_k^2\tau} e^{-\tau} d\tau \\ &= a_k e^{-\nu_k^2 t} + \frac{b_k}{\nu_k^2 - 1}\left(e^{-t} - e^{-\nu_k^2 t}\right) \end{aligned}$$

because $\nu_k^2 \neq 1$ for any $k$. The formal solution reads

$$u(x,t) = \sum_{k=1}^{\infty}\left(a_k e^{-\nu_k^2 t} + \frac{b_k}{\nu_k^2 - 1}\left(e^{-t} - e^{-\nu_k^2 t}\right)\right)\cos(\nu_k(x+1))$$

after restoring the original variable $y = x + 1$.

Existence and smoothness of the formal solution: If $f = 0$, then the formal solution is obtained from one above by setting $b_k = 0$. In this case, the smoothness of the formal solution is determined by the smoothness of the initial data $u_0(x) = (x+1)^2(x-1)$. This function is a polynomial and, hence, from the class $C^\infty$ on any interval. Therefore $u_0$ belongs to the domain of the Sturm-Liouville operator if it satisfies the required boundary condition: $u_0'(-1) = u_0(1) = 0$, which is the case. Thus, the hypotheses of Theorem **37.2** are fulfilled are the formal solution is the classical one. □

EXAMPLE **37.3**. *Solve the initial value problem for the heat equation*

$$\begin{aligned} &u_t' = u_{xx}''\,, \quad (x,t) \in (-1,1)\times(0,\infty)\,, \\ &u(x,0) = \cos^2(\pi x/2)\,, \quad x \in [-1,1]\,, \\ &u_x'(\pm 1, t) = 0\,, \quad t \geq 0\,. \end{aligned}$$

SOLUTION: It is convenient to shift the variable $y = x + 1$ so that the problem can be cast in the interval $[0,2]$:

$$\begin{aligned} &u_t' = u_{yy}''\,, \quad (y,t) \in (0,2)\times(0,\infty)\,, \\ &u\Big|_{t=0} = \cos^2[\pi(y-1)/2)\,, \quad y \in [0,2]\,, \\ &u_y'\Big|_{y=0} = u_y'\Big|_{y=2} = 0\,, \quad t \geq 0\,. \end{aligned}$$

The associated Sturm-Liouville problem is

$$-X''(y) = \lambda X(y)\,, \quad X'(0) = X'(2) = 0$$

The zero is an eigenvalue according to the general theory and the corresponding eigenfunction can be set to one:

$$\lambda = 0 \quad \Rightarrow \quad X(y;0) = 1$$

For $\lambda = \nu^2$, $\nu > 0$, a solution that satisfies the boundary condition at the left endpoint is

$$X(y;\nu) = \cos(\nu x)$$

The eigenvalues are found from the boundary condition at the right endpoint:

$$X'(2;\nu) = -\nu\sin(2\nu) = 0 \quad \Rightarrow \quad \nu = \nu_k = \frac{\pi k}{2}, \quad k = 1, 2, ...$$

The functions

$$X_0(y) = 1, \quad X_k(y) = X(y;\nu_k) = \cos(\nu_k x)$$

form an orthogonal basis (not orthonormal). One can normalize them to make an orthonormal basis. However, this is not needed for this particular problem because the initial data is a *linear combination* (not a Fourier series) of the above functions:

$$\begin{aligned}\cos^2[\pi(y-1)/2] &= \frac{1}{2} + \frac{1}{2}\cos(\pi y - \pi) = \frac{1}{2} - \frac{1}{2}\cos(\pi y) \\ &= \frac{1}{2}X_0(y) - \frac{1}{2}X_2(y)\end{aligned}$$

so that the solution must also be a linear combination of $X_0$ and $X_2$

$$u = V_0(t)X_0(y) + V_2(t)X_2(y)$$

where the coefficients are solutions to the initial value problems:

$$\begin{aligned}V_0'(t) = 0, \quad V_0(0) = \frac{1}{2} \quad &\Rightarrow \quad V_0(t) = \frac{1}{2} \\ V_2'(t) = -\nu_2^2 V_2(t), \quad V_2(0) = -\frac{1}{2} \quad &\Rightarrow \quad V_2(t) = -\frac{1}{2}\, e^{-\nu_2^2 t}\end{aligned}$$

Thus the solution reads

$$u(x,t) = \frac{1}{2} + \frac{1}{2}e^{-\pi^2 t}\cos(\pi x)$$

where the shift to the original variable was done: $X_2(y) = X_2(x+1) = -\cos(\pi x)$. □

**37.6. Exercises.**

**1**. Find a formal solution to the problem:

$$u'_t = \alpha^2 u''_{xx} + f_0 e^{-t/t_0} \sin(\pi x/l)\,, \quad (x,t) \in (0,l) \times (0,\infty)\,,$$
$$u(x,0) = Ax(l-x)\,, \quad 0 \le x \le l\,,$$
$$u(0,t) = u(l,t) = 0\,, \quad t \ge 0\,.$$

Is the formal solution a classical solution of the problem? Explain.

**2**. Find a formal solution to the problem:

$$u'_t = \alpha^2 u''_{xx}\,, \quad (x,t) \in (0,l) \times (0,\infty)\,,$$
$$u(x,0) = v(x)\,, \quad 0 \le x \le l\,,$$
$$u'_x(0,t) = 0\,, \quad u(l,t) = 0\,, \quad t \ge 0\,.$$

Give an example of a polynomial function $u_0$ such that the formal solution is a classical solution.

**3**. Find a formal solution to the problem:

$$u'_t = \alpha^2 u''_{xx}\,, \quad (x,t) \in (-l,l) \times (0,\infty)\,,$$
$$u(x,0) = Ax\,, \quad -l \le x \le l\,,$$
$$u'_x(-l,t) = 0\,, \quad u'_x(l,t) = 0\,, \quad t \ge 0\,.$$

Is the formal solution a classical one? Investigate the limit of the solution as $t \to \infty$.

**4**. Find the formal solution of the equation with periodic boundary conditions:

$$u'_t = u''_{xx} + f_0 \sin^2(\pi x) e^{-t/t_0}\,, \quad t > 0\,,$$
$$u(x,0) = 0\,, \quad u(x+1,t) = u(x,t)\,.$$

**5**. Find the formal solution to the initial and boundary value problem

$$u'_t = 4u''_{xx}\,, \quad (x,t) \in (0,1) \times (0,\infty)\,,$$
$$u(x,0) = x^2(x-1)^2\,, \quad x \in [0,2]\,,$$
$$u'_x(0,t) = u(1,t) = 0\,, \quad t \ge 0\,.$$

Is the formal solution a classical one?

**6**. Solve the initial value problem for the heat equation

$$u'_t = u''_{xx}\,, \quad (x,t) \in (-2,2) \times (0,\infty)\,,$$
$$u(x,0) = \sin^2(\pi x)\,, \quad x \in [-2,2]\,,$$
$$u'_x(\pm 2,t) = 0\,, \quad t \ge 0\,.$$

**Hints and answers to selected problems.**
**1**. The classical solution is

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} a_k e^{-\alpha^2\pi^2 t/l^2} \sin\left(\frac{\pi k}{l}\,x\right) \\ &\quad + \frac{f_0 l^2 t_0}{\pi^2\alpha^2 t_0 - l^2}\,\sin\left(\frac{\pi}{l}\,x\right)\left(e^{-t/t_0} - e^{-\alpha^2\pi^2 t/l^2}\right)\,, \\ a_{2m-1} &= \frac{8Al^2}{\pi^3(2m-1)^3}\,, \quad a_{2m} = 0\,, \quad m = 1,2,\ldots\,. \end{aligned}$$

If $t_0 = l^2/(\pi^2\alpha^2)$, then l'Hospital's rule can be used to evaluate the limit $t_0 \to l^2/(\pi^2\alpha^2)$ in the second term.

**2**. The formal solution is

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} a_k e^{-\alpha^2\lambda_k t} \sin\left(\frac{\pi(2k-1)}{2l}(x-l)\right)\,, \\ \lambda_k &= \left(\frac{\pi(2k-1)}{2l}\right)^2\,, \\ a_k &= \frac{2}{l}\int_0^l v(x)\sin\left(\frac{\pi(2k-1)}{2l}(x-l)\right)\,dx \end{aligned}$$

**3**. Shift the variable: $y = x+l$ so that $0 < y < 2l$. Restate the problem in the variable $y$. The basis is

$$X_0(x) = \frac{1}{\sqrt{2l}}\,, \quad X_k(x) = \frac{1}{\sqrt{l}}\cos(\nu_k y)\,, \quad \nu_k = \frac{\pi k}{2l}\,,\ k = 1,2,\ldots$$

where $\lambda_k = \nu_k^2$ and $\lambda_0 = 0$ are the eigenvalues of the Sturm-Liouville operator.
**4**. The orthonormal set is obtained by solving the boundary value problem with periodic boundary conditions

$$X''(x) + \lambda X(x) = 0\,, \quad X(x+1) = X(x)$$

Its solutions form a trigonometric orthonormal Fourier basis in the interval $[0,1]$:

$$X_0(x) = 1\,, \quad X_k^c(x) = \sqrt{2}\cos(2\pi kx)\,, \quad X_k^s = \sqrt{2}\sin(2\pi kx)\,,$$

where $k = 1,2,\ldots$ and $\lambda = \lambda_k = (2\pi k)^2$, $\lambda_0 = 0$. The formal solution has the same form (**37**.**4**) where $a_k$ and $F_k(t)$ are the Fourier coefficients in the above basis on the interval $[0,1]$. For example, if $f = 0$, the formal solution is

$$u(x,t) = a_0 + \sum_{k=1}^{\infty}(a_k^c + a_k^s)e^{-(2\pi k)^2 t}$$

$$a_0 = \int_0^1 u_0(x)dx\,,\;\; a_k^c = \int_0^1 u_0(x)X_k^c(x)dx\,,\;\; a_k^s = \int_0^1 u_0(x)X_k^s(x)dx$$

**5**. Follow Example **37.2**.

**6**. Follow Example **37.3**

## 38. Fourier method for 2D hyperbolic equations

**38.1. Formulation of the problem.** Let $u(x,t)$ be a function (real or complex) of two real variables $x$ and $t$. Let $L_x$ be the Sturm-Liouville operator acting on the variable $x$:

$$L_x u(x,t) = -\frac{\partial}{\partial x}\left(p(x)\frac{\partial u(x,t)}{\partial x}\right) + q(x)u(x,t)\,, \quad a < x < b\,.$$

Consider the following initial and boundary value problem in a rectangle

$$\Pi_\infty = (a,b)\times(0,\infty)$$

for a 2D hyperbolic equation:

$$(\mathbf{38.1}) \qquad \frac{\partial^2 u(x,t)}{\partial t^2} = -L_x u(x,t) + f(x,t)\,, \qquad (x,t)\in\Pi_\infty\,,$$

$$(\mathbf{38.2}) \qquad u\Big|_{t=0} = v_1(x)\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = v_2(x)\,, \quad x\in[a,b]\,,$$

$$(\mathbf{38.3}) \qquad \left(\alpha_a u - \beta_a\frac{\partial u}{\partial x}\right)\Big|_{x=a} = \left(\alpha_b u + \beta_b\frac{\partial u}{\partial x}\right)\Big|_{x=b} = 0\,, \quad t\geq 0\,,$$

The problem is to find a function $u(x,t)$ that has continuous second order partial derivatives, satisfies the equation (**38.1**) in $\Pi_\infty$, and the initial and boundary conditions, which require that $u$ has first-order partial derivative to exist at the boundary of $\Pi_\infty$. Such a function is called a *classical solution*:

$$u(x,t)\in C^2(\Pi_\infty)\cap C^1(\,\overline{\Pi}_\infty\,)$$

A classical solution does not exist for arbitrary choice of functions $f$, $u_0$, and $u_1$. It will be assumed that $f$ and $v_1$ are continuous, while $v_0$ is continuously differentiable:

$$(\mathbf{38.4}) \qquad f\in C^0(\Pi_\infty)\,, \quad u_0\in C^1[a,b]\,, \quad u_1\in C^0[a,b]\,.$$

The first condition ensures that $f$ lies in the same class as $L_x u$ and $u''_{tt}$, while the third one is required to make a solution $u(x,t)$ compatible with the second initial condition ($u'_t(x,t)$ is continuous on the boundary of $\Pi_\infty$ and, in particular, $u'_t(x,0)$ is continuous). Next, there is also a consistency condition for the initial and boundary conditions. By taking $t=0$ in the boundary conditions, it is concluded that the initial data must satisfy the boundary conditions:

$$(\mathbf{38.5}) \qquad \left(\alpha_a v_1 - \beta_a\frac{dv_1}{dx}\right)\Big|_{x=a} = \left(\alpha_l v_1 + \beta_l\frac{dv_1}{dx}\right)\Big|_{x=b} = 0\,.$$

The consistency conditions depend on $v_1$ and $v_1'$ for this reason $v_1$ is required to be from the class $C^1[a,b]$.

It is possible to prove that *if the classical solution exists, it is unique and depends continuously on the parameters* $v_1$*,* $v_2$*, and* $f$, similarly to the Cauchy problem for the corresponding parabolic equation. In other words, small variations of $v_{1,2}$ and $f$ produce small variations of the solution. This means that the Cauchy problem is well-posed. However, an accurate formulation of the continuity property is rather technical and, for this reason, is omitted.

It is argued that a two-dimensional hyperbolic equation describes a general wave process in one spatial dimension. If this is a good mathematical model for such processes, then it is natural to expect that the solution should be unique (there is only one physical wave process corresponding to given initial data) and small variations of the initial data should not lead to large variations of the solution.

**38.2. Physical significance of the problem.** Note that if $p(x) = c^2 > 0$ is constant and $q(x) = 0$, then this is the initial and boundary value problem for the wave equation that describes a motion of an elastic string of a finite length $l = b - a$ subject to an external force $f(x, t)$. If $t$ is interpreted as a physical time and $x$ as the variable labeling points of the string, then a solution $u(x, t)$ to (**38.1**) describes a general wave process in an interval (like vibrations of an elastic string). The boundary conditions have the following physical interpretation

- The condition $\beta = 0$ means that the corresponding endpoint of an elastic string is rigidly fixed.
- The condition $\alpha = 0$ means that the corresponding endpoint of an elastic string is left loose and can slide along the vertical line $x = a$ or $x = b$. Think of a string whose end point has a small loop put in a vertical rod, and the loop can slide up and down without any resistance or friction.
- If $\alpha$ and $\beta$ do not vanish simultaneously, then the corresponding endpoint of an elastic string is not rigidly fixed but it cannot move freely either as there is a force acting in the direction opposite to the motion of the endpoint (one can think of a spring attached to the endpoint of the string).

If, in addition, the physical properties of the string are not uniform (its mass distribution and elastic properties depend on position), then these effect can be accommodated by a non-constant function $p(x)$.

**38.3. Formal solution to the homogeneous equation.** Suppose that

$$f(x,t) = 0\,.$$

The corresponding equation is called a *homogeneous* hyperbolic equation. It can formally be solved by separating variables as follows.

Let us look for a solution in the form

$$u(x,t) = V(t)X(x)$$

Then separating variables

$$V''(t)X(x) = -V(t)L_xX(x)$$
$$\frac{V''(t)}{V(t)} = -\frac{L_xX(x)}{X(x)}$$

from which it follows that $X$ must be an eigenfunction of the Sturm-Liouville operator

$$L_xX(x) = \lambda X(x)\,,$$

while the function $V$ satisfies the equation

$$V''(t) + \lambda V(t) = 0\,.$$

Let $X_k(x)$, $k = 1, 2, ...$, be a real orthonormal set of eigenfunctions of the Sturm-Liouville operator and $\lambda_k \geq 0$ be the corresponding eigenvalues (recall Theorem **36.4**). Since the eigenfunctions form an orthogonal basis, the Fourier series of the initial data converge at least in the mean (see Theorems **36.5** and **36.6**)

$$u_0(x) = \sum_{k=0}^{\infty} a_kX_k(x)\,, \quad a_k = \langle v, X_k\rangle = \int_0^l v_1(x)X_k(x)\,dx$$
$$u_1(x) = \sum_{k=0}^{\infty} b_kX_k(x)\,, \quad b_k = \langle v_2, X_k\rangle = \int_0^l v_2(x)X_k(x)\,dx$$

Suppose that the initial data $v_1$ and $v_2$ are replaced by the corresponding partial sums of their Fourier series:

$$v_1(x) \to v_{1n}(x) = \sum_{k=1}^{n} a_kX_k(x) \tag{38.6}$$

$$v_2(x) \to v_{2n}(x) = \sum_{k=1}^{n} b_kX_k(x)\,. \tag{38.7}$$

In this case, the initial data $v_{1n}(x)$ and $v_{2n}(x)$ satisfy all the conditions for the classical solution to exist because $X_k \in C^2[0, l]$ (see Theorem **36.4**, Part (iv)) and satisfy the boundary conditions. As noted before, the classical solution is unique if it exists. Since $L_x$ maps any linear combination of eigenfunctions into a linear combinations of the same eigenfunctions, the classical solution exists and has the form

$$u_n(x,t) = \sum_{k=1}^{n} V_k(t) X_k(x)\,, \tag{38.8}$$

where the expansion coefficients are to be determined. The substitution of this expansion into the equation yields

$$\sum_{k=1}^{n} V_k''(t) X_k(x) = -\sum_{k=1}^{n} \lambda_k V_k(t) X_k(x)\,.$$

Owing to the linear independence of the functions $X_k(x)$, the equality is only possible if the coefficients in the right and left side match:

$$V_k''(t) = -\lambda_k V_k(t)\,, \quad t > 0\,, \quad k = 1, 2, ..., n\,.$$

The general solution of this equation can be written in the form

$$\begin{aligned} V_k(t) &= A_k \cos\Big(\sqrt{\lambda_k}\, t\Big) + \frac{B_k}{\sqrt{\lambda_k}} \sin\Big(\sqrt{\lambda_k}\, t\Big)\,, \quad \lambda_k > 0 \\ V_1(t) &= A_1 + B_1 t \quad \text{if } \lambda_1 = 0\,, \end{aligned}$$

where $A_k$ and $B_k$ are constants. The reason not to include the factor $\lambda_k^{-1/2}$ into an arbitrary constant $B_k$ in the second term is the following. Recall that $\lambda_k$ form an unbounded increasing sequence and $\lambda_1$ may or may not be zero. If $\lambda_1 = 0$, then the corresponding solution $V_1(t)$ can be obtained by taking the limit $\lambda_1 \to 0^+$ of the solution with $\lambda_1 > 0$:

$$A_1 + B_1 t = \lim_{\lambda_1 \to 0^+} A_1 \cos\Big(\sqrt{\lambda_1}\, t\Big) + \lim_{\lambda_1 \to 0^+} \frac{B_1}{\sqrt{\lambda_1}} \sin\Big(\sqrt{\lambda_1}\, t\Big)$$

In what follows, the case when $\lambda_1 = 0$ will not be considered separately because, whenever necessary, it can be obtained by the above limiting procedure.

Using the initial conditions

$$\begin{aligned} u_n(x,0) = v_{1n}(x) \quad &\Rightarrow \quad \sum_{k=1}^{n} V_k(0) X_k(x) = \sum_{k=1}^{n} a_k X_k(x) \\ \frac{\partial u_n(x,t)}{\partial t}\Big|_{t=0} = u_{1n}(x) \quad &\Rightarrow \quad \sum_{k=1}^{n} V_k'(0) X_k(x) = \sum_{k=1}^{n} b_k X_k(x) \end{aligned}$$

it is concluded by linear independence of the basis functions $X_k(x)$ that the constants $A_k$ and $B_k$ satisfy the system of linear equations:

$$\begin{aligned} V_k(0) &= A_k = a_k \\ V_k'(0) &= B_k = b_k\,. \end{aligned}$$

Thus, the classical solution for every $n$ is given by the sum

$$(\mathbf{38.9}) \quad u_n(x,t) = \sum_{k=1}^{n}\Big(a_k\cos(\sqrt{\lambda_k}t) + \frac{b_k}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t)\Big)X_k(x)\,,$$

$$a_k = \langle u_0, X_k\rangle = \int_0^l u_0(x)X_k(x)\,dx\,,$$

$$b_k = \langle u_1, X_k\rangle = \int_0^l u_1(x)X_k(x)\,dx\,.$$

If one takes a formal limit $n \to \infty$, then if the parameters $v_{1,2}$ are smooth enough (recall Steklov's theorems), then their Fourier series would converge to their values, one one might expect that the sequence of classical solutions $u_n(x,t)$ would converge to a solution of the original Cauchy problem. This is in general not so because the limit function may not be smooth enough. But the formal limit

$$(\mathbf{38.10}) \qquad u(x,t) = \sum_{k=1}^{\infty}\Big(a_k\cos(\sqrt{\lambda_k}t) + \frac{b_k}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t)\Big)X_k(x)$$

is called a *formal solution* to the Cauchy problem. Its existence and smoothness can be investigated by means of basic theorems for convergence of functional series.

**38.4. Formal and classical solutions.** It is possible to formulate *sufficient* conditions on the initial data so that the formal solution is a classical solution. The result was established by V.A. Steklov in 1922.

THEOREM **38.1**. (Formal and classical solutions)
*Let $\mathcal{M}_L$ be the domain of the Sturm-Liouville operator $L$ in the hyperbolic initial and boundary value problem (**38.1**)–(**38.3**). Suppose that*

$$v_1 \in \mathcal{M}_L\,, \quad Lv_1 \in \mathcal{M}_L\,, \quad v_2 \in \mathcal{M}_L\,.$$

*Then the Fourier series (**38.10**) is the classical solution to the problem and moreover the sum of (**38.10**) is from the class $C^2(\overline{\Pi}_\infty)$.*

The second condition is highly restrictive. Even if it is not fulfilled the formal solution can be proved to be a classical one by simpler means based on estimates of the upper bounds of terms of the formal solution as stated in Theorems **34.2** and **34.3**.

EXAMPLE **38.1**. *Find the formal solution to the initial and boundary value problem for the following homogeneous hyperbolic (wave) equation*

$$u''_{tt} = u''_{xx}\,, \quad 0 < x < 1\,,\ t > 0\,,$$
$$u(x,0) = 0\,, \quad u'_t(x,0) = x(1-x)\,, \quad 0 \le x \le 1\,,$$
$$u(0,t) = 0\,, \quad u(1,t) = 0\,, \quad t \ge 0\,.$$

*Investigate whether the formal solution given by the Fourier series is also the classical solution to the problem.*

SOLUTION: The associated Sturm-Liouville problem reads

$$L_x X_k = -X''_k = \lambda_k X_k\,, \quad X_k(0) = X_k(1) = 0$$

The corresponding eigenvalues and orthonormal eigenfunctions are:

$$X_k(x) = \sqrt{2}\sin(\nu_k x)\,, \quad \nu_k = \pi k\,, \quad \lambda_k = \nu_k^2\,, \quad k = 1, 2, \ldots$$

Fourier coefficients of the initial data: In this case, $v_1(x) = 0$ and $v_2(x) = x(1-x)$ so that

$$\begin{aligned} a_k &= \langle u_0, X_k\rangle = 0\,,\\ b_k &= \langle u_1, X_k\rangle = \sqrt{2}\int_0^1 x(1-x)\sin(\nu_k x)\,dx\\ &= \frac{\sqrt{2}}{\nu_k}\int_0^1 (1-2x)\cos(\nu_k x)\,dx\\ &= \frac{2\sqrt{2}}{\nu_k^3}\Big((-1)^k - 1\Big)\,, \end{aligned}$$

where the integration by parts is used twice to calculate the integral and the boundary terms vanish each time. Note that $b_k = 0$ for even $k$.

The formal solution is given by the Fourier series

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \frac{b_k}{\nu_k}\sin(\nu_k t)X_k(x)\\ &= \sum_{k=1}^{\infty} \frac{4}{\nu_k^4}\Big((-1)^k - 1\Big)\sin(\nu_k t)\sin(\nu_k x)\,. \end{aligned}$$

Smoothness of the formal solution. The simplest way to check if the formal solution is a classical one is verify in the initial data fulfill the hypotheses of Steklov's theorem. The domain of the associated Sturm-Liouville operator consists of functions that have continuous second derivatives in $(0, 1)$ and vanish at the endpoints. Evidently, $v_1(x) = 0$ and $Lv_1(x) = -v''_1(x) = 0$ are from the domain. The function $v_2(x) =$

$x^2(1-x)$ is differentiable any number of times because it is a polynomial in $(0,1)$, and $v_2(0) = v_2(1) = 0$. So, $v_2$ is from the domain. By Steklov's Theorem **38.1**, the formal solution is the classical one and has continuous second partial derivatives even at the boundary of the rectangle $[0,1] \times [0,\infty)$.

Another, a more general, way is to use Theorems **34.2** and **34.3**. The series is majorated by a convergent numerical series. Indeed

$$|b_k| \le \frac{2\sqrt{2}}{\nu_k^3}|(-1)^k - 1| \le \frac{4\sqrt{2}}{\nu_k^3}$$

and since $|\sin(y)| \le 1$ for all real $y$, it is concluded that

$$\left|\frac{b_k}{\nu_k}\sin(\nu_k t)X_k(x)\right| \le \frac{|b_k|\sqrt{2}}{\nu_k} \le \frac{8}{\nu_k^4} = M_k$$

and the series

$$\sum_{k=1}^{\infty} M_k = \frac{8}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4} < \infty$$

converges as a $p-$series with $p = 4$. Recall that the series $\sum_k k^{-p}$ converges for all reap $p > 1$. By Theorem **34.2**, the Fourier series converges and the sum is a continuous function on $[0,1]\times[0,\infty)$ because the terms of the series are continuous.

By carrying out term-by-term differentiation of the Fourier series two *formal* relations are obtained:

$$u_t'(x,t) \sim \sum_{k=1}^{\infty} b_k\sqrt{2}\cos(\nu_k t)\sin(\nu_k x)\,,$$
$$u_x'(x,t) \sim \sum_{k=1}^{\infty} b_k\sqrt{2}\sin(\nu_k t)\cos(\nu_k x)\,,$$

The series are majorated by a convergent numerical $p-$series for all $0 \le x \le 1$ and all $t \ge 0$. Indeed,

$$\sqrt{2}|b_k\cos(\nu_k t)\sin(\nu_k x)| \le \sqrt{2}|b_k| \le \frac{8}{\pi^3}\,\frac{1}{k^3} \quad \Rightarrow \quad \sum_{k=1}^{\infty}\frac{1}{k^3} < \infty\,,$$

and similarly for the other functional series. Thus, by Theorem **34.3** the partial derivatives of the formal solution are continuous in $[0,1]\times[0,\infty)$ and the sign $\sim$ can be replaced by the equality sign (the partial derivatives can be obtained by term-by-term differentiation of the Fourier series).

Finally, the *formal* second partial derivatives are given by the series

$$u''_{tt}(x,t) \sim -\sqrt{2}\sum_{k=1}^{\infty} b_k\nu_k \sin(\nu_k t)\sin(\nu_k x) \sim u''_{xx}(x,t)\,,$$

$$u''_{tx}(x,t) \sim \sqrt{2}\sum_{k=1}^{\infty} b_k\nu_k \cos(\nu_k t)\cos(\nu_k x) \sim u''_{xt}(x,t)\,.$$

Note that the $u''_{tt}$ and $u''_{xx}$ have the same Fourier series, which is not a surprise because partial sums of a Fourier series are classical solutions to the wave equation. The above series are also majorated by a convergent $p-$series for all $0 \le x \le 1$ and $t \ge 0$ because

$$|b_k\nu_k \sin(\nu_k t)\sin(\nu_k x)| \le |b_k|\nu_k = \frac{4\sqrt{2}}{\nu_k^2} = \frac{4\sqrt{2}}{\pi^2}\,\frac{1}{k^2} \quad \Rightarrow \quad \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$$

and similarly for the other series. By Theorem **34.3**, the second partial derivatives are continuous in $\overline{\Pi}_\infty = [0,l]\times[0,\infty)$ and are given by the corresponding term-by-term differentiation of the Fourier series representing the formal solution to the wave equation. Thus, the formal solution given by the Fourier series is from the class $C^2(\overline{\Pi}_\infty)$, satisfies the wave equation, the initial conditions, and the boundary conditions. Therefore it is the classical solution to the problem. □

EXAMPLE **38.2**. *Find the formal solution to the mixed problem for the following homogeneous hyperbolic (wave) equation and show the existence of the formal solution.*

$$\begin{aligned}
&u''_{tt} = u''_{xx}\,, \quad 0 < x < 1\,,\ t > 0\,,\\
&u(x,0) = x^2(1-x)\,, \quad u'_t(x,0) = 0\,, \quad 0 \le x \le 1\,,\\
&u'_x(0,t) = 0\,, \quad u(1,t) = 0\,, \quad t \ge 0\,.
\end{aligned}$$

SOLUTION: The associated Sturm-Liouville problem:

$$LX(x) = -X''(x) = \lambda X(x)\,, \quad X'(0) = 0\,, \quad X(1) = 0\,,$$

The zero, $\lambda = 0$, is not an eigenvalue of the Sturm-Liouville operator because in the boundary conditions $\alpha_0 = 1 \ne 0$. Since $\lambda > 0$, put $\nu = \sqrt{\lambda}$. The general solution of the equation is

$$X(x) = A\cos(\nu x) + B\sin(\nu x)$$

A solution that satisfies the boundary condition at the left point is

$$X(x;\nu) = \cos(\nu x)$$

The eigenvalues are found from the second boundary condition:

$$X(1;\nu) = \cos(\nu) = 0 \quad \Rightarrow \quad \nu = \nu_k = \pi(k - \tfrac{1}{2})\,, \quad k = 1, 2, ...$$

Since

$$\int_0^1 \cos^2(\nu_k x)dx = \frac{1}{2}$$

the corresponding orthonormal eigenfunctions are

$$X_k(x) = \sqrt{2}\cos(\nu_k x)\,.$$

Fourier coefficients of the initial data: Next, one has to calculate the Fourier coefficient of the initial data

$$\begin{aligned} a_k &= \langle v_1, X_k\rangle = \sqrt{2}\int_0^1 x^2(1-x)\cos(\nu_k x)\,dx \\ &= \sqrt{2}\Big(I_k(2) - I_k(3)\Big) \\ I_k(s) &= \int_0^1 x^s\cos(\nu_k x)dx \end{aligned}$$

while $b_k = 0$ because $u_1(x) = 0$. Using integration by part twice, the following recurrence relation for the integrals $I_k(s)$ can be inferred:

$$\begin{aligned} I_k(s) &= \frac{\sin(\nu_k)}{\nu_k} - \frac{s}{\nu_k}\int_0^1 x^{s-1}\sin(\nu_k x)dx \\ &= \frac{(-1)^{k+1}}{\nu_k} - \frac{s(s-1)}{\nu_k^2}\int_0^1 x^{s-2}\cos(\nu_k x)\,dx \\ &= \frac{(-1)^{k+1}}{\nu_k} - \frac{s(s-1)}{\nu_k^2}\,I_k(s-2) \end{aligned}$$

where $s \geq 2$. It allows one to reduce the integral for any positive integer $s \geq 2$ to one of the following two integrals:

$$\begin{aligned} I_k(s) &= I_k(0) - \frac{s(s-1)}{\nu_k^2}\,I_k(s-2)\,, \quad s \geq 2\,, \\ I_k(0) &= \int_0^1 \cos(\nu_k x)dx = \frac{\sin(\nu_k)}{\nu_k} = \frac{(-1)^{k+1}}{\nu_k}\,, \\ I_k(1) &= \int_0^1 x\cos(\nu_k x)dx = \frac{\sin(\nu_k)}{\nu_k} - \frac{1}{\nu_k}\int_0^1 \sin(\nu_k x)dx \\ &= I_k(0) - \frac{1}{\nu_k^2}\,, \end{aligned}$$

where the relation $\cos(\nu_k) = 0$ has been used to compute the last integral. Therefore

$$\begin{aligned}
I_k(2) &= I_k(0) - \frac{2}{\nu_k^2}\, I_k(0) = I_k(0)\left(1 - \frac{2}{\nu_k^2}\right) \\
I_k(3) &= I_k(0) - \frac{6}{\nu_k^2}\, I_k(1) = I_k(0)\left(1 - \frac{6}{\nu_k^2}\right) + \frac{6}{\nu_k^4} \\
a_k &= \sqrt{2}\Big(I_k(2) - I_k(3)\Big) = \sqrt{2}\left(\frac{4I_k(0)}{\nu_k^2} - \frac{6}{\nu_k^4}\right) \\
&= \frac{4\sqrt{2}}{\nu_k^3}\left((-1)^{k+1} - \frac{3}{2\nu_k}\right).
\end{aligned}$$

The formal solution is given by the Fourier series

$$\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty} a_k \cos(\nu_k t) X_k(x) \\
&= \sum_{k=1}^{\infty} \frac{8}{\nu_k^3}\left((-1)^{k+1} - \frac{3}{2\nu_k}\right)\cos(\nu_k t)\cos(\nu_k x).
\end{aligned}$$

The existence of the formal solution: Here $v_1$ and $v_2$ are from the domain of the Sturm-Liouville operator, but $Lv_1(x) = -v_1''(x) = -2 + 6x$ is not because it does not satisfy the boundary conditions. Therefore Steklov's theorem cannot be used.

