Solution Manual

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25.6 Exercises

25.6.1

Recall that in order to determine if the critical point $\mathbf{r}_0$ is a local minimum, local maximum, or a saddle point we must find the roots λ_1, λ_2 to the characteristic polynomial (in this case a quadratic) given by

$$P_2(\lambda) = \det \begin{pmatrix} a-\lambda & c \\ c & b-\lambda \end{pmatrix} = \lambda^2 - (a+b)\lambda + (ab - c^2)$$

where

$$a = f''_{xx}(\mathbf{r}_0)\ ,\ b = f''_{yy}(\mathbf{r}_0)\ ,\ c = f''_{xy}(\mathbf{r}_0)$$

Determining the roots itself is not necessary. (See Corollary 25.1) Sufficient information comes from computing $D = ab - c^2$ and the sign of a (positive or negative).

(i) $D = (-3)(-2)-(2)^2 = 2$, $a < 0 \implies$ The point $(0,0)$ is a local maximum.
(ii) $D = (3)(2) - (2)^2 = 2$, $a > 0 \implies$ The point $(0,0)$ is a local minimum.
(iii) $D = (1)(2) - (2)^2 = -2 \implies$ The point $(0,0)$ is a saddle point.
(iv) $D = (2)(2) - (2)^2 = 0 \implies$ Inconclusive. Investigate higher order differentials.

25.6.2

From $f(x,y) = x^2 + (y-2)^2$, we get that $f'_x(x,y) = 2x$ and $f'_y(x,y) = 2(y-2)$. As a result, to make both f'_x and f'_y equal to 0, $x = 0$ and $y = 2$.Therefore, our only critical point is (0,2). We then find the roots of the characteristic polynomial to find the behavior of f(x,y) around this point. The characteristic polynomial is given by:

$$\det \begin{pmatrix} f''_{xx} - \lambda & f''_{xy} \\ f''_{yx} & f''_{yy} - \lambda \end{pmatrix} = 0$$

In this case, this equals:

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$$

From this it is easy to see that each solution is greater than 0, meaning that the critical point is a local minimum.

25.6.3

25.6.4

Because $f(x, y) = x^4 - 2x^2 - y^3 + 3y$ is a polynomial, the gradient

$$\vec{\nabla} f(x, y) = \langle 4x^3 - 4x, -3y^2 + 3\rangle$$

can never be undefined. We solve for points that cause the gradient to be equal to $\vec{0}$ by finding (x, y) that solve the following system of equations:

$$4x^3 - 4x = 0$$

$$-3y^2 + 3 = 0$$

Evidently any combination of $x = 0, 1, -1$ and $y = 1, -1$ form critical points. So computing $a = f''_{xx}(x, y) = 12x^2-4$, $b = f''_{yy}(x, y) = -6y$, and $c = f''_{xy}(x, y) = 0$ we analyze 6 cases by computing $D = ab - c^2$ and noting the sign of a:

$(0, 1) :\ D = (-4)(-6)-(0)^2 = 24,\ a < 0 \implies$ The point is a local maximum.
$(0, -1) :\ D = (-4)(6) - (0)^2 = -24 \implies$ The point is a saddle point.
$(1, 1) :\ D = (6)(-6) - (0)^2 = -36 \implies$ The point is a saddle point.
$(1, -1) :\ D = (6)(6) - (0)^2 = 36,\ a > 0 \implies$ The point is a local minimum.
$(-1, 1) :\ D = (6)(-6) - (0)^2 = -36 \implies$ The point is a saddle point.
$(-1, -1) :\ D = (6)(6)-(0)^2 = 36,\ a > 0 \implies$ The point is a local minimum.

25.6.5

25.6.6

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$

$$\vec{\nabla} f(x, y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle 2x - y - 2, 2y - x + 1\rangle$$

A point (x_0, y_0) is a critical point of $f(x, y)$ if $\vec{\nabla} f(x_0, y_0) = 0$ or is undefined. There are no points at which $\vec{\nabla} f$ is undefined. Setting $\vec{\nabla} f(x, y) = 0$ gives:

$$2x - y - 2 = 0$$

$$2y - x + 1 = 0$$

The only solution to this system of equations is $(1, 0)$. To determine whether this point is a local minimum, local maximum, or saddle point, let:

$$a = \frac{\partial^2 f}{\partial x^2}(1, 0) = 2$$

$$b = \frac{\partial^2 f}{\partial y^2}(1, 0) = 2$$

$$c = \frac{\partial^2 f}{\partial x \partial y}(1, 0) = -1$$

The characteristic equation is given by:

$$P_2(\lambda) = \begin{vmatrix} a-\lambda & c \\ c & b-\lambda \end{vmatrix}$$

$$= \lambda^2 - (a+b)\lambda + (ab - c^2) = \lambda^2 - 4\lambda + 3 = 0$$

The roots of this equation are $\lambda_1 = 1$, $\lambda_2 = 3$. Because $\lambda_1,\ \lambda_2 > 0$, the Second Derivative Test tells us that:

$$\boxed{(1,0) \text{ is a local minimum of } f(x,y)}$$

25.6.7

Because $f(x,y) = \frac{1}{3}x^3 + y^2 - x^2 - 3x - y + 1$ is a polynomial, the gradient

$$\vec{\nabla} f(x,y) = \langle x^2 - 2x - 3, 2y - 1\rangle$$

can never be undefined. We solve for points that cause the gradient to be equal to $\vec{0}$ by solving for points (x,y) that satisfy the following system of equations:

$$x^2 - 2x - 3 = 0$$

$$2y - 1 = 0$$

Evidently we have two critical points $(3, \frac{1}{2})$ and $(-1, \frac{1}{2})$. Proceed for each case by computing $a = f''_{xx}(x,y) = 2x-2$, $b = f''_{yy}(x,y) = 2$, and $c = f''_{xy}(x,y) = 0$ and then computing $D = ab - c^2$ and noting the sign of a:

$(3, \frac{1}{2}) :\ D = (4)(2) - (0)^2 = 8,\ a > 0 \implies$ The point is a local minimum.

$(-1, \frac{1}{2}) :\ D = (-4)(2) - (0)^2 = -8 \implies$ The point is a saddle point.

25.6.8

From $f(x,y) = x^3 + y^3 - 3xy$ we get that

$$f'_x = 3x^2 - 3y \qquad\qquad f'_y = 3y^2 - 3x$$

To solve for each of the critical points, we set each f'_x and f'_y equal to 0. Rearranging f'_x gives us $3(x^2 - y) = 0$ and that $x^2 = y$. A similar process allows us to manipulate f'_y to get $y^2 = x$. Substituting the first equation into the second, we get that:

$$(x^2)^2 = x^4 = x$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

This leads us to two scenarios for the critical points, one where x=0 and another where x=1. Since $x^2 = y$, the critical points are (0,0) and (1,1). To find the behavior of the function of the critical points, you evaluate the roots of:

$$\begin{vmatrix} 6x-\lambda & -3 \\ -3 & 6y-\lambda \end{vmatrix} = 0$$

Evaluating this at (0,0) yields the equation $\lambda^2 - 9 = 0 \implies$ a saddle point. At (1,1) the characteristic polynomial is $(\lambda - 3)(\lambda - 9) = 0 \implies$ a local min.

25.6.9

$$f(x,y) = xy + \frac{50}{x} + \frac{20}{y} \ ; \ x, y > 0$$

$$\vec{\nabla} f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle y - \frac{50}{x^2}, x - \frac{20}{y^2} \rangle$$

A point (x_0, y_0) is a critical point of $f(x,y)$ if $\vec{\nabla} f(x_0, y_0) = 0$ or is undefined. The gradient of $f(x,y)$ is undefined at $(0,0)$, but that point is not in the domain. Setting $\vec{\nabla} f(x,y) = 0$ gives:

$$y = \frac{50}{x^2}$$

$$x = \frac{20}{y^2}$$

The only solution to this system of equations is $(5,2)$. Therefore, the only critical point is $(5,2)$. To determine whether this point is a local minimum, local maximum, or saddle point, let:

$$a = \frac{\partial^2 f}{\partial x^2}(5,2) = \frac{100}{x^3} = \frac{100}{5^3} = \frac{4}{5}$$

$$b = \frac{\partial^2 f}{\partial y^2}(5,2) = \frac{40}{y^3} = \frac{40}{2^3} = 5$$

$$c = \frac{\partial^2 f}{\partial x \partial y}(5,2) = 1$$

The characteristic equation is given by:

$$P_2(\lambda) = \begin{vmatrix} a - \lambda & c \\ c & b - \lambda \end{vmatrix} = \lambda^2 - (a+b)\lambda + (ab - c^2) = 0$$

Rather than plugging in the values of a, b, c and solving, which may be difficult, examine the equation. By Vieta's formulas:

$$\lambda_1 \lambda_2 = ab - c^2 = 3 \tag{1}$$

$$\lambda_1 + \lambda_2 = a + b = \frac{29}{5} \tag{2}$$

(1) implies that the roots are both positive or both negative. (2) concludes that both roots are positive. This information is sufficient to conclude that

$$\boxed{(5,2) \text{ is a local minimum of } f(x,y)}$$

25.6.10

On all points outside of the origin $f(x, y) = x^2 + y^2 + \frac{1}{x^2y^2}$ is defined and the gradient

$$\vec{\nabla} f(x, y) = \langle 2x - \frac{2}{x^3y^2}, 2y - \frac{2}{x^2y^3}\rangle$$

is as well. Like usual find critical points for which the gradient is equal to the zero vector, and those are any combination of $x = \pm 1$ and $y = \pm 1$ (4 points).

Proceed for each point by computing

$$a = f''_{xx}(x, y) = 2 + \frac{6}{y^2x^4}$$

$$b = f''_{yy}(x, y) = 2 + \frac{6}{y^4x^2}$$

$$c = f''_{xy}(x, y) = \frac{4}{x^3y^3}$$

and then computing $D = ab - c^2$ and noting the sign of a. Note that at all 4 points, $a, b = 8$ due to the product of only even powers of x, y and $x, y = \pm 1$. Similarly $c = 4$ because it contains the product of odd powers of x, y which forces it to be positive, and $x, y = \pm 1$ it attains that value. Hence $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$ are all local minima ($ab - c^2 = 48$ and $a > 0$).

25.6.11

f(x,y)=cos(x)cos(y)

$$\nabla f(x, y) =< -\sin x \cos y, -\cos x \sin y >$$

Since sin(x) and cos(x) cannot be 0 at the same time, the two conditions for a critical point are:

$$\sin x = \sin y = 0$$

or

$$\cos x = \cos y = 0$$

As a result, the critical points are $(\pi m, \pi n)$ or $(\frac{\pi}{2}(2m + 1), \frac{\pi}{2}(2n + 1))$, where $m, n \in \mathbb{Z}$. The characteristic polynomial is given by

$$\begin{vmatrix} -\cos x \cos y - \lambda & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y - \lambda \end{vmatrix} = 0$$

One can see that depending on the critical point being chosen, each of $-\cos x \cos y$ and $\sin x \sin y$ are either equal to 1, -1, or 0. Specifically, we have 3 scenarios for the critical points.

1. (x,y) = $(\pi m, \pi n)$, m+n are odd

2. (x,y) = $(\pi m, \pi n)$, m+n are even

3. (x,y) = $(\frac{\pi}{2}(2m+1), \frac{\pi}{2}(2n+1))$

For number 3, the characteristic polynomial gives $\lambda^2 = \sin x \sin y$ which gives a saddle point due to the roots having mixed signs. Meanwhile, for 1, both λs are equal to 1, meaning its a local min. For 2, each are -1, giving us a local max.

25.6.12

$$f(x,y) = \cos x + y^2$$

$$\vec{\nabla} f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle -\sin x, 2y \rangle$$

A point (x_0, y_0) is a critical point of $f(x,y)$ if $\vec{\nabla} f(x_0, y_0) = 0$ or is undefined. There are no points at which $\vec{\nabla} f$ is undefined. Setting $\vec{\nabla} f(x,y) = 0$ gives:

$$-\sin x = 0$$

$$2y = 0$$

The solution to this system of equations is $(k\pi, 0)$ for all integers k. To determine whether each of these points is a local minimum, local maximum, or saddle point, let:

$$a = \frac{\partial^2 f}{\partial x^2}(k\pi, 0) = -\cos(k\pi) = \begin{cases} 1 & k \text{ odd} \\ -1 & k \text{ even} \end{cases}$$

$$b = \frac{\partial^2 f}{\partial y^2}(k\pi, 0) = 2$$

$$c = \frac{\partial^2 f}{\partial x \partial y}(k\pi, 0) = 0$$

$$D = ab - c^2 = \begin{cases} 2 & k \text{ odd} \\ -2 & k \text{ even} \end{cases}$$

So, our solution splits into two cases, where k is either odd or even. When k is odd, $D > 0$ and $a > 0$, which is sufficient to conclude that these points are local minima. When k is even, $D < 0$, which is sufficient to conclude that these points are saddle points.

$$\boxed{\text{Local minima at } (k\pi, 0) \text{ for odd k ; Saddle points at } (k\pi, 0) \text{ for even k}}$$

25.6.13

The function $f(x,y) = y^3 + 6xy + 8x^3$ is a polynomial so its gradient

$$\vec{\nabla} f(x,y) = \langle 6y + 24x^2, 3y^2 + 6x \rangle$$

will not be undefined. We find critical points by finding points that satisfy $\vec{\nabla} f(x,y) = \vec{0}$, or alternatively, the following system of equations:

$$6y + 24x^2 = 0$$

$$3y^2 + 6x = 0$$

From the second equation, deduce that all critical points lie on the line

$$x = -\frac{1}{2}y^2$$

and substitute this for x into the first equation to find the equation (either form):

$$6y + 6y^4 = 0 \leftrightarrow y(y^3 + 1) = 0$$

There are only two real solutions, $y = -1$ and $y = 0$. Substituting these back into $3y^2 + 6x = 0$ it is apparent that the critical points occur at $(0,0)$ and $(-\frac{1}{2}, -1)$. Then compute point the following: $a = f''_{xx}(x,y) = 48x$, $b = f''_{yy}(x,y) = 6y$, and $c = f''_{xy}(x,y) = 6$ and then computing $D = ab - c^2$ and noting the sign of a:

$(0,0) : D = (0)(0) - (6)^2 = -36 \implies$ The origin is a saddle point.

$(-\frac{1}{2}, -1) : D = (-24)(-6) - (6)^2 = 108$, $a < 0 \implies$ This point is a local maximum.

25.6.14

f(x,y)=$x^3 - 2xy + y^2$

$$f'_x = 3x^2 - 2y = 0$$

$$f'_y = -2x + 2y = 0$$

From f'_y, we get that 2x=2y, or x=y. Using this fact to substitute into f'_x, $3x^2 - 2x = 0$ or $x(3x - 2) = 0 \implies x = y = 0$ or $x = y = \frac{2}{3}$. The characteristic polynomial is given by:

$$\begin{vmatrix} 6x - \lambda & -2 \\ -2 & 2 - \lambda \end{vmatrix} = 0$$

Testing the critical point (0,0) into the characteristic polynomial gives us that

$$\lambda^2 - 2\lambda - 4 = 0$$

Since the constant term is negative, we know that the roots of the polynomial have mixed signs, meaning (0,0) is a saddle point. Solving the characteristic polynomial for $(\frac{2}{3}, \frac{2}{3})$ gives us $\lambda = 3 \pm \sqrt{5} > 0 \implies$ local min.

25.6.15

$$f(x,y) = xy(1-x-y) = xy - x^2y - xy^2$$

$$\vec{\nabla} f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle y - y^2 - 2xy, x - x^2 - 2xy \rangle = \langle y(1-y-2x), x(1-x-2y) \rangle$$

A point (x_0, y_0) is a critical point of $f(x,y)$ if $\vec{\nabla} f(x_0, y_0) = 0$ or is undefined. There are no points at which $\vec{\nabla} f$ is undefined. Setting $\vec{\nabla} f(x,y) = 0$ gives:

$$y(1-y-2x) = 0$$

$$x(1-x-2y) = 0$$

This system has many solutions: $(0,0)$, $(1,0)$, $(0,1)$, $(\frac{1}{3}, \frac{1}{3})$. To determine whether this point is a local minimum, local maximum, or saddle point, let:

$$a = \frac{\partial^2 f}{\partial x^2} = -2y = \begin{cases} 0 & (0,0) \\ 0 & (1,0) \\ -2 & (0,1) \\ -\frac{2}{3} & (\frac{1}{3}, \frac{1}{3}) \end{cases}$$

$$b = \frac{\partial^2 f}{\partial y^2} = -2x = \begin{cases} 0 & (0,0) \\ -2 & (1,0) \\ 0 & (0,1) \\ -\frac{2}{3} & (\frac{1}{3}, \frac{1}{3}) \end{cases}$$

$$c = \frac{\partial^2 f}{\partial x \partial y} = 1 - 2x - 2y = \begin{cases} 1 & (0,0) \\ -1 & (1,0) \\ -1 & (0,1) \\ -\frac{1}{3} & (\frac{1}{3}, \frac{1}{3}) \end{cases}$$

$$D = ab - c^2 = \begin{cases} -1 & (0,0) \\ -1 & (1,0) \\ -1 & (0,1) \\ \frac{1}{3} & (\frac{1}{3}, \frac{1}{3}) \end{cases}$$

At $(0,0)$, $(1,0)$, and $(0,1)$, $D < 0$, which is sufficient to conclude that these are saddle points. At $(\frac{1}{3}, \frac{1}{3})$,$D > 0$ and $a < 0$, which is sufficient to conclude that this point is a local maximum.

$$\boxed{\text{Saddle Points at: } (0,0), (1,0), (0,1) \text{ ; Local maximum at } (\frac{1}{3}, \frac{1}{3})}$$

25.6.16

The function $f(x, y) = x\cos(y)$ is a product of a monomial and the cosine, which is a smooth function whose gradient

$$\vec{\nabla} f(x, y) = \langle \cos(y), -x\sin(y)\rangle$$

will not be undefined. We can find all points that cause the gradient to vanish by observation. From the first component it is clear that all y values that would make the gradient vanish are $y = \frac{k\pi}{2}$ for any integer k. Then notice that those values of y cause the second component of the gradient to become $\pm x$, which of course is only equal to 0 when $x = 0$. So all critical points come in the form $(0, \frac{k\pi}{2})$.

Then at all critical points $a = f''_{xx}(x, y) = 0$, $b = f''_{yy}(x, y) = -x\cos(y)$, and $c = f''_{xy}(x, y) = -\sin(y)$ and then computing $D = ab - c^2$ and noting the sign of a. However, because $f''_{xx}(x, y)$ is zero, and $c^2 = \sin^2(y)$, D will always be negative at all critical points. So all critical points $(0, \frac{k\pi}{2})$ are saddle points.

25.6.17

25.6.18

25.6.19

The function $f(x, y) = (5x + 7y - 25)e^{-x^2-xy-y^2}$ is a product of a polynomial and an exponential which is nice and smooth. The gradient

$$\vec{\nabla} f(x, y) = \langle e^{-x^2-xy-y^2}(5 + (-2x - y)(5x + 7y - 25))$$
$$, e^{-x^2-xy-y^2}(7 + (-2y - x)(5x + 7y - 25))\rangle$$

will also be nice and smooth and more importantly never undefined. To find critical points it is sufficient to solve the following system of equations since the exponential is always nonzero:

$$(5 + (-2x - y)(5x + 7y - 25)) = -10x^2 - 19xy + 50x - 7y^2 + 25y + 5 = 0$$

$$(7 + (-2y - x)(5x + 7y - 25)) = -5x^2 - 17xy + 25x - 14y^2 + 50y + 7 = 0$$

INCOMPLETE

25.6.20

25.6.21

25.6.22

The function $f(x, y) = \frac{1}{3}y^3 + xy + \frac{8}{3}x^3$ is a polynomial so the gradient

$$\vec{\nabla} f(x, y) = \langle 8x^2 + y, y^2 + x\rangle$$

cannot be undefined anywhere. To solve for points where the gradient will vanish use the fact that since $f'_y = y^2 + x = 0$ then $x = -y^2$ and we can substitute this back into $f'_x = 8x^2 + y = 0$ to find that we have critical points at y values that solve:

$$8y^4 + y = 0 \leftrightarrow y((2y)^3 + 1^3) = 0$$

$$\text{sum of cubes} \rightarrow y(2y+1)(4y^2 - 2y + 1) = 0$$

The real roots are $y = 0, -\frac{1}{2}$. Plug these back into the equation $x = -y^2$ to find that the critical points are $(0,0)$ and $(-\frac{1}{4}, -\frac{1}{2})$.

Then for each point compute $a = f''_{xx}(x,y) = 16x$, $b = f''_{yy}(x,y) = 2y$, and $c = f''_{xy}(x,y) = 1$ and then computing $D = ab - c^2$ and noting the sign of a:

$(0,0): D = (0)(0) - (1)^2 = -1 \implies$ This point is a saddle point.

$(-\frac{1}{4}, -\frac{1}{2}): D = (-4)(-1) - (1)^2 = 3,\ a < 0 \implies$ This point is a local maximum.

25.6.23

25.6.24

25.6.25

The function $f(x,y) = x + y + \sin(x)\sin(y)$ is a polynomial plus the product of two sinusoids so it is a smooth function where its gradient

$$\vec{\nabla} f(x,y) = \langle 1 + \cos(x)\sin(y), 1 + \sin(x)\cos(y) \rangle$$

will not be undefined anywhere. We want to find critical points where:

$$1 + \cos(x)\sin(y) = 0$$

$$1 + \sin(x)\cos(y) = 0$$

We may try with the first equation and find that we would like the cosine term to be equal to one and likewise the sine term as well, so a natural guess may be to give critical points as $(\pi + 2\pi j, \frac{-\pi}{2} + 2\pi k)$ for integers j, k. However this automatically fails satisfying the second equation, and likewise any points that satisfy the second equation fail the first one. We may instead directly show this by adding both equations together to form the following, using the fact that $\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$:

$$2 + \cos(x)\sin(y) + \sin(x)\cos(y) = 0 \leftrightarrow 2 + \sin(x+y) = 0$$

Evidently since the sine function is bounded below by -1 there are no critical points. Hence there are no local extrema.

25.6.26

25.6.27

25.6.28

Give $F(x,y,z) = x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$ (a spherical level set) and use the implicit differentiation equations

$$z'_x = \frac{-F'_x}{F'_z},\ z'_y = \frac{-F'_y}{F'_z}$$

to find the gradient of such a surface $z(x,y)$, which will be

$$\langle \frac{-(2x-2)}{2z-4}, \frac{-(2y+2)}{2z-4} \rangle$$

and will be valid and not undefined where $2z-4 \neq 0$ as per the Implicit Function Theorem. This essentially then boils down to finding points (x,y) that solve the following system:

$$2x - 2 = 0$$

$$2y + 2 = 0$$

Since the equations are functions of one variable each, we can simply choose the critical point to be $(1,-1)$. Then also note that we will need a z value, so evaluate $F(1,-1,z) = 0$ and solve for z:

$$F(1,-1,z) = 1 + 1 + z^2 - 2 - 2 - 4z - 10 = 0$$

$$\rightarrow z^2 - 4z - 12 = 0 \implies z = 2 \pm 4$$

So really we have two critical points, $(1,-1,6)$ and $(1,-1,-2)$. At each point we compute $a = f''_{xx}(x,y) = \frac{-2}{2z-4}$, $b = f''_{yy}(x,y) = \frac{-2}{2z-4}$, and $c = f''_{xy}(x,y) = 0$ and then compute $D = ab - c^2$ and noting the sign of a:

$(1,-1,6)$: $D = (-\frac{1}{4})(-\frac{1}{4}) - (0)^2 = \frac{1}{16}$, $a < 0 \implies$ The point is a local maximum.

$(1,-1,-2)$: $D = (\frac{1}{4})(\frac{1}{4}) - (0)^2 = \frac{1}{16}$, $a > 0 \implies$ The point is a local minimum.

An important observation is the geometrical significance of both critical points as it relates to the implicitly defined $z(z,y)$. Consider where the implicitly defined $z(x,y)$ *cannot* be determined - that is at $z = 2$, as there is where F'_z vanishes. Then also consider that since $F(x,y,z) = 0$ was the level set of a sphere, we really have two distinct surfaces $z(x,y)$, which are the hemispheres of the sphere remove the circle that forms at the intersection of $z = 2$ and the sphere given by $F(x,y,z) = 0$.

Geometrically each critical point refers to the peak or the trough of each respective hemisphere, which is a neat observation and makes sense geometrically. Being spherical also explains why at each critical point a and b have the same values.

25.6.29

25.6.30

25.6.31

Rearrange the equation of the plane given into the function $z(x, y) = 4 - x + y$. Using this, we can make a function $d(x, y)$ that gives the square of the distance between generic points on the plane $(x, y, 4 - x + y)$ and $(1, 2, 3)$. Define it like so:

$$d(x, y) = (1 - x)^2 + (2 - y)^2 + (x - y - 1)^2$$

The reasoning behind using the square of the distance is because it makes computation of the partial derivatives easier. It is justified because distance is a non-negative quantity, and so squaring such a function would not change the location of the local minumum we seek to find.

Continuing, we have the gradient as

$$\langle -2(1 - x) + 2(x - y - 1), -2(2 - y) - 2(x - y - 1)\rangle$$
$$= \langle 4x - 2y - 4, -2x + 4y - 2\rangle$$

which will never be undefined. We seek to solve the following system of equations to find the critical point:

$$4x - 2y - 4 = 0$$
$$-2x + 4y - 2 = 0$$

Using the bottom equation, find that $x = 2y - 1$, and substitute this into the top equation to get $6y - 8 = 0$. Evidently $y = \frac{4}{3}$, and so $x = \frac{5}{3}$.

Then compute $a = f''_{xx}(x, y) = 4$, $b = f''_{yy}(x, y) = 4$, and $c = f''_{xy}(x, y) = -2$ and then compute $D = ab - c^2$ and noting the sign of a:

$(\frac{5}{3}, \frac{4}{3}) :$ $D = (4)(4) - (-2)^2 = 12$, $a > 0 \implies$ This point is a local minimum.

Using the definition of the plane the point that minimizes the distance is $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$.

An alternative solution requires no minimization at all but to simply find a real value s such that the vector $\langle 1, 2, 3\rangle + s\langle 1, -1, 1\rangle$ satisfies the equation of the plane ($s = \frac{2}{3}$). This takes advantage of the fact that the line segment between $(1, 2, 3)$ and the point that minimizes the distance to the plane is at a right angle to the plane itself, so we can use the normal vector that determines the plane to find the minimizing point.

