CHAPTER 7

Multi-variable PDEs

43. Basic problems of mathematical physics

As before, Ω denotes a region (an open connected set) in $\mathbb{R}^N$ and, by assumption, the boundary $\partial\Omega$ is piecewise smooth. Recall that a boundary of a region is smooth if it is a level set of a function whose gradient does not vanish in a neighborhood of the level set. Suppose that a function $u : \ \Omega \subset \mathbb{R}^N \ \to \ \mathbb{C}$ has continuous partial derivatives, then the *gradient* of u at a point $\mathbf{x} \in \Omega$ is the vector

$$\boldsymbol{\nabla} u = \operatorname{grad} u = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}\, \mathbf{e}_j$$

where $\mathbf{e}_j$ is the standard basis in $\mathbb{R}^N$. In what follows a vector between vertical bars denotes the Euclidean length of the vector

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2} = \sqrt{\mathbf{x}\cdot\mathbf{x}}$$

Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^N$. The Euclidean dot product

$$\mathbf{n}\cdot\boldsymbol{\nabla} u = \sum_{j=1}^{N} n_j \frac{\partial u}{\partial x_j} \equiv \frac{\partial u}{\partial \mathbf{n}}$$

is called the *direction derivative* of u or the *derivative of* u *along the vector* $\mathbf{u}$. Its value determine the rate of change of u in the direction of $\mathbf{n}$. It follows from the inequality $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$ that the maximal rate of change of a function occurs in the direction of its gradient and is equal to the length $\|\operatorname{grad} u\|$ of the gradient. Among other properties of the gradient, it is worth noting that the non-zero vector $\boldsymbol{\nabla} u$ is *normal* to level sets of the function, $u(\mathbf{x}) = \text{const}$. This property follows from the fact the function has no rate of change in any direction tangential to the surface on which the function has a constant value (a level surface) and, hence, the directional derivative in any direction tangent to a level surface vanishes which implies that $\boldsymbol{\nabla} u \ne \mathbf{0}$ is orthogonal to any tangent vector. If the level set $u(\mathbf{x}) = k$ defines the boundary of a region, then a unit normal vector on the boundary is

$$\mathbf{n}(\mathbf{x}) = \pm\frac{1}{|\boldsymbol{\nabla} u|}\, \boldsymbol{\nabla} u\,, \qquad \mathbf{x} :\ u(\mathbf{x}) = k$$

The sign is chosen in accord with a required orientation of the boundary (e.g., inward or outward for a closed surface).

The gradient is a particular example of *vector fields*. A vector-valued function $\mathbf{F}:\ \Omega \to \mathbb{R}^N$ (or $\mathbb{C}^N$) is called a *vector field*. A vector field is defined by its components in the standard basis

$$\mathbf{F}(\mathbf{x}) = F_1(\mathbf{x})\mathbf{e}_1 + F_2(\mathbf{x})\mathbf{e}_2 + \cdots + F_N(\mathbf{x})\mathbf{e}_N\,.$$

Suppose that the components of a vector field have continuous partial derivatives. The *divergence* of a vector field is a scalar function defined by the rule

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_N}{\partial x_N}\,.$$

In particular, the differential operator

$$Lu = \operatorname{div}\operatorname{grad} u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} = \Delta u$$

is called the *Laplace* operator.

43.1. Basic equations of mathematical physics. Let the functions ρ and p be strictly positive in the closure $\overline{\Omega}$, that is,

$$\rho(\mathbf{x}) > 0\,, \quad p(\mathbf{x}) > 0\,, \quad \mathbf{x} \in \overline{\Omega}\,.$$

Let the function q be non-negative in $\overline{\Omega}$:

$$q(\mathbf{x}) \geq 0\,, \quad \mathbf{x} \in \overline{\Omega}\,.$$

Let a real variable $0 < t < T \leq \infty$ denote time. In theoretical physics it is shown that the second-order partial differential equation

$$\rho\frac{\partial^2 u}{\partial^2 t} = \operatorname{div}(p\operatorname{grad} u) - qu + f(\mathbf{x}, t)\,, \tag{43.1}$$

describes various vibrational processes and wave propagation. The equation

$$\rho\frac{\partial u}{\partial t} = \operatorname{div}(p\operatorname{grad} u) - qu + f(\mathbf{x}, t)\,, \tag{43.2}$$

describes various diffusion processes and heat conductance. The equation

$$-\operatorname{div}(p\operatorname{grad} u) + qu = f(\mathbf{x}) \tag{43.3}$$

describes the corresponding stationary processes (e.g., standing waves or a stationary temperature field). Equation (**43.3**) is defined in Ω and the region Ω is said to be the *domain* of Eq. (**43.3**). Equations (**43.1**) and (**43.2**) are defined in the open cylinder

$$\Pi_T = \Omega \times (0, T) \subset \mathbb{R}^{N+1}\,.$$

The boundary of the cylinder contains three parts, the side $\partial\Omega \times (0, T)$, the top $\overline{\Omega} \times \{T\}$, and the bottom $\overline{\Omega} \times \{0\}$. To make sense from the mathematical point of view, it is also necessary to assume that

$$\rho \in C^0(\overline{\Omega}), \quad p \in C^1(\overline{\Omega}), \quad q \in C^0(\overline{\Omega}).$$

Under the above assumptions, the generalized wave equation (**43**.**1**) is said to be of the *hyperbolic type*, the diffusion equation (**43**.**2**) is of the *parabolic type*, and the stationary equation (**43**.**3**) is the *elliptic type*. The classification of PDEs into the hyperbolic, parabolic, and elliptic types will be discussed in detail later. At this point, it is only noted that different types of PDEs are related to different types of physical processes described by the equations. In order to have a unique description of the physical process by a PDE, in addition to the very equation one has to specify the initial state of that process (*initial conditions*) and the state of the process at the boundary of the region in which the process is considered (*boundary conditions*). Three following problems are usually considered:

(i) The Cauchy problem for equations of the hyperbolic and parabolic types. In this case, $\Omega = \mathbb{R}^N$ and initial conditions, conditions on $u(\mathbf{x}, t)$ at $t = 0$, are set.
(ii) The boundary value problem for equations of the elliptic type in which boundary conditions, conditions on $u(\mathbf{x})$ where $\mathbf{x} \in \partial\Omega$, are imposed, while there are no initial conditions.
(iii) The mixed problem for equations of the hyperbolic and parabolic types in which both initial and boundary conditions are set and $\Omega \subset \mathbb{R}^N$.

43.2. The Cauchy problem.

Wave (hyperbolic) equation. For the wave equation (**43**.**1**) the Cauchy problem is formulated as follows: Find a function

$$u(\mathbf{x}, t) \in C^2(t > 0) \cap C^1(t \geq 0)$$

satisfying (**43**.**1**) in the half-space $t > 0$ and the initial conditions at $t = 0^+$:

$$u\Big|_{t=0} = u_0(\mathbf{x}), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = u_1(\mathbf{x}), \tag{43.4}$$

where the value at $t = 0$ is understood in the sense of the limit $t \to 0^+$ (the limit from the right) because the equation and its solution are defined for $t > 0$. In addition, it is required that

$$f \in C^0(t > 0), \quad u_0 \in C^1(\mathbb{R}^N), \quad u_1 \in C^0(\mathbb{R}^N),$$

The Cauchy problem for the wave equation also admits the following generalization. Let Σ be a piecewise smooth surface in $\mathbb{R}^N$ defined by the equation $t = \sigma(\mathbf{x})$. Suppose that the Cauchy data u_0 and u_1 are functions on Σ. The Cauchy problem for the wave equation (**43.1**) is to find a solution to (**43.1**) in a part of the region $t > \sigma(\mathbf{x})$ that is adjacent to the surface Σ and the solution must also fulfill the initial conditions:

$$u\Big|_{\Sigma} = u_0\,, \qquad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\Sigma} = u_1\,, \tag{43.5}$$

where $\mathbf{n}$ is the unit normal to Σ direction into the region $t > \sigma(\mathbf{x})$.

Diffusion (parabolic) equation. For the diffusion equation (**43.2**) the Cauchy problem is to find a function

$$u(x,t) \in C^2(t > 0) \cap C^0(t \geq 0)$$

satisfying (**43.2**) in the half-space $t > 0$ and the initial condition at $t = 0^+$:

$$u\Big|_{t=0} = u_0(\mathbf{x})\,. \tag{43.6}$$

In addition, it is required that

$$f \in C^0(t > 0)\,, \quad u_0 \in C^0(\mathbb{R}^{\mathbb{N}})\,.$$

43.3. Boundary value problem for elliptic equations. The problem is stated as: Find a function

$$u(\mathbf{x}) \in C^2(\Omega) \cap C^1(\overline{\Omega})$$

satisfying Eq. (**43.3**) in a region Ω and the boundary condition

$$\Big(\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = v\,, \qquad \alpha,\ \beta,\ v \in C^0(\partial\Omega)\,, \tag{43.7}$$

where $\mathbf{n}$ is the unit outward normal to the surface $\partial\Omega$, and α, β, and v are given continuous functions on the boundary $\partial\Omega$ such that

$$\alpha(\mathbf{x}) \geq 0\,, \quad \beta(\mathbf{x}) \geq 0\,, \quad \alpha(\mathbf{x}) + \beta(\mathbf{x}) > 0\,, \quad \mathbf{x} \in \partial\Omega\,.$$

If $\alpha = 1$ and $\beta = 0$, the boundary condition has the form

$$u\Big|_{\partial\Omega} = u_0 \tag{43.8}$$

and is called the *boundary condition of type I*. If $\alpha = 0$ and $\beta = 1$, then the boundary condition

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = u_1 \tag{43.9}$$

is called the *boundary condition of type II*. The *boundary condition of type III* is defined by $\beta = 1$ and $\alpha \geq 0$,

$$\left(\frac{\partial u}{\partial \mathbf{n}} + \alpha u\right)\Big|_{\partial\Omega} = u_2 \tag{43.10}$$

For the Laplace ($f = 0$) and Poisson equation ($f \neq 0$), the boundary value problem of type I

$$\Delta u = -f\,, \qquad u\Big|_{\partial\Omega} = u_0 \tag{43.11}$$

is called the *Dirichlet problem*; the boundary value problem of type II,

$$\Delta u = -f\,, \qquad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = u_1 \tag{43.12}$$

is called the *Neumann problem*.

Let Ω be a bounded region. Consider the region Ω_c that is the complement of the closure $\overline{\Omega}$ in $\mathbb{R}^N$, that is, $\Omega_c = \mathbb{R}^N \setminus \overline{\Omega}$. For the boundary value problem in Ω_c for an elliptic equation, some conditions at spatial infinity are often imposed in addition to boundary conditions at $\partial\Omega_c = \partial\Omega$. Conditions at infinity can, for example, define the behavior of a solution at infinity. For instance, for the *Helmholtz equation*

$$\Delta u + k^2 u = -f\,, \tag{43.13}$$

where k is a non-negative constant, in addition to the aforementioned boundary conditions at $\partial\Omega_c$, one can impose the so called *scattering* or *Sommerfeld radiation* conditions at infinity: For large $|x| = r \to \infty$, a solution should behave as

$$u(x) = e^{ik\mathbf{a}\cdot\mathbf{x}} + v(\mathbf{x})\,, \quad v(\mathbf{x}) = O(r^{-1})\,, \quad \frac{\partial v}{\partial r} - ikv = o(r^{-1})\,,$$

where $r = |\mathbf{x}|$ and $\mathbf{a}$ is a unit vector. The symbol $O(r^{-1})$ means that v falls to zero inversely proportional to r as $r \to \infty$, while the symbol $o(r^{-1})$ means that the corresponding quantity falls to zero *faster* than r^{-1} as $r \to \infty$, that is, $ro(r^{-1}) \to 0$ as $r \to \infty$. With $f = 0$, such a solution describe a scattering of a monochromatic plane wave on an obstacle of the shape $\partial\Omega_c$. The function v obeying the Sommerfeld radiation conditions describes the wave outgoing from the scattering region, while the part $e^{ik\mathbf{a}\cdot\mathbf{x}}$ describe an incident plane wave propagating in the direction of a. The Sommerfeld radiation conditions can also be added to the *stationary Schrödinger* equation

$$-\frac{\hbar^2}{2m}\Delta\psi + V\psi = E\psi$$

where $\hbar$ is the Planck constant, m is the mass of a particle, $E = \hbar^2 k^2/(2m) > 0$ is the energy of the particle, and $V = V(\mathbf{x})$ is the potential energy such that $V(x) \to 0$ as $|\mathbf{x}| \to \infty$. Such a solution describes a scattering of a quantum particle on a system described by the potential V. If $E < 0$, then instead of the Sommerfeld radiation condition, a solution of the stationary Schrödinger equation is required to be from $\mathcal{L}_2$. The conditions of the form

$$u(\mathbf{x}) = O(1) \quad \text{or} \quad u(\mathbf{x}) = o(1)\,, \quad |x| = r \to \infty\,,$$

where $O(1) = O(r^0)$ and similarly for $o(1)$, are often used for the Poisson equation.

43.4. Mixed (initial value) problems.

Wave (hyperbolic) equation. For the wave equation (**43**.**1**), the mixed problem is to find a function

$$u(\mathbf{x}, t) \in C^2(\Pi_T) \cap C^1(\overline{\Pi}_T)$$

that satisfies Eq. (**43**.**1**), the initial conditions (**43**.**4**), and the boundary condition (**43**.**7**) on the side boundary of the cylinder Π_T (that is, (**43**.**7**) should hold for all $0 < t < T$). In addition, the following smoothness conditions are required

$$f \in C^0(\Pi_T)\,, \quad u_0 \in C^1(\overline{\Omega})\,, \quad u_1 \in C^0(\overline{\Omega})\,, \quad v \in C(\partial\Omega \times [0, T])$$

and the self-consistency conditions (the consistency between the initial and boundary data)

$$\left(\alpha u_0 + \beta \frac{\partial u_0}{\partial \mathbf{n}}\right)\Big|_{\partial\Omega} = v\Big|_{t=0}\,. \tag{43.14}$$

Diffusion or heat (parabolic) equation. For the diffusion equation (**43**.**2**) the mixed problem is to find a function

$$u(\mathbf{x}, t) \in C^2(\Pi_T) \cap C^0(\overline{\Pi}_T)\,, \quad \boldsymbol{\nabla}_x u(\mathbf{x}, t) \in C^0(\overline{\Pi}_T)\,,$$

where $\boldsymbol{\nabla}_x$ denote the gradient with respect to $\mathbf{x}$, that satisfies Eq. (**43**.**2**) in Π_T, the initial condition (**43**.**5**), and the boundary condition (**43**.**7**) on the side boundary of the cylinder Π_T (in the same sense as in the case of the mixed problem for the wave equation).

Remarks of smoothness of the solution. It should be noted that solutions of the stated boundary value problems that are C^1 smooth up to the boundary of the region in which the equation is formulated *do not always exist*. The way out is to required mere continuity of solutions up to the boundary of the region. This formulation of the boundary value problem is natural if the problem does not involve partial derivatives in the boundary conditions. For example, this is suitable for Eqs. (**43**.**2**) and (**43**.**3**) with boundary conditions of type I. If the boundary conditions contain partial derivatives, then in each particular case one should define in what sense these conditions are to be fulfilled. For example, for the mixed problem for the wave equation (**43**.**1**), the second initial condition (**43**.**4**) can be understood in the sense of $\mathcal{L}_2(\Omega)$ topology:

$$\left\| \frac{\partial u}{\partial t} - u_1 \right\| \to 0\,, \quad t \to 0^+\,.$$

In the Neumann problem for the Laplace equation, the boundary condition (**43**.**9**) can be required to be satisfied in the following sense. Let $\mathbf{x}' \in \Omega$ be a point on the line through the point $\mathbf{x} \in \partial\Omega$ and parallel to the normal $\mathbf{n}_x$ at $\mathbf{x}$. In other words, if n_x is the outward normal, then $\mathbf{x}' = \mathbf{x} - s\mathbf{n}_x$ for some $s > 0$ and $\mathbf{x}' \to \mathbf{x}$ means $s \to 0^+$. Then the condition (**43**.**9**) is understood in the sense of *uniform convergence* or in the supremum norm:

$$g(s) \equiv \sup_{\mathbf{x}\in\partial\Omega} \left| \frac{\partial u}{\partial \mathbf{n}_x}(\mathbf{x}') - u_1(\mathbf{x}) \right| \to 0\,, \qquad s \to 0^+\,.$$

44. General Fourier method

44.1. The eigenvalue problem for elliptic operators. Unless stated otherwise, a region Ω is always assumed to be bounded throughout this section. Consider a linear differential operator

$$Lu = -\mathrm{div}\,(p\,\mathrm{grad}\,u) + qu\,.$$

The domain $\mathcal{M}_L$ of L consists of all functions from $C^2(\Omega)\cap C^1(\overline{\Omega})$ that satisfy the boundary condition

$$\Big(\alpha u + \beta\frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = 0\,. \tag{44.1}$$

The parameters of L and those of the boundary conditions are required to satisfy the conditions:

$$\begin{aligned} &p\in C^1(\overline{\Omega})\,,\quad q\in C^0(\overline{\Omega})\,;\quad p(x)>0\,,\quad q(x)\ge 0\,,\quad x\in\Omega \\ &\alpha,\,\beta\in C^0(\partial\Omega)\,;\quad \alpha(x)\ge 0\,,\ \beta(x)\ge 0\,,\ \alpha(x)+\beta(x)>0\,,\ x\in\partial\Omega\,. \end{aligned} \tag{44.2}$$

If $\mathcal{P}$ is a set of all polynomials (of N variables if $\Omega\subset\mathbb{R}^N$), then

$$\mathcal{P}\subset C^2(\overline{\Omega})\subset\mathcal{M}_L\subset C^0(\overline{\Omega})\subset\mathcal{L}_2(\Omega)$$

Recall the discussion after Theorem **??**: The set of continuous functions is dense in the space of square integrable functions, while by the Weierstrass theorem $\mathcal{P}$ is dense in the space of continuous functions. Therefore $\mathcal{P}$ is dense in the space of square integrable functions. Since $\mathcal{M}_L$ is larger than $\mathcal{P}$, it must be dense in $\mathcal{L}_2(\Omega)$

The eigenvalue problem for an *elliptic* operator L is to find all values of λ (eigenvalues of L) at which the equation

$$Lu = \lambda u\,,\quad u\in\mathcal{M}_L\,,$$

has a non-trivial solution as well as to find all such solutions (eigenfunctions of L).

It should be noted that eigenfunctions of smoothness $C^1(\overline{\Omega})$ may not always exist and the smoothness conditions on the boundary $\partial\Omega$ need to be relaxed to smoothness $C^0(\overline{\Omega})$ (for example, it is natural for boundary value problems of type I when $\beta = 0$).

44.2. Green's formulas.

Theorem **44.1**. (Gauss-Ostrogradsky or divergence theorem) *Let Ω be a bounded region in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ oriented outward by the unit normal $\mathbf{n}$. Suppose that components* *of a vector field* $\mathbf{F}$ *have continuous partial derivatives in a region that contains* $\overline{\Omega}$. *Then*

$$\int_\Omega \operatorname{div} \mathbf{F}\, dx = \int_{\partial\Omega} (\mathbf{F}, \mathbf{n})\, dS\,,$$

where dS *denotes the Lebesgue measure on* $\partial\Omega$ *(the surface area element).*

The theorem can also be extended to unbounded regions if $\mathbf{F}$ decreases fast enough as $|x| \to \infty$ to ensure the existence of the integrals. The following consequences of the divergence theorem can be proved.

THEOREM **44.2.** (Green's formulas)
Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *and* $v \in C^1(\overline{\Omega})$. *Then*
(**44.3**)

$$\int_\Omega v\, Lu\, dx = \int_\Omega p\,(\operatorname{grad} v, \operatorname{grad} u)\, dx - \int_{\partial\Omega} pv\, \frac{\partial u}{\partial \mathbf{n}}\, dS + \int_\Omega quv\, dx\,.$$

If, in addition, $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, *then*

$$\int_\Omega v\, Lu\, dx - \int_\Omega v\, Lu\, dx = \int_{\partial\Omega} p\, \left(v\, \frac{\partial u}{\partial \mathbf{n}} - u\, \frac{\partial v}{\partial \mathbf{n}} \right)\, dS \qquad (\mathbf{44.4})$$

The relations (**44.3**) and (**44.4**) are known as the first and second Green's formulas, respectively. If $p = 1$ and $q = 0$, then they become

$$\int_\Omega v\, \Delta\, u\, dx = -\int_\Omega (\operatorname{grad} v, \operatorname{grad} u)\, dx - \int_{\partial\Omega} v\, \frac{\partial u}{\partial \mathbf{n}}\, dS\,,$$

$$\int_\Omega (v\, \Delta u - u\, \Delta v)\; dx = \int_{\partial\Omega} \left(v\, \frac{\partial u}{\partial \mathbf{n}} - u\, \frac{\partial v}{\partial \mathbf{n}} \right)\; dS\,.$$

A proof of Green's formulas is based on the divergence theorem and the absolute continuity of the Lebesgue integral. Let Ω' be a proper subregion of Ω. It is easy to verify the identity

$$v \operatorname{div} \mathbf{F} = \operatorname{div}(v\mathbf{F}) - (\operatorname{grad} v,\, \mathbf{F})$$

by calculating the divergence in the left side explicitly in terms of partial derivatives of v and components of $\mathbf{F}$. By mean of this identity for $\mathbf{F} = p \operatorname{grad} u$, one infers that

$$\begin{aligned}\int_{\Omega'} v\, Lu\, dx &= \int_{\Omega'} v \Big[-\operatorname{div}\,(p \operatorname{grad} u) + qu \Big] dx \\ &= -\int_{\Omega'} \operatorname{div}(pv \operatorname{grad} u) dx + \int_{\Omega'} \Big[p\,(\operatorname{grad} v, \operatorname{grad} u) + quv \Big]\, dx\end{aligned}$$

The first term in the right side of this relation is transformed by means of the divergence theorem (the hypotheses of the theorem are fulfilled for Ω' by its definition as a proper subregion of Ω):

$$\int_{\Omega'} \mathrm{div}(pv\,\mathrm{grad}\,u)dx = \int_{\partial\Omega'} pv\,\frac{\partial u}{\partial \mathbf{n}}\,dS\,.$$

Finally, one can take successively larger subregions of Ω, that is, consider the limit $\Omega' \to \Omega$. By the absolute continuity of the Lebesgue integral, the integrals over $\Omega \setminus \Omega'$ and $\partial\Omega \setminus \partial\Omega'$ can be made arbitrary small for a large enough proper subregion of Ω. Thus, the above two relations hold for Ω, too, and the first Green's formula follows.

If u and v are both from $C^2(\Omega) \cap C^1(\overline{\Omega})$, then u and v can be swapped in Eq. (**44.3**). The obtained equation is subtracted from (**44.3**). The resulting relation is the second Green's formula (**44.4**).

44.3. Eigenvalues and eigenfunctions of the operator L.

THEOREM **44.3**. (Properties of the operator L)
The operator is hermitian and positive semi-definite:

$$\langle Lu, v\rangle = \langle u, Lv\rangle\,, \quad u, v \in \mathcal{M}_L\,, \tag{44.5}$$

$$\langle Lu, u\rangle \geq 0\,, \quad u \in \mathcal{M}_L\,. \tag{44.6}$$

PROOF. As noted before $\mathcal{M}_L$ is dense in $\mathcal{L}_2(\Omega)$. For any u and v from $\mathcal{M}_L$, the images Lv and $L\bar{u} = \overline{Lu}$ under the action of L are from $\mathcal{L}_2(\Omega)$. Therefore the second Green's formula (**44.4**) can be written in the form

$$\begin{aligned}\int_\Omega \left(\bar{u}\,Lv - v\,\overline{Lu}\right)\,dx &= \langle Lv, u\rangle - \langle v, Lu\rangle \\ &= \int_{\partial\Omega} p\left(v\,\frac{\partial \bar{u}}{\partial \mathbf{n}} - \bar{u}\,\frac{\partial v}{\partial \mathbf{n}}\right)\,dS\end{aligned}$$

The function u and v satisfy the boundary conditions

$$\left(\alpha v + \beta\,\frac{\partial v}{\partial \mathbf{n}}\right)\Big|_{\partial\Omega} = 0\,, \quad \left(\alpha \bar{u} + \beta\,\frac{\partial \bar{u}}{\partial \mathbf{n}}\right)\Big|_{\partial\Omega} = 0$$

By the assumption, $\alpha(x) + \beta(x) > 0$ for any $x \in \partial\Omega$. This implies that this *homogeneous* linear system of equations has a nontrivial solution (α and β do not vanish simultaneously anywhere in $\partial\Omega$), which, in turn, is possible if the determinant of the matrix composed of the coefficients of α and β vanishes:

$$v\,\frac{\partial \bar{u}}{\partial \mathbf{n}} - \bar{u}\,\frac{\partial v}{\partial \mathbf{n}} = 0\,.$$

Thus, the integrand in the integral over $\partial\Omega$ is identically zero and, hence, $\langle Lv, u\rangle - \langle v, Lu\rangle = 0$ which, by definition, means that L is hermitian.

Put $v = \bar{u}$ in the first Green's formula (**44.3**). Since $Lu \in \mathcal{L}_2(\Omega)$,

$$\langle Lu, u\rangle = \int_\Omega p|\mathrm{grad}\, u|^2 dx - \int_{\partial\Omega} pu\, \frac{\partial \bar{u}}{\partial \mathbf{n}}\, dS + \int_\Omega q|u|^2\, dx$$

The first and third terms in the right side are non-negative because $p > 0$ and $q \geq 0$ in Ω. The second term can be transformed by means of the boundary condition (**44.1**),

$$\frac{\partial u}{\partial \mathbf{n}} = -\frac{\alpha}{\beta}\, u\,, \quad \text{if} \quad \beta(x) > 0\,, \quad x \in \partial\Omega\,,$$
$$u = 0\,, \quad \text{if} \quad \beta(x) = 0\,, \quad x \in \partial\Omega\,,$$

and, hence,

$$(\mathbf{44.7}) \qquad \langle Lu, u\rangle = \int_\Omega \Big(p|\mathrm{grad}\, u|^2 + q|u|^2\Big)\, dx + \int_{\partial\Omega_0} \frac{\alpha}{\beta}\, p\, |u|^2\, dS \geq 0\,,$$

where $\partial\Omega_0$ is the part of Ω where $\alpha(x) > 0$ and $\beta(x) > 0$ (both α and β do not vanish). Thus, the operator L is positive definite. This also implies that L is hermitian. □

COROLLARY **44.1**. (Eigenvalues and eigenfunctions of the operator L) *Let the operator L be defined as in Section* **44.1**. *Then*

(i) *All eigenvalues of L are non-negative;*
(ii) *Eigenfunctions of L corresponding to distinct eigenvalues are orthogonal;*
(iii) *Eigenfunctions of L can be chosen to be real.*

PROOF. Let λ_0 be an eigenvalue of L. Then $Lu_0 = \lambda_0 u_0$ where $u_0 \neq 0$. Since L is hermitian, λ_0 is real. By Theorem **44.3** the operator is also positive definite and, hence,

$$0 \leq \langle Lu_0, u_0\rangle = \lambda_0 \langle u_0, u_0\rangle = \lambda_0 \|u_0\|^2 \quad \Rightarrow \quad \lambda_0 \geq 0\,.$$

By Theorem **36.2** eigenfunctions of L corresponding to different eigenvalues are orthogonal (as for any hermitian operator). If u_0 is a complex-valued eigenfunction of L, then put $u_0 = u_1 + iu_2$ where $u_{1,2}$ are real-valued functions. Since the corresponding eigenvalue λ_0 is real and $Lu_{1,2}$ are real-valued functions,

$$Lu_0 = \lambda_0 u_0 \quad \Rightarrow \quad L(u_1 + iu_2) = \lambda_0(u_1 + iu_2)$$
$$\Rightarrow \quad Lu_j = \lambda u_j\,,\ j = 1, 2\,.$$

This shows that the real-valued functions u_j are also eigenfunctions of L corresponding to the eigenvalue λ_0. □

THEOREM **44.4**. (Zero eigenvalue of the operator L)
Let Ω be a bounded region and L be the operator defined in Section **44.1**. *In order for $\lambda = 0$ to be an eigenvalue of L, it is necessary and sufficient that $q = 0$ and $\alpha = 0$, and in this case $\lambda = 0$ is a simple eigenvalue and the corresponding eigenfunction is a constant function.*

PROOF. *Necessity.* Let $\lambda = 0$ be an eigenvalue of L and u_0 be the corresponding eigenfunction so that $Lu_0 = 0$, $u_0 \in \mathcal{M}_L$. It follows from (**44.7**) that

$$0 = \langle Lu_0, u_0 \rangle = \int_\Omega \Big(p|\mathrm{grad}\, u_0|^2 + q|u_0|^2 \Big)\, dx + \int_{\partial\Omega_0} \frac{\alpha}{\beta}\, p\, |u_0|^2\, dS$$

This requires that $p|\mathrm{grad}\, u_0|^2 = 0$ and $q|u_0|^2 = 0$. Since $p(x) > 0$ and $q(x) \geq 0$ in Ω, it is concluded that $\mathrm{grad}\, u_0 = 0$ or u_0 must be a non-zero constant function in Ω. The latter implies that $q = 0$. Furthermore, if u_0 is a constant function, then the boundary condition (**44.1**) is fulfilled if $\alpha u_0 = 0$ or $\alpha = 0$ (since $u_0 \neq 0$). The above line of arguments also shows that $u_0 = \mathrm{const}$ is the only (linearly independent) eigenfunction corresponding the zero eigenvalue, that is, if $\lambda = 0$ is eigenvalue, then it is a simple eigenvalue.

Sufficiency. Let $q = 0$ and $\alpha = 0$. Then by (**44.2**) $\beta > 0$ and the eigenvalue problem for $\lambda = 0$ reads

$$Lu = -\mathrm{div}\,(p\, \mathrm{grad}\, u_0) = 0\,, \quad \frac{\partial u_0}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0\,.$$

A constant function $u_0 = \mathrm{const}$ is a solution to this problem and, hence, an eigenfunction of L corresponding to the eigenvalue $\lambda = 0$. □

In previous sections, it was found that an orthogonal set of eigenfunctions of the Laplace operator ($p = 1$, $q = 0$) form a complete set in the space of square integrable functions over a bounded region. The eigenvalues form an unbounded sequence without limit points and each eigenvalue has a finite multiplicity. They have been established by separating variables and reducing a multidimensional eigenvalue problem to several one-dimensional (Sturm-Liouville) problems. On other hand, even for regions of relatively simple shapes, the method of separating variables appears to be inapplicable. Do these properties remain valid for general elliptic operators in arbitrary regions? The following theorem answers this important question.

Let Ω be a bounded region in $\mathbb{R}^N$. For $N \geq 2$ it is assumed that either $\beta = 1$ or $\beta = 0$ in the boundary condition (**44.1**). In other words, the boundary condition can only have one of the following two forms:

$$\text{either} \qquad u\Big|_{\partial\Omega} = 0 \qquad \text{or} \qquad \Big(\frac{\partial u}{\partial \mathbf{n}} + \alpha u\Big)\Big|_{\partial\Omega} = 0\,, \quad \alpha \geq 0\,.$$

If in addition the boundary $\partial\Omega$ is a smooth enough surface and the coefficients $p > 0$, $q \geq 0$, and $\alpha \geq 0$ are sufficiently smooth functions, the following theorem holds

THEOREM **44.5**. (Fourier series over eigenfunctions of the operator L)

(i) *The set of all eigenvalues of L does not have limit points and each eigenvalue has a finite multiplicity;*
(ii) *For any $u \in \mathcal{M}_L$, the Fourier series of u over an orthogonal set of eigenfunctions of L converges uniformly to u;*
(iii) *The set of eigenfunctions of L is complete in $\mathcal{L}_2(\Omega)$*

The exact smoothness conditions on the parameters of the operator L under which the theorem holds have been stated for the case $N = 1$ (the Sturm-Liouville problem). The exact smoothness conditions on $\partial\Omega \subset R^N$, $N \geq 2$, will be stated later for the Dirichlet problem ($L = -\Delta$). If the boundary $\partial\Omega$ is from the class C^2 (that is, it is a level set of a twice-continuously differentiable function whose gradient vanishes nowhere), then it is typically sufficient for the theorem to hold. However, specific examples studied by separating variables show that in many problems this condition of the smoothness of $\partial\Omega$ may be relaxed.

Theorem **44.5** implies that the set of eigenvalues of L is countable. Indeed, since the set of eigenvalues does not have a limit point in $\mathbb{R}$, for every eigenvalue λ there is an open interval of a *finite* length that contains λ and no other (distinct) eigenvalues. Thus, distinct eigenvalues form a countable set. Furthermore, each eigenvalue has a finite multiplicity. Therefore all eigenvalues of L can be ordered:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_k \to \infty, \quad k \to \infty,$$

where each λ_k is repeated in this sequence as many times as is its multiplicity. Note that $\mathcal{M}_L$ is an infinite dimensional linear set. If the Fourier series of $f \in \mathcal{M}_L$ over eigenfunctions of L converges to f, then the set of linearly independent eigenfunctions must be countable (not finite), which means that the sequence $\{\lambda_k\}$ must have infinitely many distinct elements and, hence, $\lambda_k \to \infty$ as $k \to \infty$ (a monotonically increasing sequence can only be bounded by its limit point which does not exists for the sequence of the eigenvalues of L). Let ϕ_k be an eigenfunction corresponding to λ_k:

$$L\phi_k = \lambda_k\phi_k, \quad k = 1, 2, ..., \quad \phi_k \in \mathcal{M}_L$$

By Corollaries **??** and **44.1** the eigenfunctions ϕ_k can be chosen real and orthonormal so that

$$\langle L\phi_k, \phi_j\rangle = \lambda_k\langle\phi_k, \phi_j\rangle = \lambda_k\delta_{jk}.$$

Theorem **44.5** states that for any $f \in \mathcal{M}_L$, the Fourier series of f over the orthonormal set $\{\phi_k\}_1^\infty$

$$f(x) = \sum_{k=1}^{\infty} f_k \phi_k(x)\,, \quad f_k = \langle f, \phi_k \rangle$$

converges *uniformly* to f in $\overline{\Omega}$, that is,

$$\left\| f - \sum_{k=1}^{n} f_k \phi_k \right\|_\infty = \sup_\Omega \left| f(x) - \sum_{k=1}^{n} f_k \phi_k(x) \right| \to 0\,, \quad n \to \infty\,.$$

Note that the supremum here is equal to the maximal value because any continuous functions on a bounded closed set in a Euclidean space attains its extreme values.

Finally, since $\mathcal{M}_L$ is dense in $\mathcal{L}_2(\Omega)$, the set of eigenfunctions $\{\phi_k\}_1^\infty$ is complete in $\mathcal{L}_2(\Omega)$ (Theorem **??** and recall also Theorem **??**). Thus, for any $f \in \mathcal{L}_2(\Omega)$ its Fourier series over the set of eigenfunctions of the operator L converges to f in the mean (almost everywhere):

$$f(x) = \sum_{k=1}^{\infty} \langle f, \phi_k \rangle\, \phi_k(x) \quad a.e.\,, \quad f \in \mathcal{L}_2(\Omega)\,.$$

Suppose $f \in \mathcal{M}_L$. Can the Fourier series of f over $\{\phi_k\}_1^\infty$ be differentiated term by term? The following theorem holds.

THEOREM **44.6**. (Differentiation of Fourier series)
Let $f \in \mathcal{M}_L$. Then its Fourier series converges uniformly

$$\max_{\overline{\Omega}} \left| \sum_{k=1}^{n} \langle f, \phi_k \rangle \phi_k(x) - f(x) \right| \to 0\,, \quad n \to \infty\,,$$

and it can be differentiated one time term by term and the obtained series converges in $\mathcal{L}_2(\Omega)$ to the corresponding derivative of f:

$$\text{grad}\, f(x) = \sum_{k=1}^{\infty} \langle f, \phi_k \rangle\, \text{grad}\, \phi_k(x) \quad a.e.\,, \quad f \in \mathcal{M}_L \tag{44.8}$$

44.4. Variational principle for eigenvalues. Suppose L is the Sturm-Liouville operator (or the elliptic operator L defined in Section 7.2). The following variational principle holds.