The terms of the series are bounded by

$$\begin{aligned}
&|a_k \cos(\nu_k t) X_k(x)| \le \sqrt{2}|a_k| \\
&|a_k| \le \frac{4\sqrt{2}}{\nu_k^3}\left(1 + \frac{3}{2\nu_k}\right) \le \frac{4\sqrt{2}}{\nu_k^3}, \\
&\sum_k \frac{1}{\nu_k^3} < \infty
\end{aligned}$$

By Theorem **34.2** the formal solution exists for all $(x,t) \in [0,1] \times [0,\infty)$ and is a continuous function. A term-by-term differentiation of the series would produce an extra factor $\nu_k$ in each term of the series:

$$\begin{aligned}
u_t'(x,t) &\sim -\sqrt{2}\sum_{k=1}^{\infty} a_k \nu_k \sin(\nu_k t)\cos(\nu_k x), \\
u_x'(x,t) &\sim -\sqrt{2}\sum_{k=1}^{\infty} a_k \nu_k \cos(\nu_k t)\sin(\nu_k x).
\end{aligned}$$

So, the terms are bounded by $C/\nu_k^2$ where $C$ is independent of $k$. The series $\sum 1/\nu_k^2 < \infty$ converges. Therefore the formal solution has continuous partial derivatives by Theorem **34.3**. Unfortunately, Theorem **34.3** does not allow to conclude that the second derivatives are continuous too. For example,

$$u''_{tt}(x,t) \sim -\sqrt{2}\sum_{k=1}^{\infty} \nu_k^2 a_k \cos(\nu_k t)\cos(\nu_k x) \sim u''_{xx}(x,t)$$

The use of the inequality $\sqrt{2}|a_k| \le 8/\nu_k^3$ shows that the above functional series is majorated by a divergent numerical series: $\sqrt{2}\nu_k^2|a_k| \le 8/\nu_k$ and $\sum_k(1/\nu_k) = \infty$. So, whether the formal solution is classical or not remains to be investigated. More elaborate tests are needed. □

**Remark.** In the above example, the term-by-term differentiation two times led to a functional series of a special type whose terms are the product of an oscillating function and a monotonically decreasing one, like $\sum_k \cos(\nu_k x)/\nu_k$. The convergence of such series can be studied by the *Dirichlet-Abel test*. However, technicalities associated with applications of the test to formal solutions of hyperbolic equations go beyond the scope of this course.

**38.5. Formal solution of a non-homogeneous problem.** The Fourier method allows us to obtain a formal solution to the problem (**38.1**)–(**38.3**). The Fourier coefficients of the external force $f(x,t)$ are

$$F_k(t) = \langle f, X_k\rangle = \int_a^b f(x,t)X_k(x)\,dx\,.$$

Let us replace the initial data by the truncated Fourier series as displayed in (**38.6**) and (**38.7**). Similarly, the external force is replaced by its truncated Fourier series:

$$f(x,t) \quad \to \quad f_n(x,t) = \sum_{k=1}^{n} F_k(t)X_k(x)\,. \tag{38.11}$$

Suppose that $f(x,t)$ is continuous on $\overline{\Pi}_\infty = [a,b]\times[0,\infty)$ so that $F_k(t)$ are continuous functions for $t \ge 0$. The corresponding classical solution to the problem has the form (**38.8**) where the expansion coefficients satisfy the initial value problem

$$V_k''(t) + \lambda_k V_k(t) = F_k(t)\,, \quad k = 1, 2, ..., n\,,$$
$$V_k(0) = a_k\,, \quad V_k'(0) = b_k$$

This initial value problem can be solved, for instance, by the Laplace transform method or by the method of variation of parameters. Put $\nu_k = \sqrt{\lambda_k}$. Then the solution has the form

$$V_k(t) = a_k \cos(\nu_k t) + \frac{b_k}{\nu_k} \sin(\nu_k t) + \frac{1}{\nu_k} \int_0^t F_k(\tau) \sin\Big(\nu_k (t-\tau)\Big)\, d\tau\,. \tag{38.12}$$

If the smallest eigenvalue of the Sturm-Liouville operator is zero, $\lambda_1 = 0$ or $\nu_1 = 0$, then the corresponding function $V_1(t)$ is obtained by taking the limit $\nu_1 \to 0^+$ in (**38.12**) so that $\cos(\nu_1 t) \to 1$ and $\sin(\nu_1 t)/\nu_1 \to t$. The first two terms in (**38.12**) give the solution of the corresponding homogeneous problem, while the last term is a particular solution that satisfies the zero initial conditions.

The continuity of $F_k$ guarantees that the particular solution is twice continuously differentiable for all $t > 0$. The reader is advised to verify this by differentiating the particular solution twice. Thus, for every $n$, the solution given by (**38.8**) and (**38.12**) is a classical solution. The formal solution is then given by the Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} V_k(t) X_k(x)$$

The existence (or convergence) and smoothness of the formal solution can be studied by means of Theorems **34.2** and **34.3**.

EXAMPLE **38.3**. *Use the Fourier method to find the formal solution describing the forced vibrations of an elastic string of length $l$ with fixed endpoints if the string was initially at rest and the force*

$$f(x,t) = f_0 x(l-x)\sin(\omega t)\,,$$

*where $f_0 \geq 0$ and $\omega > 0$ are constants. Investigate whether the formal solution is also the classical solution.*

SOLUTION: The problem is to find a formal solution to the Cauchy problem:

$$\begin{aligned}
&c^{-2} u''_{tt}(x,t) = u''_{xx}(x,t) + f(x,t)\,, \quad 0 < x < l\,, \quad t > 0\,,\\
&u(x,0) = u'_t(x,0) = 0\,, \quad 0 \leq x \leq l\,,\\
&u(0,t) = u(l,t) = 0\,, \quad t \geq 0
\end{aligned}$$

where a constant $c$ depends on elastic properties of the string (one can always set $c = 1$ if one agrees to measure the speed in units of $c$).

The associated Sturm-Liouville problem

$$-X'' = \lambda X\,, \quad X(0) = X(l) = 0$$

has the solution

$$\lambda = \lambda_k = \nu_k^2\,, \quad \nu_k = \frac{\pi k}{l}\,, \quad X_k(x) = \sqrt{\frac{2}{l}}\sin(\nu_k x)\,, \quad k = 1, 2, ...$$

where $X_k$ are orthonormal,

$$\langle X_k, X_m\rangle = \frac{2}{l}\int_0^l \sin(\nu_k x)\sin(\nu_m x)\,dx = \delta_{km}\,.$$

The Fourier coefficients of the external force are

$$\begin{aligned} F_k(t) &= \langle f, X_k\rangle = \sqrt{\frac{2}{l}}\int_0^l f(x,t)\sin(\nu_k x)dx \\ &= \sqrt{\frac{2}{l}}\, f_0\sin(\omega t)\int_0^l x(l-x)\sin(\nu_k x)\,dx \\ &= \sqrt{\frac{2}{l}}\, f_0\sin(\omega t)\,\frac{2}{\nu_k^2}\int_0^l \sin(\nu_k x)dx \\ &= f_0\sin(\omega t)\,\frac{2\sqrt{2}}{\nu_k^3\sqrt{l}}\Big(1-(-1)^k\Big) \end{aligned}$$

where the integration by parts has been done twice to calculate the integral. The Fourier coefficients vanish for even $k = 2m$, $m = 1, 2, ....$

The formal solution. The expansion coefficients in the formal solution satisfy the initial value problem

$$c^{-2}V_k''(t) + \nu_k^2 V_k(t) = F_k(t)\,, \quad V_k(0) = V_k'(0) = 0\,.$$

because the the string was initially at rest, $u(x,0) = u_t'(x,0) = 0$. Its solution is given by the convolution integral in (**38.12**):

$$V_k(t) = \frac{c^2}{\omega_k}\int_0^t F_k(\tau)\sin\Big(\omega_k(t-\tau)\Big)d\tau\,, \qquad \omega_k = c\nu_k\,.$$

The integral is evaluated with the help of the trigonometric identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

so that

$$(\mathbf{38.13})\quad \int_0^t \sin(\omega t)\sin\Big(\omega_k(t-\tau)\Big)\,d\tau = \frac{\omega\sin(\omega_k t)-\omega_k\sin(\omega t)}{\omega^2-\omega_k^2}$$

$$V_k(t) = \frac{4\sqrt{2}cf_0}{\sqrt{l}\nu_k^4}\Big(1-(-1)^k\Big)\frac{\omega\sin(\omega_k t)-\omega_k\sin(\omega t)}{\omega^2-\omega_k^2}$$

provided $\omega \neq \omega_k$. If $\omega$ coincides with one of $\omega_k$, then the value of the integral is still obtained by means of the above trigonometric identity. It can also be found by taking the limit $\omega \to \omega_k$ using l'Hospital's rule:

$$\lim_{\omega\to\omega_k}\frac{\omega\sin(\omega_k t)-\omega_k\sin(\omega t)}{\omega^2-\omega_k^2} = \lim_{\omega\to\omega_k}\frac{\sin(\omega_k t)-t\omega_k\cos(\omega t)}{2\omega} = \frac{\sin(\omega_k t)}{2\omega_k}-\frac{t}{2}\cos(\omega_k t)\,.$$

In this case, the amplitude of the corresponding vibrational mode of the string grows linearly with time and can reach *any* value no matter how small the amplitude $f_0$ of the applied force. This phenomenon is knows as a *resonance*. It will be discussed in the next section in detail.

Existence and smoothness of the formal solution. Suppose first that $\omega$ does not coincide with any of $\omega_k = c\nu_k$. In this case

$$\sqrt{\frac{2}{l}}\,|V_k(t)| \leq \frac{8c\omega f_0}{l\nu_k^4}\frac{|\omega\sin(\omega_k t)-\omega_k\sin(\omega t)|}{|\omega^2-\omega_k^2|} \leq \frac{8c\omega f_0}{l\nu_k^4}\frac{\omega+\omega_k}{|\omega^2-\omega_k^2|} = \frac{8c\omega f_0}{l\nu_k^4}\frac{1}{|\omega-\omega_k|}\,.$$

For large values of $k$ so that $\omega_k$ is much larger than the constant $\omega$,

$$\sqrt{\frac{2}{l}}\,|V_k(t)| \sim \frac{\text{const}}{k^5}\,,\qquad k\to\infty\,,$$

for all $t \geq 0$. This shows that the Fourier series representing the formal solution is majorated by a $p-$series:

$$|\sum_k V_k(t)X_k(x)| \leq \sum_k |V_k(t)||X_k(x)| = \sqrt{\frac{2}{l}}\sum_k |V_k(t)| \leq C\sum_k \frac{1}{k^5}$$

for some constant $C$ and all $t \geq 0$ and $x \in [0,l]$ because

$$|X_k(x)| = \sqrt{\frac{2}{l}}\,|\sin(\nu_k x)| \leq \sqrt{\frac{2}{l}}\,.$$

Thus, by Theorem **34.2** the formal solution exists and is a function continuous in the rectangle $[0,l]\times[0,\infty)$.

The continuity of partial derivatives can be established by Theorem **34.3**. Differentiation of the term $\sin(\omega t)$ in $V_k(t)$ produces the factor $\omega$ which does not affect the behavior of $|V_k(t)|$ as $k \to \infty$. But differentiation of the term $\sin(\omega_k t)$ produces the factor $\omega_k$ that grows with increasing $k$. Therefore for large $k$, this term gives a dominant contribution to $V_k'(t)$ and $V_k''(t)$. Each differentiation gives an extra factor $\omega_k \sim k$. Similarly each differentiation of $X_k(x)$ produces the factor $\nu_k \sim k$. Therefore the series of the first partial derivatives is majorated by the $p-$series with $p = 5 - 1 = 4$, while the series of second partial derivatives is majorated by the $p-$series with $p = 5 - 2 = 3$. For example,

$$|V_k'(t)X_k'(x)| \sim \frac{\text{const}}{k^3}, \quad k \to \infty .$$

Therefore the series obtained by taking term-by-term first and second partial derivatives of the Fourier series are majorated by a convergent numerical series for all points in the rectangle $\overline{\Pi}_\infty = [0, l] \times [0, \infty)$. By Theorem **34.3** the sum of the Fourier series has continuous second partial derivatives in $\overline{\Pi}_\infty$ and, hence, is the classical solution to the problem.

If $\omega = \omega_m$ for some $m$, then the analysis of the convergence does not change because the convergence is determined by the behavior of $V_k(t)$ for large values of $k$ which does not change. In other words, only the convergence of the series made of terms with $k > m$ has to be studied which is the same as in the previous case. Thus, the Fourier series gives the classical solution to the problem in this case, too. □

**38.6. The resonance phenomenon revisited.** The resonance phenomenon has already been discussed with an example of a vibrating string using d'Alembert's formula for solving the wave equation. Let us analyze this phenomenon by the Fourier method as the latter allows us also to extend the analysis to more general hyperbolic equations.

The solution of the wave equation describing forced vibrations of an elastic string under a periodic external force obtained in Example **38.3** has interesting feature: If the frequency $\omega$ of the external force coincide with one of the eigenfrequencies $\omega_k = c\nu_k$, then the amplitude $V_k(t)$ of the corresponding vibrational eigenmode of the string increases linearly with increasing time,

$$V_k(t) = C_k \left( \frac{\sin(\omega_k t)}{2\omega_k} - \frac{t}{2} \cos(\omega_k t) \right) .$$

while as shown in Example **38.3**, all other $V_k(t)$ are bounded for all $t \geq 0$. Since $u(x, t)$ represent deviation of the string from its equilibrium shape at a position $x$ and time $t$, a physical string is going to break as any physical string has a threshold value for a maximal stretching. Note if $x$ does not coincides with a node of $X_m(x) = \sin(\nu_m x)$, then $u(x,t)$ oscillates about 0 with ever increasing amplitude, producing an unbounded stretching of the string with increasing time $t$:

$$|u(x,t)| \approx |V_k(t)||X_k(x)| \approx \frac{C_k|t|}{2}|\cos(\omega_k t)|\sin(\nu_k x)| \quad \text{as} \quad t \to \infty$$

The value of $|u(x,t)|$ grow arbitrary large and this is going to happen no matter how small the amplitude $f_0$ of the applied force is! One says that *the applied force is in resonance with one of the vibrational modes of a string.*

- How general is this phenomenon?
- How to suppress a potential resonance?

The answer to the first question is that any vibrational process described by a hyperbolic (wave) equation may have a resonance. For example, a person jumping up and down on a bridge with one of the eigenfrequencies of the bridge may collapse the bridge. There is a reason to believe that soldiers marching in step caused the collapse of the Angers Bridge (France) in 1850. A famous collapse of the Tacoma Narrows Bridge was collapsed in 1940 due to *aeroelastic flutter* which can be attributed to forces caused by wind resonating with "*swinging*" modes of the bridge. The motion of pedestrians walking over the London Thames Millennium Bridge, that was open in 2000, happened to be in resonance with one of vibrational eigenmode of the bridge, and this caused the bridge to vibrate with an increasing amplitude. This problem was fixed in the next two years. How?

A general approach to suppress potential resonances is to introduce damping into a vibrating system, that is, a mechanism that damps mechanical energy into heat. In other words, if a vibrating system is left along it eventually looses its energy and comes to rest. A simple mathematical model of a vibrating system with damping is described by the (*telegraph*) equation

$$u''_{tt} + 2\gamma u'_t = -Lu + f\,, \quad 0 < x < l\,, \quad t > 0$$

where $\gamma > 0$ is a damping constant and $L$ is a Sturm-Liouville operator in an interval $(0, l)$. Here $c = 1$ as compared with Example **38.3**. Note that the no damping case $\gamma = 0$ corresponds to a general hyperbolic (wave) equation. Suppose that zero in not an eigenvalue of $L$. Physically, this means that a vibrating system cannot move as a whole. Put

$$f(x,t) = g(x)\sin(\omega t)$$

If $X_k(x)$ are eigenfunctions of $L$ and $\lambda_k = \nu_k > 0$ are the corresponding eigenvalues, then the expansion coefficients in the Fourier series for the formal solution

$$u(x,t) = \sum_{k=1}^{\infty} V_k(t) X_k(x)$$

satisfy the initial value problem

$$V_k''(t) + 2\gamma V_k'(t) + \nu_k^2 V_k(t) = g_k \sin(\omega t)\,, \quad V_k(0) = V_k'(0) = 0\,,$$

where $g_k$ are the Fourier coefficients $g(x)$:

$$g_k = \langle g, X_k \rangle\,.$$

The zero initial conditions mean that the system is initially at rest and it is set into motion by the periodic external force $f$. As Example **38.3** shows, the function $g(x)$ can always be chosen smooth enough so that the Fourier coefficients would decrease fast enough to ensure that the formal solution given by the Fourier series is the classical solution.

Suppose first that the system has no damping, $\gamma = 0$. Then the solution has the same form as in Example **38.3** ($\omega_k = \nu_k$ as $c = 1$):

$$V_k(t) = \frac{g_k}{\nu_k} \frac{\omega \sin(\nu_k t) - \nu_k \sin(\omega t)}{\omega^2 - \nu_k^2}\,, \quad \gamma = 0\,,\ \omega \neq \nu_k\,.$$

If the frequency of the external force is in resonance with one of the vibrational eigenmodes, $\omega = \nu_k$ for some $k$, then

$$V_k(t) = \frac{g_k}{2\nu_k} \left( \frac{\sin(\nu_k t)}{\nu_k} - t\,\cos(\nu_k t) \right)\,, \quad \gamma = 0\,,\ \omega = \nu_k\,.$$

This shows that the resonance phenomenon exists in a general wave equation. As soon as the frequency of the external force matches one of the eigenfrequencies of a vibrating system, the amplitude of the corresponding mode grows unboundedly with increasing time.

Suppose now that the damping parameter is small so that $0 < \gamma < \nu_1$. The solution to the initial value problem can be obtained by the Laplace transform method:

$$V_k(t) = \frac{g_k}{\Omega_k} \int_0^t e^{-\gamma\tau} \sin(\Omega_k \tau) \sin\Big(\omega(t-\tau)\Big) d\tau\,, \quad \gamma < \nu_k\,,$$

where $\Omega_k = \sqrt{\nu_k^2 - \gamma^2}$. An explicit form is obtained by evaluation of the integral

$$V_k(t) = e^{-\gamma t}\Big(a_k \cos(\Omega_k t) + \frac{b_k}{\Omega_k}\sin(\Omega_k t)\Big) + A_k\cos(\omega t) + B_k \sin(\omega t)\,,$$
$$A_k = -\frac{2\gamma\omega g_k}{(\omega^2-\nu_k^2)^2 + (2\gamma\omega)^2}\,, \quad a_k = -A_k\,,$$
$$B_k = -\frac{(\omega^2-\nu_k^2)g_k}{(\omega^2-\nu_k^2)^2 + (2\gamma\omega)^2}\,, \quad b_k = -B_k\omega\Big(1 + \frac{2\gamma^2}{\omega^2-\nu_k^2}\Big)\,.$$

Note that the two terms, those with the exponential factor $e^{-\gamma t}$ and without it, are a solution of the homogeneous equation and a particular solution of the non-homogeneous equation. This form could be used to find the solution by the method of undetermined coefficients instead of the Laplace method. The solution without damping is recovered if one sets $\gamma = 0$ in the above solution. The effect of the damping is twofold. First the eigenfrequencies are shifted $\Omega_k = \sqrt{\nu_k^2 - \gamma^2} < \nu_k$ as compared with the undamped case. Second, the amplitude of all vibration eigenmodes decays exponentially. When $t$ is much larger than the characteristic damping time $1/\gamma$ so that $e^{-\gamma t} \approx 0$, vibrations of all eigenmodes disappear and only forced vibrations of frequency $\omega$ remain. They have bounded amplitudes. The largest value of $A_k$ is attained at $\omega_k = \nu_k$ so that

$$V_k(t) \sim -\frac{g_k}{2\gamma\nu_k}\cos(\nu_k t)\,, \quad \omega = \nu_k\,,\ t \to \infty\,.$$

Thus, the resonance is not possible when a vibrating system has damping. Note, however, that the asymptotic amplitude is inversely proportional to the damping parameter $\gamma$. So, if $\gamma$ is too small, the amplitude can still be large enough to cause a break down of realistic vibrational systems (like bridges). In practice, one would want to create a damping mechanism in a real system whose parameter $\gamma$ is close to the frequencies of eigenmodes that can most likely be in resonance with external forces. Ideally, one wants these potentially dangerous modes to be *over damped*.

If $\gamma > \nu_k$ for some $k$, then the corresponding mode is *over damped* and for $\omega = \nu_k < \gamma$ (resonance) its amplitude decays exponentially without oscillations:

$$V_k(t) = \frac{g_k}{\Gamma_k}\int_0^t e^{-\gamma\tau}\sinh(\Gamma_k\tau)\sin\Big(\omega(t-\tau)\Big)d\tau\,, \quad \gamma > \nu_k\,,$$

where $\Gamma_k = \sqrt{\gamma^2 - \nu_k^2}$. This expression is obtained from the previous one by the substitution

$$\Omega_k = \sqrt{\nu_k^2 - \gamma^2} = \sqrt{-(\gamma^2 - \nu_k^2)} = i\Gamma_k$$

and the Euler formula:

$$\frac{\sin(\Omega_k t)}{\Omega_k} = \frac{e^{it\Omega_k} - e^{-it\Omega_k}}{2i\Omega_k} = \frac{e^{t\Gamma_k} - e^{-t\Gamma_k}}{2\Gamma_k} = \frac{\sinh(\Gamma_k t)}{\Gamma_k} .$$

An explicit form of the integral can also be obtained from the case $\gamma < \nu_k$ by the substitution $\Omega_k = i\Gamma_k$ so that

$$\frac{\sin(\Omega_k t)}{\Omega_k} = \frac{\sinh(\Gamma_k t)}{\Gamma_k}, \quad \cos(\Omega_k t) = \cosh(\Gamma_k t)$$

in the expression for $V_k(t)$, while the constants $A_k$, $B_k$, $a_k$, and $b_k$ have the same form. Since $\gamma \pm \Gamma_k > 0$, oscillations of all the eigenmodes with $\nu_k < \gamma$ are exponentially suppressed (they are over damped). In other words, the modes dye out without a singe oscillation. Physically, one can say that over damped modes would not contribute to vibrations of the system (e.g., of a bridge).

If $\gamma = \nu_k$ for some $k$, then for this mode $\Omega_k = \Gamma_k = 0$ and the corresponding amplitude can be obtained by taking the limit $\Omega_k \to 0$ or $\Gamma_k \to 0$:

$$V_k(t) = g_k \int_0^t e^{-\gamma\tau} \tau \sin\Big(\omega(t-\tau)\Big) d\tau , \quad \gamma = \nu_k .$$

Note that $\sin(\Omega_k t)/\Omega_k \to t$ as $\Omega_k \to 0$. The explicit form of $V_k(t)$ is obtained from the case $\gamma < \nu_k$ by taking the limit $\Omega_k \to 0$ so that

$$\frac{\sin(\Omega_k t)}{\Omega_k} \to t , \quad \cos(\Omega_k t) \to 1 ,$$

while $\gamma = \nu_k$ in the coefficients $A_k$, $B_k$, $b_k$, and $a_k$ in the expression for $V_k(t)$.

In either of these cases, the expansion coefficients $V_k(t)$ remains bounded for all $t \geq 0$ and any $\omega$. For example, for *under damped* eigenmodes:

$$\begin{aligned} |V_k(t)| &\leq \frac{|g_k|}{\Omega_k} \int_0^t e^{-\gamma\tau} \Big| \sin(\Omega_k \tau) \sin\Big(\omega(t-\tau)\Big) \Big| d\tau \\ &\leq \frac{|g_k|}{\Omega_k} \int_0^t e^{-\gamma\tau} d\tau = \frac{|g_k|}{\Omega_k} \frac{1 - e^{-\gamma t}}{\gamma} \leq \frac{|g_k|}{\Omega_k \gamma} \end{aligned}$$

for all $t \geq 0$. The other two cases can be analyzed similarly. The boundedness of the amplitudes follows from the convergence of the integrals of $e^{-\gamma t}\sinh(\Gamma_k t) \geq 0$ and $e^{-\gamma t}t \geq 0$ over $[0,\infty)$. The needed calculations are left to the reader as an exercise.

**38.7. Example with explicitly unknown eigenfrequencies.** For general boundary conditions, the eigenfrequencies of a vibrating string are roots of a transcendental equation which cannot be found explicitly. Nevertheless, the formal solution can still be obtained and its smoothness can be studied. The following example illustrates some basic techniques.

EXAMPLE **38.4**. *Find the formal solution to the problem*

$$\begin{aligned} &u''_{tt} = u''_{xx} + f_0\cos(\omega t)\,, \quad 0 < x < 1\,,\ t > 0\,,\\ &u(x,0) = 0\,, \quad u'_t(x,0) = g_0 x\,, \quad 0 \leq x \leq 1\,,\\ &u(0,t) = u'_x(0,t)\,, \quad u'_x(1,t) = 0\,, \quad t \geq 0\,, \end{aligned}$$

*where $f_0$, $\omega$, and $g_0$ are constants, which describes forced vibrations of an elastic string with one end rigidly fixed, while the other end is loose. Investigate the smoothness of the formal solution.*

SOLUTION: The associated Sturm-Liouville problem: Since $\alpha_0 = 1 > 0$, zero is not an eigenvalue. Put $\lambda = \nu^2 > 0$, $\nu > 0$. The eigenvalue problem reads

$$-X''(x) = \nu^2 X(x)\,, \quad X(0) = X'(0)\,, \quad X'(1) = 0\,.$$

To solve it, the method of Section **36.6** is used. The function

$$X_1(x;\nu) = \cos(\nu x)$$

is the solution of this equation satisfying the initial conditions $X_1(0;\nu) = 1$ and $X_1'(0;\nu) = 0$. The functions

$$X_2(x;\nu) = \frac{\sin(\nu x)}{\nu}$$

is the solution of this equation satisfying the initial conditions $X_2(0;\nu) = 0$ and $X_1'(0;\nu) = 1$. Then the combination (**36.8**),

$$X(x;\nu) = \cos(\nu x) + \frac{\sin(\nu x)}{\nu}\,,$$

satisfies the first boundary condition. The eigenvalues are solutions of the equation (**36.9**):

$$X'(1;\nu) = 0 \quad \Rightarrow \quad -\nu\sin(\nu) + \cos(\nu) = 0 \quad \Rightarrow \quad \tan(\nu) = \frac{1}{\nu}$$

By graphing the tangent function and $1/\nu$, it is clear that the equation has one root in each interval

$$\pi(k-1) < \nu_k < \frac{\pi}{2} + \pi(k-1)\,, \quad k = 1, 2, ...$$

Asymptotically $k \to \infty$, the roots $\nu_k$ are approaching $\frac{\pi}{2} + \pi k$ from the right. The orthogonal (*not orthonormal*) eigenfunctions are

$$X_k(x) = \sin(\nu_k x + \varphi_k)\,, \quad \sin(\varphi_k) = \frac{\nu_k}{\sqrt{\nu_k^2 + 1}}\,, \quad \cos(\varphi_k) = \frac{1}{\sqrt{\nu_k^2 + 1}}$$

Note that the function $X(x;\nu_k)$ is proportional to $X_k(x)$ and, hence, the latter are orthogonal just as $X(x;\nu_k)$. The normalization coefficient of the eigenfunction is not relevant for the Fourier method. By the trigonometric identities,

$$\begin{aligned} X(x;\nu_k) &= \cos(\nu_k x) + \frac{\sin(\nu_k x)}{\nu_k} \\ &= \frac{\sqrt{\nu_k^2+1}}{\nu_k}\Big(\sin(\varphi_k)\cos(\nu_k x) + \cos(\varphi_k)\sin(\nu_k x)\Big) \\ &= \frac{\sqrt{\nu_k^2+1}}{\nu_k}\sin(\nu_k x + \varphi_k) = \frac{\sqrt{\nu_k^2+1}}{\nu_k}\, X_k(x) \end{aligned}$$

There are a few useful properties of the eigenfunctions that can be deduced from simple trigonometry and the condition $\tan(\nu_k) = 1/\nu_k$:

$$\begin{aligned} \frac{X_k'(1)}{\nu_k} &= \cos(\nu_k + \varphi_k) = 0\,, \\ X_k(1) &= \sin(\nu_k + \varphi_k) = (-1)^{[\frac{k}{2}]} \end{aligned}$$

where $[y]$ denotes the integer part of $y$. Note that the first relation is merely the boundary condition, while the second one follows from the positions of the roots $\nu_k$ at which the signs of $\sin(\nu_k)$ and $\cos(\nu_k)$ are the same have the periodic pattern $+,-,-,+,+,-,-,+,+,\cdots$. The norm of the eigenfunctions is

$$\begin{aligned} \|X_k\|^2 &= \langle X_k, X_k\rangle = \int_0^1 \sin^2(\nu_k x + \varphi_k)dx \\ &= \frac{1}{2} - \frac{1}{2}\int_0^1 \cos(2\nu_k x + 2\varphi_k)dx \\ &= \frac{1}{2} - \frac{1}{4\nu_k}\Big(\sin(2\nu_k + 2\varphi_k) - \sin(2\varphi_k)\Big) \\ &= \frac{1}{2} + \frac{\sin(\varphi)\cos(\varphi_k)}{2\nu_k} = \frac{\nu_k^2 + 2}{2(\nu_k^2+1)}\,, \end{aligned}$$

where the property $\cos(\nu_k + \varphi_k) = 0$ has been used. In the asymptotic region, $k \to \infty$, $\nu_k \to \infty$, and $\|X_k\|^2 \to \frac{1}{2}$.

The Fourier coefficients of the initial data. The functions $X_k(x)/\|X_k\|$

form an orthonormal basis. Integrating by parts and using the property $\cos(\nu_k + \varphi_k) = 0$, one infers that

$$
\begin{aligned}
b_k &= \frac{\langle u_1, X_k\rangle}{\|X_k\|} = \frac{g_0}{\|X_k\|}\int_0^1 x\sin(\nu_k x + \varphi_k)dx \\
&= \frac{g_0}{\nu_k\|X_k\|}\int_0^1 \cos(\nu_k x + \varphi_k)dx \\
&= \frac{g_0}{\nu_k^2\|X_k\|}\Big(\sin(\nu_k + \varphi_k) - \sin(\varphi_k)\Big) \\
&= \frac{g_0}{\nu_k^2\|X_k\|}\left((-1)^{[\frac{k}{2}]} - \frac{\nu_k}{\sqrt{\nu_k^2+1}}\right).
\end{aligned}
$$

The Fourier coefficients of the external force are

$$
\begin{aligned}
F_k(t) &= \frac{\langle f, X_k\rangle}{\|X_k\|} = \frac{f_0}{\|X_k\|}\sin(\omega t)\int_0^1 \sin(\nu_k x + \varphi_k)dx \\
&= \frac{f_0}{\|X_k\|}\,\sin(\omega t)\,\frac{\cos(\varphi_k) - \cos(\nu_k + \varphi_k)}{\nu_k} \\
&= \frac{f_0}{\|X_k\|\nu_k\sqrt{\nu_k^2+1}}\,\sin(\omega t)\,,
\end{aligned}
$$

where the property $\cos(\nu_k + \varphi_k) = 0$ has again been used.

The formal solution to the problem is given by the Fourier series

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty} V_k(t)\frac{X_k(x)}{\|X_k\|} = \sum_{k=1}^{\infty} T_k(t)\sin(\nu_k x + \varphi_k) \\
T_k(t) &= \frac{b_k}{\nu_k\|X_k\|}\sin(\nu_k t) + \frac{1}{\|X_k\|\nu_k}\int_0^t F_k(\tau)\sin\Big(\nu_k(t-\tau)\Big)\,d\tau \\
&= \frac{b_k}{\nu_k\|X_k\|}\sin(\nu_k t) + \frac{f_0}{\|X_k\|^2\nu_k^2\sqrt{\nu_k^2+1}}\cdot\frac{\omega\sin(\nu_k t) - \nu_k\sin(\omega t)}{\omega^2 - \nu_k^2}\,.
\end{aligned}
$$

The integral in $T_k(t)$ is calculated in Example **38.3**.

The existence of the formal solution. The smoothness of the formal solution can be studied by examining the asymptotic behavior of $|T_k(t)|$ for large $k$:

$$
|T_k(t)\sin(\nu_k x + \phi_k)| \le |T_k(t)| \le M_k\,, \quad (x,t) \in \overline{\Pi}_T
$$

If $\sum M_k < \infty$, then the formal solution is continuous. Using the basic inequalities $|A \pm B| \leq |A| + |B|$, $|\sin\theta| \leq 1$, and $|\cos\theta| \leq 1$, one has

$$\begin{aligned}
|T_k(t)| &\leq \left|\frac{b_k}{\nu_k\|X_k\|}\sin(\nu_k t)\right| + \\
&\quad + \frac{|f_0|}{\|X_k\|\nu_k^2\sqrt{\nu_k^2+1}} \cdot \frac{|\omega\sin(\nu_k t) - \nu_k\sin(\omega t)|}{|\omega^2-\nu_k^2|} \\
&\leq \frac{b_k}{\nu_k\|X_k\|} + \frac{|f_0|}{\|X_k\|^2\nu_k^2\sqrt{\nu_k^2+1}}\frac{\omega+\nu_k}{|\omega^2-\nu_k^2|} \\
&= \frac{b_k}{\nu_k\|X_k\|} + \frac{|f_0|}{\|X_k\|^2\nu_k^2\sqrt{\nu_k^2+1}}\frac{1}{|\omega-\nu_k|} \\
&\leq \frac{2|g_0|}{\nu_k^3\|X_k\|^2} + \frac{|f_0|}{\|X_k\|^2\nu_k^2\sqrt{\nu_k^2+1}}\frac{1}{|\omega-\nu_k|} = M_k
\end{aligned}$$

For large values of $k$,

$$\nu_k \sim k \quad \Rightarrow \quad \|X_k\|^2 \sim \frac{1}{2}$$

Therefore

$$\begin{aligned}
&\frac{2|g_0|}{\nu_k^3\|X_k\|^2} \sim \frac{C_1}{k^3}, \\
&\frac{|f_0|}{\|X_k\|^2\nu_k^2\sqrt{\nu_k^2+1}}\frac{1}{|\omega-\nu_k|} \sim \frac{C_2}{k^4} \\
&\Rightarrow \quad \sum_{k=1}^{\infty} M_k < \infty
\end{aligned}$$

Thus the formal solution exists and is a continuous function by Theorem **34.2**. □

**38.8. Exercises.**

**1**. Give an example of polynomial initial data $u_0$ and $u_1$ of the least degree that satisfy the hypotheses of Theorem **38.1** if $p = 1$ and $q = 0$ and $u(0,t) = u(l,t) = 0$, $t \geq 0$.