25.6.32

26.6 Exercises

26.6.1

The function $f(x, y, z) = x^2 + y^2 + z^2 + 2x + 4y - 8z$ has the gradient

$$\vec{\nabla} f(x, y, z) = \langle 2x + 2, 2y + 4, 2z - 8\rangle$$

which implies a critical point are located at $(-1,-2,4)$. Compute the second derivatives $f''_{xx}=2$, $f''_{yy}=2$, $f''_{zz}=2$, $f''_{xy}=0$, $f''_{yz}=0$, $f''_{zx}=0$ at the point. Then use the characteristic polynomial for the second derivative matrix given by

$$P_3(\lambda)=\det\begin{pmatrix}2-\lambda & 0 & 0\\ 0 & 2-\lambda & 0\\ 0 & 0 & 2-\lambda\end{pmatrix}=(2-\lambda)^3=0$$

whose roots are obviously $\lambda=2$ with multiplicity 3. All of these roots are positive so it is apparent that we have a local minimum at $(-1,-2,4)$. Interestingly, this point is the center of all of the level sets (spheres) that this function takes on, so this is intuitively true.

26.6.2

$$f(x,y)=x^2+y^3+z^2+12xy-2z$$
$$\nabla f(x,y)=\langle 2x+12y,3y^2+12x,2z-2\rangle=\vec{0}$$

From the f'_z term we get that z=1. From the f'_x term we get that x=-6y. Substituting that into f'_y yields $y^2-24y=y(y-24)=0 \implies (0,0,1)$ or (-144,24,1) are the critical points. The characteristic polynomial is defined by:

$$\begin{vmatrix}2-\lambda & 12 & 0\\ 12 & 6y-\lambda & 0\\ 0 & 0 & 2-\lambda\end{vmatrix}=0$$

To make this easier to compute, one can use the last row of the matrix to take the determinant from, since it is an alternating row from the top, giving us

$$(2-\lambda)\begin{vmatrix}2-\lambda & 12\\ 12 & 6y-\lambda\end{vmatrix}=0$$

At (0,0), the polynomial becomes $(2-\lambda)(\lambda^2-2\lambda-144)$, which based off the sign of the constant term of the quadratic is a saddle point. At (-144,24,1), the polynomial becomes $(2-\lambda)(\lambda^2-146\lambda+144)$ and using Descartes' rule of signs we can quickly gather that the roots of the polynomial are all greater than 0, meaning (-144, 24, 1) is a local min.

26.6.3

$$f(x,y,z)=x^2+y^3-z^2+12xy+2z$$
$$\vec{\nabla}f(x,y,z)=\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\rangle=\langle 2x+12y,3y^2+12x,-2z+2\rangle$$

A point (x_0,y_0) is a critical point of $f(x,y)$ if $\vec{\nabla}f(x_0,y_0)=0$ or is undefined. There are no points at which $\vec{\nabla}f(x,y,z)$ is undefined. Setting $\vec{\nabla}f(x,y)=0$ gives:

$$2x+12y=0$$

$$3y^2 + 12x = 0$$
$$-2z + 2 = 0$$

The solutions to this system are $(0,0,1)$ and $(-144,24,1)$. To determine whether these points are local minima, local maxima, or saddle points, calculate the second-order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial y^2} = 6y = \begin{cases} 0 & (0,0,1) \\ 144 & (-144,24,1) \end{cases}$$
$$\frac{\partial^2 f}{\partial z^2} = -2$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2$$
$$\frac{\partial^2 f}{\partial y \partial z} = 0$$
$$\frac{\partial^2 f}{\partial y \partial z} = 0$$

The characteristic equation for $(0,0,1)$ is given by:

$$P_3(\lambda) = \begin{vmatrix} 2-\lambda & 12 & 0 \\ 12 & -\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)(\lambda^2 - 2\lambda - 144) = 0$$

One of the roots of this equation is $-\lambda_2$. Rather than factoring the quadratic, use Vieta's formulas to show that:

$$\lambda_2 \lambda_3 = -144 \tag{1}$$
$$\lambda_2 + \lambda_3 = 2 \tag{2}$$

(1) implies that two of the roots have opposite signs. This information is sufficient to conclude that $(0,0,1)$ is a saddle point.
The characteristic equation for $(-144,24,1)$ is given by:

$$P_3(\lambda) = \begin{vmatrix} 2-\lambda & 12 & 0 \\ 12 & 144-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$
$$= (-2-\lambda)(\lambda^2 - 146\lambda + 144) = 0$$

Again, $\lambda_1 = -2$ is one root of the equation. Vieta's formulas show that:

$$\lambda_2 \lambda_3 = 144 \tag{1}$$
$$\lambda_2 + \lambda_3 = 146 \tag{2}$$

Together, (1) and (2) imply that $\lambda_1, \lambda_2 > 0$. They do not have the same sign as -2, which is sufficient to conclude that $(-144,24,1)$ is also a saddle point.

$$\boxed{\text{Saddle points at } (0,0,1) \text{ and } (-144,24,1)}$$

26.6.4

The function $f(x, y, z) = \sin(x) + z\sin(y)$ has the gradient

$$\langle \cos(x), z\cos(y), \sin(y) \rangle$$

which tells us that critical points are located at $(\frac{\pi}{2} + n\pi, m\pi, 0)$ for integers n, m. Compute the second derivatives $f''_{xx} = -\sin(x)$, $f''_{yy} = -z\sin(y)$, $f''_{zz} = 0$, $f''_{xy} = 0$, $f''_{yz} = \cos(y)$, $f''_{zx} = 0$ at those points. Then use the characteristic polynomial for the second derivative matrix given by

$$P_3(\lambda) = \det \begin{pmatrix} -\sin\left(\frac{\pi}{2} + n\pi\right) - \lambda & 0 & 0 \\ 0 & -(0)\sin(m\pi) - \lambda & \cos(m\pi) \\ 0 & \cos(m\pi) & 0 - \lambda \end{pmatrix}$$

$$= (\mp 1 - \lambda)(\lambda^2 - 1) = 0$$

From the $(\lambda^2 - 1)$ term it is apparent that two roots will not have the same sign, so automatically all points $(\frac{\pi}{2} + n\pi, m\pi, 0)$ are saddle points.

26.6.5

$$f(x, y) = x^2 + \frac{5}{3}y^3 + z^2 - 2xy - 4zy$$

$$\nabla f(x, y) = \langle 2x - 2y, 5y^2 - 2x - 4z, 2z - 4y \rangle = \vec{0}$$

From f'_x we can get that x=y and from f'_z that z=2y. If one were to make this substitutions into f'_y, one would see that $5y^2 - 2y - 8y = 5y(y - 2) = 0$, meaning the critical points occur at y=0 and y=2. The characteristic polynomial is:

$$\begin{vmatrix} 2 - \lambda & -2 & 0 \\ -2 & 10y - \lambda & -4 \\ 0 & -4 & 2 - \lambda \end{vmatrix} = 0$$

Simplifying this gives

$$(2 - \lambda)[\lambda^2 - 2(2 + 10y)\lambda + 20(y - 1)] = 0$$

At y=0, the constant term of the quadratic is negative, meaning the roots are mixed $\implies$ a saddle at (0,0,0). At y=2, the quadratic gives 2 positive solutions by Descartes' rule of signs. That along with the factor $(2 - \lambda)$ $\implies$ (2,2,4) is a local min.

26.6.6

26.6.7

26.6.8

26.6.9

26.6.10

26.6.11

26.6.12

26.6.13

$$f(x,y) = x^2 + xy^2 + y^4$$
$$df(x,y) = (2x + y^2)dx + (2xy + 4y^3)dy$$

$df(x,y)$ vanishes at $(0,0)$, so $(0,0)$ is a critical point of $f(x,y)$.
In this form, the behavior of $f(x,y)$ is unclear because the xy^2 term can be either positive or negative in a neighborhood of $(0,0)$. Completing the square gives:

$$f(x,y) = (x + \frac{1}{2}y)^2 + \frac{3}{4}y^2$$

In all neighborhoods of $(0,0)$, both terms are positive, so

$$\boxed{(0,0) \text{ is a local minimum of } f(x,y)}$$

26.6.14

For $f(x,y) = \ln\big(1 + x^2y^2\big)$ it is apparent that $df(0,0)$ vanishes, so the origin is a critical point. Put $u = x^2y^2$, and then use the known Maclaurin polynomial for $\ln(1+u)$

$$T_n = \sum_1^n (-1)^{n+1}\frac{u^n}{n} = u + O(u^2)$$

to investigate $f(x,y) - f(0,0)$:

$$f(x,y) - f(0,0) = f(x,y) = u + O(u^2)$$

It is evident that since $u = x^2y^2$ is always positive for $(x,y) \neq (0,0)$, in any neighborhood around the origin the function is positive and so origin is a local minimum.

26.6.15

$$f(x,y) = x^2 \ln 1 + x^2y^2$$
$$\nabla f(x,y) = \langle 2x \ln 1 + x^2y^2 + \frac{2x^3y^2}{1+x^2y^2}, \frac{2x^4y}{1+x^2y^2} \rangle$$

One can verify that $\nabla f(0,0) = \vec{0}$. Next, to see the behavior of the function near (0,0), one can replace $\ln 1 + x^2y^2$ in f(x,y) with its Taylor polynomial with the substitution that $u = x^2y^2$. When we do so, we get that

$$f(x,y) = x^2[u + O(u^2)]$$

Since the sign of the function is determined by x^2 and u, and both are strictly positive around the origin, (0,0) is a local min.

26.6.16

$$f(x,y) = xy(\cos\bigl(x^2y\bigr) - 1)$$

$df(0,0)$ vanishes, so $(0,0)$ is a critical point of $f(x,y)$.
Let $u = x^2y$:

$$\begin{aligned} f(x,y) - f(0,0) = f(x,y) &= xy(\cos u - 1) \\ &= xy((1 - u^2/2 + O(u^4)) - 1) \\ &= xy(-u^2/2 + O(u^4)) \\ &= -\frac{x^5y^3}{2} + O(x^9y^5) \end{aligned}$$

The $-x^5y^3/2$ term can take on either positive or negative values in all neighborhoods of $(0,0)$, so $(0,0)$ must not be an extremum of $f(x,y)$.

$$\boxed{(0,0) \text{ is not an extremum of } f(x,y)}$$

26.6.17

For $f(x,y) = (x^2 + 2y^2)\arctan(x+y)$ it is apparent that $df(0,0)$ vanishes so the origin is a critical point. Put $u = x + y$, and then use the known Maclaurin polynomial for $\arctan u$

$$T_n = \sum_0^n (-1)^n \frac{u^{2n+1}}{2n+1} = u + O(u^3)$$

to investigate $f(x,y) - f(0,0)$:

$$f(x,y) - f(0,0) = f(x,y) = (x^2 + 2y^2)(u + O(u^3)) = (x^2 + 2y^2)(x+y) + O(u^5)$$

It is evident that since in a neighborhood around $(0,0)$ the quantity $u = x+y$ can take on both positive and negative values, the first term will as well, thus there cannot be a local extremum at the origin.

26.6.18

$$f(x, y) = \cos(e^{xy} - 1)$$
$$\nabla f(x, y) = \langle -\sin(e^{xy} - 1) \cdot ye^{xy}, -\sin(e^{xy} - 1) \cdot xe^{xy} \rangle$$

It is evident that the origin is a critical point of the function. To analyze the behavior of the function near the origin, replace the cosine function with its Taylor polynomial, using the substitution $u = e^{xy} - 1$ to find f(x,y)-f(0,0).

$$f(x, y) = \cos(e^{xy} - 1) = \cos(u) = 1 - \frac{u^2}{2} + O(u^4)$$

$$f(x, y) - f(0, 0) = -\frac{u^2}{2} + O(u^4)$$

The sign of f(x,y)-f(0,0) is dependent on the term $-\frac{u^2}{2}$ which is negative for all values close to the origin, yielding a local max.

26.6.19

$$f(x, y) = \ln\big(y^2 \sin^2 x + 1\big)$$

$df(0, 0)$ vanishes, so $(0, 0)$ is a critical point of $f(x, y)$.
Let $u = y^2 \sin^2 x$:

$$\begin{aligned} f(x, y) - f(0, 0) = f(x, y) &= \ln(u + 1) \\ &= u - u^2/2 + O(u^3) \\ &= u(1 - u/2 + O(u^2)) \end{aligned}$$

$u = y^2 \sin^2 x$ is positive for all $(x, y) \neq (0, 0)$ and $1 - u/2 + O(u^2)$ is positive in sufficiently small neighborhoods of $(0, 0)$. Because $f(x, y) - f(0, 0)$ is positive at all surrounding points around $(0, 0)$, it can be concluded that

$$\boxed{(0, 0) \text{ is a local minimum of } f(x, y)}$$

26.6.20

For $f(x, y) = e^{x+y^2} - 1 - \sin\big(x - y^2\big)$ it is apparent that $df(0, 0)$ vanishes so the origin is a critical point. Put $u = x + y^2$ and $v = x - y^2$ and use known Maclaurin polynomials for e^u and $\sin(v)$

$$e^u = 1 + u + O(u^2)$$

$$\sin(v) = v + O(v^3)$$

to investigate $f(x, y) - f(0, 0)$:

$$f(x, y) - f(0, 0) = f(x, y) = u - v + O(u^2) + O(v^3) \text{ (neglect these terms)}$$
$$\approx x + y^2 - x + y^2 + \varepsilon(x, y) \approx 2y^2$$

Evidently we will always have a positive number, so we have a local minimum at the origin.

26.6.21

26.6.22

For $f(x,y,z) = 2 - 2\cos(x+y+z) - x^2 - y^2 - z^2$ it is apparent that $df(0,0,0)$ vanishes so the origin is a critical point. Put $u = x+y+z$ and use the known Maclaurin series for the cosine

$$\cos(u) = 1 - u^2 + O(u^4)$$

to investigate $f(x,y,z) - f(0,0,0)$:

$$f(x,y,z) - f(0,0,0) = f(x,y) = 2 - 2(1 - u^2 + O(u^4)) - x^2 - y^2 - z^2$$

$$\approx 2 - 2 + (x+y+z)^2 - x^2 - y^2 - z^2 = 2xy + 2xz + 2yz$$

In a small neighborhood around the origin it is apparent that possible to get negative and positive values depending on the octant the portion of the neighborhood occupies (consider when $x, y, z > 0$ versus when $y, z > 0$ and $x < 0$). Therefore the origin cannot be a local extremum.

26.6.23

26.6.24

26.6.25

26.6.26

26.6.27

$$f(x,y) = 1 + 2x - 3y$$

Since the function is linear both within D and on the borders of D, the absolute

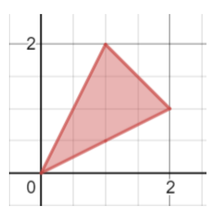


Figure 1: Domain

extrema must be found on the corners. The absolute minimum occurs at (1,2) with a value of -3, while the absolute maximum occurs at (2,1) with a value of 2.

26.6.28

$$f(x,y) = x^2 + y^2 + xy^2 - 1 \ ; \ |x| \leq 1, |y| \leq 2$$

First, find all critical points of $f(x,y)$.

$$df(x,y) = (2x + y^2)dx + (2y + 2xy)dy$$

vanishes at $(0,0)$ and $(-1, \pm\sqrt{2})$. These critical points are possible candidates for the extrema of $f(x,y)$.

Next, find the maximum and minimum values of $f(x,y)$ along its boundary. The four sides of the rectangular boundary, must be analyzed separately.

$$f(-1,y) = 1 + y^2 - y^2 - 1 \equiv 0 \ ; \ |y| \leq 2$$

So all points $(-1,y)$ along the left side of the boundary are candidates for extremum.

$$f(1,y) = 2y^2 \ ; \ |y| \leq 2$$

This one-variable function reaches a minimum at $(1,0)$ and a maximum at both $(1,-2)$ and $(1,2)$, so these are new candidates for extremum.

$$f(x,-2) = x^2 + 4x - 3 \ ; \ |x| \leq 1$$

This one-variable function reaches has extremum at $(1,0)$ and $(-1,0)$, so these are candidates for extremum of f.

$$f(x,2) = x^2 + 4x - 3 \ ; \ |x| \leq 1$$

This one-variable function yields no new extremum.

Finally, calculate the value of $f(x,y)$ at all possible extremum.

$$f(0,0) = -1$$

$$f(-1,\sqrt{2}) = f(-1,y) = 0$$

$$f(1,0) = 0$$

$$f(1,\pm 2) = 8$$

$$f(0,1) = 0$$

The largest values at $(1,\pm 2)$ are the maxima. The smallest value at $(0,0)$ is the minimum.

$$\boxed{\text{Maxima: } f(1,\pm 2) = 8, \text{ Minimum: } f(0,0) = -1}$$

26.6.29

26.6.30

$$f(x, y) = xy^2 + z$$

D is the cube with $|x|, |y|, |z| \leq 1$. The first is to check for critical points within D; however: $f'_z = 1 \implies$ there are no critical points within D. Next is to check for critical points on the border of D. One can see that the sign of y does not matter because of there is only a y^2 term, leaving the points to check to be

$$(1, \pm 1, 1), (1, \pm 1, -1), (-1, \pm 1, 1), (-1, \pm 1, -1)$$

The value of f at those coordinates are, respectively, 2, 0, 0, -2, meaning the max is $f(1, \pm 1, 1) = 2$ and min $f(-1, \pm 1, -1) = -2$.

26.6.31

Give the gradient of $f(x, y, z) = xy^2 + z$ as

$$\langle y^2, 2xy, 1\rangle$$

and set it equal to the zero vector to find that there are no critical points. We must then investigate the values it attains on the boundary of the set $D = \{(x, y, z)|\ 1 \leq x^2 + y^2 \leq 4, -2 + x \leq z \leq 2 - x\}$

The first condition for D is a cylindrical washer with included inner radius 1 and included outer radius 2. The second condition is equivalent to $|z| \leq 2 - x$, so the cylindrical washer is bounded above and below by the planes given by $z = \pm(2 - x)$. So there are four surfaces that make up the boundary, and we can give some equations for x, y, z for each surface:

$$\text{top skewed washer } S_t = \{(x, y, x)|\ z = 2 - x,\ 1 \leq x^2 + y^2 \leq 4\}$$

$$\text{bottom skewed washer } S_b = \{(x, y, x)|\ z = -2 + x,\ 1 \leq x^2 + y^2 \leq 4\}$$

$$\text{outer shell } S_o = \{(x, y, x)|\ x = 2\cos(t),\ y = 2\sin(t), |z| \leq 2 - 2\cos(t)\}$$

$$\text{inner shell } S_i = \{(x, y, x)|\ x = \cos(t),\ y = \sin(t), |z| \leq 2 - \cos(t)\}$$

For S_t and S_b the problem becomes a two-variable extrema problem on that surface, which we know how to do. We find that for S_t

$$f(x, y, 2 - x) = f_t(x, y) = xy^2 + 2 - x$$

which has the absolute maximum 4 at $(-2, 0)$ and an absolute minimum 0 at $(2, 0)$ (use parametric equations for (x, y) on the washer to find them). Using the definition for z, we may want to rewrite these points as $(-2, 0, 4)$ and $(2, 0, 0)$. Likewise for S_b we need to find extrema on $f(x, y, 2 - x) = xy^2 - 2 + x$, which will be (there are more maximum points (6 on this surface), come back to this)

INCOMPLETE

26.6.32

26.6.33

26.6.34

The volume of the first octant that is cut off by the plane is determined by the intersection of the plane with the three coordinate axes in the first octant and is given by

$$V = \frac{x_i y_i z_i}{2}$$

where $(x_i, 0, 0)$, $(0, y_i, 0)$, and $(0, 0, z_i)$ are where the plane passing through the point $(3, 2, 1)$ intersect with the coordinate axes (this is clear through construction and symmetry of the division of the rectangular prism whose largest diagonal is the vector from the origin to (x_i, y_i, z_i))

The values of $x_i,\ y_i,\ z_i$ are determined by the choice of the normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$, and the relationship is given by the definition of points that satisfy the plane:

$$\vec{n} \cdot \langle x - 3, y - 2, z - 1 \rangle = 0$$

This means the three points where the plane intersected with the coordinate axes form the following equalities:

$$n_1(x_i - 3) + n_2(-2) + n_3(-1) = 0 \rightarrow x_i = \frac{n_3 + 2n_2}{n_1} + 3$$

$$n_1(-3) + n_2(y_i - 2) + n_3(-1) = 0 \rightarrow y_i = \frac{n_3 + 3n_1}{n_2} + 2$$

$$n_1(-3) + n_2(-2) + n_3(z_i - 1) = 0 \rightarrow z_i = \frac{2n_2 + 3n_1}{n_3} + 1$$

Using the above equations for $x_i,\ y_i,\ z_i$ we can substitute back into the volume equation to find:

$$V(n_1, n_2, n_3) = \frac{\left(\frac{n_3+2n_2}{n_1} + 3\right)\left(\frac{n_3+3n_1}{n_2} + 2\right)\left(\frac{2n_2+3n_1}{n_3} + 1\right)}{2}$$

$$= \frac{1}{2}\left(\right)$$

The goal now is to find $n_1,\ n_2,\ n_3$ that minimize the volume function given above.

INCOMPLETE

26.6.35

26.6.36

26.6.37

26.6.38

26.6.39

26.6.40

26.6.41

27.8 Exercises

27.8.1

Find the gradient of $f(x, y) = xy$ and the gradient of $g(x, y) = x + y$:

$$\vec{\nabla} f(x, y) = \langle y, x \rangle$$
$$\vec{\nabla} g(x, y) = \langle 1, 1 \rangle$$

It is known that $f(x, y)$ attains extrema when $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$, that is at points (x, y) that satisfy the following system using Lagrange multipliers (recall one of the equations is the constraint itself):

$$y = \lambda$$
$$x = \lambda$$
$$x + y = 1$$

Evidently the only critical point is $(\frac{1}{2}, \frac{1}{2})$ - but because it is the only one we may choose to use the second derivative test or to use the properties of the function f under the constraint (for instance, parameterize (x, y) as $(t, 1-t)$ as given by the constraint to show that we have a downwards opening parabola) to determine that it is a maximum. Thus $f(x, y)$ constrained to $g(x, y) = 1$ has a maximum $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ at that point.

27.8.3

Find the gradient of $f(x, y) = xy^2$ and the gradient of $g(x, y) = 2x^2 + y^2$:

$$\vec{\nabla} f(x, y) = \langle y^2, 2xy \rangle$$
$$\vec{\nabla} g(x, y) = \langle 4x, 2y \rangle$$

It is known that $f(x, y)$ attains extrema when $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$, that is at points (x, y) that satisfy the following system using Lagrange multipliers (recall one of the equations is the constraint itself):

$$y^2 = 4\lambda x$$
$$2xy = 2\lambda y$$
$$2x^2 + y^2 = 6$$

With some algebra (resolve the middle equation for x, then substitute x for lambda in the first equation, and then substitute $4x^2$ for y^2 and solve for x), it is apparent that $x = \pm 1$ $y = \pm 2$ (both still satisfy the constraint). Thus we have critical points at $(1, \pm 2)$, where $f(1, \pm 2) = 4$, and at $(-1, \pm 2)$, where $f(-1, \pm 2) = -4$. Hence we have $\max f = 4$ and $\min f = -4$

27.8.4

$$f(x, y) = y^2$$
$$g(x, y) = x^2 + y^2 - 4$$

The critical points of $f(x, y)$ on $g(x, y)$ must satisfy:

$$\begin{cases} \vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y) \\ g(x, y) = 0 \end{cases}$$

which becomes:

$$\begin{cases} 0 = 2\lambda x \\ 2y = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

λ is either 0 or 1. The critical points when $\lambda = 0$ are $(\pm 2, 0)$. The critical points when $\lambda = 1$ are $(0, \pm 2)$. Plugging in the critical points:

$$f(\pm 2, 0) = 0$$
$$f(0, \pm 2) = 4$$

Therefore:

$$\boxed{\text{Minimum: } (\pm 2, 0, 0), \text{ Maximum: } (0, \pm 2, 4)}$$

27.8.5

27.8.6

First is to find the gradient of $f(x, y) = 2x^2 - 2y^2$ and $g(x, y) = x^4 + y^4 - 16$

$$\nabla f(x, y) = \langle 4x, -4y \rangle$$
$$\nabla g(x, y) = \langle 4x^3, 4y^3 \rangle$$

For the Lagrange multipliers, we get this system of equations:

$$4x = 4\lambda x^3$$

$$-4y = 4\lambda y^3$$
$$x^4 + y^4 = 16$$

Rearranging the first two equations yields $x(\lambda x^2 - 1) = 0$ and $y(\lambda y^2 + 1) = 0$. Then, there are four separate conditions.

1. x=0
2. $x^2 = \frac{1}{\lambda}$
3. y=0
4. $y^2 = \frac{1}{\lambda}$

From 1 and 3 we get that either $x^2 = 4$ or $y^2 = 4$ respectively, giving us f=8 and -8 respectively. The only thing to try out now is a combination of both 2 and 4, meaning $x^2 = y^2$. There is no need to go further than this as making this substitution into f yields f=0. As a result, the absolute min in the region is -8 and max 8.

27.8.7

Find the gradient of $f(x, y) = Ax^2 + 2Bxy + Cy^2$ and the gradient of $g(x, y) = x^2 + y^2$:

$$\vec{\nabla} f(x, y) = \langle 2Ax + 2By, 2Bx + 2Cy \rangle$$
$$\vec{\nabla} g(x, y) = \langle 2x, 2y \rangle$$

It is known that $f(x, y)$ attains extrema when $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$, that is at points (x, y) that satisfy the following system using Lagrange multipliers (recall one of the equations is the constraint itself):

$$2Ax + 2By = 2\lambda x$$
$$2Bx + 2Cy = 2\lambda y$$
$$x^2 + y^2 = 1$$

There is a bit of algebra we have to do to find critical values. First, we can add the first two equations together and divide through by 2 to find

$$Ax + By + Bx + Cy = \lambda x + \lambda y \to (A + B)x + (B + C)y = \lambda x + \lambda y$$

This tells us that $A + B = \lambda$ and $B + C = \lambda$, furthermore that $A = C$ and A or C $= \lambda - B$. Then substitute that last equality in for A and C in the first equations like so:

$$(\lambda - B)x + By = \lambda x$$
$$Bx + (\lambda - B)y = \lambda y$$

Deduce then that $x = y$. Using the third equation it is apparent that critical points occur at $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$, and what kind of extrema they form entirely depend on the choice of $A,\ B,\ C$.