THEOREM **44.7**. (Variational principle for eigenvalues)
Let $\{\lambda_k\}_0^\infty$ be the sequence of the eigenvalues of L ordered in the increasing order (in which each eigenvalue is repeated as many times as is its multiplicity). Let $\mathcal{M}_L^k \subset \mathcal{M}_L$ be the subset of functions from the

domain $\mathcal{M}_L$ which are orthogonal to the first $k-1$ eigenfunctions, that is, $f \in \mathcal{M}_L^k$ if $\langle f, \phi_j\rangle = 0$, $j = 1, 2, ..., k-1$, where $L\phi_j = \lambda_j \phi_j$. Then

$$\lambda_k = \inf_{f\in\mathcal{M}_L^k} \frac{\langle Lf, f\rangle}{\|f\|^2}$$

and the infimum is reached on any eigenfunction corresponding to the eigenvalue λ_k.

Proof. Let $f \in \mathcal{M}_L^k$. Then the first $k-1$ its Fourier coefficients over the set $\{\phi_j\}_1^\infty$ vanishes $f_j = 0$, $j = 1, 2, ..., k-1$. Using the result (**??**)

$$\langle Lf, f\rangle = \sum_{j=k}^{\infty} \lambda_j |f_j|^2 \geq \lambda_k \sum_{j=k}^{\infty} |f_j|^2$$

because $\lambda_k \leq \lambda_j$ if $j \geq k$. By the Parseval-Steklov equality (**??**)

$$\|f\|^2 = \sum_{j=k}^{\infty} |f_j|^2$$

as the set of eigenfunctions in complete in $\mathcal{M}_L$. Therefore

$$\lambda_k \leq \frac{\langle Lf, f\rangle}{\|f\|^2}, \quad f \in \mathcal{M}_K^k .$$

In particular, $\phi_k \in \mathcal{M}_K^k$ because $\langle \phi_k, \phi_j\rangle = 0$ if $j = 1, 2, ..., k-1$, and $L\phi_k = \lambda_k\phi_k$. Therefore the equality can actually be reached for $f = \phi_k$ because

$$\frac{\langle Lf, f\rangle}{\|f\|^2} = \lambda_k \frac{\langle \phi_k, \phi_k\rangle}{\|\phi_k\|^2} = \lambda_k .$$

The proof is complete. □

Note that the variational principle uses only the completeness of the set of eigenfunctions and the monotonicity of the sequence of eigenvalues. Therefore it is also valid for hermitian operators acting on periodic functions (for example, the Laplace operator on a torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$).

The lowest eigenvalue of L is

$$\lambda_1 = \inf_{f\in\mathcal{M}_L} \frac{\langle Lf, f\rangle}{\|f\|^2} .$$

It is zero if the minimum of $\langle Lf, f\rangle$ given by (**44.7**) over *normalized* functions from $\mathcal{M}_L$, that is, $\|f\| = 1$ is zero. The minimum value vanishes if and only if $q = 0$ and $\alpha = 0$ according to (**44.7**) and, in this case, it is achieved for a constant function. By the variational principle, the minimum is achieved on an eigenfunction and, hence, a non-zero constant function is an eigenfunction of L. In this regard, recall the proof of Theorem **44.4**.

44.5. The boundary and initial value problem for a hyperbolic equation. Let Ω be a bounded region in $\mathbb{R}^N$. Consider the boundary and initial value problem for the hyperbolic equation. For simplicity, let us take first $\rho = 1$:

$$\begin{aligned} &u''_{tt}(\mathbf{x},t) = -L_x u(\mathbf{x},t) + f(\mathbf{x},t)\,,\\ &u(\mathbf{x},t) \in \mathcal{M}_L\,, \quad t \geq 0\,,\\ &u(\mathbf{x},0) = u_0(\mathbf{x})\,, \quad u'_t(\mathbf{x},0) = u_1(\mathbf{x})\,. \end{aligned}$$

Note that the boundary conditions are included into the domain of the Sturm-Liouville operator L_x. It does not depend on the variable t so that its action on $u(\mathbf{x},t)$ is calculated for each (fixed) value of t. This also implies that $u(\mathbf{x},t)$ must be from the domain of L_x at every value of $t \geq 0$.

Let $X_{\mathbf{k}}(\mathbf{x})$ be a real orthonormal basis in $\mathcal{L}_2(\Omega)$

$$\langle X_{\mathbf{k}}, X_{\mathbf{k}'}\rangle = \int_\Omega X_{\mathbf{k}}(\mathbf{x}) X_{\mathbf{k}'}(\mathbf{x})\, d\mathbf{x} = \delta_{\mathbf{k}\mathbf{k}'}$$

made of linearly independent eigenfunctions of the Sturm-Liouville operator L_x, where the boldface index $\mathbf{k}$ denotes a collection of integers needed to label the basis functions. For example, spherical harmonics Y_l^m are labeled by two integers $l = 0, 1, 2, ...$ and $|m| \leq l$. In this case, $\mathbf{k} = (l, m)$ and $\delta_{\mathbf{k}\mathbf{k}'} = \delta_{ll'}\delta_{mm'}$. Since each eigenvalue of L_x has a finite multiplicity, one can always make a specific ordering in labels of the basis functions. For example, one can start with the smallest eigenvalue and count all basis eigenfunctions corresponding to it, then take the second smallest eigenvalue and count all basis functions corresponding to it, etc. The sum

$$\sum_{|\mathbf{k}| \leq n} a_{\mathbf{k}} X_{\mathbf{k}}$$

denotes a linear combination of the basis functions with n first terms.

Suppose that the parameters of the problem are linear combinations of the basis functions

$$\begin{aligned} f(\mathbf{x},t) &= \sum_{|\mathbf{k}| \leq n} F_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x})\,,\\ u_0(\mathbf{x}) &= \sum_{|\mathbf{k}| \leq n} a_{\mathbf{k}} X_{\mathbf{k}}(\mathbf{x})\,,\\ u_1(\mathbf{x}) &= \sum_{|\mathbf{k}| \leq n} b_{\mathbf{k}} X_{\mathbf{k}}(\mathbf{x})\,, \end{aligned}$$

In this case, the solution to the problem is *a linear combination of all basis functions which enter into the expansions of the parameters*:

$$u(\mathbf{x},t) = \sum_{|\mathbf{k}|\le n} T_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x}) \,,$$

Indeed, the substitution of the above expansion in to the equation yields

$$\sum_{|\mathbf{k}|\le n} T''_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x}) = -\sum_{|\mathbf{k}|\le n} \lambda_{\mathbf{k}} T_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x}) + \sum_{|\mathbf{k}|\le n} F_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x})$$

where the relation $L_x X_{\mathbf{k}} = \lambda_{\mathbf{k}} X_{\mathbf{k}}$ was taken into account. Since $X_{\mathbf{k}}$ are linearly independent, the equation holds if the coefficients at $X_{\mathbf{k}}$ in the left and right side match. Similarly, the initial conditions require that

$$\begin{aligned} \sum_{|\mathbf{k}|\le n} T_{\mathbf{k}}(0) X_{\mathbf{k}}(\mathbf{x}) &= \sum_{|\mathbf{k}|\le n} a_{\mathbf{k}} X_{\mathbf{k}}(\mathbf{x}) \,, \\ \sum_{|\mathbf{k}|\le n} T'_{\mathbf{k}}(0) X_{\mathbf{k}}(\mathbf{x}) &= \sum_{|\mathbf{k}|\le n} b_{\mathbf{k}} X_{\mathbf{k}}(\mathbf{x}) \,. \end{aligned}$$

Thus, the expansion coefficients satisfy the initial value problem:

$$T''_{\mathbf{k}}(t) + \lambda_{\mathbf{k}} T_{\mathbf{k}}(t) = F_{\mathbf{k}}(t) \,, \quad T_{\mathbf{k}}(0) = a_{\mathbf{k}} \,, \quad T'_{\mathbf{k}}(0) = b_{\mathbf{k}} \tag{44.9}$$

which is identical to the initial value problem for the expansion coefficients in a two-dimensional wave equation describing vibrations of an elastic string. Put $\omega_{\mathbf{k}} = \sqrt{\lambda_{\mathbf{k}}}$. Its solution reads

$$T_{\mathbf{k}}(t) = a_{\mathbf{k}} \cos(\omega_{\mathbf{k}} t) + \frac{b_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sin(\omega_{\mathbf{k}} t) + \frac{1}{\omega_{\mathbf{k}}} \int_0^t \sin[\omega_{\mathbf{k}}(t-\tau)] F_{\mathbf{k}}(\tau)\, d\tau \,.$$

If the parameters of the problem are not linear combination of the basis functions, then they can be expanded into *Fourier series* over the basis $X_{\mathbf{k}}$:

$$\begin{aligned} f(\mathbf{x},t) &= \sum_{\mathbf{k}} F_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x}) \,, \quad F_k(t) = \langle f, X_{\mathbf{k}} \rangle = \int_\Omega f(\mathbf{x},t) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x} \,, \\ u_0(\mathbf{x}) &= \sum_{\mathbf{k}} a_{\mathbf{k}} X_{\mathbf{k}}(\mathbf{x}) \,, \quad a_{\mathbf{k}} = \langle u_0, X_{\mathbf{k}} \rangle = \int_\Omega u_0(\mathbf{x}) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x} \,, \\ u_1(\mathbf{x}) &= \sum_{\mathbf{k}} b_{\mathbf{k}} X_{\mathbf{k}}(\mathbf{x}) \,, \quad b_{\mathbf{k}} = \langle u_1, X_{\mathbf{k}} \rangle = \int_\Omega u_1(\mathbf{x}) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x} \,, \end{aligned}$$

The solution can also be sought as a Fourier series

$$u(\mathbf{x},t) = \sum_{\mathbf{k}} T_{\mathbf{k}}(t) X_{\mathbf{k}}(\mathbf{x}) \,. \tag{44.10}$$

However, the crucial difference with the case when the parameters are linear combination of the basis functions is that for a Fourier series converging in the mean the order of summation and differentiation cannot be rearranged:

$$\frac{\partial^2}{\partial t^2}\sum_{\mathbf{k}} \neq \sum_{\mathbf{k}} \frac{\partial^2}{\partial t^2}\,, \quad L_x \sum_{\mathbf{k}} \neq \sum_{\mathbf{k}} L_x$$

in contrast to sums with *finitely many terms.* For this reason is not generally possible to conclude that the expansion coefficients satisfy the stated initial value problem. On the other hand, if the Fourier expansions of the parameters are changed to the corresponding *partial sums*, then the solution can always be found. Since the partial sums of the parameters converge to the actual parameters, one can hope that the associated sequence of solutions would converge to the actual solution. So, the series (**44.10**), where the expansion coefficients satisfy the initial value problem (**44.9**), is called a *formal solution* to the problem. If the sum of the series is proved to be from the class $C^2(\Omega) \cap C^1(\overline{\Omega})$, then it is the classical solution.

44.6. The boundary and initial value problem for a parabolic equation. Let Ω be a bounded region in $\mathbb{R}^N$. Consider the boundary and initial value problem for the hyperbolic equation:

$$\begin{aligned} &u_t'(\mathbf{x}, t) = -L_x u(\mathbf{x}, t) + f(\mathbf{x}, t)\,, \\ &u(\mathbf{x}, t) \in \mathcal{M}_L\,, \quad t \geq 0\,, \\ &u(\mathbf{x}, 0) = u_0(\mathbf{x})\,. \end{aligned}$$

The boundary conditions are included into the domain of the Sturm-Liouville operator L_x. It does not depend on the variable t so that its action on $u(\mathbf{x}, t)$ is calculated for each (fixed) value of t. This also implies that $u(\mathbf{x}, t)$ must be from the domain of L_x at every value of $t \geq 0$.

The analysis of the preceding section can be repeated for this problem. The only difference is that the expansion coefficients $T_{\mathbf{k}}(t)$ in the formal solution (**44.10**) must satisfy the first-order initial value problem:

(**44.11**)
$$\begin{aligned} &T_{\mathbf{k}}'(t) + \lambda_{\mathbf{k}} T_{\mathbf{k}}(t) = F_{\mathbf{k}}(t)\,, \quad T_{\mathbf{k}}(0) = a_{\mathbf{k}}\,, \\ &a_{\mathbf{k}} = \langle u_0, X_{\mathbf{k}} \rangle = \int_\Omega u_0(\mathbf{x}) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}\,. \end{aligned}$$

The solution to the initial value problem can be found by the method of variation of parameters:

$$T_{\mathbf{k}}(t) = a_{\mathbf{k}} e^{-\lambda_{\mathbf{k}} t} + \int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)} F_{\mathbf{k}}(\tau)\, d\tau\,.$$

If the sum of the series (**44**.**10**) with the above coefficients $T_{\mathbf{k}}$ is proved to be smooth enough, then it is the classical solution to the problem.

44.7. The boundary value problem for an elliptic equation in a cylinder. Some boundary value problems for elliptic equations can be solved by separation of variables in combination with the Fourier method. Let Π be a bounded region in $\mathbb{R}^{N+1}$. Suppose that the Sturm-Liouville operator has the following form

$$Lu(\mathbf{x}, z) = -\frac{\partial^2 u}{\partial z^2} + L_x u$$

where $\mathbf{x} \in \mathbb{R}^N$ and L_x is a general Sturm-Liouville operator that depends and acts only on the variables $\mathbf{x}$. For example, a three-dimensional Laplace operator can viewed as the sum

$$-\Delta u(x, y, z) = -\frac{\partial^2 u}{\partial z^2} + L_x u\,, \qquad L_x u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$$

Let Ω be a bounded region in $\mathbb{R}^N$. Consider the boundary value problem for the elliptic equation $Lu = f$ in the cylinder $\Pi = (0, l) \times \Omega$ where $z \in (0, l)$ and $\mathbf{x} \in \Omega$:

$$\begin{aligned} &-u''_{zz}(\mathbf{x}, t) = -L_x u(\mathbf{x}, z) + f(\mathbf{x}, z)\,,\\ &u(\mathbf{x}, z) \in \mathcal{M}_{L_x}\,, \quad 0 \le z \le l\,,\\ &u(\mathbf{x}, 0) = u_0(\mathbf{x})\,, \quad u(\mathbf{x}, l) = u_1(\mathbf{x})\,. \end{aligned}$$

The boundary conditions of the side surface of the cylinder are included into the definition of the domain of the operator L_x. For example, if Ω is a disk, $r^2 = x^2 + y^2 < a^2$, on a plane spanned by (x, y) and $L_x = -\Delta$ in the plane, then the boundary condition on the side of the cylinder $x^2 + y^2 = a^2$, $0 < z < l$ is

$$u(x, y, z) \in \mathcal{M}_{L_x} \quad \Rightarrow \quad \Big(\alpha u + \beta\, \frac{\partial u}{\partial r}\Big)\Big|_{r=a} = 0$$

where the constants α and β satisfy the usual conditions for a Sturm-Liouville operator.

This boundary value problem can be solved by separating variables so in exactly the same way as the initial and boundary value problem for the hyperbolic equation (here z used in place of t). Let $X_{\mathbf{k}}$ be an orthonormal basis in $\mathcal{L}_2(\Omega)$ obtained from the eigenfunctions of the operator L_x. Then the formal solution is given by the Fourier series (compare with (**44.10**))

$$u(z,\mathbf{x}) = \sum_{\mathbf{k}} Z_{\mathbf{k}}(z) X_{\mathbf{k}}(\mathbf{x})$$

in which the expansion coefficients $Z_{\mathbf{k}}(z)$ satisfy the *boundary value problem*:

(**44.12**)
$$\begin{aligned} -Z''_{\mathbf{k}}(z) + \lambda_{\mathbf{k}} Z_{\mathbf{k}}(z) &= F_{\mathbf{k}}(t)\,, \\ Z_{\mathbf{k}}(0) = a_{\mathbf{k}}\,, \qquad Z_{\mathbf{k}}(l) &= b_{\mathbf{k}} \\ a_{\mathbf{k}} = \langle u_0, X_{\mathbf{k}}\rangle &= \int_\Omega u_0(\mathbf{x}) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}\,, \\ b_{\mathbf{k}} = \langle u_1, X_{\mathbf{k}}\rangle &= \int_\Omega u_0(\mathbf{x}) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}\,, \\ F_{\mathbf{k}}(z) = \langle f, X_{\mathbf{k}}\rangle &= \int_\Omega f(\mathbf{x}, z) X_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}\,, \end{aligned}$$

It can be solved by the Green's function method. The needed Green's function was found in Section **11**:

$$\begin{aligned} Z_{\mathbf{k}}(z) &= a_{\mathbf{k}} \frac{\sinh[\sqrt{\lambda_{\mathbf{k}}}(l-z)]}{\sinh(l\sqrt{\lambda_{\mathbf{k}}})} + b_{\mathbf{k}} \frac{\sinh(z\sqrt{\lambda_{\mathbf{k}}})}{\sinh(l\sqrt{\lambda_{\mathbf{k}}})} \\ &\quad + \int_0^l G_{\mathbf{k}}(z,z') F_{\mathbf{k}}(z')\, dz'\,, \end{aligned}$$

$$G_{\mathbf{k}}(z,z') = \frac{1}{\sqrt{\lambda_{\mathbf{k}}}\sinh(l\sqrt{\lambda_{\mathbf{k}}})} \begin{cases} \sinh(z\sqrt{\lambda_{\mathbf{k}}}) \sinh\left(\sqrt{\lambda_{\mathbf{k}}}(l-z')\right), & z \le z'\,, \\ \sinh\left(\sqrt{\lambda_{\mathbf{k}}}(l-z)\right) \sinh(z'\sqrt{\lambda_{\mathbf{k}}}), & z' \le z\,. \end{cases}$$

If the sum of the Fourier series is proved to be smooth enough, then it is the classical solution to the problem.

45. The Laplace operator in a rectangle

The Laplace operator is the simplest elliptic operator (see Section **43** to review notations). Let Ω be a bounded region in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ oriented outward by the unit vector $\mathbf{n}$. The eigenvalue problems for the Laplace operator read

$$(\mathbf{45.1}) \qquad -\Delta u = \lambda u\,, \quad x \in \Omega\,; \qquad u\big|_{\partial\Omega} = 0\,,$$

$$(\mathbf{45.2}) \qquad -\Delta u = \lambda u\,, \quad x \in \Omega\,; \qquad \Big(\frac{\partial u}{\partial \mathbf{n}} + \alpha u\Big)\Big|_{\partial\Omega} = 0\,, \quad \alpha \geq 0$$

where α is a continuous function on the boundary $\partial\Omega$. One can view Problem (**45.1**) as the (formally) limit case of Problem (**45.2**) as $\alpha \to \infty$ everywhere on the boundary $\partial\Omega$ (divide the boundary condition by $\alpha > 0$ and take the limit).

Let Ω be a rectangle in $\mathbb{R}^2$. Suppose that the coordinate system is chosen so that $\Omega = (0,a) \times (0,b)$. Let the first and second coordinates are denoted x and y, respectively. Then the boundary of Ω consists of four straight line segments parallel to the coordinate axes. If $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ are the standard basis in $\mathbb{R}^2$, then the outward normal on the boundary lines $x = a$ and $y = b$ are $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, while the outward normal on the boundary lines $x = 0$ and $y = 0$ are the opposite vectors, $-\mathbf{e}_1$ and $-\mathbf{e}_2$, respectively. Note that the line $x = a$ is a level set $g(x,y) = x - a$ so that $\operatorname{grad} g = (1,0) = \mathbf{e}_1$, and similarly for $y = b$.

The analysis is readily extended to rectangular regions in higher dimensional spaces. For example, if $\Omega = (0,a) \times (0,b) \times (0,c) \subset \mathbb{R}^3$, then $\partial\Omega$ are faces of the rectangular box that are parts of six planes parallel to the coordinate planes. The outward normals on the faces that are in the planes $x = a$, $y = b$, and $z = c$ are $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$, respectively. The outward normals on the faces that lie in the planes $x = 0$, $y = 0$, and $z = 0$ are the opposite vectors $-\mathbf{e}_1$, $-\mathbf{e}_2$, and $-\mathbf{e}_3$, respectively.

45.1. Rectangular domains in Problem (45.1). Consider the eigenvalue problem for the Laplace operator in a rectangle

$$\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$$

Let $0 < x < a$ and $0 < y < b$. Solutions to (**45.1**) are found by separating variable

$$u(x,y) = X(x)Y(y) \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = X''Y\,, \quad \frac{\partial^2 u}{\partial y^2} = XY''\,,$$

By dividing the equation by XY it assumes the form

$$-\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = \lambda$$

from which it follows that

$$-X''(x) = \mu X(x)\,, \quad -Y''(y) = \nu Y(y)\,, \quad \mu + \nu = \lambda$$

where μ and ν are constants. The boundary condition requires that

$$\begin{aligned} u(0,y) &= 0\,, \quad u(a,y) = 0\,, \quad 0 \le y \le b \\ u(x,0) &= 0\,, \quad u(x,b) = 0\,, \quad 0 \le x \le a\,. \end{aligned}$$

Therefore the two-dimensional eigenvalue problem is reduced to two one-dimensional eigenvalue (Sturm-Liouville) problems

$$(\mathbf{45.3}) \qquad -X''(x) = \mu X(x)\,, \quad X(0) = X(a) = 0\,;$$

$$(\mathbf{45.4}) \qquad -Y''(y) = \nu Y(y)\,, \quad Y(0) = Y(b) = 0\,.$$

The eigenvalues and the corresponding orthonormal eigenfunctions are found in Section **??**:

$$(\mathbf{45.5}) \quad \mu = \mu_n = \frac{\pi^2 n^2}{a^2}\,, \quad X_n(x) = \sqrt{\frac{2}{a}}\sin\Big(\frac{\pi n}{a}\,x\Big)\,, \quad n = 1, 2, \dots\,;$$

$$(\mathbf{45.6}) \quad \nu = \nu_m = \frac{\pi^2 m^2}{a^2}\,, \quad Y_m(y) = \sqrt{\frac{2}{b}}\sin\Big(\frac{\pi m}{a}\,y\Big)\,, \quad m = 1, 2, \dots\,.$$

Therefore the eigenvalues and the corresponding eigenfunctions are

$$(\mathbf{45.7}) \quad \lambda = \lambda_{nm} = \mu_n + \nu_m = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\,, \quad n, m = 1, 2, \dots\,,$$

$$(\mathbf{45.8}) \quad u = \phi_{nm}(x,y) = X_n(x)Y_n(y) = \frac{2}{\sqrt{ab}}\sin\Big(\frac{\pi n}{a}\,x\Big)\sin\Big(\frac{\pi m}{a}\,y\Big)$$

The eigenfunctions are orthonormal in $\mathcal{L}_2(\Omega)$

$$\int_\Omega \phi_{nm}\phi_{n'm'}\,dxdy = \int_0^a X_n(x)X_{n'}(x)\,dx \int_0^b Y_m(y)Y_{m'}(y)\,dy = \delta_{nn'}\delta_{mm'}$$

where the Fubini theorem was used to calculate the integral.

The region $\Omega = D_1 \times D_2$ is the direct product of two intervals $D_1 = (0,a)$ and $D_2 = (0,b)$. The functions $X_n(x)$ form a complete orthonormal set (basis) in $\mathcal{L}_2(D_1)$, while the functions $Y_m(y)$ form a complete orthonormal set (basis) in $\mathcal{L}_2(D_2)$. By Theorem **??**, the eigenfunctions $\phi_{nm}(x,y) = X_n(x)Y_m(y)$ form a complete orthonormal set in

$\mathcal{L}_2(\Omega)$. Thus, for any function $f \in \mathcal{L}_2(\Omega)$, the Fourier series

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle f, \phi_{nm}\rangle\, \phi_{nm}(x,y) \quad a.e.$$

converges in the mean (the sum coincides with $f(x,y)$ almost everywhere).

Evidently, the method can readily be extended to a rectangular domain in $\mathbb{R}^N$. If

$$\Omega = (0,a_1) \times (0,a_2) \times \cdots \times (0,a_N) \subset \mathbb{R}^N\,,$$
$$0 < x_j < a_j\,, \quad j = 1,2,...,N\,,$$

then the eigenvalues and the corresponding orthonormal eigenfunctions are

$$\lambda = \lambda_{\mathbf{n}} = \pi^2 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \cdots + \frac{n_N^2}{a_N^2}\right)\,, \quad n_j = 1,2,...\,,$$
$$u = \phi_{\mathbf{n}}(x) = \left(\frac{2^N}{a_1 a_2 \cdots a_N}\right)^{1/2} \prod_{j=1}^{N} \sin\left(\frac{\pi n_j}{a_j}\, x_j\right)\,,$$

where the vectors $\mathbf{n} = (n_1, n_2, ..., n_N)$ with positive integer-valued components are used to label the eigenvalues and the corresponding eigenfunctions. The details of derivation of these equations are left to the reader as an exercise.

45.2. Rectangular domains in Problem (45.2). Let $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$. The boundary of this rectangle consists of four smooth pieces (straight line segments):

$$\partial\Omega = S_{1L} \cup S_{1R} \cup S_{2L} \cup S_{2R}\,,$$
$$S_{1L} = \{x=0\} \times [0,b]\,, \quad S_{1R} = \{x=a\} \times [0,b]\,,$$
$$S_{2L} = [0,a] \times \{y=0\}\,, \quad S_{2R} = [0,a] \times \{y=b\}\,.$$

Here the subscripts 1 and 2 stand for the "first" and "second" variables in the ordered pair (x,y), while the subscripts L and R stand for the "left" and "right" endpoints of the interval spanned by the corresponding (first or second) variable. If $\partial\Omega$ is oriented outward by the unit normal vector $\mathbf{n}$, then

$$\mathbf{n}\Big|_{S_{1L}} = -\mathbf{e}_1\,, \quad \mathbf{n}\Big|_{S_{1R}} = \mathbf{e}_1\,, \quad \mathbf{n}\Big|_{S_{2L}} = -\mathbf{e}_2\,, \quad \mathbf{n}\Big|_{S_{2R}} = \mathbf{e}_2\,,$$

where $\mathbf{e}_j$, $j = 1,2$, is the standard basis in $\mathbb{R}^2$. Note that the normal $\mathbf{n}$ is not defined at the four corners of the rectangle because its boundary is not smooth at these points. In the above relations, the corner points are not included. Since

$$\operatorname{grad} u(x,y) = \mathbf{e}_1 u'_x(x,y) + \mathbf{e}_2\, u'_y(x,y)\,,$$

the normal derivative is

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{S_{1L}} = -u'_x(0,y)\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{S_{1R}} = u'_x(a,y)\,, \quad 0 \le y \le b\,;$$
$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{S_{2L}} = -u'_y(x,0)\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{S_{2R}} = u'_y(x,b)\,, \quad 0 \le x \le a\,,$$

where the normal derivative is continuously extended to the endpoints of the specified (closed) intervals, assuming, of course that $u \in C^1(\overline{\Omega})$.

*If $\partial\Omega$ is piecewise smooth, then the boundary condition (**45.2**) is understood as the set of boundary conditions for each smooth piece of $\partial\Omega$ and in each boundary condition the normal derivative is continuously extended to the boundary of the corresponding smooth piece of $\partial\Omega$.* In the case considered, there are for smooth pieces in $\partial\Omega$. In particular

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{S_{1L}} + \alpha u\Big|_{S_{1L}} = -u'_x(0,y) + \alpha(0,y)u(0,y)\,, \quad 0 \le y \le b$$

and the other three boundary conditions are obtained similarly. Even though the variables can be separated in the Laplace operator, the boundary conditions may not admit the existence of a solution in the form $u(x,y) = X(x)Y(y)$. This is a limitation of the method. If $u(x,y) = X(x)Y(y)$, then the condition at S_{1L} yields

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{S_{1L}} + \alpha u\Big|_{S_{1L}} = -X'(0)Y(y) + \alpha(0,y)X(0)Y(y) = 0\,, \quad y \in [0,b]\,.$$

It cannot be reduced is reduced to a boundary condition for X *independent of* y unless $\alpha(0,y) = \alpha = \text{const}$, $0 \le y \le b$, in which case

$$-X'(0) + \alpha X(0) = 0\,.$$

A similar analysis shows that α must be a constant function on the other three smooth pieces of the boundary. If, in addition, the function α is required to be continuous in $\partial\Omega$, then the problem admits a separation of variables if the function α is a constant function on the boundary $\partial\Omega$, that is, $\alpha(x,y) = \alpha$ for all $(x,y) \in \partial\Omega$. In the next section, it is shown that the continuity condition may be relaxed in some cases.

Thus, if α is a non-negative constant on $\partial\Omega$, then the original two-dimensional eigenvalue problem is equivalent to two one-dimensional

(Sturm-Liouville) problems:

$$(\mathbf{45.9}) \qquad -X''(x) = \mu X(x)\,, \quad \begin{cases} X'(0) - \alpha X(0) = 0 \\ X'(a) + \alpha X(a) = 0 \end{cases}$$

$$(\mathbf{45.10}) \qquad -Y''(y) = \nu Y(y)\,, \quad \begin{cases} Y'(0) - \alpha Y(0) = 0 \\ Y'(b) + \alpha Y(b) = 0 \end{cases}$$

where $\lambda = \mu + \nu$. Each of these problems can be solved by the method of Section 8.2 (see also Example 8.1). By setting $l = a$ (or $l = b$), $\alpha_0 = \alpha_l = \alpha$ and $\beta_0 = \beta_l = 1$ in (**36.3**), the boundary conditions in (**45.9**) (or (**45.10**) are obtained. By Theorem **44.4**, $\mu = 0$ is an eigenvalue if and only if $\alpha = 0$ and in this case the corresponding eigenfunction is a constant function. The details of the case $\alpha = 0$ are left to the reader as an exercise. In what follows it is assumed that $\alpha > 0$. In this case all eigenvalues must be strictly positive, $\mu > 0$. Equation (**36.8**) determines the solution of the equation in (**45.9**) that satisfies the first boundary condition is

$$X(x;\mu) \equiv Z(x;\mu) = \cos(\sqrt{\mu}\,x) + \frac{\alpha}{\sqrt{\mu}}\sin(\sqrt{\mu}\,x)$$

The second boundary condition yields an equation for the eigenvalues:

$$-\sqrt{\mu}\sin(\sqrt{\mu}\,a) + \alpha\cos(\sqrt{\mu}\,a) + \alpha\Big[\cos(\sqrt{\mu}\,a) + \frac{\alpha}{\sqrt{\mu}}\sin(\sqrt{\mu}\,a)\Big] = 0$$

or

$$(\mathbf{45.11}) \qquad \cot(a\theta) = \frac{1}{2\alpha}\left(\theta - \frac{\alpha^2}{\theta}\right), \qquad \theta = \sqrt{\mu}\,.$$

The function in the right side of this transcendent equation is strictly monotonic because its derivative is positive

$$\frac{1}{2\alpha}\left(\theta - \frac{\alpha^2}{\theta}\right)' = \frac{1}{2\alpha}\left(1 + \frac{\alpha^2}{\theta^2}\right) > 0\,, \qquad \theta > 0$$

The graph of this function has a vertical asymptote at $\theta = 0$ and tends to $-\infty$ as $\theta \to 0^+$. It also has slant asymptote given by the graph of the linear function $\theta/(2\alpha)$ which is approached from below. Therefore the graph intersects the graph of $\cot(a\theta)$ only once in each interval $\pi(n-1)/a < \theta < \pi n/a$, $n = 1, 2, \ldots$. Thus, Eq. (**45.11**) has countably many simple roots:

$$\theta = \theta_n(a)\,, \qquad \frac{\pi(n-1)}{a} < \theta_n(a) < \frac{\pi n}{a}\,, \quad n = 1, 2, \ldots\,.$$

A similar analysis holds for the problem (**45.10**). The eigenvalues and the corresponding eigenfunctions of the problems (**45.9**) and (**45.10**) are, respectively,

$$(45.12)\quad \mu = \mu_n = \theta_n^2(a)\,, \quad X = X_n(x) = Z(x;\mu_n)\,, \quad n = 1, 2, \dots\,,$$

$$(45.13)\quad \nu = \nu_m = \theta_m^2(b)\,, \quad Y = Y_m(y) = Z(y;\nu_m)\,, \quad m = 1, 2, \dots\,.$$

The eigenvalues and the corresponding eigenfunctions for Problem (**??**) in a two-dimensional rectangle are

$$(45.14)\qquad \lambda = \lambda_{nm} = \mu_n + \nu_m\,,$$

$$(45.15)\qquad u = \phi_{nm}(x,y) = X_n(x)Y_m(y)$$

*The functions (**45.15**) form an orthogonal complete set in* $\mathcal{L}_2(\Omega)$. Indeed, by the general analysis of the Sturm-Liouville problem, the set $\{X_n\}_1^\infty$ is a complete orthogonal set in $\mathcal{L}_2(0,a)$, while the set $\{Y_m\}_1^\infty$ is a complete orthogonal set in $\mathcal{L}_2(0,b)$. Since $\Omega = (0,a)\times(0,b)$, the set $\{\phi_{nm}\}$ has to be complete and orthogonal in $\mathcal{L}_2(\Omega)$ by Theorem **??**. An orthonormal complete set is obtained by normalizing ϕ_{nm}. A calculation of the normalization constant $\|\phi_{nm}\|$ is left to the reader as an exercise.

45.3. More general boundary conditions. Let the eigenvalue problem for the Laplace operator in $\Omega = (0,a)\times(0,b)$ with the boundary condition

$$(45.16)\qquad \left(\beta(x)\frac{\partial u}{\partial \mathbf{n}} + \alpha(x)u\right)\Big|_{\partial\Omega} = 0\,.$$

The boundary $\partial\Omega$ consists of four smooth pieces S_{1L}, S_{1R}, $S2L$ and S_{2R}. The functions α and β restricted to the *interior* of these pieces are denoted by α_s and β_s, $s = 1L, 1R, 2L, 2R$, respectively. For example,

$$\alpha\Big|_{S_{1L}} = \alpha_{1L}(y) = \alpha(0,y)\,, \quad \beta\Big|_{S_{1L}} = \beta_{1L}(y) = \beta(0,y)\,, \quad 0 < y < b\,,$$

$$\alpha\Big|_{S_{2R}} = \alpha_{2R}(x) = \alpha(x,b)\,, \quad \beta\Big|_{S_{2R}} = \beta_{2R}(y) = \beta(x,b)\,, \quad 0 < x < a\,.$$

Note the *open* intervals in the domain of these functions; they correspond to the interior of the intervals S_{1L} and S_{2R}.

Then eigenvalue problem for the Laplace operator subject to the boundary condition (**45.16**) in a rectangle can be written in the form:

$$-u''_{xx} - u''_{yy} = \lambda u\,, \qquad (x,y)\in\Omega\,,$$

$$(45.17)\qquad \begin{cases} \alpha_{1L}u(0,y) - \beta_{1L}u'_x(0,y) = 0 \\ \alpha_{1R}u(a,y) + \beta_{1R}u'_x(a,y) = 0 \end{cases}, \quad 0 \le y \le b\,,$$

$$(45.18)\qquad \begin{cases} \alpha_{2L}u(x,0) - \beta_{2L}u'_y(x,0) = 0 \\ \alpha_{2R}u(x,b) + \beta_{2R}u'_y(x,b) = 0 \end{cases}, \quad 0 \le x \le a\,.$$

In the boundary conditions (**45.17**), it is assumed that the functions $\alpha_s(y)$ and $\beta_s(y)$, $s = 1L, 1R$, are continuous in $(0, b)$ and can be continuously extended to the endpoints of the interval $[0, b]$ (the limits of these functions as $y \to 0^+$ and $y \to b^-$ exist). Similarly, in (**45.18**), it is assumed that the functions $\alpha_s(x)$ and $\beta_s(x)$, $s = 2L, 2R$, are continuous in $(0, a)$ and can be continuously extended to the endpoints $x = 0$ and $x = a$. Note that under these assumptions, the functions α and β in (**45.16**) may have jump discontinuities at the four corners of the rectangle, while they are continuous elsewhere.

If a solution is sought in the form $u(x, y) = X(x)Y(y)$, then the boundary conditions cannot be fulfilled, *unless the functions α_s and β_s are constant functions*. Indeed, take, for instance, the first condition in (**45.17**):

$$\alpha_{1L}(y)X(0)Y(y) - \beta_{1L}(y)X'(0)Y(y) = 0\,, \quad 0 \le y \le b\,.$$

If α_{1L} and β_{1L} vary in S_{1L}, then the boundary condition is impossible to fulfill by choosing the constants $X(0)$ and $X'(0)$. A similar observation can be made for the other three boundary conditions.