**2**. Give an example of polynomial initial data $u_0$ and $u_1$ of the least degree that satisfy the hypotheses of Theorem **38.1** if $p = 1$ and $q = 0$ and $u'_x(0,t) = u'_x(l,t) = 0$, $t \geq 0$.

**3**. Give an example of polynomial initial data $u_0$ and $u_1$ of the least degree that satisfy the hypotheses of Theorem **38.1** if $p = 1$ and $q = 0$ and $u(0,t) = u'_x(0,t)$ and $u'_x(l,t) = 0$, $t \geq 0$.

**4**. Find the formal solution to the problem

$$u''_{tt} = u''_{xx}\,, \quad 0 < x < 1\,,\ t > 0\,,$$
$$u(0,t) = u(1,t) = 0\,,\ t \geq 0\,,$$
$$u(x,0) = u_0(x) = x^2(1-x)^2\,,$$
$$u'_t(x,0) = u_1(x) = 2x^2(1-x) - 2x(1-x)^2\,,\ 0 \leq x \leq 1\,.$$

Use Theorems **34.2** and **34.3** to investigate the convergence of the Fourier series and its smoothness. Is the formal solution also the classical solution? *Hint*: to simplify computation of the Fourier coefficients, note that $u_1(x) = u'_0(x)$

**5**. Use Theorem **38.1** to show that the formal solution to the following problem coincides with the classical solution and find it:

$$c^{-2}u''_{tt} = u''_{xx}\,, \quad (x,t) \in (0,l) \times (0,\infty)\,,$$
$$u(x,0) = 0\,, \quad u'_t(x,0) = Ax^2(l-x)\,, \quad 0 \leq x \leq l\,,$$
$$u'_x(0,t) = 0\,, \quad u(l,t) = 0\,,\ t \geq 0\,.$$

**6**. Find a formal solution to the problem

$$u''_{tt} = u''_{xx} + x\cos(\omega\pi t/2)\,, \quad (x,t) \in (0,1) \times (0,\infty)\,,$$
$$u(x,0) = x^2(1-x)\,, \quad u'_t(x,0) = x, \quad 0 \leq x \leq l\,,$$
$$u'_x(0,t) = 0\,, \quad u(1,t) = 0\,, \quad t \geq 0\,.$$

**7**. Find a formal solution to the problem

$$u''_{tt} = u''_{xx} + x(1-x)\cos(\omega 2\pi t)\,, \quad (x,t) \in (0,1) \times (0,\infty)\,,$$
$$u(x,0) = 0\,, \quad u'_t(x,0) = x, \quad 0 \leq x \leq l\,,$$
$$u'_x(0,t) = 0\,, \quad u'_x(1,t) = 0\,, \quad t \geq 0\,.$$

**8**. Find formal solutions to the problem

$$u''_{tt} = u''_{xx}\,, \quad (x,t) \in (0,1) \times (0,\infty)\,,$$
$$u(x,0) = 0\,, \quad u'_t(x,0) = u_1(x), \quad 0 \leq x \leq 1\,,$$
$$u'_x(0,t) = 0\,, \quad u'_x(1,t) = 0\,, \quad t \geq 0\,,$$

where $u_1(x) = 1$, $u_1(x) = \frac{1}{2} - |x - \frac{1}{2}|$, $u_1(x) = x(1-x)$, and $u_1(x) = x^2(1-x)^2$ Investigate and compare the asymptotic behavior of the Fourier coefficients in the formal solutions. Is any of the formal solutions a classical one?

**9**. A closed vibrating string of length $l$ can be modeled by periodic boundary conditions. Find a formal solution to the periodic initial value problem

$$\begin{aligned} &u''_{tt} = c^2 u''_{xx} + f(x)\cos(\omega t)\,, \quad (x,t) \in \mathbb{R} \times (0,\infty)\,, \\ &u(x,0) = u_0(x)\,, \quad u'_t(x,0) = u_1(x), \quad 0 \le x \le l\,, \\ &u(x+l,t) = u(x,t)\,, \quad t \ge 0\,. \end{aligned}$$

where $f$, $u_0$, and $u_1$ are continuous periodic functions with period $l$. In particular, let $u_0(x) = u_1(x) = 0$ and $f(x) = x(l-x)$, $0 \le x \le l$. Is the formal solution a classical one for any value of $\omega$? Next, consider the periodic extension of the initial data $u_{0,1}(x+l) = u_{0,1}(x)$. Verify the periodicity of the solution given by d'Alembert's formula for periodic initial data.

**10**. Find a formal solution to the following problem:

$$\begin{aligned} &u''_{tt} + 2u'_t = u''_{xx} + f(x,t)\,, \quad (x,t) \in (0,1) \times (0,\infty)\,, \\ &u(x,0) = 0\,, \quad u'_t(x,0) = 0, \quad 0 \le x \le 1\,, \\ &u(0,t) = 0\,, \quad u(1,t) + u'_x(0,t) = 0\,, \quad t \ge 0\,. \end{aligned}$$

if $f(x,t) = x$ if $t \in [0,1]$ and $f(x,t) = 0$ if $t > 1$.

**Selected answers and hints.**
**1**. $u_0(x) = x^p(l-x)^q$, $q,p > 2$; $u_1(x) = x^n(l-x)^m$, $n,m > 0$
**2**. $u_0(x) = x^p(l-x)^q$, $q,p > 3$; $u_1(x) = x^n(l-x)^m$, $n,m > 1$
**5**. The orthonormal basis for the problem

$$X_k(x) = \left(\frac{2}{l}\right)\cos(\nu_k x)\,, \quad \nu_k = \frac{\pi}{2l} + \frac{\pi k}{l}\,, \quad k = 0,1,2,...$$

The answer follows from the procedure of Sec. **??** with $f = 0$ and $u_1 = Ax^2(l-x)$.
**6**. The orthonormal basis for this problem is the same in Problem 5 where $l = 1$ The answer follows from the procedure of Sec. **??** with $f = x\cos(\pi\omega t/2)$, $u_0(x) = x^2(1-x)$, and $u_1 = x$. Note that the Fourier coefficients of $f$ are proportional to those of $u_1$, and the Fourier coefficients of $u_0$ can also be extracted from a solution to Problem 5.
**9**. The orthonormal basis:

$$X_k^s(x) = \left(\frac{2}{l}\right)^{1/2}\sin(\nu_k x)\,, \quad X_k^c(x) = \left(\frac{2}{l}\right)^{1/2}\cos(\nu_k x)\,, \quad X_0(x) = \left(\frac{1}{l}\right)^{1/2}$$

where $\nu_k = 2\pi k/l$, $k = 1, 2, \ldots.$ The formal solution

$$u(x,t) = A_0(t)X_0(x) + \sum_{k=1}^{\infty} \Big( A_k(t)X_k^c(x) + B_k(t)X_k^s(x) \Big)$$

where the Fourier coefficients are solutions to the initial value problems (they can be found above in the text of the section)

$$A_k'' + \omega_k^2 A_k = F_n^c \cos(\omega t)\,,\ F_n^c = \int_0^l f(x)X_n^c(x)\,dx$$

$$A_k(0) = \int_0^l u_0(x)X_k^c(x)dx\,,\ A_k'(0) = \int_0^l u_1(x)X_k^c(x)dx\,,$$

$$B_k'' + \omega_k^2 B_k = F_n^s \cos(\omega t)\,,\ F_n^s = \int_0^l f(x)X_n^s(x)\,dx$$

$$B_k(0) = \int_0^l u_0(x)X_k^s(x)dx\,,\ B_k'(0) = \int_0^l u_1(x)X_k^s(x)dx\,,$$

where $\omega_k = c\nu_k$; and for $A_0(t)$, put $k = 0$ in the first equation.

## 39. Laplace equation in rectangles

**39.1. The internal Dirichlet problem in a rectangle.** It is convenient to shift the variables so that a solution to the Laplace equation is sought in the rectangle $\Omega = (0,a) \times (0,b)$. The problem is to find a harmonic function that has prescribed values on the boundary of the rectangle:

$$\begin{aligned} &u''_{xx} + u''_{yy} = 0\,, \quad (x,y) \in \Omega \\ &u(0,y) = v_0(y)\,, \quad u(a,y) = v_a(y)\,, \quad 0 \le y \le b \\ &u(x,0) = h_0(x)\,, \quad u(x,b) = h_b(x)\,, \quad 0 \le x \le b \end{aligned}$$

The letter $v$ is used for vertical boundary data, and $h$ denotes the horizontal boundary data. The solution is required to be continuous along the boundary $\partial\Omega$ (to be from the class $C^2(\Omega) \cap C^0(\bar{\Omega})$). For this reason, the boundary values should satisfy the consistency conditions:

$$\begin{aligned} u(0,0) &= v_0(0) = h_0(0)\,, \\ u(0,b) &= v_0(b) = h_b(0)\,, \\ u(a,0) &= v_a(0) = h_0(a)\,, \\ u(a,b) &= v_a(b) = h_b(a)\,. \end{aligned}$$

**Remark.** If the consistency conditions are not possible to meet, the Fourier method introduced below can still be applied but the convergence of the formal solution at the corners of the rectangle is generally lost. This problem is known as corner singularities of the formal solution. A further discussion of corner singularities will be given later.

**General idea for solving by the Fourier method.** The solution can always be written as the sum

$$u(x,y) = u_h(x,y) + u_v(x,y)$$

where $u_v(x,y)$ is the solution to the associated problem in which *the vertical boundary data are set to zero*:

$$u_v(0,y) = v_0(y) = 0\,, \quad u_v(a,y) = v_a(y) = 0$$

while $u_h(x,y)$ is the solution of the associated problem in which *the horizontal boundary data are set to zero*:

$$u_h(x,0) = h_0(x) = 0\,, \quad u_h(x,b) = h_b(y) = 0$$

The sum $u_v + u_h$ is a harmonic function that satisfies the required boundary conditions of the original Dirichlet problem.

Let us separate variables in the first (vertical) associated problem

$$u_v(x,y) = \tilde{Y}(y)X(x)$$

Then the zero boundary condition on the vertical lines $x = 0$ and $x = a$ requires that

$$X(0) = X(a) = 0$$

and the Laplace equation becomes:

$$\frac{X''(x)}{X(x)} + \frac{\tilde{Y}''(y)}{\tilde{Y}(y)} = 0$$

Therefore $X$ must be a solution to the eigenvalue problem for the Sturm-Liouville operator with the Dirichlet type boundary conditions:

$$L_x X(x) = -X''(x) = \lambda X(x)\,, \qquad X(0) = X(a) = 0\,.$$

whereas the function $\tilde{Y}$ is a solution to the equation

$$\tilde{Y}''(y) - \lambda \tilde{Y}(y) = 0\,.$$

The eigenfunctions

$$X(x) = X_n(x)\,, \quad \lambda = \lambda_n\,,\ n = 1, 2, ...$$

form an orthogonal basis in the interval $[0, a]$. Therefore a formal solution can be written as a Fourier series

$$u_v(x, y) = \sum_{n=1}^{\infty} \tilde{Y}_n(y) X_n(x)$$

The horizontal boundary conditions requires that

$$u_v(x, 0) = h_0(x) = \sum_{n=1}^{\infty} \tilde{Y}_n(0) X_n(x)$$

$$u_v(x, b) = h_b(x) = \sum_{n=1}^{\infty} \tilde{Y}_n(b) X_n(x)$$

By expanding the boundary data into the Fourier series

$$h_0(x) = \sum_{n=1}^{\infty} h_{0n} X_n(x)\,, \quad h_{0n} = \frac{\langle h_0, X_n\rangle}{\|X_n\|^2}\,,$$

$$h_b(x) = \sum_{n=1}^{\infty} h_{bn} X_n(x)\,, \quad h_{bn} = \frac{\langle h_b, X_n\rangle}{\|X_n\|^2}$$

and comparing the expansions with boundary conditions it is concluded that the functions $\tilde{Y}_n$ are solutions to the boundary value problem

$$\tilde{Y}_n''(y) - \lambda_n \tilde{Y}_n(y) = 0\,, \quad \tilde{Y}_n(0) = h_{0n}\,, \quad \tilde{Y}_n(b) = h_{bn}$$

If this problem has a solution, then it is unique. Indeed, by linearity of the equation the difference of any two solutions $\tilde{Y}_n = \tilde{Y}_{1n} - \tilde{Y}_{2n}$ also satisfies the equation and zero boundary conditions:

$$\tilde{Y}_n''(y) - \lambda_n \tilde{Y}_n(y) = 0\,, \quad \tilde{Y}_n(0) = \tilde{Y}_n(b) = 0$$

This implies that $-\lambda_n < 0$ is an eigenvalue of the Sturm-Liouville operator $L_y = -\frac{d^2}{dy^2}$ with the Dirichlet boundary conditions, which was proved to have only strictly positive eigenvalues. Therefore, $\tilde{Y}_n = 0$. The unique solution to the boundary value problem will be found explicitly below. Thus, the obtained Fourier series is a formal solution to the problem. Of course, the convergence of the series must be investigated to show the existence of the formal solution. This investigation will be carried out later showing the formal solution does exist.

**Remark.** A useful analogy to the above procedure is a separation of variables in polar coordinates for a Dirichlet problem in a disk. The basis $X_n$ is an analog of the trigonometric Fourier harmonics, whereas the functions $\tilde{Y}_n$ are analogous to the coefficients in the formal solution given by the trigonometric Fourier series that satisfy the boundary value problem for the Cauchy-Euler equation.

The solution to the second (horizontal) associated problem can be found in a similar fashion because the roles of $x$ and $y$ are swapped. The solution is a formal Fourier series

$$u_h(x, y) = \sum_{n=1}^{\infty} \tilde{X}_n(x) Y_n(y)$$

over the basis of eigenfunctions of the Sturm-Liouville operator in the $y$ variable:

$$L_y Y_n(y) = -Y_n''(y) = \lambda_n Y_n(y)\,, \quad Y(0) = Y(b) = 0$$

where the expansion coefficients are solutions to the boundary value problems

$$\tilde{X}_n'' - \lambda_n \tilde{X}_n = 0\,, \quad \tilde{X}_n(0) = \frac{\langle v_0, Y_n\rangle}{\|Y_n\|^2}\,, \quad \tilde{X}_n(a) = \frac{\langle v_a, Y_n\rangle}{\|Y_n\|^2}\,.$$

**39.2. Formal solution to the Dirichlet problem.**

Step 1: Two associate eigenvalue problems. Let us solve the two associate Sturm-Liouville problems (for the vertical and horizontal operators, $L_x$

and $L_y$). In this case, the horizontal eigenvalue problem is

$$L_x X = -X''(x) = \lambda X(x)\,, \qquad X(0) = X(a) = 0$$

$$\Rightarrow \quad \lambda = \nu_k^2\,,\ \nu_k = \frac{\pi k}{a}\,, \quad X(x) = X_k(x) = \sin(\nu_k x)\,,\ k = 1, 2, ...$$

$$\langle X_k, X_n\rangle = \int_0^a X_n(x) X_k(x) dx = \frac{a}{2}\,\delta_{kn}$$

The functions $X_k$ form an orthogonal basis in the space of continuous functions in the interval $[0, a]$. Similarly, the vertical eigenvalue problem is

$$L_y Y = -Y''(y) = \lambda Y(y)\,, \qquad Y(0) = Y(b) = 0$$

$$\Rightarrow \quad \lambda = \mu_k^2\,,\ \mu_k = \frac{\pi k}{b}\,, \quad Y(y) = Y_k(y) = \sin(\mu_k x)\,,\ k = 1, 2, ...$$

$$\langle Y_k, Y_n\rangle = \int_0^b Y_n(y) Y_k(y) dy = \frac{b}{2}\,\delta_{kn}$$

The functions $Y_k$ form an orthogonal basis in the space of continuous functions in the interval $[0, b]$.

Step 2: The first associate boundary value problem. Let us set the boundary data in the original problem to zero on the vertical lines:

$$v_0(y) = v_a(y) = 0\,, \quad 0 \le y \le b\,.$$

The horizontal boundary data are expanded into the Fourier series over the "horizontal" basis $X_n$:

$$h_0(x) = \sum_{n=1}^{\infty} h_{0n} X_n(x)\,, \quad h_{0n} = \frac{2}{a}\int_0^a h_0(x) X_n(x)\,dx$$

$$h_b(x) = \sum_{n=1}^{\infty} h_{bn} X_n(x)\,, \quad h_{bn} = \frac{2}{a}\int_0^a h_b(x) X_n(x)\,dx$$

Then the formal solution has the form

$$u_v(x, y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x)\,,$$

where the expansion coefficients $\tilde{Y}_k(y)$ satisfies the boundary value problem

$$L_y \tilde{Y}_k + \nu_k^2 \tilde{Y}_k = -\tilde{Y}_k''(y) + \nu_k^2 \tilde{Y}_k(y) = 0\,, \quad 0 < y < b$$

$$\tilde{Y}_k(0) = h_{0k}\,, \quad \tilde{Y}_k(b) = h_{bk}$$

The equation has two linearly independent solutions $e^{\pm\nu_k y}$ or $\sinh(\nu_k y)$ or $\cosh(\nu_k y)$. A general solution is their linear combination. One always choose two linearly independent solutions so that one of them vanishes at one end of the interval, while the other vanishes at the other end:

$$\tilde{Y}_1(y) = \sinh(\nu y)\,, \quad \tilde{Y}_1(0) = 0\,,$$
$$\tilde{Y}_2(y) = \sinh(\nu(b-y))\,, \quad \tilde{Y}_2(b) = 0$$

Recall that

$$\sinh(\nu b - \nu y) = \sinh(\nu b)\cosh(\nu y) - \cosh(\nu b)\sinh(\nu y)$$

so that $\tilde{Y}_1$ and $\tilde{Y}_2$ are not proportional to one another and, hence, are linearly independent. Thus, the general solution has the form

$$\tilde{Y}_k(y) = A_k \sinh(\nu_k y) + B_k \sinh(\nu_k(b-y))\,.$$

The boundary conditions yield

$$\begin{aligned}
\tilde{Y}_k(0) = h_{0k} \quad &\Rightarrow \quad B_k \sinh(\nu_k b) = h_{0k} \quad \Rightarrow \quad B_k = \frac{h_{0k}}{\sinh(\nu_k b)} \\
\tilde{Y}_k(b) = h_{bk} \quad &\Rightarrow \quad A_k \sinh(\nu_k b) = h_{bk} \quad \Rightarrow \quad A_k = \frac{h_{bk}}{\sinh(\nu_k b)} \\
\tilde{Y}_k(y) &= h_{bk}\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} + h_{0k}\frac{\sinh(\nu_k(b-y))}{\sinh(\nu_k b)} \\
u_v(x,y) &= \sum_{k=1}^{\infty}\left(h_{bk}\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} + h_{0k}\frac{\sinh(\nu_k(b-y))}{\sinh(\nu_k b)}\right)\sin(\nu_n x)
\end{aligned}$$

The simplicity of equations for $A_k$ and $B_k$ is the reason for the above specific choice of two linearly independent solutions. Of course, one could use any two linearly independent solution and fix the values of arbitrary constant in their linear combination using the boundary conditions. The technicalities would not be so elegant and simple, though.

Step 3: The second associate boundary value problem. Let us set the boundary data in the original problem to zero on the horizontal lines:

$$h_0(x) = h_b(x) = 0\,, \quad 0 \le x \le a\,.$$

The vertical boundary data are expanded into the Fourier series over the "vertical" basis $Y_n$:

$$v_0(y) = \sum_{n=1}^{\infty} v_{0n} Y_n(y) \,, \quad v_{0n} = \frac{2}{b}\int_0^b v_0(y) Y_n(y)\, dy$$
$$v_a(y) = \sum_{n=1}^{\infty} v_{an} X_n(x) \,, \quad v_{an} = \frac{2}{b}\int_0^b v_a(y) Y_n(y)\, dy$$

Then the formal solution has the form

$$u_h(x, y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y) \,,$$

where the expansion coefficients $\tilde{X}_k(x)$ satisfies the boundary value problem

$$L_x \tilde{X}_k + \mu_k^2 \tilde{X}_k = -\tilde{X}_k''(x) + \nu_k^2 \tilde{X}_k(x) = 0 \,, \quad 0 < x < a$$
$$\tilde{X}_k(0) = v_{0k} \,, \quad \tilde{X}_k(x) = v_{ak}$$

Using the same method as in the previous case, the solution is

$$\tilde{X}_k(x) = v_{ak} \frac{\sinh(\mu_k x)}{\sinh(\mu_k a)} + v_{0k} \frac{\sinh(\mu_k (a - x))}{\sinh(\mu_k a)}$$
$$u_h(x, y) = \sum_{k=1}^{\infty} \left( v_{ak} \frac{\sinh(\mu_k x)}{\sinh(\mu_k a)} + v_{0k} \frac{\sinh(\mu_k (a - x))}{\sinh(\mu_k a)} \right) \sin(\mu_n y)$$

Step 4: The formal solution to the Dirichlet problem is the sum of the solutions to the two associated problems:

$$u(x, y) = u_v(x, y) + u_h(x, y)$$

**The existence and smoothness of the formal solution.** Let us investigate the convergence of the formal series. Consider first the formal solution $u_v(x, y)$. Owing to that the function $\sinh(y)$ is monotonically increasing, the terms of the series are bounded by

$$|\tilde{Y}_k(y) X_k(x)| \le |\tilde{Y}_k(y)| \le |h_{bk}| \frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} + |h_{0k}| \frac{\sinh(\nu_k (b - y))}{\sinh(\nu_k b)}$$
$$\le |h_{bk}| + |h_{0k}| \,, \quad 0 \le y \le b$$

Therefore if the series of Fourier coefficients of the boundary data converge absolutely

$$\sum_{k=1}^{\infty} |h_{bk}| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |h_{bk}| < \infty$$

then the formal solution $u_v$ exists and is a continuous function in the rectangle $[0,a]\times[0,b]$. A similar conclusion holds for $u_h$. The formal solution $u_h$ is a continuous function in the rectangle $[0,a]\times[0,b]$ if the series of Fourier coefficients of boundary data converge absolutely:

$$\sum_{k=1}^{\infty}|v_{ak}|<\infty \quad \text{and} \quad \sum_{k=1}^{\infty}|v_{0k}|<\infty$$

To investigate continuity of partial derivatives in the open rectangle, observe the following inequalities

$$\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} = e^{\nu_k(y-b)}\,\frac{1-e^{-2\nu_k y}}{1-e^{-2\nu_k b}} \le 2e^{\nu_k(y-b)} \le 2e^{-\nu_k\delta}\,, \quad 0\le y\le b-\delta\,,$$

for all large enough $k$ because the sequence

$$\lim_{k\to\infty}\frac{1-e^{-2\nu_k y}}{1-e^{-2\nu_k b}}=1$$

converges to 1 and, hence, all its terms cannot exceed 2 for all large enough $k$. Similarly

$$\frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \le 2e^{-\nu_k\delta}\,, \quad 0\le y\le b-\delta$$

for any arbitrary small delta. Then it follows that

$$|X_k'(x)|\le \nu_k\,, \quad |X_k''(x)|\le \nu_k^2$$
$$\left|\frac{d}{dy}\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)}\right| = \nu_k\frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \le 2\nu_k e^{-\nu_k\delta}$$
$$\left|\frac{d^2}{dy^2}\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)}\right| = \nu_k^2\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} \le 2\nu_k^2 e^{-\nu_k\delta}$$

Therefore the terms of the series obtained by term-by-term differentiation of the formal solution $p$ times are bounded by

$$|\tilde{Y}_k^{(p-s)}(y)X_k^{(s)}(x)| \le 2\nu_k^p\Big(|h_{bk}|+|h_{0k}|\Big)e^{-\nu_k\delta} \le M\nu_k^p e^{-\nu_k\delta}$$

where a constant $M$ is independent of $k$. Here is assumed that the series of Fourier coefficients converge absolutely, and, hence, their terms must be bounded because $|h_{bk}|\to 0$ and $|h_{0k}|\to 0$ as $k\to\infty$. The series of the bounds converges by the root test for any $\delta>0$

$$\lim_{k\to\infty}\sqrt[k]{\nu_k^p e^{-\nu_k\delta}} = \lim_{k\to\infty} e^{-\nu_k\delta/k} = e^{-\pi\delta/a} < 1$$
$$\sum_{k=1}^{\infty}\nu_k^p e^{-\nu_k\delta} < \infty$$

because $\sqrt[k]{c} \to 1$ and $\sqrt[k]{k} \to 1$ as $k \to \infty$. Since the terms of the formal solution have continuous derivatives of any order, the formal series also have continuous partial derivatives of any order for all $(x, y) \in [0, a] \times [0, b)$ (because $\delta > 0$ is arbitrary). Thus, $u_v \in C^\infty(\Omega)$. A similar analysis can be done for $u_h$ (one has to swap $x$ with $y$ and $a$ with $b$).

PROPOSITION **39.1**. (Formal and classical solution)
*If the series of Fourier coefficients of the boundary data converge absolutely*

$$\sum_{k=1}^{\infty} |h_{bk}| < \infty\,, \quad \sum_{k=1}^{\infty} |h_{bk}| < \infty\,, \quad \sum_{k=1}^{\infty} |v_{ak}| < \infty\,, \quad \sum_{k=1}^{\infty} |v_{0k}| < \infty$$

*then the formal solution to the Dirichlet problem is the classical solution and any partial derivative of the solution can be obtained by term-by-term differentiation in the open rectangle* $(0, a) \times (0, b)$

EXAMPLE **39.1**. *Find the formal solution to the Dirichlet problem:*

$$\begin{aligned}
&\Delta u(x, y) = 0\,, \quad (x, y) \in (-1, 1) \times (0, 2)\,,\\
&u(-1, y) = 0\,, \quad u(1, y) = y(2 - y)\,, \quad y \in [0, 2]\,,\\
&u(x, 0) = 0\,, \quad u(x, 2) = 1 - x^2\,, \quad x \in [-1, 1]\,.
\end{aligned}$$

*Investigate if the formal solution is a classical one.*

SOLUTION: The associate eigenvalue problems. The horizontal eigenvalue problem is

$$\begin{aligned}
-X_k''(x) &= \nu_k^2 X_k(x)\,, \quad X_k(-1) = X_k(1) = 0\,,\\
X_k(x) &= \sin(\nu_k(x + 1))\,, \quad \nu_k = \frac{\pi k}{2}\,,\ k = 1, 2, \dots\,,\\
\langle X_k, X_j\rangle &= \int_{-1}^{1} \sin(\nu_k(x + 1)) \sin(\nu_j(x + 1))\, dx\\
&= \int_{0}^{2} \sin(\nu_k s) \sin(\nu_j s)\, ds = \delta_{jk}\,,
\end{aligned}$$

where $s = x+1$. This problem can be solved by making a shift $s = x+1$ so that the interval $x \in [-1, 1]$ is mapped onto $s \in [0, 2]$. Solving the problem in the $s$ variable and substituting $s = x + 1$ gives the eigenfunctions. Alternatively, note that $X(x; \nu) = \sin(\nu(x + 1))$ is a solution that satisfies the boundary condition at the left endpoint, $X(-1; \nu) = 0$. It follows from the second boundary condition that

$X(1;\nu) = \sin(2\nu) = 0$ and, hence, $\nu = \nu_k = \pi k/2$. Similarly, the vertical eigenvalue problem is

$$
\begin{aligned}
&-Y_k''(y) = \mu_k^2 Y_k(x)\,, \quad Y_k(0) = Y_k(2) = 0\,,\\
&Y_k(y) = \sin(\mu_k y)\,, \quad \mu_k = \frac{\pi k}{2}\,,\ k = 1, 2, \dots\,,\\
&\langle Y_k, Y_j\rangle = \int_0^2 \sin(\mu_k y)\sin(\mu_j y)\,dy = \delta_{jk}\,.
\end{aligned}
$$

The fist associate boundary value problem. Setting all boundary data to zero on the vertical lines $x = \pm 1$, one gets

$$
\begin{aligned}
&\Delta u_v(x, y) = 0\,, \quad (x, y) \in (-1, 1) \times (0, 2)\,,\\
&u_v(-1, y) = 0\,, \quad u_v(1, y) = 0\,, \quad y \in [0, 2]\,,\\
&u_v(x, 0) = 0\,, \quad u_v(x, 2) = 1 - x^2\,, \quad x \in [-1, 1]\,.
\end{aligned}
$$

The boundary data on the horizontal lines are expanded into the Fourier series

$$
\begin{aligned}
1 - x^2 &= \sum_{k=1}^{\infty} h_k X_k(x)\\
h_k &= \frac{1}{\|X_n\|^2}\int_{-1}^{1}(1 - x^2)X_n(x)dx = \int_{-1}^{1}(1 - x^2)\sin(\nu_n(x+1))dx\\
&= \int_0^2 (2s - s^2)\sin(\nu_k s)\,ds = \frac{2(1 - (-1)^k)}{\nu_k^3}
\end{aligned}
$$

where the integral was evaluated by integrating by parts twice (the boundary terms vanish in each integration). The solution is sought in the form

$$
u_v(x, y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x)\,,
$$

where the expansion coefficients are found by solving the boundary value problem:

$$
-\tilde{Y}_k''(y) + \nu_k^2 \tilde{Y}_k(y) = 0\,, \qquad \tilde{Y}_k(0) = 0\,, \quad \tilde{Y}_k(2) = h_k
$$

The general solution is a linear combination of a solution satisfying the zero boundary condition at $y = 0$ and a solution satisfying the zero boundary condition at $y = 2$:

$$
\tilde{Y}_k(y) = A_k \sinh(\nu_k y) + B_k \sinh(\nu_k(y - 2))
$$

Then

$$\tilde{Y}_k(0) = 0 \quad \Rightarrow \quad B_k = 0$$
$$\tilde{Y}_2(2) = h_k \quad \Rightarrow \quad A_k = \frac{h_k}{\sinh(2\nu_k)}$$

The solution to the second problem reads

$$u_v(x, y) = \sum_{k=1}^{\infty} \frac{2(1-(-1)^k)}{\nu_k^3} \frac{\sinh(\nu_k y)}{\sinh(2\nu_k)} \sin[\nu_k(x+1)]$$

The second associate boundary value problem. Setting the boundary data to zero on the horizontal lines $y = 0$ and $y = 2$, one get

$$\Delta u_h(x, y) = 0\,, \quad (x, y) \in (-1, 1) \times (0, 2)\,,$$
$$u_h(-1, y) = 0\,, \quad u_h(1, y) = y(2-y)\,, \quad y \in [0, 2]\,,$$
$$u_h(x, 0) = 0\,, \quad u_h(x, 2) = 0\,, \quad x \in [-1, 1]\,.$$

The vertical boundary data are expanded into the Fourier series over the basis $Y_n$:

$$y(2-y) = \sum_{k=1}^{\infty} v_k Y_k(y)\,,$$
$$v_k = \frac{1}{\|Y_k\|^2} \int_0^2 y(2-y) Y_k(y)\, dy = \int_0^2 y(2-y) \sin(\mu_k y)\, dy$$
$$= \frac{2(1-(-1)^k)}{\mu_k^3}$$

where the integral was evaluated by integrating by parts twice (the boundary terms vanish in each integration). The formal solution has the form

$$u_h(x, y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) \sin(\mu_k y)\,,$$

where the expansion coefficients satisfy the boundary value problem in the interval $[-1, 1]$

$$-\tilde{X}_k''(x) + \mu_k^2 \tilde{X}_k(x) = 0\,, \quad \tilde{X}_k(-1) = 0\,, \quad \tilde{X}_k(1) = v_k$$

A general solution to the equation reads

$$\tilde{X}_k(x) = A_k \sinh(\mu_k(x+1)) + B_k \sinh(\mu_k(x-1))$$

which is a linear combination of two linearly independent solution one of which satisfy the zero boundary condition on the left endpoint $x = -1$, which is $\sinh(\mu_k(x+1))$, while the second one, $\sinh(\mu_k(x-1))$, satisfies the zero boundary condition at the right endpoint $x = 1$. The boundary conditions require that

$$\begin{aligned} \tilde{X}_k(-1) &= -B_k \sinh(2\mu_k) = 0 \quad \Rightarrow \quad B_k = 0\,, \\ \tilde{X}_k(1) &= A_k \sinh(2\mu_k) = v_k \quad \Rightarrow \quad A_k = \frac{v_k}{\sinh(2\mu_k)} \end{aligned}$$

Therefore

$$u_h(x,y) = \sum_{k=1}^{\infty} \frac{2(1-(-1)^k)}{\mu_k^3} \, \frac{\sinh[\mu_k(x+1)]}{\sinh(2\mu_k)} \, \sin(\mu_k y)\,.$$

The formal solution to the Dirichlet problem is the sum

$$u(x,y) = u_v(x,y) + u_h(x,y)\,.$$

It follows that

$$|v_k| = |h_k| \le \frac{4}{\nu_k^3} = \frac{32}{\pi^3}\,\frac{1}{k^3} \quad \Rightarrow \quad \sum_k |h_k| < \infty$$

By Proposition **39.1** the formal solution is the classical solution. □

**39.3. Mixed problems in a rectangle.** Any mixed problem for the Laplace equation in a rectangle can be solved by the discussed method. The procedure is illustrated by an example.