27.8.8

$$f(x,y,z) = xyz$$
$$g(x,y,z) = 3x^2 + 2y^2 + z^2 - 6$$

The critical points of $f(x,y,z)$ on $g(x,y,z)$ must satisfy:

$$\begin{cases} \vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$$

which becomes:

$$\begin{cases} yz = 6\lambda x \\ xz = 4\lambda y \\ xy = 2\lambda z \\ 3x^2 + 2y^2 + z^2 = 6 \end{cases}$$

Solving the first and second equations for z and setting them equal to each other gives:

$$\frac{6\lambda x}{y} = \frac{4\lambda y}{x} \to 6x^2 = 4y^2 \to y = \pm x\frac{\sqrt{6}}{2}$$

Solving the first and third equations for y and setting them equal to each other gives:

$$\frac{6\lambda x}{z} = \frac{2\lambda z}{x} \to 6x^2 = 2z^2 \to z = \pm x\sqrt{3}$$

Now, having solved for y and z in terms of x, examine the fourth equation.

$$3x^2 + 2y^2 + z^2 = 3x^2 + 2(\frac{3}{2}x^2) + 3x^2 = 9x^2 = 6$$

$$x = \pm\sqrt{\frac{2}{3}}$$

Hence,

$$y = \pm 1$$
$$z = \pm\sqrt{2}$$

The eight points to be tested are $(\pm\sqrt{\frac{2}{3}}, \pm 1, \pm\sqrt{2})$. The maxima of $f(x,y,z) = xyz$ will be the positive product of the magnitudes of $(\pm\sqrt{\frac{2}{3}}, \pm 1, \pm\sqrt{2})$ and the minima will be the negative product. Therefore,

$$\boxed{\text{Maximum: } \frac{2}{\sqrt{3}}, \text{ Minimum: } -\frac{2}{\sqrt{3}}}$$

27.8.9

27.8.11

The gradients of f,g, and h are:

$$\nabla f(x,y,z) = \langle -1, 3, -3 \rangle$$

$$\nabla g(x,y,z) = \langle 1, 1, -1 \rangle$$

$$\nabla h(x,y,z) = \langle 0, 2y, 4z \rangle$$

The system of equations one gets from the Lagrangian multipliers are:

$$\begin{cases} -1 = \lambda_1 \\ 3 = \lambda_1 + \lambda_2(2y) \\ -3 = \lambda_1(-1) + \lambda_2(4z) \\ x + y - z = 0 \\ y^2 + 2z^2 = 1 \end{cases}$$

Which can be simplified from the first equation to:

$$\begin{cases} 4 = 2\lambda_2 y \\ -4 = 4\lambda_2 z \\ x + y - z = 0 \\ y^2 + 2z^2 = 1 \end{cases}$$

Rearranging the first two equations gives that

1. $y = \frac{2}{\lambda_2}$
2. $z = \frac{-1}{\lambda_2}$

Since both y and z can be represented in terms of λ_2, and h(x,y,z) is only in terms of y and z, we can find the values of λ_2 that satisfy our conditions, giving that $(\frac{2}{\lambda_2})^2 + 2(\frac{-1}{\lambda_2})^2 = \frac{4+2}{(\lambda_2)^2} = 1$. As a result, $\lambda_2 = \pm\sqrt{6}$, and 2 possible values for y and z.

1. $(y, z) = (\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}})$
2. $(y, z) = (\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$

Plugging in these values into the condition associated with g(x,y,z) gives $x = \frac{-3}{\sqrt{6}}$ for 1 and $x = \frac{3}{\sqrt{6}}$ for 2. Plugging in these coordinates into f gives f=$2\sqrt{6}$ for 1 and $-2\sqrt{6}$ for 2, the max and mins respectively.

27.8.12

$$f(x,y,z) = xy + yz$$
$$g_1(x,y,z) = xy - 1$$
$$g_2(x,y,z) = y^2 + 2z^2 - 1$$

The critical points of $f(x,y,z)$ on $g_1(x,y,z), g_2(x,y,z)$ must satisfy:

$$\begin{cases} \vec{\nabla} f(x,y,z) = \lambda_1 \vec{\nabla} g_1(x,y,z) + \lambda_2 \vec{\nabla} g_2(x,y,z) \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}$$

which becomes:

$$\begin{cases} y = \lambda_1 y \\ x + z = \lambda_1 x + \lambda_2 2y \\ y = \lambda_2 4z \\ xy = 1 \\ y^2 + 2z^2 = 1 \end{cases}$$

The fourth equation indicates that $x = \frac{1}{y}$ and that x, y are non-zero. Therefore, it is found that $\lambda_1 = 1$ from the first equation. The second equation becomes: $x + z = x + \lambda_2 2y \to z = 2\lambda_2 y \to \lambda_2 = \frac{z}{2y}$. Plugging this value of λ_2 into the third equation: $y = \frac{z}{2y} 4z \to 2y^2 = 4z^2 \to z = \pm\frac{1}{\sqrt{2}} y$. Plug this value of z into the last equation:

$$y^2 + 2(\pm\frac{1}{\sqrt{2}} y)^2 \to y = \pm\frac{1}{\sqrt{2}}$$

Therefore, the candidates for extrema are $(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}), (\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}), (-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}), (-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2})$. Plug each of these values into $f(x,y,z)$:

$$f(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}) = f(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}) = 1 + \frac{1}{2\sqrt{2}}$$
$$f(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}) = f(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}) = 1 - \frac{1}{2\sqrt{2}}$$

Finally,

$$\boxed{\text{Maximum: } 1 + \frac{1}{2\sqrt{2}}, \text{ Minimum: } 1 - \frac{1}{2\sqrt{2}}}$$

27.8.25

27.8.26

Since the problem asks us to minimize the sum of the reciprocals, we will take that to be our $f(\vec{x})$. Meanwhile, we will take the product of the numbers as the boundary condition. In summary,

$$f(\vec{x}) = \sum_{i=1}^{m} \frac{1}{x_i}$$

$$g(\vec{x}) = \prod_{i=1}^{m} x_i = K$$

where K is some constant. The Lagrange multiplier for some arbitrary x_i is $f'_{x_i} = \lambda g'_{x_i}$. One can see that $f'_{x_i} = \frac{-1}{(x_i)^2}$ since f is a sum of the reciprocals. Meanwhile, $g'_{x_i} = \frac{K}{x_i}$, which is just the product of every variable except for x_i, taking advantage of the fact that the product of all the variables is just K. As a result, the Lagrange multiplier condition is:

$$\frac{-1}{(x_i)^2} = \lambda \frac{K}{x_i}$$

Rearranging this gives $x_i = \frac{-1}{\lambda K}$. Since x_i can be any variable, the product of everything, $x_1 x_2 x_3 .... x_m$ is just $(\frac{-1}{\lambda K})^m = K$. Solving this for λ gives $\lambda = \frac{-1}{KK^{1/m}}$ or $\frac{1}{\lambda} = -KK^{1/m}$. Substituting this back into what we solved x_i to be gives that $x_i = \frac{KK^{1/m}}{K} = K^{1/m}$.

27.8.28

$$V(r,h) = \frac{1}{2}\pi r^2 h$$

$$SA(r,h) = \pi r^2 + \pi r h = S$$

The critical points of V given SA must satisfy:

$$\begin{cases} \vec{\nabla} V(r,h) = \lambda \vec{\nabla} SA(r,h) \\ SA(r,h) = S \end{cases}$$

which becomes:

$$\begin{cases} \pi r h = \lambda_1 (2\pi r + \pi h) \\ \frac{1}{2}\pi r^2 = \lambda_1 \pi r \\ \pi r^2 + \pi r h = S \end{cases}$$

Solve the second equation for λ_1:

$$\frac{1}{2}\pi r^2 = \lambda \pi r \rightarrow \lambda_1 = \frac{r}{2}$$

Note that the case where $r = 1$ because it is obviously does not maximize the volume. Plug this value for λ_1 into the first equation:

$$\pi r h = (\frac{r}{2})(2\pi r + \pi h) = \pi r^2 + \frac{\pi r h}{2}$$

$$\frac{\pi r h}{2} = \pi r^2$$
$$h = 2r$$

Plug the value of h into the last equation:

$$\pi r^2 + \pi r(2r) = S$$
$$3\pi r^2 = S$$
$$r = \sqrt{\frac{S}{3\pi}}$$

Backtrack to find that:

$$h = 2\sqrt{\frac{S}{3\pi}}$$

The final answer is then:

$$\boxed{r = \sqrt{\frac{S}{3\pi}}, h = 2\sqrt{\frac{S}{3\pi}}}$$

27.8.31

27.8.32

28.6 Exercises

28.6.1

Because the function is constant on D, the lower and upper sums are equal (thus the double integral converges). The double integral is equal to (by geometrical means) $(b-a)(d-c)k$.

28.6.2

Split D into two sections over $y = 0$ (we will use the additivity of the double integral) so that we have $D_1 = [0,1] \times [-1,0]$ and $D_2 = [0,1] \times [0,1]$. In taking the double integral over both of those regions, we may split the integral like so:

$$\iiint_D f(x,y)dA = \iiint_{D_1} f(x,y)dA + \iiint_{D_2} f(x,y)dA$$

The problem degenerates into the same kind of problem as **1.**

Since the value of the function is constant on both of those sets, the lower and upper Riemann sums on those sets are equal to each other (they converge) and by geometrical means the first integral over D_1 is equal to k_2, and the second integral over D_2 is equal to k_1. The sum (the value of the original integral) is $k_1 + k_2$.

28.6.3

Given that $f(x,y) = xy^2$ and $D = [0,1] \times [0,1]$, $x_j = \frac{j}{N_1}$, $y_k = \frac{k}{N_2}$. Additionally, $m_{jk} = x_{j-1}y_{k-1}^2 = \frac{(j-1)(k-1)^2}{N_1 N_2^2}$ and $M_{jk} = x_j y_k^2 = \frac{jk^2}{N_1 N_2^2}$. U is defined by $\sum_{j=1}^{N_1} \sum_{k=1}^{N_2} M_{jk}\Delta A$.Substituting what we have for M_{jk} and factoring out what we can, we have that:

$$U = \frac{\Delta x \Delta y}{N_1 N_2^2} (\sum_{j=1}^{N_1} j)(\sum_{k=1}^{N_2} k^2)$$

Using the fact that $\Delta x = \frac{1}{N_1}$ and $\Delta y = \frac{1}{N_2}$ and expanding out the sums for each summation, we can simplify it out to

$$\frac{1}{12}(\frac{1}{N_1^2 N_2^3})N_1 N_2 (N_1+1)(N_2+1)(2N_2+1)$$

Taking the limit as $N_1, N_2 \to \infty$ makes every term except those without N_1 or N_2 in the denominator, giving $\frac{1}{12}(2) = \frac{1}{6}$ as the limit of U. Similarly, L is defined as

$$L = \frac{1}{N_1 N_2^2} \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} (j-1)(k-1)^2 \Delta x \Delta y$$

Expanding this out in a similar way for U gives:

$$\frac{1}{12} \frac{1}{N_1 N_2^2} (N_1 - 1)(N_2 - 1)(2N_2 - 1)$$

and taking the limit of this as $N_1, N_2 \to \infty$ gives $\frac{1}{6}$ as well. Since L=U=$\frac{1}{6}$, the integral of the function over the domain is one sixth.

28.6.4

The set D describes a right triangle whose legs have lengths 1 that lie on the x and y axes with one vertex at the origin. The hypotenuse is a line segment from $(0,1)$ to $(1,0)$. Since $f(x,y) = 1-(x+y) = z$, it is apparent that we can rearrange it into the standard form for a continuous plane with normal vector $\langle 1,1,1 \rangle$ passing through the line $y + x = 1$ ($z = 0$ on the plane).

Since the plane is continuous the upper and lower sums converge to the same value.

So the volume bounded is a triangular pyramid that we can geometrically find the integral. We just need to find out where the plane intersects with the z-axis, which is at $z = 1$. The other three important vertices are $(0,1,0)$, $(1,0,0)$, and the origin. The double integral is given by $V = \frac{1}{2}(1)(1)(1) = \frac{1}{2}$.

28.6.6

$$f(x,y) = x^2 + y^2 \ ; \ D = [1,2] \times [1,3]$$

Because x changes by 1 in total, $\Delta x = \frac{1}{N_1}$ for some constant number of x partitions N_1. Meanwhile, y changes by 2 in total, so $\Delta y = \frac{2}{N_2}$ for some constant number of y partitions N_2. Let:

$$x_j = 1 + j\Delta x = 1 + \frac{j}{N_1}$$

$$y_k = 1 + k\Delta x = 1 + \frac{2k}{N_2}$$

for $j = 1, 2, ..., N_1$ and $k = 1, 2, ..., N_2$. For each partition of the domain, $[x_{j-1}, x_j] \times [y_{k-1}.y_k]$, a minimum m_{jk} and a maximum M_{jk} must be determined. It is clear that

$$M_{jk} = x_j^2 + y_k^2$$

$$m_{jk} = x_{j-1}^2 + y_{k-1}^2$$

because x, y are increasing with increasing j, k. These formulas can be further simplified to:

$$M_{jk} = x_j^2 + y_k^2 = (1 + \frac{j}{N_1})^2 + (1 + \frac{2k}{N_2})^2$$

$$= 2 + \frac{2j}{N_1} + \frac{j^2}{N_1^2} + \frac{4k}{N_2} + \frac{4k^2}{N_2^2}$$

and

$$m_{jk} = x_j^2 + y_k^2 = (1 + \frac{j-1}{N_1})^2 + (1 + \frac{2(k-1)}{N_2})^2$$

$$= 2 + \frac{2(j-1)}{N_1} + \frac{(j-1)^2}{N_1^2} + \frac{4(k-1)}{N_2} + \frac{4(k-1)^2}{N_2^2}$$

Evaluating the upper sum yields:

$$U(f, N_1, N_2) = \sum_{k=1}^{N_2} \sum_{j=1}^{N_1} M_{jk} \Delta A$$

$$= \sum_{k=1}^{N_2} \sum_{j=1}^{N_1} (x_j + y_k)^2 \Delta x \Delta y = \sum_{k=1}^{N_2}$$

$$= \frac{2}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{j=1}^{N_1} (2 + \frac{2j}{N_1} + \frac{j^2}{N_1^2} + \frac{4k}{N_2} + \frac{4k^2}{N_2^2})$$

$$= 4 + \frac{2}{N_1} \sum_{j=1}^{N_1} (\frac{2j}{N_1} + \frac{j^2}{N_1^2}) + \frac{2}{N_2} \sum_{k=1}^{N_2} (\frac{4k}{N_2} + \frac{4k}{N_2^2})$$

This expression can be simplified using the identities $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

$$4 + \frac{2N_1(N_1+1)}{N_1^2} + \frac{N_1(N_1+1)(2N_1+1)}{3N_1^3} + \frac{4N_2(N_2+1)}{N_2^2} + \frac{4N_2(N_2+1)(2N_2+1)}{3N_2^3}$$

Taking the limit as $N_1, N_2 \to \infty$ simply gives:

$$\lim_{N_1,N_2\to\infty} U(f, N_1, N_2) = 4 + 2 + \frac{2}{3} + 4 + \frac{8}{3} = \frac{40}{3}$$

Repeating this process for the lower sum gives a similar expression:

$$L(f, N_1, N_2) =$$

$$4+\frac{2N_1(N_1-1)}{N_1^2}+\frac{N_1(N_1-1)(2N_1-1)}{3N_1^3}+\frac{4N_2(N_2-1)}{N_2^2}+\frac{4N_2(N_2-1)(2N_2-1)}{3N_2^3}$$

Again, taking the limit as $N_1, N_2 \to \infty$ gives:

$$\lim_{N_1,N_2\to\infty} L(f, N_1, N_2) = 4 + 2 + \frac{2}{3} + 4 + \frac{8}{3} = \frac{40}{3}$$

Because the value of the true Riemann sum $R(f, N_1, N_2)$ is bounded between $L(f, N_1, N_2)$ and $U(f, N_1, N_2)$:

$$L(f, N_1, N_2) \leq R(f, N_1, N_2) \leq U(f, N_1, N_2)$$

$$\lim_{N_1,N_2\to\infty} L(f, N_1, N_2) \leq \lim_{N_1,N_2\to\infty} R(f, N_1, N_2) \leq \lim_{N_1,N_2\to\infty} U(f, N_1, N_2)$$

$$\frac{40}{3} \leq R(f, N_1, N_2) \leq \frac{40}{3}$$

$$\boxed{\text{The integral converges to } \frac{40}{3}}$$

28.6.7

For the partition, give $\Delta x = 1$ and $\Delta y = 2$. So there are really only a few points where we need to find the value of the function (since N_1 and N_2 are comparable in size with the dimensions of the rectangular domain).

Those coordinates are: $(0, 0)$, $(1, 0)$, $(0, 2)$ and $(1, 2)$. The function values at those locations are 0, 1, 4, and 5, respectively. We can just sum those up and multiply by the area of each partition rectangle, which is 2. The estimated volume is thus 10.

28.6.10

This integral represents the volume of a cylinder with radius 1 and height k. The double integral is equal to $k\pi$.

28.6.11

To evaluate $\iint_D \sqrt{1 - x^2 - y^2}$ over the domain of the unit circle centered at the origin is the same as calculating the volume of a hemisphere. The function $f(x, y) = \sqrt{1 - x^2 - y^2}$ maps out the unit hemisphere with a positive z-coordinate, which intersects perfectly with the domain of the unit sphere. The volume of this solid is $\frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi$

28.6.12

The shape of $1-x-y$ in D is a right triangular pyramid. Its base is a a triangle with area $B = \frac{1}{2}lw = \frac{1}{2}$ and a height of $h = 1$. The volume of the pyramid is $V = \frac{1}{3}Bh = \frac{1}{6}$.

$$\boxed{V = \frac{1}{6}}$$

28.6.13

The integrand represents a plane whose normal vector is given by $\langle cb, ca, ab\rangle$. This plane intersects the z-axis at the point $(0, 0, c)$, and coincides with the line given by $bx+ay = ab$ (which contains the hypotenuse of the triangular domain's boundary triangle, and intersects at the points $(0, b)$ and $(a, 0)$).

The figure that the double integral finds the volume of is a pyramid whose base vertices are given in the problem and whose upper vertex is $(0, 0, c)$. By geometry the volume of such a pyramid is given by $V = \frac{1}{2}(a)(b)(c) = \frac{abc}{2}$.

28.6.17

The integral of $\iint_D \sqrt{1-x^2-y^2} - (\sqrt{x^2+y^2} - 1)$, where D is the unit circle centered at the origin in the first quadrant, can be interpreted as $\frac{1}{4}$ the volume of the solid with an upper bound of $\sqrt{1-x^2-y^2}$ and lower bound of $\sqrt{x^2+y^2}-1$. The first function maps out the top half of a sphere of radius 1. The second function can be though of the upper part of the function

$$z^2 = x^2 + y^2 = 1$$

which defines the top part of the cone. The solid bounded by these two together is simply the volume of a cone at height 1 and half the unit sphere, which is:

$$\frac{1}{2}\frac{4}{3}\pi r^3 \cdot \frac{\pi r^2 h}{3} = \frac{2\pi^2}{9}$$

and $\frac{1}{4}$ of this volume is $\frac{\pi^2}{18}$

28.6.18

The first term of the integral in D describes the unit half-sphere above the z-axis. Its volume is $V_1 = \frac{1}{2}(\frac{4}{3}\pi r^3) = \frac{2\pi}{3}$. The negative of the second and third terms $1 - \sqrt{x^2+y^2}$ in D describes a cone with radius 1 and height 1. The volume of the cone is $V_2 = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}$. Hence, the total volume is

$$\boxed{V = V_1 - V_2 = \frac{\pi}{3}}$$

29.1 Exercises

29.1.1

We may give D as the set difference of the square $[-2,2] \times [-2,2]$ and of the open disk $\{(x,y)|\ x^2+y^2 < 1\}$. Then it becomes apparent that due to the additivity of the double integral we may simply take the double integral of the function $f(x,y)=k$ over the square and subtract from that value the value of the double integral of the same function over the open disk. Using geometry it is apparent that the value of the double integral is $16k - \pi k$.

29.1.4

The linearity of the double integral allows us to take the double integral of each term in the integrand and add them together. Using this fact, observe that the term $\sqrt{4-x^2-y^2}$ represents the top hemisphere of a sphere of radius 2 centered at the origin, and on D we are taking a quarter of that volume. Likewise notice that the term given by 2 represents a constant value, so on D it looks like a slice of a cylinder of radius 2 and height 2.

Hence the double integral is equal to $\frac{4}{3}\pi - 2\pi$.

29.1.5

$\iint_D (4-x-y)dA = \iint_D 3dA + \iint_D (1-x-y)dA$ by properties of integrals. In this problem, D is the triangle with vertices at the origin, (1,0) and (0,1). The first integral is 3 times the area of the triangle, or $\frac{3}{2}$. The second is the volume of the region bounded by the plane x+y+z=1 and the coordinate planes in the first octant. The volume of this is 1/3 the volume of the triangular prism of height 1, giving $\frac{1}{3} \cdot (\frac{1}{2})(1) = \frac{1}{6}$. The total value of the integral is then $\frac{3}{2} + \frac{1}{6} = \frac{5}{3}$

29.1.6

Prove that

$$\iint_D \frac{\sin(xy)}{xy} dA \leq A(D)$$

for $x, y > 0$. Recall that $\sin(u) \leq u$ for $u \geq 0$. Therefore, by the positivity of the double integral, it can also be concluded that:

$$\iint_D \frac{\sin(xy)}{xy} dA \leq \iint_D \frac{xy}{xy} dA = \iint_D dA = A(D)$$

29.1.7

We will first want to partition the disk using polar coordinates. Give

$$\Delta r = \frac{1}{N},\ r_k = k\Delta r$$

and

$$\Delta\theta = \frac{2\pi}{M},\ \theta_j = j\Delta\theta$$

for natural numbers k, j, N, M. Following the method of Example **29.1** we can give each area element in the partition as $\overline{r}_k \Delta r \Delta\theta$ where $\overline{r}_k = \frac{1}{2}(r_k + r_{k+1})$.

Given that $r^2 = x^2 + y^2$, $|x| \le r$, and $|y| \le r$, it is apparent that

$$ax^2 + by^2 \le r^2(a+b)$$

and so we may proceed by using a midpoint (meaning the sample points occur at $(\overline{r}_k, \theta_j)$) Riemann sum to find the double integral:

$$\iint_D r^2(a+b)dA = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{k=1}^{N}\sum_{j=1}^{M} (\overline{r}_k)^2(a+b)\overline{r}_k \Delta r \Delta\theta$$

We can pull out the quantity $(a+b)$ from both sums since it is a constant. Then replace $\Delta\theta$ with $\frac{2\pi}{M}$ (definition we gave above).

Then we can sum with respect to j first and find

$$(a+b) \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{k=1}^{N} (\overline{r}_k)^3 \Delta r \frac{2\pi}{M}(M)$$

which we then sum over k using the definition of the single variable Riemann sum:

$$2\pi(a+b) \lim_{N\to\infty} \sum_{k=1}^{N} (\overline{r}_k)^3 \Delta r = 2\pi(a+b)\int_0^1 r^3 dr = (a+b)\frac{\pi}{2}$$

So it is true that

$$\iint_D (ax^2 + by^2)dA \le (a+b)\frac{\pi}{2}$$

29.1.8

We know that $mA(D) \le \iint_D f dA \le MA(D)$. In this case, the region D is the rectangle [1,2] x [1,2], which has an area of 1. The minimum of f in D, m, is 1 at (1,1) while the max, 16, occurs at (2,2). You can quickly figure that the maxes and mins occur at the extreme values of x and y since x and y are only multiplied together. Therefore, the double integral $\iint_D f dA$ lies between 1 and 16.

29.1.10

It is known that the sine function takes on a maximal value of 1 when its argument is $\frac{\pi}{2} + 2\pi k$ for integers k and minimal values of -1 when its argument is $-\frac{\pi}{2} + 2\pi j$ for integers j. The sine function also takes on the value of 0 when its argument is $0 + \pi k\ell$ for integers ℓ.

Evidently for the integrand the maximum values occur where $x+y = \frac{\pi}{2}+2\pi k$, and minimal ones where $x+y = -\frac{\pi}{2}+2\pi j$, and zero values where $x+y = 0+\pi k\ell$. Since values on D are bounded by the x and y axes and by a line passing through $(0, \pi)$ and $(\frac{\pi}{4}, 0)$, it is apparent that we will never have negative values of $x+y$, and we may omit the auxiliary integer variables and simply find where the sum of x and y form principal angles where the max/zero values are.

$$x_m + y_m = \frac{\pi}{2},\ x_0 + y_0 = 0$$

The first equation indicates that maximum values are given by the line passing through $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, 0)$, which partially lies in D, which is sufficient to say that the maximum value of f on D is 1. Likewise at the origin the second equation is satisfied which is sufficient to know that the minimum value of f on D is 0. Thus the integral is bounded above and below by the area of D (which you can get using elementary geometry) times the min/max values:

$$0 \le \iint_D \sin(x+y)dA \le \frac{\pi^2}{8}$$

29.1.12

If the double integral of f over a region D of non-zero area is 0, then there must be a point on D where f is also 0. This can be proven from the intermediate value theorem or the mean value theorem for integrals. We know that the average value of the function is $\frac{1}{A(D)}\iint_D f dA$, and that that value is equal to $A(D)f(x_0, y_0)$ for some $(x_0, y_0) \in D$, given that f is continuous within D. However, since the double integral is 0, so is the average value. That, therefore makes the equation $0 = A(D)f(x_0, y_0)$, and since the area isn't 0, the value of the function at that point must be.

29.1.13

We will first want to partition the disk using polar coordinates. Give

$$\Delta r = \frac{1}{N},\ r_k = k\Delta r$$

and

$$\Delta\theta = \frac{2\pi}{M},\ \theta_j = j\Delta\theta$$

for natural numbers k, j, N, M. Following the method of Example **29.1** we can give each area element in the partition as $\overline{r}_k \Delta r \Delta\theta$ where $\overline{r}_k = \frac{1}{2}(r_k + r_{k+1})$.

Given that $r^2 = x^2 + y^2$, $|x| \le r$, and $|y| \le r$, it is apparent that we may take the midpoint Riemann sum to find the double integral:

$$\iint_D e^{x^2+y^2} dA = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{k=1}^{N}\sum_{j=1}^{M} e^{(\overline{r}_k)^2}\overline{r}_k \Delta r \Delta\theta$$

Using the defintions of Δr and $\Delta\theta$ and taking the sum in j first, we obtain the following sum:

$$\lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{k=1}^{N} e^{(\overline{r}_k)^2}\overline{r}_k \Delta r \frac{2\pi}{M}(M) = \lim_{N\to\infty} 2\pi \sum_{k=1}^{N} \overline{r}_k e^{(\overline{r}_k)^2}\Delta r$$

The sum on the right is a one dimensional Riemann sum that is by definition the following integral:

$$2\pi \int_0^1 re^{r^2}dr = (e-1)\pi$$

29.1.15

$\iint_D ln(x^2+y^2)dA$ where D is the region $|x|+|y| \leq 1$. One can see that the region D is the rhombus with vertices 1 along each direction of the coordinate axes. As a result, we know that $x^2+y^2 \leq 1$. The natural log of any number $x \in (0,1] \leq 0$. Since the function itself is negative for every value in the domain, the double integral must also be negative.

29.1.16

Notice that the graph of the integrand is unlike any geometrical shape we are used to dealing with, so an alternative argument is needed to determine the sign of the double integral. Notice that the integrand takes on both positive and negative values on D. Particularly for (x,y) beyond the disk $x^2+y^2 \leq 1$ (that is, (x,y) where $1 < x^2+y^2 \leq 4$), the function takes on negative values while anywhere on that disk the function takes on nonnegative values.