Suppose that α_s and β_s are constants. This means that the function α and β in (**45.16**) are piecewise constant along the boundary and may have jump discontinuities at the four corners of the rectangle. In this case, *the eigenvalue problem can be solved by separating variables.* Moreover if α_s and β_s are constants such that

$$\alpha_s \ge 0\,, \quad \beta_s \ge 0\,, \quad \alpha_s + \beta_s > 0\,, \quad s = 1L,\ 1R,\ 2L,\ 2R\,,$$

then the eigenvalues are non-negative. Indeed, in this case the problem can be reduced to two Sturm-Liouville problems

$$-X''(x) = \mu X(x)\,, \qquad \begin{cases} \alpha_{1L}X(0) - \beta_{1L}X'(0) = 0 \\ \alpha_{1R}X(a) + \beta_{1R}X'(a) = 0 \end{cases}$$

$$-Y''(y) = \nu Y(y)\,, \qquad \begin{cases} \alpha_{2L}Y(0) - \beta_{2L}Y'(0) = 0 \\ \alpha_{2R}Y(b) + \beta_{2R}Y'(b) = 0 \end{cases}$$

$$\lambda = \mu + \nu\,.$$

Let $\mu = \mu_n$ and $X = X_n(x)$, $n = 1, 2, \ldots$, be the eigenvalues and the corresponding orthonormal eigenfunctions in the first problem and, similarly, $\nu = \nu_m$ and $Y = Y_m(y)$. The sets $\{X_n\}_1^\infty$ and $\{Y_m\}_1^\infty$ are complete in $\mathcal{L}_2(0, a)$ and $\mathcal{L}_2(0, b)$, respectively. Then the eigenvalues of the Laplace operator and the corresponding eigenfunctions are

$$\lambda_{nm} = \mu_n + \nu_m\,, \quad \phi_{nm}(x, y) = X_n(x)Y_m(y)\,, \quad n, m = 1, 2, \ldots\,.$$

By Theorem **??**. the set $\{\phi_{nm}\}$ is a complete orthogonal set in $\mathcal{L}_2(\Omega)$.

Clearly, the above analysis is straightforward to extend to the boundary value problem for the Laplace operator for a rectangular domain in $\mathbb{R}^N$.

45.4. Periodic boundary conditions. Consider a straight line segment. The endpoints of the segment are glued so that the segment becomes a circle. A circle is denoted $\mathbb{S}^1$. A function on $\mathbb{S}^1$ is a periodic function on $\mathbb{R}$. Consider a rectangle $(0,a)\times(0,b)$ in $\mathbb{R}^2$. Let us glue together two opposite sides of the rectangle The result is a cylindrical shell, e.g., $\Omega=(0,a)\times\mathbb{S}^1$. Functions on a cylindrical shell are functions of two real variables that are periodic in one of the variables. If two edges of a cylindrical shell are glued together, a two-dimensional torus, $\mathbb{S}^1\times\mathbb{S}^1$, is obtained. Function on a torus are periodic functions of two real variables. Similarly, functions on an $N-$dimensional torus are periodic functions of n real variables.

Two-dimensional cylindrical shell. One can pose the eigenvalue problem for the Laplace operator on a cylindrical shell:

$$
\begin{aligned}
&-u''_{xx}-u''_{yy}=\lambda u\,, \quad (x,y)\in(0,a)\times\mathbb{S}^1\,,\\
&\begin{cases}\alpha_0 u(0,y)-\beta_0 u'_x(0,y)=0\\ \alpha_a u(a,y)+\beta_a u'_x(a,y)=0\end{cases}, \quad y\in\mathbb{R}\,,\\
&u(x,y+b)=u(x,y)\,,\quad x\in[0,a]\,,\quad y\in\mathbb{R}\,.
\end{aligned}
$$

Here the variable y spans a circle $\mathbb{S}^1$ and b is the circumference so that the cylindrical shell has radius $b/(2\pi)$. If the parameters in the boundary conditions are constants (independent of the variable y), the problem can be solved by separating variables

$$
\begin{aligned}
&u(x,y)=X(x)Y(y)\,,\quad \lambda=\mu+\nu\,,\\
&-Y''(y)=\nu Y(y)\,,\quad Y(y+b)=Y(y)\,,\quad y\in\mathbb{R}\\
&-X''(x)=\mu X(x)\,,\quad \begin{cases}\alpha_0 X(0)-\beta_0 X'(0)=0\\ \alpha_a X(a)+\beta_a X'(a)=0\end{cases}
\end{aligned}
$$

Using the results of Section **??**,

$$
\begin{aligned}
&u(x,y)=\phi_{km}(x,y)=X_k(x)Y_m(y)\,,\quad \lambda=\lambda_{km}=\mu_k+\nu_m\,,\\
&Y_m(y)=e^{2i\pi my/b}\,,\quad \nu=\nu_m=\left(\frac{2\pi m}{b}\right)^2\,,\quad m=0,\pm1,\pm2,...,
\end{aligned}
$$

and $\mu=\mu_k$ and $X(x)=X_k(x)$, $k=1,2,...$, are the eigenvalues and the corresponding eigenfunctions of the Sturm-Liouville operator $L=-d^2/dx^2$. The functions X_k form a complete orthogonal set in $\mathcal{L}_2(0,a)$, while the functions $Y_m(y)$ form a complete orthogonal set in $\mathcal{L}_2(0,b)$.

By Theorem **??**, the functions $\phi_{km}(x,y)$ form a complete orthogonal set in $\mathcal{L}_2(\Omega)$ where $\Omega = (0,a)\times\mathbb{S}^1$. Note also that eigenfunctions of the operator $L = -d^2/dy^2$ on $\mathbb{S}^1$ can be chosen real

$$Y(y) = Y_n(y) = \Big\{1\,,\ \cos\Big(\frac{2\pi n}{b}\,y\Big)\,,\ \sin\Big(\frac{2\pi n}{b}\,y\Big)\Big\}\,,\quad n = 1,2,...$$

Each eigenvalue, except $\nu = 0$, has multiplicity two.

Two-dimensional torus. The following example illustrates the method of solving the eigenvalue problem on a two-dimensional torus. The method is readily extended to a higher-dimensional torus.

EXAMPLE **45.1**. (Eigenvalue problem on a torus)
Find the eigenvalues and eigenfunctions of the Laplace operator on a two-dimensional torus $\mathbb{S}^1\times\mathbb{S}^1$*:*

$$-\Delta u = \lambda u\,,\qquad u\in C^2(\mathbb{S}^1\times\mathbb{S}^1)\,.$$

Show that eigenfunctions form a complete orthogonal set in $\mathcal{L}_2(\mathbb{S}^1\times\mathbb{S}^1)$.

SOLUTION: Let a and b be the dimensions of the torus. Solutions satisfy the periodicity conditions:

$$u(x+a,y) = u(x,y)\,,\quad u(x,y+b) = u(x,y)\,.$$

Let $u(x,y) = X(x)Y(y)$. Then the problem is reduced to two one-dimensional problems

$$\begin{aligned} -X''(x) &= \mu X(x)\,, \qquad X(x+a) = X(x)\,;\\ -Y''(y) &= \nu Y(y)\,, \qquad Y(y+b) = Y(y)\,, \end{aligned}$$

where $\lambda = \mu+\nu$. Therefore using the results of Section **??**

$$\mu = k_n^2,\quad k_n = \frac{2\pi n}{a}\,,\quad X = X_n(x) = e^{ik_nx}\,,\quad n = 0,\pm1,\pm2,...,;$$
$$\nu = p_m^2,\quad p_m = \frac{2\pi m}{b}\,,\quad Y = Y_m(y) = e^{ip_my}\,,\quad m = 0,\pm1,\pm2,....$$

Note that X_{-n} and X_n, $n\neq 0$ correspond to the same eigenvalue k_n^2 (which has multiplicity 2). Any linear combination of them is also an eigenfunction corresponding to k_n^2. The functions X_{-n} and X_n are orthogonal. The eigenvalues and eigenfunctions the Laplace operator are

$$\lambda_{nm} = k_n^2 + p_m^2\,,\quad \phi_{nm}(x,y) = X_n(x)Y_m(y)\,,\quad n,m = 0,\pm1,\pm2,....$$

Each set of functions X_n, $n = 0,\pm1,...$, and Y_m, $m = 0,\pm1,...$, is a complete orthogonal set in $\mathcal{L}_2(\mathbb{S}^1)$. By Theorem **??**, the functions ϕ_{nm}

form a complete orthogonal set in $\mathcal{L}_2(\mathbb{S}^1 \times \mathbb{S}^1)$:

$$\langle \phi_{nm}, \phi_{n'm'} \rangle = \int_0^a \int_0^b \phi_{nm}(x,y)\overline{\phi_{n'm'}(x,y)}\, dxdy = ab\, \delta_{nn'}\delta_{mm'}$$

Fourier series over ϕ_{nm} are two-dimensional trigonometric Fourier series. □

45.5. Physical interpretation. Consider a wave equation in a Euclidean space $\mathbb{R}^N$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta u(x,t)\,, \qquad x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N\,.$$

Let $k = (k_1, k_2, ..., k_N) \in \mathbb{R}^N$ be a (fixed) vector and

$$(k,x) = k_1x_1 + k_2x_2 + \cdots + k_Nx_N$$

denote the dot product in $\mathbb{R}^N$ so that the length of a vector is

$$|k| = \sqrt{k_1^2 + k_2^2 + \cdots + k_N^2} = \sqrt{(k,k)}\,.$$

The function

$$u(x,t;k) = Ae^{-i\omega t + i(k,x)}\,, \tag{45.19}$$

where A is a constant, is a solution to the wave equation if the parameters ω and k satisfy the relation

$$\omega^2 = c^2|k|^2\,. \tag{45.20}$$

Indeed,

$$\frac{\partial^2 u}{\partial x_j^2} = (ik_j)^2 u \quad \Rightarrow \quad \Delta u = -(k_1^2 + k_2^2 + \cdots + k_N^2)u = -|k|^2 u$$

$$\frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} = \frac{(-i\omega)^2}{c^2}\, u = -\frac{\omega^2}{c^2}\, u$$

so that the wave equation is satisfied, provided (**45**.**20**) holds. Note also that the real and imaginary parts of (**45**.**19**) also satisfy the wave equation

$$u(x,t) = B\cos(\omega t - (k,x)) \quad \text{or} \quad u(x,t) = C\sin(\omega t - (k,x))\,.$$

If $u(x,t)$ denotes a "deviation" of some quantity from its equilibrium value (e.g., air density for sound waves, or a component of the electric (or magnetic) field for electromagnetic waves (radio waves, light)), then the deviation has a constant value A in space on the set

$$(k,x) = \omega t = c|k|t \quad \Rightarrow \quad (\hat{k}, x) = ct$$

where $\hat{k} = k/|k|$ is the unit vector in the direction of the vector k. For every t, the above equation describes a plane in space perpendicular to the vector $\hat{k}$. The plane moves parallel in the direction of $\hat{k}$ at a constant rate (speed) c with increasing t.

Thus, the amplitude of the solution remains constant on a plane in space that moves in the direction of the vector k at a constant rate c. At any fixed point x, the amplitude of the solution oscillates with frequency ω. If x is moved along the vector k by the distance $\lambda = 2\pi/|k|$, then the dot product (k, x) is changed by 2π so that $u(x,t)$ does not change. In other words, at a fixed t, $u(x,t)$ is a periodic function along the line parallel to the vector k with period λ. For this reason the solution (**45**.**19**) (or its real or imaginary parts) is called a *plane wave* propagating in the direction k, the vector k is called the *wave vector* of a plane wave, ω is the *frequency*, $\lambda = 2\pi/|k|$ is the *wave length*, the relation (**45**.**20**) is called the *dispersion relation* of the wave, and a constant A is an *amplitude* of the wave.

Visible light is a superposition of electromagnetic waves (c is the speed of light in a medium through which the light is propagating). The color of light is determined by wavelength (or frequency) of the wave. For example, variations of red, green, and blue colors correspond to electromagnetic waves of wavelength ranges: $620 \div 750$ *nm* (dark red to light red), $495 \div 570$ *nm* , and $450 \div 495$ *nm*, respectively. So, every color can be precisely defined by the corresponding numerical value of the wavelength or frequency of light. This is why solutions of the wave equation with a fixed frequency are also called *monochromatic*.

The most general form of a *monochromatic solution* is

$$u(x,t) = u_\omega(x) e^{\pm i\omega t}\,, \tag{45.21}$$

where $u_\omega(x)$ is the oscillation amplitude at a point x in space. For example, there can be several plane waves passing through the point x in different directions or with different wave vectors k_n, $n = 1, 2, ..., m$. In this case,

$$u_\omega(x) = \sum_{n=1}^{m} A_n e^{i(k_n, x)}\,, \qquad |k_n| = \frac{\omega}{c}\,.$$

Note that all wave vectors should have the same length by the dispersion relation (**45**.**20**). The substitution of (**45**.**21**) into the wave equation yields an equation for the amplitude of the monochromatic wave:

$$-\Delta u_\omega = \lambda^2 u_\omega\,, \quad \lambda = \frac{\omega}{c}\,.$$

This equation looks like the eigenvalue problem for the Laplace operator where the wavelength of the monochromatic wave (or its frequency) plays the role of the spectral parameter (the eigenvalue). If the wave equation is considered in the entire space, then the corresponding equation is known as the Helmholtz equation which has solutions for any λ or any wavelength. An example is provided by a superposition of plane waves of the same frequency.

The situation changes if the wave equation is considered in a bounded spatial region Ω. If the wave cannot propagate beyond the boundary $\partial\Omega$, then any (physical) solution is required to satisfy some boundary conditions. The boundary conditions depend on the interaction of waves with the boundary. For example, if the boundary acts like a mirror, a perfect reflector, then the amplitude of the wave must vanish on the boundary (recall the reflection principle for the wave equation). In this case, a bounded region Ω is called a *wave resonator*. A resonator supports only waves of certain wavelengths (frequencies). An amplitude of a monochromatic wave in a resonator is a solution to the eigenvalue problem of the Laplace operator is a bounded region:

$$-\Delta u_\omega = \lambda^2 u_\omega\,, \quad \lambda = \frac{\omega}{c}\,, \quad u_\omega\Big|_{\partial\Omega} = 0\,,$$

In particular, in a rectangular resonator (domain)

$$\Omega = (0, a_1) \times (0, a_2) \times \cdots \times (0, a_N)$$

the solution exists only for discrete values of the frequency ω or wavelength λ:

$$\omega = \omega_{\mathbf{n}} = \frac{c}{\pi}\left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \cdots + \frac{n_N^2}{a_N^2}\right)^{1/2}, \quad n_j = 1, 2, \ldots\,,$$
$$u_\omega(x) = \phi_{\mathbf{n}}(x) = \sin\left(\frac{\pi n_1}{a_1}\,x_1\right)\sin\left(\frac{\pi n_2}{a_2}\,x_2\right)\cdots\sin\left(\frac{\pi n_N}{a_N}\,x_N\right)\,.$$

The functions $\phi_{\mathbf{n}}$ form a complete orthogonal set in $\mathcal{L}_2(\Omega)$ by Theorem **??**:

$$\begin{aligned}\langle\phi_{\mathbf{n}}, \phi_{\mathbf{n}'}\rangle &= \int_0^{a_1}\int_0^{a_2}\cdots\int_0^{a_N} \phi_{\mathbf{n}}(x)\phi_{\mathbf{n}'}(x)\,dx_1 dx_2\cdots dx_N \\ &= 2^{-N} a_1 a_2 \cdots a_N\,\delta_{\mathbf{n}\mathbf{n}'}\end{aligned}$$

where the Kronecker symbol $\delta_{\mathbf{n}\mathbf{n}'} = 1$ if $\mathbf{n} = \mathbf{n}'$ (or $n_j = n_j'$ for all $j = 1, 2, \ldots, N$) and it vanishes otherwise.

A solution to the wave equation in a resonator is a superposition of *standing waves*:

$$u(x,t)=\sum_{\mathbf{n}}\Big(A_{\mathbf{n}}\cos(\omega_{\mathbf{n}}t)+\frac{B_{\mathbf{n}}}{\omega_{\mathbf{n}}}\sin(\omega_{\mathbf{n}}t)\Big)\phi_{\mathbf{n}}(x)\,, \tag{45.22}$$

where the summation is carried out over all possible distinct sets of positive integers $\mathbf{n}=(n_1,n_2,...,n_N)$. For physical solutions the coefficients $A_{\mathbf{n}}$ and $B_{\mathbf{n}}$ should decay fast enough with increasing the length $|\mathbf{n}|$ of the vector $\mathbf{n}$ in order for the Fourier to converge to a sufficiently smooth function. The solution satisfies the initial conditions:

$$u(x,0)=u_0(x)=\sum_{\mathbf{n}}A_{\mathbf{n}}\phi_{\mathbf{n}}(x)\quad\Rightarrow\quad A_{\mathbf{n}}=\frac{\langle u_0,\phi_{\mathbf{n}}\rangle}{\langle\phi_{\mathbf{n}},\phi_{\mathbf{n}}\rangle}$$

$$u_t'(x,0)=u_1(x)=\sum_{\mathbf{n}}B_{\mathbf{n}}\phi_{\mathbf{n}}(x)\quad\Rightarrow\quad B_{\mathbf{n}}=\frac{\langle u_1,\phi_{\mathbf{n}}\rangle}{\langle\phi_{\mathbf{n}},\phi_{\mathbf{n}}\rangle}$$

Other boundary conditions. In a similar fashion, the superposition principle can be used to obtain a formal solution to the wave equation with periodic boundary condition for some of the variables and general Sturm-Liouville boundary conditions for the other variables.

EXAMPLE **45.2**. *Find the formal solution to the wave equation in a three-dimensional space spanned by real variables* (x,y,z)

$$u''_{tt}=c^2(u''_{xx}+u''_{yy}+u''_{zz})\,,\quad (x,y,z)\in(0,a)\times(0,b)\times\mathbb{S}^1\,,\ t>0\,,$$

that satisfies the initial conditions

$$u(x,y,z,0)=0\,,\quad u_t'(x,y,z)=Bx(a-x)\sin^2(z)$$

for all $(x,y,z)\in[0,a]\times[0,b]\times\mathbb{R}$ *and the boundary conditions*

$$\begin{aligned}&u(0,y,z,t)=u(a,y,z,t)=0\,,\quad (y,z)\in[0,b]\times\mathbb{R}\,,\\&u_y'(x,0,z,t)=u_y'(x,b,z,t)=0\,,\quad (x,z)\in[0,a]\times\mathbb{R}\,,\\&u(x,y,z+2\pi,t)=u(x,y,z,t)\,,\quad (x,y,z)\in[0,a]\times[0,b]\times\mathbb{R}\,,\end{aligned}$$

for all $t\geq 0$. *Determine whether the formal solution is also a classical solution to the problem.*

SOLUTION: The fist step is to find the eigenfunctions of the Laplace operator on $\Omega=(0,a)\times(0,b)\times\mathbb{S}^1$. The eigenfunctions are sought in the form

$$u_\omega(x,y,z)=X(x)Y(y)Z(z)$$

The equation

$$-\frac{\partial^2 u_\omega}{\partial x^2} - \frac{\partial^2 u_\omega}{\partial y^2} - \frac{\partial^2 u_\omega}{\partial z^2} = \lambda^2 u_\omega$$

is reduced to three one-dimensional eigenvalue problems:

$$\begin{cases} -X''(x) = \mu^2 X(x) \\ X(0) = X(a) = 0 \end{cases} \Rightarrow \begin{cases} \mu = \mu_n = \pi n/a\,,\ n = 1, 2, ... \\ X(x) = X_k(x) = \sin(\mu_n x) \end{cases}$$

$$\begin{cases} -Y''(y) = \nu^2 Y(y) \\ Y'(0) = Y'(b) = 0 \end{cases} \Rightarrow \begin{cases} \nu = \nu_m = \pi m/b\,,\ m = 0, 1, ... \\ Y(y) = Y_m(y) = \cos(\nu_m x) \end{cases}$$

$$\begin{cases} -Z''(z) = \beta^2 Z(z) \\ Z(z+2\pi) = Z(z) \end{cases} \Rightarrow \begin{cases} \beta = \beta_k = k\,,\ k = 0, \pm 1, \pm 2... \\ Z(z) = Z_k(z) = e^{ikz} \end{cases}$$

where the eigenfrequencies of the standing waves are

$$\omega_{\mathbf{n}} = \frac{\lambda}{c} = \frac{1}{c}\sqrt{\mu_n^2 + \nu_m^2 + \beta_k^2}\,, \quad \mathbf{n} = (n, m, k)$$

and the corresponding eigenmodes are

$$\phi_{\mathbf{n}}(x, y, z) = X_n(x) Y_m(y) Z_k(z) = \sin(\mu_n)\cos(\nu_m y) e^{ikz}$$

The modes are orthogonal and form a complete set in $\mathcal{L}_2(\Omega)$, where $\Omega = (0, a) \times (0, b) \times \mathbb{S}^1$. They have the norm:

$$\begin{aligned} \langle \phi_{\mathbf{n}}, \phi_{\mathbf{n}} \rangle &= \int_0^a \int_0^b \int_0^{2\pi} \phi_{\mathbf{n}}(x,y,z)\overline{\phi_{\mathbf{n}}(x,y,z)}\,dxdydz \\ &= \int_0^a X_n(x)\overline{X_n(x)}\,dx \int_0^b Y_m(y)\overline{Y_m(y)}\,dy \int_0^{2\pi} Z_k(z)\overline{Z_k(z)}\,dz \\ &= \int_0^a \sin^2(\mu_n x)\,dx \int_0^b \cos^2(\nu_m y)\,dy \int_0^{2\pi} dz \\ &= \frac{a}{2}\cdot\frac{b}{2}\cdot 2\pi = \frac{\pi ab}{2}\,, \end{aligned}$$

if $\nu_m \neq 0$ or $m \neq 0$. For all $\mathbf{n} = (n, 0, k)$ (in this case $Y_0(y) = 1$) the integral over $[0, b]$ contributes the factor b so that

$$\langle \phi_{\mathbf{n}}, \phi_{\mathbf{n}} \rangle = \begin{cases} \pi ab\,, & m = 0 \\ \frac{1}{2}\pi ab\,, & m > 0 \end{cases}$$

Next, the Fourier coefficients of the initial data have to be calculated. Note that $A_{\mathbf{n}} = 0$ in the series (**45.22**) because $u_0(x) = 0$ in this example. The other initial data has the form of the product:

$$\begin{aligned} &u_1(x) = f(x)g(y)h(z)\,, \\ &f(x) = x(a-x)\,,\ g(y) = B\,,\ h(z) = \sin^2(z) \end{aligned}$$

Therefore the triple integral for the Fourier coefficients becomes the product of three ordinary integrals:

$$\begin{aligned}\langle u_1, \phi_{\mathbf{n}}\rangle &= \int_0^a \int_0^b \int_0^{2\pi} u_1(x,y,z)\overline{\phi_{\mathbf{n}}(x,y,z)}\,dxdydz \\ &= \int_0^a f(x)\overline{X_n(x)}\,dx \int_0^b g(y)\overline{Y_m(y)}\,dy \int_0^{2\pi} h(z)\overline{Z_k(z)}\,dz\end{aligned}$$

The integrals are easy to calculate (see Section **37.5** for the first integral):

$$\begin{aligned}\int_0^a f(x)\overline{X_n(x)}\,dx &= \int_0^a x(a-x)\sin(\mu_n x)\,dx = \frac{2(1-(-1)^n)}{\mu_n^3} \\ \int_0^b g(y)\overline{Y_m(y)}\,dy &= B\int_0^1 Y_0(y)\overline{Y_m(y)}\,dy = Bb\delta_{0m} \\ \int_0^{2\pi} \sin^2(z)\overline{Z_k(z)}\,dz &= \pi\delta_{0k} - \frac{\pi}{2}\delta_{2k} - \frac{\pi}{2}\delta_{-2k}\end{aligned}$$

where the identity

$$\begin{aligned}\sin^2(z) &= \frac{1}{2} - \frac{1}{2}\cos(2z) = \frac{1}{2} - \frac{1}{4}\left(e^{2ix} + e^{-2iz}\right) \\ &= \frac{1}{2}Z_0(z) - \frac{1}{4}\Big(Z_2(z) + Z_{-2}(z)\Big)\end{aligned}$$

and the orthogonality of $Z_k(z)$ were used to compute the last integral. The Fourier coefficients vanish for all $m > 0$, $k \neq 0, \pm 2$, and even $n = 2p$, $p = 1, 2, ...$, so that

$$\begin{aligned}B_{\mathbf{n}} &= \frac{\langle u_1, \phi_{\mathbf{n}}\rangle}{\langle \phi_{\mathbf{n}}, \phi_{\mathbf{n}}\rangle} = \frac{4B}{a\mu_{2p-1}^3}\,, \quad \mathbf{n} = (2p-1, 0, 0) \\ B_{\mathbf{n}} &= -\frac{2B}{a\mu_{2p-1}^3}\,, \quad \mathbf{n} = (2p-1, 0, \pm 2)\,.\end{aligned}$$

The triple Fourier sum in (**45.22**) is reduced to three single sums over $p = 1, 2, ...$ for pairs $(k, m) = (0, 0), (0, \pm 2)$. The corresponding eigenfrequencies of the contributing modes are

$$\omega_{0p} = \frac{\mu_{2p-1}}{c}\,, \quad \omega_{2p} = \frac{1}{c}\sqrt{\mu_{2p-1}^2 + 4}\,.$$

The formal solution to the stated problem reads:

$$
\begin{aligned}
u(x,y,z,t) &= \sum_{p=1}^{\infty} \frac{4B\sin(\omega_{0p}t)}{a\omega_{0p}\mu_{2p-1}^3} X_{2p-1}(x)Y_0(y)Z_0(z) \\
&\quad - \sum_{p=1}^{\infty} \frac{2B\sin(\omega_{2p}t)}{a\omega_{0p}\mu_{2p-1}^3} X_{2p-1}(x)Y_0(y)\Big(Z_2(z)+Z_{-2}(z)\Big) \\
&= \frac{4B}{a}\sum_{p=1}^{\infty} \frac{\sin(\mu_{2p-1}x)}{\mu_{2p-1}^3}\left(\frac{\sin(\omega_{0p}t)}{\omega_{0p}} - \frac{\sin(\omega_{2p}t)}{\omega_{2p}}\cos(2z)\right) .
\end{aligned}
$$

To analyze the convergence of the series, let us estimate the growth of μ_{2p-1}, ω_{0p}, and ω_{2p} as $p \to \infty$. Since $|sin(\theta)| \le 1$ and $|\cos(\theta)| \le 1$ for all θ, the series is majorated by the series

$$
\sum_{p=1}^{\infty} \frac{1}{\mu_{2p-1}^3}\left(\frac{1}{\omega_{0p}} + \frac{1}{\omega_{2p}}\right) = \sum_{p=1}^{\infty} \frac{c}{\mu_{2p-1}^4}\left(1 + \frac{\omega_{0p}}{\omega_{2p}}\right) =
$$

for all t and all (x, y, z). Furthermore by the inequality

$$
\frac{\omega_{0p}}{\omega_{2p}} = \frac{\mu_{2p-1}}{\sqrt{\mu_{2p-1}^2+4}} < 1
$$

the above series is majorated by the convergent series

$$
\sum_{p=1}^{\infty} \frac{2c}{\mu_{2p-1}^4} = \frac{2ca^4}{\pi^4}\sum_{p=1}^{\infty} \frac{1}{(2p-1)^4} .
$$

Thus, the series converges uniformly for all (x, y, z, t) and therefore the sum is a continuous function. Furthermore, each differentiation with respect to t gives factors $\omega_{0p} \sim 2p-1$ and $\omega_{2p} < \omega_{0p}$. Therefore the series obtained by the double term-by-term differentiation with respect to t is also majorated by the convergent series

$$
\sum_{p=1}^{\infty} \frac{(2p-1)^2}{(2p-1)^4} = \sum_{p=1}^{\infty} \frac{1}{(2p-1)^2} < \infty
$$

and, hence, converges uniformly for all (x, y, z, t). The double differentiation with respect to x gives the factor $\mu_{2p-1}^2 \sim (2p-1)^2$ in each term of the series. Therefore, the obtained series also converges uniformly for all (x, y, z, t). Finally, the differentiation with respect y and z does not produce any growing factors in the Fourier series. Thus, the sum of the Fourier series is has continuous partial derivatives up to the second order and, therefore, is a classical solution of the problem. □

The wave equation in two spatial dimensions one of which is wrapped into a circle $\mathbb{S}^1$ describes vibrations of a thin cylindrical shell embedded into three dimensional Euclidean space. The function $u(t,x,y)$ is a deviation of the shell from its equilibrium cylindrical shape $\mathbb{S}^1\times(0,b)$ at a position $(x,y)\in\mathbb{S}^1\times(0,b)$ in the normal direction. If the shell has the (circular) boundaries fixed, then $u(t,x,0)=u(t,x,b)=0$. Similarly, the two spatial dimensions can be wrapped into torus. In this case, the wave equation describes vibrations of a thin toroidal shell. A solution of the wave equation describing free vibrations of such shells is a linear combinations of standing waves (defined by the corresponding eigenvalue problem for the Laplace operator).

45.6. Superposition principle in quantum mechanics. Quantum mechanics postulates that a particle moving in a conservative force field with a potential $V(x)$ (recall that a conservative force field is a force field of the form $F=-\mathrm{gradV}$) is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi\,,\quad \psi(0,x)=\psi_0(x)\,,$$
$$H\psi=-\frac{\hbar^2}{2m}\,\Delta\psi+V(x)\psi\,,$$

where $\hbar$ is the Planck constant, m is the mass of particle, and $\psi(t,x)$ is the *wave function* or *probability amplitude* whose physical significance is that the integral

$$P_\Omega(t)=\int_\Omega|\psi(t,x)|^2dx$$

is the probability to find the particle in a spatial region Ω at time t in any experiment. If the particle is allowed to be anywhere in space, then $P_\Omega(t)=1$ if $\Omega=\mathbb{R}^N$. This condition is the *normalization condition* for the wave function. Therefore quantum theory, by its very physical interpretation, requires the use of the space of square integrable functions and linear (differential) operators in it. In fact, the linear operator H mapping its domain $\mathcal{M}_H\subset\mathcal{L}_2$ to $\mathcal{L}_2$ is called the *energy operator* or the *Hamiltonian* of the physical system and the number

$$E_\psi=\langle H\psi,\psi\rangle=\int\overline{\psi}H\psi\,dx\,,\quad \int|\psi|^2dx=1\,,$$

is called the expectation value of the *energy of the system* in a state ψ. Since the energy of a physical system cannot be negative, the Hamiltonian is necessarily hermitian

$$\langle H\psi,\psi\rangle\geq0\,,\quad \psi\in\mathcal{M}_L\subset\mathcal{L}_2$$

Similarly to the wave equation, one can try to use the superposition principle to find a solution to the Schrödinger equation as a linear combination of eigenfunctions of H:

$$\psi(t,x) = \sum_E A_E e^{-iEt/\hbar}\psi_E(x)\,, \qquad H\psi_E(x) = E\psi_E(x)\,,$$

where constants A_E are chosen so that the initial condition is satisfied. An apparent problem with the superposition principle is the "sum" does not make any sense if the eigenvalues do not form a countable set. In fact, eigenvalues of physical Hamiltonian are quite often form non-countable subsets or real numbers when the problem is formulated in $\mathbb{R}^N$. If, however, a physical system (a particle) is confined in a bounded region Ω, then the superposition principle is easy to use to obtain the solution. First, note that if a particle cannot be found in any region Ω' that does not overlap with Ω, the probability to find the particle in Ω' should be zero. The latter implies that the wave function must be zero almost everywhere outside Ω:

$$\int_{\Omega'} |\psi(x)|^2 dx = 0\,,\ \forall \Omega' \cap \Omega = \emptyset \quad \Rightarrow \quad \psi(x) = 0 \ \ a.e.\,,\ x \notin \Omega\,.$$

In this case, the eigenvalue problem can be formulated in $\mathcal{L}_2(\Omega)$:

$$H\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi = E\psi\,, \quad \psi\Big|_{\partial\Omega} = 0\,.$$

One can prove that, if $V(x) \geq 0$, the eigenvalues are positive ($E > 0$) and form a countable set without any limit point, $E = E_n$, $n = 1, 2, ...$, and each eigenvalue has a finite multiplicity (see the last section in this chapter). The corresponding eigenfunctions form an orthonormal set in $\mathcal{L}_2(\Omega)$. If the initial wave function is a linear combination of the eigenfunctions $\psi_{E_n}(x)$

$$\psi_0(x) = \sum_{n=1}^{M} A_n \psi_{E_n}(x)$$

for some integer M, then the solution of the Schrödinger equation is

$$\psi(t,x) = \sum_{n=1}^{M} A_n e^{-itE_n/\hbar}\psi_{E_n}(x)\,.$$

The initial data ψ_0 can be any square integrable function. In this case, the finite sum becomes the Fourier series, $M \to \infty$, and

$$A_n = \frac{\langle \psi_0, \psi_{E_n}\rangle}{\langle \psi_{E_n}, \psi_{E_n}\rangle}$$

are the Fourier coefficients of the initial probability amplitude. By the very interpretation of quantum mechanics, the convergence in the mean is sufficient as only integrals of $|\psi(x,t)|^2$ over subregions of Ω have a physical significance (they define the probability to find the particle in portions of Ω). Suppose that the eigenfunction are normalized so that they form a complete *orthonormal* set. Let $\psi_M(x,t)$ be the sequence of partial sums of the Fourier series, $M = 1, 2, \ldots$. Then for any t

$$\|\psi_M\|^2 = \int_\Omega |\psi_M(x,t)|^2 dx = \sum_{n=1}^{M} \left| A_n e^{-itE_n/\hbar} \right|^2 = \sum_{n=1}^{M} |A_n|^2 .$$

Therefore by the Parseval-Steklov equality

$$\|\psi\|^2 = \lim_{M\to\infty} \|\psi_M\|^2 = \sum_{n=1}^{\infty} |A_n|^2 = \|\psi_0\|^2 = 1 .$$

This shows that the Fourier series always converges in the mean and the $\mathcal{L}_2$ norm of the solution does not change in due course of time evolution. The physical interpretation of this conclusion is straightforward. The particle cannot leave the region Ω and, therefore, the probability to find it in the whole Ω must remain 1 for all $t > 0$ if it were 1 at $t = 0$. Of course, the integral of $|\psi(x,t)|^2$ over subregions of Ω will generally depend on time t as a particle can travel from one part of Ω to another.

45.7. Exercises.

1. Solve the eigenvalue problem by separating variables

$$\begin{aligned}
&-u''_{xx} - u''_{yy} = \lambda u , \quad (x,y) \in \Omega = (0,a) \times \mathbb{S}^1 \\
&u(0,y) = u(a,y) = 0 , \ y \in \mathbb{R} \\
&u(x, y+b) = u(x,y) , \quad 0 \le x \le a , \quad y \in \mathbb{R} .
\end{aligned}$$

Verify hypotheses of relevant theorems to prove that eigenfunctions form an orthonormal complete set in $\mathcal{L}_2(\Omega)$.

2. Solve the eigenvalue problem by separating variables

$$\begin{aligned}
&-u''_{xx} - u''_{yy} - u''_{zz} = \lambda u , \quad (x,y,z) \in \Omega = (0,a) \times (0,b) \times \mathbb{S}^1 \\
&u'_x(0,y,z) = u'_x(a,y,z) = 0 , \ (y,z) \in [0,b] \times \mathbb{R} \\
&u(x,0,z) = u'_y(x,b,z) = 0 , \ (x,z) \in [0,a] \times \mathbb{R} , \\
&u(x,y,z+l) = u(x,y,z) , \quad (x,y,z) \in [0,a] \times [0,b] \times \mathbb{R} .
\end{aligned}$$

Verify hypotheses of relevant theorems to prove that eigenfunctions form an orthonormal complete set in $\mathcal{L}_2(\Omega)$.

3. Solve the eigenvalue problem by separating variables for the Laplace operator in an N dimensional torus:

$$-\Delta u(x) = \lambda u(x)\,, \quad x \in \Omega = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1\,.$$

Verify hypotheses of relevant theorems to prove that eigenfunctions form an orthonormal complete set in $\mathcal{L}_2(\Omega)$.

4. Use the superposition principle to solve the initial value (mixed) problem for a vibrating elastic rectangular membrane

$$\begin{aligned}
&c^{-2}u''_{tt} = \Delta u\,, \quad (t,x) \in (0,\infty)\times\Omega\,, \quad \Omega = (0,a_1)\times(0,a_2)\\
&u(0,x) = u_0(x) = 2\sin(\pi x_1/a_1)\sin(2\pi x_2/a_2)\,,\\
&u'_t(0,x) = u_1(x) = -3\sin(2\pi x_1/a_1)\sin(\pi x_2/a_2)\\
&u(t,x)\Big|_{\partial\Omega} = 0\,,
\end{aligned}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$.