EXAMPLE **39.2**. *Find the formal solution of the mixed problem*

$$\begin{aligned} &\Delta u(x,y) = 0\,, \quad (x,y) \in (-1,1) \times (0,2)\,, \\ &u(-1,y) = y(2-y)\,, \quad u'_x(1,y) = 0\,, \quad y \in [0,2]\,, \\ &-u'_y(x,0) = 1 - x^2\,, \quad u(x,2) = 0\,, \quad x \in [-1,1]\,. \end{aligned}$$

*Show that the formal solution exists and is continuous on* $[-1,1]\times[0,2]$.

SOLUTION: Two associate eigenvalue problems. The eigenvalue problem associated with the horizontal variable is

$$\begin{aligned} &-X''(x) = \lambda X(x)\,, \quad -1 < x < 1\,, \\ &X(-1) = 0\,, \quad X'(1) = 0 \end{aligned}$$

The zero is not an eigenvalue for a mixed problem. Put $\lambda = \nu^2$, $\nu > 0$. A solution that satisfies the boundary condition at the left endpoint is

$$X(x;\nu) = \sin(\nu(x+1))$$

It can also be obtained by shifting variable $s = x + 1$ so that the interval becomes $0 < s < 2$ and $X = \sin(\nu s)$ satisfies the zero boundary condition at $s = 0$. The eigenvalues are found from the other boundary condition:

$$X'(1;\nu) = 0 \;\Rightarrow\; \nu\cos(2\nu) = 0 \;\Rightarrow\; \nu = \nu_k = \frac{\pi}{4}(2k-1)\,,\;\; k = 1, 2, ...$$

The corresponding orthogonal eigenfunctions are:

$$\begin{aligned} X(x;\nu_k) &= X_k(x) = \sin[\nu_k(x+1)]\,, \\ \langle X_k, X_n\rangle &= \int_{-1}^{1} X_k(x)X_n(x)\,dx = \delta_{nk} \end{aligned}$$

Note the integral must be evaluated for $k = n$ to show that $\|X_k\|^2 = 1$.

The eigenvalue problem associated with the vertical variables is

$$-Y''(y) = \lambda Y(y)\,, \quad Y'(0) = Y(2) = 0\,.$$

The zero is not an eigenvalue for this problem. Put $\lambda = \mu^2$, $\mu > 0$. The function $Y(y;\mu) = \cos(\mu y)$ is a solution satisfying the left boundary condition $Y'(0) = 0$. Therefore the eigenvalues are determined by the right endpoint boundary condition:

$$Y(2;\mu) = 0 \quad \Rightarrow \quad \cos(2\mu) = 0 \quad \Rightarrow \quad \mu = \nu_k$$

where $\nu_k$ are the same as in the previous problem. The corresponding eigenfunctions form an orthonormal basis in the space of continuous functions on $[0, 2]$:

$$Y_k(y) = Y(y;\nu_k) = \cos(\nu_k y)\,, \quad \langle Y_k, Y_n\rangle = \int_0^2 Y_k(y)Y_n(y)\,dy = \delta_{kn}$$

The integral was evaluated for $k = n$ to show that $\|Y_k\|^2 = 1$.

The first associate boundary value problem. Setting the vertical boundary data to zero the following problem is obtained

$$\begin{aligned} &\Delta u_v(x,y) = 0\,, \quad (x,y) \in (-1,1)\times(0,2)\,, \\ &u_v(-1,y) = 0\,, \quad u'_{vx}(1,y) = 0\,, \quad y \in [0,2]\,, \\ &-u'_{vy}(x,0) = 1 - x^2\,, \quad u_v(x,2) = 0\,, \quad x \in [-1,1]\,. \end{aligned}$$

The horizontal boundary data are expanded over the horizontal basis

$$
\begin{aligned}
1-x^2 &= \sum_{k=1}^{\infty} h_k X_k(x)\,,\\
h_k &= \frac{\langle x^2-1, X_k\rangle}{\|X_k\|^2} = \int_{-1}^{1}(x^2-1)\sin[\nu_k(x+1)]\,dx\\
&= \int_0^2 (t^2-2t)\sin(\nu_k t)\,dt = \frac{1}{\nu_k}\int_0^2 (2t-2)\cos(\nu_k t)\,dt\\
&= \frac{2\sin(2\nu_k)}{\nu_k^2} - \frac{2}{\nu_k^2}\int_0^2 \sin(\nu_k t)\,dt\\
&= -\frac{2}{\nu_k^2}\left((-1)^k + \frac{1}{\nu_k}\right)\,.
\end{aligned}
$$

where the integral was evaluated by integration by parts two times. Therefore the formal solution is given by a formal Fourier series

$$
u_v(x,y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x)
$$

where the expansion coefficients are solutions to the boundary value problems:

$$
-\tilde{Y}_k''(y) + \nu_k^2 \tilde{Y}_k(y) = 0\,, \quad \tilde{Y}_k'(0) = h_k\,, \quad \tilde{Y}_k(2) = 0
$$

The functions

$$
\cosh(\nu_k y)\,, \quad \sinh[\nu_k(2-y)]
$$

are linearly independent solutions that satisfy the corresponding zero boundary conditions at $y=0$ and $y=2$, that is, the first function has the vanishing derivative at $y=0$, while the second one has the zero value at $y=2$. So, a general solution is convenient to take in the form

$$
\tilde{Y}_k(x) = A_k\cosh(\nu_k y) + B_k\sinh[\nu_k(2-y)]
$$

The boundary conditions requires that

$$
\begin{aligned}
\tilde{Y}_k'(0) = h_k \quad &\Rightarrow \quad B_k = -\frac{h_k}{\nu_k\cosh(2\nu_k)}\,,\\
\tilde{Y}_k(2) = 0 \quad &\Rightarrow \quad A_k = 0
\end{aligned}
$$

The formal solution reads

$$
u_v(x,y) = -\sum_{k=1} \frac{h_k}{\nu_k}\,\frac{\sinh[\nu_k(2-y)]}{\cosh(2\nu_k)}\,\sin[\nu_k(x+1)]\,.
$$

The second associate boundary value problem. Setting the boundary data to zero at the horizontal lines $x = \pm 1$, the following problem is obtained:

$$\begin{aligned}
&\Delta u_h(x,y) = 0\,, \quad (x,y) \in (-1,1)\times(0,2)\,,\\
&u_h(-1,y) = y(2-y)\,, \quad u'_{hx}(1,y) = 0\,, \quad y \in [0,2]\,,\\
&-u'_{hy}(x,0) = 0\,, \quad u_h(x,2) = 0\,, \quad x \in [-1,1]\,.
\end{aligned}$$

The vertical boundary data are expanded over the basis $Y_n$

$$\begin{aligned}
2y - y^2 &= \sum_{k=1}^{\infty} v_k Y_k(y)\,,\\
v_k &= \frac{\langle 2y-y^2, Y_k\rangle}{\|Y_k\|^2} = \int_0^2 (2y-y^2)\cos(\nu_k y)\,dy\\
&= -\frac{1}{\nu_k}\int_0^2 (2-2y)\sin(\nu_k y)\,dy\\
&= \frac{2}{\nu_k^2}\left(1 - \frac{(-1)^k}{\nu_k}\right)\,.
\end{aligned}$$

The formal solution is then given by a formal Fourier series

$$u_h(x,y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y)$$

where the expansion coefficients satisfy the boundary value problems

$$-\tilde{X}''(x) + \nu_k^2 \tilde{X}(x) = 0\,, \quad \tilde{X}_k(-1) = \beta_k\,, \quad \tilde{X}'_k(1) = 0\,.$$

The functions

$$\sinh[\nu_k(x+1)]\,, \quad \cosh[\nu_k(1-x)]$$

are linearly independent solutions that satisfy the corresponding zero boundary conditions at the endpoints, that is, the first function vanishes at $x = -1$ and the derivatives of the second function vanishes at $x = 1$. The solution to the boundary value problem is convenient to seek in the form of their linear combination:

$$\tilde{X}_k(x) = A_k \sinh[\nu_k(x+1)] + B_k \cosh[\nu_k(1-x)]$$

The boundary conditions require that

$$\begin{aligned}
\tilde{X}_k(-1) = v_k \quad &\Rightarrow \quad B_k = \frac{v_k}{\cosh(2\nu_k)}\\
\tilde{X}'_k(1) = 0 \quad &\Rightarrow \quad A_k = 0
\end{aligned}$$

The formal solution to the second problem reads

$$u_h(x,y) = \sum_{k=1}^{\infty} v_k \, \frac{\cosh[\nu_k(1-x)]}{\cosh(2\nu_k)} \, \cos(2\nu_k y)$$

The formal solution to the original problem is given by the sum

$$u(x,y) = u_v(x,y) + u_h(x,y)$$

An analysis of the existence and smoothness of the formal solution can be carried out in the same way as for the Dirichlet problem. Note that due to different boundary condition the Fourier coefficients $h_k$ are scaled by a factor in the formal solution, $h_k \to h_k/\nu_k$, as compared to the Dirichlet case. Therefore the formal solution exists and is continuous on the closed rectangle because

$$\sum_k \frac{|h_k|}{\nu_k} < \infty\,, \quad \sum_k |v_k| < \infty$$

□

**39.4. Generalizations of the method.** Let an elliptic equation in a rectangle have the form

$$L_x u(x,y) + L_y u(x,y) = f(x,y) \qquad (x,y) \in \Omega = (a,b)\times(c,d)$$
$$\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = v(x,y)\,, \qquad (x,y) \in \partial\Omega$$

where $L_x$ is a Sturm-Liouville operator acting on the variable $x$, and $L_y$ is a second order differential operator acting on the variable $y$ that is positive in the inner product space of continuous functions in $[c,d]$:

$$\langle L_y Y, Y\rangle = \int_c^d L_y Y(y)\overline{Y(y)}\,dy \geq 0$$

for any function $Y(y)$ from the domain of $L_y$.

The boundary conditions are restated using the values of the parameters $\alpha$, $\beta$, and $v$ on the *vertical boundary lines*:

$$\begin{cases} \alpha_a u(a,y) - \beta_b u'_x(a,y) = v(a,y) \\ \alpha_b u(b,y) + \beta_b u'_x(b,y) = v(b,y)\,, \end{cases} \qquad c \leq y \leq d$$

where

$$\alpha\Big|_{x=a} = \alpha_a\,, \quad \alpha\Big|_{x=b} = \alpha_b\,, \quad \beta\Big|_{x=b} = \beta_b\,, \quad \beta\Big|_{x=b} = \beta_b\,,$$

and the *horizontal boundary lines*

$$\begin{cases} \alpha_c u(x,c) - \beta_c u'_y(x,c) = v(x,c) \\ \alpha_d u(x,d) + \beta_d u'_y(x,d) = v(x,d) \end{cases}, \qquad a \leq x \leq b$$

where

$$\alpha\Big|_{y=c} = \alpha_c\,, \quad \alpha\Big|_{y=d} = \alpha_d\,, \quad \beta\Big|_{y=c} = \beta_c\,, \quad \beta\Big|_{y=d} = \beta_d\,.$$

The Sturm-Liouville operator $L_x$ in an interval $(a, b)$ generates an orthogonal basis:

$$L_x X(x) = \lambda X(x)\,, \quad a < x < b\,, \quad \begin{cases} \alpha_a X(a) - \beta_b X'(a) = 0 \\ \alpha_b X(b) + \beta_b X'(b) = 0 \end{cases}$$

The eigenfunctions form an orthogonal basis

$$\lambda = \lambda_n\,, \quad X = X_n(x)\,,$$
$$\langle X_n, X_m\rangle = \int_a^b X_n(x)X_m(x)dx = 0\,,\ n \neq m$$

Hence, the inhomogeneity can be expanded into the Fourier series which is converges to $f$ for all values of the arguments in the rectangle if $f$ is smooth enough:

$$f(x, y) = \sum_{n=1}^{\infty} F_n(y)X_n(x)\,,$$
$$F_n(y) = \frac{\langle f, X_n\rangle}{\|X_n\|^2} = \frac{1}{\|X_n\|^2}\int_a^b f(x, y)X_n(x)\,dx$$

Let us assume for a time being that the boundary data vanish on the vertical lines:

$$v(a, y) = v(b, y) = 0$$

and the horizontal boundary data are expanded into a Fourier series over the found basis:

$$v(x, c) = \sum_{n=1}^{\infty} c_n X_n(x)\,, \quad c_n = \frac{1}{\|X_n\|^2}\int_a^b v(x, c)X_n(x)\,dx$$
$$v(x, d) = \sum_{n=1}^{\infty} d_n X_n(x)\,, \quad d_n = \frac{1}{\|X_n\|^2}\int_a^b v(x, d)X_n(x)\,dx$$

A *formal solution* can be sought in the form of a formal Fourier series:

$$u(x, y) = \sum_{n=1}^{\infty} \tilde{Y}_n(y)X_n(x)$$

Let us substitute the series into the equation and *formally* differentiate the series term-by-term:

$$\begin{aligned} L_x u + L_y u &= \sum_{n=1}^{\infty} \tilde{Y}_n(y) L_x X_n(x) + \sum_{n=1}^{\infty} X_n(x) L_y \tilde{Y}_n(y) \\ &= \sum_{n=1}^{\infty} X_n(x) \Big( L_y \tilde{Y}_n(y) + \lambda_n \tilde{Y}_n(y) \Big) \\ &= \sum_{n=1}^{\infty} F_n(y) X_n(x) \end{aligned}$$

Assuming that the above formal manipulations can be justified later and owing to that $X_n$ is an orthogonal basis, the series in the left and right sides are equal only if the coefficients at the corresponding basis functions are equal. Therefore the expansion coefficients satisfy the following differential equations:

$$L_y \tilde{Y}_n(y) + \lambda_n \tilde{Y}_n(y) = F_n(y)\,, \quad c < y < d$$

Let us see if the boundary condition is satisfied by such a solution. First note that, by construction, the formal solution satisfies the Sturm-Liouville boundary conditions one the lines $x = a$ and $x = b$ because the boundary data was assumed to have zero value on these lines.

A substitution of the formal solution into the boundary conditions at $y = c$ and $y = d$ shows that the expansion coefficients must satisfy the *non-homogeneous boundary value problem*

$$\begin{gathered} L_y \tilde{Y}_n(y) + \lambda_n \tilde{Y}_n(y) = F_n(y)\,, \quad c < y < d\,, \\ \begin{cases} \alpha_c \tilde{Y}_n(c) - \beta_c \tilde{Y}_n'(c) = c_n \\ \alpha_d \tilde{Y}_n(d) + \beta_d \tilde{Y}_n'(d) = d_n \end{cases} . \end{gathered}$$

This problem is analogous to that for the operator $L_r$ in polar coordinates.

What to do if the data $v$ does not vanish on the vertical lines, $x = a$ and $x = b$? This obvious drawback of the method can be eliminated if the *second operator $L_y$ is also a Sturm-Liouville operator*. Note the ellipticity of $L_x + L_y$ requires that $L_y$ is positive and, hence, is hermitian. So, this is not a too restrictive assumption that $L_y$ is a Sturm-Liouville operator. Consider an *associate homogeneous problem*

$$L_x u + L_y u = 0\,, \quad \alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = v(x, y)$$

where the boundary data *vanish at the horizontal lines*:

$$v(x, c) = v(x, d) = 0\,, \quad a \le x \le b$$

Consider an orthonormal basis obtained by solving the Sturm-Liouville problem for the operator $L_y$:

$$L_y Y_n(y) = \mu_n Y_n(y)\,, \quad c < y < d\,,$$
$$\begin{cases} \alpha_c Y_n(c) - \beta_c Y_n'(c) = 0 \\ \alpha_d Y_n(d) + \beta_d Y_n'(d) = 0 \end{cases}.$$

where

$$\langle Y_n, Y_m\rangle = \int_c^d Y_n(y) Y_m(y)\, dy = 0\,, \quad n \neq m$$

The formal solution is sought in the form of a formal Fourier series

$$w(x,y) = \sum_{n=1}^{\infty} \tilde{X}_n(x) Y_n(y)$$

To find the expansion coefficients $\tilde{X}_n(x)$, let us expand the boundary data on the lines $x = a$ and $x = b$ into the Fourier series over the eigenfunctions of $L_y$:

$$v(a,y) = \sum_{n=1}^{\infty} a_n Y_n(y)\,, \quad a_n = \frac{1}{\|Y_n\|^2} \int_c^d v(a,y) Y_n(y)\, dy$$
$$v(b,y) = \sum_{n=1}^{\infty} b_n Y_n(y)\,, \quad b_n = \frac{1}{\|Y_n\|^2} \int_c^d v(b,y) Y_n(y)\, dy$$

Then substituting the formal solution into the equation and into the boundary conditions and carrying out term-by-term differentiation (under the assumption that this operation is to be justified later), it is concluded in the same way as for the problem studied before that the expansion coefficients are solutions to the boundary value problems:

$$L\tilde{X}_n(x) + \mu_n \tilde{X}_n(x) = 0\,, \quad a < x < b\,;$$
$$\begin{cases} \alpha_a \tilde{X}_n(a) - \beta_a \tilde{X}_n'(a) = a_n \\ \alpha_b \tilde{X}_n(b) + \beta_b \tilde{X}_n'(b) = b_n \end{cases}$$

Let $u_V(x,y)$ be the formal solution to the problem where *the boundary data were set to zero on the vertical boundary lines* $x = a$ and $x = b$, and $u_H(x,y)$ be the formal solution to the problem where *the boundary data were set to zero on the horizontal boundary lines*, $y = c$ and $y = d$, then the sum

$$u(x,y) = u_V(x,y) + u_H(x,y) = \sum_{n=1}^{\infty} \tilde{Y}_n(y) X_n(x) + \sum_{n=1}^{\infty} \tilde{X}_n Y_n(y)$$

is the formal solution to the boundary value problem for a 2D elliptic equation with generic boundary data. By construction, $u$ satisfies (formally) the equation

$$L_x u + L_y u = (L_x + L_y)u_V + (L_x + L_y)u_H = f + 0 = 0$$

It also satisfies the boundary conditions. For example, for the vertical line $x = a$

$$\begin{aligned}\alpha_a u(a,y) - \beta_a u_x'(a,y) &= \alpha_a u_H(a,y) - \beta_a u_{Hx}'(a,y) \\ &\quad + \alpha_a u_V(a,y) - \beta_a u_{Vx}'(a,y) \\ &= v(a,y) + 0 = v(a,y)\end{aligned}$$

and similarly for the other three boundary lines.

It should be pointed out that if the elliptic operator is the sum of two Sturm-Liouville operators, the inhomogeneity can be included either into $u_V$ or $u_H$ (in the latter case, $f$ is expanded over $Y_n$). It can be split into a sum $f = f_1 + f_2$ so that $f_1$ included into $u_V$ and $f_2$ is included into $u_H$. This freedom in solving the problem can be used to simplify technicalities.

**39.5. Exercises.**

**1**. Solve the Dirichlet problem

$$\begin{aligned}&\Delta u(x,y) = 0\,, \quad (x,y) \in (-1,1) \times (-2,2)\,,\\ &u(-1,y) = \sin(\pi y/2)\,, \quad u(1,y) = 2\sin(\pi y)\,, \quad y \in [-2,2]\\ &u(x,-2) = -2\sin(\pi x)\,, \quad u(x,2) = \sin(2\pi x)\,, \quad x \in [-1,1]\,.\end{aligned}$$

**2**. Solve the Dirichlet problem

$$\begin{aligned}&\Delta u(x,y) = 0\,, \quad (x,y) \in (-1,1) \times (-1,0)\,,\\ &u(-1,y) = y(1+y)\,, \quad u(1,y) = 0\,, \quad y \in [-1,0]\\ &u(x,-1) = 0\,, \quad u(x,0) = x^2 - 1\,, \quad x \in [-1,1]\,.\end{aligned}$$

**3**. Find a formal solution to the mixed problem:

$$\begin{aligned}&\Delta u(x,y) = 0\,, \quad (x,y) \in (0,1) \times (1,2)\,,\\ &-u_x'(0,y) = 0\,, \quad u_x'(1,y) = 0\,, \quad y \in [1,2]\,,\\ &u(x,1) = x^2(1-x^2)\,, \quad u(x,2) = 0\,, \quad x \in [0,1]\,.\end{aligned}$$

**4**. Find a formal solution to the mixed problem:

$$\begin{aligned}&\Delta u(x,y) = 0\,, \quad (x,y) \in (0,1) \times (1,2)\,,\\ &-u_x'(0,y) = 0\,, \quad u_x'(1,y) = (y-1)(2-y)\,, \quad y \in [1,2]\,,\\ &u(x,1) = x^2(1-x^2)\,, \quad u(x,2) = 0\,, \quad x \in [0,1]\,.\end{aligned}$$

*Hint*: Use the solution from Problem 3.

## 40. The Neumann problem in rectangles

**40.1. The solvability condition.** Let us find the normal derivative on the boundary of a rectangle $\Omega = (0,a)\times(0,b)$ in terms of partial derivatives:

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = v(x,y)\Big|_{\partial\Omega}$$

The boundary $\partial\Omega$ consists of four lines. Let $\mathbf{e}_x$ and $\mathbf{e}_y$ be unit vectors in the directions of the $x$ and $y$ axes, respectively. Then the outward normal on the boundary lines is

$$\begin{aligned}
\mathbf{n}\Big|_{x=0} = -\mathbf{e}_x \quad &\Rightarrow \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{x=0} = -v(0,y) = v_0(y)\\
\mathbf{n}\Big|_{x=a} = \mathbf{e}_x \quad &\Rightarrow \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{x=a} = -v(a,y) = v_a(y)\\
\mathbf{n}\Big|_{y=0} = -\mathbf{e}_y \quad &\Rightarrow \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{y=0} = -v(x,0) = h_0(x)\\
\mathbf{n}\Big|_{y=b} = \mathbf{e}_y \quad &\Rightarrow \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{y=b} = v(x,b) = h_b(x)\,.
\end{aligned}$$

Consider the Neumann problem for the Laplace equation in a rectangle $\Omega = (0,a)\times(0,b)$

$$\begin{aligned}
&\Delta u(x,y) = 0\,, \quad (x,y)\in(0,a)\times(0,b)\\
&-u'_x(0,y) = v_0(y)\,, \quad u'_x(a,y) = v_a(y)\,, \quad y\in[0,b]\,,\\
&-u'_y(x,0) = h_0(x)\,, \quad u'_y(x,b) = h_b(x)\,, \quad x\in[0,a]\,,
\end{aligned}$$

Recall that the problem has a solution if the boundary data satisfy the solvability condition:

$$\oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\,ds = \oint_{\partial\Omega} v(x,y)ds = 0$$

For a rectangular region it can be cast in the following form

$$\textbf{(40.1)} \quad \begin{aligned}
&\int_0^b \Big(u'_x(a,y) - u'_x(0,y)\Big)\,dy = \int_0^b \Big(v_0(y) + v_a(y)\Big)\,dy = c\,,\\
&\int_0^a \Big(u'_y(x,b) - u'_y(x,0)\Big)\,dx = \int_0^a \Big(h_0(x) + h_b(x)\Big)\,dx = -c
\end{aligned}$$

The first equation gives the integral of the normal derivative over the vertical boundary lines, while the second one is the line integral of the normal derivative over the horizontal lines. Their sum must vanish and, hence, they must have the opposite values, $c$ and $-c$.

An attempt to use the same method as in the case of the Dirichlet or mixed problem, that is, to represent the solution as the sum of solutions to two associate Neumann problems in one of which the boundary data are set to zero on the horizontal boundary lines, while in the other the boundary data are set to zero on the vertical lines, is generally impossible because none of these problems has a solution if $c \neq 0$. Indeed, if $h_0 = h_b = 0$ or $v_0 = v_a = 0$, and $c \neq 0$, the solvability conditions for the associate Neumann problems are not fulfilled. Thus, the approach can only work if $c = 0$.

**40.2. Formal solution.** Let us assume for a moment that the boundary data are such that their mean values vanish

$$\frac{1}{a}\int_0^a h_{0,b}(x)\,dx = 0\,, \quad \frac{1}{b}\int_0^b v_{0,a}(y)\,dy = 0$$

This implies that $c = 0$ and the solvability condition holds for the horizontal and vertical data separately (in fact, it holds for any associated problem where any three of the four boundary data are set to zero). Then the formal solution can written as the sum

$$u(x,y) = u_v(x,y) + u_h(x,y)$$

of solutions to two associated Neumann problems. The first one, $u_v$, is obtained by setting the boundary data at the vertical edges $x = 0$ and $x = a$ to zero

$$v_0(y) = v_a(y) = 0$$

and the second one, $u_h$, is obtained by setting the boundary data at the horizontal edges $y = 0$ and $y = b$ to zero

$$h_0(x) = h_b(x) = 0$$

The solution is unique up to an additive constant.

**The first associate Neumann problem.** The formal solution is sought as a formal Fourier series

$$u_v(x,y) = \tilde{Y}_0(y)X_0(x) + \sum_{k=1}^{\infty} \tilde{Y}_k(y)X_k(x)\,.$$

over an orthogonal basis generated by the horizontal Sturm-Liouville problem

$$-X_k''(x) = \nu_k^2 X_k(x)\,, \quad X_k'(0) = X_k'(a) = 0\,.$$

Zero is an eigenvalue so that

$$\nu_0 = 0\,, \quad X_0(x) = 1\,, \quad \|X_0\|^2 = a\,,$$
$$\nu_k = \frac{\pi k}{a}\,, \quad X_k(x) = \cos(\nu_k x)\,, \quad k = 1, 2, \ldots,\,, \quad \|X_k\|^2 = \frac{a}{2}\,.$$

By the assumption the Fourier coefficient of the boundary data for the basis function $X_0(x) = 1$ vanishes

$$h_{00} = \frac{\langle h_0, X_0\rangle}{\|X_0\|^2} = \frac{1}{a}\int_0^a h_0(x)\,dx = 0\,,$$
$$h_{b0} = \frac{\langle h_b, X_0\rangle}{\|X_0\|^2} = \frac{1}{a}\int_0^a h_b(x)\,dx = 0$$

Therefore the function $\tilde{Y}_0$ is a solution to the boundary value problem

$$\tilde{Y}_0'' = 0\,, \quad \tilde{Y}_0'(0) = -h_{00} = 0\,, \quad \tilde{Y}_0'(b) = h_{b0} = 0$$

Its general solution is a constant function. Therefore the first term in the formal solution $\tilde{Y}_0(y)X_0(x)$ is an additive constant which can be omitted as the solution ii unique up to an additive constant which can be added at the very end, that is, without loss of generality

$$u_v(x,y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y)X_k(x)\,.$$

The expansion coefficients are solutions to the boundary value problems (here $k$ can take zero value):

$$\tilde{Y}_k''(y) - \nu_k^2\tilde{Y}_k(y) = 0\,, \quad 0 < y < b\,, \quad \tilde{Y}_k'(0) = -h_{0k}\,, \quad \tilde{Y}_k'(b) = h_{bk}$$

where $h_{0k}$ and $h_{bk}$ are Fourier coefficients of the horizontal boundary data:

$$h_0(x) = \sum_{k=1}^{\infty} h_{0k}X_k(x)\,, \quad h_{0k} = \frac{1}{\|X_k\|^2}\int_0^a h_0(x)X_k(x)\,dx$$
$$h_b(x) = \sum_{k=1}^{\infty} h_{bk}X_k(x)\,, \quad h_{bk} = \frac{1}{\|X_k\|^2}\int_0^a h_b(x)X_k(x)\,dx$$

To find the solution to the boundary value problem, let us write a general solution to the equation as a linear combination of solutions satisfying the zero boundary condition at left and right endpoints of the interval, respectively. The function $\cosh(\nu_k y)$ satisfies the equation and has the vanishing derivative at $y = 0$, while the linearly independent function $\cosh(\nu_k(b-y))$ satisfies the equation and has the vanishing derivative at $y = b$. Then the solution is sought as the linear combination

$$\tilde{Y}_k(y) = A_k\cosh(\nu_k y) + B_k\cosh(\nu_k(b-y))\,.$$

It follows that

$$\tilde{Y}_k'(0) = -\nu_k B_k \sinh(\nu_k b) \quad \Rightarrow \quad B_k = \frac{h_{0k}}{\nu_k \sinh(\nu_k b)}$$
$$\tilde{Y}_k'(b) = \nu_k A_k \sinh(\nu_k b) \quad \Rightarrow \quad A_k = \frac{h_{bk}}{\nu_k \sinh(\nu_k b)}$$

Thus, the formal solution reads

$$u_v(x,y) = \sum_{k=1}^{\infty} \left( \frac{h_{bk}}{\nu_k} \frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} + \frac{h_{0k}}{\nu_k} \frac{\cosh(\nu_k (b-y))}{\sinh(\nu_k b)} \right) \cos(\nu_k x)$$

**The second associated Neumann problem.** The solution to the second problem is obtained along a similar line of arguments in which the roles of variables $x$ and $y$ are swapped. The vertical Sturm-Liouville problem

$$-Y_k''(y) = \mu_k^2 Y_k(y)\,, \quad Y_k'(0) = Y_k'(b) = 0\,,$$

has the following solution

$$\mu_0 = 0\,, \quad Y_0(y) = 1\,, \quad \|Y_0\|^2 = b\,,$$
$$\mu_k = \frac{\pi k}{b}\,, \quad Y_k(y) = \cos(\mu_k x)\,, \quad k = 1, 2, ..., \quad \|Y_k\|^2 = \frac{b}{2}\,.$$

The vertical boundary data are expanded over the vertical orthogonal basis

$$v_0(y) = v_{00} + \sum_{k=1}^{\infty} v_{0k} Y_k(y)\,, \quad v_{0k} = \frac{1}{\|Y_k\|^2} \int_0^b v_0(y) Y_k(y)\, dy$$
$$v_a(y) = v_{a0} + \sum_{k=1}^{\infty} v_{ak} Y_k(y)\,, \quad v_{ak} = \frac{1}{\|Y_k\|^2} \int_0^b v_b(y) Y_k(y)\, dy$$

The formal solution is given by a formal Fourier series over the vertical basis:

$$u_h(x,y) = \tilde{X}_0(x) Y_0(y) + \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y)\,.$$

where the expansion coefficients are solutions to the boundary value problems:

$$\tilde{X}_k''(x) - \mu_k^2 \tilde{X}_k(x) = 0\,, \quad \tilde{X}_k'(0) = -v_{0k}\,, \quad \tilde{X}_k'(a) = v_{ak}$$

Just like in the first problem, the solution $\tilde{X}_0(x)$ is an arbitrary constant because by the assumption

$$v_{00} = v_{a0} = 0$$

Since the formal solution is defined modulo an additive constant, the constant term $\tilde{X}_0(x)Y_0(y)$ can be omitted now and added at the very end.

For $k > 0$, the solution is found using the same principle:

$$\begin{aligned}\tilde{X}_k(x) &= A_k \cosh(\mu_k x) + B_k \cosh[\mu_k(a-x)]\,,\\ A_k &= \frac{v_{ak}}{\mu_k \sinh(\mu_k a)}\,, \quad B_k = \frac{v_{0k}}{\mu_k \sinh(\mu_k a)}\,,\ k = 1, 2, ...\end{aligned}$$

are solutions to the equation Thus, the formal solution has the form

$$u_h(x,y) = \sum_{k=1}^{\infty} \left( \frac{v_{ak}}{\mu_k} \frac{\cosh(\mu_k x)}{\sinh(\mu_k a)} + \frac{v_{0k}}{\mu_k} \frac{\cosh(\mu_k(a-x))}{\sinh(\mu_k a)} \right) \cos(\mu_k u)$$

**General boundary data.** Let write the solvability condition in terms of the Fourier coefficients of a general boundary data:

$$\oint_{\partial\Omega} v(x,y)\, ds = a(h_{00} + h_{b0}) + b(v_{00} + v_{a0}) = 0$$

This shows that if the boundary data altered by any function that is orthogonal to $X_0$ on the horizontal boundaries, and to $Y_0$ on the vertical boundaries, the solvability condition does not changed. This implies that the Neumann problem with *constant* boundary data obtained by setting all the Fourier coefficients to zero, except those for $k = 0$:

$$h_{0k} = h_{bk} = v_{0k} = v_{bk} = 0\,, \quad k = 1, 2, ...$$

has the same solvability condition as the original one. Therefore the solution to the Neumann problem is the sum of the solution to the problem with the stated constant boundary data and the solution with the boundary data obtained from the original ones by subtracting the corresponding constant boundary data included into the first problem:

$$\begin{aligned}h_0(x) &\to h_0(x) - h_{00}\,, \quad h_b(x) \to h_b(x) - h_{b0}\,,\\ v_0(y) &\to v_0(y) - v_{00}\,, \quad v_a(y) \to v_a(y) - v_{a0}\,,\end{aligned}$$

The average value of the shifted boundary data is equal to zero by construction. For example,

$$\frac{1}{a}\int_0^a \Big(h_0(x) - h_{00}\Big)\, dx = \frac{1}{a}\int_0^a h_0(x)\, dx - h_{00} = h_{00} - h_{00} = 0$$

by definition of the Fourier coefficient $h_{00}$. The solution to this Neumann problem has already been found. It remains to find the solution to the Neumann problem with the constant boundary data.