So we will try to compare the signed volumes that each portion of the graph forms on D. Suppose that the overall sign of the integral is negative. That means that the signed volume under the surface on the washer $1 < x^2+y^2 \leq 4$ is negative and is of greater magnitude than the signed volume of the surface on the disk $x^2+y^2 \leq 1$. We do not have the tools currently to find these volumes directly, but I will pose the following idea.

Suppose we find an underestimate for the volume (V_2) under the xy plane by applying known methods of calculating volumes of solids of revolution, and then compare that value with an overestimate of the volume (V_1) above the xy plane found using geometry. If $|V_2| \geq |V_1|$ then the overall sign of the double integral is negative. Keep in mind that if we had instead taken an overestimate for V_2 and an underestimate for V_1 and compared them that way, then we do not have a way to determine if the sign of the double integral is negative.

To find V_1, it is easy to give it as the cylinder of height 1 with its base as the disk $x^2+y^2 \leq 1$. By geometry V_1 equals π.

To find V_2, we need to investigate a radial cross section of the surface. We can accomplish this by giving $r^2 = x^2+y^2$, which transforms the integrand (a function of (x,y)) into a function of one variable, r. The function becomes $z(r) = \sqrt[3]{1-r^2}$ We may plot this function where r is the horizontal axis and z is the vertical axis (form points (r,z)) and find that the curve passes through the points $(1,0)$ and $(2,-\sqrt[3]{3})$ in this system. Using the linear function $r=1-3^{-\frac{1}{3}}z$ that passes through those points, we can use it to find V_2 by using the washer method for finding the volume of the solid of revolution that forms by revolving $z(r)$ through 2π radians.

The integral for the solid of revolution (which is identical to the volume of the original integrand under the xy plane) is:

$$\pi\int_0^{-\sqrt[3]{3}}\left[(2)^2-(1-3^{-\frac{1}{3}}z)^2\right]dz$$

$$=\pi\int_0^{-\sqrt[3]{3}}\left[3+\frac{2z}{3^{\frac{1}{3}}}-\frac{z^2}{3^{\frac{2}{3}}}\right]dz=\pi\left[3z+\frac{z^2}{3^{\frac{1}{3}}}-\frac{z^3}{3^{\frac{5}{3}}}\right]_0^{-\sqrt[3]{3}}$$

$$=\left(3^{-\frac{2}{3}}-2\sqrt[3]{3}\right)\pi=-\frac{5}{3^{\frac{2}{3}}}\pi$$

We can verify however we like that indeed $|-\frac{5}{3^{\frac{2}{3}}}\pi|\geq|\pi|$, which means that the sign of the double integral is indeed negative.

30.3 Exercises

30.3.1

We have a continuous function over a rectangular region so we may apply Fubini's theorem directly. The integral becomes

$$\int_0^2\int_0^1(x+y)dxdy$$

which we can integrate iteratively to find

$$\int_0^2\int_0^1(x+y)dxdy\to\int_0^2\left(\frac{x^2}{2}+xy\right)\Big|_{x=0}^{x=1}dy\to\int_0^2\left(\frac{1}{2}+y\right)dy$$

$$\int_0^2\left(\frac{1}{2}+y\right)dy\to\left(\frac{y}{2}+\frac{y^2}{2}\right)\Big|_0^2=3$$

30.3.2

$$\iint_D xy^2dA\ ,\ D=[0,1]\times[-1,1]$$

$$=\int_{-1}^1\int_0^1 xy^2dxdy$$

$$=(\int_{-1}^1 y^2dy)(\int_0^1 xdx)$$

$$= (\frac{2}{3})(\frac{1}{2})$$

$$\boxed{= 1/3}$$

30.3.3

$$\iint_D \sqrt{x+2y}dA \text{ , } D = [1,2] \times [0,1]$$

$$\int_0^1 \int_1^2 \sqrt{x+2y}dxdy$$

This integral can be solved using substitution. Let:

$$u = x + 2y$$

$$du = dx$$

The integral then becomes:

$$\int_0^1 \int_{1+2y}^{2+2y} \sqrt{u}dudy$$

$$= \int_0^1 \frac{2}{3}[u^{3/2}]_{1+2y}^{2+2y}dy$$

$$= \frac{2}{3}\int_0^1 (2+2y)^{3/2} - (1+2y)^{3/2}dy$$

Now use another substitution and let:

$$v = 2+2y, w = 1+2y$$

$$dv = dw = 2dy$$

The integral now becomes:

$$\frac{1}{3}(\int_2^4 v^{3/2}dv - \int_1^3 w^{3/2}dw)$$

$$= \frac{2}{15}([v^{5/2}]_2^4 - [w^{5/2}]_1^3)$$

$$= \frac{2}{15}(32 - 4\sqrt{2} - 9\sqrt{3} + 1)$$

$$\boxed{\frac{2}{15}(33 - 4\sqrt{2} - 9\sqrt{3})}$$

30.3.4

We have a continuous function over a rectangular region so we may apply Fubini's theorem directly. The integral becomes

$$\int_0^2 \int_0^1 (1 + 3x^2 y) dx dy$$

which we can integrate iteratively to find

$$\int_0^2 \int_0^1 (1 + 3x^2 y) dx dy \to \int_0^2 (x + x^3 y)\Big|_{x=0}^{x=1} dy \to \int_0^2 (1 + y) dy = 4$$

30.3.5

Integrate with respect to y first and then x to make it easier:

$$\int_0^1 \int_0^1 x e^{yx} dy dx \to \int_0^1 (e^x - 1) dx = e - 2$$

30.3.6

$$\iint_D \cos(x + 2y) dA \ , \ D = [0, \pi] \times [0, \pi/4]$$

$$\int_0^{\pi/4} \int_0^{\pi} \cos(x + 2y) dx dy$$

Let:

$$u = x + 2y$$

$$du = dx$$

The integral becomes:

$$\int_0^{\pi/4} \int_{2y}^{\pi+2y} \cos(u) du dy$$

$$= \int_0^{\pi/4} [\sin u]_{2y}^{\pi+2y} dy$$

$$= \int_0^{\pi/4} \sin(2y + \pi) - \sin(2y) dy$$

Using the sine addition formulas, this becomes:

$$-2 \int_0^{\pi/4} \sin(2y) dy$$

Now let:

$$v = 2y$$

$$dv = 2dy$$

The integral finally becomes:

$$-\int_0^{\pi/2} \sin(v)dv$$

$$= [\cos v]_0^{\pi/2}$$

$$\boxed{= -1}$$

30.3.7

We have a continuous function over a rectangular region so we may apply Fubini's theorem directly. The integral becomes

$$\int_0^1 \int_0^1 \frac{1+2x}{1+y^2} dxdy$$

which we can integrate iteratively to find

$$\int_0^1 \int_0^1 \frac{1+2x}{1+y^2} dydx \to \int_0^1 (1+2x)\arctan(y)\Big|_{y=0}^{y=1} dx \to \frac{\pi}{4}\int_0^1 (1+2x)dx$$

$$= \frac{\pi}{2}$$

30.3.8

$$\iint_D \frac{y}{x^2+y^2}$$

D is the rectangle [0,1]x[1,2]. The easiest order of integration would be dx dy.

$$\int_1^2 \int_0^1 \frac{y}{x^2+y^2} dxdy$$

$$\int_1^2 y(1/y)\tan^{-1}(\frac{x}{y})|_0^1 dy = \int_1^2 \tan^{-1}(1/y)dy$$

To complete this integral, you should use the substitution 1/y $= \tan\theta$, or y=$\cot\theta \implies dy = -\csc^2\theta d\theta$

$$-\int_{\pi/4}^{\cot^{-1}(2)} \theta\csc^2\theta d\theta$$

After doing integration by parts, I personally did it via the tabular method, you get an expression that evaluates to:

$$2\tan^{-1}(1/2) + \frac{1}{2}\ln 5/2 - \pi/4$$

30.3.9

$$\int_0^1 \int_0^1 (x-y)^n dxdy$$

Let:

$$u = x - y$$

$$du = dx$$

The integral becomes:

$$\int_0^1 \int_{-y}^{1-y} u^n dxdy$$

$$\int_0^1 [\frac{u^{n+1}}{n+1}]_{-y}^{1-y} dy$$

$$\frac{1}{n+1} \int_0^1 (1-y)^{n+1} - (-y)^{n+1} dy$$

By the linearity of double integrals, the above can be split into

$$\frac{1}{n+1}(\int_0^1 (1-y)^{n+1} dy - \int_0^1 (-y)^{n+1})$$

For the first integral, let:

$$v = 1 - y$$

$$dv = -dy$$

And for the second integral, let:

$$w = -y$$

$$dw = -dy$$

So the integral becomes:

$$-\frac{1}{n+1}(\int_1^0 v^{n+1} dv - \int_0^{-1} w^{n+1} dw)$$

Simply by the power rule:

$$-\frac{1}{(n+1)(n+2)}([v^{n+2}]_1^0 - [w^{n+2}]_0^{-1})$$

$$-\frac{1}{(n+1)(n+2)}((-1) - (-1)^{n+2})$$

$$\frac{1 + (-1)^{(n+1)(n+2)}}{n+2}$$

Because $(-1)^{n+2} = (-1)^n$ for all integers n, the answer can be more simply written as:

$$\boxed{\frac{1 + (-1)^n}{(n+1)(n+2)}}$$

30.3.10

We have a continuous function over a rectangular region so we may apply Fubini's theorem directly. The integral becomes

$$\int_0^2 \int_0^1 e^x \sqrt{y + e^x} dxdy$$

which we can integrate iteratively to find

$$\int_0^2 \int_0^1 e^x \sqrt{y + e^x} dxdy \to \int_0^2 \frac{2}{3}(y + e^x)^{\frac{3}{2}} \Big|_{x=0}^{x=1} dy \to \frac{2}{3} \int_0^2 ((y + e)^{\frac{3}{2}} - (y + 1)^{\frac{3}{2}})dy$$

$$\frac{2}{3} \int_0^2 ((y + e)^{\frac{3}{2}} - (y + 1)^{\frac{3}{2}})dy = \frac{4}{15}(1 - 9\sqrt{3} - e^{\frac{5}{2}} + (2 + e)^{\frac{5}{2}})$$

30.3.13

We have a continuous function over a rectangular region so we may apply Fubini's theorem directly. The integral becomes

$$\int_1^2 \int_0^1 \frac{1}{2x + y} dxdy$$

which we can integrate iteratively to find

$$\int_1^2 \int_0^1 \frac{1}{2x + y} dxdy \to \frac{1}{2} \int_1^2 \ln(2x + y) \Big|_{x=0}^{x=1} dy \to \frac{1}{2} \int_1^2 (\ln(2 + y) - \ln(y))dy$$

(use integration by parts)

$$\frac{1}{2} \int_1^2 (\ln(2 + y) - \ln(y))dy = \left(-\frac{1}{2} y \ln(y) + \ln(2 + y) + \frac{1}{2} y \ln(2 + y) \right) \Big|_1^2$$

$$= \frac{1}{2} \ln \left(\frac{64}{27} \right)$$

30.3.16

The volume of such a solid is given by the double integral

$$\int_0^2 \int_{-1}^1 (1 + 3x^2 + 6y^2) dxdy$$

which taken iteratively we find

$$\int_0^2 \int_{-1}^1 (1 + 3x^2 + 6y^2) dxdy \to \int_0^2 (4 + 12y^2) dy = 40$$

30.3.17

According to the given statements, x can vary between 0 and 3 while y can vary between 0 and 2 so that $z = 4 - y^2$ remains positive. Therefore, the volume of E is

$$\int_0^2 \int_0^3 (4 - y^2) dx dy$$

$$= 3 \int_0^2 4 - y^2 dy$$

$$= 3[4y - \frac{y^3}{3}]_0^2$$

$$= 3(8 - \frac{8}{3})$$

$$\boxed{= 16}$$

30.3.19

The surface $z = xy$ is symmetric across the lines $y = \pm x$, but more importantly the parts of the surface in each quadrant are really just reflected versions of the surface in other quadrants. For example, the portion of the surface in the first quadrant is really just a vertically flipped (negated) version of the part of the surface in the second or fourth quadrant. So the signed volume captured by a double integral over any square $[-a, a] \times [-a, a]$ is going to be by symmetry 0.

We are asked to find the value of the double integral on the part of the square $[-1, 1] \times [-1, 1]$ that is not in the first quadrant, so using what we know of the symmetry of the integrand, it is sufficient to find the double integral over the square $[-1, 0] \times [0, 1]$ (alternatively choose the similar square in the fourth quadrant). The value of such an integral is given by the double integral

$$\int_{-1}^0 \int_0^1 (xy) dy dx = -\frac{1}{4}$$

30.3.21

Consider the integral:

$$\int_a^b \int_a^b [f(x) - f(y)]^2 dy dx$$

We argue that the value of the integral must be strictly non-negative because the integrand $(f(x) - f(y))^2$ is strictly non-negative.

$$= \int_a^b \int_a^b [f(x) - f(y)]^2 dy dx \geq 0$$

We can now simplify the integral using some basic algebraic manipulation.

$$= \int_a^b \int_a^b (f(x)^2 + f(y)^2 - 2f(x)f(y))dydx \geq 0$$

$$= \int_a^b \int_a^b f(x)^2 dydx + \int_a^b \int_a^b f(y)^2 dydx - 2\int_a^b f(x)f(y)dydx \geq 0$$

$$= 2\int_a^b f(x)f(y)dydx \leq \int_a^b \int_a^b f(x)^2 dydx + \int_a^b \int_a^b f(y)^2 dydx$$

$$= 2(\int_a^b f(x)dx)(\int_a^b f(y)dy) \leq (b-a)(\int_a^b f(x)^2 dx + \int_a^b f(y)^2 dy)$$

The two definite integrals on the l.h.s and the two integrals on the r.h.s. only differ in their variables of integration, so their values are the same.

$$= 2(\int_a^b f(x)dx)^2 \leq 2(b-a)\int_a^b (f(x))^2 dx$$

$$\boxed{= (\int_a^b f(x)dx)^2 \leq (b-a)\int_a^b (f(x))^2 dx}$$

30.3.22

First we will change coordinates by applying the transformation at every point: $(x, y) \to (x + a, y + b)$. Then the problem becomes finding the average squared distance from the point (a, b) to every point on a disk of radius R centered at the new origin. We will then change some coordinates into polar coordinates. The points (x, y) on the disk centered at the origin are represented alternatively as $(r\cos(\theta), r\sin(\theta))$. It is clear from construction that $0 \leq r \leq R$ and $0 \leq \theta < 2\pi$.

The squared distance between points on the disk and (a, b) is $(r\cos(\theta) - a)^2 + (r\sin(\theta) - b)^2$. Taking the sum over all points on the disk by varying r and θ the total sum of the squared distance (recall $dA = rdrd\theta$) is

$$\int_0^{2\pi} \int_0^R \left((r\cos(\theta) - a)^2 + (r\sin(\theta) - b)^2\right) rdrd\theta$$

and the average value is that integral divided by the area of the disk:

$$\frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left((r\cos(\theta) - a)^2 + (r\sin(\theta) - b)^2\right) rdrd\theta$$

$$= \frac{1}{2}(2a^2 + 2b^2 + R^2)$$

31.6 Exercises

31.6.1

Integrating over y first and then over x: it is apparent that the bounds in x are just going to be $-2 \leq x \leq 2$. For the bounds in y we will need to make two, since the two legs of the triangle that intersect at the origin form line segments of different slopes. We should find that while x is nonnegative we have $\frac{1}{2}x \leq y \leq 1$ and while x is negative we have $-\frac{1}{2}x \leq y \leq 1$. So the double integral becomes

$$\int_{-2}^{0} \int_{-\frac{1}{2}x}^{1} f(x, y) dy dx + \int_{0}^{2} \int_{\frac{1}{2}x}^{1} f(x, y) dy dx$$

When integrating with respect to x first and then y it is apparent that the bounds in y are $0 \leq y \leq 1$, and that for x we have $-2y \leq x \leq 2y$. The integral is

$$\int_{0}^{1} \int_{-2y}^{2y} f(x, y) dx dy$$

31.6.2

If D is the trapezoid with vertices (0,0) (1,0) (1,2) (0,1), then we can say that $x \in [0, 1]$. For any value of x within its domain, y goes from 0 to the line y=x+1, so the bounds for y are $y \in [0, x+1]$, do the double integral would be

$$\int_{0}^{1} \int_{0}^{x+1} dy dx$$

To switch the bounds, you would first define hard bounds for y being $y \in [0, 1]$ and $y \in [1, 2]$, separating the bounds into 2 regions because the x bounds changes at y=1. In the first region for y, x simply goes from 0 to 1, while past y=1 x has a lower bound of the line x=y-1. As a result, the double integral is in the form of:

$$\int_{0}^{1} \int_{0}^{1} dx dy + \int_{1}^{2} \int_{y-1}^{1} dx dy$$

31.6.3

x can vary between -1 and 1. Given this information, y varies between $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$.

$$\boxed{\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx}$$

Switching the order, y can vary between -1 and 1. Given this information, x varies between $-\sqrt{1-y^2}$ and $\sqrt{1-y^2}$.

$$\boxed{\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx dy}$$

31.6.4

The region describes a disk of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$ (note that the left hand side forces y to be positive so this conclusion is legitimate). Knowing this the bounds of integration are not too bad to find. Evidently the boundary $x^2+(y-\frac{1}{2})^2 = \frac{1}{4}$ can be rewritten as $x = \pm\sqrt{\frac{1}{4} - (y - \frac{1}{2})^2}$ or $y = \pm\sqrt{\frac{1}{4} - x^2}+\frac{1}{2}$. Since we know that both $|x|$ or $|y - \frac{1}{2}|$ may not exceed the value of the radius of the disk we find that the integral can be represented as

$$\int_0^1 \int_{-\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}}^{\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}} f(x,y)dxdy$$

or

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\sqrt{\frac{1}{4}-x^2}+\frac{1}{2}}^{\sqrt{\frac{1}{4}-x^2}+\frac{1}{2}} f(x,y)dydx$$

31.6.5

The region defined is the circle of radius 2 with the circle of radius 1 taken out from the middle of it. It is easy to see that the bounds are not simple, not matter the direction you come from. Additionally, because the region has complete rotational symmetry, switching the order of the variables will not change the bounds, meaning the bounds for dydx will be similar to the bounds for dxdy. If we choose dydx for the order of integration, x has a hard bound of $x \in [-2, 2]$. However, the y bounds do vary depending on x. When $x \in [-2, -1]$, y is bounded by $y \in [0, \sqrt{4 - x^2}]$, $x \in [-1, 1] y \in [\sqrt{1 - x^2}, \sqrt{4 - x^2}$, and when $x \in [1, 2]$ $y \in [0, \sqrt{4 - x^2}]$. Additionally, since those bounds only account for the top half of the region, the area would be 2 times the sum of the double integrals defined by those bounds, or:

$$\iint_D dA = 2(\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} dydx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} dydx + \int_1^2 \int_0^{\sqrt{4-x^2}} dydx)$$

To reverse the order of integration, simply switch x and y.

31.6.6

$x^2 \leq x$ for $x \in [0, 1]$

$$\int_0^1 \int_{x^2}^{x} xydydx$$

$$= \int_0^1 x[\frac{y^2}{2}]_{x^2}^{x} dx$$

$$= \frac{1}{2} \int_0^1 x(x^2 - x^4)dx$$

$$= \frac{1}{2}\int_0^1 x^3 - x^5 dx$$

$$= \frac{1}{2}[\frac{x^4}{4} - \frac{x^6}{6}]_0^1$$

$$= \frac{1}{2}(\frac{1}{4} - \frac{1}{6})$$

$$\boxed{= \frac{1}{24}}$$

31.6.7

It is sensible to integrate the function with respect to x first and then y. Note that the intersection of the line $x = 3$ and the quadratic $x = 4 - y^2$ occur at $y = \pm 1$. The integral becomes

$$\int_{-1}^{1}\int_{3}^{4-y^2}(2+y)dxdy \to \int_{-1}^{1}(2x+yx)\Big|_3^{4-y^2} dy \to \int_{-1}^{1}(-y^3 - 2y^2 + y + 2)dy$$

$$= \frac{8}{3}$$

31.6.8

Evaluate $\iint_D (x+y)dA$ where D is the region bounded by the curves $x = y^4, x = y$. When sketching out the curves, one can see that the region is bounded in the y direction by $y \in [0,1]$ and x by $x \in [y^4, y]$, meaning the iterated integral is

$$\int_0^1\int_{y^4}^{y}(x+y)dxdy$$

$$= \int_0^1 (\frac{x^2}{2} + yx)|_{y^4}^{y} dy$$

$$= \int_0^1 \frac{1}{2}(y^2 - y^8) + y(y - y^4)dx = \int_0^1 \frac{1}{2}(y^4 - y^8) + y^2 - y^5 dx$$

$$= \frac{-1}{2}(\frac{y^9}{9} + \frac{y^6}{3} - y^3)|_0^1 = \frac{5}{18}$$

31.6.9

The lines $y = -x$ and $y = x$ intersect at $(0,0)$, then the x values are limited by $x = 3$. The integral is:

$$\int_0^3\int_{-x}^{x}(2+y)dydx$$

$$= \int_0^3 [2y + y^2/2]_{-x}^{x} dx$$

$$= \int_0^3 4x dx$$

$$= [2x^2]_0^3$$

$$\boxed{= 18}$$

31.6.10

It is sensible to integrate the function with respect to y first and then x. Note that the intersection between the curves given for the bounds in y occur at $x = \pm 1$. The integral becomes

$$\int_{-1}^{1} \int_{2+x^2}^{4-x^2} (x^2 y) dy dx \rightarrow \int_{-1}^{1} (x^2 \frac{y^2}{2})\Big|_{2+x^2}^{4-x^2} dx \rightarrow \int_{-1}^{1} (6x^2 - 6x^4) dx$$

$$= \frac{8}{5}$$

31.6.11

Evaluate the integral $\iint_D \sqrt{1-y^2} dA$ where D is the triangle with vertices (0,0) (0,1) and (1,0). First, notice that it would be useful to integrate in the order of dxdy, as it would make the integral easier to do, as having a y term with $\sqrt{1-y^2}$ would be easier to integrate with. So, the bounds for integration are $y \in [0,1]$ as can be seen with the vertices, and $x \in [0,y]$, which is best seen when you graph the bounds yourself. As a result, the integral is:

$$\int_0^1 \int_0^y \sqrt{1-y^2} dx dy$$

$$= \int_0^1 x\sqrt{1-y^2}|_0^y dy = \int_0^1 y\sqrt{1-y^2} dy$$

$$= -\frac{1}{2} \int_1^0 \sqrt{u} du = \frac{1}{2} \int_0^1 \sqrt{u} du = \frac{1}{3} u^{3/2}|_0^1 = 1/3$$

31.6.12

The lines $x = -3y$ and $x = 2y$ intersect at $(0,0)$.

$$\int_0^1 \int_{-3y}^{2y} xy dx dy$$

$$= \frac{1}{2} \int_0^1 y[x^2]_{-3y}^{2y} dy$$

$$= \frac{1}{2} \int_0^1 y(4y^2 - 9y^2) dy$$

$$= \frac{1}{2}\int_0^1 -5y^3 dy$$

$$= \frac{-5}{8}[y^4]_0^1$$

$$\boxed{= \frac{-5}{8}}$$

31.6.13

The triangle is convenient in the sense that we do not have to split the double integral up into multiple double integrals. This is because we can give the hypotenuse as the line segment given by the line $y = x$ for $0 \leq x \leq 1$. The leg of the triangle on the x axis can be our lower y bound.

So it is apparent that we should integrate with respect to y first (it is also easier that way) and then x. The integral becomes

$$\int_0^1 \int_0^x y\sqrt{x^2 - y^2} dydx \rightarrow -\frac{1}{3}\int_0^1 ((x^2 + y^2)^{\frac{3}{2}})\Big|_0^x \rightarrow \int_0^1 \left(\frac{1}{3}|x|^3\right) dx$$

$$= \frac{1}{12}$$

31.6.14

Choose the vertically simple region of integration. Then $0 \leq x \leq a$, and then $0 \leq y \leq -\sqrt{2ax - x^2} + a$. The double integral becomes:

$$\int_0^a \int_0^{-\sqrt{2ax-x^2}+a} (2a - x)^{-\frac{1}{2}} dydx \rightarrow \int_0^a -\sqrt{x} + a(2a - x)^{-\frac{1}{2}} dx$$

$$= \left[2\left(\sqrt{2} - 1\right) - \frac{2}{3}\right] a^{\frac{3}{2}}$$

31.6.15

By symmetry, the value of the integral in each of the four quadrants will be the same as four times the integral in the first quadrant.

$$\iint_D |xy| dA = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} xydydx$$

$$= 2\int_0^a x[y^2]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_0^a x(a^2 - x^2) dx$$

$$= 2\int_0^a xa^2 - x^3 dx$$
$$= 2[\frac{a^2x^2}{2} - \frac{x^4}{4}]_0^a$$
$$= 2a^4(\frac{1}{2} - \frac{1}{4})$$
$$\boxed{= \frac{a^4}{2}}$$

31.6.16

Upon sketching this parallelogram it is apparent that some divisions need to be made such that we can integrate on this region correctly. The four vertices of the parallelogram are given by $(0, a)$, (a, a), $(2a, 3a)$, and $(3a, 3a)$. Split up the parallelogram into two triangles and a middle section.

Give the left triangular section as the triangle bounded by the line segment (lying in $y = a$) connecting $(0, a)$ and (a, a), the line given by $y = x + a$ for $0 \leq x \leq a$, and a line segment starting at (a, a) going in the positive y direction until it intersects the line given by $y = x + a$. The double integral over this region is given by

$$\int_0^a \int_a^{x+a} (x^2 + y^2) dy dx \rightarrow \int_0^a \left(a^2x + ax^2 + \frac{4x^3}{3} \right) dx = \frac{7a^4}{6}$$

Then the middle section of the parallelogram is given by a parallelogram itself, but the sides to the left and right are just straight lines occupying only the y axis. The bounds in x for this section are $a \leq x \leq 2a$, and the bounds in y are $x \leq y \leq x + a$. The double integral here is

$$\int_a^{2a} \int_x^{x+a} (x^2 + y^2) dy dx \rightarrow \int_a^{2a} \left(\frac{a^3}{3} + a^2x + 2ax^2 \right) dx = \frac{13a^4}{2}$$

Finally the right triangle is given using bounds in x as $2a \leq x \leq 3a$, where the upper y curve is given by $y = 3a$ and the lower curve is given by the line $y = x$. The double integral is given by

$$\int_{2a}^{3a} \int_x^{3a} (x^2 + y^2) dy dx \rightarrow \int_{2a}^{3a} \left(9a^3 + 3ax^2 - \frac{4x^3}{3} \right) dx = \frac{19a^4}{3}$$

We must sum up the three results, and find that the double integral is equal to $14a^4$.