5. Use the superposition principle to solve the mixed problem for the wave equation describing vibrations of a thin cylindrical shell of radius a and of length l with fixed boundaries:

$$\begin{aligned}
&c^{-2}u''_{tt} = u''_{xx} + u''_{yy}\,,\\
&u(t, x+2\pi a, y) = u(t,x,y)\,, \quad u(t,x,0) = u(t,x,l) = 0\,,\\
&u(0,x,y) = u_0\cos(x/a)\sin(\pi y/l)\,, \quad u'_t(0,x,y) = u_1\sin(2\pi y/l)\,,
\end{aligned}$$

where u_0 and u_1 are constants.

6. Find the formal solution to the wave equation in a two-dimensional space (plane) spanned by real variables (x, y) that satisfies the given boundary and initial conditions:

$$\begin{aligned}
&u''_{tt} = u''_{xx} + u''_{yy}\,, \quad (x,y) \in (0,1)\times\mathbb{S}^1\,,\\
&u'_x(0,y,t) = u(1,y,t)\,, \quad y \in \mathbb{R}\,,\ t \geq 0\,,\\
&u(x, y+2\pi, t) = u(x,y,t)\,, \quad (x,y) \in [0,1]\times\mathbb{R}\,,\ t \geq 0\,,\\
&u(x,y,0) = 0\,, \quad u'_t(x,y,0) = Bx^2(1-x)\cos^2(y)\,.
\end{aligned}$$

Investigate whether the formal solution is also a classical solution to the problem.

7. Find eigenvalues of the Hamiltonian for a free particle ($V = 0$) is a rectangular box $\Omega = (-a_1, a_1)\times(-a_2, a_2)\times(-a_3, a_3) \subset \mathbb{R}^3$. Solve the mixed (inital value) problem for the Schrödinger equation for a free particle in the box Ω if the initial wave function of the particle is

$$\psi(0,x)=\psi_0(x)=c_0\,\cos\Big(\frac{\pi x_1}{2a_1}\Big)\sin\Big(\frac{\pi x_2}{a_2}\Big)\sin\Big(\frac{3\pi x_3}{a_3}\Big)\,,$$

where c_0 is the normalization constant (such that $\|\psi_0\|=1$). Find the normalization constant.

8. Consider a quantum particle of mass m in a square $\Omega=(0,l)\times(0,l)$. Assume that the particle cannot penetrate the boundary of the square. Suppose that the particle is initially localized in a smaller square $\Omega_0=(0,l/2)\times(0,l/2)$ with a uniform probability density so that the initial wave function has a constant value in Ω_0

$$\psi(x,y,0)=\psi_0(x,y)=C\,,\quad (x,y)\in\Omega_0$$

and vanishes otherwise, $\psi_0=0$ if $(x,y)\notin\Omega_0$. Find the probability amplitude $\psi(x,y,t)$ for $t>0$
Hint: Find the value of C from the condition $\|\psi_0\|^2=1$ in $\mathcal{L}_2(\Omega)$. Impose the zero boundary condition $\psi(x,y,t)=0$ if $(x,y)\in\partial\Omega$ and use the Fourier method to find the formal solution of the initial value problem for the Schrödinger equation.

46. Eigenvalue problem for the Bessel operator

46.1. Bessel equation. The equation

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0$$

is called *the Bessel equation*. Consider its solutions for real ν. The coefficient at the second derivative vanishes at $x = 0$. According to the theory of linear homogeneous equations, the Bessel equation has two linearly independent solutions on a single interval of continuity of the coefficient where the coefficient at the second derivative does not vanish. The standard form of the equation reads

$$u'' + p(x)u' + q(x)u = 0\,, \quad p(x) = \frac{1}{x}\,, \quad q(x) = \frac{x^2 - \nu^2}{x^2}\,, \quad x > 0$$

Note that $x = 0$ is the so called *regular singular* point of the equation because $xp(x) = 1$ and $x^2q(x) = x^2 - \nu^2$ are functions analytic everywhere (in particular, at $x = 0$). So two linearly independent solutions for $x > 0$ can be found by the method of Frobenius:

$$u(x) = x^r \sum_{k=0}^{\infty} a_0 x^k$$

where r and the coefficients a_n are to be found by the substitution of the power series into the Bessel equation. Collecting the coefficients at $x^0 = 1$, the *indicial equation* is obtained

$$r(r-1)+p_0r+q_0 = 0\,, \quad p_0 = \lim_{x\to 0^+} xp(x) = 1\,, \quad q_0 = \lim_{x\to 0^+} x^2q(x) = -\nu^2$$

from which it follows that

$$r = \pm\nu\,.$$

It is not then difficult to find a recurrence relation for the coefficients in the power series for each root $r = \pm\nu$, solve it, and obtain two linearly independent solutions

$$u_1(x) = a_0 \sum_{k=0} \frac{(-1)^n}{4^k k!(1+\nu)_k}\, x^{2k+\nu}\,,$$

$$u_2(x) = b_0 \sum_{k=0} \frac{(-1)^n}{4^k k!(1-\nu)_k}\, x^{2k-\nu}\,,$$

$$(t)_k = t(t+1)(t+2)\cdots(t+k-1)\,.$$

The constants a_0 and b_0 are arbitrary (the sum $u_1 + u_2$ is a general solution of the Bessel equation).

DEFINITION **46.1**. (Gamma function)
For $x > 0$, the function defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

is called the Gamma function.

Note that the convergence of the integral on the upper limit is guaranteed for all $x > 0$ by the exponential decay of the integrand as $t \to \infty$. For $0 < x < 1$, the convergence of the integral on the lower limit is guaranteed by integrability of the power function t^n on $(0, 1)$ for $n > -1$.

THEOREM **46.1**. (Properties of the Gamma function)
The gamma function has the following properties:

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x)\,, \quad x > 0\,, \\ \Gamma(n+1) &= n!\,, \quad n = 1, 2, 3, \ldots\,, \\ \Gamma(1/2) &= \sqrt{\pi}\,, \end{aligned}$$

The first property is proved by integration by parts. Since $\Gamma(1) = 1$, the second property follows from the first one. The last property is obtained by means of the substitution $s = \sqrt{t}$ in the integral representation of $\Gamma(1/2)$ that converts the integral into a Gaussian integral. Note that the stated properties allow one to compute the values of Γ at a half-integer argument. For example $\Gamma(5/2) = (3/2)\Gamma(3/2) = (3/2)(1/2)\Gamma(1/2) = 3\sqrt{\pi}/4$ or, in general,

$$\begin{aligned} \Gamma(k + \tfrac{1}{2}) &= (k - \tfrac{1}{2})\Gamma(k - 1 + \tfrac{1}{2}) \\ &= (k - \tfrac{1}{2})(k - 1 - \tfrac{1}{2})\Gamma(k - 2 + \tfrac{1}{2}) \\ &= (k - \tfrac{1}{2})(k - 1 - \tfrac{1}{2}) \cdots (\tfrac{1}{2})\Gamma(\tfrac{1}{2}) \\ &= \frac{(2k-1)(2k-3)\cdots 1}{2^k}\sqrt{\pi} \\ &= \frac{(2k-1)!!}{2^k}\sqrt{\pi} \end{aligned}$$

It follows from the properties of the Gamma function that

$$(t)_k = \frac{\Gamma(t+k)}{\Gamma(t)}$$

It is then convenient to choose the coefficient $a_0 = 1/(2^\nu \Gamma(1+\nu))$ so that the first linearly independent solution u_1 assumes the form

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1+\nu+k)} \left(\frac{x}{2}\right)^{2k+\nu} . \qquad (\mathbf{46.1})$$

This function is called the *Bessel function of order* ν. If $\nu \geq 0$, it is a *regular* solution of the Bessel equation. If x is replaced by a complex variable z, then the series (**46.1**) defines *an analytic extension of the Bessel function to the complex plane*. Using the ratio test, it is not difficult to show that the series converges absolutely for all complex z. In other words, the series has infinite radius of convergence. The Bessel functions are analytic in the entire complex plane and their derivatives and integrals can be obtained by term-by-term differentiation and integration of the series (**46.1**).

In particular, it follows from the relation

$$k!\Gamma(1+\tfrac{1}{2}+k) = \frac{k!(2k+1)!!}{2^{k+1}}\sqrt{\pi} = \frac{(2k+1)!}{2^{2k+1}}$$

that the Bessel functions of a half-integer order are expressed in elementary functions:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{k=0} \frac{(-1)^k}{(2k+1)!}\, x^{2n+1} = \sqrt{\frac{2}{\pi x}}\,\sin(x) \tag{46.2}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{k=0} \frac{(-1)^k}{(2k)!}\, x^{2n} = \sqrt{\frac{2}{\pi x}}\,\cos(x) \tag{46.3}$$

Furthermore, if $\nu > 0$ and ν is not an integer, then J_ν and $J_{-\nu}$ are linearly independent solutions of the Bessel equation. The linear combination

$$N_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}}{\sin(\nu\pi)} \tag{46.4}$$

is also a solution of the Bessel equation, known as the *Neumann (or Weber) function of order* ν. If ν is not an integer, then a general solution of the Bessel equation is a linear combination of the Bessel and Neumann functions of order ν.

If $\nu = n \geq 0$ (an integer), then by using the property of the gamma function $\Gamma(-k) = \infty$, $k = 0, 1, 2, ...$, it follows that

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k-n+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k-n} = (-1)^n J_n(x)$$

where the last equality is obtained by changing the summation index $k = m + n$, $m = 0, 1, 2, ...$, if $k = n, n+1, ...$ and by using the property $\Gamma(m+1) = m!$. Therefore J_{-n} is not linearly independent of J_n. If ν is a non-negative integer, then the indicial equation has two roots $r = \pm n \neq 0$ or one root $r = 0$ of multiplicity two and, by the method of Frobenius, a second linearly independent solution can be found in the form

$$u_2(x) = C_n J_n(x) \ln(x) + \sum_{k=0}^{\infty} b_k x^{k-n}$$

where C_n and b_k are to be determined from the Bessel equation. In particular one show that the following limit exists

$$\begin{aligned} N_n(x) &= \lim_{\nu\to n} N_\nu(x) = \lim_{\nu\to n} \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \\ &= \frac{1}{\pi}\left[\frac{\partial J_\nu(x)}{\partial\nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial\nu}\right]_{\nu=n} \end{aligned} \tag{46.5}$$

and defines a second linearly independent solution of the Bessel equation, which is called the *Neumann function of order* n.

Another choice of two linearly independent solutions of the Bessel equation is known as the *Hankel functions of the first kind of order* ν:

$$H_\nu^{(1)} = \frac{i}{\sin(\nu\pi)}\left[J_\nu(x)e^{-i\pi\nu} - J_{-\nu}(x)\right], \quad \nu \neq n,$$

$$H_n^{(1)} = J_n(x) + \frac{i}{\pi}\left[\frac{\partial J_\nu(x)}{\partial\nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial\nu}\right]_{\nu=n}$$

and the *Hankel functions of the second kind of order* ν:

$$H_\nu^{(1)} = \frac{1}{i\sin(\nu\pi)}\left[J_\nu(x)e^{i\pi\nu} - J_{-\nu}(x)\right], \quad \nu \neq n,$$

$$H_n^{(2)} = J_n(x) - \frac{i}{\pi}\left[\frac{\partial J_\nu(x)}{\partial\nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial\nu}\right]_{\nu=n}$$

Evidently, $H_\nu^{(2)}$ is the complex conjugation of $H_\nu^{(1)}$. A general solution of the Bessel equation can also be written as a linear combination of the Hankel functions of the first and second kind. All solutions of the Bessel equation are also called *cylindrical functions*.

46.2. Generalized eigenvalue problem. Let L be a differential operator in an open bounded connected region Ω in $\mathbb{R}^N$. Consider the *generalized eigenvalue problem*

$$Lu(x) = \lambda\sigma(x)u(x), \quad x \in \Omega \subset \mathbb{R}^N,$$

where $\sigma \in C^0(\overline{\Omega})$ and $\sigma(x) > 0$ in Ω. The objective is to find all (complex) values of $\lambda \in \mathbb{C}$ (eigenvalues of L) at which the equation has a non-trivial solution and all such solutions as well (eigenfunctions of L). The difference with the standard eigenvalue problem is a particular case when $\sigma(x) = 1$.

Suppose that L is hermitian in $\mathcal{L}_2(\Omega)$, that is,

$$\langle Lu, v\rangle = \langle u, Lv\rangle \,.$$

Then eigenvalues λ are real. Indeed, let u be an eigenfunction for some λ, $Lu = \lambda\sigma u$. Then using the inner product in the space $\mathcal{L}_2(\Omega;\sigma)$:

$$\begin{aligned}\langle Lu, u\rangle &= \lambda \int_\Omega |u(x)|^2\sigma(x)\,dx \\ \langle u, Lu\rangle &= \int_\Omega u(x)\overline{Lu(x)}\,dx = \overline{\lambda}\int_\Omega |u(x)|^2\sigma(x)\,dx\end{aligned}$$

Since the left sides of these relation are equal by hermiticity of L, it follows that $\lambda = \overline{\lambda}$.

Eigenfunctions corresponding to distinct eigenvalues are orthogonal in the space $\mathcal{L}_2(\Omega,\sigma)$. Let $Lu_j = \lambda_j\sigma u_j$, $j = 1, 2$, and $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned}\lambda_1\langle u_1, u_2\rangle_\sigma &= \int_\Omega \lambda u_1 u_2 \sigma dx = \int_\Omega (Lu_1)u_2 dx = \langle Lu_1, u_2\rangle \\ &= \langle u_1, Lu_2\rangle = \langle u_1, \lambda_2 u_2\rangle_\sigma = \lambda_2\langle u_1, u_2\rangle_\sigma \\ &\Rightarrow\ (\lambda_1 - \lambda_2)\langle u_1, u_2\rangle_\sigma = 0 \\ &\Rightarrow\ \langle u_1, u_2\rangle_\sigma = 0\end{aligned}$$

because $\lambda_1 \neq \lambda_2$.

If $\sigma(x) > 0$ in the closure $\overline{\Omega}$, then the problem is equivalent to a *regular* eigenvalue problem

$$L_\sigma u \equiv \frac{1}{\sigma} Lu = \lambda u\,.$$

If L is hermitian in $\mathcal{L}_2(\Omega)$, then the operator $L_\sigma = (1/\sigma)L$ is hermitian in $\mathcal{L}_2(\Omega;\sigma)$ because

$$\langle L_\sigma u, v\rangle_\sigma = \langle Lu, v\rangle\,.$$

So, all the results established for a hermitian operator L in $\mathcal{L}_2(\Omega)$ are valid for the operator L_σ in $\mathcal{L}_2(\Omega;\sigma)$. In particular, one can consider the Sturm-Liouville problem in $\mathcal{L}_2((a,b);\sigma)$, which is to find solutions

$$u(x) \in C^2(a,b)\times C^1([a,b])\,,\quad u'' \in \mathcal{L}_2(a,b)\,,$$

to the generalized Sturm-Liouville equation

$$Lu \equiv -(pu')' + qu = \lambda\sigma u\,,\quad a < x < b\,,$$

that satisfy the boundary conditions

$$\alpha_a u(a) - \beta_a u'(a) = 0\,,\quad \alpha_b u(b) + \beta_b u'(b) = 0\,,$$

where all the parameters are non-negative and $\alpha_a + \beta_a > 0$, $\alpha_b + \beta_b > 0$. The functions p and q are from the same class as the corresponding functions in the Sturm-Liouville problem. The weight $\sigma(x)$ is required to be continuous and positive:

$$\sigma \in C^0[a,b]\,, \qquad \sigma(x) > 0\,,\ x \in [a,b]\,.$$

In this case, the problem is called a *regular* Sturm-Liouville problem. Theorem **36**.**4** also hold for the regular Sturm-Liouville problem. In particular, the eigenvalues λ are non-negative and simple; they form a countable set with no limit points. The corresponding eigenfunctions form a complete orthogonal set in $\mathcal{L}_2((a,b);\sigma)$.

If the positivity condition for $p(x)$ and $\sigma(x)$ in the *closed* interval $[a,b]$ is relaxed to the positivity in the *open* interval (a,b), then the corresponding problem is called *singular* Sturm-Liouville problem. The following eigenvalue problem gives an example of a singular Sturm-Liouville problem. The eigenvalue problem for the Bessel operator is an example of a singular Sturm-Liouville problem.

46.3. Eigenvalue problem for the Bessel operator. Let $\nu \geq 0$. Consider the following (generalized) eigenvalue problem

$$L_\nu u \equiv -(xu')' + \frac{\nu^2}{x}\,u = \lambda\, x\, u\,, \quad 0 < x < 1\,, \tag{46.6}$$

$$u(x) = O(x^\gamma) \text{ as } x \to 0^+\,; \quad \alpha u(1) + \beta u'(1) = 0\,, \tag{46.7}$$
$$\gamma = \min(\nu,1)\,, \quad \alpha \geq 0\,, \quad \beta \geq 0\,, \quad \alpha + \beta > 0\,.$$

The condition (**46**.**7**) means that the function $u(x) \approx cx^\gamma$ for some constant c as $x \to 0^+$ (it vanishes as a power function). The domain of the operator L_ν consists of functions from $C^2((0,1])$ that satisfy the boundary conditions (**46**.**7**) and the condition that the function $x^{-1/2}L_\nu u$ is square integrable:

$$u \in \mathcal{M}_{L_\nu} \quad \text{if } u \in C^2((0,1])\,,\ x^{-1/2}L_\nu u \in \mathcal{L}_2(0,1)\,,\ u \text{ satisfies } (46.\mathbf{7})\,.$$

The set of p times continuously differentiable functions on $[0,1]$, $p > 2$, that vanish at the endpoints of the interval is contained in $\mathcal{M}_L$ and also dense in $\mathcal{L}_2(0,1)$. Therefore $\mathcal{M}_L$ is dense in $\mathcal{L}_2(0,1)$. The operator L_ν defined by the rule (**46**.**6**) on the domain $\mathcal{M}_{L_\nu}$ is called the *Bessel operator*.

THEOREM **46**.**2**. (Hermiticity of the Bessel operator)
The Bessel operator is positive semidefinite and, hence, hermitian:

$$\langle L_\nu u, u\rangle \geq 0\,, \quad u \in \mathcal{M}_{L_\nu}\,.$$

PROOF. Using the integration by parts in the improper integral

$$\begin{aligned}\langle L_\nu u, u\rangle &= -\lim_{a\to 0^+}\int_a^1 (xu')'\bar{u}\,dx + \nu^2\lim_{a\to 0^+}\int_a^1 \frac{|u|^2}{x}\,dx\\ &= \lim_{a\to 0^+}\int_a^1 x|u'|^2dx - \lim_{a\to 0^+} xu'\bar{u}\Big|_a^1 + \nu^2\lim_{a\to 0^+}\int_a^1 \frac{|u|^2}{x}\,dx\end{aligned}$$

If $\nu = 0$, then the last term vanishes. If $\nu > 0$, then $u(x) = O(x^\gamma)$ where $0 < \gamma \leq 1$ and, hence, $u'(x) = O(x^{\gamma-1})$ as $x \to 0^+$. Therefore $x|u'(x)|^2 = O(x^{2\gamma-1})$ where $-1 < 2\gamma - 1$ and therefore the limit in the first term exists. Similarly, $xu'\bar{u} = O(x^{2\gamma}) \to 0$ as $x \to 0^+$ in the boundary term. The value of the boundary term at $x = 1$ is transformed by means of the boundary condition (**46.7**):

$$-xu'\bar{u}\Big|_0^1 = -u'(1)\overline{u(1)} = \begin{cases} (\alpha/\beta)|u'(1)|^2\,, & \beta \neq 0\\ 0\,, & \beta = 0\end{cases}$$

The last term $|u|^2/x = O(x^{2\gamma-1})$ and, hence, it is integrable. Thus,

$$\langle L_\nu u, u\rangle = \int_0^1 x|u'|^2\,dx + \nu^2\int_0^1 \frac{|u|^2}{x}\,dx + \frac{\alpha}{\beta}|u'(1)|^2 \geq 0\,.$$

Thus, the operator L_ν is positive and therefore hermitian because its domain is dense in $\mathcal{L}_2(0,1)$. □

THEOREM **46.3**. (Eigenvalues of the Bessel operator)
Let L_ν be the Bessel operator. Let λ_0 be an eigenvalue of L_ν, $L_\nu u_0(x) = \lambda_0 x u_0(x)$, $u_0 \in \mathcal{M}_L$. Then

(i) *λ_0 is simple and non-negative;*
(ii) *In order for $\lambda_0 = 0$, it is necessary and sufficient that $\nu = 0$ and $\alpha = 0$, and in this case the corresponding eigenfunction is a constant function, $u_0(x) = \text{const}$.*

A proof of this theorem is analogous to the proof of Parts (i) and (iii) of Theorem **36.4**. Part (ii) is proved in the same way as Theorem **36.3**. The details are left to the reader as an exercise.

46.4. Eigenfunctions of the Bessel operator. Theorem **46.3** states that if $\lambda = 0$ is an eigenvalue than the corresponding eigenfunction is a constant function. So without loss of generality suppose that $\lambda > 0$. By multiplying Eq. (**46.6**) by x, it can be rewritten

$$x^2u'' + xu' + (\lambda x^2 - \nu^2)u = 0$$

By changing the variable x to $z = \sqrt{\lambda}x$ the above equation becomes the Bessel equation whose general solution is a linear combination of

$J_\nu(z)$ and $N_\nu(z)$ and therefore any eigenfunction must have the forms

$$u(x;\lambda) = C_1 J_\nu(\sqrt{\lambda}\,x) + C_2 N_\nu(\sqrt{\lambda}\,x)$$

where the constants $C_{1,2}$ and λ are chosen so that $u(x;\lambda) \in \mathcal{M}_{L_\nu}$. The first boundary condition in (**46.7**) (the regularity condition at $x = 0$) requires that $C_2 = 0$. The second condition yields that $\lambda = \mu^2$ where μ is a positive root of the equation

$$\alpha J_\nu(\mu) + \beta\mu J_\nu'(\mu) = 0\,, \quad \alpha, \beta \geq 0\,, \quad \alpha + \beta > 0\,. \tag{46.8}$$

THEOREM **46.4**. (Roots of Eq. (**46.8**))
*Let $\nu > -1$. Then the roots of Eq. (**46.8**) are real and simple except possibly the zero root. The set of roots is countable and symmetric about zero and has no limit points*

A proof is omitted. That the roots are real and simple can be also understood from the fact that L_ν is a positive operator. The fact that there are countably many roots may be anticipated from the graph of $J_\nu(x)$; it has an oscillatory behavior like trigonometric functions with the amplitude of oscillations decreasing with increasing x (see, e.g., $J_{1/2}(x)$). A more accurate consideration is based on the theorem from the theory of functions of complex variable: *The set of zeros of an entire function does not have limit points.*

It follows from this theorem that the roots of Eq. (**46.8**) can be enumerated and arranged in the increasing order

$$\mu = \mu_n^{(\nu)}(\alpha,\beta)\,,\ n = 1, 2, \dots\,, \quad \mu_1^{(\nu)}(\alpha,\beta) < \mu_2^{(\nu)}(\alpha,\beta) < \cdots$$

The Bessel function has an asymptotic behavior

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\nu - \frac{\pi}{2}\right) + O(x^{-3/2})\,, \quad x \to \infty\,,$$

from which an approximate expression for the roots of $J_\nu(\mu)$ can be deduced

$$\mu = \mu_n^{(\nu)}(1,0) \approx \frac{3\pi}{4} + \frac{\pi}{2}\,\nu + \pi n\,.$$

THEOREM **46.5**. (Orthogonality of Bessel functions)
*Let μ_1 and μ_2 be two roots of Eq. (**46.8**) and $\nu > -1$. Then*

$$\int_0^1 x J_\nu(\mu_1 x) J_\nu(\mu_2 x)\,dx = 0\,, \quad \mu_1 \neq \mu_2$$

$$\int_0^1 x J_\nu^2(\mu_1 x)\,dx = \frac{1}{2}\Big[J_\nu'(\mu_1)\Big]^2 + \frac{1}{2}\left(1 - \frac{\nu^2}{\mu_1^2}\right) J_\nu^2(\mu_1) \tag{46.9}$$

PROOF. The orthogonality of eigenfunctions follows from hermiticity of the Bessel operator as this is a particular case of the generalized eigenvalue problem discussed in Section 9.2. The normalization relation is established as follows. Let μ_1 and μ_2 be real numbers. The functions $J_\nu(\mu_1 x)$ and $J_\nu(\mu_2 x)$ satisfy the Bessel equation:

$$\frac{d}{dx}\left[x\,\frac{dJ_\nu(\mu_1 x)}{dx}\right] + \left(\mu_1^2 x - \frac{\nu^2}{x}\right) J_\nu(\mu_1 x) = 0$$
$$\frac{d}{dx}\left[x\,\frac{dJ_\nu(\mu_2 x)}{dx}\right] + \left(\mu_2^2 x - \frac{\nu^2}{x}\right) J_\nu(\mu_2 x) = 0$$

Let us multiply the second equation by $J_\nu(\mu_1 x)$ and subtract the result from the first equation multiplied by $J_\nu(\mu_2 x)$. The resulting relation is then integrated over the interval $(0, 1)$ to obtain

$$\int_0^1 \frac{d}{dx}\left[x\left(J_\nu(\mu_1 x)\frac{dJ_\nu(\mu_2 x)}{dx} - J_\nu(\mu_2 x)\frac{dJ_\nu(\mu_1 x)}{dx}\right)\right] dx$$
$$= (\mu_2^2 - \mu_1^2)\int_0^1 x J_\nu(\mu_1 x) J_\nu(\mu_2 x)\, dx$$

The integral of the derivative in the right side has a contribution only from the upper limit. Indeed, it follows from (**46.1**) that

$$J_\nu(\mu x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{\mu x}{2}\right)^\nu + O(x^{\nu+2})$$
$$x\mu J'_\nu(\mu x) = \frac{\nu}{\Gamma(\nu+1)}\left(\frac{\mu x}{2}\right)^\nu + O(x^{\nu+2})$$

Therefore

$$\mu_1 x J_\nu(\mu_2 x) J'_\nu(\mu_1 x) - \mu_2 x J_\nu(\mu_1 x) J'_\nu(\mu_2 x) = O(x^{2\nu+2}) \to 0\,,\ x \to 0^+$$

that is, the contribution of the lower limit in the integral of the derivative vanishes. Thus, if $\mu_1 \neq \mu_2$, then

$$\int_0^1 x J_\nu(\mu_1 x) J_\nu(\mu_2 x) dx = \frac{1}{\mu_2^2 - \mu_1^2}\Big[\mu_1 J_\nu(\mu_2) J'_\nu(\mu_1) - \mu_2 J_\nu(\mu_1) J'_\nu(\mu_2)\Big]$$

Suppose that $\mu_{1,2}$ are roots of Eq. (**46.8**) (they are eigenvalues of the Bessel operator). Then the numbers α and β satisfy the linear *homogeneous* system

$$\alpha J_\nu(\mu_1) + \beta\mu_1 J'_\nu(\mu_1) = 0$$
$$\alpha J_\nu(\mu_2) + \beta\mu_2 J'_\nu(\mu_2) = 0$$

But $\alpha+\beta>0$ and therefore they cannot vanish simultaneously. This is only possible if the determinant of the linear system vanishes:

$$\det\begin{pmatrix} J_\nu(\mu_1) & \mu_1 J_\nu'(\mu_1) \\ J_\nu(\mu_2) & \mu_2 J_\nu'(\mu_2)\end{pmatrix} = \mu_2 J_\nu(\mu_1)J_\nu'(\mu_2) - \mu_1 J_\nu(\mu_2)J_\nu'(\mu_1) = 0\,.$$

Note that gives an alternative proof of the orthogonality of the eigenfunctions of the Bessel operator. Suppose μ_1 is an eigenvalue (it satisfies Eq. (**46.8**)), while μ_2 is not (it is a real variable). The right side of (**46.9**) is given by the following limit which is calculated by by means of l'Hospital's rule

$$\begin{aligned}&\lim_{\mu_2\to\mu_1}\frac{\mu_1 J_\nu(\mu_2)J_\nu'(\mu_1) - \mu_2 J_\nu(\mu_1)J_\nu'(\mu_2)}{\mu_2^2-\mu_1^2}\\ &= \frac{1}{2}\Big[J_\nu'(\mu_1)\Big]^2 - \frac{1}{2\mu_1}\Big[J_\nu'(\mu_1)+\mu_1 J_\nu''(\mu_1)\Big]\\ &= \frac{1}{2}\Big[J_\nu'(\mu_1)\Big]^2 + \frac{1}{2}\left(1-\frac{\nu^2}{\mu_1^2}\right)J_\nu^2(\mu_1)\,,\end{aligned}$$

where the second derivative $J_\nu''(\mu)$ has been expressed in terms of the Bessel function and its derivative using the Bessel equation. The proof is complete. □

Theorems **46.4** and **46.5** allow us to conclude that the Bessel operator (**46.5**) has:

(i) *simple eigenvalues* $\lambda_n = (\mu_n^{(\nu)})^2$, $n=1,2,\ldots$, *such that* $\lambda_1 < \lambda_2 < \cdots$ *where* $\mu_n^{(\nu)}$ *are positive roots of Eq.(***46.8***);*

(ii) *eigenfunctions* $u(x;\lambda_n) = J_\nu(\mu_n^{(\nu)}x)$ *corresponding to* λ_n *form an orthogonal set in the space* $\mathcal{L}_2(\Omega,\sigma)$ *with* $\Omega=(0,1)$ *and weight* $\sigma(x)=x$.

Let $u\in\mathcal{M}_L$. Then one can define a Fourier series

$$u(x)\sim\sum_{n=1}^{\infty} a_n^{(\nu)} J_\nu(\mu_n^{(\nu)}x)\,,\quad a_n^{(\nu)} = \frac{1}{c_n}\int_0^1 xu(x)J_\nu(\mu_n^{(\nu)}x)dx\,,$$

where c_n is given by the integral (**46.9**). This series is called the *Fourier-Bessel series* of a function u. The following theorem addresses the question about convergence of Fourier-Bessel series.

THEOREM **46.6**. (Convergence of Fourier-Bessel series) *Let* $u\in\mathcal{M}_L$. *Then the Fourier series of the function* $\sqrt{x}\,u(x)$ *over the set* $\sqrt{x}\,J_\nu(\mu_n^{(\nu)}x)$, $n=1,2,\ldots$,

$$\sqrt{x}\,u(x) = \sum_{n=1}^{\infty} a_n^{(\nu)}\sqrt{x}\,J_\nu(\mu_n^{(\nu)}x)\,,\quad a_n^{(\nu)} = \frac{\langle u, J_\nu(\mu_n^{(\nu)}x)\rangle_\sigma}{\|J_\nu(\mu_n^{(\nu)}x)\|_\sigma}\,,\quad \sigma(x)=x$$

converges uniformly.

The set of functions $\sqrt{x}\, u(x)$, $u \in \mathcal{M}_{L_\nu}$, is dense in $\mathcal{L}_2(0,1)$. By the above theorem each such function can be approximated in $\mathcal{L}_2(0,1)$ by linear combinations of $\sqrt{x}\, J_\nu(\mu_n^{(\nu)}(x)$. Therefore the set of functions $\{\sqrt{x}\, J_\nu(\mu_n^{(\nu)}(x)\}_1^\infty$ is complete in $\mathcal{L}_2(0,1)$ and the following assertion is true

COROLLARY **46.1**. (Completeness of Bessel functions)
The set $J_\nu(\mu_n^{(\nu)}x)$, $n = 1, 2, ...,$ *is complete in* $\mathcal{L}_2(\Omega;\sigma)$ *with* $\Omega = (0,1)$ *and weight* $\sigma(x) = x$.

46.5. Further properties of the Bessel functions. Recall that a convergent power series can be differentiated and integrated term-by-term and the obtained series have the same radius of convergence. In particular, let us multiply the series (**46.1**) by $x^{\pm\nu}$ and then differentiate it. The obtained series can be expressed via the series for the Bessel function of the orders $\nu \pm 1$,

$$\frac{d}{dx}\Big[x^\nu J_\nu(x)\Big] = x^\nu J_{\nu-1}(x)\,,$$
$$\frac{d}{dx}\Big[\frac{J_\nu(x)}{x^\nu}\Big] = -\frac{J_{\nu+1}(x)}{x^\nu}\,,$$

using the property $\Gamma(a+1) = a\Gamma(a)$. It follows from the first relation that

$$\int x^\nu J_{\nu-1}(x)\, dx = x^\nu J_\nu(x) + C$$

This indefinite integral is useful for calculation of the Fourier-Bessel coefficients of a power function (see Example below). By taking the derivatives in the first two relations in this section, they can be restated in an alternative form:

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x}\, J_\nu(x)\,,$$
$$J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x)\,.$$

Taking the difference of these relations, one infers that

$$J_{\nu+1}(x) - \frac{2\nu}{x}\, J_\nu(x) + J_{\nu-1}(x) = 0\,.$$

Suppose that $\mu_k^{(\nu)}$ are positive zeros of $J_\nu(x)$. Then

$$J_\nu\left(\mu_k^{(\nu)}\right) = 0 \quad\Rightarrow\quad J'_\nu\left(\mu_k^{(\nu)}\right) = J_{\nu-1}\left(\mu_k^{(\nu)}\right) = -J_{\nu+1}\left(\mu_k^{(\nu)}\right)$$

EXAMPLE **46.1**. *Let $f(x) = x^\nu$, $\nu \geq 0$, and $\phi_k(x) = J_\nu(\mu_k^{(\nu)} x)$, $k = 1, 2, \ldots$, be a complete orthonormal Bessel set on $\mathcal{L}_2((0,1);\sigma)$, where $\sigma(x) = x$. Find the Fourier-Bessel coefficients of f. In particular, consider the case when the boundary conditions in the eigenvalue problem for the Bessel operator are chosen so that $\mu_k^{(\nu)}$ are roots of $J_\nu(x)$, that is, $J_\nu(\mu_k^{(\nu)}) = 0$.*

SOLUTION: Then

$$\begin{aligned}\langle f, \phi_k\rangle_\sigma &= \int_0^1 x^{\nu+1} J_\nu\left(\mu_k^{(\nu)} x\right) dx = \frac{1}{[\mu_k^{(\nu)}]^{\nu+2}} \int_0^{\mu_k^{(\nu)}} z^{\nu+1} J_\nu(z)\, dz \\ &= \frac{1}{[\mu_k^{(\nu)}]^{\nu+2}} z^{\nu+1} J_{\nu+1}(z)\Big|_0^{\mu_k^{(\nu)}} = \frac{J_{\nu+1}(\mu_k^{(\nu)})}{\mu_k^{(\nu)}} \\ &= -\frac{J_\nu'(\mu_k^{(\nu)})}{\mu_k^{(\nu)}}\end{aligned}$$

The squared norm of the eigenfunctions is given by (**46**.**9**). In particular if $\mu_k^{(\nu)}$ are roots of J_ν, then the second term in the right side of (**46**.**9**) vanishes so that the Fourier-Bessel coefficients are given by

$$c_k = \frac{\langle f, \phi_k\rangle_\sigma}{\langle \phi_k, \phi_k\rangle_\sigma} = \frac{2J_{\nu+1}(\mu_k^{(\nu)})}{\mu_k^{(\nu)}[J_\nu'(\mu_k^{(\nu)})]^2} = \frac{2}{\mu_k^{(\nu)} J_{\nu+1}(\mu_k^{(\nu)})}$$

□

46.6. Exercises.

1. Prove Theorem 9.2 following the guidelines given after it in the text.

2. Consider the (singular) eigenvalue problem

$$Lu(x) \equiv -\Big((1-x^2)u'(x)\Big)' = \lambda u(x)\,, \quad -1 < x < 1\,.$$

Note that L resembles the Sturm-Liouville operator with one important difference that $p(x) = 1 - x^2$ is not strictly positive in the closure $\overline{\Omega} = [-1, 1]$ because $p(\pm 1) = 0$. Formulate regularity conditions at $x = \pm 1$ to obtain a domain $\mathcal{M}_L \subset \mathcal{L}_2(-1, 1)$ of L that is dense in $\mathcal{L}_2(-1, 1)$ and such that L is positive in it. Show that if $\lambda = \lambda_n = n(n+1)$, $n = 0, 1, 2, \ldots$, the corresponding eigenfunctions are given by the Legendre polynomials.

3. Use the power series of the exponential to prove that

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)=\sum_{n=-\infty}^{\infty} J_n(x)t^n$$

Hint: Recall the binomial expansion:

$$(a+b)^n=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\,a^{n-k}b^k$$

Let $a=t$ and $b=-1/t$. Regroup the terms in the power series representation of e^u where $u=(x/2)(t-1/t)$ to convert it to a series over powers t^k, $k=0,\pm1,\pm2,...$ and use (**46.1**).