The boundary data has four parameters and, hence, one can try to find a solution as a general harmonic polynomial of degree 2 which also has four parameters:

$$u(x,y) = Ax + By + C(x^2 - y^2) + Dxy$$

The constant term is omitted because the solution, if it exists in this form, is unique up to an an additive constant. The vertical boundary conditions yields

$$u'_x\Big|_{x=0} = A + Dy = -v_{00}\,, \quad 0 \le y \le b\,,$$
$$u'_x\Big|_{x=a} = A + 2aC + Dy = v_{a0}\,, \quad 0 \le y \le b\,.$$

Therefore $D = 0$ as the equation must hold for any $y$ in the specified interval, and

$$A = -v_{00}\,, \quad C = \frac{1}{2a}\Big(v_{a0} + v_{00}\Big)$$

With $D = 0$, the horizontal boundary condition are

$$u'_y\Big|_{y=0} = B = -h_{00}\,, \quad 0 \le x \le a\,,$$
$$u'_y\Big|_{y=b} = B - 2bC = h_{b0}\,, \quad 0 \le x \le a\,.$$

so that

$$B = -h_{00}\,, \quad C = -\frac{1}{2b}\Big(h_{b0} + h_{00}\Big)$$

The solvability condition ensures that the two expressions obtained for the constant $C$ are equal. Indeed, by dividing the solvability condition by $2ab$ one infers that

$$C = \frac{1}{2a}\Big(v_{a0} + v_{00}\Big) = -\frac{1}{2b}\Big(h_{b0} + h_{00}\Big)$$

Thus, the formal solution to the Neumann problem with general boundary data satisfying the solvability condition reads

$$u(x,y) = u_v(x,y) + u_h(x,y) - v_{00}x - h_{00}y + C(x^2 - y^2) + \text{const}$$

Note that, in accord with the general analysis of the Neumann problem, the found solution is unique up to an additive constant.

**40.3. Existence and smoothness of the formal solution.** The existence and smoothness of the formal solution can be analyzed in the same way as the formal solution to the Dirichlet problem. In fact, a sufficient condition for the formal solution to be a classical one is the same as in the Dirichlet case.

PROPOSITION **40.1**. (Formal and classical solution)
*If the series of Fourier coefficients of the boundary data converge absolutely*

$$\sum_{k=1}^{\infty} |h_{bk}| < \infty\,, \quad \sum_{k=1}^{\infty} |h_{bk}| < \infty\,, \quad \sum_{k=1}^{\infty} |v_{ak}| < \infty\,, \quad \sum_{k=1}^{\infty} |v_{0k}| < \infty$$

*then the formal solution to the Neumann problem is the classical solution and any partial derivative of the solution can be obtained by term-by-term differentiation in the open rectangle* $(0,a)\times(0,b)$

The formal solution consists of four similar series. Let us prove each series is from the class

$$C^1(\bar{\Omega}) \cap C^\infty(\Omega)$$

and so must be their sum. Let us investigate the convergence of the first series in the formal solution $u_v(x,y)$:

$$w(x,y) \sim \sum_{k=1}^{\infty} \frac{h_{bk}}{\nu_k} \frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \cos(\nu_k x)$$

The sign $\sim$ is used to emphasize that the series is formal. Taking the term-by-term partial derivatives of $w$ gives the following formal series

$$w'_x(x,y) \sim -\sum_{k=1}^{\infty} h_{bk} \frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \sin(\nu_k x)$$

$$w'_y(x,y) \sim \sum_{k=1}^{\infty} h_{bk} \frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} \cos(\nu_k x)$$

Owing to the monotonicity of $\cosh(z)$ and $\sinh(z)$ in the interval $z \geq 0$, one has

$$\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} \leq 1\,, \quad 0 \leq y \leq b$$

$$\frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \leq \frac{\cosh(\nu_k b)}{\sinh(\nu_k b)} \leq 2\,, \quad 0 \leq y \leq b$$

where the latter inequality hold for all large enough $k$. Note that $\coth(\nu_k b) > 1$ and $\coth(\nu_k b) \to 1$ as $k \to \infty$. Therefore starting with some $k = N$, the values of $\coth(\nu_k b)$ will be in the interval $(1,2)$ for all $k > N$. The stated inequalities imply that the term of the series for $w$ are bounded for all $(x,y) \in \bar{\Omega}$ and the series of the bounds converges by the hypothesis:

$$\left| \frac{h_{bk}}{\nu_k} \frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \cos(\nu_k x) \right| \leq \frac{2|h_{bk}|}{\nu_k} \leq \frac{1}{\nu_1} |h_{bk}| \quad \text{and} \quad \sum_k |h_{bk}| < \infty$$

Since the terms of the series are continuous everywhere, the series converges for all $(x, y) \in \bar{\Omega}$, that is, $w(x, y)$ exists, and, moreover, $w(x, y)$ is continuous on $\bar{\Omega}$. The terms for the series for partial derivatives are continuous everywhere and they are bounded for all $(x, y) \in \bar{\Omega}$, and the series of the bounds converges:

$$\left| h_{bk} \frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \sin(\nu_k x) \right| \le 2|h_{bk}| \quad \text{and} \quad \sum_k |h_{bk}| < \infty$$
$$\left| h_{bk} \frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} \cos(\nu_k x) \right| \le |h_{bk}| \quad \text{and} \quad \sum_k |h_{bk}| < \infty$$

Therefore $w$ has continuous partial derivatives in $\bar{\Omega}$ and they can be obtained by term-by-term differentiation of the formal solution. Thus,

$$w \in C^1(\bar{\Omega})$$

Let us restrict the range of $y$ by a smaller interval

$$0 < \delta \le y \le b - \delta$$

where $\delta$ can be arbitrary small, but not zero. In this case, the ratio of the hyperbolic functions in the series for $w$ can have have exponentially small bounds for large $\nu_k$:

$$\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)} \le \frac{\sinh(\nu_k (b-\delta))}{\sinh(\nu_k b)} = e^{-\nu_k \delta} \, \frac{1 - e^{-\nu_k(2b-\delta))}}{1 - e^{-2\nu_k b}} \le 2e^{-\delta \nu_k}$$
$$\frac{\cosh(\nu_k y)}{\sinh(\nu_k b)} \le \frac{\cosh(\nu_k (b-\delta))}{\sinh(\nu_k b)} = e^{-\nu_k \delta} \, \frac{1 + e^{-\nu_k(2b-\delta))}}{1 - e^{-2\nu_k b}} \le 2e^{-\delta \nu_k}$$

because the ratios in the upper bounds converge to 1 and, hence, cannot exceed 2 for all $k$ large enough. Each term-by-term differentiation of $w$ produces a factor $\nu_k$. Therefore, if one takes a partial derivative of order $p$ (regardless with respect to which variable), the terms of such a series are abounded by

$$|h_{bk}| \nu_k^{p-1} e^{-\delta \nu_k} \le \nu_k^{p-1} e^{-\delta \nu_k}$$

for all large enough $k$ because $|h_{bk}| \to 0$ as $k \to \infty$ by the convergence of the series $\sum |h_{bk}| < \infty$. The sries of the bounds converges by the root test

$$\lim_{k\to\infty} \sqrt[k]{\nu_k^{p-1} e^{-\delta \nu_k}} = e^{-\delta \pi / a} < 1 \quad \Rightarrow \quad \sum_k \nu_k^{p-1} e^{-\delta \nu_k} < \infty$$

because $\sqrt[k]{c} \to 1$ ($c$ is any positive number) and $\sqrt[k]{k} \to 1$ as $k \to \infty$. This implies that all partial derivatives of $w$ of any order are continuous in the open rectangle (because $\delta$ is arbitrary small) and can be obtained by term-by-term differentiation of the formal solution. Thus

$$w \in C^1(\bar{\Omega}) \cap C^\infty(\Omega)$$

The same analysis can be repeated for the other three series in the formal solution with the same conclusion.

EXAMPLE **40.1**. *Determine whether the Neumann problem has a solution and, if it does, find its formal solution*

$$\begin{aligned}
&\Delta u(x,y) = 0\,, \quad (x,y) \in (0,2) \times (-1,1)\,,\\
&-u'_x(0,y) = y^2 - 1\,, \quad u'_x(2,y) = 0\,, \quad y \in [-1,1]\,,\\
&-u'_y(x,-1) = x(2-x)\,, \quad u'_y(x,1) = 0\,, \quad x \in [0,2]\,.
\end{aligned}$$

*Determine whether the formal solution is a classical one.*

SOLUTION: Solvability condition:

$$\begin{aligned}
\int_{-1}^{1} \Big( -u'_x(0,y) + u'_x(2,y) \Big)\, dy &= \int_{-1}^{1} (y^2-1)\, dy = -\frac{4}{3}\\
\int_{0}^{2} \Big( -u'_y(x,-1) + u'_y(x,1) \Big)\, dx &= \int_{0}^{2} (2x-x^2)\, dx = \frac{4}{3}
\end{aligned}$$

The boundary data satisfy the solvability condition. A solution exists and is unique up to an additive constant.

The associated boundary eigenvalue problems: An orthogonal basis generated by the horizontal Sturm-Liouville operator:

$$\begin{aligned}
&-X''_k(x) = \nu_k^2 X_k(x)\,, \quad X'_k(0) = X'_k(2) = 0\,,\\
&\nu_0 = 0\,, \quad X_0(x) = 1\,, \quad \|X_0\|^2 = 2\,,\\
&\nu_k = \frac{\pi k}{2}\,, \quad X_k(x) = \cos(\nu_k x)\,, \quad k = 1,2,\ldots, \quad \|X_k\|^2 = 1
\end{aligned}$$

Expanding the horizontal boundary data at $y = -1$ over the horizontal basis $X_n$

$$x(2-x) = h_0 + \sum_{k=1}^{\infty} h_k X_k(x)\,,$$

$$h_0 = \frac{1}{\|X_0\|^2}\int_0^2 (2x - x^2)dx = \frac{2}{3}$$

$$\begin{aligned} h_k &= \frac{1}{\|X_k\|^2}\int_0^2 (2x-x^2)X_k(x)dx = \int_0^2 (2x-x^2)\cos(\nu_k x)\,dx \\ &= \frac{1}{\nu_k}\int_0^2 (2-2x)\sin(\nu_k x)dx \\ &= -\frac{2-2x}{\nu_k^2}\cos(\nu_k x)\Big|_0^2 + \frac{2}{\nu_k^2}\int_0^2 \cos(\nu_k x)\,dx \\ &= \frac{2((-1)^{k+1}-1)}{\nu_k^2} \end{aligned}$$

An orthogonal basis generated by the vertical Sturm-Liouville operator:

$$-Y_k''(y) = \mu_k^2 Y_k(y)\,, \quad Y_k'(-1) = Y_k'(1) = 0\,,$$

$$\mu_0 = 0\,, \quad Y_0(y) = 1\,, \quad \|Y_0\|^2 = 2\,,$$

$$\mu_k = \frac{\pi k}{2}\,, \quad Y_k(y) = \cos[\mu_k(y+1)]\,, \quad k = 1, 2, ..., \quad \|Y_k\|^2 = 1$$

Note that the shift $y \to y+1$ maps $[-1,1]$ to $[0,2]$ so that the eigenfunctions can be obtained from the previous case by this shift. The vertical boundary data on the line $x = 0$ are expanded into a Fourier series over the vertical basis:

$$y^2 - 1 = v_0 + \sum_{k=1}^{\infty} v_k Y_k(y)\,,$$

$$v_0 = -\frac{1}{\|Y_0\|^2}\int_{-1}^1 (1-y^2)\,dy = -\frac{2}{3}$$

$$\begin{aligned} v_k &= -\frac{1}{\|Y_k\|^2}\int_{-1}^1 (1-y^2)Y_k(y)dy = -\int_{-1}^1 (1-y^2)\cos[\mu_k(y+1)]\,dy \\ &= -\int_0^2 x(2-x)\cos(\mu_k x)\,dx = -\frac{2((-1)^{k+1}-1)}{\mu_k^2} \end{aligned}$$

The change of the integration variable $x = y + 1$ reduces the integral to the one already calculated in the horizontal case for the coefficients $h_k$ (just replace $\nu_k$ by $\mu_k$).

The first associate Neumann problem. Setting the new boundary data to zero at the vertical lines $x = 0$ and $x = 2$, and shifting the horizontal data by additive constants so that their average values vanish, the first associated Neumann problem is obtained. Its formal solution reads

$$u_v(x, y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x)\,,$$

The expansion coefficients satisfy the boundary value problems

$$\tilde{Y}_k''(y) - \nu_k^2 \tilde{Y}_k(y) = 0\,, \quad \tilde{Y}_k'(-1) = -h_k\,, \quad \tilde{Y}_k'(1) = 0$$

A solution is convenient to seek in the form

$$\tilde{Y}_k(y) = A_k \cosh[\nu_k(y+1)] + B_k \cosh[\nu_k(y-1)]$$

The two linearly independent solutions are chosen so that the first one has the vanishing derivative at $y = -1$, while the other at $y = 1$. The coefficients $A_k$ and $B_k$ are determined by the boundary condition

$$\tilde{Y}_k(y) = \frac{h_k}{\nu_k} \frac{\cosh(\nu_k(y-1))}{\sinh(2\nu_k)}$$

The second associate Neumann problem. Setting the new boundary data at the horizontal lines $y = \pm 1$ to zero, and changing the vertical data by additive constants so that their averages vanish, the second associated Neumann problem is obtained whose formal solution is given by the Fourier series:

$$u_h(x, y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y)$$

The expansion coefficients satisfy the following boundary value problems:

$$\tilde{X}_k''(x) - \mu_k^2 \tilde{X}_k(x) = 0\,, \quad \tilde{X}_k'(0) = -v_k\,, \quad \tilde{X}_k'(2) = 0$$

A general solution to the equation is convenient to write in the form

$$\tilde{X}_k(x) = A_k \cosh(\mu_k x) + B_k \cosh[\mu_k(2-x)]$$

The two linearly independent solutions are chosen so that the first one has vanishing derivative at $x = 0$, while the other at $x = 2$. The coefficients $A_k$ and $B_k$ are determined by the boundary condition:

$$\tilde{X}_k(x) = v_k \frac{\cosh[\mu_k(2-x)]}{\mu_k \sinh(2\mu_k)}$$

The associated Neumann problem with constant coefficients. One has to find a harmonic functions $u(x, y)$ in the rectangle that satisfies the following Neumann boundary conditions

$$-u'_x(0,y) = v_0 = -\frac{2}{3}\,,\quad u'_x(2,y) = 0\,,\quad -1 \le y \le 1\,,$$
$$-u'_y(x,-1) = h_0 = \frac{2}{3}\,,\quad u'_y(x,1) = 0\,,\quad 0 \le x \le 2\,.$$

A solution is sought in the form

$$u(x,y) = Ax + B(y+1) + C\Big(x^2 - (y+1)^2\Big)$$

where the variable $y+1$ spans the interval $[0,2]$. The vertical boundary conditions yield

$$A = \frac{2}{3}\,,\quad A + 4C = 0 \quad \Rightarrow \quad C = -\frac{1}{6}$$

The horizontal boundary conditions yield

$$B = -\frac{2}{3}\,,\quad B - 4C = 0 \quad \Rightarrow \quad C = -\frac{1}{6}$$

Note that the constant $C$ is found to have the same value as required.

Formal solution. Since $\mu_k = \nu_k = \frac{\pi k}{2}$ (the rectangle is a square) so that $v_k = -h_k$, the solution of the Neumann problem reads (up to an additive constant)

$$u(x,y) = u_v(x,y) + u_h(x) + \frac{2}{3}(x-y-1) - \frac{1}{6}\,x^2 + \frac{1}{6}\,(y+1)^2\,,$$
$$u_v(x,y) = \sum_{k=1}^{\infty} \frac{h_k}{\nu_k}\,\frac{\cosh[\nu_k(y-1)]}{\sinh(2\nu_k)}\cos(\nu_k x)\,,$$
$$u_h(x,y) = \sum_{k=1}^{\infty} \frac{h_k}{\nu_k}\,\frac{\cosh[\nu_k(2-x)]}{\sinh(2\nu_k)}\cos[\nu_k(y+1)]\,.$$

The existence and smoothness of the formal solution. The series of the Fourier coefficients of the boundary data converges absolutely because

$$|h_k| \le \frac{4}{\nu_k^2} = \frac{16}{k^2}\,\frac{1}{k^2}\,,\qquad \sum_k \frac{1}{k^2} < \infty$$

By Proposition **40.1**, the formal solution exists and its first partial derivatives are continuous in the rectangle and its boundary, and partial derivatives of the formal solution of any order are continuous in the interior of the rectangle, that is,

$$u \in C^1(\bar{\Omega}) \cap C^\infty(\Omega)\,,\quad \Omega = (0,2)\times(-1,1)$$

and, hence, $u$ is a classical solution of the said Neumann problem. □

**40.4. Exercises.**

**1**. Solve the Neumann problem or show that no solution exists

$$\begin{aligned}
&\Delta u(x,y) = 0\,, \quad (x,y) \in (-1,1)\times(-2,2)\,,\\
&-u'_x(-1,y) = \cos^2(\pi y)\,, \quad u'_x(1,y) = 3\cos(2\pi y)\,, \quad y\in[-2,2]\\
&-u'_y(x,-2) = -2\sin^2(\pi x)\,, \quad u'_y(x,2) = -4\cos(2\pi x)\,, \quad x\in[-1,1]\,.
\end{aligned}$$

*Hint*: Use the double angle formula to transform the boundary data containing the squares of trigonometric functions.

**2**. Find a formal solution to the Neumann problem and investigate its convergence or show that no solution exists

$$\begin{aligned}
&\Delta u(x,y) = 0\,, \quad (x,y) \in (-1,1)\times(0,4)\,,\\
&-u'_x(-1,y) = \tfrac{1}{4}y - 1\,, \quad u'_x(1,y) = 1 - \tfrac{1}{4}y\,, \quad y\in[0,4]\\
&-u'_y(x,0) = -x\,, \quad u'_y(x,4) = 0\,, \quad x\in[-1,1]\,.
\end{aligned}$$

**3**. Find a formal solution to the Neumann problem and investigate its convergence or show that no solution exists

$$\begin{aligned}
&\Delta u(x,y) = 0\,, \quad (x,y) \in (-1,1)\times(0,4)\,,\\
&-u'_x(-1,y) = \tfrac{1}{4}y - 1\,, \quad u'_x(1,y) = 0\,, \quad y\in[0,4]\\
&-u'_y(x,0) = 1 + 2x\,, \quad u'_y(x,4) = 0\,, \quad x\in[-1,1]\,.
\end{aligned}$$

## 41. Green's function for the Sturm-Liouville operator

Suppose $A$ is an $N \times N$ matrix. Consider a general linear equation

$$A\mathbf{u} = \mathbf{f}$$

in an $N-$dimensional Euclidean space (complex or real). The problem is to find a vector $\mathbf{u}$ that satisfies this equation for a given vector $\mathbf{f}$. Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions. Put

$$\mathbf{u}_0 = \mathbf{u}_1 - \mathbf{u}_2 \,.$$

Then

$$A\mathbf{u}_0 = A(\mathbf{u}_1 - \mathbf{u}_2) = A\mathbf{u}_1 - A\mathbf{u}_2 = \mathbf{f} - \mathbf{f} = \mathbf{0}$$

Therefore a general solution is the sum of a particular solution and a *general* solution of the associated homogeneous equation (with $\mathbf{f} = 0$). Furthermore, *if a solution exists, then it is unique, if the homogeneous equation has only the trivial solution*:

$$A\mathbf{u}_0 = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{u}_0 = \mathbf{0} \,.$$

In this case, the matrix $A$ is *invertible* and the solution has the form

$$\mathbf{u} = A^{-1}\mathbf{f} \,.$$

Suppose $A$ is symmetric or hermitian. This means that

$$\langle A\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, A\mathbf{v}\rangle$$

for any two vectors $\mathbf{u}$ and $\mathbf{v}$, where $\langle \cdot, \cdot\rangle$ denotes the dot product. Let $\lambda$ be an eigenvalue and $\mathbf{u}_\lambda$be a corresponding eigenvector:

$$A\mathbf{u}_\lambda = \lambda\mathbf{u}_\lambda \,.$$

Then a solution $\mathbf{u}$ to the linear problem must satisfy the condition

$$\langle \mathbf{f}, \mathbf{u}_\lambda\rangle = \langle A\mathbf{u}, \mathbf{u}_\lambda\rangle = \langle \mathbf{u}, A\mathbf{u}_\lambda\rangle = \lambda\langle \mathbf{u}, \mathbf{u}_\lambda\rangle$$

because the eigenvalues are real $\overline{\lambda} = \lambda$. In particular, this condition shows that, if $A$ has the zero eigenvalue $\lambda = 0$, then *the linear problem has no solution, unless the vector* $\mathbf{f}$ *is orthogonal to all corresponding eigenvectors*:

$$\langle \mathbf{f}, \mathbf{u}_0\rangle = 0 \,, \qquad A\mathbf{u}_0 = \mathbf{0} \,, \quad \mathbf{u}_0 \neq \mathbf{0}$$

Suppose that $A$ has the zero eigenvalue and $\mathbf{f}$ satisfies the above condition. How to find the solution to the linear problem in this case? Recall that the set of linearly independent eigenvectors of a hermitian $A$ form an orthonormal basis in the Euclidean space. Therefore the solution can be expanded over this basis:

$$\mathbf{u} = \sum_\lambda c_\lambda \mathbf{u}_\lambda \,, \qquad \langle \mathbf{u}_\lambda, \mathbf{u}_{\lambda'}\rangle = \delta_{\lambda\lambda'}$$

For the sake of simplicity of notations all eigenvalues are assumed to be simple in this equation. This is not a limitation. If $\lambda$ has a multiplicity $m_\lambda$, then the *index* $\lambda$ has to be changed to a *double index* $\lambda j$, $j = 1, 2, ..., m_\lambda$ and

$$\sum_\lambda \quad \to \quad \sum_\lambda \sum_{j=1}^{m_\lambda} \qquad \text{and} \qquad \delta_{\lambda\lambda'} \quad \to \quad \delta_{\lambda\lambda'}\delta_{jj'}$$

in all equations. Similarly

$$\mathbf{f} = \sum_\lambda f_\lambda \mathbf{u}_\lambda \,, \qquad f_\lambda = \langle \mathbf{f}, \mathbf{u}_\lambda \rangle$$

The substitution of the expansions of $\mathbf{u}$ and $\mathbf{f}$ into the linear equation yields

$$A\mathbf{u} = \sum_\lambda c_\lambda A\mathbf{u}_\lambda = \sum_\lambda \lambda c_\lambda \mathbf{u}_\lambda = \sum_\lambda f_\lambda \mathbf{u}_\lambda$$

from which it follows (due to linear independence of the basis vectors) that

$$\lambda c_\lambda = f_\lambda = \langle \mathbf{f}, \mathbf{u}_\lambda \rangle \quad \Rightarrow \quad c_\lambda = \frac{1}{\lambda}\langle \mathbf{f}, \mathbf{u}_\lambda \rangle \,, \;\; \lambda \neq 0 \,.$$

Note that $f_\lambda = 0$ for $\lambda = 0$ so that $c_0$ remains arbitrary while all $c_\lambda$ for $\lambda \neq 0$ are uniquely determined. Therefore

$$\mathbf{u} = c_0\mathbf{u}_0 + \sum_{\lambda\neq 0} \frac{1}{\lambda} \langle \mathbf{f}, \mathbf{u}_\lambda \rangle \, \mathbf{u}_\lambda \equiv c_0\mathbf{u}_0 + G\,\mathbf{f} \,.$$

The linear operator $G$ is the analog of the inverse $A^{-1}$ in this case. Note well that its *domain* is reduced to the subspace *orthogonal* to the set of all zero eigenvectors of $A$. If $A$ has no zero eigenvalue, then $G = A^{-1}$.

A similar linear problem exists in functional spaces for differential operators. Although its general analysis goes beyond the scope of this course, a particular case relevant to the Fourier method for partial differential equations, namely, the linear problem for the Sturm-Liouville operator can be solved by means of a basic theory of ordinary differential equations, namely, by the *method of variations of parameters*. In doing so, one can find an analog of the *inverse* $G$ for a linear differential operator, known as a *Green's function* of the differential operator.

**41.1. Linear problem for the Sturm-Liouville operator.** Let $\mathcal{M}_L$ be the domain of a Sturm-Liouville operator $L$

$$(41.1)\qquad u \in \mathcal{M}_L : \quad \begin{cases} u \in C^2(0,l) \cap C^1[0,l]\,, \\ \alpha_0 u(0) - \beta_0 u'(0) = 0\,, \quad \alpha_l u(l) + \beta_l u(l) = 0 \end{cases}$$

$$(Lu)(x) = -\Big(p(x)u'(x)\Big)' + q(x)u(x)$$

where the functions $p$ and $q$ and the constants $\alpha_{0,l}$ and $\beta_{0,l}$ satisfy the standard conditions for a Sturm-Liouville operator. Consider a general linear equation

$$(41.2)\qquad Lu = f\,, \qquad f \in C^0([0,l])$$

The problem is to find a function from the domain $\mathcal{M}_L$ of the operator $L$ that satisfies the equation for all $x \in (0,l)$ (in the *open* interval). Since $L$ maps any function from $\mathcal{M}_L$ into a function that is continuous in $(0,l)$ and square integrable in $(0,l)$, the function $f$ is required to be of that class in order for the problem to make sense.

**The existence and uniqueness of the solution.** Suppose that the problem has a solution. Let us analyze its uniqueness. Just like in the finite dimensional case, let $u_1$ and $u_2$ be two solutions, then $u_0 = u_1 - u_2$ must be an eigenfunction of $L$ corresponding to the zero eigenvalue:

$$Lu_0 = L(u_1 + u_2) = Lu_1 - Lu_2 = f - f = 0\,,$$

because $L$ is a linear operator. Now recall that $L$ has the zero eigenvalue if and only if $\alpha_0 = \alpha_l = 0$ and $q(x) = 0$, and in this case any such eigenfunction is proportional to $u_0(x) = 1$. Therefore, if the Sturm-Liouville operator has the zero eigenvalue, then the linear problem has no solution, unless

$$\langle f, u_0 \rangle = \int_0^l f(x)dx = 0\,.$$

Indeed, if $u$ is a solution, then by hermiticity of $L$

$$\langle f, u_0 \rangle = \langle Lu, u_0 \rangle = \langle u, Lu_0 \rangle = 0\,.$$

The following assertion has been proved

THEOREM **41.1**. *Let $L$ be a Sturm-Liouville operator with domain $\mathcal{M}_L$. In order for the equation*

$$Lu = f\,, \quad u \in \mathcal{M}_L\,, \quad f \in C^0([0,l])$$

*to have a solution it is necessary that $\lambda = 0$ is not an eigenvalue of $L$, and, in this case the solution is unique if it exists. If $\lambda = 0$ is an* *eigenvalue of $L$, then it is necessary that $f$ is orthogonal to the unit function $u_0(x) = 1$:*

$$\langle u_0, f\rangle = \int_0^l f(x)\,dx = 0\,.$$

*In the latter case, any solution has the form $u(x) = cu_0(x) + u_p(x)$ where $c$ is a constant and $u_p$ is a particular solution.*

**Finding a particular solution.** According to a general theory of ordinary differential equations, a particular solution of a linear differential equation can be found by the *method of variations of parameters*. Let us start with an example of the simplest Strum-Liouville operator which has no zero eigenvalue.

EXAMPLE **41.1**. *Solve the boundary value problem*

$$Lu(x) = -u''(x) = f(x)\,, \quad x \in (0,1)$$
$$u(0) = u(1) = 0$$

*Represent the solution in the form*

$$u(x) = G\,f(x) = \int_0^1 G(x,y)f(y)dy\,.$$

SOLUTION: The general solution of the associated homogeneous equation $Lu_0 = 0$ is

$$u_0(x) = c_1 + c_2 x$$

Let us choose two linearly independent solutions, $u_1(x)$ and $u_2(x)$, of the homogeneous equation so that

$$u_1(0) = 0 \quad\Rightarrow\quad u_1(x) = x$$
$$u_2(1) = 0 \quad\Rightarrow\quad u_2(x) = 1 - x$$

In other words,

- the first solution $u_1(x)$ is chosen to satisfy the boundary condition on the left endpoint of the interval $x = 0$, $u_1(0) = 0$
- the second solution $u_2(x)$ satisfies the boundary condition at the right endpoint $x = 1$, $u_2(1) = 0$

Note that the solutions $u_1(x)$ and $u_2(x)$ are linearly independent. This is not by accident. It will be proved below that *two solutions to the Sturm-Liouville equation, one satisfying the boundary condition at the left endpoint of the interval and the other at the right endpoint, are linearly independent*. According to the method of variations of parameters, a particular solution is sought in the form

$$u(x) = u_1(x)v_1(x) + u_2(x)v_2(x) = xv_1(x) + (1-x)v_2(x)\,,$$

where the functions $v_1$ and $v_2$ are to be determined and, in addition, they are also required to satisfy the condition

$$u_1(x)v_1'(x) + u_2(x)v_2'(x) = 0 \quad \text{or} \quad xv'(x) + (1-x)v_2'(x) = 0$$

With this choice of $u_1(x)$ and $u_2(x)$, the functions $v_1$ and $v_2$ must satisfy the boundary conditions

$$v_1(1) = 0\,, \quad v_2(0) = 0$$

in order for $u(x)$ to satisfy $u(0) = u(1) = 0$. The substitution of $u(x)$ into the equation yields

$$\begin{aligned} Lu &= -[xv_1 + (1-x)v_2]'' = -[xv_1' + (1-x)v_2' + v_1 - v_2]' \\ &= -v_1'(x) + v_2'(x) = f(x) \end{aligned}$$

This equation and the additional condition form a system of two equations for the derivatives $v_1'$ and $v_2'$:

$$\begin{cases} xv_1'(x) + (1-x)v_2'(x) = 0 \\ \qquad -v_1'(x) + v_2'(x) = f(x) \end{cases} \qquad v_1(1) = 0\,, \quad v_2(0) = 0$$

Solving the above system for the derivatives, one infers that

$$\begin{cases} v_1'(x) = -(1-x)f(x) \\ v_1(1) = 0 \end{cases} \quad \Rightarrow \quad v_1(x) = \int_x^1 (1-y)f(y)dy$$

$$\begin{cases} v_2'(x) = xf(x) \\ v_2(0) = 0 \end{cases} \quad \Rightarrow \quad v_2(x) = \int_0^x yf(y)dy$$

Thus, the solution has the form

$$u(x) = x\int_x^1 (1-y)f(y)dy + (1-x)\int_0^x yf(y)dy$$

Define the function $G(x,y)$ on the rectangle $[0,1]\times[0,1]$:

$$G(x,y) = \begin{cases} x(1-y)\,,\; y \ge x \\ (1-x)y\,,\; y < x \end{cases}$$

The function is continuous. Note that $G(x,x) = x(1-x)$. Then the solution can be written in the form

$$u(x) = \int_0^1 G(x,y)f(y) \equiv G\,f(x)$$

The *integral operator* $G$ plays the role of the inverse operator for the differential operator $L$. The function $G(x,y)$ is called *the Green's function* of $L$. □

Let us try to apply this method to find a solution in the case when $L$ has the zero eigenvalue. The simplest case is provided by the following example.

EXAMPLE **41.2**. *Solve the boundary value problem*

$$Lu(x) = -u''(x) = f(x)\,, \quad x \in (0,1)$$
$$u'(0) = u'(1) = 0$$

*Represent the solution in the form*

$$u(x) = c_0 u_0(x) + \int_0^1 G(x,y)f(y)dy\,.$$

*where* $u_0(x) = 1$ *is an eigenfunction of* $L$ *corresponding to the zero eigenvalue, and* $f$ *satisfies the condition* $\langle f, u_0\rangle = 0$.

SOLUTION: The difference with the previous example is that in this case *it is impossible to choose two linearly independent solutions so that* $u_1(x)$ *satisfies the boundary condition at* $x = 0$, $u_1'(0) = 0$, *while the other satisfies the boundary condition at* $x = 1$, $u_2'(1) = 0$, because the general solution of the homogeneous equation is a linear function $u_0(x) = c_1 + c_2 x$ and its derivative is constant, $u_0'(x) = c_2$. An attempt to satisfy the boundary condition either at the left or right endpoints leads to constant functions, which implies that $u_1$ is proportional to $u_2$ and, hence, they are not linearly independent.