31.6.17

Evaluate $\iint_D y^2 dA$ where D is the cycloid defined by $x = a(t - sint), y = a(1 - cost), t \in [0, 2\pi]$. If you were to graph the cycloid, you would see that x goes from 0 to $2\pi a$, while y goes from 0 to $a(1 - cost)$. Because of x(t) differentiates, it will be easiest to integrate in the order of dydx. As a result, the integral is:

$$\int_0^{2\pi a} \int_0^{a(1-cost)} y^2 dydx$$

$$= \int_0^{2\pi a} \frac{y^3}{3}|_0^{a(1-cost)} dx = 1/3 \int_0^{2\pi a} (a(1 - cost))^3 dx$$

Even though now we have an integral where the integrand and differential do not match, we do have a relation between x and t, namely, $x(t) = a(t - sint)$. As a result, we can differentiate each side to get $dx = a(1-cost)dt$, now making the integral

$$1/3 \int_0^{2\pi} (a(1 - cost))^4 dt$$

making sure to change the bounds to be in terms of t, not x (x=0 corresponds to t=0, and x=2πa corresponds to t=2π). Integrating this is now just a matter of integrating powers of trigonometric functions; however there are some tricks that can make this easier. The integral expands out to:

$$\frac{a^4}{3} \int_0^{2\pi} \cos^4 t - 4\cos^3 t + 6\cos^2 t - 4\cos t + 1dt$$

which can be written as a sum of the integrals of each term. However, because of symmetry with the cosine function, any odd power of cosine along these bounds evaluates to 0, reducing the integral to just:

$$\frac{a^4}{3} \int_0^{2\pi} (1/2(1 - \cos 2t))^2 + 6\cos^2 tdt + \frac{2\pi a^4}{3}$$

Continuing the trend of using the power reduction identity and removing any term that is just an odd power of cos along the bounds will eventually yield

$$\frac{35\pi a^4}{12}$$

31.6.21

The integrand is a circular paraboloid ($z = x^2 + y^2$). The graph of $|x| + |y| \leq 1$ is a rectangle with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$.

$$\boxed{\text{A circular paraboloid bounded by a rectangle with vertices}(1, 0), (0, 1), (-1, 0), (0, -1).}$$

31.6.22

The integrand represents the top half of an ellipsoid (as in the half occupying the positive z axis). Such an ellipsoid intersects the z axis at $z = 1$, the y axis at $y = 3$, and the x axis at $x = 2$. The boundary of the set D is conveniently also the trace of the integrand where $z = 0$. It is apparent then that the solid region is just the positive half of an ellipsoid including the inside of that half.

31.6.24

Evaluate the integral $\iint_D 3x^2 + y^2 dA$ where D is the region bounded by the parabola $x = y^2$ and the line x=1. The bounds for integration are best defined by $y \in [-1, 1]$ and $x \in [y^2, 1]$.

$$\int_{-1}^{1}\int_{y^2}^{1} 3x^2 + y^2 dxdy$$

$$\int_{-1}^{1} (x^3 + y^2x)|_{y^2}^{1} dy = \int_{-1}^{1} (1 - y^6) + y^2(1 - y^2)dy$$

$$\int_{-1}^{1} -y^6 - y^4 + y^2 + 1dy = 2\int_{0}^{1} -y^6 - y^4 + y^2 + 1dy$$

$$2(-\frac{1}{7} - \frac{1}{5} + \frac{1}{3} + 1) = \frac{208}{105}$$

31.6.25

The integrand is $z = y$. x varies between 0 and 1 while y varies between 0 and $\sqrt{1 - x^2}$.

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} ydydx$$

$$\frac{1}{2}\int_{0}^{1} [y^2]_0^{\sqrt{1-x^2}} dx$$

$$\frac{1}{2}\int_{0}^{1} 1 - x^2 dx$$

$$\frac{1}{2}[x - x^3/3]_0^1$$

$$\frac{1}{2}(\frac{2}{3})$$

$$\boxed{= \frac{1}{3}}$$

31.6.26

Because of symmetry in all 3 directions, we can simplify the integral to:

$$8 \iint_{D'} \sqrt{a^2 - y^2} dA$$

where D' is the quarter of the unit circle in the first quadrant. We got 8 from multiplying 2 three times, one time because of symmetry in the z-direction, as it has the same volume when z is negative as positive, and from dividing the original region into fourths. The easiest order of integration is dxdy as it gets rid of the square root.

$$8 \int_0^1 \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - y^2} dx dy$$

$$8 \int_0^1 a^2 - y^2 dy = \frac{16a^3}{3}$$

31.6.29

$$\int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y) dy dx$$

The region defined by the bounds of integration is just the area between $y = x^3$ and $\sqrt{x}$. The inverse of these functions are $x = y^2$ and $x = \sqrt[3]{y}$. The intersection of these curves occur at y=0 and y=1. The new iterated integral would then be:

$$\int_0^1 \int_{y^2}^{\sqrt[3]{y}} f(x, y) dx dy$$

31.6.30

$$\int_0^1 \int_{y^2}^{y} f(x, y) dx dy$$

The line $x = y$ and the parabola $x = y^2$ intersect at $(0, 0)$ and $(1, 1)$. Solving each equation for y, the bounds are found to be $y = x$ and $y = \sqrt{x}$. $\sqrt{x} > x$ for $x \in (0, 1)$. Hence, the integral becomes

$$\boxed{\int_0^1 \int_{x}^{\sqrt{x}} f(x, y) dy dx}$$

31.6.31

The region is the area between the constant function $y = 1$ and the curve $y = e^x$ when $0 \leq x \leq 1$. We may invert the exponential function into $x = \ln(y)$ (omit the absolute value bars since we are in the first quadrant). Knowing the bounds from before, it is apparent that the new bounds are $\ln(y) \leq x \leq 1$ while $1 \leq y \leq e$. The double integral becomes

$$\int_1^e \int_{\ln(y)}^1 f(x,y)dxdy$$

31.6.32

$$\int_1^4 \int_{\sqrt{y}}^2 f(x,y)dxdy$$

The region defined by the bounds is between the curve $y = x^2$ and the lines y=1 and x=2. Redefining these bounds in terms of dydx gives:

$$\int_1^2 \int_1^{x^2} f(x,y)dydx$$

31.6.33

$$\int_0^3 \int_0^y f(x,y)dxdy + \int_3^6 \int_0^{6-y} f(x,y)dxdy$$

The region of integration of the first integral is the region bound by the lines $y = x$, $y = 3$, and $x = 0$. Reversing the order of integration gives:

$$\int_0^3 \int_x^3 f(x,y)dydx$$

The region of integration of the second integral is the region bound by the lines $x + y = 6$, $x = 0$, $y = 3$, and $y = 6$. Reversing the order of integration gives:

$$\int_0^3 \int_3^{6-x} f(x,y)dydx$$

The sum of these two integrals is:

$$\int_0^3 \int_x^3 f(x,y)dydx + \int_0^3 \int_3^{6-x} f(x,y)dydx$$

$$= \int_0^3 (\int_x^3 f(x,y)dydx + \int_3^{6-x} f(x,y)dydx)$$

By the additive properties of integrals, this can be simplified to:

$$\boxed{\int_0^3 \int_x^{6-x} f(x,y)dydx}$$

31.6.34

The region of integration is given as the inequalities $\sqrt{x} \leq y \leq 2$ while $0 \leq x \leq 4$. This is essentially the region above the square root curve under the constant $y = 2$ until they intersect, for x and y being positive.

To swap the order of integration we need to notice that the region expressed differently is the region under the parabola $x = y^2$ so long as y does not exceed 2. Evidently x and y have to be bounded below by zero.

So the integral can be rewritten as

$$\int_0^2 \int_0^{y^2} (1+y^3)^{-1} dx dy \rightarrow \int_0^2 y^2(1+y^3)^{-1} dy = \frac{2}{3}\ln(3)$$

31.6.35

$$\int_{-6}^{2} \int_{\frac{x^2}{4}-1}^{2-x} f dy dx$$

The bounds of the integral define the region between the curves y=2-x and $y = \frac{x^2}{4} - 1$. The inverse of each of these functions are $x = 2 - y$ and $x = \pm\sqrt{4y+4}$. However, the horizontal bounds for the region is defined by multiple functions. The first is the region in $y \in [0, 8]$, where x is bounded by $-\sqrt{4y+y}$ to the left and 2-y to the right, and second is the region bounded by the same function to the left but $\sqrt{4y-4}$ to the right, going from $y \in [0, -1]$. As a result the iterated integrals in the order of dxdy is:

$$\int_{-1}^{0} \int_{-\sqrt{4y-4}}^{\sqrt{4y-4}} f dx dy + \int_0^8 \int_{-\sqrt{4y-4}}^{2-y} f dx dy$$

31.6.37

The curves $y = \sqrt{2ax}$ and $y = \sqrt{2ax - x^2}$ represent a parabola and a hemisphere of radius a, respectively. The region itself is just the region between those two curves while x does not exceed $2a$. Both are located in the first quadrant, and we may rewrite each as $x = \frac{y^2}{2a}$ and $x = \pm\sqrt{a^2 - y^2} + a$.

Evidently in this perspective we do not have an x-simple region, so we must split the integral in 3 pieces around the point where it ceases to be simple, at (a, a). So now we may give three sets of bounds: $0 \leq y \leq a$ and $\frac{y^2}{2a} \leq x \leq -\sqrt{a^2 - y^2} + a$ for the first section, and then $0 \leq y \leq a$ and $\sqrt{a^2 - y^2} + a \leq x \leq 2a$ for the section adjacent to it. Finally the last section has bounds $a \leq y \leq 2a$ and $\frac{y^2}{2a} \leq x \leq 2a$. The double integral is thus equal to

$$\int_0^a \int_{\frac{y^2}{2a}}^{-\sqrt{a^2-y^2}+a} f(x, y) dx dy + \int_0^a \int_{\sqrt{a^2-y^2}+a}^{2a} f(x, y) dx dy$$

$$+ \int_a^{2a} \int_{\frac{y^2}{2a}}^{2a} f(x, y) dx dy$$

31.6.39

The region of integration is the triangle with vertices $(0,1)$, $(0,-1)$, and $(1,0)$. This region is symmetric with respect to the x-axis. Therefore, for each point (x,y) in the region that contributes $e^{x^2}\sin(y^3)dA$, there will be a point $(x,-y)$ in the region that contributes $e^{x^2}\sin((-y)^3)dA = -e^{x^2}\sin(y^3)dA$. These contributes cancel out, so we can conclude that the integral will be

$$\boxed{=0}$$

31.6.40

It is not really necessary to consider the shape of the set D itself, rather we may consider each quadrant individually. It is also important to note that the integrand is a saddle raised to the 9^{th} power, where the axes of symmetry are the x and y axes. Since 9 is an odd number, the sign of the values of the function are retained on the whole domain, so it is only necessary to investigate the saddle itself and we may omit the exponentiation entirely.

With that in mind, consider how the function changes sign around the lines given by $|x| = |y|$ - this information is sufficient (due to symmetry in all quadrants) to know that the double integral on D is going to vanish, it equals 0.

31.6.41

31.6.42

$$\iint_D (\cos(x^2) - \cos(y^2))dA$$

The region of integration is symmetric with respect to the line $y = x$. Therefore, for each point (x,y) in the region that contributes $(\cos(x^2) - \cos(y^2))dA$ to the integral, there is another point (y,x) that contributes $(\cos(y^2) - \cos(x^2))dA$ to the integral. These contributions cancel out, so we can conclude that the integral will be

$$\boxed{=0}$$

32.5 Exercises

32.5.1

From the bounds it is apparent that the radius r is bounded above by 2 and below by 1, while the angle θ is bounded above by π and below by 0. This means that we have points that lie on the top (positive) half of an annulus (a ring or a washer) lying in the xy plane.

The area is (by geometry) $\frac{3}{2}\pi$.

32.5.2

$$\int_{-\pi/2}^{\pi/2} \int_{0}^{2a\cos\theta} rdrd\theta$$

The polar graph $r = 2a\cos\theta$ is the circle of radius a centered at the point (a,0), and the range $\theta \in [-\pi/2, \pi/2]$ goes through the entire circle. Therefore, the area of the region is just the area of the circle, or $\pi(a)^2$

32.5.3

$$\int_{-\pi/4}^{\pi/4} \int_{0}^{1/\cos\theta} rdrdtheta$$

The bounds of the integral describe the region bounded by the lines:

$$\theta = \arctan(y/x) = -\pi/4 \rightarrow y = -x$$

$$\theta = \arctan(y/x) = \pi/4 \rightarrow y = x$$

$$r = 1/\cos\theta \rightarrow r\cos\theta = 1 \rightarrow x = 1$$

These lines form a triangle with vertices at $(0,0)$, $(1,-1)$, and $(1,1)$. This region is a right triangle with leg lengths $\sqrt{2}$,$\sqrt{2}$ and hypotenuse 2. The area of this triangle is

$$\frac{1}{2}ab\sin\theta = \frac{1}{2}\sqrt{2}\sqrt{2} = \boxed{1}$$

32.5.4

The bounds indicate that the region is the full cardioid given by $r = 1 + \cos(\theta)$. The area of the region is given by evaluating the double integral into the single integral and continuing the computation:

$$\frac{1}{2}\int_{-\pi}^{\pi} (1+\cos(\theta))^2\, d\theta \rightarrow \frac{1}{4}\int_{-\pi}^{\pi} (3 + 4\cos(\theta) + \cos(2\theta))\, d\theta = \frac{3}{2}\pi$$

32.5.9

The shape generated from the inequality $a^2 \leq x^2 + y^2 \leq b^2$ is the same as the interval $r \in [a,b]$ for polar coordinates, where r is the radius. The restriction on the quadrant restricts the range to $\theta \in [0, \pi/2]$. Using the bounds defined for r and θ along with converting x and y into polar coordinates transforms the integral into:

$$\int_{0}^{\pi/2} \int_{a}^{b} (r\cos\theta)(r\sin\theta)(rdrd\theta)$$

$$= \int_{a}^{b} r^3 dr \int_{0}^{\pi/2} \frac{1}{2}\sin 2\theta d\theta$$

$$\frac{1}{16}(b^4 - a^4)(2) = \frac{1}{8}(b^4 - a^4)$$

32.5.10

It is apparent from the disk D that the radius is bounded above by a and bounded below by 0, and that we have the full range in θ, that is 0 to 2π. Using the known equations for transforming coordinates from the cartesian coordinate system to the polar coordinate system (not forgetting the extra r in the integrand), the double integral becomes

$$\int_0^{2\pi}\int_0^a \left(\sin\left(r^2\right)\right) r dr d\theta \rightarrow \int_0^{2\pi} \frac{1}{2}\left(1-\cos\left(a^2\right)\right) d\theta = \pi(1-\cos\left(a^2\right))$$

32.5.11

$$\iint_D \arctan(y/x) dA$$

Converting to polar coordinates, the bounds of D become

$$a^2 \leq r^2 \leq b^2 \rightarrow a \leq r \leq b$$

$$y = x\sqrt{3} \rightarrow y/x = \sqrt{3} \rightarrow \arctan(y/x) = \theta = \pi/3$$

$$y = x/\sqrt{3} \rightarrow y/x = 1/\sqrt{3} \rightarrow \arctan(y/x) = \theta = \pi/6$$

Lastly, the integrand is:

$$\arctan(y/x) = \theta$$

and the Jacobian is $J = r$. Therefore, the transformed integral is:

$$\int_a^b \int_{\pi/6}^{\pi/3} r\theta d\theta dr$$

$$= (\int_a^b r dr)(\int_{\pi/6}^{\pi/3} \theta d\theta)$$

$$= \frac{1}{2}(b-a)[\theta^2/2]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{4}(b-a)(\pi^2/9 - \pi^2/36)$$

$$\boxed{= \frac{\pi^2}{48}(b-a)}$$

32.5.17

$$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} e^{x^2+y^2} dx dy$$

Graphing the region defined by the integrals shows that the region is a semicircle with radius 1 on the right side of the y-axis. This converts into a rectangular region in polar coordinates, with $r \in [0, 1]$ and $\theta \in [-\pi/2, \pi/2]$. Additionally, using the fact that $r^2 = x^2 + y^2$, the integral becomes:

$$\int_{-\pi/2}^{\pi/2} \int_0^1 e^{r^2} r dr d\theta$$

$$\pi/2 \int_0^1 e^u du$$

$$\frac{\pi}{2}(e - 1)$$

32.5.18

$$\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + y) dy dx$$

The region of integration is the left half of the unit circle. In polar coordinates, the integral becomes:

$$\int_{\pi/2}^{3\pi/2} \int_0^1 r(r\cos\theta + r\sin\theta) dr d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \int_0^1 r^2(\cos\theta + \sin\theta) dr d\theta$$

$$= \int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta)[r^3/3]_0^1 d\theta$$

$$= \frac{1}{3} \int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta) d\theta$$

$$= \frac{1}{3}[\sin\theta - \cos\theta]_{\pi/2}^{3\pi/2}$$

$$= \boxed{-\frac{2}{3}}$$

32.5.19

From the bounds it is found that $0 \le x \le \sqrt{1 - (y-1)^2}$ while y is in the closed interval $[0, 2]$. It is apparent that the shape of the region is the right half of a disk centered at the point $(0, 1)$ with radius 1.

Converting the integral to polar coordinates, the bounds change such that $0 \le r \le 2\sin(\theta)$ so long as $0 \le \theta \le \frac{\pi}{2}$. Knowing that $x^2 + y^2 = r$ in polar coordinates, and changing $dxdy$ to $rdrd\theta$, the integral becomes

$$\int_0^{\frac{\pi}{2}} \int_0^{2\sin(\theta)} (r)\, r dr d\theta \to \int_0^{\frac{\pi}{2}} \frac{8}{3}\sin^3(\theta) d\theta$$

$$\to -\frac{8}{3} \int_0^{\frac{\pi}{2}} (1 - \cos^2(\theta))(-\sin(\theta)) d\theta = \frac{16}{9}$$

32.5.20

$$\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xydydx$$

Looking at the bounds for the first integral, we see that it is bounded at the bottom for the positive half of the unit circle and at the top by the line y=x from where the two intersect to x=1. The second integral is just bounded by the line y=x and the x-axis from x=1 to $x = \sqrt{2}$. The final integral is bounded by the circle with radius 2 and the x-axis, starting at $x = \sqrt{2}$. Putting all of these regions together onto one coordinate plane reveals that the integral is:

$$\iint_D xydA$$

where D is the region bounded by the annulus with radii 1 and 2 from the x-axis to the line y=x. This region corresponds to the polar bounds of $r \in [1, 2], \theta \in [0, \pi/4]$, making the integral:

$$\int_0^{\pi/4}\int_1^2 (r\cos\theta)(r\sin\theta)rdrd\theta$$

$$\int_0^{\pi/4} \frac{1}{2}\sin 2\theta d\theta \int_1^2 r^3 dr$$

$$\frac{-1}{4}\cos 2\theta|_0^{\pi/4} \cdot \frac{1}{4}r^4|_1^2$$

$$\frac{15}{16}$$

32.5.27

$$r = 1 + \cos\theta$$

While θ varies between 0 and 2π, r varies between 0 and 1. Therefore, the area enclosed by the cardioid $r = 1 + \cos\theta$ can be calculated using:

$$\int_0^{2\pi}\int_0^1 rdrd\theta$$

$$= \int_0^{2\pi} [r^2/2]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}2\pi$$

$$\boxed{= \pi}$$

32.5.28

From the statement of the problem we find that $\theta/4 \le r \le \theta/2$, which we can use to express the original region in polar coordinates. Directly transforming, the region becomes the triangle bounded by the same values given in the problem (think two linear functions starting from the origin with different slopes meeting at a vertical line, and the triangle that forms from that relationship). The double integral that finds the area of the original region is

$$\int_0^{2\pi}\int_{\frac{\theta}{4}}^{\frac{\theta}{2}} r dr d\theta \to \frac{1}{2}\int_0^{2\pi}\left(\frac{\theta}{2}\right)^2 - \left(\frac{\theta}{4}\right)^2 d\theta = \frac{1}{4}\pi^3$$

32.5.29

To solve for the bounds you are integrating over, it is best to draw out the circle and cardioid and find the bounds of r in terms of theta. Solving for when $1 + \sin\theta = \frac{3}{2}$ gives that the angle is bounded by $\theta \in [\pi/6, 5\pi/6]$. The picture also shows that r is bounded from the top by the cardioid and at the bottom by the circle. As a result, the integral for the area becomes:

$$\int_{\pi/6}^{5\pi/6}\int_{3/2}^{1+\sin\theta} r dr d\theta$$

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6}(1+\sin\theta)^2 - \frac{9}{4} d\theta$$

$$\frac{9\sqrt{3}}{8} - \frac{\pi}{4}$$

32.5.34

Observing the two functions it becomes apparent that a solid forms where the surface $z_{top} = 4 - \sqrt{x^2 - y^2}$ is greater than or equal to $z_{bot} = 3\sqrt{x^2 - y^2}$. The region of integration in the rectangular plane is given by the closed set whose boundary is the level set $z_{top} - z_{bot} = 0$, which is when $x^2 + y^2 = 1$, meaning the region of integration is the disk given by $x^2 + y^2 \le 1$. We may convert to polar coordinates by bounding r from 0 to 1 and by bounding θ from 0 to 2π. The double integral becomes

$$\int_0^{2\pi}\int_0^1 (4 - r - 3r)\, r dr d\theta \to \int_0^{2\pi}\frac{2}{3}d\theta = \frac{4}{3}\pi$$

32.5.35

The region of integration is the annulus with inner radius 1 and outer radius 2. The integrand is simply the equation for the cone, since we are captuiring the signed volume under it.

Since we have the annulus as defined above, it is evident that $1 \leq r \leq 2$ and that θ takes on its natural range.

Replacing the equation for the cone with $z = r$, the double integral becomes

$$\int_0^{2\pi} \int_1^2 r(r) dr d\theta \rightarrow \int_0^{2\pi} \frac{7}{3} d\theta = \frac{14}{3}\pi$$

32.5.36

Looking at the graph, the base of the region of integration is the circle of radius 2 centered at the origin. One can derive this from setting z=-3 to the first equation, getting $4 = x^2 + y^2$. This region corresponds the the rectangular region $r \in [0, 2], \theta \in [0, 2\pi]$ in polar coordinates. Additionally, the volume of the solid is the double integral of upper bound - lower bound dA, or:

$$\iint_D (1 - x^2 - y^2) - 3dA$$

$$2\pi \int_0^2 (4 - r^2) r dr$$

$$8\pi$$

32.5.37

Evidently the solid is formed when values attained by the plane are greater than or equal to the values attained by the upper sheet of the hyperboloid. This means that the region of integration happens on the set bounded by the level set where both surfaces attain the same value. So from $x^2 + y^2 - 4 = -1$ we deduce that $x^2 + y^2 \leq 3$ is the region of integration. It will help to rewrite the upper sheet of the hyperboloid as the surface $z = \sqrt{x^2 + y^2 + 1}$

Converting to polar coordinates, we find that r is bounded from 0 to $\sqrt{3}$, and then θ can take on the full range of values from 0 to 2π. The integral becomes

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \left(2 - \sqrt{r^2 + 1}\right) r dr d\theta \rightarrow \int_0^{2\pi} \frac{2}{3} d\theta = \frac{4}{3}\pi$$

32.5.38

The volume of such solids is given by the general equation:

$$\iint (z_{top} - z_{bottom}) dA$$

In polar coordinates, $z_{top} = r^2$ and $z_{bottom} = 0$. The bounds become:

$$z = x^2 + y^2 \rightarrow z = r^2$$

$$x^2 + y^2 = 2x \rightarrow r^2 = 2r\cos\theta \rightarrow r = 2\cos\theta$$

Therefore, r varies from 0 to $2\cos\theta$ and θ varies from 0 to π because the cylinder $r = 2\cos\theta$ makes one full revolution between $\theta = 0$ and $\theta = \pi$. The Jacobian is $J = r$. The transformed integral is:

$$\int_0^{\pi}\int_0^{2\cos\theta} r^3 dr d\theta$$

$$= \frac{1}{4}\int_0^{\pi} [r^4]_0^{2\cos\theta} d\theta$$

$$= 4\int_0^{\pi} \cos^4\theta d\theta$$

$$= 4\int_0^{\pi} \cos^2\theta(1-\sin^2\theta) d\theta$$

$$= 4\int_0^{\pi} (\cos^2\theta - \cos^2\theta\sin^2\theta) d\theta$$

Using the sine double angle formula:

$$= 4\int_0^{\pi} (\cos^2\theta - \frac{1}{4}\sin^2(2\theta)) d\theta$$

Simplifying and using the identities $\cos^2 x = \cos(2x)/2 + 1/2$ and $\sin^2 x = -\cos(2x)/2 + 1/2$:

$$\int_0^{\pi} (4\cos(2x)/2 + 2) d\theta - \int_0^{\pi} (-\cos(4\theta)/2 + 1/2) d\theta$$

Finally, integrate $\cos(ax)$ normally to get:

$$\boxed{= 3\pi/2}$$

32.5.39

$$\lim_{x\to 0} \frac{1}{\pi a^2} \iint_D f(x,y) dA$$

33.5 Exercises

33.5.7

We are asked to transform from the (u, v) coordinates into the (x, y) coordinates. Since the region D' is a square, we may represent the boundary of the square with four line segments lying on these four lines: $u = 0$, $u = 1$, $v = 0$, and $v = 1$.

Using the equations given in the problem we may solve for the boundary of D in the (x, y) coordinate system. It is immediate that $x = 0$ and $x = 1$ are lines that form part of the new boundary.

Since y depends on both u and v, we may want to hold one of those variables constant and observe what happens if we vary the other (while still maintaining the bounds given by D'). Trying with $v = 0$, it is apparent that $y = 0$. That is fine. Then if we try with $v = 1$, we can still vary u. According to the definition of D', u varies from 0 to 1, exactly as x does (as per $x = u$). Therefore we may give $y = 1 - x^2$.

Hence D is the area bounded by the line $x = 0$, $y = 0$, and $y = 1 - x^2$. (the area under the parabola in the first quadrant)

33.5.8

D' is the triangle with vertices (0,0) (1,1) and (1,0). This can be interpreted as the region defined by the bounds $u \in [0, 1]$ and $v \in [0, u]$. Applying the fact that $x = v^2, y = u$ transforms those bounds to $y \in [0, 1]$ and $x \in [0, (u)^2] = [0, y^2]$. As a result, the region D is defined by those bounds in the xy-plane.