4. Use the result of Problem 3 to prove that

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\theta-x\sin\theta)}d\theta\,,\quad n=0,1,2,....$$

Hint: Put $t=e^{in\theta}$ and use the orthogonality of $e^{in\theta}$, $n=0,\pm1,\pm2,...$

5. Show that the power series (**46.1**) has infinite radius of convergence.

6. Find the Fourier-Bessel coefficients of the function $f(x)=x$ over the basis $\phi_k(x)=J_1(\mu_k x)$, $k=1,2,...$, where μ_k are roots of the equation $J_1(\mu)+\mu J_1(\mu)=0$. Explain why do these function form a complete orthogonal set in $\mathcal{L}_2((0,1);\sigma)$, $\sigma(x)=x$.

47. The Laplace operator in axially symmetric regions

47.1. The eigenvalue problem for the Laplace operator in a disk. Consider the eigenvalue problem for the Laplace operator

$$-\Delta u = \lambda u\,, \quad (x,y) \in \Omega = \{(x,y)\,|\,x^2+y^2<a^2\}\,, \quad u\Big|_{\partial\Omega} = 0\,.$$

Note that the transformation that defines polar coordinates maps the rectangle $\Omega' = (0,a)\times(0,2\pi)$ onto Ω with the interval $y=0$, $0 \le x < a$, removed. So the idea is to reformulate the problem in the polar coordinates in the rectangle Ω' by imposing appropriate boundary conditions at $\partial\Omega'$.

Let $u(x,y)$ be a solution. Then in the polar coordinates it is a function of two variables

$$U(r,\varphi) = u(r\cos\theta,\, r\sin\theta)$$

It follows from this representation that the solution is 2π periodic for every $0<r<a$ so that the interval $(0,2\pi)$ can be wrapped into a circle $\mathbb{S}^1$. Thus,

$$T:\ (0,a)\times\mathbb{S}^1 \ \to\ \Omega\setminus\{(0,0)\}$$

So, the function U must be periodic in φ

$$U(r,\varphi+2\pi) = U(r,\varphi)\,, \quad 0<r\le a\,.$$

The boundary condition requires that

$$u\Big|_{\partial\Omega} = U(a,\varphi) = 0\,, \quad 0\le\theta<2\pi\,.$$

The Jacobian $J=r$ of the transformation vanishes at the origin $(x,y)=(0,0)$ and therefore differential operators are generally not defined at the origin. Indeed, the equation becomes

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial U}{\partial r}\right) - \frac{1}{r^2}\,\frac{\partial^2 U}{\partial\theta^2} = \lambda U\,, \tag{47.1}$$

which makes no sense for $r=0$. Since any solution u is from $C^2(\Omega)$ and the origin is an interior point of Ω, second partial derivatives of u exist and are continuous at the origin. This implies that $U(r,\varphi)$ must at least be *regular* at $r=0$. In other words, $U(0,\varphi)$ is defined by the limit of $U(r,\varphi)$ as $r\to 0^+$ so that

$$|U(0,\varphi)| < \infty\,.$$

A solution $U(r,\varphi)$ that satisfy the regularity condition can be extended to the origin by transforming it back to the rectangular coordinates which is then used to verify the equation in rectangular coordinates at the origin.

Since the domain Ω' of Eq. (**47.1**) is the direct product $(0,a)\times\mathbb{S}^1$, one can try to find a solution by separating variables:

$$U(r,\varphi)=R(r)\Phi(\varphi)\,.$$

The substitution of this relation into (**47.1**) and the subsequent division of the result by U yields

$$-\frac{1}{rR}(rR')'-\frac{1}{r^2}\frac{\Phi''}{\Phi}=\lambda$$

The equation is satisfied if and only if $\Phi''/\Phi=\nu$ is a constant. As a result, the original two-dimensional problem is reduced to two one-dimensional problems in $\mathbb{S}^1$ and in $(0,a)$:

(**47.2**) $$-\Phi''=\nu\Phi\,,\quad \Phi(\varphi+2\theta)=\Phi(\varphi)\,;$$

(**47.3**) $$r(rR')'+(\lambda r^2-\nu)R=0\,,\quad |R(0)|<\infty\,,\quad R(a)=0\,.$$

The eigenvalue problem for the second derivative operator with periodic boundary conditions (**47.2**) was already solved. The eigenvalues and the corresponding eigenfunctions orthonormal in $\mathcal{L}_2(\mathbb{S}^1)$ are

$$\nu=\nu_m=m^2\,,\quad \Phi_{\pm m}(\varphi)=\frac{1}{\sqrt{2\pi}}\,e^{\pm im\theta}\,,\quad m=0,1,2,\ldots\,.$$

Equation (**47.3**) is the Bessel equation whose general solution for $\nu=m^2$ is

$$R(r)=c_1J_m(\sqrt{\lambda}r)+c_2N_m(\sqrt{\lambda}r)\,,\quad m=0,1,2,\ldots\,.$$

Note that eigenvalues are positive $\lambda>0$ by Theorem **46.3**. The regularity condition demands that $c_2=0$ and the boundary condition gives an equation for λ:

$$J_m(\sqrt{\lambda}\,a)=0\quad\Rightarrow\quad \lambda=\lambda_{mj}=\frac{(\mu_j^{(m)})^2}{a^2}\,,\quad j=1,2,\ldots\,,$$

where $\mu=\mu_j^{(m)}$ are positive roots of the Bessel function $J_m(\mu)=0$. The corresponding orthonormal functions are obtained by the orthogonality of Bessel functions stated in Theorem **46.5**

$$R_{mj}(r)=c_{mj}J_m\Big(\mu_j^{(m)}\frac{r}{a}\Big)\,,$$

$$\frac{1}{c_{mj}}=\left(\int_0^a J_m^2\Big(\mu_j^{(m)}\frac{r}{a}\Big)r\,dr\right)^{1/2}=\frac{a}{\sqrt{2}}|J_m'(\mu_j^{(m)})|\,.$$

Since the Bessel functions form a complete set in $\mathcal{L}_2((0,a);\sigma)$, where $\sigma(r)=r$, and the functions Φ_n form a complete in $\mathcal{L}_2(\mathbb{S}^1)$, by Theorem **??** the eigenfunctions functions

(**47.4**) $$U_{mj}^{\pm}(r,\varphi)=R_{mj}(r)\Phi_{\pm m}(\varphi)\,,\quad m=0,1,\ldots\,,\ j=1,2,\ldots\,,$$

form a complete orthonormal set in $\mathcal{L}_2(\Omega'; J)$, where $J = r$ is the Jacobian and $\Omega' = (0, a) \times (0, 2\pi)$. When going over back to the rectangular coordinates, Ω' is mapped onto Ω safe for a set of zero measure and, hence, the set (**47.4**) is complete and orthonormal in $\mathcal{L}_2(\Omega)$ by Theorem **??**. Note that the eigenfunction can be chosen real in accord with the general theory:

$$R_{mj}(r)\cos(m\theta)\,, \quad R_{mj}(r)\sin(m\theta)\,.$$

For $m = 0$, the second eigenfunction vanishes. So the eigenvalues λ_{mj} are simple for $m = 0$ and have multiplicity 2 for $m = 1, 2, ...$

It is still left to verify that the regular solutions (**47.4**) obtained for $\lambda = \lambda_{mj}$ satisfy the original equation in rectangular coordinates at the origin. By the Euler formula

$$x \pm iy = r(\cos\theta \pm i\sin\theta) = re^{\pm i\theta} \quad \Rightarrow \quad (x \pm iy)^m = r^m e^{\pm im\theta}\,.$$

Using the power series representation of the Bessel function (**46.1**), where $\Gamma(k + m + 1) = (k + m)!$, and that $(x \pm iy)^m = r^m e^{\pm im\theta}$ and $r^2 = x^2 + y^2$, it is easy to see that for any $\mu > 0$ the product

$$J_m(\mu r)e^{\pm im\theta} = \mu^m \left(\frac{x \pm iy}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{2k}(x^2 + y^2)^k}{4^k k!(k + m)!}$$

is a power series in the original rectangular coordinates x and y. Hence, all eigenfunctions $U_{mj}(r, \varphi) = u_{mj}(x, y)$ are from $C^\infty(\Omega)$ for all $m = 0, 1, ...$ and $j = 1, 2, ...$ (as any function represented by a power series). Therefore the obtained solutions also fulfill the equation at the origin.

47.2. Eigenvalue problem for the Laplace operator in an annulus. Consider the eigenvalue problem

$$\begin{aligned} &-\Delta u(x, y) = \lambda u(x, y)\,, \quad (x, y) \in \Omega\,, \\ &u(x, y)\Big|_{\partial\Omega} = 0\,, \\ &\Omega = \{(x, y)\,|\, 0 < a^2 < x^2 + y^2 < b^2\} \end{aligned}$$

The annulus Ω is the image of the rectangle $\Omega' = (a, b) \times [0, 2\pi)$ in the plane spanned by the polar coordinates (r, φ). Since the solution has to satisfy zero boundary conditions, the problem can be solved by separating variables in polar coordinates

$$u(x, y) = u(r\cos\theta, r\sin\theta) = U(r, \varphi) = R(r)\Phi(\varphi)$$

The technicalities of separating variables in the equation are essentially the same as in the example in the previous section (see the derivation of (**47.2**) and (**47.3**)). The original eigenvalue problem is reduced to two one-dimensional eigenvalue problems

$$-\Phi'' = \nu\Phi\,, \quad \Phi(\varphi + 2\theta) = \Phi(\varphi)\,;$$
$$-(rR')' + \frac{\nu}{r}R = \lambda rR\,, \quad R(a) = R(b) = 0\,.$$

Note that the origin where the Jacobian $J = r$ vanishes in not in Ω. In contrast to (**47.3**), the last eigenvalue problem is a regular Sturm-Liouville problem in an interval (a, b) with weight $\sigma(r) = r$. By the general theory for regular Sturm-Liouville problems presented in Section **46.2**, the eigenvalues are positive (because of the zero boundary conditions), simple and they form a countable set with no limit point. The corresponding eigenfunctions form a complete orthonormal set in $\mathcal{L}_2((a, b); \sigma)$ where $\sigma = r$. Since

$$\nu = \nu_m = m^2\,, \quad \Phi(\varphi) = \Phi_{\pm m}(\varphi) = e^{\pm im\theta}\,, \quad m = 0, 1, 2, ...,$$

The general solution of the equation for $R(r)$ is given by

$$R(r; \mu) = c_1 J_m(\mu r) + c_2 N_m(\mu r)\,, \quad \mu = \sqrt{\lambda} > 0\,.$$

The boundary conditions require

$$\begin{cases} c_1 J_m(\mu a) + c_2 N_m(\mu a) = 0 \\ c_1 J_m(\mu b) + c_2 N_m(\mu b) = 0 \end{cases}$$

The system must have a non-zero solution for the pair (c_1, c_2), otherwise $R(r) = 0$. *A homogeneous system of linear equations has a non-zero solution if and only if the determinant of the matrix of coefficients in the system vanishes.* This yields a desired equation for the eigenvalues:

$$\det\begin{pmatrix} J_m(\mu a) & N_m(\mu a) \\ J_m(\mu b) & N_m(\mu b) \end{pmatrix} = J_m(\mu a)N_m(\mu b) - J_m(\mu b)N_m(\mu a) = 0\,.$$

Although one cannot solve this equation explicitly, but a general theory of Section **46.2** guarantees that this equation has countably many simple positive roots that have have no limit point, $\mu = \mu_{jm}$, $j = 1, 2, ...,$ and $\mu_{jm} \to \infty$ as $j \to \infty$ for every non-negative integer m. When the determinant vanishes, the equations are no longer independent and one of them may be used to find a relation between c_1 and c_2. It follows from the first equation that

$$c_1 J_m(\mu_{jm} a) + c_2 N_m(\mu_{jm} a) = 0$$
$$\Rightarrow \quad c_1 = cN_m(\mu_{jm} b)\,, \quad c_2 = -cJ_m(\mu_{jm} b)$$

for any $c \neq 0$. Note that the substitution of c_1 and c_2 into the equation turns it into the determinant of the system. The latter vanishes as μ_{jm}

is its root. The corresponding eigenfunctions are

$$R_{jm}(r) = N_m(\mu_{jm}b)J_m(\mu_{jm}r) - J_m(\mu_{jm}b)N_m(\mu_{jm}r)\,.$$

By the general theory these functions are orthogonal in $\mathcal{L}_2((a,b);\sigma)$:

$$\langle R_{jm}, R_{j'm}\rangle_\sigma = \int_a^b R_{jm}(r)R_{j'm}(r)r dr = 0\,, \quad j \neq j'$$

and form a complete (orthogonal) set in $\mathcal{L}_2((a,b);\sigma)$, $\sigma = r$, for every $m = 0, 1, 2, ...$ Every f that is square integrable with weight σ on (a,b) can be expanded into the Fourier series

$$f(r) = \sum_{j=1}^{\infty} f_{jm}R_{jm}(r)\,, \quad f_{jm} = \frac{\langle f, R_{jm}\rangle_\sigma}{\langle R_{jm}, R_{jm}\rangle_\sigma}$$

that converges in the mean. Furthermore, let $\mathcal{M}_{L_m}$ be the set of functions from $C^2(a,b)\cap C^0([a,b])$ that vanish at a and b (the domain of the Sturm-Liouville operator in the equation for $R(r)$). Then if $f \in \mathcal{M}_{L_m}$, then the Fourier series converges uniformly (according to the general analysis of the Sturm-Liouville problem).

Thus, the eigenvalues and the corresponding eigenfunctions in the Laplace operator in an annulus are

$$\lambda = \lambda_{jm} = \mu_{jm}^2\,, \quad j = 1, 2, ...,\ m = 0, 1, ...,$$
$$U(r,\varphi) = U_{jm}^{\pm}(r,\varphi) = R_{jm}(r)\Phi_{\pm m}(\varphi)\,.$$

By theorem **??**, they form a complete orthogonal set in the space $\mathcal{L}_2(\Omega';\sigma)$ where $\Omega' = (a,b)\times\mathbb{S}^1$ and $\sigma = r$. If so desired, the eigenfunctions can be expressed in the original rectangular coordinates (x,y) by means of the power series representation of Bessel and Neumann functions and by the Euler formula $r^m e^{\pm im\theta} = (x \pm iy)^m$ just like in the previous section. By the change of variable theorem, the functions

$$u_{jm}^{\pm}(x,y) = u_{jm}^{\pm}(r\cos\theta, r\sin\theta) = U_{jm}^{\pm}(r,\varphi)$$

form a complete orthonormal set in $\mathcal{L}_2(\Omega)$. Every function $g(x,y)$ that is square integrable on the annulus Ω can be expanded into the Fourier series

$$
\begin{aligned}
g(x,y) &= \sum_{j=1}^{\infty} g_{j0}u_{j0}(x,y) + \sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\Big[g_{jm}^{+}u_{jm}^{+}(x,y) + g_{jm}^{-}u_{jm}^{-}(x,y)\Big]\,,\\
g_{jm}^{\pm} &= \frac{\langle g, u_{jm}^{\pm}\rangle}{\langle u_{jm}^{\pm}, u_{jm}^{\pm}\rangle}\,, \quad \langle u_{jm}^{\pm}, u_{jm}^{\pm}\rangle = 2\pi\langle R_{jm}, R_{jm}\rangle_{\sigma}\,,\\
\langle g, u_{jm}^{\pm}\rangle &= \int_{\Omega} g(x,y)\,\overline{u_{jm}^{\pm}(x,y)}dxdy\\
&= \int_0^{2\pi}\int_a^b g(r\cos\theta, r\sin\theta)\,\overline{U_{jm}^{\pm}(r,\varphi)}\,rdrd\theta
\end{aligned}
$$

The Fourier series converges in the mean.

47.3. Eigenvalue problem for the Laplace operator in a cylinder. Let us begin with the eigenvalue problem for the Laplace operator on a three-dimensional cylinder with the zero boundary conditions. The cylinder has radius a and height l.

EXAMPLE **47.1**. *Solve the eigenvalue problem*

$$
\begin{aligned}
&-\Delta u(x,y,z) = \lambda u(x,y,z)\,, \quad (x,y,z)\in\Omega\,,\\
&u\Big|_{\partial\Omega} = 0\,,\\
&\Omega = \{(x,y,z)\,|\,x^2+y^2<a^2\,,\ 0<z<h\}\,,
\end{aligned}
$$

by separating variables in cylindrical coordinates

SOLUTION: Under the transformation defined by cylindrical coordinates Ω is the image of a rectangular box $\Omega' = [0,a)\times[0,2\pi)\times(0,l)$ where $0\le r<a$, $0\le\theta<2\pi$, and $0<z<l$. So, one can try to find solutions in the form

$$
u(x,y,z) = u(r\cos\theta, r\sin\theta, z) = U(r,\varphi,z) = R(r)\Phi(\varphi)Z(z)
$$

As in polar coordinates, U must be periodic in φ:

$$
U(r,\varphi+2\pi,z) = U(r,\varphi,z) \quad\Rightarrow\quad \Phi(\varphi+2\pi) = \Phi(\varphi)
$$

for all (r,h) and, as before, the periodicity in φ will be indicated as $\varphi\in\mathbb{S}^1$ (the variable φ spans a circle). The Jacobian of the cylindrical coordinates $J=r$ vanishes on the z axis. The z axis is intersecting Ω and its boundary. Therefore the regularity conditions must be imposed:

$$
|U(0,\varphi,z)|<\infty\,,\quad \theta\in\mathbb{S}^1\,,\quad z\in[0,l] \quad\Rightarrow\quad |R(0)|<\infty\,.
$$

The boundary $\partial\Omega$ consists of three surfaces, a cylindrical shell S_1 and two disks, S_2 and S_3:

$$S_1 = \{(x,y,z)\,|\,x^2+y^2=a^2\,,\ 0\le z\le l\}\,,$$
$$S_2 = \{(x,y,z)\,|\,x^2+y^2\le a^2\,,\ z=0\}\,,$$
$$S_3 = \{(x,y,z)\,|\,x^2+y^2\le a^2\,,\ z=l\}\,.$$

In cylindrical coordinates, the zero boundary on $\partial\Omega$ reads

$$u\Big|_{S_1} = U(a,\varphi,z)=0\,,\quad \theta\in\mathbb{S}^1\,,\ z\in[0,l] \quad\Rightarrow\quad R(a)=0\,,$$
$$u\Big|_{S_2} = U(r,\varphi,0)=0\,,\quad \theta\in\mathbb{S}^1\,,\ r\in[0,a] \quad\Rightarrow\quad Z(0)=0\,,$$
$$u\Big|_{S_3} = U(r,\varphi,l)=0\,,\quad \theta\in\mathbb{S}^1\,,\ r\in[0,a] \quad\Rightarrow\quad Z(l)=0\,.$$

The equation to solve is written in cylindrical coordinates:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}-\frac{\partial^2 U}{\partial z^2}=\lambda U\,,$$
$$-\frac{\Phi Z}{r}\Big(rR'\Big)'-\frac{RZ}{r^2}\,\Phi''-R\Phi Z''=\lambda R\Phi Z\,,$$
$$-\frac{(rR')'}{rR}-\frac{1}{r^2}\frac{\Phi''}{\Phi}-\frac{Z''}{Z}=\lambda$$

The only way to fulfill this equation for all (admissible) values of independent variables r, φ, and z is to require that the fractions Φ''/Φ and Z''/Z are constants. This case the eigenvalue problem for the three-dimensional Laplace operator is equivalent to three one-dimensional boundary value problems:

$$-\Phi''(\varphi)=\nu\Phi(\varphi)\,,\quad \Phi(\varphi+2\pi)=\Phi(\varphi)\,,$$
$$-Z''(z)=\eta Z(z)\,,\quad Z(0)=Z(l)=0\,,$$
$$-\Big(rR'(r)\Big)'-\frac{\nu}{r}\,R(r)=\gamma rR(r)\,,\quad |R(0)|<\infty\,,\ R(a)=0\,,$$
$$\gamma=\lambda-\eta\,.$$

The first two problems have been solved before. So the solutions are

$$\nu=m^2\,,\quad \Phi(\varphi)=\Phi_{\pm m}(\varphi)=e^{\pm im\theta}\,,\quad m=0,\pm1,\pm2,...$$
$$\eta=\left(\frac{\pi n}{l}\right)^2,\quad Z(z)=Z_n(z)=\sin\!\left(\frac{\pi n}{l}\,z\right),\quad n=1,2,....$$

The third problem is the eigenvalue problem for the Bessel operator

$$-\Big(rR'(r)\Big)'-\frac{m^2}{r}\,R(r)=\gamma rR(r)\,,\quad |R(0)|<\infty\,,\ R(a)=0$$

Its solution is found in the previous section:

$$\gamma = \gamma_{jm} = \frac{(\mu_j^{(m)})^2}{a^2}\,, \quad R(r) = R_{jm}(r) = J_m\Big(\mu_j^{(m)}\frac{r}{a}\Big)\,, \quad j = 1, 2, ...$$

where $\mu_j^{(m)}$ are roots of the Bessel function J_m. Thus, the eigenvalues and the corresponding eigenfunctions are

$$\lambda = \lambda_{jmn} = \gamma_{jm} + \nu_n = \Big(\frac{\mu_j^{(m)}}{a}\Big)^2 + \Big(\frac{\pi n}{l}\Big)^2\,,$$
$$U(r,\varphi,z) = U_{jmn}^{\pm}(r,\varphi,z) = R_{jm}(r)\Phi_{\pm m}(\varphi)Z_n(z)\,,$$

where $j, n = 1, 2, ...$ and $m = 0, 1, 2, ...$ Using the power series representation for the Bessel function, it is not difficult to see that these functions are regular at on the z axis (when $(x, y) = (0, 0)$) in the original rectangular variables. The analysis is identical to that carried out in the case of the eigenvalue problem for the Laplace operator in a disk. By Theorem **??** these functions are orthogonal in $\mathcal{L}_2(\Omega'; \sigma)$ where $\sigma = r$ for distinct triples of integers (j, m, n) and different indices $\pm$. The norm of the eigenfunctions is computed by Fubini's theorem for a triple integral over rectangular box Ω':

$$\begin{aligned}\|U_{jmn}^{\pm}\|^2 &= \int_{\Omega'} |U_{jmn}(r,\varphi,z)|^2 r dr d\theta dz \\ &= \int_0^a |R_{jm}(r)|^2 r dr \int_0^{2\pi} |\Phi_{\pm m}(\varphi)|^2 d\theta \int_0^l |Z_n(z)|^2 dz \\ &= \frac{a^2}{2}\Big(J_m'(\mu_j^{(m)})\Big)^2 \cdot 2\pi \cdot \frac{l}{2} = \frac{\pi a^2 l}{2}\Big(J_m'(\mu_j^{(m)})\Big)^2 .\end{aligned}$$

Changing the variables back to the original coordinates, it is concluded that the obtained eigenfunctions form an orthogonal complete set in $\mathcal{L}_2(\Omega)$. □

General boundary conditions on a cylinder. Let Ω be a cylinder in $\mathbb{R}^3$:

$$\Omega = \{(x, y, z)\,|\, x^2 + y^2 < a^2\,,\ 0 < z < l\} = (0, l) \times D$$

where D is a disk of radius a. Consider the eigenvalue problem for the Laplace operator with general boundary conditions for an elliptic operator:

$$\textbf{(47.5)} \qquad -\Delta u = \lambda u\,, \quad (x, y, z) \in \Omega\,, \quad \Big(\alpha u + \beta\frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = 0\,,$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta > 0$ on the boundary $\partial\Omega$. Let us analyze under what conditions on the functions α and β the problem can be solved by separating variables in cylindrical coordinates.

Since the region of the eigenvalue problem is the direct product of two regions, one can try to solve the problem by separating the variable z first. Put

$$u(x, y, z) = V(x, y)Z(z) , \qquad (x, y) \in D , \quad z \in (0, l) ,$$

and

$$Lu = L_{xy}u + L_z u , \qquad L_{xy}u = -u''_{xx} - u''_{yy} , \quad L_z u = -u''_{zz} .$$

The boundary $\partial\Omega$ consists of three pieces: the disks $S_0 = \{z = 0\} \times \bar{D}$ and $S_l = \{z = l\} \times \bar{D}$ and the cylinder $S = [0, l] \times \partial D$, where ∂D is the boundary of D (the circle of radius a). The outward unit normal on the boundary $\partial\Omega$ is

$$\mathbf{n}(x, y, z)\Big|_{S_0} = \mathbf{n}(x, y, 0) = -\mathbf{e}_z , \quad x^2 + y^2 \le a ,$$
$$\mathbf{n}(x, y, z)\Big|_{S_l} = \mathbf{n}(x, y, l) = \mathbf{e}_z , \quad x^2 + y^2 \le a ,$$
$$\mathbf{n}(x, y, z)\Big|_{S} = \mathbf{n}(x, y, z) = \mathbf{e}_r , \quad x^2 + y^2 = a , \ 0 \le z \le l ,$$

where $\mathbf{e}_z = \mathbf{e}_3 = (0, 0, 1)$ is the unit vector in the direction of the z axis and $\mathbf{e}_r = (\cos\theta, \sin\theta, 0)$ is the unit vector normal to a circle centered at the origin in the xy plane (φ is the polar angle). The boundary condition for each smooth piece of $\partial\Omega$ reads

$$\left(\alpha u + \beta\frac{\partial u}{\partial \mathbf{n}}\right)\Big|_{S_0} = \left(\alpha u - \beta\frac{\partial u}{\partial z}\right)\Big|_{z=0} = 0 , \quad (x, y) \in \overline{D} ,$$
$$\left(\alpha u + \beta\frac{\partial u}{\partial \mathbf{n}}\right)\Big|_{S_l} = \left(\alpha u + \beta\frac{\partial u}{\partial z}\right)\Big|_{z=l} = 0 , \quad (x, y) \in \overline{D} ,$$
$$\left(\alpha u + \beta\frac{\partial u}{\partial \mathbf{n}}\right)\Big|_{S} = \left(\alpha u + \beta\frac{\partial u}{\partial r}\right)\Big|_{r=a} = 0 , \quad 0 \le z \le l , \ (x, y) \in \partial D .$$

where $r = (x^2 + y^2)^{1/2}$ (the radial variable of polar coordinates). The normal is not defined on two circles along which the cylinder S is attached to the disks S_0 and S_l. The first and second equations above are defined at the circle $x^2 + y^2 = a^2$ in the sense of the limit $r \to a^-$ (they are continuously extended to the boundary ∂D). Similarly, the third equation is continuously extended to the boundary circles of S by the limits $z \to 0^+$ and $z \to l^-$.

Let α_0 and β_0 be the functions α and β restricted to the open disk S_0 continuously extended to its boundary circle. Similarly, α_l and β_l be the functions α and β restricted to the open disk S_l and continuously extended to its boundary circle, and α_a and β_a are α and β restricted to the open cylindrical shell $x^2 + y^2 = a^2$, $0 < z < l$, and continuously extended to its boundary circles in the planes $z = 0$ and $z = l$. Then separating the variables in the original equation

$$-\Delta(VZ) = -Z(V''_{xx} + V''_{yy}) - VZ'' = \lambda VZ$$

$$-\frac{1}{V}(V''_{xx} + V''_{yy}) - \frac{Z''}{Z} = \lambda$$

the original problem is reduced to two eigenvalue problems:

$$(\mathbf{47.6}) \qquad -Z'' = \mu Z\,, \qquad \begin{cases} \alpha_0 Z(0) - \beta_0 Z'(0) = 0 \\ \alpha_l Z(l) + \beta_l Z'(l) = 0 \end{cases},$$

$$(\mathbf{47.7}) \qquad -V''_{xx} - V''_{yy} = \eta V\,, \qquad \alpha_a V\Big|_{r=a} + \beta_a \frac{\partial V}{\partial r}\Big|_{r=a} = 0\,,$$

where μ and η are separation constants, $\lambda = \mu + \eta$. Thus, the variable can indeed be separated, *provided*

- the functions α_0, α_l, β_0, and β_l are *independent of* (x, y) and are constants;
- the functions α_a and β_a are *independent* of the variable z.

The problem (**47.6**) is the eigenvalue problem for the Sturm-Liouville operator. The eigenvalues form an infinite countable set $\mu = \mu_n$, $n = 1, 2, \ldots$, and the corresponding (normalized) eigenfunctions $Z(z) = Z_n(z)$ form a complete orthonormal set in $\mathcal{L}_2(0, l)$.

If, *in addition*,

- the functions α_a and β_a are constant on S_a

then the problem (**47.7**) can be solved by separating variables in polar coordinates. Put

$$V = R(r)\Phi(\varphi)\,.$$

Then the problem (**47.7**) is equivalent to two eigenvalue problems

$$(\mathbf{47.8}) \qquad -\Phi'' = \xi\Phi\,, \qquad \Phi(\varphi + 2\pi) = \Phi(\varphi)\,,$$

$$(\mathbf{47.9}) \qquad r(rR')' + (\eta r^2 - \xi)R = 0\,, \qquad \begin{cases} |R(0)| < \infty \\ \alpha_a R(a) + \beta_a R'(a) = 0 \end{cases},$$

where ξ is a separation constant. The eigenvalues and eigenfunctions for the problem (**47.8**) are

$$\xi = \xi_m = m^2\,, \quad \Phi_{\pm m}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{\pm im\theta}\,, \quad m = 0, 1, 2, \ldots\,.$$

The coefficient in $\Phi_{\pm m}$ is chosen to make them orthonormal in $\mathcal{L}_2(\mathbb{S}^1)$. The problem (**47.9**) is the eigenvalue problem for the Bessel operator where $\nu^2 = \xi^2 = m^2$. The eigenvalues and the corresponding eigenfunctions are

$$\eta = \eta_{jm} = \frac{[\mu_j^{(m)}(\alpha_a, \beta_a)]^2}{a^2}, \quad R_{jm} = c_{jm} J_m\Big(\mu_j^{(m)} \frac{r}{a}\Big), \quad j = 1, 2, \dots,$$

where $\mu_j^{(m)} = \mu_j^{(m)}(\alpha_a, \beta_a)$ are roots of Eq. (**46.8**) with $\alpha = \alpha_a$ and $\beta = \beta_a$. The normalization constants c_{jm} are chosen to make R_{jm} orthonormal in accord with Theorem **46.5**.

As shown above, the functions $R_{jm}(r)\Phi_{\pm m}(\varphi) = V_{jm}(x, y)$ are from C^∞ in the disk $x^2 + y^2 < a^2$ and, hence, the functions $V_{jm}(x, y)Z_n(z)$ satisfy the original equation at all points $(0, 0, z)$ in Ω where the Jacobian of cylindrical coordinates vanishes. Thus, the eigenvalues and the corresponding eigenfunctions of the original problem

$$\begin{aligned} \lambda &= \lambda_{jmn} = \mu_n + \frac{[\mu_j^{(n)}(\alpha_a, \beta_a)]^2}{a^2}, \\ u &= U_{mnj}(x, y, z) = Z_m(z) R_{jn}(r) \Phi_n(\varphi). \end{aligned}$$

The functions U_{mnj} form a complete orthonormal set in $\mathcal{L}_2(\Omega)$.

47.4. More complex planar regions. The image Ω of the rectangle

$$\Omega' = (r_1, r_2) \times (\varphi_1, \varphi_2) \subset (0, \infty) \times (0, 2\pi),$$

under the transformation that defines polar coordinates is the part of the ring $0 < r_1^2 < x^2 + y^2 < r_2^2$ between two rays $\varphi = \varphi_1 > 0$ and $\varphi = \varphi_2 < 2\pi$. Consider the eigenvalue problem for the Laplace operator in Ω:

$$-\Delta u = \lambda u, \quad (x, y) \in \Omega, \quad \Big(\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = 0,$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta > 0$. Let $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be the orthonormal basis of polar coordinates in the xy plane. The boundary $\partial\Omega$ is the union of four smooth pieces: S_{r_1} and S_{r_2} are, respectively, the parts (arcs) of the circles $x^2 + y^2 = r_1^2$ and $x^2 + y^2 = r_2^2$ that lie in the sector $\varphi_1 \leq \theta \leq \varphi_2$, and S_{φ_1} and S_{φ_2} are, respectively, the parts (straight line segments) of the rays $\varphi = \varphi_1$ and $\varphi = \varphi_2$ that lie between the circles $(r_1 \leq r \leq r_2)$. The outward unit normal on $\partial\Omega$ is then

$$\begin{aligned} &\mathbf{n}\Big|_{S_{r_1}} = -\mathbf{e}_r, && \mathbf{n}\Big|_{S_{r_2}} = \mathbf{e}_r, && \varphi_1 \leq \theta \leq \varphi_2 \\ &\mathbf{n}\Big|_{S_{\varphi_1}} = -\mathbf{e}_\theta\Big|_{\varphi=\varphi_1}, && \mathbf{n}\Big|_{S_{\varphi_2}} = \mathbf{e}_\theta\Big|_{\varphi=\varphi_1}, && r_1 \leq r \leq r_2. \end{aligned}$$

Therefore by (**??**)

$$\frac{\partial U}{\partial \mathbf{n}}\Big|_{S_{\varphi_1}} = -\frac{1}{r}\frac{\partial U}{\partial \theta}\Big|_{\varphi=\varphi_1}, \qquad \frac{\partial U}{\partial \mathbf{n}}\Big|_{S_{\varphi_2}} = \frac{1}{r}\frac{\partial U}{\partial \theta}\Big|_{\varphi=\varphi_2}.$$

In the polar coordinates the problem becomes the eigenvalue problem in the rectangle Ω'. If $u(x,y) = U(r,\varphi)$, then

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial U}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = -\lambda U\,,$$

$$\begin{cases} \alpha_{\varphi_1} U(r,\varphi_1) - (\beta_{\varphi_1}/r)U'_\theta(r,\varphi_1) = 0 \\ \alpha_{\varphi_2} U(r,\varphi_2) + (\beta_{\varphi_2}/r)U'_\theta(r,\varphi_2) = 0 \end{cases},$$

$$\begin{cases} \alpha_{r_1} U(r_1,\varphi) - \beta_{r_1}U'_r(r_1,\varphi) = 0 \\ \alpha_{r_2} U(r_2,\varphi) + \beta_{r_2}U'_r(r_2,\varphi) = 0 \end{cases},$$

where α_p and β_p, $p = r_1, r_2, \varphi_1, \varphi_2$ are the reductions of α and β to S_p (with a continuous extension to the end points of each smooth piece S_p). Note that Ω does not contain the origin ($r_1 > 0$) where the change of variables is singular and, hence, no regularity condition is required. Furthermore, solutions are no longer 2π periodic as φ does not sweep its full range in Ω. Even if α_p and β_p are constants, the boundary conditions for φ do not admit a separation of variables by the substitution

$$U(r,\varphi) = R(r)\Phi(\varphi)\,,$$

because of the factor $1/r$. If the functions β_{φ_1} and β_{φ_2} are proportional to r, then the method works. Another case that can easily be solved by separation of variables is when either $\alpha_{\varphi_1} = \alpha_{\varphi_2} = 0$ or $\beta_{\varphi_1} = \beta_{\varphi_2} = 0$. The further discussion is limited to this latter case. The problem is equivalent to two one-dimensional eigenvalue problems

$$(\mathbf{47.10}) \qquad -\Phi'' = \xi\Phi\,, \qquad \Phi(\varphi_1) = \Phi(\varphi_2) = 0\,,$$

$$(\mathbf{47.11}) \qquad -(rR')' + \frac{\xi}{r}R = \lambda r R\,, \quad \begin{cases} \alpha_{r_1} R(r_1) - \beta_{r_1}R'(r_1) = 0 \\ \alpha_{r_2} R(r_2) + \beta_{r_2}R'(r_2) = 0 \end{cases},$$

where the functions α and β are assumed to have constant values on the circles $|x| = r_1$ and $|x| = r_2$. The problem (**47.10**) is the Sturm-Liouville problem discussed in Section 8. Its solution reads

$$\xi = \nu_n^2\,, \quad \nu_n = \frac{\pi n}{\Delta\theta}\,, \quad \Phi_n(\varphi) = \sin\Big(\nu_n(\varphi - \varphi_1)\Big)\,, \quad n = 1, 2, \dots\,,$$

where $\Delta\theta = \varphi_2 - \varphi_1$. The details are left to the reader as an exercise.