So, let us just use the method of variation of parameter without any conditions on the choice of two linearly independent solutions. Put

$$u_1(x) = 1\,, \quad u_2(x) = x$$

they are two linearly independent solutions of the equation $u''(x) = 0$. The solution of the linear problem is sought in the form

$$u(x) = u_1(x)v_1(x) + u_2(x)v_2(x) = v_1(x) + xv_2(x)$$

where

$$u_1(x)v_1'(x) + u_2(x)v_2'(x) = v_1'(x) + xv_2'(x) = 0$$

and

$$-u''(x) = f(x) \quad \Rightarrow \quad -v_2'(x) = f(x)$$

Therefore

$$v_1'(x) = xf(x) \quad \Rightarrow \quad v_1(x) = c_1 + \int_0^x yf(y)dy$$
$$-v_2'(x) = f(x) \quad \Rightarrow \quad v_2(x) = c_2 - \int_0^x f(y)dy$$
$$u(x) \;=\; c_1 + c_2 x + \int_0^x yf(y)dy - x\int_0^x f(y)dy$$

The solution contains two integration constants which can be used to satisfy the boundary conditions:

$$u'(x) = c_2 + xf(x) - \int_0^x f(y)dy - xf(x) = c_2 - \int_0^x f(y)dy$$
$$u'(0) = 0 \quad \Rightarrow \quad c_2 = 0$$
$$u'(1) = 0 \quad \Rightarrow \quad \int_0^1 f(y)dy = 0$$

Thus, it is impossible to fulfill the boundary conditions and the problem has no solution unless $\langle f, 1\rangle = 0$ (in full accord with the general analysis). Yet, $c_1$ remains arbitrary. Suppose that $f$ is orthogonal to the unit function. Then

$$u(x) = c_1 + \int_0^x yf(y)dy - x\int_0^x f(y)dy = c_1 + \int_0^x yf(y)dy + x\int_x^1 f(y)dy$$

where it was used that

$$\int_0^x f(y)dy = \int_0^1 f(y)dy - \int_x^1 f(y)dy = -\int_x^1 f(y)dy$$

Define a function $G(x, y)$ on the rectangle $[0, 1] \times [0, 1]$

$$G(x, y) = \begin{cases} y\,, \ y \le x \\ x\,, \ y > x \end{cases}$$

Then the solution can be cast in the form

$$u(x) = c_0 u_0(x) + G\,f(x) = c_0 u_0(x) + \int_0^1 G(x, y)f(y)dy$$

where $u_0(x) = 1$, $c_0$ is an arbitrary constant, and the integral operator $G$ has the domain

$$\mathcal{M}_G = \Big\{ f \in C^0([0, 1]) \,|\, \langle f, u_0\rangle = 0 \Big\}$$

in full accord with Theorem **41.1**. □

**41.2. Green's function in the general case.** Let $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$ be two linearly independent solutions to the associated homogeneous equation $Lu = 0$. Since $L$ is a linear operator, one can always construct another pair of solutions by taking linear combinations of $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$. For example, put

$$u_1(x) = \tilde{u}_1(x) - a\tilde{u}_2(x)\,, \quad u_2(x) = \tilde{u}_2(x) - b\tilde{u}_1(x)$$

Demand that the new pair $u_1(x)$ and $u_2(x)$ satisfies the boundary conditions

$$\alpha_0 u_1(0) - \beta_0 u_1'(0) = 0\,, \quad \alpha_l u_2(l) + \beta_l u_2'(l) = 0\,. \qquad (\mathbf{41.3})$$

By solving these equations for $a$ and $b$

$$a = \frac{\alpha_0 \tilde{u}_1(0) - \beta_0 \tilde{u}_1'(0)}{\alpha_0 \tilde{u}_2(0) - \beta_0 \tilde{u}_2'(0)}\,, \quad b = \frac{\alpha_l \tilde{u}_2(l) + \beta_l \tilde{u}_2'(l)}{\alpha_l \tilde{u}_1(l) + \beta_l \tilde{u}_1'(l)}$$

If the denominator in the equation for $a$ vanishes, then take $u_1 = \tilde{u}_2$. Similarly, for $u_2$. This procedure always makes sense unless a constant is a solution to $Lu = 0$, that is, zero is an eigenvalue of $L$ (see Example **41.2**). Let us assume that $L$ has no zero eigenvalue. The case when $u_1(x) = 1$ ($\alpha_0 = \alpha_l = 0$ and $q(x) = 0$) is left to the reader as an exercise.

According to the method of variation of parameters, a (general) solution to Eq. (**41.2**) can be found in the form (as in Examples **41.1** and **41.2**)

$$u(x) = v_1(x)u_1(x) + v_2(x)u_2(x)\,,$$

where the derivatives of the functions $v_1$ and $v_2$ satisfy the linear system

$$\begin{aligned} u_1 v_1' + u_2 v_2' &= 0 \\ u_1' v_1' + u_2' v_2' &= -\frac{f}{p} \end{aligned}$$

The first equation is just an additional condition of $v_1$ and $v_2$, while the second equation in obtained by the substitution of $u$ into the equation $Lu = f$ and using the conditions that $Lu_1 = 0$ and $Lu_2 = 0$ (the technical details are similar to Examples **41.1** and **41.2**). The system has a unique solution because the determinant of the system is the Wronskian of two linearly independent solutions of the associated homogeneous system:

$$W(x) = \det \begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \neq 0\,, \quad 0 \leq x \leq l\,.$$

By the Liouville-Ostrogradsky theorem (see (**36.6**)),

$$W(x)p(x) = W(0)p(0)\,, \quad 0 \leq x \leq l\,.$$

Therefore

$$v_1'(x) = \frac{f(x)u_2(x)}{p(0)W(0)}\,, \quad v_2'(x) = -\frac{f(x)u_1(x)}{p(0)W(0)}\,.$$

In order for the solution $u(x)$ to satisfy the boundary conditions, it is demanded that

$$v_2(0) = 0\,, \quad v_1(l) = 0\,. \tag{41.4}$$

Indeed

$$\begin{aligned} u'(0) &= v_1'(0)u_1(0) + v_2'(0)u_2(0) + v_1(0)u_1'(0) + v_2(0)u_2'(0) \\ &= v_1(0)u_1'(0) + v_2(0)u_2'(0) \end{aligned}$$

where the first equation $u_1v_1' + u_2v_2' = 0$ in the above linear system was used. Therefore

$$\begin{aligned} \alpha_0 u(0) - \beta_0 u'(0) = v_1(0)\Big(\alpha_0 u_1(0) - \beta_0 u_1'(0)\Big) \\ + v_2(0)\Big(\alpha_0 u_2(0) - \beta_0 u_2'(0)\Big) = 0 \end{aligned}$$

thanks to the first relation in (**41.3**) and the first condition in (**41.4**). The second boundary condition (at $x = l$) in (**41.1**) is verified along similar lines. By integrating the solutions for the derivatives $v_1'$ and $v_2'$ with the initial conditions (**41.4**), one infers that

$$\begin{aligned} v_1(x) &= -\frac{1}{p(0)W(0)} \int_x^l f(y)u_2(y)\,dy \\ v_2(x) &= -\frac{1}{p(0)W(0)} \int_0^x f(y)u_1(y)\,dy \end{aligned}$$

Therefore the solution to the problem (**41.2**), (**41.1**) can be written in the form

$$u(x) = \int_0^l G(x,y)f(y)dy\,, \tag{41.5}$$

$$G(x,y) = -\frac{1}{p(0)W(0)} \begin{cases} u_1(x)u_2(y)\,, & 0 \le x \le y\,, \\ u_2(x)u_1(y)\,, & y \le x \le l\,. \end{cases} \tag{41.6}$$

The function $G(x,y)$ is called the *Green's function* the boundary value problem (**41.2**), (**41.1**).

**41.3. Summary of the procedure to find Green's function.** Let us summarize the procedure to calculate Green's function for the Sturm-Liouville operator.

Step 1: Checking for the zero eigenvalue. Check if zero is an eigenvalue of the Sturm-Liouville operator. If it is not, go to Step 2a, otherwise go to Step 2b.

Step 2a: Linearly independent solutions. Find any two linearly independent solutions to the Sturm-Liouville equation. Take a linear combination of them such that it satisfies the boundary condition at the left endpoint of the interval to get $u_1(x)$. Take another linear combination such that it satisfies the boundary condition at the right endpoint of the interval to get $u_2(x)$. Compute Green's function using (**41**.**6**).

Step 2b: Linearly independent solutions. If zero is an eigenvalue of the Sturm-Liouville operator, take $u_1(x) = 1$ and $u_2(x)$ as another linearly independent solution. Apply the method of variation of parameters to find a particular solution. The result is given in Problem 7 in Exercises.

EXAMPLE **41**.**3**. *Find Green's function for the following boundary value problem*

$$Lu \equiv -u'' + \omega^2 u = f(x)\,, \quad u(0) = u(l) = 0\,. \tag{41.7}$$

SOLUTION: The operator $L$ is a particular case of the Sturm-Liouville operator with $p = 1$, $q = \omega^2 = \text{const} > 0$, and $\beta_0 = \beta_l = 0$. The operator $L$ has no zero eigenvalue because $q = \omega^2 > 0$ (and, hence, the quadratic form $\langle Lu, u\rangle > 0$ is strictly positive for all $u \in \mathcal{M}_L$.

The homogeneous equation has a general solution

$$u(x) = Ae^{\omega x} + Be^{-\omega x}$$

To find $u_1(x)$, one has to choose $A$ and $B$ so that $u(0) = 0$. To find $u_2(x)$, one has to choose $A$ and $B$ so that $u(l) = 0$. For example:

$$u_1(x) = \frac{1}{2}\left(e^{\omega x} - e^{-\omega x}\right)\,, \quad u_2(x) = \frac{1}{2}\left(e^{\omega(l-x)} - e^{-\omega(l-x)}\right)$$

It is technically more convenient to use the hyperbolic functions:

$$\sinh(z) = \frac{1}{2}\Big(e^{z} - e^{-z}\Big)\,, \quad \cosh(z) = \frac{1}{2}\Big(e^{z} + e^{-z}\Big)\,.$$

The general solution can also be written in the form

$$u(x) = A\sinh(\omega x) + B\cosh(\omega x)$$

The advantage of using the hyperbolic functions is that

$$\begin{aligned}&\sinh(0) = 0\,, \quad \cosh(0) = 1\,,\\ &(\sinh(z))' = \cosh(z)\,, \quad (\cosh(z))' = \sinh(z)\end{aligned}$$

These properties allows to easily find $u_1$ and $u_2$. If one demands that $u(a) = 0$ for some $x = a$, then

$$u(a) = 0 \quad \Rightarrow \quad u(x) = \sinh(\omega(x-a))$$

If one demands that $u'(a) = 0$ at some $a$, then

$$u'(a) = 0 \quad \Rightarrow \quad u(x) = \cosh(\omega(x-a))$$

Thus, in this problem let us take

$$u_1(x) = \sinh(\omega x)\,, \quad u_2(x) = \sinh\Big(\omega(l-x)\Big)\,.$$

Note that $u_2$ is a linear combination of $\sinh(\omega x)$ and $\cosh(\omega x)$ because

$$\sinh(a+b) = \cosh(a)\sinh(b) + \cosh(b)\sinh(a)\,.$$

Therefore

$$W(0) = \det\begin{pmatrix} 0 & \sinh(\omega l) \\ \omega & -\omega\cosh(\omega l)\end{pmatrix} = -\omega\sinh(\omega l)$$

and the Green's function of the problem reads
(**41.8**)

$$G(x,y) = \frac{1}{\omega\sinh(\omega l)}\begin{cases} \sinh(\omega x)\sinh\Big(\omega(l-y)\Big)\,, & 0 \le x \le y\,, \\ \sinh\Big(\omega(l-x)\Big)\sinh(\omega y)\,, & y \le x \le l\,. \end{cases}$$

The solution to the problem (**41.7**) is then given by (**41.5**). □

**Remark.** If $\omega = 0$, then two linearly independent solutions of $u'' = 0$ are 1 and $x$. Therefore one can take $u_1(x) = x$ and $u_2(x) = l - x$ to obtain the Green's function in this case. Alternatively, one can take the limit $\omega \to 0$ in (**41.8**).

Example **41.4**. *Solve the boundary value problem using the method of Green's functions:*

$$-u''(x) + 4u(x) = x\,, \quad u'(0) = u(1) = 0\,.$$

Solution: The uniqueness of the solution. In this case, the parameters of the Sturm-Liouville operator are $p(x) = 1$, $q(x) = 0$, $\alpha_0 = 0$, and $\alpha_1 \neq 0$. Thus, $\lambda = 0$ is not an eigenvalue of the considered Sturm-Liouville operator. So the solution exists and is unique. It can be obtained by the Green's function method.
Finding the Green's function. Using the general method proposed above:

$$\begin{aligned} -u_1''(x) + 4u_1(x) = 0\,, \quad u_1'(0) = 0 \quad &\Rightarrow \quad u_1(x) = \cosh(2x) \\ -u_2''(x) + 4u_2(x) = 0\,, \quad u_2(1) = 0 \quad &\Rightarrow \quad u_2(x) = \sinh[2(1-x)] \end{aligned}$$

The Wronskian reads

$$W(0) = \det\begin{pmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0)\end{pmatrix} = \det\begin{pmatrix} 1 & \sinh(2) \\ 0 & -2\cosh(2)\end{pmatrix} = -2\cosh(2)$$

Therefore, taking into account $p(x) = 1$, the Green's function is

$$G(x,y) = \frac{1}{2\cosh(2)}\begin{cases} \cosh(2x)\sinh[2(1-y]\,, & 0 \le x \le y \\ \sinh[2(1-x)]\cosh(2y)\,, & y \le x \le 1 \end{cases}$$

Solving the problem. The solution to the boundary value problem is then

$$u(x) = \int_0^1 G(x,y)\, y\, dy = \frac{\sinh[2(1-x)]}{2\cosh(2)} \int_0^x \cosh(2y)\, y\, dy + \frac{\cosh(2x)}{2\cosh(2)} \int_x^1 \sinh[2(1-y)]\, y\, dy$$

The integrals involved are calculated by integration by parts

$$\begin{aligned}\int_0^x \cosh(2y)\, y\, dy &= \frac{1}{2}\int_0^x y\, d\sinh(2y)\\ &= \frac{x}{2}\sinh(2x) - \frac{1}{2}\int_0^x \sinh(2y)\, dy\\ &= \frac{x}{2}\sinh(2x) - \frac{1}{4}\Big(\cosh(2x) - 1\Big)\\ \int_x^1 \sinh[2(1-y)]\, y\, dy &= -\frac{1}{2}\int_x^1 y\, d\cosh[2(1-y)]\\ &= \frac{x}{2}\cosh[2(1-x)] - \frac{1}{2} + \int_x^1 \cosh[2(1-y)]\, dy\\ &= \frac{x}{2}\cosh[2(1-x)] - \frac{1}{2} - \frac{1}{4}\sinh[2(1-x)]\,.\end{aligned}$$

□

**41.4. More general boundary conditions.** Let $L_v$ be the differential operator that has the same action of any twice differentiable function as the Sturm-Liouville operator

$$L_v u(x) = Lu(x) = -[p(x)u'(x)]' + q(x)u(x)$$

with the functions $p$ and $q$ having the same properties as in the Sturm-Liouville operator, but with a different domain

$$\begin{aligned}&u \in \mathcal{M}_{L_v}: \quad u \in C^2(0,l) \cap C^1[0,l]\,,\\ &\alpha_0 u(0) - \beta_0 u'(0) = v_0\,, \quad \alpha_l u(l) + \beta_l u(l) = v_1\,,\end{aligned} \tag{41.9}$$

where $v_0$ and $v_1$ are given constants. Thus, $L_v \neq L$. The operators $L_v$ and $L$ coincide if $v_0 = v_l = 0$. Recall that (differential) operators are determined by ($i$) the rule by which they act and ($ii$) by the domain (the class of functions on which the rule applies). Evidently, the operators $L_v$ and $L$ have different domains, unless $v_0 = v_l = 0$.

Consider the linear problem for the operator $L_v$

$$L_v u = f\,, \qquad u \in \mathcal{M}_{L_v}\,, \quad f \in C^0([0,l]) \tag{41.10}$$

Suppose that the associated boundary value problem with $v_0 = v_l = 0$ has the unique solution given by (**41.5**). Let $u_1(x)$ and $u_2(x)$ be two linearly independent solutions that are used to construct the Green's function (**41.6**). Then the problem (**41.10**), (**41.9**) also has the unique solution of the form

$$
\begin{aligned}
u(x) &= Au_1(x) + Bu_2(x) + u_p(x)\,, \\
u_p(x) &= \int_0^l G(x,y)f(y)dy \\
A &= \frac{v_l}{\alpha_l u_1(l) + \beta_l u_1'(l)} \\
B &= \frac{v_0}{\alpha_0 u_2(0) - \beta_0 u_2'(0)}
\end{aligned}
\tag{41.11}
$$

To prove the assertion note that, arbitrary $A$ and $B$, the function (**41.11**) is the general solution to the second order differential equation (**41.10**). The particular solution $u_p(x)$ satisfies the boundary conditions (**41.1**) or (**41.9**) with $v_0 = v_l = 0$. The linear combination $Au_1 + Bu_2$ is a general solution to the associated homogeneous equation (**41.10**) ($f = 0$) and, in addition, $u_1(x)$ and $u_2(x)$ satisfy, respectively, the first and second conditions in (**41.1**). The assertion is reduced to verifying whether there is a unique choice of $A$ and $B$ at which the boundary conditions (**41.9**) are satisfied:

$$
\begin{aligned}
v_0 &= \alpha_0 u(0) - \beta_0 u'(0) \\
&= A[\alpha_0 u_1(0) - \beta_0 u_1'(0)] + B[\alpha_0 u_2(0) - \beta_0 u_2'(0)] + \alpha_0 u_p(0) - \beta_0 u_p'(0) \\
&= B[\alpha_0 u_2(0) - \beta_0 u_2'(0)]
\end{aligned}
$$

because $u_1$ and $u_p$ satisfy the first condition in (**41.1**). Similarly,

$$
\begin{aligned}
v_l &= \alpha_l u(l) + \beta_l u'(l) \\
&= A[\alpha_l u_1(l) + \beta_l u_1'(l)] + B[\alpha_l u_2(l) + \beta_l u_2'(l)] + \alpha_l u_p(l) - \beta_l u_p'(l) \\
&= A[\alpha_l u_1(l) + \beta_l u_1'(l)]
\end{aligned}
$$

because $u_2$ and $u_p$ satisfy the second condition in (**41.1**). Thus, $A$ and $B$ are uniquely determined.

EXAMPLE **41.5**. *Use the solution in Example* **41.4** *to solve the boundary value problem*

$$-u''(x) + 4u(x) = x\,, \quad u'(0) = 2\,, \quad u(1) = 1\,.$$

SOLUTION: Let $u_p(x)$ be the solution found in Example **41.4**. Then the solution to the problem in question is unique (as $\alpha_1 = 1 \neq 0$) and should have the form

$$\begin{aligned} u(x) &= Au_1(x) + Bu_2(x) + u_p(x) \\ &= A\cosh(2x) + B\sinh[2(1-x)] + u_p(x) \\ u'(x) &= 2A\sinh(2x) - 2B\cosh[2(1-x)] + u_p'(x) \end{aligned}$$

The constants $A$ and $B$ must be chosen so that the boundary conditions are satisfied. Using that $u_p'(0) = 0$ and $u_p(1) = 0$ by construction,

$$2 = u'(0) = -2B\cosh(2) \quad \Rightarrow \quad B = -\frac{1}{\cosh(2)},$$

$$1 = u(1) = A\cosh(2) \quad \Rightarrow \quad A = \frac{1}{\cosh(2)}$$

$$u(x) = \frac{\cosh(2x)}{\cosh(2)} + \frac{2\cosh[2(1-x)]}{\cosh(2)} + u_p(x)$$

□

**Remark.** Suppose that in the boundary value problem studied $\alpha_0 = \alpha_l = 0$ and $q(x) = 0$. Then the associated homogeneous differential equation

$$-[p(x)u'(x)]' = 0$$

has two linearly independent solutions

$$u_1(x) = 1, \quad u_2(x) = \int_0^x \frac{dy}{p(y)}.$$

A particular solution can be found by the double integration:

$$\begin{aligned} -[p(x)u_p'(x)]' = 0 \quad &\Rightarrow \quad u_p'(x) = -\frac{1}{p(x)}\int_0^x f(y)dy \\ &\Rightarrow \quad u_p(x) = -\int_0^x \frac{1}{p(z)}\int_0^z f(y)dydz. \end{aligned}$$

so that the general solution reads

$$\begin{aligned} u(x) &= Au_1(x) + Bu_2(x) + u_p(x) = A + B\int_0^x \frac{dy}{p(y)} + u_p(x) \\ u'(x) &= \frac{B}{p(x)} - \frac{1}{p(x)}\int_0^x f(y)dy \end{aligned}$$

In this case, the boundary conditions yields

$$\begin{cases} u'(0) = v_0 \\ u'(l) = v_l \end{cases} \quad \Rightarrow \quad \begin{cases} \frac{B}{p(0)} = v_0 \\ \frac{B}{p(l)} - \frac{1}{p(l)}\int_0^1 f(y)\,dy = v_l \end{cases}$$

which is not possible to satisfy in general by a single parameter $B$, unless some additional conditions are imposed on $f$. Indeed, the first equation gives

$$B = v_0 p(0)$$

so that the second equations yields an additional condition on $f$:

$$\int_0^l f(y)dy = v_0 p(0) - v_l p(l) \quad \text{or} \quad \langle f, 1\rangle = v_0 p(0) - v_l p(l)\,.$$

If $f$ does not satisfy this condition, then *the problem has no solution*. If the condition is fulfilled, then $u(x)$ *exists but is not unique* because the constant $A$ remains arbitrary just like in the case of the Sturm-Liouville operator stated in Theorem **41.1**. Note that the condition on $f$ is reduced to the one in Theorem **41.1** if $v_0 = v_l = 0$.

### 41.5. Exercises.

**1**. Find the Green's function for the problem

$$-u''(x) = f(x)\,, \quad -l < x < l\,, \quad u(-l) = u(l) = 0$$

Use it to solve the boundary value problem

$$-u''(x) = f_0 \sin(\pi x/l)\,, \quad u(-l) = u_0\,, \quad u(l) = u_1\,,$$

where $u_0$, $u_1$, and $f_0$ are constants.
*Hint*: Take $u_1(x) = x + l$ and $u_2(x) = x - l$ to construct Green's function. Use Eq. (**41.11**) to solve the problem where the interval $(0, l)$ is to be changed to $(-l, l)$, $\beta_{-l} = \beta_l = 0$, and $\alpha_{-l} = \alpha_l = 1$.

**2**. Find the Green's function for the problem

$$-u''(x) = f(x)\,, \quad 0 < x < l\,,$$
$$\alpha_0 u(0) - \beta_0 u'(0) = 0\,, \quad \alpha_l u(l) + \beta_l u'(l) = 0\,,$$

where $\alpha_0 > 0$ and $\alpha_l > 0$.

**3**. Find the Green's function for the problem

$$u'' + \omega^2 u = f(x)\,, \quad 0 < x < l\,, \quad u(0) = u'(l) = 0\,.$$

Use it to find the solution to the boundary value problem

$$u'' + \omega^2 u = f(x)\,, \quad 0 < x < l\,, \quad u(0) = u_0\,,\ u'(l) = u_l\,.$$

*Hint:* Take $u_1(x) = \sin(\omega x)$ and $u_2(x) = \cos(\omega(l - x))$. Explain why!

**4**. Find the Green's function for the problem

$$-u'' + \omega^2 u = f(x)\,, \quad 0 < x < l\,, \quad u(-l) = u'(l) = 0\,.$$

Use it to find the solution to the boundary value problem

$$-u'' + \omega^2 u = f(x)\,, \quad -l < x < l\,, \quad u(-l) = u_0\,,\ u'(l) = u_l\,.$$

*Hint*: Take $u_1(x) = \sinh[\omega(l + x)]$ and $u_2(x) = \cosh[\omega(l - x)]$. Explain why!

**5**. Find the general solution to the boundary value problem

$$-u'' + \omega^2 u = f(x)\,, \quad 0 < x < l\,, \quad u'(0) = u'(l) = 0\,.$$

*Hint*: Take $u_1(x) = \cosh(\omega x)$ and $u_2(x) = \cosh[\omega(l - x)]$. Explain why!

**6**. Show the eigenvalue problem in the interval $(0, 2\pi)$

$$-u''(x) = \lambda u(x)\,, \quad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)$$

has the same solution as the eigenvalue problem for the operator $L = -d^2/dx^2$ on the circle of unit radius. Find the general solution to the boundary value problem in the interval $(0, 2\pi)$:

$$u''(x) + n^2 u(x) = f(x)\,, \quad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)$$

where $n$ is an integer, or show that it has no solution.
*Hint*: Start with the case $n = 0$. Integrate the equation over the interval $[0, 2\pi]$. Does the solution exist for any $f$? Recall Theorem **41.1**. Suppose $f$ is such that the solution exists. Put $u(x) = xv_1(x) + (2\pi - x)v_2(x)$ and use the method of variation of parameters. Show that $v_1'(2\pi) = 0$, $v_2'(0) = 0$, and $v_1(2\pi) = v_2(0) = C$ where $C$ is some constant. Show that

$$v_1(x) = C - \frac{1}{2\pi}\int_x^{2\pi} (2\pi - y) f(y) dy\,,$$
$$v_2(x) = C - \frac{1}{2\pi}\int_0^{x} y f(y)\, dy$$

Find the Green's function and the general solution. Next, analyze the case when $f$ is not orthogonal to $\cos(nx)$ or $\sin(nx)$ or both, $n \neq 0$. Does the solution exist? Recall Theorem **41.1**. Suppose that $f$ is orthogonal to $\cos(nx)$ and $\sin(nx)$. Find the solution using the method of variation of parameters with $u_1(x) = \cos(nx)$ and $u_2(x) = \sin(nx)$, following a similar line of arguments as in the case $n = 0$.

**7**. Prove that the following boundary value problem for a Sturm-Liouville operator with zero eigenvalue

$$Lu(x) = -[p(x)u'(x)]' = f(x)\,, \quad x \in (0, l)$$
$$u'(0) = u'(l) = 0$$

has the general solution

$$u(x) = c_0 + \int_0^l G(x, y) f(y)\, dy\,,$$
$$G(x, y) = \begin{cases} u_2(y)\,, & y \le x \\ u_2(x)\,, & y > x \end{cases}\,, \qquad u_2(x) = \int_0^x \frac{dy}{p(y)}$$

if $f$ is orthogonal to the unit function, $\langle f, 1\rangle = 0$, where $c_0$ is an arbitrary constant, and the problem has no solution if $\langle f, 1\rangle \ne 0$.
*Hint*: As in Example **41.2**, use the method of variation of parameters to find a particular solutions with two linearly independent solutions of the associated homogeneous equation $u_1(x) = 1$ and $u_2(x)$.

**8**. Change the boundary conditions in the previous problem to

$$u'(0) = v_0\,, \quad u'(l) = v_l\,.$$

Prove that the general solution has the form

$$u(x) = c_0 + v_l p(l) u_2(x) + \int_0^l G(x, y) f(y)\, dy$$

where $c_0$ is an arbitrary constant, if $f$ satisfies the condition $\langle f, 1\rangle = v_0 p(0) - v_l p(l)$, and no solution exists otherwise.

## 42. Poisson equation in rectangles

**42.1. The Dirichlet problem for the Poisson equation.** Let $u_0(x,y)$ be a solution to the Dirichlet problem for the Laplace equation:

$$\Delta u_0(x,y) = 0\,, \quad (x,y) \in \Omega\,,$$
$$u_0\Big|_{\partial\Omega} = v(x,y)\,, \quad (x,y) \in \partial\Omega\,.$$

The solution to this problem is known to be unique. Let us seek the solution to the Dirichlet problem for the Poisson equation in the form

$$u = u_0 + u_f$$

Then the unknown function $u_f$ is the solution to the Dirichlet problem with the zero boundary condition for the Poisson equation:

$$-\Delta u_f(x,y) = f(x,y)\,, \quad (x,y) \in \Omega\,, \qquad (\mathbf{42.1})$$
$$u_f\Big|_{\partial\Omega} = 0\,.$$

The latter problem is proved to have a solution from the class $C^2(\Omega) \cap C^0(\overline{\Omega})$ if the function $f$ is sufficiently smooth. In particular, for a rectangular region $\Omega$, the solution can be obtained by the Fourier method.

**42.2. The Dirichlet problem in a rectangle.** Let $\Omega = (0,a) \times (0,b)$. Let $X_k(x)$ be the orthonormal basis of the eigenfunctions of the Sturm-Liouville operator

$$-X_k''(x) = \nu_k^2 X_k(x)\,, \quad 0 < x < a\,,$$
$$X_k(0) = X_k(a) = 0\,,$$
$$\nu_k = \frac{\pi k}{a}\,, \quad X_k = \sin(\nu_k x)\,, \quad k = 1, 2, \ldots\,,$$
$$\langle X_k, X_j \rangle = \int_0^a X_k(x) X_j(x)\, dx = \frac{a}{2}\delta_{jk}\,.$$

A solution to the problem (**42.6**) is sought in the form

$$u_f(x,y) = \sum_{k=1}^{n} \tilde{Y}_k(y) X_k(x)$$

Then the boundary conditions at $x = 0$ and $x = a$ are automatically satisfied:

$$u_f(0,y) = u_f(a,y) = 0\,, \quad 0 \le y \le b\,.$$

The zero boundary conditions on the other two edges of the rectangle

$$u_f(x,0) = u_f(x,b) = 0\,, \quad 0 \le x \le a$$

require that

$$\tilde{Y}_k(0) = \tilde{Y}_k(b) = 0\,,$$

for any $k$, because of the linear independence of the basis functions $X_k$. The substitution of $u_f$ into the Poisson equation yields

$$-\Delta u_f(x,y) = \sum_{k=1}^{n} \Big(-\tilde{Y}_k''(y) + \nu_k^2 \tilde{Y}_k(y)\Big) X_k(x) = f(x,y)$$

where the relation $X_k'' = -\nu_k^2 X_k$ was used. If $f(x,y)$ is a linear combination of the basis functions:

$$f(x,y) = \sum_{k=1}^{n} F_k(y) X_k(x)\,,$$

$$F_k(y) = \frac{\langle f, X_k\rangle}{\|X_k\|^2} = \frac{2}{a}\int_0^a f(x,y) X_k(x)\,dx\,,$$

then owing to the linear independence of the basis functions, the coefficients $\tilde{Y}_k(y)$ must be solutions to the boundary value problems for a Sturm-Liouville operator:

$$-\tilde{Y}_k''(y) + \nu_k^2 \tilde{Y}_k(y) = F_k(y)\,, \qquad Y_k(0) = Y_k(b) = 0\,.$$

This problem is solved by the method of variation of parameters as explained in Section **41.3** and the answer can be written with the help of Green's function of the Sturm-Liouville operator:

$$\tilde{Y}_k(y) = \int_0^b G_k(y,y') F_k(y')\,dy'\,, \tag{42.2}$$

$$G_k(y,y') = \frac{1}{\nu_k \sinh(\nu_k b)} \begin{cases} \sinh\Big(\nu_k(b-y)\Big)\sinh(\nu_k y')\,, & y' \le y \\ \sinh(\nu_k y)\sinh\Big(\nu_k(b-y')\Big)\,, & y \le y'\,. \end{cases}$$

The problem is solved.

If $f(x,y)$ is not a linear combination of the basis functions, then the solution is sought a formal Fourier series

$$u_f(x,y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x) = \sum_{k=1}^{\infty} \tilde{Y}_k(y)\sin(\nu_k x) \tag{42.3}$$

where $Y_k(y)$ are given by (**42.2**). The existence and smoothness of the formal solution can be studied by standard means.

**Alternative approach.** Instead of using the horizontal basis $X_k$ in the interval $0 \le x \le a$, one can use the basis in the interval $0 \le y \le b$

$$Y_k(y) = \sin(\mu_k y)\,, \quad \mu_k = \frac{\pi k}{b}\,, \quad k = 1, 2, \dots\,,$$
$$\langle Y_k, Y_j\rangle = \int_0^b Y_k(y) Y_j(y) = \frac{b}{2}\,\delta_{jk}$$

to expand the solution into the Fourier series

$$u_f(x,y) = \sum_{k=1}^{\infty} \tilde{X}_x(x) Y_k(y) = \sum_{k=1}^{\infty} \tilde{X}_k(x)\sin(\mu_k y) \tag{42.4}$$

This expansion automatically ensures $u_f(x,0) = u_f(x,b) = 0$, the Poisson equation and the other two boundary conditions require that the coefficients $\tilde{X}_k$ satisfy the boundary value problems

$$-\tilde{X}_k'' + \mu_k^2 \tilde{X}_k(x) = H_k(x)\,, \quad \tilde{X}_k(0) = \tilde{X}_k(a) = 0\,,$$
$$H_k(x) = \frac{\langle f, Y_k\rangle}{\|Y_k\|^2} = \frac{2}{b}\int_0^b f(x,y)\sin(\mu_k y)\,dy\,.$$

The solution to the boundary value problem is found by the method of variation of parameters and can be written using the Green's function:

$$\tilde{X}_k(x) = \int_0^a G_k(x,x') H_k(x')\,dx'\,, \tag{42.5}$$

$$G_k(x,x') = \frac{1}{\mu_k \sinh(\mu_k a)} \begin{cases} \sinh\Big(\mu_k(a-x)\Big)\sinh(\mu_k x')\,, & x' \le x \\ \sinh(\mu_k x)\sinh\Big(\mu_k(a-x')\Big)\,, & x \le x'\,. \end{cases}$$

The sums of the series (**42.5**) and (**42.3**) converge in the mean and therefore may differ in the rectangle $[0,a]\times[0,b]$ only in a set of measure zero. If $f$ is smooth enough, the series give the same classical solution. The use of either (**42.5**) or (**42.3**) for solving the Poisson equation is a matter of technical convenience. One can even split $f(x,y)$ into a sum

$$f(x,y) = f_1(x,y) + f_2(x,y)$$

and solve the Poisson problem for $f = f_1$ by the Fourier series (**42.3**) while for $f = f_2$ by the Fourier series (**42.5**). The sum of the two formal solutions is the formal solution of the original problem. The choice of $f_1$ and $f_2$ is a matter of convenience.