33.5.9

The region D' in the uv plane defined by the inequality $|u|+|v| \leq 1$ is equivalent to the region bounded by the points $u + v = x = -1$, $u + v = x = 1$ and $u - v = y = -1$, $u - v = y = 1$. Therefore,

$$\boxed{\text{The region D in the } xy \text{ plane is the square with vertices (0,0), (1,0), (1,1), and (0,1).}}$$

33.5.11

We first find equations of lines passing through those four coordinate points. They are given by $y = x + 4$, $y = x - 4$, $y = -\frac{1}{3}x + \frac{8}{3}$, and $y = -\frac{1}{3}x$. We may convert each of these lines to their corresponding lines in the (u, v) coordinate system using the transformation given in the problem. Simply substitute the given equations for y and x into each of the lines and resolve into coordinate curves.

We find the following curves: $u = \pm 4$, $v = 0$, and $v = 8$. This is a rectangular region so it is convenient in the integral. The Jacobian of transformation for this change of variables is given by the determinant:

$$\left| \det \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \right| = \frac{1}{4}$$

The integral becomes

$$\int_0^8 \int_{-4}^4 \left(8 \left(\frac{v - 3u}{4} \right) + 4 \left(\frac{u + v}{4} \right) \right) \left(\frac{1}{4} \right) dudv \to 6 \int_0^8 vdv = 192$$

33.5.12

$$\iint_D x^2 - xy + y^2 dA$$

Using the change of variables defined, the Jacobian is:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix} = \frac{2}{\sqrt{3}}$$

Substituting the redefined variables into the expression $x^2 - xy + y^2$ shows that the equivalent expression is $u^2 + v^2$, meaning D' is the region bounded by the circle $u^2 + v^2 = 1$ and the double integral becomes:

$$\frac{2}{\sqrt{3}} \iint_{D'} u^2 + v^2 dA'$$

It would probably be best to change the variables again into polar, giving the bounds as $r \in [0, 1], \theta \in [0, 2\pi]$ and the double integral:

$$\frac{2}{\sqrt{3}} \int_0^{2\pi} \int_0^1 r^2 r dr d\theta$$

$$\frac{4\pi}{\sqrt{3}} (\frac{1}{4} r^4)|_0^1$$

$$\frac{\pi}{\sqrt{3}}$$

33.5.13

$$\iint (x^2 - y^2)^{-1/2} dA$$

It is important to recall the identity $\cosh^2 \theta - \sinh^2 \theta = 1$. The original region is bounded by $x^2 - y^2 = u^2 = 1$ and $x^2 - y^2 = u^2 = 4$, so u is bounded by $u = 1$ and $u = 2$. Similarly, the original region is bounded by $x = 2y$ and $x = 4y$, which is equivalent to

$$u \cosh v = 2u \sinh v \to \tanh v = \frac{1}{2} \to v = \tanh^{-1}(1/2)$$

$$u \cosh v = 4u \sinh v \to \tanh v = \frac{1}{4} \to v = \tanh^{-1}(1/4)$$

Now, we must compute the Jacobian $J(u, v)$.

$$J(u, v) = |\det \begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix}|$$

$$= |\det \begin{pmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \end{pmatrix}|$$

$$= u \cosh^2 v - u \sinh^2 v = u$$

Therefore, the transformed integral is:

$$\int_{\tanh^{-1} 1/4}^{\tanh^{-1} 1/2} \int_{1}^{2} (u)(u^2 \cosh^2 v - u^2 \sinh^2 v)^{-1/2} du dv$$

$$= \int_{\tanh^{-1} 1/4}^{\tanh^{-1} 1/2} \int_{1}^{2} du dv$$

$$= \int_{\tanh^{-1} 1/4}^{\tanh^{-1} 1/2} dv$$

$$= \int_{\tanh^{-1} 1/4}^{\tanh^{-1} 1/2} dv$$

$$\boxed{= \tanh^{-1}(1/2) - \tanh^{-1}(1/4)}$$

33.5.14

From the original bounds we can rewrite them as $\frac{x}{y} = 1$, $\frac{x}{y} = \frac{1}{2}$, $x + y = 1$, and $x + y = 2$. Then knowing that $u = \frac{x}{y}$ and $v = x + y$, it is apparent that $\frac{1}{2} \le u \le 1$ and $1 \le v \le 2$. The Jacobian of transformation is found by observing the following:

$$dudv = Jdxdy \to \frac{1}{J} dudv = dxdy$$

$$J = \left| \det \begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{1}{y} & 1 \\ \frac{-x}{y^2} & 1 \end{pmatrix} \right| = \frac{x+y}{y^2}$$

Remembering to take the reciprocal of J, the integral becomes

$$\iint e^{\frac{x}{y}} \frac{(x+y)^3}{y^2} \left(\frac{y^2}{x+y} \right) dudv \to \int_1^2 \int_{\frac{1}{2}}^{1} e^u v^2 dudv \to \left(e^u|_1^2\right) \left(\frac{v^3}{3} \Big|_1^2 \right)$$

$$= \frac{7}{3}(e - \sqrt{e})$$

33.5.19

The lazy way out is sometimes the easiest way out. Give $u = xy$ and $v = xy^2$. Then it is evident that $1 \le u \le 2$ and $1 \le v \le 2$ from the definitions of the bounding curves in the (x, y) coordinate system. Then to compute the Jacobian we may want to compute it for the reverse transformation, that is to compute J for

$$dudv = Jdxdy$$

So compute partial derivatives of u and v and compute the following determinant:

$$J = \left| \det \begin{pmatrix} y & 2xy \\ x & x^2 \end{pmatrix} \right| = yx^2$$

So $dudv = (yx^2)dxdy$, which is already in the integral. Conveniently the integral becomes

$$\int_1^2 \int_1^2 dudv = 1$$

33.5.20

$$\iint_D e^{x-y} dA$$

As discussed in several previous problems, the region defined by the inequality $|x|+|y| \leq 1$ is equivalent to the region bounded by the lines $x+y=1$, $x+y=-1$ and $x-y=1$, $x-y=-1$. Therefore, we make the substitution $u = x+y$ and $v = x-y$. The Jacobian of this substitution is given by:

$$J(u,v) = |\det \begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix}|$$

$$= |\det \begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix}|^{-1}$$

$$= |\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}|^{-1}$$

$$= |-2|^{-1}$$

$$= \frac{1}{2}$$

Therefore, the transformed integral is:

$$\frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} e^v dvdu$$

$$= \frac{1}{2} \int_{-1}^{1} [e^v]_{-1}^{1} du$$

$$= \frac{e - \frac{1}{e}}{2} \int_{-1}^{1} du$$

$$\boxed{= e - \frac{1}{e}}$$

33.5.21

Give $u = x+y$ and $v = y - x^3$, so that $1 \leq u \leq 2$ and $0 \leq v \leq 1$. Computing the Jacobian J that satisfies $\frac{1}{J}dudv = dydx$ by taking the partial derivatives of u and v, we find

$$J = \left| \det \begin{pmatrix} 1 & -3x^2 \\ 1 & 1 \end{pmatrix} \right| = 1 + 3x^2$$

This makes the integral convenient since it makes the new integrand 1. The double integral becomes (by geometry)

$$\int_0^1 \int_1^2 dudv = 1$$

33.5.22

With some rearranging of the parabola equations it becomes apparent that we may give $u = xy$ and $v = y - x^2$ to find that $-1 \leq u \leq 1$ and $1 \leq v \leq 2$. Then we may compute the Jacobian where

$$dudv = Jdxdy$$

This is given by the following determinant:

$$J = \left| \det \begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix} \right| = \left| \det \begin{pmatrix} y & -2x \\ x & 1 \end{pmatrix} \right| = y + 2x^2$$

So $dudv = (y + 2x^2)dxdy$, which is conveniently already in the integral. The integral then becomes

$$\int_1^2 \int_{-1}^1 dudv = 2$$

33.5.23

$$\iint_D (x+y)^2/x^2 dA = \iint_D (1 + y/x)^2 dA$$

The bounds of the region of integration are $x + y = 1$, $x + y = 2$ and $y = x$, $y = 2x$. Rearranging the latter pair to $y/x = 1$, $y/x = 2$ makes it clear that we should substitute $u = x + y$ and $v = y/x$, or equivalently $x = \frac{u}{v+1}$ and $y = \frac{uv}{v+1}$. The Jacobian of this substitution is:

$$J(u, v) = |\det \begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix}|$$

$$= |\det \begin{pmatrix} u'_x & v'_x \\ u'_y & v'_y \end{pmatrix}|^{-1}$$

$$= |\begin{pmatrix} 1 & -y/x^2 \\ 1 & 1/x \end{pmatrix}|^{-1}$$

$$= |\frac{1}{x} + \frac{y}{x^2}|^{-1}$$

$$= \frac{x^2}{x+y}$$

$$= \frac{u^2}{u(1+v)^2}$$

$$= \frac{u}{(1+v)^2}$$

Therefore, the transformed integral is:

$$\int_1^2 \int_1^2 (\frac{u}{(1+v)^2})(1+v)^2 dudv$$

$$= \int_1^2 \int_1^2 ududv$$

$$= \int_1^2 udu$$

$$= \frac{1}{2}[u^2]_1^2$$

$$\boxed{= 3/2}$$

34.6 Exercises

34.6.1

Since the solid region E is given by a rectangular prism, the bounds of the integral are easy to set up. The integral becomes

$$\int_0^2 \int_1^2 \int_0^1 (xy - 3z^2) dxdydz \to \int_0^2 \int_1^2 \left(\frac{1}{2}y - 3z^2\right) dydz$$

$$\to \int_0^2 \left(\frac{3}{4} - 3z^2\right) dz = -\frac{13}{2}$$

34.6.2

$$\iiint_E 6xzdV$$

The inequalities given define the bounds for each of the variables, making them $y \in [0, x+z], x \in [0, z], z \in [0, 1]$. The bounds also help us find the best order to integrate in, dy dx dz, since the bounds on y are dependent on each of the other variables, and x dependent only on z. The iterated integral is then:

$$\int_0^1 \int_0^z \int_0^{x+z} 6xzdydxdz$$

$$\int_0^1 \int_0^z 6xz(x+z)dxdz = 6 \int_0^1 \int_0^z x^2z + xz^2 dxdz$$

$$\int_0^1 3x^2z^2 + 2x^3z|_0^z dz = \int_0^1 3z^4 + 2z^4$$

$$\int_0^1 5z^4 dz = z^5|_0^1 = 1$$

34.6.3

$$\iiint_D ze^{y^2}dV$$
$$= \int_0^1 \int_0^z \int_0^y ze^{y^2}dxdydz$$
$$= \int_0^1 \int_0^z yze^{y^2}dydz$$
$$= \frac{1}{2}\int_0^1 z[e^{y^2}]_0^z dz$$
$$= \frac{1}{2}\int_0^1 ze^{z^2} - zdz$$
$$= \frac{1}{4}[e^{z^2} - z^2]_0^1$$
$$= \frac{1}{4}((e-1)-(1))$$
$$\boxed{= \frac{1}{4}(e-2)}$$

34.6.4

The region is vertically simple (vertical meaning parallel to the $\vec{e}_3$ basis vector). We can find the bounds for z in the vertically simple manner quickly from the problem statement. So the following inequality holds: $0 \le z \le x + y + 1$.

The planar region in the xy plane that we wish to find bounds for in x and y is part of the interior side of a parabola. Note that since $x = 0$ and $y = 1$ are part of the bounds, it follows that the parabola given by $x = \sqrt{y}$ can only be traced out so long as $0 \le y \le 1$. It is also known that $0 \le x \le \sqrt{y}$. We may set up the triple integral as follows:

$$\int_0^1 \int_0^{\sqrt{y}} \int_0^{x+y+1} (6xy)\, dzdxdy \to \int_0^1 \int_0^{\sqrt{y}} (6x^2y + 6xy^2 + 6xy)dxdy$$
$$\to \int_0^1 \left(3y^2 + 2y^{\frac{5}{2}} + 3y^3\right) dy = \frac{65}{28}$$

34.6.7

We want to find out what the region of integration is. Imagine the positive portion of the cylinder $z = \sqrt{1-y^2}$ is sliced by the planes $x = y$ and $x = 0$. we essentially have just a slice of the cylinder that lies in the first octant, much like the image in Figure **34.3 Right** (for Study Problem **34.1**).

So we know that for z we have $0 \le z \le \sqrt{1-y^2}$. Then for the planar region we need to find bounds. We know that the cylinder intersects the line $x = y$

when $y = 1$. We can also deduce that y may only drop to 0 and nothing less. So bounds on y are $0 \le y \le 1$. As for x, we have a line $x = 0$ and another $x = y$ that give us bounds $0 \le x \le y$.

So the triple integral becomes

$$\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} (zx)\, dzdxdy \to \frac{1}{2} \int_0^1 \int_0^y \left(x - xy^2\right) dxdy$$

$$\to \frac{1}{4} \int_0^1 \left(y^2 - y^4\right) dy = \frac{1}{30}$$

34.6.8

$$\iiint_E z(x^2 + y^2) dV$$

The region is bounded by $z = 0$ and $z = 1 - x^2 - y^2$. Doing a polar substitution, this becomes $z = 0$ and $z = 1 - r^2$. Therefore, the transformed integral is (Note: Remember to include the Jacobian) :

$$\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} zr^3 dzdrd\theta$$

$$= 2\pi \int_0^1 \int_0^{1-r^2} zr^3 dzdr$$

$$= \pi \int_0^1 r^3 [z^2]_0^{1-r^2} dr$$

$$= \pi \int_0^1 r^3 (1 - r^2)^2 dr$$

$$= \pi \int_0^1 r^3 (r^4 - 2r^2 + 1)$$

$$= \pi \int_0^1 r^7 - 2r^5 + r^3$$

$$= \pi [r^8/8 - r^6/3 + r^4/4]_0^1$$

$$\boxed{= \pi/24}$$

34.6.12

The plane $z = 1 - x$ intersects with the plane $z = 0$ when $x = 1$. This constitutes an upper bound for x, where y^2 is the lower bound. So $y^2 \le x \le 1$. Then we wish to find out how much y can vary. Notice that due to $x = 1 = y^2$,

$-1 \leq y \leq 1$. It is also known from the start that $0 \leq z \leq 1-x$. So we may set up the integral and evaluate it as follows:

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x}(1)dzdxdy \to \int_{-1}^{1}\int_{y^2}^{1}(1-x)dxdy \to \int_{-1}^{1}\left(\frac{1}{2}-y^2+\frac{y^4}{2}\right)dy$$

$$= \frac{8}{15}$$

34.6.13

To find the region of integration it is easiest to find it in the vertically simple way. The bounds in z are straightforward, they are $0 \leq z \leq 4-y^2$. We may find the bounds for the x values by rearranging the equations for the lines given to find out the bounds. So $\frac{1}{2}y \leq x \leq y$. The parabolic sheet intersects the xy plane where $y = 2$ (omitting the negative root since we are in the first octant) and the lines $y = x$ and $y = 2x$ intersect at the origin so bounds on y are $0 \leq y \leq 2$.

We have the following triple integral:

$$\int_{0}^{2}\int_{\frac{1}{2}y}^{y}\int_{0}^{4-y^2} dzdxdy \to \int_{0}^{2}\int_{\frac{1}{2}y}^{y}(4-y^2)dxdy \to \frac{1}{2}\int_{0}^{2} 4y-y^3dy = 2$$

34.6.18

In the 2-D region bounded by $x = 0$, $y = 0$, and $x + y = 1$: $xy \leq x + y$. Therefore, we can write the integral as:

$$\int_{0}^{1}\int_{0}^{1-x}\int_{xy}^{x+y} dzdydx$$

$$= \int_{0}^{1}\int_{0}^{1-x}(x+y-xy)dydx$$

$$= \int_{0}^{1}[xy+y^2/2(1-x)]_0^{1-x}dx$$

$$= \int_{0}^{1} x(1-x)+(1-x)^3/2dx$$

$$= \int_{0}^{1}(x-x^2-x^3/2+3x^2/2-3x/2+1/2)dx$$

$$= \int_{0}^{1}(-x^3/2+x^2/2-x/2+1/2)dx$$

$$= -1/8+1/6-1/4+1/2$$

$$\boxed{= 7/24}$$

34.6.21

The surface $z = 6 - x^2 - y^2$ is a paraboloid and the surface $z = \sqrt{x^2 + y^2}$ is a single cone opening upwards. The intersection of these surfaces happens where $z = 2$ (you may combine the equations for the surfaces into $z = 6 - z^2$ and solve for the positive root). At $z = 2$, the intersection is a circle of radius 2 centered at the origin. So the region of integration is given by $x^2 + y^2 \leq 4$ where $\sqrt{x^2 + y^2} \leq z \leq 6 - (x^2 + y^2)$.

We will opt to use polar coordinates to evaluate this integral. Knowing that $x^2 + y^2 \leq 4$, it is apparent that $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. The triple integral becomes

$$\int_0^{2\pi} \int_0^2 \int_{\sqrt{x^2+y^2}}^{6-(x^2+y^2)} (1) dz(r) dr d\theta \to \int_0^{2\pi} \int_0^2 (6r - r^3 - r^2) dr d\theta \to \int_0^{2\pi} \frac{16}{3} d\theta$$

$$= \frac{32}{3}\pi$$

34.6.22

The region E is symmetric axross the planes $z = 0$ and $x = 0$, and we may use this to our advantage. Notice that the integrand contains terms multiplied by x or z^3, which is skew symmetric across those planes. Even when multiplied together, skew symmetry holds. So automatically the integral over the region E with the rectangular cavity not made vanishes.

Notice that by construction the integral we wish to find can be represented like so:

$$\iiint_E 24xy^2z^3 dA$$

$$= \iiint_{\text{E without cavity}} 24xy^2z^3 dA - \iiint_{[0,1]\times[-1,1]\times[0,1]} 24xy^2z^3 dA$$

$$= 0 - \iiint_{[0,1]\times[-1,1]\times[0,1]} 24xy^2z^3 dA$$

So really all we are tasked to do is to find out what the integral over the rectangular cavity (as a solid) would have been.

$$-\int_0^1 \int_{-1}^1 \int_0^1 (24xy^2z^3) dx dy dz \to -\int_0^1 \int_{-1}^1 (12y^2z^3) dy dz \to -\int_0^1 (8z^3) dz$$

$$= -2$$

34.6.23

The region E is the ball centered at the origin with radius 2 less the unit ball centered at the origin. This region is symmetrical about the plane $y = z$. Therefore, for each point (x, y, z) that contributes $(\sin^2(xz) - \sin^2(xy))dV$ to the integral, there is a point (x, z, y) that contributes $(\sin^2(xy) - \sin^2(xz))dV$. These contributions cancel out, so the integral is:

$$\boxed{= 0}$$

34.6.24

Rewite the integrand as $1 + \sin^2(xz) - \sin^2(xy)$. Apply linearity to find the two integrals given by

$$\iiint_E dV + \iiint_E (\sin^2(xz) - \sin^2(xy))dV$$

Notice that the integrand of the second integral is skew symmetric about the same plane $z = y$, because if we apply the transformation $(x, y, z) \to (x, z, y)$ the sign of the integrand is flipped. The region E itself is a region between two spheres, which is symmetric across the plane $z = y$, so we can conclude that the second integral will vanish.

We simply compute the first integral by geometry:

$$\iiint_E dV = \frac{4}{3}\pi(2^3 - 1^3) = \frac{28}{3}\pi$$

34.6.32

Find the region E for which:

$$\iiint_E (1 - x^2/a^2 - y^2/b^2 - z^2/c^2)dV$$

is maximized. Looking at the function you can see that each level set of it is an ellipsoid. However, the function is positive for only a certain region E. Therefore, it makes the most sense that the region E is all values where $f(x, y, z) = 1 - x^2/a^2 - y^2/b^2 - z^2/c^2 \geq 0$. Solving this yields the solid ellipsoid:

$$1 \geq x^2/a^2 + y^2/b^2 + z^2/c^2$$

.

35.6 Exercises

35.6.4

We do not need to change anything for the z bounds since they are given directly save for just applying $r = x^2 + y^2$. Then $0 \leq z \leq r^2$. Then since the region is bounded by a cylinder of unit radius it is apparent that $0 \leq r \leq 1$ and that $0 \leq \theta \leq 2\pi$.

35.6.5

There is no restriction on θ, so we can say that theta goes on its natural range of $\theta \in [0, 2\pi]$. Additionally, the cylinder $x^2 + y^2 = 1$ translates directly to the bound $r \in [0, 1]$. Finally, the cone $(z-1)^2 = x^2 + y^2$ becomes $z - 1 = \pm r$, giving us the bound of z to be $z \in [1 - r, 1 + r]$. If one were to sketch out each of the restrictions, the bounds themselves would also make more intuitive sense, so if possible, try to sketch it out and make sure you can visually see where each of the bounds come from.

35.6.6

The transformation to cylindrical coordinates results in the following change in the bounds of E:

$$z = 0 \to z = 0$$

$$z = x^2 + y^2 \to z = r^2$$

$$x^2 + y^2 = 2x \to r^2 = 2r\cos\theta \to r = 2\cos\theta$$

Because a lower bound for r is not specified, there is an implicit lower bound of $r = 0$. Additionally $r = 2\cos\theta$ traces out one rotation of the unit circle centered at $(x, y) = (1, 0)$ on the interval $\theta \in (-\pi/2, \pi/2]$, so these are the bounds on θ. Therefore, E' is the region with bounds:

$$0 \leq z \leq r^2$$

$$0 \leq r \leq 2\cos\theta$$

$$-\pi/2 \leq \theta \leq \pi/2$$

35.6.7

Since E is the sphere of radius a in the first octant, it is easy to see that θ may only vary from 0 to $\frac{\pi}{2}$. Furthermore since the radius of the sphere is a, the quantity r may only vary from 0 to a. The surfaces for z that bound the surface is the plane $z = 0$ and $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$. So $0 \leq z \leq \sqrt{a^2 - r^2}$

35.6.8

$$\iiint_E |z| dV$$

The polar angle is unbounded, making its bound its natural range, $\theta \in [0, 2\pi]$. Additionally, the cylinder $x^2 + y^2 = 1$ gives the restriction of r to be $r \in [0, 1]$. Finally, one can manipulate the equation of the sphere to be $z^2 = 4 - r^2$, making the bounds for z $z \in [-\sqrt{4 - r^2}, \sqrt{4 - r^2}]$. The iterated integral then becomes:

$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} |z| r dz dr d\theta$$

However, the function —z— is even, which simplifies the integral to:

$$4\pi \int_0^1 \int_0^{\sqrt{4-r^2}} zrdzdr$$

$$2\pi \int_0^1 r(\sqrt{4-r^2})^2 dr = 2\pi \int_0^1 4r - r^3 dr$$

$$2\pi(2r^2 - \frac{r^4}{4})|_0^1 = 2\pi(2 - \frac{1}{4})$$

$$\frac{7\pi}{2}$$

35.6.9

$$\iiint_E (x^2y + y^3)dV = \iiint_E y(x^2 + y^2)dV$$

Using the substitution $r^2 = x^2 + y^2$ and $\tan\theta = y/x$, the bound $z = 1 - x^2 - y^2$ becomes $z = 1 - r^2$. The constraint that the region is in the first octant implies $0 \le \theta \le \pi/2$ and $0 \le r \le 1$. The Jacobian is, of course, $J = r$. Therefore, the transformed integral is:

$$\int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} r(r\sin\theta)(r^2)dzdrd\theta$$

$$= \int_0^{\pi/2} \int_0^1 \int_0^{1-r^2} r^4 \sin\theta dzdrd\theta$$

$$= (\int_0^{\pi/2} \sin\theta d\theta)(\int_0^1 r^4(1 - r^2)dr)$$

$$= \int_0^1 (r^4 - r^6)dr$$

$$= [\frac{r^5}{5} - \frac{r^7}{7}]_0^1$$

$$= 1/5 - 1/7$$

$$\boxed{= 2/35}$$

35.6.10

The region of integraion is a cylinder with a cylindrical cut out, where its base sits on the xy plane and is cut off above by the plane $z = x + y + 5$. This plane does not cut off the cylinder short on the xy plane, so the base is given by an annulus of inner radius 1 and outer radius 2. This is enough to deduce that $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

To find bounds in z apply the transformation $(x, y) \to (r\cos(\theta), r\sin(\theta))$ to the plane $z = x+y+5$. The bottom bound is still 0. The upper bound becomes $r\cos(\theta) + r\sin(\theta) + 5$. Applying the same transformation to the integrand, the triple integral becomes:

$$\int_0^{2\pi} \int_1^2 \int_0^{r\cos(\theta)+r\sin(\theta)+5} r\sin(\theta)dz(r)drd\theta$$

$$\to \int_0^{2\pi} \int_1^2 \left(r^3\sin(\theta)\cos(\theta) + r^3\sin^2(\theta) + 5r^2\sin(\theta)\right) drd\theta$$

$$\to \int_0^{2\pi} \int_1^2 \left(\frac{r^3}{2} + \frac{r^3}{2}\left(\sin(2\theta) - \cos(2\theta)\right) + 5r^2\sin(\theta)\right) drd\theta$$

The last two terms will vanish due to the periodicity of the sine and cosine. Then the remaining integral is

$$\int_0^{2\pi} \int_1^2 \frac{r^3}{2} drd\theta \to \pi \frac{r^4}{4}\bigg|_1^2 = \frac{15}{4}\pi$$

35.6.11

Converting the cylinder $x^2+y^2 = 2x$ into polar coordinates yields $r = 2\cos\theta$, as in the xy-plane it is the circle with radius 1 with its center at (1,0). Alternatively, using the substitutions $x^2+y^2 = r^2$ and $x = r\cos\theta$ also yields the same results. since there is no other bound on r, $r \in [0, 2\cos\theta]$. The plane and cone both directly give the restriction on z to be $z \in [0, r]$. However, there is a restriction on the polar angle, as the circle only exists on the right side of the y-axis, making the bound $\theta \in [-\pi/2, \pi/2]$. The iterated integral then becomes:

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r rdzdrd\theta$$

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 drd\theta$$

$$1/3 \int_{-\pi/2}^{\pi/2} (2\cos\theta)^3 d\theta = \frac{16}{3} \int_0^{\pi/2} \cos^3\theta d\theta$$

$$\frac{16}{3} \int_0^{\pi/2} (1 - \sin^2\theta)\cos\theta$$

Using u=$\sin\theta$, du = $\cos\theta d\theta$:

$$\frac{16}{3}\int_0^1 (1-u^2)du = \frac{16}{3}(u - \frac{u^3}{3})|_0^1$$

$$\frac{32}{9}$$

35.6.12

$$\iiint_E yzdV$$

Using the substitution $r^2 = x^2 + y^2$ and $\tan\theta = y/x$, the bound $z = a^2 - x^2 - y^2$ becomes $z = a^2 - r^2$. The constraint that the region is in the first octant implies $0 \le \theta \le \pi/2$ and $0 \le r \le a$. The Jacobian is, of course, $J = r$. Therefore, the transformed integral is:

$$\int_0^{\pi/2}\int_0^a\int_0^{a^2-r^2} r(r\sin\theta)zdzdrd\theta$$

$$(\int_0^{\pi/2}\sin\theta d\theta)(\frac{1}{2}\int_0^a r^2[z^2]_0^{a^2-r^2}dr)$$

$$\frac{1}{2}\int_0^a r^2(a^2-r^2)^2dr$$

$$\frac{1}{2}\int_0^a r^2(r^4 - 2a^2r^2 + a^4)dr$$

$$\frac{1}{2}\int_0^a (r^6 - 2a^2r^4 + a^4r^2)dr$$

$$\frac{1}{2}[\frac{1}{7}r^7 - \frac{2}{5}a^2r^5 + \frac{1}{3}a^4r^3]_0^a$$

$$\frac{a^7}{2}(1/7 - 2/5 + 1/3)$$

$$\frac{a^7}{2}(8/105)$$

$$\boxed{4a^7/105}$$

35.6.13

The region is bounded above by the plane and below by the paraboloid. Rewrite the paraboloid equation as $z = \frac{1}{2}r^2$. The region of integration in the xy plane is the disk of radius 2 (find the boundary by solving $2 = \frac{1}{2}r^2$). Then it follows that $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. The triple integral becomes

$$\int_0^{2\pi}\int_0^2\int_{\frac{1}{2}r^2}^2 (r^2)dz(r)drd\theta \to \int_0^{2\pi}\int_0^2\left(2r^3 - \frac{1}{2}r^5\right)drd\theta \to \int_0^{2\pi}\frac{8}{3}d\theta = \frac{16}{3}\pi$$

35.6.18

The spheres restrict ρ to be $\rho \in [1, 2]$. The restriction of the first octant makes $\theta \in [0, \pi/2]$ and $\phi \in [0, \pi/2]$.