With $\xi = \nu_n \geq 0$, the problem (**47.11**) is a regular Sturm-Liouville problem in $\mathcal{L}_2((a,b);\sigma)$ with $p(r) = \sigma(r) > 0$ in $[a,b]$ (if $a > 0$) and $q(r) = \nu_n^2/r \geq 0$ (see Section 9.2). The corresponding operator is the

Bessel operator. A general solution of the Bessel equation in (**47.11**) for $\xi = \nu_n^2$ reads

$$R(r) = C_1 J_{\nu_n}(\sqrt{\lambda}\, r) + C_2 N_{\nu_n}(\sqrt{\lambda}\, r) .$$

By a basic theory of ordinary differential equations, there is a unique choice of the constants C_1 and C_2 with which

$$R(r) = R_1(r;\lambda) , \qquad R_1(r_1;\lambda) = 1 , \quad R_1'(r_1;\lambda) = 0 ,$$

and there is another (unique) choice of C_1 and C_2 at which the solution satisfies the following conditions:

$$R(r) = R_2(r;\lambda) , \qquad R_2(r_2;\lambda) = 0 , \quad R_2'(r_2;\lambda) = 1 .$$

An explicit form of C_1 and C_2 is obtained by the substitution of the general solution into the initial conditions at $r = r_1$ and $r = r_2$ and solving for C_1 and C_2. The uniqueness of the solution follows from the linear independence of the Bessel and Neumann functions (their Wronskian does not vanish). Note that a linear combination of $R_1(r;\lambda)$ and $R_2(r;\lambda)$ is a general solution to the Bessel equation (a general solution is a linear combination of *any two* linearly independent solutions). Then the boundary condition at $r = r_1$ is satisfied if the coefficients in a linear combination of R_1 and R_2 are chosen so that it is proportional to

$$R(r;\lambda) = \beta_{r_1} R_1(r;\lambda) + \alpha_{r_1} R_2(r;\lambda)$$

The proportionality coefficient drops out from the second boundary condition and the latter becomes an equation for the eigenvalues

$$\alpha_{r_2} R(r_2;\lambda) + \beta_{r_2} R'(r_2;\lambda) = 0 .$$

Roots of this equation is generally difficult to analyze, not to mention finding its values. In practice, it is solved numerically. But from the general theory of Section **46.2** it follows that there are countably many positive simple roots and their set has no limit points. If $\lambda = \lambda_{nj}$, $j = 1, 2, ...$, are roots for a given n, then the corresponding eigenfunctions are $R_{nj}(r) = R(r;\lambda_{nj})$. If $\Phi_n(\varphi)$ are eigenfunctions in the problem (**47.10**), then orthogonal eigenfunctions of the original problem are

$$U(r,\varphi) = U_{nj}(r,\varphi) = \Phi_n(\varphi) R(r;\lambda_{nj}) .$$

By Theorem **??**, they form a complete orthogonal set in $\mathcal{L}_2(\Omega)$.

EXAMPLE **47.2**. *Solve the eigenvalue problem by separating variables in polar coordinates*

$$-\Delta u = \lambda u , \quad (x,y) \in \Omega , \quad u\Big|_{\partial\Omega} = 0 ,$$

where Ω is the part of the ring in the first quadrant

$$\Omega = \{(x,y)\,|\,0 < a^2 < x^2 + y^2 < b^2\,,\ x > 0\,,\ y > 0\}\,.$$

SOLUTION: Here $\alpha = 1$ and $\beta = 0$ on $\partial\Omega$. The region Ω is the image of the rectangle $(r,\varphi) \in (a,b) \times (0,\pi/2)$. The problem (**47.10**) is

$$-\Phi'' = \xi\Phi\,, \qquad \Phi(0) = \Phi(\pi/2) = 0\,.$$

The eigenvalues and the corresponding orthonormal eigenfunctions are

$$\xi = \xi_n = 4n^2\,, \qquad \Phi_n(\varphi) = \frac{2}{\sqrt{\pi}}\sin(2n\theta)\,, \quad n = 1, 2, \dots\,.$$

Here the coefficient $2/\sqrt{\pi}$ is the normalization constant such that $\|\Phi_n\| = 1$ in $\mathcal{L}_2(0,\pi/2)$. A general solution of the Bessel equation in the problem (**47.11**) (here $\nu_n = 2n$) is

$$R(r) = C_1 J_{2n}(\sqrt{\lambda}\,r) + C_2 N_{2n}(\sqrt{\lambda}\,r)$$

Although the procedure outlined above can be used, the simplicity of the boundary conditions allows for a simpler (equivalent) procedure to find an equation for eigenvalues. The substitution of the solution into the boundary condition yields a homogeneous linear system

$$\begin{aligned} C_1 J_{2n}(\sqrt{\lambda}\,a) + C_2 N_{2n}(\sqrt{\lambda}\,a) &= 0 \\ C_1 J_{2n}(\sqrt{\lambda}\,b) + C_2 N_{2n}(\sqrt{\lambda}\,b) &= 0 \end{aligned}$$

The constants C_1 and C_2 cannot vanish simultaneously (the system must have non-trivial solutions). This can happen if and only if the determinant of the system vanishes:

$$J_{2n}(\sqrt{\lambda}\,a)N_{2n}(\sqrt{\lambda}\,b) - N_{2n}(\sqrt{\lambda}\,a)J_{2n}(\sqrt{\lambda}\,a) = 0\,.$$

It follows from the general theory of the eigenvalue problem for the Sturm-Liouville operator that this equation has simple roots and the set of roots is countable and has no limit point. Let $\lambda = \lambda_{nj} = \mu_{nj}^2$, $j = 1, 2, \dots$, be the roots of this equation. Then for $\lambda = \lambda_{nj}$, the equations in the linear system for C_1 and C_2 are not independent, and only one of the equations has to be solved to find the corresponding eigenfunction. Either of the equations implies that C_1 is proportional to C_2 with the proportionality coefficient determined from the equation. For example, using the first equation to determine the proportionality coefficient, the normalized eigenfunction corresponding to the eigenvalue $\lambda = \lambda_{nj}$ can be taken in the form

$$R_{nj}(r) = c_{nj}\Big(N_{2n}(\mu_{nj}a)J_{2n}(\mu_{nj}r) - J_{2n}(\mu_{nj}a)N_{2n}(\mu_{nj}r)\Big)$$

where c_{nj} is the normalization constant fixed by the condition

$$\|R_{nj}\|_\sigma = \left(\int_a^b |R_{nj}(r)|^2 r dr\right)^{1/2} = 1\,,$$

in $\mathcal{L}_2((a,b);\sigma)$, where $\sigma(r) = r$. Note that $R_{nj}(a) = 0$ identically, while $R_{nj}(b) = 0$ by the equation for the eigenvalues. The orthogonality of R_{nj} corresponding to different pairs of indices nj is guaranteed by hermiticity of the Sturm-Liouville operator. The set of eigenfunctions

$$U_{nj}(r,\varphi) = \Phi_n(\varphi)R_{nj}(r)$$

form a complete orthonormal set in $\mathcal{L}_2(\Omega';\sigma)$ by Theorem **??** and the functions obtained from U_{nj} by transformation to the original rectangular coordinates (x,y) form a complete orthonormal set in $\mathcal{L}_2(\Omega)$. □

47.5. General scheme for separation of variables. Suppose that rectangular coordinates in $\mathbb{R}^{N+M}$ are divided into two sets so that $x \in \Omega \subset \mathbb{R}^N$ and $y \in D \subset \mathbb{R}^M$, where Ω and D are regions. The boundary of the region $\Omega \times D$ is the union

$$\partial(\Omega \times D) = (\partial\Omega \times \overline{D}) \cup (\overline{\Omega} \times \partial D)$$

For example, let $\Omega = (0,a)$ and $D = (0,b)$. Then $\partial\Omega = \{x = 0\} \cup \{x = a\}$ and $\overline{\Omega} = [0,a]$. Similarly, $\partial D = \{y = 0\} \cup \{y = b\}$ and $\overline{D} = [0,b]$. Then

$$\partial\Omega \times \overline{D} = \Big(\{x = 0\} \cup \{x = a\}\Big) \cup [0,b] = S_{1L} \cup S_{1R}$$

$$\overline{\Omega} \times \partial D = [0,a] \cup \Big(\{y = 0\} \cup \{y = b\}\Big) = S_{2L} \cup S_{2R}\,,$$

where S_{1L}, S_{1R}, S_{2L}, and S_{2R} are four straight line segments whose union is the boundary of the rectangle $(0,a) \times (0,b)$ (see Section 10.2).

Let $\Omega \subset \mathbb{R}^2$ be a disk of radius a and centered at the origin, $x^2+y^2 < a^2$, and let $D = (0,b)$. Then $\Omega \times D \subset \mathbb{R}^3$ is the solid cylinder or radius a and height b. It is obtained by attaching a straight line segment of length b to each point of the disk so that the segment is perpendicular to the disk. If z spans D, then $\partial\Omega \times \overline{D}$ is the cylindrical shell

$$\partial\Omega \times \overline{D} = \{(x,y,z) \mid x^2 + y^2 = a^2,\ 0 \le z \le b\}$$

which is the side boundary of the solid cylinder. Similarly, $\overline{\Omega} \times \partial D$ is the union of two disks $x^2 + y^2 \le a^2$, one is in the xy plane ($z = 0$) and the other is in the parallel plane $z = b$; they are the top and bottom boundaries of the solid cylinder:

$$\partial\Omega \times \overline{D} = \{(x,y,z) \mid x^2 + y^2 \le a^2,\ z = 0,b\}$$

The union of the cylindrical shell and the two disks is the boundary of the solid cylinder $\Omega \times (0, b)$.

In the region $\Omega \times D$, consider the eigenvalue problem for an elliptic operator

$$L_x u + L_y u = \lambda u\,, \quad (x, y) \in \Omega \times D\,, \tag{47.12}$$

$$\left(\alpha_1 u + \beta_1 \frac{\partial u}{\partial \mathbf{n}}\right)\bigg|_{\partial\Omega\times\overline{D}} = 0\,, \quad \left(\alpha_2 u + \beta_2 \frac{\partial u}{\partial \mathbf{n}}\right)\bigg|_{\overline{\Omega}\times\partial D} = 0\,, \tag{47.13}$$

where L_x and L_y are elliptic operators independent of y and x, respectively, and the functions α_1 and β_1 are independent of y, while α_2 and β_2 are independent of x.

Eigenfunctions are sought in the form $u(x, y) = X(x)Y(y)$. The substitution into Eq. (**47.12**) yields

$$Y(y)L_x X(x) + X(x)L_y Y(y) = \lambda X(x)Y(y)$$

from which it follows that

$$\frac{L_x X(x)}{X(x)} = \lambda - \frac{L_y Y(y)}{Y(y)}\,.$$

The left side of this relation depends on x only, whereas the right side depends on y. The equality is possible if there exist two constants μ and ν such that $\mu + \nu = \lambda$ and

$$L_x X(x) = \mu X(x)\,, \quad x \in \Omega\,, \tag{47.14}$$

$$L_y Y(y) = \nu Y(y)\,, \quad y \in D\,. \tag{47.15}$$

The boundary conditions on X and Y are deduced from (**47.13**) by substituting $u = XY$ into it:

$$\left(\alpha_1 X + \beta_1 \frac{\partial X}{\partial \mathbf{n}}\right)\bigg|_{\partial\Omega} = 0\,, \tag{47.16}$$

$$\left(\alpha_2 Y + \beta_2 \frac{\partial Y}{\partial \mathbf{n}}\right)\bigg|_{\partial D} = 0\,. \tag{47.17}$$

This, the original eigenvalue problem is reduced to two eigenvalue problems with a lesser number of variables. If the $\mu = \mu_n$ and $X = X_n(x)$, $n = 1, 2, ...$, are the eigenvalues and the corresponding eigenfunctions of the problem (**47.14**), (**47.16**), and $\nu = \nu_m$ and $Y = Y_m(y)$, $m = 1, 2, ...$, are the eigenvalues and the corresponding eigenfunctions of the problem (**47.15**), (**47.17**), then the eigenvalues and the corresponding eigenfunctions of the original problem (**47.12**), (**47.13**) are

$$\lambda = \lambda_{nm} = \mu_n + \nu_m\,, \qquad u = \phi_{nm}(x, y) = X_n(x)Y_m(y)\,.$$

If in addition the sets $\{X_n\}_1^\infty$ and $\{Y_m\}_1^\infty$ are orthogonal and complete in in $\mathcal{L}_2(\Omega)$ and $\mathcal{L}_2(D)$, respectively, then the functions ϕ_{nm} form an orthogonal complete set in $\mathcal{L}_2(\Omega \times D)$ by Theorem **??**.

If Ω and D are rectangles or tori, then, a multivariable eigenvalue problem (with appropriate boundary conditions) can be reduced to several Sturm-Liouville problems and/or eigenvalue problems on a circle by separating variables. Can the method be extended to more generals domains like a ball or cylinder? The idea is to use transformations (or *curvilinear coordinates*) that change a given region to a rectangular region and then to try to separate variables.

47.6. Exercises.

1. Solve the eigenvalue problem for the Laplace operator

$$-\Delta u = \lambda u\,, \quad (x,y) \in \Omega = \{(x,y)\,|\,x^2+y^2<a^2\} \subset \mathbb{R}^2\,,$$
$$\Big(\alpha u + \beta\frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = 0\,, \quad \alpha \geq 0\,,\ \beta \geq 0\,,\ \alpha+\beta>0\,,$$

where α and β are constants.

2. Solve the eigenvalue problem by separating variables in polar coordinates

$$-\Delta u = \lambda u\,, \quad (x,y) \in \Omega\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0\,,$$

where Ω is the part of the ring in the first quadrant

$$\Omega = \{(x,y)\,|\,0<a^2<x^2+y^2<b^2\,,\ x>0\,,\ y>0\}\,.$$

3. Solve the eigenvalue problem by separating variables in cylindrical coordinates

$$-\Delta u = \lambda u\,, \quad (x,y,z) \in \Omega\,, \quad u\Big|_{\partial\Omega} = 0\,,$$

where Ω is the part of the solid cylinder in the first octant

$$\Omega = \{(x,y,z)\,|\,x^2+y^2<a^2\,,\ x>0\,,\ y>0\,, 0<z<l\}\,.$$

4. Solve the eigenvalue problem by separating variables in cylindrical coordinates

$$-\Delta u = \lambda u\,, \quad (x,y,z) \in \Omega\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0\,,$$

where Ω is the part of the solid cylinder

$$\Omega = \{(x,y,z)\,|\,x^2+y^2<a^2\,,\ x>0\,,\ -h<z<h\}\,.$$

5. Solve the eigenvalue problem by separating variables in cylindrical coordinates

$$-\Delta u = \lambda u\,, \quad (x,y,z)\in\Omega\,, \quad u\Big|_{\partial\Omega} = 0\,,$$

where Ω is the cylindrical shell

$$\Omega = \{(x,y,z)\,|\,0 < a^2 < x^2+y^2 < b^2\,,\ -h < z < h\}\,.$$

6. Solve the eigenvalue problem by separating variables in cylindrical coordinates

$$-\Delta u = \lambda u\,, \quad (x,y,z)\in\Omega\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0\,,$$

where Ω is the wedge in the solid cylinder

$$\Omega = \{(x,y,z)\,|\,x^2+y^2 < a^2\,,\ 0 < y < x\,,\ -h < z < h\}\,.$$

Answers.

1. Eigenvalues and corresponding eigenfunctions are

$$\lambda = \lambda_{nj} = \left(\frac{\mu_{nj}}{a}\right)^2\,, \quad j = 1,2,\dots\,,\ n = 0,1,\dots\,,$$
$$u(r\cos\theta, r\sin\theta) = U^{\pm}_{nj}(r,\varphi) = c_{nj}J_n\Big(\frac{\mu_{nj}}{a}\,r\Big)e^{\pm in\theta}\,,$$

where c_{nj} are normalization constants and for every $n = 0,1,\dots$ the numbers μ_{nj}, $j = 1,2,\dots$, are roots of the equation

$$\alpha J_n(\mu) + \beta\mu J_n'(\mu) = 0$$

2. There is zero eigenvalue $\lambda = 0$ and the corresponding normalized eigenfunction is a constant function $u_0 = 2[\pi(b^2-a^2)]^{-1/2}$. Positive eigenvalues and corresponding eigenfunctions are

$$\lambda = \lambda_{nj} = \left(\frac{\mu_{nj}}{a}\right)^2\,, \quad j = 1,2,\dots\,,\ n = 0,1,\dots\,,$$
$$u(r\cos\theta, r\sin\theta) = U_{nj}(r,\varphi)$$
$$= c_{nj}\Big[N_{2n}'(\mu_{nj}b)J_{2n}(\mu_{nj}r) - J_{2n}'(\mu_{nj}b)N_{2n}(\mu_{nj}r)\Big]\cos(2n\theta)\,,$$

where c_{nj} are normalization constants and for every $n = 0,1,\dots$ the numbers μ_{nj}, $j = 1,2,\dots$, are positive roots of the equation

$$J_{2n}'(\mu a)N_{2n}'(\mu b) - J_{2n}'(\mu b)N_{2n}'(\mu a) = 0\,.$$

3. Eigenvalues and corresponding eigenfunctions are

$$\lambda = \lambda_{nj} = \left(\frac{\mu_{nj}}{a}\right)^2 + \left(\frac{\pi k}{l}\right)^2, \quad j = 1, 2, \dots, \; n = 0, 1, \dots, \; k = 1, 2, \dots$$
$$u(r\cos\theta, r\sin\theta, z) = U_{nj}(r, \varphi, z)$$
$$= c_{njk} J_{2n}\Big(\frac{\mu_{nj}}{a}\, r\Big) \sin(2n\theta) \sin\Big(\frac{\pi k}{l}\, z\Big),$$

where c_{njk} are normalization constants and for every $n = 0, 1, \dots$ the numbers μ_{nj}, $j = 1, 2, \dots$, are roots of the equation

$$J_{2n}(\mu) = 0\,.$$

4. There is the zero eigenvalue $\lambda = 0$ and the corresponding normalized eigenfunction is a constant function $u_0 = (\pi a^2 h)^{-1/2}$. Positive eigenvalues and the corresponding eigenfunctions are

$$\lambda = \lambda_{nj} = \left(\frac{\mu_{nj}}{a}\right)^2 + \left(\frac{\pi k}{2h}\right)^2, \quad j = 1, 2, \dots, \; n = 0, 1, \dots, \; k = 1, 2, \dots$$
$$u(r\cos\theta, r\sin\theta, z) = U_{nj}(r, \varphi, z)$$
$$= c_{njk} J_{n}\Big(\frac{\mu_{nj}}{a}\, r\Big) \cos(n\theta) \cos\Big(\frac{\pi k}{2h}\, (z+h)\Big),$$

where c_{njk} are normalization constants and for every $n = 0, 1, \dots$ the numbers μ_{nj}, $j = 1, 2, \dots$, are positive roots of the equation

$$J_n'(\mu) = 0\,.$$

5. Positive eigenvalues and the corresponding eigenfunctions are

$$\lambda = \lambda_{nj} = \left(\frac{\mu_{nj}}{a}\right)^2 + \left(\frac{\pi k}{2h}\right)^2, \quad j = 1, 2, \dots, \; n = 0, 1, \dots, \; k = 1, 2, \dots$$
$$u(r\cos\theta, r\sin\theta, z) = U^{\pm}_{nj}(r, \varphi, z)$$
$$= c_{njk}\Big[N_n(\mu_{nj} b) J_n(\mu_{nj} r) - J_n(\mu_{nj} b) N_n(\mu_{nj} r)\Big] e^{\pm in\theta} \sin\left(\frac{\pi k}{2h}(z+h)\right),$$

where c_{njk} are normalization constants and for every $n = 0, 1, \dots$ the numbers $\mu = \mu_{nj}$, $j = 1, 2, \dots$, are positive roots of the equation

$$J_n(\mu a) N_n(\mu b) - J_n(\mu b) N_n(\mu a) = 0\,.$$

6. There is the zero eigenvalue $\lambda = 0$ and the corresponding normalized eigenfunction is a constant function $u_0 = 2/(\pi a^2 h)^{1/2}$. Positive eigenvalues and the corresponding eigenfunctions are

$$\lambda = \lambda_{nj} = \left(\frac{\mu_{nj}}{a}\right)^2 + \left(\frac{\pi k}{2h}\right)^2, \quad j = 1, 2, \ldots, \; n = 0, 1, \ldots, \; k = 1, 2, \ldots$$

$$u(r\cos\theta, r\sin\theta, z) = U_{nj}(r, \varphi, z)$$

$$= c_{njk} J_{4n}\left(\frac{\mu_{nj}}{a}\, r\right) \cos(4n\theta) \cos\left(\frac{\pi k}{2h}\,(z+h)\right),$$

where c_{njk} are normalization constants and for every $n = 0, 1, \ldots$ the numbers μ_{nj}, $j = 1, 2, \ldots$, are positive roots of the equation

$$J'_{4n}(\mu) = 0\,.$$

48. Spherical harmonics

DEFINITION **48.1**. (Harmonic polynomials)
A polynomial $Q(\mathbf{x})$*,* $\mathbf{x} \in \mathbb{R}^N$*, is called a harmonic polynomial if it is annihilated by the Laplace operator,* $\Delta Q(\mathbf{x}) = 0$*.*

For example, the polynomials of two real variables x and y

$$Q_1(x,y) = x^2 - y^2 + 3xy - 5x\,, \quad Q_2(x,y) = x^3 - 3xy^2$$

are harmonic polynomials in $\mathbb{R}^2$. The polynomials

$$Q_1(x,y,z) = x^2 + y^2 - 2z^2 + x + 1\,, \quad Q_2(x,y,z) = x^3 + 2xy^2 - 5xz^2$$

are harmonic polynomials in $\mathbb{R}^3$. A polynomial $Q_l(\mathbf{x})$ is said to be *homogeneous* of degree l if for any real s

$$Q_l(s\mathbf{x}) = s^l Q_l(\mathbf{x})\,.$$

For example, the polynomials $Q_1(x,y)$ and $Q_1(x,y,z)$ are not homogeneous, while the polynomials $Q_2(x,y)$ and $Q_1(x,y,z)$ are homogeneous of degree $l = 3$:

$$Q_2(sx,sy) = (sx)^3 + 3sx(sy)^2 = s^3(x^3 + 3xy^2) = s^3 Q_1(x,y)$$

and similarly for $Q_3(x,y,z)$.

A sphere in $\mathbb{R}^N$ is described by the equation

$$\sum_{j=1}^{N} x_j^2 = a^2 \quad \Leftrightarrow \quad |\mathbf{x}| = a$$

where the origin of the coordinate system is set at the center of the sphere, and the vertical bars are used denote the Euclidean length of a vector. If the length scale on coordinate axes is chosen in units of radius a, then the equation becomes $|\mathbf{x}| = 1$; the sphere is called a *unit sphere* and denoted $\mathbb{S}^{N-1}$. Unless the radius a is specified, $\mathbb{S}^{N-1}$ means a unit sphere.

DEFINITION **48.2**. (Spherical harmonics)
A spherical harmonic Y_l *of degree* l *is a harmonic homogeneous polynomial* Q_l *of degree* l *reduced to the unit sphere:*

$$Y_l(\mathbf{n}) = Q_l\left(\frac{\mathbf{x}}{r}\right) = \frac{Q_l(\mathbf{x})}{r^l}\,, \qquad \mathbf{n} = \frac{\mathbf{x}}{r}\,, \quad |\mathbf{n}| = 1\,,\ r = |\mathbf{x}|$$

A complex (or real) valued function f on $\mathbb{S}^{N-1}$ is called *square integrable* if $|f|^2$ is integrable on $\mathbb{S}^{N-1}$; and the space of all such functions is denoted $\mathcal{L}_2(\mathbb{S}^{N-1})$:

$$f:\ \mathbb{S}^{N-1} \ \to\ \mathbb{C}\,, \quad f \in \mathcal{L}_2(\mathbb{S}^{N-1}) \quad \Leftrightarrow \quad \int_{\mathbb{S}^{N-1}} |f(\mathbf{n})|^2 dS < \infty\,,$$

where dS denote an infinitesimal surface area element of $\mathbb{S}^{N-1}$. For example, dS for $\mathbb{S}^1$ is the arclength on the unit circle. If $x = \cos\theta$, $y = \sin\theta$ are parametric equations of the circle, then $dS = d\theta$ and

$$\int_{\mathbb{S}^1} |f(\mathbf{n})|^2 dS = \int_0^{2\pi} |f(\mathbf{n})|^2 d\theta\,, \quad \mathbf{n} = (\cos\theta, \sin\theta)\,.$$

Using the spherical coordinates

$$x = r\sin\theta\cos\varphi\,,\ y = r\sin\theta\sin\varphi\,,\ z = r\cos\theta\,,$$

to parameterize the unit sphere $\mathbb{S}^2$,

$$\mathbf{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

one has

$$\int_{\mathbb{S}^2} |f(\mathbf{n})|^2 dS = \int_0^{2\pi}\int_0^{\pi} |f(\mathbf{n})|^2 \sin\theta\, d\theta\, d\varphi\,.$$

There is a generalization of spherical coordinates to $\mathbb{R}^N$, $N > 3$, which can be used to obtain dS for $\mathbb{S}^{N-1}$.

THEOREM 48.1. (Orthogonality of spherical harmonics) *Spherical harmonics Y_l and $Y_{l'}$ are orthogonal in $\mathcal{L}_2(\mathbb{S}^{N-1})$ if $l \neq l'$,*

$$\int_{\mathbb{S}^{N-1}} Y_l(\mathbf{n}) Y_{l'}(\mathbf{n})\, dS = 0\,, \quad l \neq l'\,.$$

PROOF. In the second Green's formula

$$\int_\Omega (v\,\Delta u - u\,\Delta v)\; dx = \int_{\partial\Omega} \left(v\,\frac{\partial u}{\partial \mathbf{n}} - u\,\frac{\partial v}{\partial \mathbf{n}} \right) dS$$

take Ω to be the unit ball $r = |\mathbf{x}| < 1$ so that $\partial\Omega = \mathbb{S}^{N-1}$ and let $u = r^l Y_l$ and $v = r^{l'} Y_{l'}$ be harmonic polynomials so that $\Delta u = \Delta v = 0$ and the right side vanishes. The outward unit normal on the sphere is $\mathbf{x}/r$ or $\mathbf{x} = r\mathbf{n}$, and the normal derivative coincides with the partial derivative with respect to r, $\partial u/\partial\mathbf{n} = \partial u/\partial r$. Since Y_l and $Y_{l'}$ are independent of the radial variable,

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{N-1}} \left(r^{l'} \frac{\partial(r^l Y_l)}{\partial r} - r^l Y_l \frac{\partial(r^{l'} Y_{l'})}{\partial r} \right)\bigg|_{r=1} dS \\ &= (l - l') \int_{\mathbb{S}^{N-1}} Y_l(\mathbf{n}) Y_{l'}(\mathbf{n})\, dS\,. \end{aligned}$$

as required. □

48.1. Spherical harmonics on a circle $\mathbb{S}^1$. All spherical harmonics on a circle $\mathbb{S}^1$ are easy to find by separating variables in the Laplace equation in polar coordinates. Let Q_l be a harmonic homogeneous polynomial. Then in polar coordinates

$$Q_l(x,y) = r^l Y_l(\varphi)\,.$$

Since Q_l is annihilated by the Laplace operator, in polar coordinates one infers

$$0 = \Delta Q_l = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Q_l}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 Q_l}{\partial \varphi^2} = r^{l-2}\Big(l^2 Y + Y''\Big)$$

from which it follows that

$$Y_l'' + l^2 Y_l = 0 \quad \Rightarrow \quad Y_l(\varphi) = A_l\cos(l\varphi) + B_l\sin(l\varphi)\,, \quad l = 0, 1, ...$$

where A_l and B_l are real constants. Put $z = x + iy$ so that $r = |z|$ and

$$z^l = (x+iy)^l = r^l e^{il\varphi} = r^l\Big(\cos(l\varphi) + i\sin(l\varphi)\Big)$$

Then any homogeneous harmonic polynomial can be written in a simple form

$$Q_l(x,y) = r^l Y_l(\varphi) = A_l \mathrm{Re}\, z^l + B_l \mathrm{Im}\, z^l\,.$$

48.2. Spherical harmonics on a sphere $\mathbb{S}^2$. It is convenient to use spherical coordinates to find spherical harmonics on $\mathbb{S}^2$. Any homogeneous harmonic polynomial has the following form in spherical coordinates

$$Q_l(x,y,z) = r^l Y_l(\theta,\varphi)$$

Writing the Laplace operator in spherical coordinates

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\varphi^2}$$

and calculating the action of its radial part on $u = Q_l$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial Q_l}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial r^l}{\partial r}\right)Y_l = l(l+1)r^l Y_l$$

in the Laplace equation $\Delta Q_l = 0$, the following equation for spherical harmonics is obtained

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y_l}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_l}{\partial\varphi^2} + l(l+1)Y_l = 0 \qquad (\mathbf{48.1})$$

So every spherical harmonics satisfies Eq. (**48.1**). It turns out that the converse is also true.

Theorem **48.2**. (Spherical harmonics on $\mathbb{S}^2$)
*In order for a function Y_l to be a spherical harmonic of degree l on $\mathbb{S}^2$, it is necessary and sufficient that Y_l is a solution of Eq. (**48.1**) of the class $C^\infty(\mathbb{S}^2)$.*

Equation (**48.1**) can be solved by separation of variables. Put

$$Y_l(\theta,\varphi) = P(\cos\theta)\Phi(\varphi)$$

Let us multiply (**48.1**) by $\sin^2\theta$ and then divide it by Y_l. Then the first and third terms in the right side of (**48.1**) depend only on θ, while the second term equal to Φ''/Φ is a function of φ and, hence, must be a constant, denoted $-\nu$. Then the functions P and Φ satisfy the equations

$$\Phi'' + \nu\Phi = 0\,, \tag{48.2}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP(\cos\theta)}{d\theta}\right) + \left(l(l+1) - \frac{\nu}{\sin^2\theta}\right)P(\cos\theta) = 0\,. \tag{48.3}$$

In order for Y_l to be uniquely defined on $\mathbb{S}^2$, it has to satisfy the periodicity condition

$$Y_l(\theta,\varphi+2\pi) = Y_l(\theta,\varphi)\,.$$

Periodic solutions of (**48.2**) exist only if $\nu = m^2$,

$$\Phi(\varphi) = a_m\cos(m\varphi) + b_m\sin(m\varphi)\,,\quad m = 0,1,2,\ldots,$$

where a_m and b_m are real constants. Thus, the problem is reduced to solving (**48.3**) with $\nu = m^2$, $m = 0, 1, \ldots$. Put $\mu = \cos\theta$ (a new variable) so that

$$\frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{d\mu} \quad\Rightarrow\quad \sin\theta\,\frac{d}{d\theta} = -\sin^2\theta\frac{d}{d\mu} = -(1-\mu^2)\frac{d}{d\mu}\,.$$

Then the function $P(\mu)$ satisfies the equation

$$-\Big((1-\mu^2)P'\Big)' + \frac{m^2}{1-\mu^2}\,P = l(l+1)P\,. \tag{48.4}$$

A solution to this equation must regular at $\mu = \pm 1$, that is,

$$|P(\pm 1)| < \infty\,.$$

Note that by Theorem **48.2**, spherical harmonics are from $C^\infty(\mathbb{S}^2)$. Equation (**48.4**) is known as *Legendre's equation* and its regular solutions are *associated Legendre functions.*

48.3. Associated Legendre functions. Consider first the case $m = 0$. Equation (**48.4**) has the form

$$\Big((1-\mu^2)P'\Big)' + l(l+1)P = 0\,, \quad |P(\pm 1)| < \infty\,. \tag{48.5}$$

THEOREM **48.3**. (Legendre polynomials)
The Legendre polynomial

$$P_l(\mu) = \frac{1}{2^l l!}\frac{d^l}{d\mu^l}\Big(\mu^2-1\Big)^l, \quad l = 0, 1, \dots, \tag{48.6}$$

*is the only linearly independent solution of (**48.5**) in* $C^2([-1,1])$.

PROOF. Put $Q_l = (\mu^2-1)^l$. It satisfy the identity

$$(\mu^2-1)Q_l' - 2l\mu Q_l = 0\,.$$

Differentiating it $l+1$ times, one infers that

$$(\mu^2-1)Q_l^{(l+2)} + 2\mu Q_l^{(l+1)} - l(l+1)Q_l^{(l)} = 0\,.$$

This relation shows that $Q_l^{(l)}$ satisfies (**48.5**) and so does P_l because P_l is proportional to $Q_l^{(l)}$.

Suppose that $P(\mu)$ is another solution from $C^2([-1,1])$. Then by the Ostrogradsky-Liouville theorem for $p(\mu) = 1-\mu^2$ (see (**36.6**)), the following relation holds

$$P_l'(\mu)P(\mu) - P_l(\mu)P'(\mu) = \frac{c}{1-\mu^2}\,, \quad |\mu| < 1\,,$$

for some constant c. The left side of this relation has the limit as $\mu \to 1^-$ or $\mu \to -1^+$ because, by assumption, both the solutions have two continuous derivatives in the *closed* interval $[-1,1]$. Therefore the corresponding limits of the right side must exist, too, which is only possible if the constant c is zero, $c = 0$. Therefore the Wronskian of P_l and P vanishes, meaning that the solutions are linearly dependent. So, P_l is the only linearly independent solution regular at $\mu = \pm 1$. □
Let us compute a few first Legendre polynomials

$$\begin{aligned} P_0 &= 1\,, \\ P_1 &= \mu\,, \\ P_2 &= \frac{3}{2}\mu^2 - \frac{1}{2}\,, \\ P_3 &= \frac{5}{2}\mu^3 - \frac{3}{2}\mu\,. \end{aligned}$$

COROLLARY **48.1**. (Orthogonality of Legendre polynomials)
Legendre polynomials form an orthogonal set in $\mathcal{L}_2(-1,1)$.

PROOF. Legendre polynomials satisfy (**48.4**) for $m = 0$. Therefore $P_l(\cos\theta)$ is a $C^\infty(\mathbb{S}^2)$ solution of Eq. (13.3) that is independent of the polar angle φ. By Theorem **48.2**, $P_l(\cos\theta)$ is a spherical function of degree l. But spherical functions of different degrees are orthogonal in $\mathcal{L}_2(\mathbb{S}^2)$ by Theorem **48.1**, and for $l \neq l'$, one has

$$0 = \int_0^{2\pi}\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta\, d\varphi = 2\pi\int_{-1}^{1} P_l(\mu)P_{l'}(\mu)\, d\mu\,,$$

which completes the proof. □

Equation (**48.6**) is called the *Rodrigues' formula* or *Ivory-Jacobi formula*. As noted in Section 4.2, the Legendre polynomials can also be obtained by the Schmidt process applied to the set of monomials μ^l, $l = 0, 1, ...$, in $\mathcal{L}_2(-1,1)$. As the set of polynomials is dense in $C^0([-1,1])$, it is also dense in $\mathcal{L}_2(-1,1)$. Hence, the Legendre polynomials form a complete orthogonal set in $\mathcal{L}_2(-1,1)$. The Fourier series for any square integrable function f in $(-1,1)$ converges to f almost everywhere (or in the mean):

$$f(\mu) = \sum_{l=0}^{\infty} \frac{2l+1}{2}\,\langle f, P_l\rangle\, P_l(\mu) \quad a.e., \qquad f \in \mathcal{L}_2(-1,1)\,.$$

Recall from Section 4.2 that $\|P_l\|^2 = 2/(2l+1)$.

DEFINITION **48.3**. (Associated Legendre functions)
The functions

$$(\mathbf{48.7})\quad P_l^m(\mu) = \Big(1-\mu^2\Big)^{\frac{m}{2}}\, \frac{d^m P_l(\mu)}{d\mu^m}\,,\quad l = 0,1,...,\quad m = 0,1,...,l\,,$$

are called associate Legendre functions.