EXAMPLE **42.1**. *Find the formal solution to the Dirichlet problem for the Poisson equation in a rectangle:*

$$-\Delta u(x,y) = y\sin(2\pi x)\,, \quad (x,y)\in(-1,1)\times(-1,0)\,,$$
$$u(-1,y) = y(1+y)\,, \quad u(1,y)=0\,, \quad y\in[-1,0]$$
$$u(x,-1)=0\,, \quad u(x,0)=x^2-1\,, \quad x\in[-1,1]\,.$$

SOLUTION: The solution is the sum

$$u(x,y) = u_0(x,y) + u_f(x,y)$$

where $u_0$ is the solution to the associated Dirichlet problem for the Laplace equation:

$$\Delta u_0(x,y) = 0\,, \quad (x,y)\in(-1,1)\times(-1,0)\,,$$
$$u_0(-1,y) = y(1+y)\,, \quad u_0(1,y)=0\,, \quad y\in[-1,0]$$
$$u_0(x,-1)=0\,, \quad u_0(x,0)=x^2-1\,, \quad x\in[-1,1]\,.$$

while $u_f$ is the solution to the associated Poisson equation with zero boundary conditions:

$$\Delta u_f(x,y) = -y\sin(2\pi x)\,, \quad (x,y)\in(-1,1)\times(-1,0)\,,$$
$$u_f(-1,y) = 0\,, \quad u_f(1,y)=0\,, \quad y\in[-1,0]$$
$$u_f(x,-1)=0\,, \quad u_f(x,0)=0\,, \quad x\in[-1,1]\,.$$

Associated Dirichlet problem for the Laplace equation. The solution is the sum

$$u_0(x,y) = U_1(x,y) + U_2(x,y)$$

where $U_1$ is the solution to the associated problem obtained by setting the boundary conditions at $x=\pm 1$ to zero:

$$\Delta U_1(x,y) = 0\,, \quad (x,y)\in(-1,1)\times(-1,0)\,,$$
$$U_1(-1,y) = 0\,, \quad U_1(1,y)=0\,, \quad y\in[-1,0]$$
$$U_1(x,-1)=0\,, \quad U_1(x,0)=x^2-1\,, \quad x\in[-1,1]\,.$$

while $U_2$ is the solution to the associated problem obtained by setting the boundary conditions at $y=-1$ and $y=0$ to zero:

$$\Delta U_2(x,y) = 0\,, \quad (x,y)\in(-1,1)\times(-1,0)\,,$$
$$U_2(-1,y) = y(1+y)\,, \quad U_2(1,y)=0\,, \quad y\in[-1,0]$$
$$U_2(x,-1)=0\,, \quad U_2(x,0)=0\,, \quad x\in[-1,1]\,.$$

The solution to the first problem is given by the Fourier series

$$U_1(x,y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x)\,,$$

over the orthonormal set in the interval $-1 \leq x \leq 1$:

$$X_k(x) = \sin[\nu_k(x+1)]\,, \quad \nu_k = \frac{\pi k}{2}\,, \quad k = 1, 2, ...$$

defined by the Sturm-Liouville eigenvalue problem:

$$-X_k''(x) = \nu_k^2 X_k(x)\,, \qquad X_k(\pm 1) = 0\,.$$

The expansion coefficients $\tilde{Y}_k$ satisfies the boundary value problem

$$\begin{aligned}
&\tilde{Y}_k''(y) - \nu_k^2 \tilde{Y}_k(y) = 0\,, \quad -1 \leq y \leq 0\,,\\
&\tilde{Y}_k(-1) = 0\,, \quad \tilde{Y}_k(0) = A_k\,,\\
&A_k = \langle x^2 - 1, X_k \rangle = \int_{-1}^{1} (x^2 - 1)\sin[\nu_k(x+1)]\,dx\\
&= \int_0^2 (t^2 - 2t)\sin(\nu_k t)\,dt = \frac{2((-1)^k - 1)}{\nu_k^3}
\end{aligned}$$

See Example **39.1** for details to calculate the integral. The expansion coefficients read

$$\tilde{Y}_k(y) = \tilde{Y}_k(0)\frac{\sinh[\nu_k(y+1)]}{\sinh(2\nu_k)} = A_k \frac{\sinh[\nu_k(y+1)]}{\sinh(2\nu_k)}$$

so that

$$U_1(x, y) = \sum_{k=1}^{\infty} A_k \frac{\sinh[\nu_k(y+1)]}{\sinh(2\nu_k)} \sin[\nu_k(x+1)]$$

The solution to the second problem is given by the Fourier series

$$U_2(x, y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y)\,,$$

over the orthonormal set in the interval $-1 \leq y \leq 0$:

$$Y_k(y) = \sin(\mu_k y)\,, \quad \|Y_k\|^2 = \frac{1}{2}\,, \quad \mu_k = \pi k\,, \quad k = 1, 2, ...$$

defined by the Sturm-Liouville eigenvalue problem:

$$-Y_k''(y) = \mu_k^2 Y_k(y)\,, \qquad Y_k(-1) = Y_k(0) = 0\,.$$

The expansion coefficients $\tilde{X}_k$ satisfies the boundary value problem

$$\tilde{X}_k''(x) - \mu_k^2 \tilde{X}_k(x) = 0\,, \quad -1 \le x \le 1\,,$$
$$\tilde{X}_k(-1) = B_k\,, \quad \tilde{X}_k(1) = 0\,,$$
$$B_k = \frac{\langle y(1+y), Y_k\rangle}{\|Y_k\|^2} = 2\int_{-1}^{0}(y+y^2)\sin(\mu_k y)\,dy$$
$$= -\frac{2}{\mu_k}\int_{-1}^{0}(1+2y)\cos(\mu_k y)\,dy = \frac{4}{\mu_k^2}\int_{-1}^{0}\sin(\nu_k x)\,dx$$
$$= \frac{4}{\mu_k^3}\Big((-1)^k - 1\Big)$$

The integral was evaluated by integration by parts twice. The expansion coefficients reads

$$\tilde{X}_k(x) = -\tilde{X}_k(-1)\frac{\sinh(\mu_k y)}{\sinh(\nu_k)} = -B_k\frac{\sinh(\mu_k y)}{\sinh(\nu_k)}$$

so that

$$U_2(x,y) = -\sum_{k=1}^{\infty} B_k\,\frac{\sinh(\mu_k y)}{\sinh(\mu_k)}\sin(\mu_k y)$$

Associated Poisson equation with zero boundary conditions. The solution $u_f(x,y)$ can be expanded into the Fourier series over either the orthonormal set in the interval $-1 \le x \le$ or in the interval $-1 \le y \le 0$. The function

$$f(x,y) = y\sin(2\pi x)$$

is proportional to one of the basis functions $X_k(x) = \sin[\nu_k(x+1)]$:

$$X_4(x) = \sin[\nu_4(x+1)] = \sin[2\pi(x+1)] = \sin(2\pi x)$$

Therefore the Fourier expansion of $f$ over the basis in the interval $-1 \le x \le 1$ consists of just one term:

$$F_k(y) = \langle f, X_k\rangle = y\delta_{4k}\,,$$
$$f(x,y) = \sum_{k=1}^{\infty}\langle f, X_k\rangle \sin[\nu_k(x+1)] = y\sin[\nu_4(x+1)]$$

It is therefore convenient to choose this basis to expand $u_f$. Then the expansion contains just one term with $k=4$:

$$u_f(x,y) = \sum_{k=1}^{\infty}\tilde{Y}_k(y)X_k(x) = \tilde{Y}_4(y)X_4(x)$$

where $\tilde{Y}_k(y)$ is the solution to the boundary value problem:

$$-\tilde{Y}_4''(y) + \nu_4^2\tilde{Y}_4(y) = y\,, \qquad \tilde{Y}_4(-1) = \tilde{Y}_4(0) = 0$$

The method of undetermined coefficients. Note that the inhomogeneity has a special form, it is a polynomial. Therefore the method of undermined coefficients can be used to solve the problem, which is technically simpler than the Green's function method. The characteristic equation is $\lambda^2 - \nu_4^2 = 0$. So, $\lambda = 0$ is not a root. Therefore a particular solution must be of the form $\tilde{Y}_p = cy$. A substitution into the equation gives

$$c\nu_4^2 y = y \quad \Rightarrow \quad c = \frac{1}{\nu_4^2}$$

A general solution has the form

$$\tilde{Y}_4(y) = A\sinh(\nu_4(y+1)) + B\sinh(\nu_4 y) + \frac{y}{\nu_4^2}$$

where two linearly independent solution to the associated homogeneous equation were chosen to satisfy the left and right zero boundary conditions, respectively, just like in the Green's function method. The condition $\tilde{Y}_4(-1) = 0$ gives

$$-B\sinh\nu_4 - \frac{1}{\nu_4^2} = 0 \quad \Rightarrow \quad B = -\frac{1}{\nu_4^2\sinh\nu_4}$$

and $\tilde{Y}_4(0) = 0$ requires that $A = 0$ so that

$$\tilde{Y}_4(y) = -\frac{\sinh(\nu_4 y)}{\nu_4^2\sinh\nu_4} + \frac{y}{\nu_4^2}$$

This method is indeed simpler, but it has a limited applicability (only for special forms of the inhomogeneity).

The Green's function method. The Green's function given in (**42.2**) (the Green's function in the interval $[0, b]$ with $b = 1$ is mapped onto the Green's function in $[-1, 0]$ if $y$ and $y'$ are changed to $-y$ and $-y'$, respectively)

$$G_4(y, y') = \frac{1}{\nu_4\sinh(\nu_4)}\begin{cases} \sinh\Big(\nu_4(1+y)\Big)\sinh(\nu_4 y'), & y \le y' \\ \sinh(\nu_4 y)\sinh\Big(\nu_4(1+y')\Big), & y' \le y. \end{cases}$$

Note that $G_4(-1, y') = G_4(0, y') = 0$ as required. Then

$$\begin{aligned}\tilde{Y}_4(y) &= \int_{-1}^{0} G_4(y, y') y' \, dy' \\ &= \frac{\sinh[\nu_4(y+1)]}{\nu_4 \sinh(\nu_4)} \int_y^0 \sinh(\nu_4 y') y' dy' \\ &\quad + \frac{\sinh(\nu_4 y)}{\nu_4 \sinh(\nu_4)} \int_{-1}^{y} \sinh[\nu_4(1+y')] y' dy'\end{aligned}$$

The integrals in this expression are evaluated by parts:

$$\begin{aligned}\int_y^0 \sinh(\nu_4 y') y' dy' &= \int_0^y \sinh(\nu_4 t) \, t \, dt \\ &= \frac{t \cosh(\nu_4 t)}{\nu_4} \Big|_0^y - \frac{1}{\nu_4} \int_0^y \cosh(\nu_4 t) \, dt \\ &= \frac{y \cosh(\nu_4 y)}{\nu_4} - \frac{\sinh(\nu_4 y)}{\nu_4^2}\end{aligned}$$

and similarly

$$\begin{aligned}&\int_{-1}^{y} \sinh[\nu_4(1+y')] y' dy' \\ &= \frac{y' \cosh[\nu_4(1+y')]}{\nu_4} \Big|_{-1}^{y} - \frac{1}{\nu_4} \int_{-1}^{y} \cosh[\nu_4(1+y')] \, dy' \\ &= \frac{1}{\nu_4} \Big( y \cosh[\nu_4(1+y)] + 1 \Big) - \frac{\sinh[\nu_4(1+y)]}{\nu_4^2}\end{aligned}$$

Using identities for hyperbolic functions, the final answer can be simplified to

$$\tilde{Y}_4(y) = -\frac{\sinh(\nu_4 y)}{\nu_4^2 \sinh(\nu_4)} + \frac{y}{\nu_4^2}$$

It is clear that the method of undermined coefficients is simpler in this case.

The formal solution. The formal solution is the sum

$$u(x, y) = U_1(x, y) + U_2(x, y) + Y_4(y) \sin(2\pi x)$$

The Fourier series for $U_1$ and $U_2$ are classical solutions to the associated Dirichlet problems by Proposition **39**.1 (the Fourier coefficients of the boundary data decay as $1/k^3$ and $\sum 1/k^3 < \infty$). The obtained solution is the classical solution. □

**42.3. The mixed problem for the Poisson equation.** Let $u_0(x, y)$ be a solution to the mixed problem for the Laplace equation:

$$\Delta u_0(x, y) = 0\,, \quad (x, y) \in \Omega\,,$$
$$\alpha u_0 + \beta \frac{\partial u_0}{\partial \mathbf{n}}\Big|_{\partial\Omega} = v(x, y)\,, \quad (x, y) \in \partial\Omega\,.$$

where $\alpha$ and $\beta$ are non-negative and cannot vanish simultaneously on $\partial\Omega$. The solution to this problem is known to be unique (if it is not a Neumann problem, $\alpha \neq 0$). Let us seek a solution to the mixed problem for the Poisson equation in the form

$$u = u_0 + u_f$$

Then the unknown function $u_f$ is the solution to the mixed problem with the zero boundary condition for the Poisson equation:

$$(\mathbf{42.6}) \qquad -\Delta u_f(x, y) = f(x, y)\,, \quad (x, y) \in \Omega\,,$$
$$\alpha u_f + \beta \frac{\partial u_f}{\partial}\Big|_{\partial\Omega} = 0\,.$$

The latter problem is proved to have a solution from the class $C^2(\Omega) \cap C^0(\overline{\Omega})$ if the function $f$ is sufficiently smooth. In particular, for a rectangular region $\Omega$, the solution can be obtained by the Fourier method in full analogy with the Dirichlet problem discussed above.

EXAMPLE **42.2**. *Solve the boundary value problem for the Poisson equation:*

$$-\Delta u(x, y) = x^2 \cos(\pi x/4)\,, \quad (x, y) \in (-1, 1) \times (0, 2)\,,$$
$$u(-1, y) = y(2 - y)\,, \quad u'_x(1, y) = 0\,, \quad y \in [0, 2]\,,$$
$$-u'_y(x, 0) = 1 - x^2\,, \quad u(x, 2) = 0\,, \quad x \in [-1, 1]\,.$$

SOLUTION: The solution is sought in the form

$$u(x, y) = U(x, y) + U_f(x, y)$$

where $U(x, y)$ is the solution of the associated mixed problem for the Laplace equation. It was found in Example **39.2**. The function $U_f$ is the solution to the associated problem with trivial boundary conditions:

$$-\Delta U_f(x, y) = x^2 \cos(\pi y/4)\,, \quad (x, y) \in (-1, 1) \times (0, 2)\,,$$
$$U_f(-1, y) = 0\,, \quad U'_{fx}(1, y) = 0\,, \quad y \in [0, 2]\,,$$
$$-U'_{fy}(x, 0) = 0\,, \quad U_f(x, 2) = 0\,, \quad x \in [-1, 1]\,.$$

The inhomogeneity of the Poisson equation is proportional to one of the basis function in the vertical interval $0 \leq y \leq 2$ used in the Fourier expansion of $U$ (see Example **39.2**). Therefore the solution has the form

$$U_f(x,y) = \tilde{X}_1(x)Y_1(y) = \tilde{X}_1(x)\cos(\pi y/4)$$

Note that the boundary condition on the vertical edges is satisfied. The expansion coefficient $X_1(x)$ is the solution to the boundary value problem:

$$-\tilde{X}_1''(x) + \mu_1^2\,\tilde{X}_1(x) = x^2\,, \qquad \tilde{X}_1(-1) = \tilde{X}_1'(1) = 0\,, \quad \mu_1 = \frac{\pi}{4}\,.$$

It can be solved by the Green's function method. The technicalities are left to the reader as an exercise.

Method of undermined coefficients. Owing to a particular type of the inhomogeneity of this linear equation, the method of undetermined coefficients for linear differential equations with constant coefficients is technically simpler. Since the inhomogeneity is a polynomial of degree 2, a particular solution is also a polynomial degree 2:

$$\tilde{X}_{1p} = a_0 + a_1x + a_2x^2$$

The substitution into the equation yields

$$\begin{aligned} &-2a_2 + \mu_1^2(a_0 + a_1x + a_2x^2) = x^2 \\ \Rightarrow \quad & a_2 = \frac{1}{\mu_1^2}\,,\ a_1 = 0\,,\ a_0 = \frac{2a_2}{\mu_1^2} \end{aligned}$$

The general solution is convenient to take in the form

$$\tilde{X}_1(x) = \frac{1}{\mu_1^2}\Big(x^2 + \frac{2}{\mu_1^2}\Big) + A\sinh[\mu_1(x+1)] + B\cosh[\mu_1(1-x)]$$

The general solution to the homogeneous equation is taken as a linear combination of a solution satisfying the boundary condition at $x = -1$ and a solution satisfying the boundary condition at $x = 1$. The constants $A$ and $B$ are determined from the boundary conditions

$$\begin{aligned} \tilde{X}_1(-1) = 0 \quad &\Rightarrow \quad B = -\frac{\mu_1^2 + 2}{\mu_1^4\cosh(2\mu_1)}\,, \\ \tilde{X}_1'(1) = 0 \quad &\Rightarrow \quad A = -\frac{2}{\mu_1^3\cosh(2\mu_1)} \end{aligned}$$

The solution is complete.

The use of the other basis. It is noteworthy that the solution can also be obtained by expanding it over the basis in the horizontal interval

$-1 \le x \le 1$ (see Example **39.2**):

$$U_f(x,y) = \sum_{k=1}^{\infty} \tilde{Y}_k(y) X_k(x)\,,$$
$$X_k(x) = \sin[\nu_k(x+1)]\,, \quad \nu_k = \frac{\pi}{4}\,(2k-1)$$

In this case, the expansion coefficients $\tilde{Y}_k$ satisfy the boundary value problem:

$$-\tilde{Y}_k''(y) + \nu_k^2 \tilde{Y}_k(y) = \alpha_k \cos(\pi y/4)\,, \quad \tilde{Y}_k'(0) = \tilde{Y}_k(2) = 0$$
$$\alpha_k = \frac{\langle x^2, X_k\rangle}{\|X_k\|^2} = \int_{-1}^{1} x^2 \sin[\nu_k(x+1)]\,dx$$
$$= \frac{1}{\nu_k}\left(1 + \frac{2(-1)^{k+1}}{\nu_k} - \frac{1}{\nu_k^2}\right)$$

Its solution can either be found by the Green's function method or by the method of undetermined coefficients (the latter is technically much simpler in this case):

$$Y_k(y) = \frac{\alpha_k}{\mu_1^2 + \nu_k^2}\,\cos(\mu_1 y)\,, \qquad \mu_1 = \frac{\pi}{4}$$

The details are left to the reader as an exercise. The formal solution reads

$$U_f(x,y) = \cos(\mu_1 y) \sum_{k=1}^{\infty} \frac{\alpha_k}{\mu_1^2 + \nu_k^2}\,\sin[\nu_k(x+1)]$$

It has a different form. Since the formal solution exists (the series converges), it is concluded that the sum of this Fourier series coincides with the solution obtained by the expansion over the basis in the vertical interval. It is equal to the expansion coefficient $\tilde{X}_1(x)$ in the previous method:

$$\tilde{X}_1(x) = \sum_{k=1}^{\infty} \frac{\alpha_k}{\mu_1^2 + \nu_k^2}\,\sin[\nu_k(x+1)]$$

The reader is advised calculate the Fourier coefficients of $\tilde{X}_1$ over the basis $X_k(x)$ and compare them with the coefficients of the above Fourier series.

Note well that the first form of the solution is obviously from the class $C^2(\Omega) \cap C^1(\overline{\Omega})$. In contrast, it is not obvious that the sum of the above Fourier series is twice continuously differentiable because $|\alpha_k| \sim 1/k$ for large $k$. This illustrate a limitation of the sufficient conditions for differentiability of the Fourier series based on the summation of the upper bounds of terms and their derivatives. □

**42.4. The Neumann problem for the Poisson equation.** The particular case $\beta = 1$ and $\alpha = 0$ of the mixed boundary conditions is known as the Neumann problem for the Poisson equation:

$$
\begin{aligned}
-\Delta u(x,y) &= f(x,y)\,, \quad (x,y) \in \Omega\,, \\
\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} &= v(x,y)\,, \quad (x,y) \in \partial\Omega\,,
\end{aligned} \tag{42.7}
$$

The Neumann problem does not a solution for any choice of parameter functions $v$ and $f$. They must satisfy the solvability condition in order for the Neumann problem to have a solution:

$$\int_{\partial\Omega} v(x,y)\,ds = -\iint_{\Omega} f(x,y)\,dxdy \tag{42.8}$$

and, in this case, the solution is unique up to an additive constant.

In contrast to the Dirichlet or mixed problem, the solution cannot be found as the sum of the solutions to two Neumann problems (one with $f = 0$ and the other with $v = 0$). In order for each problem to have a solution, the solvability condition must be fulfilled in each problem, which gives a more restrictive condition on $v$ and $f$ than the solvability condition of the original problem (**42**.**8**):

$$\int_{\partial\Omega} v\,ds = 0\,, \qquad \iint_{\Omega} f\,dxdy = 0$$

The first one is the solvability condition for the Neumann problem with $f = 0$, while the second one is the solvability condition for the Neumann problem with $v = 0$. If, however, $v$ and $f$ happen to satisfy the restricted solvability condition, then the solution of the problem can be found as the sum of the solutions to two aforementioned Neumann problems.

The idea for solving is similar to the Neumann problem for the Laplace equation. The solution is sought in the form

$$u(x,y) = u_0(x,y) + U(x,y)$$

where $u_0$ is a solution to the Neumann problem with constant boundary data and constant inhomogeneity:

$$
\begin{aligned}
-\Delta u(x,y) &= f_0\,, \quad (x,y) \in \Omega\,, \\
\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} &= v_0\,, \quad (x,y) \in \partial\Omega\,,
\end{aligned}
$$

where $f_0$ and $v_0$ are the average values of $f$ and $v$, respectively:

$$f_0 = \frac{1}{A(\Omega)} \iint_{\Omega} f(x,y)\,dxdy\,, \qquad v_0 = \frac{1}{L(\partial\Omega)} \oint_{\partial\Omega} v(x,y)\,ds$$

where $A(\Omega)$ and $L(\partial\Omega)$ are the area of $\Omega$ and the arclength of the boundary $\partial\Omega$. By construction this Neumann problem satisfy the solvability condition. Its solution depends on the shape of $\Omega$. For simple regions like a rectangle or a disk, its solution is easy to find as shown below.

The function $U(x, y)$ is a solution to the associated Neumann problem whose data have zero average value:

$$-\Delta U(x,y) = f(x,y) - f_0\,, \quad (x,y)\in\Omega\,,$$
$$\left.\frac{\partial U}{\partial \mathbf{n}}\right|_{\partial\Omega} = v(x,y) - v_0\,, \quad (x,y)\in\partial\Omega\,,$$

and, hence, they satisfy the restricted solvability condition by construction:

$$\iint_\Omega \Big(f(x,y) - f_0\Big)\,dxdy = 0\,, \quad \oint_{\partial\Omega} \Big(v(x,y) - v_0\Big)\,ds = 0$$

The problem can therefore be solved by the same method as the Dirichlet or mixed problem. The case of a rectangular region is analyzed in the next section.

**42.5. The Neumann problem in a rectangle.** Let

$$\Omega = (0,a)\times(0,b)\,.$$

The Neumann boundary conditions read

$$-u'_x(0,y) = v_0(y)\,, \quad u'_x(a,y) = v_a(y)\,,$$
$$-u'_y(x,0) = h_0(x)\,, \quad u'_y(x,b) = h_b(x)\,.$$

The average values of the boundary data and the inhomogeneity are

$$h_{00} = \frac{1}{a}\int_0^a h_0(x)\,dx\,, \quad h_{b0} = \frac{1}{a}\int_0^a h_b(x)\,dx$$
$$v_{00} = \frac{1}{b}\int_0^b v_0(y)\,dy\,, \quad v_{a0} = \frac{1}{b}\int_0^b h_b(y)\,dy$$
$$f_0 = \frac{1}{ab}\int_0^a\int_0^b f(x,y)\,dy\,dx$$

The solvability condition (**42.8**) for a rectangle can be cast in the following form

$$a(h_{00} + h_{b0}) + b(v_{00} + v_{a0}) = -abf_0$$

Consider the associated Neumann problem with constant boundary data and a constant inhomogeneity:

$$\begin{aligned} &\Delta w_0(x,y) = -f_0\,, \quad (x,y) \in (0,a)\times(0,b)\,,\\ &-w'_x(0,y) = v_{00}\,, \quad w'_x(a,y) = v_{a0}\,, \quad y \in [0,b]\,,\\ &-w'_y(x,0) = h_{00}\,, \quad w'_y(x,b) = h_{b0}\,, \quad x \in [0,a]\,. \end{aligned}$$

Let us try a general polynomial of degree two for a solution

$$w(x,y) = Ax + By + Cx^2 + Dy^2$$

Note that a general constant can be omitted as a solution is unique up to an additive constant, and the term $xy$ can also be omitted because is a harmonic function and, hence, it can have an effect only on the boundary condition, but from the homogeneous case it is known that this term is not compatible with Neumann boundary conditions for a rectangle. Then the equation is satisfied if

$$2C + 2D = -f_0$$

The boundary conditions are satisfied if the coefficients in the polynomial are solutions to the linear systems

$$\begin{cases} A = -v_{00} \\ A + 2aC = v_{a0} \end{cases} \qquad \begin{cases} B = -h_{00} \\ B + 2bD = h_{b0} \end{cases}$$

It follows that

$$C = \frac{1}{2a}\Big(v_{00} + v_{a0}\Big)\,, \qquad D = \frac{1}{2b}\Big(h_{00} + h_{b0}\Big)$$

However, the coefficients $C$ and $D$ are found to satisfy yet another condition. It turns out that this condition is nothing but the solvability condition to the original Neumann problem:

$$ab(2C + 2D) = b\Big(v_{00} + v_{a0}\Big) + a\Big(h_{00} + h_{b0}\Big) = -abf_0$$

Thus, the Neumann problem with constant boundary data and inhomogeneity has a solution

$$w(x,y) = -v_{00}x - h_{00}y + \frac{x^2}{2a}\Big(v_{00} + v_{a0}\Big) + \frac{y^2}{2b}\Big(h_{00} + h_{b0}\Big)$$

that is unique up to an additive constant. The associated Neumann problem for the Laplace equation was discussed in the previous section. Note that the shifted boundary data in this problem, $h_0(x) - h_{00}$, $h_b(x) - h_{b0}$, $v_0(y) - v_{00}$, and $v_a(y) - v_{a0}$ have zero average values, and, hence, the problem can be solved by the method in the previous section.

The associated Neumann problem for the Poisson equation with the trivial boundary conditions is

$$\begin{aligned}
&\Delta u_f(x,y) = -f(x,y) + f_0\,, \qquad (x,y) \in (0,a) \times (0,b)\,,\\
&-u'_{fx}(0,y) = 0\,, \quad u'_{fx}(a,y) = 0\,, \quad 0 \le y \le b\,,\\
&-u'_{fy}(x,0) = 0\,, \quad u'_{fy}(x,b) = 0\,, \quad 0 \le x \le a\,.
\end{aligned}$$

The right side of the Poisson equation has zero mean value over the rectangle and, hence, the problem has a solution. Since the boundary conditions are trivial, the solution can be expanded over the corresponding eigenfunctions of the Sturm-Liouville operator either in the interval $0 \le x \le a$ or in the interval $0 \le y \le b$. Just like in the case of a similar Dirichlet problem for the Poisson equation (see Section **42.2**), the choice of basis is a matter of technical convenience (to simplify the calculation of the Fourier coefficients of $f(x,y)$).

Let us expand the solution over the basis in the interval $0 \le x \le a$ obtained by solving the Sturm-Liouville problem:

$$\begin{aligned}
&-X_k(x) = \nu_k^2 X_k(x)\,, \quad 0 < x < a\,,\\
&X'_k(0) = X'_k(a) = 0\,.
\end{aligned}$$

The eigenvalues and the corresponding orthogonal basis functions are

$$\begin{aligned}
\nu_k &= \frac{\pi k}{a}\,, \quad k = 0,1,2,\ldots\,,\\
X_k(x) &= \cos(\nu_k x)\,, \quad \|X_k\|^2 = \frac{a}{2}\,, \quad k = 1,2,\ldots\,,\\
X_0(x) &= 1\,, \quad \|X_0\|^2 = a\,.
\end{aligned}$$

The solution is sought in the form of the Fourier series

$$u_f(x,y) = \tilde{Y}_0(y)X_0 + \sum_{k=1}^{\infty} \tilde{Y}_k(y)X_k(x)$$

The boundary conditions at the vertical edges $x = 0$ and $x = a$ are automatically satisfied because the derivatives of the basis functions vanish at the end points $X'_k(0) = X'_k(a) = 0$. The boundary conditions at the horizontal edges $y = 0$ and $y = b$ require that

$$\tilde{Y}'_k(0) = \tilde{Y}'_k(b) = 0\,, \quad k = 0,1,2,\ldots$$

Continuing the analogy with the similar problem Dirichlet problem, the right side of the Poisson equation is expanded into the Fourier series

$$f(x,y) - f_0 = F_0(y)X_0 + \sum_{k=1}^{\infty} F_k(y)X_k(x)$$

$$F_0(y) = \frac{\langle f - f_0, X_0\rangle}{\|X_0\|^2} = \frac{1}{a}\int_0^a f(x,y)\,dx - f_0$$

$$F_k(y) = \frac{\langle f, X_k\rangle}{\|X_k\|^2} = \frac{2}{a}\int_0^a f(x,y)\cos(\nu_k x)\,dx\,, \quad k > 0$$

Note that the constant term in the right side does not contribute to $F_k$, $k > 0$, because $\langle 1, X_k\rangle = \langle X_0, X_k\rangle = 0$ (the unit function is equal to $X_0$ which is orthogonal to all $X_k$, $k > 0$). The Fourier coefficients $F_k(y)$ are used to obtain the boundary value problems for the expansion coefficients $\tilde{Y}_k$:

$$-\tilde{Y}_0''(y) = F_0(y)\,, \quad \tilde{Y}_0'(0) = \tilde{Y}_0'(b) = 0\,,$$
$$-\tilde{Y}_k''(y) + \nu_k^2\tilde{Y}_k(y) = F_k(y)\,, \quad \tilde{Y}_k'(0) = \tilde{Y}_k'(b) = 0\,, \quad k > 0\,.$$

They are solved by the Green's function method (or by the method of variation of parameters). For $k = 0$, the corresponding Sturm-Liouville operator has the zero eigenvalue and, hence, the boundary value problem has a solution only if $F_0(y)$ satisfy the solvability condition (see Theorem **41.1**). This condition is easy to understand in this particular case. Let us integrate the equation over the interval $[0, b]$:

$$\int_0^b F_0(y)dy = -\int_0^b \tilde{Y}_0''(y)dy = \tilde{Y}_0'(0) - \tilde{Y}_0'(b) = 0\,,$$

by the boundary condition. This condition is, in fact, equivalent to the solvability condition for the discussed Neumann problem (the right side of the Poisson equation has zero mean value over the rectangle). Indeed, using the explicit form of $F_0(y)$, and the definition of the constant $f_0$:

$$\int_0^b F_0(y)dy = \frac{1}{a}\int_0^b\int_0^a f(x,y)\,dxdy - bf_0 = bf_0 - bf_0 = 0\,.$$

The solution $Y_0(y)$ is obtained by integrating twice the function $F_0(y)$. For example,

$$\tilde{Y}_0'(y) = -\int_0^y F_0(y')dy'\,, \quad \tilde{Y}_0(y) = \int_y^b \int_0^{y''} F_0(y')dy'dy''$$

where the solution is defined up an additive constant. Integrating by parts in the integral with respect to the variable $y''$, the answer can also be written in the form of Green's function found in Example **41.2**:

$$
\begin{aligned}
\tilde{Y}_0(y) &= A_0 - \int_0^b G_0(y, y')F_0(y')\,dy' \\
&= A_0 - \int_y^b y' F_0(y')\,dy' - y\int_0^y F_0(y')\,dy'\,, \\
G(y, y') &= \begin{cases} y\,, & y' \le y \\ y'\,, & y' > y \end{cases}
\end{aligned}
$$

The boundary value problem for $Y_k$, $k > 0$, is also solved by the Green's function method. Following the general procedure of Section **41.1** (see Eqs. (**41.5**) and (**41.6**)), let us first verify whether the solution is unique. The operator in the boundary value problem is the Sturm-Liouville operator with $p = 1$, $q = \nu_k^2 > 0$, and the parameters $\alpha_0 = \alpha_b = 0$ in the boundary conditions. Since $q > 0$, this operator has no zero eigenvalue and therefore, by Theorem **41.1**, the solution is unique for any choice of $F_k(y)$. Next, let us find two solutions to the associated homogeneous equation one of which satisfies the boundary condition on the left side of the interval, while the other does so on the right side of the interval. They are

$$
\cosh(\nu_k y)\,, \quad \cosh[\nu_k(b - y)]
$$

Their Wronskian is

$$
\begin{aligned}
W_k(y) &= \det\begin{pmatrix} \cosh(\nu_k y) & \cosh[\nu_k(b-y)] \\ \nu_k \sinh(\nu_k y) & -\nu_k \sinh[\nu_k(b-y)] \end{pmatrix} \\
&= -\nu_k \cosh(\nu_k y)\sinh[\nu_k(b-y)] - \nu_k \cosh[\nu_k(b-y)]\sinh(\nu_k y) \\
&= -\nu_k \sinh(\nu_k b)
\end{aligned}
$$

where the identity

$$
\sinh(\alpha + \beta) = \cosh(\alpha)\sinh(\beta) + \cosh(\beta)\sinh(\alpha)
$$

was used. The identity can be verified by expressing the hyperbolic functions via the exponential function. Using (**41.5**) and (**41.6**) the solution is obtained:

$$\begin{aligned}
\tilde{Y}_k(y) &= \int_0^b G_k(y,y')F_k(y')\,dy' \\
&= \frac{\cosh[\nu_k(b-y)]}{\nu_k\sinh(\nu_k b)}\int_0^y \cosh(\nu_k y')F_k(y')\,dy' \\
&\quad + \frac{\cosh(\nu_k y)}{\nu_k\sinh(\nu_k b)}\int_y^b \cosh[\nu_k(b-y')]F_k(y')\,dy' \\
G_k(y,y') &= \frac{1}{\nu_k\sinh(\nu_k b)}\begin{cases}\cosh[\nu_k(b-y)]\cosh(\nu_k y')\,, & y'\le y\\ \cosh(\nu_k y)\cosh[\nu_k(b-y')]\,, & y'>y\end{cases}
\end{aligned}$$

The convergence of the Fourier series (**??**) can be studied by the standard means discussed earlier.