35.6.19

Make the substitution

$$x = \rho \cos\theta \sin\phi$$

$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\phi$$

Therefore, the bounds of the region E' become:

$$x^2 + y^2 + z^2 \leq a^2$$

$$0 \leq \rho \leq a$$

and

$$z^2 \leq 3(x^2 + y^2)$$

$$\rho^2 \cos^2\phi \leq 3\rho^2 \sin^2\phi$$

$$\tan^2\phi \leq 1/3$$

$$-1/\sqrt{3} \leq \tan\phi \leq \pm 1/\sqrt{3}$$

$$-\pi/6 \leq \phi \leq \pi/6$$

There are no bounds on θ, so $0 \leq \theta \leq 2\pi$. Therefore, E' is the region such that:

$$0 \leq \rho \leq a$$

$$-\pi/6 \leq \phi \leq \pi/6$$

$$0 \leq \theta \leq 2\pi$$

35.6.20

The sphere indicates that ρ varies from 0 to a. Then the half planes (since $x \geq 0$) may be rewritten as

$$\frac{1}{2}y = \frac{\sqrt{3}}{2}x \implies \cos(\theta) = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

$$\frac{\sqrt{3}}{2}y = \frac{1}{2}x \implies \sin(\theta) = \frac{1}{2} \implies \theta = \frac{\pi}{6}$$

This means that $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$. These half planes make it so that it intersects a half arc of the greatest circle of the sphere of radius a. This means that ϕ varies from 0 to π, since only half of the full circumference is traced out by the intersection of the half planes and the sphere.

35.6.21

Completing the square for z for the equation $x^2+y^2+z^2 = 4z$ gives the redefined equation as $x^2+y^2+(z-2)^2 = 4$, which is the equation for the sphere centered at (0,0,2) with radius 2. This shape is similar to the shapes of $r = a\cos\theta$ or $r = a\sin\theta$ in polar coordinates, where the sphere is "sitting" on top of one of the axes and that the origin lies on the surface. As a result, one can think of this shape taking a similar form in spherical coordinates:

$$\rho = a\cos\theta$$

especially since z $= \rho\cos\theta$. As a result, we get that ρ is bounded by $\rho \in [0, 4\cos\theta]$. Since the polar angle goes through the entire surface, it has its natural range as its bound. However, for the azimuthal angle, ϕ, the sphere never goes below the xy-plane, restricting it to $\phi \in [0, \pi/2]$.

35.6.22

$$\iiint_E (x^2+y^2+z^2)^3 dV$$

Make the typical spherical coordinate substitution $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$. The Jacobian is, as always, $J = \rho^2\sin\phi$. If region E is the sphere with radius a, then the bounds of the integral are $0 \le \rho \le a$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. Therefore, the transformed integral is:

$$\int_0^a \int_0^{2\pi} \int_0^{\pi} (\rho^2\sin\phi)(\rho^6) d\phi d\theta d\rho$$

$$= (\int_0^a \rho^8 d\rho)(\int_0^{2\pi} d\theta)(\int_0^{\pi} \sin\phi d\phi)$$

$$= (a^9/9)(2\pi)(2)$$

$$\boxed{= \frac{4}{9}\pi a^9}$$

35.6.23

Give $x = \rho\cos(\phi)$, $y = \rho\sin(\phi)\cos(\theta)$, and $z = \rho\sin(\phi)\sin(\theta)$, where ϕ is the angle from the x axis outwards and θ is the angle swept from the positive y axis around towards the positive z axis. Also give $r = \sqrt{y^2+z^2}$ Then $x = \sqrt{1-r^2}$ and $x = \sqrt{4-r^2}$ are the hemispheres, of radius 1 and 2 respectively. So $1 \le \rho \le 2$. Then since these are positive hemispheres, they stop forming the rest of the sphere where x is negative. So ϕ varies from 0 to $\frac{\pi}{2}$. And conveniently since the hemispheres are fully formed about the x axis, θ takes on its natural range.

The triple integral becomes

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 (\rho\sin(\phi)\cos(\theta))^2 \left(\rho^2\sin(\phi)\right) d\rho d\phi d\theta$$

$$\rightarrow \left(\int_0^{2\pi} \cos^2(\theta) d\theta\right) \left(\int_0^{\frac{\pi}{2}} \sin^3(\phi) d\phi\right) \left(\int_1^2 \rho^4 d\rho\right)$$

$$\rightarrow (\pi) \left(\frac{2}{3}\right) \left(\frac{31}{5}\right) = \frac{62}{15}\pi$$

35.6.24

$$\iiint_E xyz dV$$

The spheres given restrict ρ so that $\rho \in [1, 2]$. Additionally, since both spheres and cones are independent of the polar angle, θ takes its natural range of $\theta \in [-\pi, \pi]$. Finally, for ϕ, solving the equation for the cone in terms of the angle it makes with the z-axis gives the angle to be $\tan^-1(\frac{1}{\sqrt{3}}$, or $\pi/6$, making $\phi \in [0, \pi/6]$. The iterated integral then becomes:

$$\iiint_E xyz\rho^2 \sin\phi dV'$$

$$\iiint_E \rho^5 \sin^3\phi \cos\phi \sin\theta \cos\theta dV'$$

Since the region E is a rectangular prism, Fubini's theorem applies:

$$\int_1^2 \rho^5 d\rho \int_0^{\pi/6} \sin^3\phi \cos\phi d\phi \int_{-\pi}^{\pi} \cos\theta \sin\theta d\theta$$

The last integral is odd over its range, making the entire integral 0.

35.6.25

$$\iiint_E z dV$$

Make the typical spherical coordinate substitution $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$. The Jacobian is, as always, $J = \rho^2\sin\phi$. The region E is bounded by $x^2+y^2+z^2 = \rho^2 \leq 1$ and $z \leq \sqrt{3x^2+3y^2} \rightarrow \rho\cos\phi \leq \sqrt{3}\rho\sin\phi \rightarrow \phi \geq \pi/6$. Therefore, the transformed integral is:

$$\int_0^1 \int_0^{2\pi} \int_{\pi/6}^{\pi} (\rho^2 \sin\phi)(\rho\cos\phi) d\phi d\theta d\rho$$

$$= (\int_0^1 \rho^3 d\rho)(\int_0^{2\pi} d\theta)(\int_{\pi/6}^{\pi} \cos\phi\sin\phi d\phi)$$

$$= (\frac{1}{4})(2\pi)(\frac{1}{2}[\sin^2\phi]_{\pi/6}^{\pi})$$

$$\boxed{-\pi/16}$$

35.6.30

Form these three inequalities directly from the bounds of integration:

$$0 \leq \rho \leq \frac{2}{\cos(\phi)} \to 0 \leq z \leq 2$$

$$0 \leq \phi \leq \frac{\pi}{4} \to z = r \text{ is a conical boundary}$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

From these we can deduce that the solid region is the part of the cone in the first octant that is bounded below by $z = r$ and above by $z = 2$ for $r = \sqrt{x^2 + y^2}$. The region of integration in the xy plane is quickly found to be the disk of radius 2 (since z=r=2). The triple integral for the volume in cylindrical coordinates is given below:

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_r^2 dz(r)drd\theta \to \int_0^{\frac{\pi}{2}} \int_0^2 (2r - r^2)drd\theta \to \int_0^{\frac{\pi}{2}} \frac{4}{3}d\theta = \frac{2}{3}\pi$$

35.6.32

Cylindrical solution:
The equations z=0 and $z = x^2 + y^2$ easily translate into cylindrical bounds for z, being $z \in [0, r^2]$, using the fact that $x^2 + y^2 = r^2$ in polar. The line y=x bounds the polar angle to $\theta \in [0, \pi/4]$, as $\tan^- 1(1) = \pi/4$. That leaves the restriction on the radius. There is no lower bound for the radius defined in the equations, so its lower bound is its natural lower bound, 0. Its upper bound is defined by the line x=1. Solving this equation for r in polar form gives the equation:

$$r \cos\theta = 1 \implies r = \sec\theta$$

. Therefore, the iterated integral becomes:

$$\int_0^{\pi/4} \int_0^{\sec\theta} \int_0^{r^2} f(r^2 + z^2) r dz dr d\theta$$

36.6 Exercises

36.6.1

This is fairly direct.

$$J = \left| \det \begin{pmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} & 0 \\ 0 & \frac{1}{w} & -\frac{v}{w^2} \\ -\frac{w}{u^2} & 0 & \frac{1}{u} \end{pmatrix} \right| = 0$$

36.6.2

The Jacobian is defined as:

$$\begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \\ 1 & 2v & 0 \end{vmatrix}$$

$$0 - (-1) + 2w(4uv)$$

$$8uvw + 1$$

36.6.3

The Jacobian $J(u, v, w)$ is generally given by:

$$J(u, v, w) = |\det \begin{pmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{pmatrix}|$$

For the substitution $x = uv\cos w$, $y = uv\sin w$, $z = (u^2 - v^2)/2$:

$$J(u, v, w) = |\det \begin{pmatrix} v\cos w & u\cos w & -uv\sin w \\ v\sin w & u\sin w & uv\cos w \\ u & -v & 0 \end{pmatrix}|$$

By row operations:

$$= |\det \begin{pmatrix} 0 & -v & u \\ uv\cos w & u\sin w & v\sin w \\ -uv\sin w & u\cos w & v\cos w \end{pmatrix}|$$

$$= v(uv^2\cos^2 w + uv^2\sin^2 w) + u(u^2 v\cos^2 w + u^2 v\sin^2 w)$$

$$= v(uv^2) + u(u^2 v)$$

$$\boxed{= uv^3 + u^3 v}$$

36.6.6

First change the variables to find that $u+v+w \leq a$, which is the region under a plane that intersects the u, v, and w axes at a (meaning the points are $(0, 0, a)$, $(0, a, 0)$, $(a, 0, 0)$). It is easiest to give the bounds in a vertically simple manner, meaning to start with $0 \leq w \leq a - u - v$. Then to find the bounds in v we may give the line where the plane intersects with the uv plane as $v = 5 - u$, so evidently $0 \leq v \leq a - u$. And u will then vary from 0 to a.

Then we compute the Jacobian as follows for $dxdy = Jdudv$. We must rewrite the equations given into $x = u^2$, $y = v^2$, and $z = w^2$. Then:

$$J = \left|\det \begin{pmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{pmatrix}\right| = 8uvw$$

Then the triple integral becomes

$$\int_0^a \int_0^{a-u} \int_0^{a-u-v} (8uvw)dwdvdu$$

$$\rightarrow \int_0^a 4u \int_0^{a-u} (a^2v - 2auv + u^2v - 2av^2 + 2uv^2 + v^3)dvdu$$

$$\frac{1}{3}\int_0^a u(a-u)^4 du = \frac{1}{90}a^6$$

It may be helpful to use auxiliary substitutions for the later integrals.

36.6.7

We want to integrate $\iiint_E dV$ where E is the surface generated by $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3}$. The most apparent change of variables to simplify this is:

$$u = (x/a)^{1/3} \ v = (y/b)^{1/3} \ w = (z/c)^{1/3}$$

The Jacobian of this is easier to calculate after solving for the inverse of each equation, giving:

$$\begin{vmatrix} 3au^2 & 0 & 0 \\ 0 & 3bv^2 & 0 \\ 0 & 0 & 3cw^2 \end{vmatrix} = 9abcu^2v^2w^2$$

. The resulting integral then becomes:

$$\iiint_{E'} 9abcu^2v^2w^2 dV'$$

The region that is being integrated over also now becomes the unit sphere $u^2 + v^2 + w^2 = 1$. As a result, it is easier, though maybe a little more work, to convert again into polar coordinates, making the integral:

$$9abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho\cos\phi)^2(\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)^2\rho^2\sin\phi d\rho d\phi d\theta$$

$$9abc \int_0^{2\pi} \sin^2\theta\cos^2\theta d\theta \int_0^{\pi} \sin^3\phi\cos^4\phi d\phi \int_0^1 \rho^8 d\rho$$

The integral with respect to ρ becomes $1/9$. For the integral with respect to ϕ, let $u_1 = \cos\phi \implies du_1 = -\sin\phi d\phi$. This makes the integral:

$$-\int_1^{-1} (1-u_1^2)(u_1^4)du_1 = 2\int_0^1 u_1^4 - u_1^6$$

$$\frac{4}{35}$$

For the integral w.r.t θ, we can use the double-angle and power reduction trig identities to make the integral easier to compute:

$$\int_0^{2\pi} (\frac{1}{2}\sin 2\theta)^2 d\theta = \frac{1}{4}\int_0^{2\pi} \sin^2\theta d\theta = \frac{1}{8}\int_0^{2\pi} 1 - \cos 4\theta d\theta$$

$$\frac{\pi}{4}$$

Put all together, the integral evaluates to:

$$\boxed{\frac{\pi}{32}abc}$$

36.6.8

$$(x/a)^{1/3} + (y/b)^{1/3} + (z/c)^{1/3} = 1$$

Consider the substitution $u = (x/a)^{1/3}$, $v = (y/b)^{1/3}$, $w = (z/c)^{1/3}$ or equivalently $x = au^3$, $y = bv^3$, $z = cw^3$. This gives a Jacobian of:

$$J(u, v, w) = |\det \begin{pmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{pmatrix}|$$

$$= |\det \begin{pmatrix} 3au^2 & 0 & 0 \\ 0 & 3bv^2 & 0 \\ 0 & 0 & 3cw^2 \end{pmatrix}|$$

$$= 27abcu^2v^2w^2$$

The volume of the solid is therefore:

$$\iiint_E dV = 27abc \int_0^1 \int_0^{1-u} \int_0^{1-u-v} u^2v^2w^2 dwdvdu$$

$$= 9abc \int_0^1 \int_0^{1-u} \int_0^{1-u-v} u^2v^2(1-u-v)^3 dvdu$$

$$= 9abc \int_0^1 u^2 \int_0^{1-u} v^2(1-u-v)^3 dvdu$$

Make the substitution $t = 1-u-v$, $dt = -dv$ or equivalently $v = 1-u-t$, $dv = -dt$

$$= 9abc \int_0^1 u^2 \int_0^{1-u} t^3(1-u-t)^2 dtdu$$

$$= 9abc \int_0^1 u^2 \int_0^{1-u} t^3(1 + u^2 + t^2 - 2u - 2t + 2ut) dtdu$$

$$= 9abc \int_0^1 u^2 \int_0^{1-u} ((1 - 2u + u^2)t^3 + (2u-2)t^4 + t^5) dtdu$$

$$= 9abc \int_0^1 u^2 (\frac{1}{4}(1-u)^6 - \frac{2}{5}(1-u)^6 + \frac{1}{6}(1-u)^6) du$$

$$= \frac{3}{20} abc \int_0^1 u^2 (1-u)^6 du$$

Make the substitution $s = 1-u$, $ds = -du$ or equivalently $u = 1-s$, $du = -ds$:

$$= \frac{3}{20} abc \int_0^1 s^6 (1-s)^2 ds$$

$$= \frac{3}{20} abc \int_0^1 (s^8 - 2s^7 + s^6) ds$$

$$= \frac{3}{20} abc (1/9 - 1/4 + 1/7)$$

$$= \frac{3}{20} abc (1/252)$$

$$= \frac{1}{(20)(84)} abc$$

$$\boxed{= abc/1680}$$

36.6.13

From the bounds give $u = \frac{x}{3}$, $v = \frac{y}{2}$, and $w = z$. The Jacobian is given by $\frac{1}{J}$ where J is given by

$$J = \left| \det \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \frac{1}{6}$$

The bound given by the paraboloid remains a paraboloid, but it becomes a circular paraboloid $w = u^2 + v^2$ that intersects with the same plane $z = w = 10$. Call this region E'.

The triple integral after substitution becomes

$$216 \iiint_{E'} (u^2 - v^2) dA'$$

However, notice that there is symmetry of both the integrand (skew symmetry) and the region of integration (geometric symmetry) across the line $u = v$, so we may apply the transformation $(u, v, w) \to (v, u, w)$ to find that the sign of the integral will flip and so the integral vanishes.

36.6.14

For this problem, it is helpful to note that the triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is the same as $\begin{vmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{vmatrix}$ where a, b, c are all horizontal vectors. If we were to take the derivative w.r.t x, you would see that it would simply be the vector a, and likewise for b and c. As a result, the Jacobian of the transformation would simply be the triple product, which was defined as 1.

36.6.15

Let $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle$. Substitute:

$$u = \vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$$v = \vec{b}\cdot\vec{r} = b_1x + b_2y + b_3z$$

$$w = \vec{c}\cdot\vec{r} = c_1x + c_2y + c_3z$$

The resulting Jacobian is:

$$J(u,v,w) = |\det\begin{pmatrix}u'_x & u'_y & u'_z\\ v'_x & v'_y & v'_z\\ w'_x & w'_y & w'_z\end{pmatrix}|^{-1}$$

$$= |\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}|^{-1}$$

There is no need to calculate this Jacobian by hand because we recognize the determinant as the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$.

$$J(u,v,w) = \frac{1}{|\vec{a}\cdot(\vec{b}\times\vec{c})|}$$

The transformed integral is:

$$\iiint_E(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})(\vec{c}\cdot\vec{r})dV$$

$$= \frac{1}{|\vec{a}\cdot(\vec{b}\times\vec{c})|}\int_0^{\alpha}\int_0^{\beta}\int_0^{\gamma}(uvw)dwdvdu$$

$$= \frac{1}{|\vec{a}\cdot(\vec{b}\times\vec{c})|}\int_0^{\alpha}udu\int_0^{\beta}vdv\int_0^{\gamma}wdw$$

$$= \frac{1}{|\vec{a}\cdot(\vec{b}\times\vec{c})|}(\alpha^2/2)(\beta^2/2)(\gamma^2/2)$$

$$\boxed{= \frac{1}{8|\vec{a}\cdot(\vec{b}\times\vec{c})|}\alpha^2\beta^2\gamma^2}$$

38.3 Exercises

38.3.1

First we need to find out the parameterization for C. So give $x = 2\cos(t)$ and $y = 2\sin(t)$, but in order to ensure that x remains nonnegative and we only trace out a half circle, we give the parameter the range $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The arclength element ds for this problem is $\sqrt{(-2\sin(t))^2 + (2\cos(t))^2}dt = 2dt$ (this parameterization is a natural one). So then we may substitute in the line integral like so:

$$16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(t)\cos(t)dt = \frac{32}{3}$$

38.3.2

Recall that a line segment with initial point $\vec{r}_0$ and terminal point $\vec{r}_1$ are most easily parameterized as $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, for $0 \le t \le 1$. Apply this definition to the line segment given in the problem to find that $\vec{r}(t) = \langle bt, (1-t)a\rangle$, for t in that same interval as before. Then find ds:

$$ds = \sqrt{(-a)^2 + (b)^2} = \sqrt{a^2 + b^2}$$

Then the integral becomes

$$\int_C x\sin(y)ds \to \sqrt{a^2+b^2}\int_0^1 bt\sin((1-t)a)dt$$

Give $u = 1-t$ so that $t = 1-u$ and $du = -dt$. Change the bounds as well (they actually remain unchanged):

$$\sqrt{a^2+b^2}\int_0^1 (1-u)b\sin(au)du \to \sqrt{a^2+b^2}\int_0^1 (b\sin(au) - bu\sin(au))\,du$$

$$= \frac{b}{a}\sqrt{a^2+b^2}\left[1 - \frac{\sin(a)}{a}\right]$$

38.3.3

$$\int_C xyzds$$

given that C is parameterized by $\vec{r}(t) = \langle 2\cos t, t, -2\sin t\rangle$ on $t \in [0, \pi]$.
Recall that $ds/dt = |\vec{r'}(t)| \to ds = |\vec{r'}(t)|dt$. We can calculate:

$$|\vec{r'}(t)| = |\langle 2\cos t, t, -2\sin t\rangle'| = |\langle -2\sin t, 1, -2\cos t\rangle| = \sqrt{5}$$

Using this information, we can transform the integral to:

$$\int_0^{\pi} (2\cos t)(t)(-2\sin t)(\sqrt{5})dt$$

$$= -2\sqrt{5}\int_0^{\pi} t\sin(2t)dt$$

Integrate by parts.

$$= -2\sqrt{5}[-t\cos(2t)/2 + \sin(2t)/4]_0^{\pi}$$

$$= -2\sqrt{5}(-\pi/2)$$

$$\boxed{= \pi\sqrt{5}}$$

38.3.4

We are given the parameterization for the curve C from the get go, and it is easy to deduce that the range of t is from 0 to 1. Then the arclength element ds is given by $\sqrt{(1)^2 + (2t)^2 + (3t^2)^2}dt = \sqrt{9t^4 + 4t^2 + 1}dt$.

Substitute into the line integral and resolve:

$$\int_0^1 (2t + 9t^3)(\sqrt{9t^4 + 4t^2 + 1})dt \to \frac{1}{4}\int_1^{14} \sqrt{u}du = \frac{1}{6}\left(14^{\frac{3}{2}} - 1\right)$$

38.3.5

$$\int_C zds$$

c is the intersection between $z = x^2 + y^2$ and z=4. Combining these bounds gives us that c is the circle defined by $4 = x^2 + y^2$, which can then be paramaterized to $x = 2\cos t, y = 2\sin t$. The integral then becomes:

$$\int_0^{2\pi} 4\sqrt{4\cos^2 + 4\sin^2}dt = \int_0^{2\pi} 8dt$$

$$16\pi$$

38.3.9

$$\int_C xyds$$

given that C is parameterized by $\vec{r}(t) = \langle a\sinh t, a\cosh t, \rangle$ on $t \in [0, T]$.
Recall that $ds/dt = |\vec{r'}(t)| \to ds = |\vec{r'}(t)|dt$. We can calculate:

$$|\vec{r'}(t)| = |\langle a\sinh t, a\cosh t\rangle'| = |\langle a\cosh t, a\sinh t\rangle| = a\sqrt{\cosh^2 t + \sinh^2 t} = a\sqrt{\cosh(2t)}$$

Using this information, we can transform the integral to:

$$\int_0^T (a\sinh t)(a\cosh t)(a\sqrt{\cosh(2t)})dt$$

$$= \frac{1}{2}a^3 \int_0^T \sinh(2t)\sqrt{\cosh(2t)}dt$$

Make the substitution $u = \cosh(2t) \rightarrow du = 2\sinh(2t)dt$.

$$= \frac{1}{4}a^3 \int_1^{\cosh(2T)} \sqrt{u}du$$

$$= \frac{1}{6}a^3[u^{3/2}]_1^{\cosh(2T)}$$

$$\boxed{= \frac{1}{6}a^3(\cosh(2T)^{3/2} - 1)}$$

38.3.18

We seek to take a line integral over the path given by an arc of the parabola in the problem where the integrand is the linear mass density.

Give $t = y$ and $x = \frac{t^2}{2a}$. Since $0 \leq \frac{t^2}{2a} \leq \frac{a}{2}$ (from substitution) it is apparent that $-a \leq t \leq a$. Find the arclength element ds as $\sqrt{(1)^2 + \left(\frac{t}{a}\right)^2}dt = \frac{1}{a}\sqrt{a^2+t^2}$. Then we may substitute to find the following:

$$\frac{1}{a}\int_{-a}^{a} |t|\sqrt{a^2+t^2}dt \rightarrow \frac{1}{a}\int_{a^2}^{2a^2} u^{\frac{1}{2}}du = \frac{2}{3}a^2(2\sqrt{2}-1)$$

The evaluation of the integral may be done in the piecewise manner or by symmetry (and substitution) as above.

38.3.19

The mass of the string can be evaluated as:

$$\int_0^1 \sigma(\vec{r}(t))||\vec{r'}(t)||dt$$

given $r\vec{(t)} = \langle at, \frac{at^2}{2}, \frac{at^3}{3}\rangle$. Substituting this in gives the integral as

$$\int_0^1 t\sqrt{a^2+a^2t^2+a^2t^4}dt$$

Using the substitution of $t^2 = u$, this becomes:

$$a/2\int_0^1 \sqrt{1+u+u^2}du$$

$$a/2\int_0^1 \sqrt{(u+1/2)^2+3/4}du$$

After the transformations v=u+1/2 and z=$\frac{2}{\sqrt{3}}v$, which are done to simplify the u+1/2 term and make the constant 1 respectively, becomes:

$$a/2\int_{1/\sqrt{3}}^{\sqrt{3}}\sqrt{z^2+1}dz$$

One can then do a trig substitution, where $\sqrt{z^2+1}$ is the hypotenuse, z is the length of the far side, and 1 is the length of the near side. The substitution is then $\tan\theta = z \implies \sec^2\theta d\theta = dz$. The integral then becomes:

$$a/2\int_{\pi/6}^{\pi/3}\sec^3\theta d\theta$$

$$a/4(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta))|_{\pi/6}^{\pi/3}$$

$$a/4(2\sqrt{3}+\ln\Big(2+\sqrt{3}\Big)-\frac{2}{3}-\ln\sqrt{3})$$

39.5 Exercises

39.5.1

Give the equation of this plane as $abz+acy+bcx=abc$, or otherwise $z=c-\frac{c}{b}y-\frac{c}{a}x$. Then find the surface area element dS by finding

$$dS=\sqrt{1+\Big(-\frac{c}{a}\Big)^2+\Big(-\frac{c}{b}\Big)^2}dA=\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}dA$$

where dA is the rectangular area element within the triangle whose vertices are $(0,0)$, $(0,b)$, and $(a,0)$. We can give the integration region here by bounding x from 0 to a and giving the curves in y to be from 0 to $b-\frac{b}{a}x$. The double integral becomes

$$\int_0^a\int_0^{b-\frac{b}{a}x}\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}dydx\rightarrow\int_0^a(b-\frac{b}{a}x)\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}dx$$

$$=\frac{1}{2}\sqrt{a^2b^2+b^2c^2+a^2c^2}$$

39.5.2

$$\iint_D dS$$

where S is the surface given by the equations $x^2+y^2=4$ and $z+3x+2y=1$. We can then set the region D to be the circle and the surface to be defined as

$z = 1 - 3x - 2y$. The integral becomes, after converting to polar because of the circle:

$$\int_0^{2\pi} \int_0^2 \sqrt{1+9+4} r dr d\theta$$
$$2\pi\sqrt{14} r^2/2|_0^2$$
$$4\pi\sqrt{14}$$

39.5.3

Recall the general formula for the surface area:

$$\iint_D \sqrt{1 + (g'_x)^2 + (g'_y)^2} dA$$

Here, $g(x, y) = y^2 - x^2$, $g'_x = -2x$, and $g'_y = 2y$.