In other words, with every Legendre polynomial P_l, one can associate $l+1$ functions defined by the rule (**48.7**). For even m, P_l^m are polynomials, while for odd m, P_l^m is a polynomial multiplied by $\sqrt{1-\mu^2}$. Since P_l are polynomials of degree l, their derivatives of orders higher than l vanish and $P_l^m(\mu) = 0$ if $l < m$. For example for $m = 0, 1, 2$, one has

$$\begin{aligned}
P_l^0(\mu) &= P_l(\mu)\,, \quad l \geq 0\,,\\
P_l^1(\mu) &= \sqrt{1-\mu^2}\, P_l'(\mu)\,, \quad l \geq 1\\
P_l^2(\mu) &= (1-\mu^2)\, P_l''(\mu)\,, \quad l \geq 2\,,\\
P_l^3(\mu) &= (1-\mu^2)^{3/2}\, P_l'''(\mu)\,, \quad l \geq 3\,,
\end{aligned}$$

Let us compute a few first associate Legendre functions with $m \leq l$. They will later be used to obtain a few first spherical harmonics.

$$l=0: \quad P_0^0 = 1$$

$$l=1: \quad P_1^0 = \mu\,, \quad P_1^1 = \sqrt{1-\mu^2}$$

$$l=2: \quad \begin{array}{l} P_2^0 = \tfrac{3}{2}\mu^2 - \tfrac{1}{2}\,, \quad P_2^1 = 3\mu\sqrt{1-\mu^2}\,, \\ P_2^2 = 3(1-\mu^2) \end{array}$$

$$l=3: \quad \begin{array}{l} P_3^0 = \tfrac{5}{2}\mu^3 - \tfrac{3}{2}\mu\,, \qquad P_3^1 = \sqrt{1-\mu^2}\left(\tfrac{15}{2}\mu^2 - \tfrac{3}{2}\right)\,, \\ P_3^2 = 15\mu(1-\mu^2)\,, \quad P_3^3 = 15(1-\mu^2)^{3/2}\,. \end{array}$$

THEOREM **48.4**. (Properties of associated Legendre functions)

(i) *Associate Legendre functions P_l^m are regular solutions of Legendre's equation (**48.4**);*
(ii) *For each $m \geq 0$ the set $\{P_l^m\}$, $l = m$, $m+1$, ..., is orthonormal in $\mathcal{L}_2(-1,1)$, and*

$$\langle P_l^m, P_{l'}^m \rangle = \frac{(l+m)!}{(l-m)!}\,\frac{2}{2l+1}\,\delta_{ll'}\,; \tag{48.8}$$

(ii) *For each $m \geq 0$ the set $\{P_l^m\}$, $l = m$, $m+1$, ..., is complete in $\mathcal{L}_2(-1,1)$.*

PROOF. Here a proof is outlined. Some minor technical details are left to the reader as an exercise.

(i). In Legendre's equation (**48.4**) put

$$P(\mu) = \Big(1-\mu^2\Big)^{\frac{m}{2}}\,Q(\mu)\,.$$

so that the function $Q(\mu)$ satisfies the equation

$$(1-\mu^2)Q'' - 2\mu(m+1)Q' + (l^2 + l - m^2 - m)Q = 0\,.$$

This equation can also be obtained by differentiating Eq. (**48.5**) m times, which shows that $Q = d^m P_l/d\mu^m$. Thus, P_l^m are solutions of Legendre's equation.

(ii). For brevity the mth derivative of P_l is denoted as $Q = P_l^{(m)}$. By multiplying the equation for Q by $(1-\mu^2)^m$, it can written in the form

$$\Big[(1-\mu^2)^{m+1}P_l^{(m+1)}\Big]' = -(l-m)(l+m+1)(1-\mu^2)^m P_l^{(m)}\,.$$

This identity is then used to establish the following recurrence relation by integration by parts

$$\begin{aligned}\langle P_l^m, P_{l'}^m\rangle &= \int_{-1}^{1} P_l^m P_{l'}^m \, d\mu = \int_{-1}^{1} (1-\mu^2)^m P_l^{(m)} P_{l'}^{(m)} \, d\mu \\ &= -\int_{-1}^{l} P_{l'}^{(m-1)} \Big[(1-\mu^2)^m P_l^{(m)}\Big]' \, d\mu \\ &= (l-m-1)(l+m) \int_{-1}^{1} (1-\mu^2)^{m-1} P_l^{(m-1)} P_{l'}^{(m-1)} \, d\mu \\ &= (l-m-1)(l+m)\langle P_l^{m-1}, P_{l'}^{m-1}\rangle\end{aligned}$$

Here the first equality is obtained by the definition of P_l^m, the second follows from integration by parts and that the boundary term vanishes owing to the factor $(1-\mu^2)^m$, the third holds by the identity for the derivatives $P_l^{(m)}$. By using the established relation recursively m times one infers that

$$\langle P_l^m, P_{l'}^m\rangle = \frac{(l+m)!}{(l-m)!}\langle P_l, P_{l'}\rangle = \frac{(l+m)!}{(l-m)!}\,\frac{2}{2l-1}\,\delta_{ll'}$$

by the orthogonality property of Legendre polynomials.

(iii). Let $f \in \mathcal{D}(-1,1)$ be a test function (a C^∞ function on $\mathbb{R}$ that vanishes outside $[-1,1]$). The space of test functions $\mathcal{D}(-1,1)$ is proved to be dense in $\mathcal{L}_2(-1,1)$. Then for any $m \geq 0$, the function

$$g(\mu) = f(\mu)\Big(1-\mu^2\Big)^{-\frac{m}{2}} \in \mathcal{D}(-1,1)$$

is also a test function because f and all its derivatives have support in $(-1,1)$ and so do g and all its derivatives. The set polynomials is dense in $C^0([-1,1])$ and hence g can be approximated by a polynomial with any desired accuracy. Since the derivatives of the Legendre polynomials $P_l^{(m)}$ are polynomials, g can be approximated by a linear combination of $P_l^{(m)}$. The latter means, by definition of associated Legendre functions, that f can be approximated by a linear combination of P_l^m. Therefore the set $\{P_l^m\}$, $l \geq m$, is dense in $\mathcal{D}(-1,1)$ and, hence, in $\mathcal{L}_2(-1,1)$ because $\mathcal{D}(-1,1)$ is dense in $\mathcal{L}_2(-1,1)$. □

48.4. Spherical harmonics as eigenfunctions. The analysis of Legendre polynomials and associated Legendre functions shows that all linearly independent spherical harmonics on $\mathbb{S}^2$ are given in spherical coordinates by

$$\textbf{(48.9)} \qquad Y_l^m(\theta,\varphi) = \begin{cases} P_l^m(\cos\theta)\cos(m\varphi)\,, & m = 0,1,...,l\,; \\ P_l^{|m|}(\cos\theta)\sin(|m|\varphi)\,, & m = -1,-2,...,-l \end{cases}$$

where $l = 0, 1, ...$, or in the complex form

$$\tilde{Y}_l^m(\theta, \varphi) = P_l^{|m|}(\cos\theta)e^{im\varphi}\,, \quad l = 0, 1, ..., \quad |m| \le l\,. \tag{48.10}$$

As a consequence of the orthogonality property of associated Legendre functions, the spherical harmonics Y_l^m are orthogonal in $\mathcal{L}_2(\mathbb{S}^2)$.

COROLLARY **48.2**. (Orthogonality of spherical harmonics) *The spherical harmonics Y_l^m, $l = 0, 1, ...$, $|m| \le l$, form an orthogonal set in $\mathcal{L}_2(\mathbb{S}^2)$ and*

$$\begin{aligned}\langle Y_m^l, Y_{l'}^{m'}\rangle &= \int_0^{2\pi}\int_0^{\pi} Y_m^l(\theta, \varphi)Y_{l'}^{m'}(\theta, \varphi)\sin\theta\, d\theta\, d\varphi \\ &= 2\pi\,\frac{1+\delta_{0m}}{2l+1}\,\frac{(l+|m|)!}{(l-|m|)!}\,\delta_{ll'}\,\delta_{mm'} = \|Y_l^m\|^2\delta_{ll'}\,\delta_{mm'}\,. \end{aligned}\tag{48.11}$$

Indeed, the integral in (**48.11**) is the product of two integrals. The integrals over the polar angle are

$$\begin{aligned}&\int_0^{2\pi}\sin(m\varphi)\cos(m'\varphi)\,d\varphi = 0\,,\\ &\int_0^{2\pi}\sin(m\varphi)\sin(m'\varphi)\,d\varphi = \int_0^{2\pi}\cos(m\varphi)\cos(m'\varphi)\,d\varphi\\ &= \pi(1+\delta_{m0})\delta_{mm'}\end{aligned}$$

Therefore the integral over the zenith angle θ has to be calculated at $m = m'$,

$$\int_0^{\pi} P_l^{|m|}(\cos\theta)P_{l'}^{|m|}(\cos\theta)\,\sin\theta\,d\theta = \int_{-1}^{1} P_l^{|m|}(\mu)P_{l'}^{|m|}(\mu)\,d\mu\,,$$

and, hence, it is given by (**48.8**).

THEOREM **48.5**. (Fourier series over spherical harmonics) *The spherical harmonics Y_l^m, $l = 0, 1, ...$, $|m| \le l$, form a complete orthogonal set in $\mathcal{L}_2(\mathbb{S}^2)$. The Fourier series of $f \in \mathcal{L}_2(\mathbb{S}^2)$ over the spherical harmonics converges to f in the mean:*

$$\begin{aligned}f(\mathbf{n}) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}Y_l^m(\mathbf{n}) \quad a.e.\,, \quad \mathbf{n}\in\mathbb{S}^2\,,\\ f_{lm} &= \frac{\langle f, Y_l^m\rangle}{\|Y_l^m\|^2} = \frac{1}{\|Y_l^m\|^2}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)Y_l^m(\theta,\varphi)\,\sin\theta\,d\theta\,d\varphi\\ \|Y_l^m\|^2 &= \frac{2l+1}{2\pi(1+\delta_{0m})}2l+1\,\frac{(l-|m|)!}{(l+|m|)!}(l-|m|)!\end{aligned}$$

Let Q_l be a spherical harmonics of degree l. Then it can be expanded into the Fourier series. Spherical harmonics of different degrees are orthogonal (Theorem **48.1**) and therefore $\langle Q_l, Y_l^m\rangle = 0$ if $l' \neq l$. This implies that *linear combinations*

$$Y_l(\mathbf{n}) = \sum_{m=-l}^{l} a_{lm} Y_l^m(\mathbf{n}) \,,$$

define all spherical harmonics of degree l.

Consider the eigenvalue problem

$$(\mathbf{48.12}) \qquad -\Delta_{\mathbb{S}^2} Y \equiv -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} = \lambda Y \,, \\ Y \in \mathcal{M}_L = C^\infty(\mathbb{S}^2)\,.$$

The operator $\Delta_{\mathbb{S}^2}$ is called the *Laplace-Beltrami operator* on a sphere $\mathbb{S}^2$. It is positive semi-definite in $\mathcal{L}_2(\mathbb{S}^2)$ (a proof is left to the reader as an exercise) and hence it is hermitian (because its domain is dense in $\mathcal{L}_2(\mathbb{S}^2)$). Its eigenvalues are real non-negative. The analysis given in this section shows that the spherical functions Y_l^m, $|m| \leq l$, are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^2$ that correspond to the eigenvalue $\lambda = l(l+1)$ which has multiplicity $2l+1$ (the number of linearly independent eigenfunctions for each l). In fact, it is possible to prove the converse that the regularity condition $Y \in C^\infty(\mathbb{S}^2)$ requires that the equation (**48.12**) has non-trivial solutions only if $\lambda = l(l+1)$, $l = 0, 1, \ldots$. Note that by separating variables the problem is reduced to two Sturm-Liouville problems (**48.2**) and (**48.2**) where $l(l+1)$ is replaced by λ. Consequently, the eigenvalues λ are eigenvalues of the (singular) Sturm-Liouville operator (the Legendre operator) in (**48.4**) where again $l(l+1)$ is replaced by λ. It can be proved that this Sturm-Liouville problem has regular solutions if $\lambda = l(l+1)$ and $|m| \leq l$. Thus, all linearly independent eigenfunctions of the Laplace-Beltrami operator are the spherical harmonics Y_l^m.

48.5. Exercises.

1. Show that the Laplace-Beltrami operator on a sphere $\mathbb{S}^2$ is positive semi-definite in $\mathcal{L}_2(\mathbb{S}^2)$

$$\langle LY, Y\rangle = \int_{\mathbb{S}^2} \overline{Y(\mathbf{n})}\, LY(\mathbf{n})\, dS \geq 0\,, \qquad Y \in C^\infty(\mathbb{S}^2)\,.$$

2. Find the explicit form of the following spherical harmonics

$$\begin{array}{rlll} m=0\ : & Y_0^m(\theta,\varphi)\,, & Y_1^m(\theta,\varphi)\,, & Y_2^m(\theta,\varphi)\,; \\ |m|=1\ : & Y_1^m(\theta,\varphi)\,, & Y_2^m(\theta,\varphi)\,; & \\ |m|=2\ : & Y_2^m(\theta,\varphi)\,, & Y_3^m(\theta,\varphi)\,. & \end{array}$$

Use spherical angles to indicate the parts on the unit sphere in which the spherical harmonic are positive and the parts in which they are negative.

3. Let $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ be a unit vector that spans $\mathbb{S}^2$ and $\mathbf{a}$ be a constant non-zero vector. Find the Fourier series over the spherical harmonics for the following functions

$$\begin{aligned} f(\mathbf{n}) &= 1\,, \\ f(\mathbf{n}) &= \sin^2\theta\cos^2\varphi \\ f(\mathbf{n}) &= \mathbf{a}\cdot\mathbf{n}\,, \\ f(\mathbf{n}) &= |\mathbf{a}\times\mathbf{n}|^2\,, \\ f(\mathbf{n}) &= \begin{cases} 1\,, & n_3 > 0 \\ 0\,, & n_3 \le 0 \end{cases}\,, \end{aligned}$$

where the cross $\times$ denotes the vector product in $\mathbb{R}^3$.

4. Prove the identity

$$P_l(\mu) = \frac{1}{\pi}\int_0^\pi \Big(\mu + i\sqrt{1-\mu^2}\,\cos\varphi\Big)^l\, d\varphi$$

by verifying that the integral satisfies Legendre's equation for $m = 0$.

Answers.

2. Using $P_l^m(\mu) = (1-\mu^2)^{m/2} P_l^{(m)}(\mu)$ for $m = 0, 1, ..., l$, and $\mu = \cos\theta$, one has

$$\begin{array}{ll} Y_0^0(\theta,\varphi) = 1\,, & Y_1^0(\theta,\varphi) = \cos\theta\,, \\ Y_2^0(\theta,\varphi) = \frac{3}{2}\cos^2\theta - \frac{1}{2}\,, & \\ Y_1^1(\theta,\varphi) = \sin\theta\cos\varphi\,, & Y_1^{-1}(\theta,\varphi) = \sin\theta\sin\varphi\,, \\ Y_2^1(\theta,\varphi) = \frac{3}{2}\sin(2\theta)\cos\varphi\,, & Y_2^{-1}(\theta,\varphi) = \frac{3}{2}\sin(2\theta)\sin\varphi\,, \\ Y_2^2(\theta,\varphi) = 3\sin^2\theta\cos(2\varphi)\,, & Y_2^{-2}(\theta,\varphi) = 3\sin^2\theta\sin(2\varphi)\,, \\ Y_3^2(\theta,\varphi) = 15\sin^2\theta\cos\theta\cos(2\varphi)\,, & Y_2^{-2}(\theta,\varphi) = 15\sin^2\theta\cos\theta\sin(2\varphi)\,, \end{array}$$

3. Using basic trigonometric identities and the results of Problem 2,

$$f(\mathbf{n}) = 1 = Y_0^0\,,$$

$$f(\mathbf{n}) = \sin^2\theta\cos^2\varphi = \frac{1}{6}Y_2^2 - \frac{1}{3}Y_2^0 + \frac{1}{3}Y_0^0$$

$$f(\mathbf{n}) = (\mathbf{a},\mathbf{n}) = a_1 Y_1^1 + a_2 Y_1^{-1} + a_3 Y_1^0\,,$$

$$f(\mathbf{n}) = |\mathbf{a}\times\mathbf{n}|^2 = \frac{2}{3}|\mathbf{a}|^2 Y_0^0 + \frac{1}{3}(a_1^2 + a_2^2 - a_3^2)Y_2^0 + \frac{1}{6}(a_2^2 - a_1^2)Y_2^2$$
$$-\frac{1}{3}a_1 a_2 Y_2^{-2} - \frac{2}{3}a_1 a_3 Y_2^1 - \frac{2}{3}a_2 a_3 Y_2^{-1}\,,$$

$$f(\mathbf{n}) = \begin{cases} 1\,, & n_3 > 0 \\ 0\,, & n_3 \le 0 \end{cases} = \sum_{l=0}^{\infty} f_l Y_l^0(\theta,\varphi)$$

$$f_l = \frac{2l+1}{2}\int_0^1 P_l(\mu)\,d\mu = \frac{2l+1}{2l(l+1)}\,P_l'(0)$$

Alternatively, the last integral can be found by expanding the integral of the generating function into the Taylor series:

$$\frac{1}{\sqrt{1-2\mu t+t^2}} = \sum_{l=0}^{\infty} t^l P_l(\mu) \;\Rightarrow\; 1 + \frac{\sqrt{1+t^2}-1}{t} = \sum_{l=0}^{\infty} t^l \int_0^1 P_l(\mu)\,d\mu$$

49. Poisson equation in spherical coordinates

Here the Laplace and Poisson equation are solved in three dimensional regions that are either balls or spherical layers. For brevity of notation, a position vector in a three-dimensional space is denoted by boldface letter $\mathbf{r}$ and its components are (x, y, z) so that

$$\mathbf{r} = (x, y, z)$$

A function on a spatial region Ω is denoted as

$$f(x, y, z) = f(\mathbf{r}), \quad \mathbf{r} \in \Omega$$

The length of position vector is denoted as

$$|\mathbf{r}| = r .$$

A boundary of a spatial region is *smooth* if it is a level set of a C^1 function $g(\mathbf{r})$ whose gradient does not vanish:

$$\partial\Omega \text{ is smooth} \quad \Leftrightarrow \quad \begin{cases} g(\mathbf{r}) = 0, & \text{for all } \mathbf{r} \in \partial\Omega \\ \boldsymbol{\nabla} g(\mathbf{r}) \neq \mathbf{0}, & \text{for all } \mathbf{r} \in \partial\Omega \end{cases}$$

A unit normal vector on $\partial\Omega$ is

$$\mathbf{n} = \pm \frac{1}{|\boldsymbol{\nabla} g(\mathbf{r})|} \boldsymbol{\nabla} g(\mathbf{r})$$

The sign is chosen to make $\mathbf{n}$ either inward or outward. For example, a sphere is smooth because it is a lever set of a polynomial function:

$$g(\mathbf{r}) = |\mathbf{r}|^2 - a^2 = x^2 + y^2 + z^2 - a^2 = 0 .$$

Its gradient does not vanish on the sphere $r = a > 0$:

$$\boldsymbol{\nabla} g = (2x, 2y, 2z) = 2\mathbf{r} \quad \Rightarrow \quad |\boldsymbol{\nabla} g| = 2r$$

If Ω is a ball $r < a$, then the outward normal reads

$$\mathbf{n} = \frac{1}{|\boldsymbol{\nabla} g(\mathbf{r})|} \boldsymbol{\nabla} g(\mathbf{r}) = \frac{\mathbf{r}}{a}$$

The inward normal has the opposite direction. Note that the normal is continuous on the sphere n (the sphere is smooth).

49.1. Formulation of the problem. Consider the Poisson equation in a three-dimensional bounded open region Ω with a smooth boundary $\partial\Omega$ that is oriented outward by the unit vector $\mathbf{n}$.

$$-\Delta u(\mathbf{r}) = f(\mathbf{r}), \qquad \mathbf{r} \in \Omega,$$

$$\Big(\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = v(\mathbf{r}), \quad \mathbf{r} \in \partial\Omega$$

If $\beta = 0$, then the problem is called the *Dirichlet problem*, if $\alpha = 0$, it is called the *Neumann problem*. If Ω is a ball or a spherical layer and the functions α and β are constant on each connected piece of the boundary, then the problem can be solved by separating variables in spherical coordinates.

49.2. Formulation of the problem in a ball. Let Ω be a ball $|\mathbf{r}| < a$. The boundary data v and the inhomogeneity f are expressed in spherical coordinates

$$f = f(r\mathbf{n})\,, \quad v = v(a\mathbf{n})\,, \quad \mathbf{n} = (\sin\theta\cos\varphi, \sin\theta, \sin\varphi, \cos\theta)$$

and then expanded over the spherical harmonics

$$f(r\mathbf{n}) = \sum_{l=0}^{n}\sum_{m=-l}^{m=l} F_{lm}(r)Y_l^m(\theta,\varphi)\,,$$

$$F_{lm}(r) = \frac{\langle f, Y_l^m\rangle}{\|Y_l^m\|^2} = \frac{1}{\|Y_l^m\|^2}\int_0^{2\pi}\int_0^{\pi} f(r\mathbf{n})Y_l^m(\theta,\varphi)\sin\theta\,d\theta\,d\varphi\,,$$

$$v(a\mathbf{n}) = \sum_{l=0}^{n}\sum_{m=-l}^{m=l} A_{lm}Y_l^m(\theta,\varphi)\,,$$

$$A_{lm} = \frac{\langle f, Y_l^m\rangle}{\|Y_l^m\|^2} = \frac{1}{\|Y_l^m\|^2}\int_0^{2\pi}\int_0^{\pi} v(a\mathbf{n})Y_l^m(\theta,\varphi)\sin\theta\,d\theta\,d\varphi\,.$$

If necessary, the limit $n \to \infty$ can be taken. The solution to the problem is sought in the form of the Fourier expansion over the spherical harmonics

$$u(\mathbf{r}) = u(r\mathbf{n}) = \sum_{l=0}^{n}\sum_{m=-l}^{m=l} R_{lm}(r)Y_l^m(\theta,\varphi)$$

The boundary conditions are

$$\alpha u(a\mathbf{n}) + \beta\frac{\partial u}{\partial r}\Big|_{r=a} = v(a\mathbf{n})$$

The substitution of the expansions of u and v into this equation gives the boundary condition for the expansion coefficients $R_{lm}(r)$:

$$\sum_{l,m}\Big(\alpha R_{lm}(a) + \beta R'_{lm}(a)\Big)Y_l^m(\theta,\varphi) = \sum_{l,m} A_{lm}Y_l^m(\theta,\varphi)$$

and owing to the linear independence of the spherical harmonics, this equality is possible if and only if

$$\alpha R_{lm}(a) + \beta R'_{lm}(a) = A_{lm}$$

The action of the Laplace operator in spherical coordinates on a function $R_{lm}(r)Y_l^m(\theta,\varphi)$ is

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\Delta_{\mathbb{S}^2}\right)R_{lm}Y_l^m = Y_l^m\left(\frac{1}{r^2}\Big(r^2R'_{lm}(r)\Big)'+\frac{l(l+1)}{r^2}R_{lm}\right)$$

because the spherical harmonics are eigenfunctions of the Laplace-Beltrami operators on a unit sphere: $-\Delta_{\mathbb{S}^2}Y_l^m = l(l+1)Y_l^m$. Therefore the substitution of the expansions of u and f into the Poisson equation yields the boundary value problem for the expansion coefficients:

$$\textbf{(49.1)}\quad -\frac{1}{r^2}\Big(r^2R'_{lm}(r)\Big)'+\frac{l(l+1)}{r^2}R_{lm}=F_{lm}(r)\,,\quad 0<r<a\,,$$
$$|R_{lm}(0)|<\infty\,,\quad \alpha R_{lm}(a)+\beta R'_{lm}(a)=A_{lm}$$

If Ω is a spherical layer, $a^2 < r^2 < b^2$, a similar analysis leads to the following boundary value problem for the expansion coefficients

$$\textbf{(49.2)}\quad -\frac{1}{r^2}\Big(r^2R'_{lm}(r)\Big)'+\frac{l(l+1)}{r^2}R_{lm}=F_{lm}(r)\,,\quad 0<r<a\,,$$
$$\alpha_aR_{lm}(a)-\beta_aR'_{lm}(a)=A_{lm}\,,$$
$$\alpha_bR_{lm}(b)+\beta_bR'_{lm}(b)=B_{lm}\,,$$

where α_a and β_a are the values of α and β on the inner boundary sphere $r=a$ (note the negative sign at β_a which stems from that the outward normal is directed toward the origin), while α_a and β_a are the values of α and β on the outer boundary sphere $r=b$, and A_{lm} and B_{lm} are the Fourier coefficients of the boundary data on the spheres:

$$v(a\mathbf{n})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_l^m(\mathbf{n})\,,$$
$$v(b\mathbf{n})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}Y_l^m(\mathbf{n})\,.$$

49.3. Solvability condition for the Neumann problem. If $\alpha = 0$ in the boundary conditions, the (Neumann) problem does not have a solution for any choice of the inhomogeneity f and the boundary data v. They must satisfy the solvability condition

$$\int_\Omega f(\mathbf{r})d\mathbf{r}+\int_{\partial\Omega}v(\mathbf{r})\,dS=0$$

The first integral is a triple integral over the spatial region Ω, while the second integral is the surface integral over the boundary of Ω. The derivation is identical to the two-dimensional case discussed earlier (the solvability condition follows from the divergence theorem). In the case of Ω being a ball $r < a$, the solvability condition can be written in the spherical coordinates:

$$\begin{aligned} 0 &= c_f + c_a\,, \\ c_f &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} f(r\mathbf{n})r^2\sin\theta\, drd\theta d\varphi\,, \\ c_a &= a^2\int_0^{2\pi}\int_0^{\pi} v(a\mathbf{n})\sin\theta\, drd\theta d\varphi\,, \\ \mathbf{n} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \end{aligned}$$

The integrals are nothing by the Fourier coefficients for the first spherical harmonic $Y_0^0 = 1$. Therefore the solvability condition only restricts parameters in the boundary value problem (**49.1**) for $l = 0$ or R_{00}:

$$\int_0^a F_{00}(r)\, r^2 dr + a^2 A_{00} = 0\,.$$

If Ω is a spherical layer $0 < a < r < b$, then

$$\begin{aligned} 0 &= c_f + c_a + c_b\,, \\ c_b &= b^2\int_0^{2\pi}\int_0^{\pi} v(b\mathbf{n})\sin\theta\, drd\theta d\varphi \end{aligned}$$

where c_f and c_a have the same form as in the case of a ball, but the integration in c_f with respect to the radial variable r is curried out over the interval $[a, b]$. If it exists, a solution to the Neumann problem is unique ap to an additive constant. The solvability condition impose a restriction on parameters of the boundary value problem (**49.2**) only for $l = 0$:

$$\int_a^b F_{00}(r)\, r^2 dr + a^2 A_{00} + b^2 B_{00} = 0\,.$$

49.4. Boundary value problem for the radial part. Let us find the general solution to the equation

$$-\frac{1}{r^2}\Big(r^2 R_l'(r)\Big)' + \frac{l(l+1)}{r^2}\, R_l = F_l(r)$$

The index m is omitted as the differential operator in the radial part does not depend on it so that the analysis is identical for any $|m| \le l$. The associated homogeneous equation is equidimensional. It is solved by the substitution $R = r^s$ so that

$$s(s+1)r^{s-2} - l(l+1)r^{s-2} = 0 \quad\Rightarrow\quad s = l\,,\; s = -l-1$$

The general solution is therefore

$$R_l(r) = C_1 r^l + \frac{C_2}{r^{1+l}} + R_p(r)$$

where $R_p(r)$ is a particular solution. A particular solution can be found by the method of variation of parameters using r^l and r^{1-l} as two linearly independent solutions. The boundary conditions (either (**49.1**) or (**49.2**)) are then fulfilled by appropriate choice of the constants C_1 and C_2. For the Neumann problem, C_1 and C_2 exist for $l = 0$ only if the solvability condition is fulfilled, and, in this case, C_1 remains arbitrary.

The method of undetermined coefficients. . If $F_l(r)$ happens to be polynomial, then a particular solution is easy to find by the method of undetermined coefficients for equidimensional equations. By multiplying the radial equation by r^2, it is reduced to the standard form of an *equidimensional equation*

$$-\Big(r^2 R_l'(r)\Big)' + l(l+1)R_l = r^2 F_l(r)$$

Let $F_l(r) = f_0 r^k$. A particular solution is sought in the form

$$R_p(r) = Cr^s\,, \quad s = k+2$$

The substitution into the equation yields

$$C[-s(s+1) + l(l+1)]r^s = f_0 r^s \quad \Rightarrow \quad C = \frac{f_0}{l(l+1) - s(s+1)}$$

This is only possible if $s \neq l$ or $s \neq -l-1$. In the latter case, the solution should be sought in the form

$$R_p(r) = Cr^s \ln(r)\,, \quad s(s+1) = l(l+1)$$

The substitution into the equation yields

$$-(2s+1)Cr^s = f_0 r^s \quad \Rightarrow \quad C = -\frac{f_0}{2s+1}\,, \quad s(s+1) = l(l+1)\,.$$

The method of the radial Green's functions. Alternatively or when F_l is not a polynomial, one can use the formalism of the Green's functions. Let us illustrate it by the Dirichlet problem in a ball (**49.1**):

$$-\Big(r^2 R'(r)\Big)' + l(l+1)R = r^2 F(r)\,,$$
$$|R_l(0)| < \infty\,, \quad R_l(a) = A_l$$

The particular solution that defines the Green's function is required to satisfy the trivial boundary conditions:

$$|R_p(0)| < \infty\,, \quad R_p(a) = 0\,.$$

Following the general strategy given in Section **41**, to find such particular solution by the method of variation of parameters, two linearly independent solutions to the associated homogeneous equation have to be constructed one of which satisfies the left boundary condition, while the other satisfies the right one, where in both the conditions any parameters are set to zero ($A_l = 0$ in this case) They are:

$$Z_{1l}(r) = (r/a)^l\,, \qquad Z_{1l}(0) = 1 < \infty\,,$$
$$Z_{2l}(r) = (r/a)^l - (a/r)^{l+1}\,, \qquad Z_{2l}(a) = 0\,.$$

The radial equation contains the Sturm-Liouville operator with $p(r) = r^2$. According the Liouville-Ostrodgradsy theorem (see (**36.6**)), the Wronskian of any two linearly independent solutions satisfies the equation

$$W'(r) = -\frac{p'(r)}{p(r)}\,W(r) \quad \Rightarrow \quad \frac{W'(r)}{W(r)} = -\frac{2}{r}$$

Integrating this equation with the initial condition $W(a)$ (the point $r = 0$ is singular), one infers

$$\int_a^r \frac{W'(\rho)}{W(\rho)}\,d\rho = -\int_r^a \frac{2}{\rho}\,d\rho \quad \Rightarrow \quad r^2W(r) = a^2W(a)$$

With the above choice of the linearly independent solutions

$$a^2W_l(a) = a^2\det\begin{pmatrix} Z_{1l}(a) & Z_{2l}(a) \\ Z'_{1l}(a) & Z'_{2l}(a)\end{pmatrix} = a^2Z'_{2l}(a) = (2l+1)a$$

The remaining technicalities are identical used to derive the radial Green's function in the case of the Dirichlet problem in a disk (**??**). The results reads

$$R_p(r) = \int_0^a G_l(r,\rho)F_l(\rho)\,\rho^2\,d\rho\,, \tag{49.3}$$
$$R_p(r) = -\frac{Z_{1l}(r)}{a^2W_l(a)}\int_r^a F_l(\rho)Z_{2l}(\rho)\rho^2\,d\rho$$
$$\qquad -\frac{Z_{2l}(r)}{a^2W_l(a)}\int_0^r F_l(\rho)Z_{1l}(\rho)\rho^2\,d\rho\,,$$
$$G_l(r,\rho) = -\frac{1}{a^2W_l(a)}\begin{cases} Z_{1l}(r)Z_{2l}(\rho)\,, & r < \rho \\ Z_{2l}(r)Z_{1l}(\rho)\,, & \rho < r\end{cases}$$

where $a^2 W_l(a) = (2l+1)a$. Note the integration weight ρ^2 in contrast to the two-dimensional case. It is related to the Jacobian in the spherical coordinates $J = r^2 \sin\theta$.

A particular solution satisfying the Dirichlet boundary conditions in a spherical layer

$$R_p(a) = R_p(b) = 0\,,$$

is given by the corresponding radial Green's function obtained analogously to the case of an annulus has the form:

$$\begin{aligned} R_p(r) &= \int_a^b G_l(r,\rho) F_l(\rho)\rho^2\, d\rho \\ &= -\frac{Z_l(r;a)}{a^2 W_l(a)} \int_r^b F_l(\rho) Z_l(\rho;b)\, \rho^2\, d\rho \\ &\quad -\frac{Z_l(r;b)}{a^2 W_l(a)} \int_a^r F_l(\rho) Z_l(\rho;a)\, \rho^2\, d\rho\,. \end{aligned} \tag{49.4}$$

where

$$\begin{aligned} Z_l(r;c) &= (r/c)^l - (c/r)^{l+1}\,, \quad Z_l(c;c) = 0\,, \quad c = a, b\,, \\ a^2 W_l(a) &= -(2l+1) a Z_l(a;b) \end{aligned}$$

The Green's functions of other problems can be derived in a similar way.

The Dirichlet problem. Using the Green's function (**49**.**3**), the solution to the Dirichlet problem in a ball is given by the expansion over the spherical harmonics

$$\begin{aligned} u(r\mathbf{n}) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} R_{lm}(r) Y_l^m(\mathbf{n})\,, \\ R_{lm}(r) &= A_{lm}\left(\frac{r}{a}\right)^l + \int_0^a G_l(r,\rho) F_{lm}(\rho)\, \rho^2\, d\rho\,. \end{aligned}$$

The solution to the Dirichlet problem in a layer $0 < a < r < b$ has the following expansion coefficients:

$$R_{lm}(r) = A_{lm}\frac{Z_l(r;b}{Z_l(a;b)} + B_{lm}\frac{Z_l(r;a)}{Z_l(b;a)} + \int_0^a G_l(r,\rho) F_{lm}(\rho)\, \rho^2\, d\rho\,,$$

where the radial Green's function is given in (**49**.**4**).