**Remark.** The method of variation of parameters (Green's function method) is technically more difficult than the method of undermined coefficients, provided the inhomogeneity has a special form (a combination of polynomials, trigonometric, and exponential functions). This is illustrated with the following example.

EXAMPLE **42.3**. *Find a formal solution to the Neumann problem for the Poisson equation in a rectangle or show that no solution exist:*

$$\begin{aligned}
&-\Delta u(x,y) = 4y\cos^2(\pi x)\,, \qquad (x,y)\in(-1,1)\times(0,1)\,,\\
&-u'_x(-1,y) = 2y-2\,,\quad u'_x(1,y)=0\,,\quad y\in[0,1]\,,\\
&-u'_y(x,0) = (x-1)/2\,,\quad u'_y(x,1)=0\,,\quad x\in[-1,1]\,.
\end{aligned}$$

*If a formal solution exists, determine whether it is a classical solution.*

SOLUTION: The solvability condition:

$$\begin{aligned}
f_0 &= \frac{1}{2}\int_0^1\int_{-1}^1 f(x,y)dxdy = 2\int_0^1 ydy\int_{-1}^1\cos^2(\pi x)\,dx = 1\,,\\
h_0 &= -\frac{1}{2}\int_{-1}^1 u'_y(x,0)\,dx = \frac{1}{4}\int_{-1}^1 (x-1)\,dx = -\frac{1}{2}\,,\\
v_0 &= -\int_0^1 u'_x(-1,y)\,dy = \int_0^1 (2y-2)\,dy = -1\,.
\end{aligned}$$

Therefore

$$2h_0 + v_0 = -2f_0 \quad\Rightarrow\quad -1-1=-2$$

as required, and the problem is solvable.

The associated problem with constant data. Let us find a solution to the problem

$$\begin{aligned}
&\Delta u_0(x,y) = -f_0 = -1\,, \qquad (x,y) \in (-1,1)\times(0,1)\,,\\
&-u'_{0x}(-1,y) = v_0 = -1\,, \quad u'_{0x}(1,y) = 0\,, \quad y \in [0,1]\,,\\
&-u'_{0y}(x,0) = h_0 = -\frac{1}{2}\,, \quad u'_{0y}(x,1) = 0\,, \quad x \in [-1,1]\,.
\end{aligned}$$

According to a general analysis a solution can be found in the form (using a shift of the interval $[-1,1]$ to $[0,2]$)

$$u_0 = A(x+1) + By + C(x+1)^2 + Dy^2$$

The equation yields

$$2C + 2D = -1$$

The boundary conditions give

$$-A = v_0 = -1\,, \quad A + 4C = 0\,, \quad -B = h_0 = -\frac{1}{2}\,, \quad B + 2D = 0$$

Note that $C = D = -\frac{1}{4}$ also satisfies the condition stemming from the Poisson equation. Therefore

$$u_0(x,y) = x + \frac{y}{2} - \frac{(x+1)^2}{4} - \frac{y^2}{4}$$

The associated homogeneous Neumann problem. Next let us solve the Neumann problem in which the inhomogeneity is set to zero, while the boundary data are shifted by the corresponding constants to make the averages vanish:

$$\begin{aligned}
&-\Delta U(x,y) = 0\,, \qquad (x,y) \in (-1,1)\times(0,1)\,,\\
&-U'_x(-1,y) = 2y - 2 - v_0 = 2y - 1\,, \quad u'_x(1,y) = 0\,, \quad y \in [0,1]\,,\\
&-U'_y(x,0) = \frac{x-1}{2} - h_0 = \frac{x}{2}\,, \quad u'_y(x,1) = 0\,, \quad x \in [-1,1]\,.
\end{aligned}$$

A solution is sought as the sum

$$U(x,y) = U_1(x,y) + U_2(x,y)$$

of solutions the associated problems with vertical and horizontal boundary conditions set to zero, respectively:

$$
\begin{aligned}
U_1: \quad & \Delta U_1(x,y) = 0\,, \qquad (x,y) \in (-1,1)\times(0,1)\,,\\
& -U'_{1x}(-1,y) = 0\,, \quad U'_{1x}(1,y) = 0\,, \quad y \in [0,1]\,,\\
& -U'_{1y}(x,0) = x/2\,, \quad U'_{1y}(x,1) = 0\,, \quad x \in [-1,1]\,.\\
U_2: \quad & \Delta U_2(x,y) = 0\,, \qquad (x,y) \in (-1,1)\times(0,1)\,,\\
& -U'_{2x}(-1,y) = 2y-1\,, \quad U'_{2x}(1,y) = 0\,, \quad y \in [0,1]\,,\\
& -U'_{2y}(x,0) = 0\,, \quad U'_{2y}(x,1) = 0\,, \quad x \in [-1,1]\,.
\end{aligned}
$$

Solving the first problem: The solution to the first problem is expanded over the corresponding *orthogonal* basis in the horizontal interval $-1 \le x \le 1$

$$
\begin{aligned}
U_1(x,y) &= \tilde{Y}_0(y)X_0(x) + \sum_{k=1}^{\infty} \tilde{Y}_k(y)X_k(x)\,,\\
X_0(x) &= 1\,, \quad \|X_0\|^2 = 2\,, \quad \nu_0 = 0\,,\\
X_k(x) &= \cos[\nu_k(x+1)]\,, \quad \|X_k\|^2 = 1\,, \quad \nu_k = \frac{\pi k}{2}
\end{aligned}
$$

The expansion coefficients satisfy the boundary value problem:

$$
\tilde{Y}''_k(y) - \nu_k^2 \tilde{Y}_k(y) = 0\,, \quad \tilde{Y}'_k(0) = -\frac{\langle x, X_k\rangle}{2\|X_k\|^2}\,, \quad \tilde{Y}'_k(1) = 0
$$

whose solution reads

$$
\tilde{Y}_k(y) = -\tilde{Y}'_k(0)\,\frac{\cosh[\nu_k(1-y)]}{\nu_k \sinh(\nu_k)}
$$

The Fourier coefficient is calculated by integration by parts:

$$
\begin{aligned}
\langle x, X_k\rangle &= \int_{-1}^{1} x\cos[\nu_k(x+1)]\,dx = \int_0^2 (s-1)\cos(\nu_k s)\,ds\\
&= \frac{1}{\nu_k^2}\Big((-1)^k - 1\Big)\,, \quad k > 0
\end{aligned}
$$

Note that by construction of the shifted boundary data there should be $\langle x, X_0\rangle = 0$, which is indeed the case:

$$
\langle x, X_0\rangle = \int_{-1}^{1} x\,dx = 0\,.
$$

Therefore $Y_0(y) = 0$ in accord with a general analysis of the Neumann problem for the Laplace equation given earlier (it could have been omitted from the very beginning). A solution to the first problem (up to an additive constant) reads

$$U_1(x,y) = \sum_{k=1}^{\infty} \frac{[1-(-1)^k]}{2\nu_k^3}\,\frac{\cosh[\nu_k(1-y)]}{\sinh(\nu_k)}\,\cos[\nu_k(x+1)]$$

By the criterion (**40.1**) this is also a classical solution.

Solving the second problem: The solution to the second problem is expanded over the corresponding *orthonormal* basis in the vertical interval $0 \le y \le 1$

$$U_2(x,y) = \sum_{k=1}^{\infty} \tilde{X}_k(x) Y_k(y)\,,$$
$$Y_k(y) = \cos(\mu_k y)\,, \quad \|Y_k\|^2 = 2\,, \quad \mu_k = \pi k\,, \quad k = 1,2,...$$

The basis also contains the constant function $Y_0(y) = 1$ corresponding to the zero eigenvalue of the associated Strum-Liouville operator. However, according to the general analysis, the term $\tilde{X}_0(x)Y_0(y)$ vanishes in the solution thanks to the shifted boundary data. The expansion coefficients satisfy the boundary value problem:

$$\tilde{X}_k''(x) - \mu_k^2 \tilde{X}_k(x) = 0\,, \quad \tilde{X}_k'(-1) = -\frac{\langle 2y-1, Y_k\rangle}{\|Y_k\|^2}\,, \quad \tilde{X}_k'(1) = 0$$

whose solution reads

$$\tilde{X}_k(x) = -\tilde{X}_k'(-1)\,\frac{\cosh[\mu_k(1-x)]}{\mu_k \sinh(2\mu_k)}\,.$$

The Fourier coefficient is calculated by integration by parts:

$$\frac{\langle 2y-1, Y_k\rangle}{\|Y_k\|^2} = \frac{1}{2}\int_0^1 (2y-1)\cos(\mu_k y)\,dy = \frac{1}{2\mu_k^2}\Big((-1)^k - 1\Big)\,.$$

The solution to the second problem reads

$$U_2(x,y) = \sum_{k=1}^{\infty} \frac{[1-(-1)^k]}{2\mu_k^3}\,\frac{\cosh[\mu_k(1-x)]}{\sinh(2\mu_k)}\,\cos(\mu_k y)$$

By the criterion (**40.1**) this is also a classical solution.

The associated Neumann problem with zero boundary data. Finally, one has to solve the Neumann problem in which the inhomogeneity is shifted by a constant so that the average vanishes, while all the boundary data are set to zero:

$$\begin{aligned} -\Delta U_f(x,y) &= 4y\cos^2(\pi x) - f_0 = 4y\cos^2(\pi x) - 1\,, \\ -U'_{fx}(-1,y) &= 0\,, \quad U'_{fx}(1,y) = 0\,, \quad y \in [0,1]\,, \\ -U'_{fy}(x,0) &= 0\,, \quad U'_{fy}(x,1) = 0\,, \quad x \in [-1,1]\,. \end{aligned}$$

The inhomogeneity in the Poisson equation is a linear combination of the basis functions $X_k(x) = \cos[\nu_k(x+1)]$ in the horizontal interval $-1 \le x \le 1$ used in the solution of the first problem

$$\begin{aligned} 4y\cos^2(\pi x) - 1 &= 2y[1+\cos(2\pi x)] - 1 = 2y - 1 + 2y\cos(2\pi x) \\ &= (2y-1)X_0(x) + 2yX_4(x)\,. \end{aligned}$$

It is therefore convenient to seek the solution to the Poisson equation with the *trivial* Neumann boundary conditions in the form of the linear combination

$$U_f(x,y) = \tilde{Y}_0(y)X_0(x) + \tilde{Y}_4(y)X_4(x)$$

where the expansion coefficients satisfy the boundary value problems:

$$\begin{aligned} -Y_0''(y) &= \sqrt{2}(2y-1) \equiv F_0(y)\,, \quad Y_0'(0) = Y_0'(1) = 0\,, \\ -Y_4''(y) + 4\pi^2 Y_4(y) &= 2y \equiv F_4(y)\,, \quad Y_4'(0) = Y_4'(1) = 0\,, \end{aligned}$$

Note that the first problem is solvable because

$$\int_0^1 (2y-1)\,dy = 0$$

which is due to the shift of the original inhomogeneity $f(x,y)$ by a constant function to make the average vanish. Therefore

$$\tilde{Y}'(y) = \int_0^y (2y'-1)dy' = y^2 - y$$

Note that $\tilde{Y}_0'(1) = 0$ as required, and

$$\tilde{Y}(y) = C + \frac{y^3}{3} - \frac{y^2}{2}$$

where $C$ is a constant. The same answer can be obtained by Green's function method:

$$\begin{aligned} \tilde{Y}_0(y) &= A_0 + \int_0^1 G_0(y,y')F_0(y')\,dy' \\ &= A_0 - \int_y^1 y'(2y'-1)\,dy' - y\int_0^y (2y'-1)\,dy' \\ &= C - \frac{y^3}{3} + \frac{y^2}{2} \end{aligned}$$

where $A_0$ is an arbitrary constant, and all constants arising from the integration were included into a constant $C$.

Method of undetermined coefficients. Owing to a special form of the right side of the equation (it is a polynomial), a particular solution is also a polynomial according to the method of undetermined coefficients for the second order linear differential equations. The general solution may be taken in the form

$$\tilde{Y}_4(y) = \tilde{Y}_p(y) + A\cosh(2\pi y) + B\cosh[2\pi(1-y)]\,,$$

that is the sum of a particular solution $\tilde{Y}_p$ and the general solution of the associated homogeneous problem. The latter is convenient to take as a linear combination of solutions one of which satisfying the boundary condition at $y = 0$, while the other at $y = 1$. A particular solution should have the form $\tilde{Y}_p(y)cy$. A subtitution into the equation gives $c = 1/(2\pi)^2$ so that

$$\begin{aligned} \tilde{Y}_4(y) &= \frac{y}{2\pi^2} + A\cosh(2\pi y) + B\cosh[2\pi(1-y)]\,, \\ \tilde{Y}_4'(y) &= \frac{1}{2\pi^2} + 2\pi A\sinh(2\pi y) - 2\pi B\sinh[2\pi(1-y)]\,. \end{aligned}$$

The constants $A$ and $B$ are found from the boundary conditions:

$$B = \frac{1}{(2\pi)^3\sinh(2\pi)} = -A$$

so that

$$\tilde{Y}_4(y) = \frac{y}{(2\pi)^2} + \frac{\cosh[2\pi(1-y)] - \cosh(2\pi y)}{(2\pi)^3\sinh(2\pi)}$$

Green's function method. The solution can also be obtained by Green's function method:

$$\begin{aligned}
Y_4(y) &= -\int_0^1 G_4(y,y')F_4(y')\,dy' \\
&= \frac{\cosh[2\pi(1-y)]}{2\pi\sinh(2\pi)}\int_0^y \cosh(2\pi y')y'\,dy' \\
&\quad + \frac{\cosh(2\pi y)}{2\pi\sinh(2\pi)}\int_y^1 \cosh[2\pi(1-y')]y'\,dy' \\
&= \frac{\cosh[2\pi(1-y)]}{(2\pi)^2\sinh(2\pi)}\left[y\sinh(2\pi y) - \frac{1}{2\pi}\Big(\cosh(2\pi y)-1\Big)\right] \\
&\quad + \frac{\cosh(2\pi y)}{(2\pi)^2\sinh(2\pi)}\left[y\sinh[2\pi(1-y)] + \frac{1}{2\pi}\Big(\cosh[2\pi(1-y)]-1\Big)\right] \\
&= \frac{y}{(2\pi)^2} + \frac{1}{(2\pi)^3\sinh(2\pi)}\Big(\cosh[2\pi(1-y)] - \cosh(2\pi y)\Big)
\end{aligned}$$

The final form of the solution to the Neumann problem with zero boundary conditions is

$$U_f = C - \frac{2y^3}{3} + y^2 + \left[\frac{y}{(2\pi)^2} + \frac{\cosh[2\pi(1-y)] - \cosh(2\pi y)}{(2\pi)^3\sinh(2\pi)}\right]\cos(2\pi y)$$

The formal solution. The formal solution is the sum of all solutions:

$$u(x,y) = u_0(x,y) + U_1(x,y) + U_2(x,y) + U_f(x,y)$$

are from the class $C^2(\Omega)\cap C^1(\overline{\Omega})$, where $\Omega = (-1,1)\times(0,1)$. Therefore $u(x,y)$ is a classical solution to the studied Neumann problem. □

**42.6. Corner singularities in the Fourier method.** A harmonic function in a rectangle is continuous along its boundary. Unless it vanishes at all four corners of the rectangle, such a function cannot be represented as a sum of four harmonic functions each of which is not identically zero on just one edge of the rectangle. For example, the harmonic polynomial

$$u(x,y) = b^2 - by + xy\,, \qquad \Delta u = 0$$

does not have zero values at all the corners:

$$\begin{aligned}
u(0,0) &= b^2\,, \quad & u(0,b) &= 0\,, \\
u(a,0) &= b^2\,, \quad & u(a,b) &= ab\,.
\end{aligned}$$

The Fourier method can *formally* be applied to the Dirichlet problem with boundary data

$$\begin{aligned} u(x,0) &= b^2 = h_0(x)\,, & u(x,b) &= bx = h_b(x)\,,\\ u(0,y) &= b^2 - by = v_0(y)\,, & u(a,y) &= ay = v_a(y)\,. \end{aligned}$$

despite that the boundary data is no longer continuous. For example, at the corner $(a,0)$,

$$h_0(a) = b^2 \neq v_b(a) = ab$$

is $a \neq b$. The Fourier coefficients of $h_b(x)$ in the horizontal basis with Dirichlet type boundary conditions are

$$\begin{aligned} \tilde{Y}_{bk} &= \frac{\langle h_b, X_k\rangle}{\|X_k\|^2} = \frac{2b}{a}\int_0^a x\sin(\nu_k x)\,dx\\ &= -\frac{2b}{a\nu_k}\,x\cos(\nu_k x)\Big|_0^a + \frac{2b}{a\nu_k}\int_0^a \cos(\nu_k)dx\\ &= \frac{2(-1)^k ab}{\nu_k}\,. \end{aligned}$$

The second integral vanishes thanks to $\sin(\nu_k a) = 0$. The series

$$U(x,y) = \sum_{k=1}^{n} Y_{bk}\,\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)}\,X_k(x)\,, \qquad n = 1,2,...$$

is the formal solution to the Dirichlet problem in which all boundary data were set to zero except for $h_b(x)$. An attempt to detect uniform convergence of the series for $U_{1b}(x,y)$ would fail because

$$|\tilde{Y}_{bk}| = \frac{c}{k} \quad \Rightarrow \quad \sum_{k=1}^{\infty} |Y_{bk}| = c\sum_{k=0}^{\infty}\frac{1}{k} = \infty\,.$$

where $c$ is a constant. Furthermore the terms of the sequence of partial sums

$$U_n(x,y) = \sum_{k=1}^{n} Y_{bk}\,\frac{\sinh(\nu_k y)}{\sinh(\nu_k b)}\,X_k(x)\,, \qquad n = 1,2,...$$

vanish at the corners of the rectangle by construction:

$$U_n(0,0) = U_n(a,0) = U_n(0,b) = U_n(a,b) = 0\,,$$

which implies that in the limit $n \to \infty$, the sum $U(x,y)$ vanishes at all four corners. One can construct three other formal solutions with non-zero $h_0$, $v_0$, and $v_a$, while other boundary data are set to zero. These solutions also vanish at all corners by the same reason. The sum of the four formal solutions is the formal solution $u_f(x,y)$ of the original Dirichlet problem. This implies that the formal solution vanishes at the corners, which contradicts the required boundary values at the corners, the classical solution $u(x, y)$ does not. For example,

$$u_f(a, b) = 0 \neq ab = u(a, b)$$

This stems from the fact that the Fourier series

$$h_b(x) \sim \sum_{k=1}^{\infty} \tilde{Y}_{bk} X_k(x)$$

converges to $h_b(x)$ pointwise for every $x$ but $x = a$ since $X_k(a) = 0$, while $h_b(a) \neq 0$. Recall that the convergence in the mean does not guarantee a pointwise convergence (a pointwise convergence may be violated on a set of measure zero). Thus, the formal solution does not converge uniformly on the boundary of the rectangle because of the corner points.

It is then natural to ask if the Fourier method is applicable at all in the case when the boundary value function does not vanish at the corners of the rectangle. Or, more generally, does the Dirichlet problem have a solution if the boundary data are not continuous and, if a solution exists, is it unique? The answer is given by the following theorem.

THEOREM **42.1**. (Singular Dirichlet problem)
*Let $u_0$ be a piecewise continuous function on the boundary $\partial\Omega$ of an open bounded region $\Omega$. Then there exists a unique harmonic function $u(x, y)$ in $\Omega$ such that*

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0(x_0, y_0)$$

*for any point $(x_0, y_0) \in \partial\Omega$ at which $u_0$ is continuous.*

In the Fourier method, four Dirichlet problems with piecewise continuous boundary value data are solved (unless the boundary data vanish at all four corners of the rectangle). By Theorem **42.1** the formal solution obtained by the Fourier method converges to a harmonic function satisfying the boundary conditions, except possibly for a small neighborhood of the corner points.

**42.7. Poisson equation in a plane.** The equation

$$\Delta u(x, y) = f(x, y)\,, \qquad (x, y) \in \mathbb{R}^2\,,$$
$$u \in C^2(\mathbb{R}^2)\,, \quad f \in C^0(\mathbb{R}^2)$$

is called *the Poisson equation*. It is a linear equation. Therefore its general solution is the sum of a harmonic function (the general solution of the Laplace equation) and a particular solution:

$$u(x,y) = u_0(x,y) + u_f(x,y)\,, \quad \Delta u_0(x,y) = 0\,.$$

It is not difficult to verify the the function

$$U(x,y) = \frac{1}{4\pi}\ln(x^2+y^2)$$

satisfies the Laplace equation everywhere in the plane but the origin $(x,y) = (0,0)$:

$$\Delta U(x,y) = 0\,, \quad (x,y) \neq (0,0)$$

It is called the *fundamental solution* to the Laplace equation. It has a useful property. Suppose that the function $f$ vanishes outside a bounded region $\Omega$ in the plane. Furthermore, the boundary of $\Omega$ is assumed to be a smooth curve, meaning that, it has a continuous unit tangent vector. Then the function given by the following double integral over $\Omega$

$$u_f(x,y) = \iint_\Omega U(x-x', y-y')f(x',y')dx'dy'$$

is a solution to the Poisson equation. A proof of this assertion goes beyond the scope of this course.

**42.8. More general elliptic equations.** The second partial derivatives in the Laplace operator can be replaced by Sturm-Liouville operators, one is acting on the variable $x$, while the other on $y$:

$$L_x u(x,y) + L_y u(x,y) = f(x,y)$$

This equation can be solved with mixed boundary conditions in a rectangle by separating variables. The basis in the Fourier series is formed by eigenfunctions of a general Sturm-Liouville operator.

**Poisson equation on a torus.** The Poisson equation can also be formulated on a torus. A torus is obtained by identifying the opposite edges of a rectangle. Any function on a torus is a periodic functions of two variables:

$$u(x+a,y) = u(x,y+b) = u(x,y)$$

The corresponding boundary conditions on $(0,a)\times(0,b)$ are not equivalent to the mixed boundary conditions:

$$\begin{aligned} u(0,y) &= u(a,y)\,, \quad u'_x(0,y) = u'_x(a,y)\,, \\ u(x,0) &= u(x,b)\,, \quad u'_y(x,0) = u'_y(x,b) \end{aligned}$$

The Laplace equation

$$\Delta u_0(x,y) = 0$$

with such boundary conditions has non-trivial solution, a constant function. For any $u$ satisfying the said boundary conditions the following identity holds

$$\int_{\partial\Omega} u\frac{\partial u}{\partial \mathbf{n}}\,ds = \int_0^a [u(x,b)u'_y(x,b) - u(x,0)u'_y(x,0)]\,dx$$
$$+ \int_0^b [u(a,y)u'_x(a,y) - u(0,y)u'_x(0,y)]\,dy = 0$$

If $u$ also satisfies the Laplace equation, then by the divergence theorem

$$0 = \iint_\Omega u\Delta u\,dxdy = \int_{\partial\Omega} u\frac{\partial u}{\partial \mathbf{n}}\,ds - \iint_\Omega (\boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} u)\,dxdy$$
$$= -\iint_\Omega (\boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} u)\,dxdy$$

But the integral can vanish only if $\boldsymbol{\nabla} u = \mathbf{0}$. So, $u$ must a constant function. As consequence, the solution to the Poisson equation

$$\Delta u(x,y) = f(x,y)$$

with the said boundary conditions is not unique (it is unique up an additive constant, similarly to the Neumann problem). Yet, it does not have a solution for any (continuous) $f$. The solvability condition requires that $f$ is orthogonal to the unit function in $\Omega$:

$$\iint f\,dxdy = \int_0^a\int_0^b f(x,y)\,dydx = 0$$

It can be obtained in the same way as the analogous integrability of the Neumann problem. If the solvability condition is fulfilled, then a formal solution is obtained by separation of variables. The solution is expanded into the trigonometric Fourier series (the Sturm-Liouville problem on a circle).

**Elliptic equation on a surface of a cylinder.** Finally, one can mix periodic boundary conditions with the Sturm-Liouville boundary conditions

$$u(0,y) = u(a,y)\,,\qquad u'_x(0,y) = u'_x(a,y)\,,$$
$$\alpha_0 u(x,0) - \beta_0 u'_y(x,0) = v_0(x)\,,\qquad \alpha_b u(x,b) + \beta_b u'_y(x,b) = v_b(x)$$

In other words, the vertical edges of the rectangle $(0,a)\times(0,b)$ are identified to obtain a surface of a cylinder. On the edges of the cylinder, the Sturm-Liouville boundary conditions are imposed. The equation

$$-u''_{xx}(x,y) + L_y u(x,y) = f(x,y)$$

can be solved by separating variables. The solution is expanded into the trigonometric Fourier series (over the eigenfunctions of the second derivative operator on a circle). In the case when $\alpha_0 = \alpha_b = 0$ (so that the values of $\beta_0$ and $\beta_b$ may be absorbed into $v_0$ and $v_b$ by setting $\beta_0 = \beta_b = 1$), all solutions to the associated homogeneous problem ($f = 0$ and $v = 0$) are constant functions. This assertion follows from the identity

$$\int_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}}\, ds = 0$$

which holds for any $u$ satisfied the boundary conditions. A proof is left to the reader as an exercise. Therefore the solution is not unique (it is unique up to an additive constant). Yet, a solution does not exist unless $f$ and $v$ satisfy the solvability condition:

$$\int_0^a\int_0^b f(x,y)\, dydx + \int_0^a [v_b(x) + v_0(x)]\, dx = 0$$

A proof is analogous to the proof of the solvability condition in the case of the Neumann problem for the Poisson equation and left to the reader as an exercise.

**42.9. Exercises.**

**1**. Solve the Dirichlet problem

$$\begin{aligned}
&-\Delta u(x,y) = 6\sin(2\pi x)\sin(\pi y)\,, \quad (x,y) \in (-1,1)\times(-2,2)\,,\\
&u(-1,y) = \sin(\pi y/2)\,, \quad u(1,y) = 2\sin(\pi y)\,, \quad y \in [-2,2]\\
&u(x,-2) = -2\sin(\pi x)\,, \quad u(x,2) = \sin(2\pi x)\,, \quad x \in [-1,1]\,.
\end{aligned}$$

*Hint*: See Exercise **6** in Section **40.4**.

**2**. Solve the Dirichlet problem

$$\begin{aligned}
&-\Delta u(x,y) = y\sin(\pi x)\,, \quad (x,y) \in (-1,1)\times(-1,0)\,,\\
&u(-1,y) = y(1+y)\,, \quad u(1,y) = 0\,, \quad y \in [-1,0]\\
&u(x,-1) = 0\,, \quad u(x,0) = x^2 - 1\,, \quad x \in [-1,1]\,.
\end{aligned}$$

*Hint*: See Exercise **7** in Section **40.4**.

**3**. Solve the Neumann problem or show that no solution exists

$$\begin{aligned}
&-\Delta u(x,y) = 6x^2\sin^2(\pi y/4)\,, \quad (x,y) \in (-1,1)\times(-2,2)\,,\\
&-u'_x(-1,y) = -\cos^2(\pi y)\,, \quad u'_x(1,y) = 3\cos(2\pi y)\,, \quad y \in [-2,2]\\
&-u'_y(x,-2) = -2\sin^2(\pi x)\,, \quad u'_y(x,2) = -4\cos(2\pi x)\,, \quad x \in [-1,1]\,.
\end{aligned}$$

*Hint*: See Exercise **8** in Section **40.4**.

**4**. Solve the Neumann problem or show that no solution exists

$$-\Delta u(x,y) = xy\,, \quad (x,y) \in (-1,1)\times(0,4)\,,$$
$$-u'_x(-1,y) = \tfrac{1}{4}y - 1\,, \quad u'_x(1,y) = 1 - \tfrac{1}{4}y\,, \quad y \in [0,4]$$
$$-u'_y(x,0) = -x\,, \quad u'_y(x,4) = 0\,, \quad x \in [-1,1]\,.$$

*Hint*: See Exercise **9** in Section **40.4**.

**5**. Solve the mixed problem:

$$-\Delta u(x,y) = x^2 y\,, \quad (x,y) \in (-1,1)\times(0,2)\,,$$
$$\begin{cases} u(-1,y) = 3\cos(\pi y/4) \\ u'_x(1,y) = -2\cos(3\pi y/4) \end{cases}, \quad y \in [0,2]\,,$$
$$\begin{cases} -u'_y(x,0) = 2\sin[\pi(x+1)/4] \\ u(x,2) = -4\sin[3\pi(x+1)/4] \end{cases}, \quad x \in [-1,1]\,.$$

*Hint*: See Exercise **11** in Section **40.4**.

**6**. Solve the mixed problem:

$$-\Delta u(x,y) = y^2\sin(2\pi x)\,, \quad (x,y) \in (0,1)\times(1,2)\,,$$
$$u(0,y) = 0\,, \quad u(1,y) = 0\,, \quad y \in [1,2]\,,$$
$$-u'_y(x,1) = x^2(1-x)^2\,, \quad u'_y(x,2) = 0\,, \quad x \in [0,1]\,.$$

*Hint*: See Exercise **12** in Section **40.4**.

**Hints.**

**1**. The associated problem with all boundary conditions set to zero has a solution in the form

$$u(x,y) = C\sin(2\pi x)\sin(\pi y)$$

with a suitable choice of the constant $C$. The associated problem in which the inhomogeneity in the Poisson equation is set to zero has the solution in the form

$$\begin{aligned} u(x,y) &= U_1(x,y) + U_2(x,y)\,, \\ U_1(x,y) &= X_1(x)\sin(\pi y/2) + X_2(x)\sin(\pi y)\,, \\ U_2(x,y) &= Y_1(y)\sin(\pi x) + Y_2(y)\sin(2\pi x) \end{aligned}$$

**2**. The associated problem with all boundary conditions set to zero has a solution in the form

$$u(x,y) = Y(y)\sin(\pi x)\,, \quad Y(-1) = Y(0) = 0\,.$$

To solve the associated boundary value problems for the Laplace equation, one should use the orthonormal set $X_k = \sin[\nu_k(x+1)]$, $\nu_k = \pi k/2$, $k = 1, 2, ...$, in the interval $[-1, 1]$, and the set $Y_k(y) = \sqrt{2}\sin(\mu_k y)$, $\mu_k = \pi k$, $k = 1, 2, ...$, in the interval $[-1, 0]$.

**3**. The solvability condition does not hold. No solution exists.

**4**. The solvability condition holds. The mean value of the inhomogeneity in the rectangle vanishes. The mean values of the boundary data over the opposite edges vanish separately. Therefore the solution is the sum of two Fourier series. The orthonormal basis in the $[-1, 1]$ is $X_k(x) = \cos[\nu_k(x + 1)]$, $\nu_k = \pi k/2$, $k = 1, 2, ....$ The orthonormal basis in $[0, 4]$ is $Y_k(y) = (1/\sqrt{2})\cos(\mu_k y)$, $\mu_k = \pi k/4$, $k = 1, 2, ....$

**5**. The orthonormal basis in $[-1, 1]$ is $X_k(x) = \sin[\nu_k(x + 1)]$, $\nu_k = \pi(2k - 1)/4$ $(\cos(2\nu_k) = 0)$, $k = 1, 2, ....$ The The orthonormal basis in $[0, 2]$ is $Y_k(y) = \cos(\mu_k y)$, $\mu_k = \pi(2k - 1)/4$, $k = 1, 2, ....$ The boundary data are proportional to the basis functions. The solution to the associated problem for the Laplace equation has the form

$$\begin{aligned} u(x, y) &= U_1(x, y) + U_2(x, y)\,, \\ U_1(x, y) &= \tilde{X}_1(x)\cos(\pi y/4) + \tilde{X}_2(x)\cos(3\pi y/4)\,, \\ U_2(x, y) &= \tilde{Y}_1(y)\sin[\pi(x + 1)] + \tilde{Y}_2(y)\sin[3\pi(x + 1)] \end{aligned}$$

**6**. The solution is the Fourier series over the basis in the interval $[0, 1]$: $X_k(x) = \sqrt{2}\sin(\nu_k x)$, $\nu_k = \pi k$, $k = 1, 2, ....$ The solution to the associated problem to the Poisson equation where all the boundary data are set to zero has the form

$$u_f(x, y) = Y(x)\sin(2\pi x)\,, \quad Y'(1) = Y'(2) = 0\,.$$

The solution to the associated problem for the Laplace equation is sought as the Fourier series over the above basis.