49.5. Some examples. When the inhomogeneity and boundary data are polynomials of some small order, their expansions over spherical harmonics can be obtained without integration, just by using their form in spherical coordinates and comparing the latter with explicit form of the spherical harmonics. Here is a list of spherical harmonics for $l = 0, 1, 2, 3$, which is sufficient to expand any polynomial of degree at most 3:

$$
\begin{aligned}
l = 0: \quad & Y_0^0 = 1\,, \\[4pt]
l = 1: \quad & \begin{cases} Y_1^0 = \cos\theta\,, \\ Y_1^1 = \sin\theta\cos\varphi\,, \\ Y_1^{-1} = \sin\theta\sin\varphi\,, \end{cases} \\[4pt]
l = 2: \quad & \begin{cases} Y_2^0 = \frac{3}{2}\cos^2\theta - \frac{1}{2}\,, \\ Y_2^1 = 3\sin\theta\cos\theta\cos\varphi\,, \\ Y_2^{-1} = 3\sin\theta\cos\theta\sin\varphi\,, \\ Y_2^2 = 3\sin^2\theta\cos(2\varphi)\,, \\ Y_2^{-2} = 3\sin^2\theta\sin(2\varphi)\,, \end{cases} \\[4pt]
l = 3: \quad & \begin{cases} Y_3^0 = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\,, \\ Y_3^1 = \left(\frac{15}{2}\cos^2\theta - \frac{3}{2}\right)\sin\theta\cos(\varphi)\,, \\ Y_3^{-1} = \left(\frac{15}{2}\cos^2\theta - \frac{3}{2}\right)\sin\theta\sin(\varphi)\,, \\ Y_3^2 = 15\cos\theta\sin^2\theta\cos(2\varphi)\,, \\ Y_3^{-2} = 15\sin^3\theta\cos(2\varphi)\,, \\ Y_3^3 = 15\sin^3\theta\cos(3\varphi)\,, \\ Y_3^{-3} = 15\sin^3\theta\sin(3\varphi)\,. \end{cases}
\end{aligned}
$$

EXAMPLE **49.1**. *Solve the boundary value problem in the ball* Ω : $x^2 + y^2 + z^2 < 1$*:*

$$-\Delta u(x, y, z) = 21xz\,, \quad u\Big|_{\partial\Omega} = z\Big|_{\partial\Omega}$$

SOLUTION: Let us obtain first the expansion of the inhomogeneity and the boundary data over the spherical harmonics. Writing the inhomogeneity and the boundary data in spherical coordinates, one infers that

$$
\begin{aligned}
z\Big|_{r=1} &= \cos\theta = Y_1^0\,, \\
xz &= 15r^2\sin\theta\cos\theta\cos\varphi = 7r^2 Y_2^1
\end{aligned}
$$

The solution is a linear combination of the spherical harmonics in the inhomogeneity and the boundary data:

$$u(r\mathbf{n}) = R_1(r)Y_1^0(\mathbf{n}) + R_2(r)Y_2^1(\mathbf{n})$$

The boundary value problem for R_1 is ($l = 1$):

$$-\frac{1}{r^2}\Big(r^2 R_1'(r)\Big)' + \frac{2}{r^2}\,R_1(r) = 0\,, \quad |R_1(0)| < \infty\,, \quad R_1(1) = 1$$

The general solution to this equidimensional equation is

$$R_1(r) = C_1 r + \frac{C_2}{r^2}$$

The regularity condition requires that $C_2 = 0$. The second boundary condition requires that $C_1 = 1$:

$$R_1(r) = r$$

The boundary value problem for R_2 is ($l = 2$):

$$-\frac{1}{r^2}\Big(r^2 R_2'(r)\Big)' + \frac{6}{r^2}\,R_2(r) = 7r^2\,, \quad |R_1(0)| < \infty\,, \quad R_1(1) = 0$$

If the equation is multiplied by r^2, it becomes an equidimensional equation with the inhomogeneity being a polynomial $7r^4$. So, the method of undetermined coefficients applies. Since $4 \neq l = 2$ or $4 \neq -3 = -l-1$, a particular solution should have the form

$$R_p(r) = Cr^4$$

The substitution of this function into the equation yields

$$-20Cr^2 + 6Cr^2 = 7r^2 \quad \Rightarrow \quad C = -\frac{1}{2}$$

The general solution reads

$$R_2(r) = C_1 r^2 + \frac{C_2}{r^3} - \frac{1}{2}\,r^4$$

The regularity condition requires that $C_2 = 0$, while the second boundary condition is fulfilled if $C_1 = \frac{1}{2}$:

$$R_2(r) = \frac{1}{2}\,r^2(1 - r^2)$$

The solution to the problem is

$$\begin{aligned} u(x, y, z) &= rY_1^0(\mathbf{n}) + \frac{1}{2}\,r^2(1 - r^2)Y_2^1(\mathbf{n}) \\ &= r\cos\theta + \frac{3}{2}\,r^2(1 - r^2)\sin\theta\cos\theta\cos\varphi \\ &= z + \frac{3}{2}\,xz\Big(1 - x^2 - y^2 - z^2\Big)\,. \end{aligned}$$

□

EXAMPLE **49.2.** *Solve the boundary value problem in a spherical layer* $\Omega:\ 1 < r < 2$*, where* r *is the distance from the origin:*

$$-\Delta u(x,y,z) = 30z^2\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{r=1} = 3(x^2+y^2)\Big|_{r=1}\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{r=2} = -16$$

SOLUTION: This is the Neumann problem. Therefore the solvability condition must be checked. Since it involves only the Y_0^0 coefficient in the expansions of the inhomogeneity and the boundary data, let us find first these expansions:

$$\begin{aligned}
z^2 &= 30r^2\cos^2\theta = 30r^2\cdot\frac{2}{3}\cdot\frac{3}{2}\cos^2\theta \\
&= 20r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2} + \frac{1}{2}\right) \\
&= 20r^2Y_2^0 + 10r^2Y_0^0 \\
3(x^2+y^2)\Big|_{r=1} &= 3\sin^2\theta = 3 - 3\cos^2\theta = 2Y_0^0 - 2Y_2^0 \\
-16 &= -16Y_0^0
\end{aligned}$$

The solvability condition is fulfilled.

$$10\int_1^2 r^4 dr - 2^2\cdot 16 + 1^2\cdot 2 = 62 - 64 + 2 = 0$$

A solution exists and is a linear combination of two spherical harmonics Y_0^0 and Y_2^0:

$$u(r\mathbf{n}) = R_0(r)Y_0^0(\mathbf{n}) + R_2(r)Y_2^0(\mathbf{n})\,.$$

The boundary value problem for $R_0(r)$ is ($l = 0$):

$$-\frac{1}{r^2}\Big(r^2R_0'(r)\Big)' = 10r^2\,, \quad -R_0'(1) = 2\,, \quad R_0'(1) = -16$$

Note the negative sing at the derivative at $r = 1$. The normal derivative on the inner boundary is $-\partial u/\partial r$. The inhomogeneity is a monomial. The solution can be found by the method of undetermined coefficients. A particular solution should have the form

$$R_p(r) = Cr^4$$

The substitution into the equation yields:

$$-20Cr^2 = 10r^2 \quad\Rightarrow\quad C = -\frac{1}{2}$$

The general solution and its derivative are

$$R_0(r) = C_1 + \frac{C_2}{r} - \frac{1}{2}r^4\,,$$
$$R_0'(r) = -\frac{C_2}{r^2} - 2r^3$$

The Neumann boundary conditions require that

$$R_0'(1) = -2 \quad\Rightarrow\quad -C_2 - 2 = -2 \quad\Rightarrow\quad C_2 = 0\,,$$
$$R_0'(2) = -16 \quad\Rightarrow\quad -\frac{C_2}{4} - 16 = -16 \quad\Rightarrow\quad C_2 = 0$$

That the both conditions are fulfilled by choosing a *single* constant is the direct consequence of the solvability condition. If one proceeded without checking the solvability condition and the latter were not fulfilled, one would have found that the boundary value problem for $R_0(r)$ has no solution. Thus,

$$R_0(r) = C_1 - \frac{1}{2}r^4$$

where C_1 remains arbitrary (if the Neumann problem has a solution, it is unique up an additive constant). The second boundary value problem is ($l = 2$)

$$-\frac{1}{r^2}\Big(r^2R_2'(r)\Big)' + \frac{6}{r^2}R_2(r) = 20r^2\,,\quad -R_2'(1) = -2\,,\quad R_2'(2) = -16$$

Using the method of undetermined coefficients, a particular solution is found:

$$R_p(r) = -\frac{10}{7}r^4$$

The general solution and its derivative are

$$R_2(r) = C_1r^2 + \frac{C_2}{r^3} - \frac{10}{7}r^4\,,$$
$$R_2'(r) = 2C_1r - \frac{3C_2}{r^4} - \frac{40}{7}r^3$$

The boundary conditions require that

$$\left\{\begin{aligned} 14C_1 - 21C_2 &= 54 \\ 7\cdot 64C_1 - 21C_2 &= 16^2\cdot 13 \end{aligned}\right. \quad\Rightarrow\quad C_1 = \frac{1637}{217}\,,\quad C_2 = \frac{1600}{651}\,.$$

□

EXAMPLE **49.3**. *Solve the boundary value problem in the ball $r < 1$, where r is the distance from the origin:*

$$-\Delta u(x, y, z) = 30xyz\,,\quad \Big(u + \frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{r=1} = 0$$

SOLUTION: The inhomogeneity has the following decomposition over the spherical harmonics

$$30xyz = 30r^3 \cos\theta \sin^2\theta \cos\varphi \sin\varphi = r^3 Y_3^2$$

The solution has the form

$$u(r\mathbf{n}) = R_3(r) Y_3^2(\mathbf{n})$$

where R_3 is the solution to the boundary value problem ($l = 3$):

$$-\frac{1}{r^2}\Big(r^2 R_3'(r)\Big)' + \frac{12}{r^2}\, R_3(r) = r^3\,,$$
$$|R_3(0)| < \infty\,, \quad R_3(1) + R_3'(1) = 0\,.$$

Using the method of undetermined coefficients, a particular solution is found:

$$R_p(r) = -\frac{1}{18}\, r^5$$

The general solution that is regular at the origin and its derivative are

$$R_3(r) = C_2 r^3 - \frac{1}{18}\, r^5\,, \quad R_3'(r) = 3C_2 r^2 - \frac{5}{18}\, r^4$$

The boundary condition requires that

$$C_2 - \frac{1}{18} + 3C_2 - \frac{5}{18} = 0 \quad \Rightarrow \quad C_2 = \frac{1}{12}$$

The solution is

$$\begin{aligned} u(x, y, z) &= \frac{1}{12}\, r^3 \Big(1 - \frac{2}{3}\, r^2\Big) Y_3^2 \\ &= \frac{15}{6}\, xyz \Big(1 - \frac{2}{3}(x^2 + y^2 + z^2)\Big) \end{aligned}$$

□

49.6. Exercises.

In what follows r denotes the distance from the origin.

1. Solve the boundary value problem in the ball $r < 2$

$$-\Delta u(x,y,z) = 0\,, \quad u\Big|_{r=2} = (30xyz - 6z^2)\Big|_{r=2}$$

2. Solve the boundary value problem in the spherical layer $1 < r < 2$

$$-\Delta u(x,y,z) = 6z^2\,, \quad u\Big|_{r=1} = 3(x^2+y^2)\Big|_{r=1}\,, \quad u\Big|_{r=2} = 1$$

3. Solve the boundary value problem in the spherical layer $1 < r < 2$ or show that no solution exists:

$$-\Delta u(x,y,z) = 0\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{r=1} = 8z^2\Big|_{r=1}\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{r=2} = -3(x^2+y^2)\Big|_{r=2}$$

4. Solve the boundary value problem in the ball $r < 1$ or show that no solution exists:

$$-\Delta u(x,y,z) = 5(x^2+y^2+z^2) - x\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{r=1} = -3z^2\Big|_{r=1}$$

5. Solve the boundary value problem in the spherical layer $1 < r < 2$:

$$-\Delta u(x,y,z) = x - 3z^2\,, \quad u\Big|_{r=1} = 1\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{r=2} = xy\Big|_{r=2}$$

6. Solve the boundary value problem in the spherical layer $1 < r < 2$:

$$-\Delta u(x,y,z) = 30x^3\,, \qquad \Big(2u + \frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{r=1} = 0 \quad u\Big|_{r=2} = y\Big|_{r=2}\,,$$

50. The Laplace operator in spherically symmetric regions

50.1. Eigenvalue problem for the Laplace operator in a ball. Let Ω be a three-dimensional ball. Consider the following eigenvalue problem in Ω:

$$(\mathbf{50.1})\qquad -\Delta u = \lambda u\,,\qquad |x| < a\,,\qquad u\Big|_{|x|=a} = 0\,.$$

In spherical coordinates the equation becomes

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) - \left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\theta^2}\right) = \lambda U\,,$$

or

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) - \frac{1}{r^2}\Delta_{\mathbb{S}^2}U = \lambda U\,,$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on a unit sphere $\mathbb{S}^2$ and

$$U(r,\theta,\varphi) = u(r\sin\theta\cos\varphi,\ r\sin\theta\cos\varphi,\ r\cos\theta)\,.$$

The ball (with its center removed) is the image of

$$\Omega' = (0,a)\times\mathbb{S}^2$$

under the transformation defined by spherical coordinates,

$$x = r\mathbf{n}\,,\quad r\geq 0\,,\quad |\mathbf{n}| = 1,\quad \mathbf{n}\in\mathbb{S}^2\,,$$

where the unit vector $\mathbf{n} = (\sin\theta\cos\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)$ spans the sphere $\mathbb{S}^2$. A solution $U(r,\theta,\varphi) = u(r\mathbf{n})$ defines a function on $\mathbb{S}^2$ for every (fixed) r, meaning that U must be a *regular function of θ and φ for every* $0 < r < a$ (in particular, U has to be 2π periodic in φ). The Jacobian $J = r^2\sin\theta$ vanishes at $r = 0$. Therefore it is necessary to impose a regularity condition at $r = 0$ so that the boundary conditions are

$$|U(0,\theta,\varphi)| < \infty\,,\qquad U(a,\theta,\varphi) = 0\,.$$

where the value of a solution U at $r = 0$ is defined as the limit $r \to 0^+$. The regularity condition does *not* yet guarantee that a function obtained from a regular solution in spherical coordinates by transforming it back to the rectangular coordinates satisfies the original equation (**50.1**) at $x = 0$. A solution of (**50.1**) is from $C^2(\Omega)$ and, hence, has to have continuous second partial derivatives at $x = 0$, which is not guaranteed by the regularity condition. *The fact that a regular solution in spherical coordinates $U(r,\theta,\varphi) = u(x)$ extended to $x = 0$ by continuity satisfies (**50.1**) at $x = 0$ must be verified in addition.*

In accord with the the general method of separating variables, a solution is sought in the form

$$U(r,\theta,\varphi) = R(r)Y(\theta,\varphi)\,.$$

Then the original problem is equivalent to two problems

$$(\mathbf{50.2})\qquad -\Delta_{\mathbb{S}^2} Y = \mu Y\,,\qquad Y \in C^\infty(\mathbb{S}^2)\,,$$

$$(\mathbf{50.3})\qquad \Big(r^2R'\Big)' + (\lambda r^2 - \mu)R = 0\,,\quad |R(0)| < \infty\,,\quad R(a) = 0\,.$$

The problem (**50.2**) is the eigenvalue problem for the Laplace-Beltrami operator on a sphere $\mathbb{S}^2$:

$$\mu = l(l+1)\,,\quad Y(\theta,\varphi) = Y_l^m(\theta,\varphi)\,,\quad l = 0,1,...,\quad |m| \le l\,.$$

where Y_l^m are spherical harmonics. By making the substitution

$$R(r) = \frac{R_1(r)}{\sqrt{r}}\,,$$

the problem (**50.3**) becomes the eigenvalue problem for the Bessel operator with $\nu = l + \frac{1}{2} > 0$:

$$r^2R_1'' + rR_1' + \left[\lambda r^2 - \Big(l + \frac{1}{2}\Big)^2\right] R_1 = 0\,.$$

By a general analysis of the eigenvalue problem for an elliptic operator, the eigenvalues in (**50.1**) are strictly positive $\lambda > 0$. Therefore solutions regular at $r = 0$ are given by the Bessel functions

$$R(r;\lambda) = \frac{1}{\sqrt{r}}\, J_{l+\frac{1}{2}}(\sqrt{\lambda}\, r)\,.$$

To find the eigenvalues λ, the second boundary condition $R(a) = 0$ has to be used. To simplify notations introduced earlier for the roots $\mu_j^{(\nu)}$ of the Bessel function J_ν, put

$$J_\nu(\mu) = 0 \quad\Rightarrow\quad \mu = \mu_j^{(\nu)} \equiv \mu_{lj}\,,\quad \nu = l + \frac{1}{2}\,,\quad j = 1,2,...\,.$$

Then the second boundary condition requires that for every l

$$\sqrt{\lambda} a = \mu_{jl}\,,\quad j = 1,2,...\,.$$

The eigenvalues and the corresponding eigenfunction of the Laplace operator in a ball are

$$(\mathbf{50.4})\quad \lambda = \lambda_{lj} = \frac{\mu_{lj}^2}{a^2}\,,\quad U_{ljm}(r,\theta,\varphi) = \frac{C_{ljm}}{\sqrt{r}}\, J_{l+\frac{1}{2}}\Big(\mu_{lj}\frac{r}{a}\Big)\, Y_l^m(\theta,\varphi)\,,$$

where $l = 0,1,...$, $j = 0,1,...$, $m = 0,\pm1,...,\pm l$, and C_{ljm} are normalization constants.

Since $r = 0$ is a singular point (where the Jacobian vanishes) one has to verify that the functions U_{ljm} also satisfy the original equation in the rectangular coordinates at $|x| = 0$. The Bessel function $J_{l+\frac{1}{2}}(r)$ is

$$J_{l+\frac{1}{2}}(r) = \left(\frac{r}{2}\right)^{l+\frac{1}{2}} \left(1 + g(r^2))\right),$$

where g is an analytic function of $r^2 = |x|^2$, that is, g is defined by a power series in r^2 and, hence, is also a power series in the original rectangular coordinates. The latter means that g is from the C^∞ class in the original rectangular coordinates. Therefore

$$U_{ljm}(r, \theta, \varphi) = r^l Y_l^m(\theta, \varphi) h_{lj}(r^2)$$

where h_{lj} is an analytic function of $r^2 = |x|^2$ defined by a power series in r^2. However the combination $r^l Y_l^m$ is a harmonic polynomial (see Section **48.2**) in the original rectangular coordinates. Therefore U_{ljm} is from the $C^\infty(\Omega)$ class in the original variables and, hence, satisfies the equation at $x = 0$.

The eigenfunctions are orthogonal in $\mathcal{L}_2(\Omega)$. The latter follows from the orthogonality property of the Bessel functions (Theorem **46.5**) and the orthogonality of spherical harmonics (**48.11**). Omitting normalization constants for the sake brevity, $U_{ljm} = R(r; \lambda_{lj}) Y_l^m(\theta, \varphi)$, one has

$$\begin{aligned}
\langle U_{ljm}, U_{l'j'm'} \rangle &= \int_\Omega U_{ljm}, U_{l'j'm'} dx \\
&= \int_0^a \int_0^\pi \int_0^{2\pi} U_{ljm}(r, \theta, \varphi) U_{l'j'm'}(r, \theta, \varphi)\, r^2 \sin\theta \, dr \, d\theta \, d\varphi \\
&= \int_0^a R(r; \lambda_{lj}) R(r; \lambda_{l'j'})\, r^2 \, dr \int_{\mathbb{S}^2} Y_l^m(\mathbf{n}) Y_{l'}^{m'}(\mathbf{n}) dS \\
&= \|Y_l^m\|^2 \delta_{ll'} \delta_{mm'} \int_0^a J_{l+\frac{1}{2}}\left(\mu_{jl}\frac{r}{a}\right) J_{l+\frac{1}{2}}\left(\mu_{j'l}\frac{r}{a}\right) r \, dr \\
&= a^2 \|Y_l^m\|^2 \delta_{ll'} \delta_{mm'} \int_0^1 J_{l+\frac{1}{2}}\left(\mu_{jl} z\right) J_{l+\frac{1}{2}}\left(\mu_{j'l} z\right) z \, dz \\
&= \frac{a^2}{2} \left(J'_{l+\frac{1}{2}}(\mu_{lj})\right)^2 \|Y_l^m\|^2 \delta_{ll'} \delta_{mm'} \delta_{jj'} .
\end{aligned}$$

By setting

$$C_{ljm} = \frac{\sqrt{2}}{a|J'_{l+\frac{1}{2}}(\mu_{lj})| \, \|Y_l^m\|}$$

in (**50.4**), the eigenfunctions become orthonormal. By completeness of the Bessel functions in $\mathcal{L}_2((0,a);\sigma)$ where $\sigma(r) = r$, and by completeness of spherical harmonics in $\mathcal{L}_2(\mathbb{S}^2)$, the function (**50.4**) form a complete orthonormal set in $\mathcal{L}_2(\Omega)$ (according to Theorem **??**), and

$$f(x) = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \langle f, U_{ljm}\rangle\, U_{ljm}(x) \quad a.e.\,, \quad f \in \mathcal{L}_2(\Omega)\,.$$

Remark. When solving the problem (**50.2**), one can formally take $\mu = l(l+1)$ and disregard the question whether or not there are other possible values of μ. The fact that the eigenfunctions (**50.4**) are proved to be a complete set in $\mathcal{L}_2(\Omega)$ implies that there are *no eigenvalues other than found with* $\mu = l(l+1)$. An eigenfunction U corresponding to $\lambda \neq \lambda_{lj}$ must be orthogonal to the functions U_{ljm} because the operator in (**50.1**) is hermitian. Therefore Fourier coefficients of U vanish. $\langle U, U_{ljm}\rangle$=0. Since the functions (**50.4**) form a complete set, any such $U \in \mathcal{L}_2(\Omega)$ must be zero almost everywhere and, hence, cannot be a nontrivial solution to the problem (**50.1**).

50.2. Bessel functions of half-integer order. Bessel functions of half-integer order can be expressed via elementary functions. For example, $J_{1/2}(z)$ and $J_{-1/2}(z)$ are given in (**46.2**) and (**46.3**). There are recurrence relations to obtains an explicit form of $J_{n+1/2}(z)$ for any integer n. Recall the recurrence relations to change the order of a Bessel function:

$$\begin{aligned} \frac{1}{z}\frac{d}{dz}\Big[z^{\nu}J_{\nu}(z)\Big] &= z^{\nu-1}J_{\nu-1}(z)\,,\\ \frac{1}{z}\frac{d}{dz}\Big[\frac{J_{\nu}(z)}{z^{\nu}}\Big] &= -\frac{J_{\nu+1}(z)}{z^{\nu+1}}\,. \end{aligned}$$

Applying them l times, one infers that

$$\begin{aligned} \left(\frac{1}{z}\frac{d}{dz}\right)^{l}\Big[z^{\nu}J_{\nu}(z)\Big] &= z^{\nu-l}J_{\nu-l}(z)\,,\\ \left(\frac{1}{z}\frac{d}{dz}\right)^{l}\Big[\frac{J_{\nu}(z)}{z^{\nu}}\Big] &= (-1)^{l}\frac{J_{\nu+l}(z)}{z^{\nu+l}}\,. \end{aligned}$$

By setting $\nu = 1/2$ and $\nu = -1/2$ in the first and second relations, respectively, an explicit form of Bessel functions of half-integer order is obtained from $J_{\pm 1/2}$:

$$J_{l+\frac{1}{2}}(z) = (-1)^l \sqrt{\frac{2}{\pi}}\, z^{l+\frac{1}{2}} \left(\frac{1}{z}\frac{d}{dz}\right)^l \frac{\sin z}{z}\,, \tag{50.5}$$

$$J_{-l-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\, z^{l+\frac{1}{2}} \left(\frac{1}{z}\frac{d}{dz}\right)^l \frac{\cos z}{z}\,, \tag{50.6}$$

where $l = 0, 1, \ldots$.

50.3. Eigenvalue problem for the Laplace operator in a spherical layer. Consider the eigenvalue problem for the Laplace operator in a three-dimensional spherical layer:

$$-\Delta u = \lambda u\,, \quad x \in \Omega\,, \quad \left(\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\right)\Big|_{\partial\Omega} = 0\,,$$
$$\Omega = \{x \in \mathbb{R}^3 \,|\, 0 < a < |x| < b\}\,,$$

where α and β are subject to the usual conditions to ensure hermiticity of the operator. The boundary $\partial\Omega$ is oriented outward:

$$\mathbf{n}\Big|_{|x|=a} = -\mathbf{e}_r\,, \qquad \mathbf{n}\Big|_{|x|=b} = \mathbf{e}_r\,,$$

where $\mathbf{e}_r = x/|x|$ in the unit vector corresponding to the radial variable of the spherical coordinate system (it is normal to any sphere $|x| =$ const). Therefore

$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{|x|=a} = -\frac{\partial u}{\partial r}\Big|_{r=a}\,, \qquad \frac{\partial u}{\partial \mathbf{n}}\Big|_{|x|=b} = \frac{\partial u}{\partial r}\Big|_{r=b}\,.$$

If α and β have constant values on $\partial\Omega$, then the problem can be solved by separation of variables in spherical coordinates.

Let $\alpha = \alpha_a$, $\beta = \beta_a$ be the values of α and β on the sphere $|x| = a$, and, constants α_b and β_b are their values on the sphere $|x| = b$. Put

$$u(r\sin\theta\cos\varphi,\, r\sin\theta\cos\varphi,\, r\cos\theta) = U(r,\theta,\varphi) = R(r)Y(\theta,\varphi)\,.$$

Writing the Laplace operator in spherical coordinates and separating the variables, the original problem leads to two eigenvalue problems, one of which is the problem (**50**.**2**) and the other is

$$\left(r^2 R'\right)' + (\lambda r^2 - \mu)R = 0\,, \quad \begin{cases} \alpha_a R(a) - \beta_a R'(a) = 0 \\ \alpha_b R(b) + \beta_b R'(b) \ = 0 \end{cases}. \tag{50.7}$$

If $\alpha \neq 0$ on $\partial\Omega$, then all eigenvalues are positive, $\lambda > 0$ (the case when $\lambda = 0$ is an eigenvalue is left to the reader to analyze as an exercise). The separation constant $\mu = l(l+1)$, $l = 0, 1, \ldots$, as shown in Section **50.1**. Just as in Section **50.1**, the problem (**50**.**7**) can be transformed to the eigenvalue problem for the Bessel operator with $\nu^2 = (l+\frac{1}{2})^2$by the substitution

$$R(r) = \frac{R_1(r)}{\sqrt{r}}\,, \quad R'(r) = \frac{R_1(r)}{\sqrt{r}} - \frac{R_1(r)}{2r\sqrt{r}}\,.$$

The boundary conditions for the functions $R_1(r)$ are deduced from the boundary conditions in (**50.7**):

$$(\mathbf{50.8}) \qquad \begin{array}{l} \left(\alpha_a + \frac{\beta_a}{2a}\right) R_1(a) - \beta_a R_1'(a) = 0 \\ \left(\alpha_b - \frac{\beta_b}{2b}\right) R_1(b) + \beta_b R_1'(b) \;\;= 0 \end{array}\,.$$

This is the eigenvalue problem for the Bessel equation in an interval $0 < a < r < b$, which is a particular case of a regular Sturm-Liouville problem discussed in Section **47.2** (no regularity condition at $r = 0$ as the origin is not in Ω); it is similar to the problem (**47.11**) and can be studied along the same lines. A general solution of the Bessel equation reads

$$R_1(r;\lambda) = C_1 J_{l+\frac{1}{2}}(\sqrt{\lambda}\, r) + C_2 J_{-l-\frac{1}{2}}(\sqrt{\lambda}\, r)\,.$$

The solution is substituted into (**50.8**) to obtain a linear homogeneous system of equations for the constants C_1 and C_2. The should have a non-trivial solution which is possible if and only if its determinant vanishes. For every $l = 0, 1, ...,$ the roots of the determinant are the eigenvalues $\lambda = \lambda_{lj}$, where the index $j = 1, 2, ...$ enumerates the roots. Recall that eigenvalues of the Sturm-Liouville operator are simple and form a countable set with no limit points (Section 8). For $\lambda = \lambda_{lj}$ the equations in (**50.8**) are linearly dependent and any of them can be used to determine the proportionality coefficient in $C_1 = \gamma_{lj} C_2$ so that the corresponding eigenfunction are

$$U_{ljm}(r,\theta,\varphi) = \frac{C_{ljm}}{\sqrt{r}} \left(\gamma_{lj} J_{l+\frac{1}{2}}(\sqrt{\lambda_{lj}}\, r) + J_{-l-\frac{1}{2}}(\sqrt{\lambda_{lj}}\, r)\right) Y_l^m(\theta,\varphi)\,,$$

where $l = 0, 1, ...,$ $j = 1, 2, ...,$ $m = 0, \pm 1, ..., \partial l$, and C_{ljm} are normalization constants defined by the condition

$$\|U_{ljm}\|^2 = \int_a^b \int_0^\pi \int_0^{2\pi} U_{ljm}^2(r,\theta,\varphi)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi = 1$$

The orthogonality of the eigenfunctions

$$\langle U_{ljm}, U_{l'j'm'}\rangle = \delta_{jj'}\delta_{ll'}\delta_{mm'}$$

follows from the orthogonality of the spherical harmonics (the factor $\delta_{ll'}\delta_{mm'}$) and from the orthogonality of eigenfunctions in a regular Sturm-Liouville problem (the factor $\delta_{jj'}$) corresponding to different eigenvalues λ_{lj} and $\lambda_{lj'}$ (Section **47.2**). The eigenfunction form a complete orthonormal set in $\mathcal{L}_2(\Omega)$ which follows from Theorem **??**, the completeness of the spherical harmonics Y_l^m in $\mathcal{L}(\mathbb{S}^2)$, and the completeness of eigenfunctions of the regular Sturm-Liouville operator in $\mathcal{L}_2((a,b);\sigma)$ where $p(r) = \sigma(r) = r > 0$ and $q(r) = (l+1/2)^2/r > 0$ in $[a,b]$ if $a > 0$.

50.4. More complex regions. If the region Ω is the image of a rectangle

$$(r,\theta,\varphi) \in (r_1,r_2) \times (\theta_1,\theta_2) \times (\varphi_1,\varphi_2)\,,$$

then the eigenvalue problem for the Laplace operator can be solved by separation of variables in spherical coordinates, provided the functions α and β have constant values on each smooth piece of the boundary $\partial\Omega$ which are coordinate surfaces of the spherical coordinates: two spheres $r = r_1$ and $r = r_2$, two cones $\theta = \theta_1$ and $\theta = \theta_2$, and two half-planes $\varphi = \varphi_1$ and $\varphi = \varphi_2$. A solution is sought in the form $R(r)Y(\theta,\varphi)$. The function $R(r)$ satisfies the eigenvalue problem (**50.7**) that can be reduced to the eigenvalue problem for the Bessel operator by the substitution $R(r) = R_1(r)/\sqrt{r}$. The difference is that the separation constant μ in (**50.7**) is no longer given by $\mu = l(l+1)$, $l = 0, 1, \ldots$, because the function $Y(\theta,\varphi)$ is not an eigenfunction of the Laplace-Beltrami operator on a sphere $\mathbb{S}^2$ (see (**50.2**) but in the part of the sphere cut out by the two cones and two half-planes (the part that corresponds to the rectangle $(\theta_1,\theta_2) \times (\varphi_1 \times \varphi_2)$).

Separating the variables in (**50.2**) by the substitution $Y(\theta,\varphi) = P(\cos\theta)\Phi(\varphi)$ into Eq. (**50.2**), the problem can be reduced to two problems (**48.2**) and (**48.3**), but in the latter equation $l(l+1)$ has to be replaced by the separation constant μ (which is still to be determined), *provided* the variables can also be separated in the boundary conditions. The vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit normals to the coordinate surfaces of θ and φ, that is, to cones and half-planes, respectively. Then by (**??**) for a cone $\theta = \theta_0$ and a half-plane $\varphi = \varphi_0$, the normal derivatives are

$$\frac{\partial U}{\partial \mathbf{n}}\Big|_{\theta=\theta_0} = \pm(\mathbf{e}_\theta, \operatorname{grad} U)\Big|_{\theta=\theta_0} = \pm\frac{R(r)}{r}\, Y'_\theta(\theta_0,\varphi)\,,$$
$$\frac{\partial U}{\partial \mathbf{n}}\Big|_{\varphi=\varphi_0} = \pm(\mathbf{e}_\theta, \operatorname{grad} U)\Big|_{\varphi=\varphi_0} = \pm\frac{R(r)}{r\sin\theta}\, Y'_\varphi(\theta,\varphi_0)\,,$$

where the two signs correspond to two possible orientations of the surfaces. Note the factors containing the Lamé coefficients of the spherical coordinates. The separation of variables does not work when these factors occur in a general boundary condition. In some particular cases when either Y or its normal derivative vanishes on the boundary of the rectangle $(\theta_1, \theta_2) \times (\varphi_1 \times \varphi_2))$, then the variables are easily separated.

For example, if Y is required to vanish on the boundary, then the problem (**50.2**) turns into the Sturm-Liouville problem (**47.10**). The eigenvalues are $\nu = \nu_m^2 \geq 0$, $\nu_m = (\pi m)/(\varphi_2 - \varphi_2)$, $m = 1, 2, \ldots$. The function $P(\cos\theta) = P(z)$ is a solution of the Sturm-Liouville problem for the operator,

$$L_m P \equiv \Big((1 - z^2)P'\Big)' + \frac{\nu_m^2}{1 - z^2}\, P = \mu P\,.$$

in an interval $-1 < z_1 < z < z_2 < 1$ where $z_1 = \cos\theta_2$ and $z_2 = \cos\theta_1$. This is a regular Sturm-Liouville problem for every ν_m. All the results of its general analysis apply. Just like the Bessel equation, the Legendre equation has two linearly independent solutions. This equation is a particular case of the so called *Papperitz equation* whose solutions can be obtained in terms of the *hypergeometric function*. If ν_m is a squared integer (e.g., $\varphi_2 - \varphi_2 = \pi/k$ where k is a positive integer), then solutions are *Legendre functions of the first and second kinds*. If, in addition, θ spans its full range ($\partial\Omega$ does not contains cones), then solutions regular at $z = \pm 1$ are given by associated Legendre functions corresponding to the squared integer ν_m^2. Given two linearly independent solutions, an equation for the eigenvalues μ is obtained by the general method developed for the Sturm-Liouville problem and illustrated with an example of the similar problem for the Bessel operator (**47.11**).

50.5. Exercises.

1. Let Ω be a ball $|x| < a$ in $\mathbb{R}^3$. Solve the eigenvalue problem

$$-\Delta u = \lambda u\,, \quad x \in \Omega\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0\,.$$

Find normalized eigenfunctions.

2. Let Ω be a spherical layer $0 < a < |x| < b$ in $\mathbb{R}^3$. Solve the eigenvalue problem

$$-\Delta u = \lambda u\,, \quad x \in \Omega\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{|x|=a} = 0\,, \quad u\Big|_{|x|=b} = 0\,.$$

3. Let Ω be a wedge of a ball

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 < a^2\,,\ x > 0\,,\ y > 0\}\,.$$

Solve the eigenvalue problem

$$-\Delta u = \lambda u\,, \quad u\Big|_{\partial\Omega} = 0\,.$$

Find normalized eigenfunctions.

4. Let Ω be a half-ball

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 < a^2\,,\ x > 0\}\,.$$

Solve the eigenvalue problem

$$-\Delta u = \lambda u\,, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0\,.$$

Answers.

1. Zero is a simple eigenvalue and the corresponding normalized eigenfunction is a constant function $U_0 = V_\Omega^{-1/2}$, where $V_\Omega = 4\pi a^3/3$ is the volume of Ω. The other eigenvalues are positive

$$\lambda = \lambda_{lj} = \frac{\mu_{lj}^2}{a^2}\,, \quad l = 0, 1, \ldots\,, \quad j = 1, 2, \ldots\,,$$

where for every l, μ_{lj}, $j = 1, 2, \ldots$, are positive roots of the equation

$$\mu J'_{l+\frac{1}{2}}(\mu) = \frac{1}{2} J_{l+\frac{1}{2}}(\mu)\,.$$

The corresponding eigenfunctions are

$$U_{ljm}(x) = C_{ljm} r^{-1/2} J_{l+\frac{1}{2}}\Big(\mu_{lj}\frac{r}{a}\Big) Y_l^m(\mathbf{n})\,, \quad x = r\mathbf{n}\,,$$

where $\mathbf{n}$ spans a unit sphere $\mathbb{S}^2$ and the normalization constants are defined by

$$|C_{ljm}|^{-2} = \frac{a^2}{2} \|Y_l^m\|^2 J_{l+\frac{1}{2}}^2(\mu_{lj}) \left(1 - \frac{l(l+1)}{\mu_{lj}^2}\right)\,.$$

2. All eigenvalues are positive $\lambda = \lambda_{lj} = \mu_{lj}^2$, $l = 0, 1, 2, \ldots$, and for every l, $\mu = \mu_{lj}$, $j = 1, 2, \ldots$, are positive (simple) roots of the equation

$$J_{l+\frac{1}{2}}(\mu b) H_{-l-\frac{1}{2}}(\mu a) = J_{-l-\frac{1}{2}}(\mu b) H_{l+\frac{1}{2}}(\mu a)$$

$$H_\nu(z) \equiv z J'_\nu(z) - \frac{1}{2} J_\nu(z)$$

The corresponding eigenfunctions are

$$U_{ljm}(x) = \frac{C_{ljm}}{\sqrt{r}} R_{lj}(r) Y_l^m(\mathbf{n})\,, \quad x = r\mathbf{n}\,,$$

$$R_{lj}(r) = H_{-l-\frac{1}{2}}(\mu_{lj}a) J_{l+\frac{1}{2}}(\mu_{lj}r) - H_{l+\frac{1}{2}}(\mu_{lj}a) J_{-l-\frac{1}{2}}(\mu_{lj}r)\,,$$

where C_{ljm} are normalization constants.

3. All eigenvalues are positive, $\lambda = \lambda_{lj} = \mu_{lj}^2$, $l = 0, 1, ...$, and for every l, $\mu = \mu_{lj}$, $j = 1, 2, ...$, are positive (simple) roots of the equation $J_{l+\frac{1}{2}}(\mu) = 0$. The corresponding eigenfunctions are

$$U_{ljm}(x) = \frac{C_{ljm}}{\sqrt{r}}\, J_{l+\frac{1}{2}}\Big(\mu_{lj}\frac{r}{a}\Big) P_l^{2m}(\cos\theta)\sin(2m\varphi)\,.$$

where $m = 1, 2, ..., M \leq l/2$ and the normalization constants are defined by

$$|C_{ljm}|^{-2} = \frac{\pi a^2}{4(2l+1)}\, |J'_{l+\frac{1}{2}}(\mu_{lj})|^2\, \frac{(l+2m)!}{(l-2m)!}\,.$$

4. The same eigenvalues as in in Problem 1 but the eigenfunctions corresponding to positive eigenvalues are

$$U_{ljm}(x) = C_{ljm} r^{-1/2} J_{l+\frac{1}{2}}\Big(\mu_{lj}\frac{r}{a}\Big) P_l^m(\cos\theta)\cos\Big(m(\varphi + \pi/2)\Big)\,,$$

where $j = 1, 2, ...$, $l = 0, 1, ...$, and $m = 0, 1, ..., l$; the normalization constants are defined by

$$|C_{ljm}|^{-2} = \frac{a^2\pi}{4l+2}\, \frac{(l+m)!}{(l-m)!}\, J^2_{l+\frac{1}{2}}(\mu_{lj}) \left(1 - \frac{l(l+1)}{\mu_{lj}^2}\right)\,.$$