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2} dA$$

Given the bounds $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, it is easiest to convert the integral to polar coordinates.

$$\int_0^{2\pi} \int_1^2 r\sqrt{1 + 4r^2} dr d\theta$$
$$= 2\pi \int_1^2 r\sqrt{1 + 4r^2} dr$$

Make the substitution $u = 1 + 4r^2$, $du = 8rdr$.

$$\frac{\pi}{4} \int_5^{17} \sqrt{u} du$$
$$\frac{\pi}{6} [u^{3/2}]_5^{17}$$
$$\boxed{\frac{\pi}{6}(17^{3/2} - 5^{3/2})}$$

39.5.4

We are already given the equation for the surface. To find the area of integration it is sufficient to equate the two values of z given for the planes with the paraboloid expression and find that the region is an annulus of inner radius 1 and outer radius 3, where θ takes on the natural range (we will be using polar coordinates).

To find the surface area elements dS find

$$dS = \sqrt{1 + (-2x)^2 + (2y)^2} dA = \sqrt{1 + 4r^2} dA$$

Then the double integral is given in the polar form:

$$\int_0^{2\pi} \int_1^3 \sqrt{1 - 4r^2} r dr d\theta \to \frac{\pi}{4} \int_5^{37} \sqrt{u} du = \frac{\pi}{6} \left(37^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

39.5.5

Since the surface given by $y = 4x+z^2$ is already defined in terms of y, we can just leave it at that and set the region D to be in terms of x and z. The triangular region, when integrating in the order of dx dz, is given by $x \in [0, z], z \in [0, 1]$. The integral then becomes:

$$\int_0^1 \int_0^z \sqrt{1 + 16 + 4z^2} dxdz$$

$$\int_0^1 z\sqrt{17 + 4z^2}$$

$$1/8 \int_{17}^{21} \sqrt{u} du$$

$$\frac{1}{8}\frac{2}{3}(21^{3/2} - 17^{3/2})$$

$$\frac{1}{12}(21^{3/2} - 17^{3/2})$$

39.5.6

Recall the general formula for the surface integral:

$$\iint_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{1 + (g'_x)^2 + (g'_y)^2} dA$$

We can rearrange the equation $x + y + z = 1$ to $z = g(x, y) = 1 - x - y$. The partial derivatives:

$$g'_x = -1$$
$$g'_y = -1$$

Therefore, the Jacobian is:

$$J = \sqrt{1 + (-1)^2 + (-1)^2}$$
$$= \sqrt{3}$$

Finally, the bounds of the integral are $x \in [0, 1]$ and $y \in [0, 1 - x]$ because this will make x,y non-negative and $z = g(x, y) = 1 - x - y$ non-negative. The transformed integral is:

$$\int_0^1 \int_0^{1-x} y(1 - x - y)\sqrt{3}dydx$$

$$= \sqrt{3} \int_0^1 \int_0^{1-x} y - xy - y^2 dydx$$

$$= \sqrt{3}\int_0^1 [y^2/2 - xy^2/2 - y^3/3]_0^{1-x} dx$$

$$= \sqrt{3}\int_0^1 ((1-x)^2/2 - x(1-x)^2/2 - (1-x)^3/3) dx$$

Expand and integrate to yield:

$$= \sqrt{3}(1/24)$$

$$\boxed{= \sqrt{3}/12}$$

39.5.7

Like before to find the region of integration simply equate the given values of z for the planes to the cone expression to find that the region of integration is the annulus of inner radius 1 and outer radius 2. It is also seen that θ ranges from 0 to 2π. We will be using polar coordinates to evaluate this surface integral.

Then the surface area element dS is found as:

$$dS = \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} dA$$

$$= \sqrt{1 + \left(\frac{r\cos(\theta)}{r}\right)^2 + \left(\frac{r\sin(\theta)}{r}\right)^2} dA = \sqrt{2} dA$$

After writing everything in polar coordinates, noting that $z = r$ due to S being a part of a cone, the double integral becomes:

$$\sqrt{2}\int_0^{2\pi}\int_1^2 (r)r^4\cos^2(\theta) dr d\theta \to \frac{2}{\sqrt{2}}\left(\int_0^{2\pi}\cos^2(\theta)d\theta\right)\left(\int_1^2 r^5 dr\right)$$

$$= \frac{21\pi}{\sqrt{2}}$$

39.5.8

$$\iint_S xy dS$$

Where S is the surface defined by the cylinder $y^2+z^2 = 1$ bounded by the planes x=0 and x+y=3. Notice that for this entire surface, it is symmetrical over the transformation $(x, y, z) \implies (x, y, -z)$, while the integrand is skew symmetric over this transformation. As a result, this integral evaluates to 0.

39.5.9

Recall the general formula for the surface integral:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{1+(g'_x)^2+(g'_y)^2}dA$$

We can rearrange the equation the equation $x^2+y^2+z^2=2$ to $z=g(x,y)=\sqrt{2-x^2-y^2}$. The partial derivatives are:

$$g'_x = \frac{-x}{\sqrt{2-x^2-y^2}}$$

$$g'_y = \frac{-y}{\sqrt{2-x^2-y^2}}$$

Therefore, the Jacobian is:

$$J = \sqrt{1+(g'_x)^2+(g'_y)^2} = \sqrt{2/(2-x^2-y^2)}$$

Using this information, we find the transformed integral is:

$$\iint_D \sqrt{2-x^2-y^2}(\sqrt{2}/\sqrt{2-x^2-y^2})dA$$

$$= \iint_D \sqrt{2}dA$$

Given the bound $z=\sqrt{2-x^2-y^2}=1$, it is easiest to convert to polar coordinates. $z=\sqrt{2-x^2-y^2}=1$ when $r=1$ and $z>1$ when $0\le r<1$. Therefore, the integral becomes:

$$\int_0^{2\pi}\int_0^1 r\sqrt{2}drd\theta$$

$$2\pi\sqrt{2}\int_0^1 rdr$$

$$\boxed{=\pi\sqrt{2}}$$

39.5.10

Find the surface area element dS as:

$$dS = \sqrt{1+4(x^2+y^2)dA}$$

where dA is an area element over the portion of the unit disk in the first octant (since the paraboloid forms the unit circle as the boundary of the disk at $z=0$). Then rewrite the double integral as

$$\iint_S (1-x^2-y^2)(\sin(x^2)-\sin(y^2))\sqrt{1-4(x^2+y^2)}dA$$

Notice that the integral exhibits skew symmetry as the integrand switches sign completely (due to the trigonometric terms) if we apply the transformation $(x, y) \to (y, x)$, otherwise a reflection over $x = y$. Note that the region of integration is symmetric across this line as well and hence the integral vanishes.

41.6 Exercises

41.6.1

Evidently it seems like the vector field involves motion that diverges from the origin. After drawing the vector field it is possible to draw flow lines by just tracing out the direction but it may be more enlightening to find an equation that models *a* flow line.

Since the vectors in the vector field are always going to be tangent to these flow line curves, we can find the following system of differential equations:

$$x'_t = ax, \; y'_t = by$$

We can avoid having to find the actual trajectory of any particles put in the vector field by combining the system into one single differential equation like so:

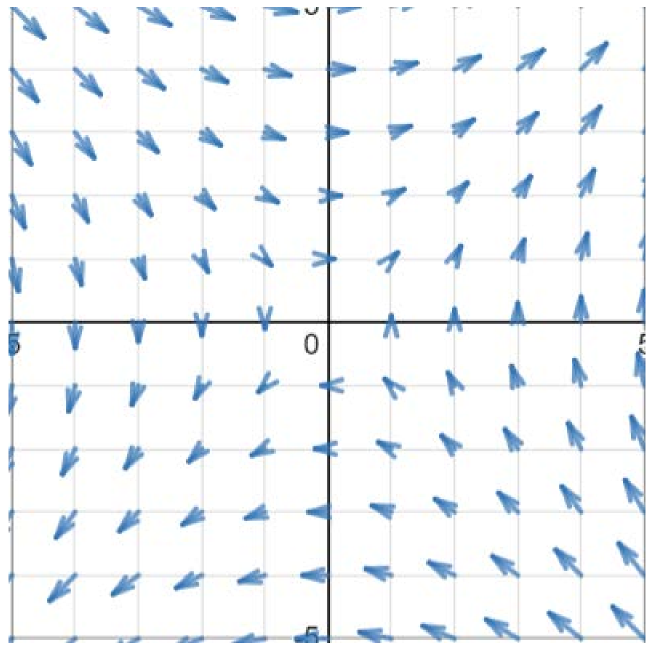
$$\frac{dy}{dx} = \frac{by}{ax} \to y = C(\pm x)^{\frac{b}{a}}$$

The explicit form here does not give us direction, but we already know the direction given by the vector field itself (outwards from the origin).

The exercise does not ask us to do this (since this process involves some unmotivated elementary differential equations theory) but it is helpful to compare with a given solution.

41.6.2

To create this vector field, simply graph the given vector centered at the point given. Evidently, the vector field tends to go towards the line y=x, where the first quadrant goes towards $(x, y) \to (\infty, \infty)$ and the third quadrant $(x, y) \to (-\infty, -\infty)$. An image of the vector field where a and b are 1 is shown below.



41.6.3

The flow lines appear to be parabolas (or degenerate lines through the origin) through the origin. This can be found by sketching a graph of the vector field $\mathbf{F}$ or by solving the differential equation $\frac{dy}{dx} = \frac{bx}{ay}$. The lines flow in the $+x$ direction if y is positive and vice versa. The lines flow in the $-y$ direction if x is positive and vice versa.

41.6.10

$$\mathbf{F} = \nabla\|\mathbf{r}\| = \langle\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\rangle$$
$$= \langle\frac{x}{\|\mathbf{r}\|}, \frac{y}{\|\mathbf{r}\|}, \frac{z}{\|\mathbf{r}\|}\rangle$$

By sketching the vector field, we find the flow lines are straight lines through the origin. Because each component of the vector field is positive, the flow lines point away from the origin.

41.6.4

The gradient is

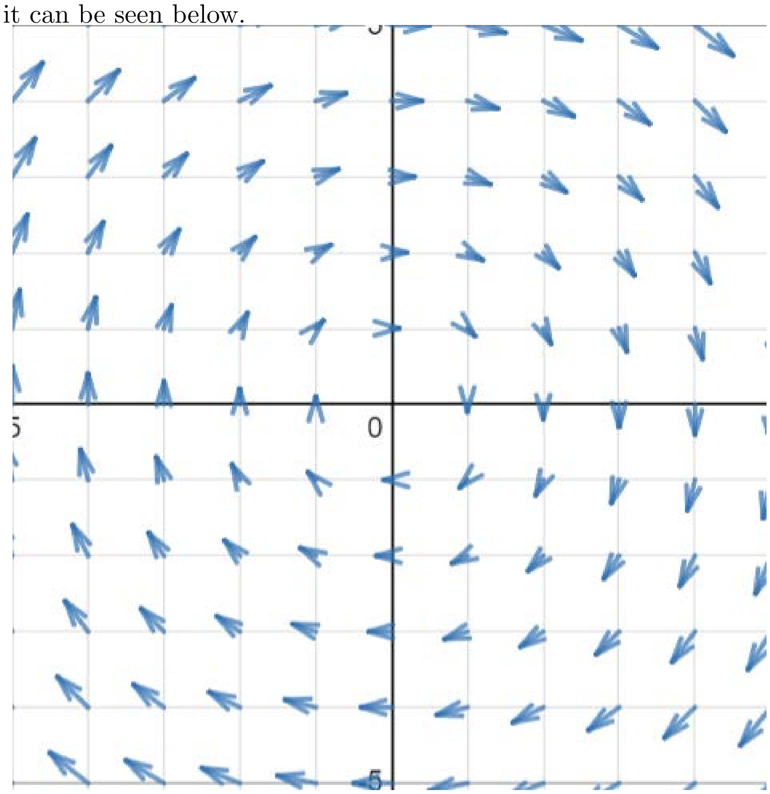
$$\left\langle \frac{-y}{r^2}, \frac{x}{r^2} \right\rangle$$

Notice that this gradient is like one we have seen before in that the motion is circular in the counterclockwise direction, except here the magnitudes of the tangent vectors to these circles shrinks with respect to squared distance from the origin.

It is also enlightening to interpret this gradient as the direction of greatest ascent of the function u at points on the domain (x, y). The function u in this problem represents the polar angle from transformations from the rectangular coordinate system to the polar coordinate system. Using this knowledge and a visual of the vector (or gradient) field, it is apparent that flow lines are circles oriented counterclockwise (as that is how the polar angle itself increases the most, assuming we take the appropriate branches of arctangent throughout motion in the path).

41.6.9

It may be helpful to think about that if one were to take planar slices of the vector field parallel to the xy-plane, that each would be identical. Additionally, the trajectory ¡y, -x¿ in the xy-plane is always going to be normal to the line y=x. As a result, the vector field will create circular flowing regions. An image of it can be seen below.



41.6.11

The gradient is

$$\left\langle -\frac{x}{||\vec{r}||^3}, -\frac{y}{||\vec{r}||^3}, -\frac{z}{||\vec{r}||^3} \right\rangle$$

It may be useful to consider that level sets of the function we took the gradient of are spheres, and the gradient is always going to be perpendicular to these spheres. Evidently paths are given by straight lines towards the origin.

41.6.16

Each point on the ball has the same angular speed, that is, they all go through the same number of radians per second. As a result, the linear speed for any point on the circle is simply the radius of its rotation multiplied by ω, since the arc-length of a sector of a circle is $\theta \cdot radius$. Additionally, the radius of rotation is the length of the vector between the point defined by $\mathbf{r}$ and the axis of rotation. The direction of motion is normal to the vector between $\mathbf{r}(t)$ and the axis of rotation and normal to $\vec{n}$. We can then take this information and create a velocity function, $\vec{v}(\mathbf{r})$. The vector between $\mathbf{r}$ and the axis of rotation can be defined as the perpendicular component of $\mathbf{r}$ onto $\vec{n}$.
Projection of $\mathbf{r}$ onto $\vec{n}$:

$$\vec{n}(\vec{n} \cdot \mathbf{r})$$

Perpendicular component:

$$\mathbf{r} - \vec{n}(\vec{n} \cdot \mathbf{r})$$

The linear speed is then $|\vec{r} - \vec{n}(\vec{n} \cdot \mathbf{r})|$. The direction of motion is normal to both the radius vector and axis of rotation, which is parallel to:

$$\vec{n} \times (\mathbf{r} - \vec{n}(\vec{n} \cdot \mathbf{r}))$$

$$\vec{n} \times \mathbf{r} - (\vec{n} \cdot \mathbf{r})\vec{n} \times \vec{n}$$

$$\vec{n} \times \mathbf{r}$$

As a result, the function $\vec{v}(\mathbf{r})$ is then:

$$\omega \frac{||\mathbf{r} - \vec{n}(\vec{n} \cdot \mathbf{r})||}{||\vec{n} \times \mathbf{r}||} \vec{n} \times \mathbf{r}$$

41.6.17

$$\mathbf{F} = \langle y, xy, 0 \rangle$$

and C is parameterized by:

$$\mathbf{r}(t) = \langle t^2, t^3, 0 \rangle$$

for $0 \leq t \leq 1$.
The line integral of $\mathbf{F}$ over the curve C is given by:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Particularly for this problem:

$$\int_0^1 \langle t^3, t^5, 0\rangle \cdot \langle 2t, 3t^2, 0\rangle dt$$

$$= \int_0^1 (2t^4 + 3t^7)dt$$

$$= \frac{2}{5} + \frac{3}{8}$$

$$\boxed{31/40}$$

41.6.18

Give the path as $\vec{r}(t) = \langle a\cos(t), b\sin(t), 0\rangle$ for $0 \leq t < 2\pi$. This path is oriented counterclockwise, so in order to do the integral just throw an extra negative sign in front. Then:

$$-\oint_C \vec{F} \cdot d\vec{r} \to -\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt \to$$

$$-\int_0^{2\pi} \langle 0, ab\cos(t)\sin(t), 0\rangle \cdot \langle -a\sin(t), b\cos(t), 0\rangle dt$$

$$\to -ab^2 \int_0^{2\pi} \cos^2(t)\sin(t)dt = 0$$

41.6.19

$$\int_C \vec{F} \cdot d\vec{r}$$

Given that $\vec{F} = \langle z, yx, zy\rangle$ and $\vec{r}(t) = \langle 2t, t + t^2, 1 + t^3\rangle$, and the integral ranges from (-2,0,0) to (2,2,2), the integral becomes:

$$\int_{-1}^1 \langle 1 + t^3, 2t^2 + 2t^3, t + t^2 + t^4 + t^5\rangle \cdot \langle 2, 1 + 2t, 3t^3\rangle dt$$

$$\int_{-1}^1 2 + 2t^3 + 2t^2 + 2t^3 + 4t^3 + 4t^4 + 3t^3 + 3t^4 + 3t^6 + 3t^7 dt$$

Because of symmetry, the integral becomes:

$$2\int_0^1 2 + 2t^2 + 7t^4 + 3t^6 dt$$

$$2(2 + 2/3 + 7/5 + 3/7) = \frac{944}{104}$$

41.6.20

$$\mathbf{F} = \langle -y, x, z \rangle$$

and C is the boundary of the part of the paraboloid $z = a^2 - x^2 - y^2$ in the first octant.
The boundary of the paraboloid C consists of three separate curves C_1, C_2, C_3, where one of x, y, z is uniformly zero. Plugging in zero and finding simple parameterizations of these curves yields:

$$C_1\text{: } \mathbf{r_1}(t) = \langle t, 0, a^2 - t^2 \rangle,\ 0 \to t \to t$$

$$C_3\text{: } \mathbf{r_2}(t) = \langle a\cos t, a\sin t, 0 \rangle,\ 0 \to t \to \pi/2$$

$$C_3\text{: } \mathbf{r_3}(t) = \langle 0, t, a^2 - t^2 \rangle,\ a \to t \to 0$$

Note the $a \to t \to 0$ in C_3 is opposite that of C_1 to maintain the counterclockwise direction of C. The integral of $\mathbf{F}$ over C is given by:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_{C_3} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

$$= \int_0^a \langle 0, t, a^2 - t^2 \rangle \cdot \langle 1, 0, -2t \rangle dt$$

$$+ \int_0^{\pi/2} \langle -a\sin t, a\cos t, 0 \rangle \cdot \langle -a\sin t, a\cos t, 0 \rangle dt$$

$$+ \int_a^0 \langle -t, 0, a^2 - t^2 \rangle \cdot \langle 0, 1, -2t \rangle dt$$

$$= \int_0^a 2t(t^2 - a^2)dt + a^2 \int_0^{\pi/2} dt + \int_a^0 2t(t^2 - a^2)dt$$

By basic integral properties, the first and third integrals cancel out. This leaves:

$$a^2 \int_0^{\pi/2} dt$$

$$\boxed{= \pi a^2/2}$$

41.6.21

Parameterize the boundary as three curves (the boundary is piecewise continuous); give $C_1 = \langle a\cos(t), a\sin(t), 0 \rangle$, $C_2 = \langle 0, a\cos(t), a\sin(t) \rangle$, and $C_3 = \langle a\sin(t), 0, a\cos(t) \rangle$. These curves are in the positive (counterclockwise) sense, so we must negate the integrals that are taken on these curves. For each curve $0 \leq t < \frac{\pi}{2}$. Then the line integral (by the additivity of Riemann integration) becomes:

$$\oint_C \vec{F} \cdot d\vec{r} = \left(-\int_{C_1} \vec{F} \cdot d\vec{r} \right) + \left(-\int_{C_2} \vec{F} \cdot d\vec{r} \right) + \left(-\int_{C_3} \vec{F} \cdot d\vec{r} \right)$$

Consider each integral:

$$-\int_{C_1} \vec{F}\cdot d\vec{r} = -\int_0^{\frac{\pi}{2}} \langle 0,0,a\cos(t)\rangle \cdot \langle -a\sin(t), a\cos(t), 0\rangle dt = 0$$

$$-\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_0^{\frac{\pi}{2}} \langle -a\sin(t),0,0\rangle \cdot \langle 0, -a\sin(t), a\cos(t)\rangle dt = 0$$

$$-\int_{C_3} \vec{F}\cdot d\vec{r} = -\int_0^{\frac{\pi}{2}} \langle -a\cos(t),0,a\sin(t)\rangle \cdot \langle a\cos(t), 0, -a\sin(t)\rangle dt = \dots$$

The last integral is all we have to evaluate. Find that it is

$$a^2 \int_0^{\frac{\pi}{2}} 1dt = \frac{\pi}{2}a^2$$

41.6.22

We can define the curve as $\vec{r}(t) = r_1 + (r_2 - r_1)t$. This makes the integral:

$$\int_0^1 (a \times (r_1 + (r_2 - r_1)t) \cdot (r_2 - r_1)dt$$

Using the distributive properties of the cross product this becomes:

$$\int_0^1 ((1-t)a \times r_1 + ta \times r_2) \cdot (r_2 - r_1)dt$$

Then distributing the dot product and cancelling anything that is multiplied by itself makes the integral:

$$\int_0^1 (1-t)r_2 \cdot a \times r_1 - (t)r_1 \cdot a \times r_2 dt$$

Since all of r_1, a, r_2 are constants, we can integrate like normal and get

$$(t - t^2/2)|_0^1 r_2 \cdot a \times r_1 - (t^2/2)|_0^1 r_1 \cdot a \times r_2$$

$$(1/2)r_2 \cdot a \times r_1 - (1/2)r_1 \cdot a \times r_2$$

$$r_2 \cdot a \times r_1$$

41.6.23

$$\mathbf{F} = \langle y\sin z, z\sin x, x\sin y\rangle$$

and C is parameterized by:

$$\mathbf{r}(t) = \langle \cos t, \sin t, \sin(5t)\rangle$$

for $0 \leq t \leq 2\pi$.
The line integral of $\mathbf{F}$ over the curve C is given by:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Particularly for this problem:

$$\int_0^{2\pi} \langle \sin(t)\sin(\sin(5t)), \sin(5t)\sin(\cos(t)), \cos(t)\sin(\sin(t))\rangle \cdot \langle -\sin t, \cos t, 5\cos(5t)\rangle dt$$

$$= -\int_0^{2\pi} \sin^2(t)\sin(\sin(5t))dt$$

$$+\int_0^{2\pi} \sin(5t)\cos(t)\sin(\cos(t))dt$$

$$+\int_0^{2\pi} 5\cos(5t)\cos(t)\sin(\sin(t))dt$$

Consider the substitution $t = u + \pi$, $dt = du$:

$$= \int_{-\pi}^{\pi} \sin^2(u+\pi)\sin(\sin(5(u+\pi)))du$$

$$+\int_{-\pi}^{\pi} \sin(5(u+\pi))\cos(u+\pi)\sin(\cos(u+\pi))du$$

$$+\int_{-\pi}^{\pi} 5\cos(5(u+\pi))\cos(u+\pi)\sin(\sin(u+\pi))du$$

All three integrals now have odd interands with symmetric bounds, so each of them cancels to 0.

$$\boxed{=0}$$

41.6.24

From the surfaces given we can parameterize the curve in steps. First give $x = \cos(t)$ and $z = \sin(t)$. Give the bounds in t as $0 \leq t \leq 2\pi$ (full period since the plane does not cut the path short). Then using the equation for the plane find that $y = 1 - \cos(t) - \sin(t)$. The integral (which is rather long in this presentation) becomes:

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{-\pi}^{\pi} \left\langle 1 - \sin(t) - \cos(t), -\cos(t)\sin(t), (1-\sin(t)-\cos(t))(\cos^2(t)+\sin^2(t))\right\rangle$$

$$\cdot\langle -\sin(t), \sin(t) - \cos(t), \cos(t)\rangle dt$$

After taking the dot product and simplifying, the integral becomes a linear combination of sines and cosines of varying frequency, which when integrated over any number of periods will return 0.

$$= \int_0^{2\pi} (\cos(t) - \sin(t) - \cos(2t) + \cos^2(t)\sin(t) - \sin^2(t)\cos(t))dt = 0$$

41.6.25

The intersection of the cone and sphere is a circle. With the restrictions $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 2$, we get that $z^2 = x^2 + y^2 \implies 2z^2 = 2 \implies z = 1$. As a result, we get $\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$ in the counterclockwise direction. $\vec{F}(\vec{r}(t)) = \langle -\sin t \sin \pi, \cos t \cos \pi, e^{\sin t \cos t} \rangle$. This makes the integral:

$$\int_0^{2\pi} \langle 0, -\cos t, e^{\sin t \cos t} \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$$

$$\int_0^{2\pi} 0 - \cos^2 t + 0 dt$$

$$-1/2 \int_0^{2\pi} (1 + \cos 2t) dt$$

$$-\pi$$

42.6 Exercises

42.6.1

⋆ The curl of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is given by the following formal determinant:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Using this definition we may compute the curl of the vector field $\vec{F} = \langle xyz, -y^2 x, 0 \rangle$:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -y^2 x & 0 \end{pmatrix} = \langle 0, xy, -y^2 - xz \rangle$$

42.6.2

Curl of the vector field $\langle \cos(xz), \sin(yz), 2\rangle$ is

$$\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(xz) & \sin(yz) & 2 \end{vmatrix}$$

$$\langle 0 - y\cos(yz), -(0 + x\sin(xz)), 0 - 0\rangle = \langle -y\cos(yz), -x\sin(xz), 0\rangle$$

42.6.3

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ h(x) & g(y) & f(z) \end{vmatrix}$$

$$\boxed{= \mathbf{0}}$$

42.6.4

The curl of a vector field $\vec{F} = \langle F_1, F_2, F_3\rangle$ is given by the following formal determinant:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Using this definition we may compute the curl of the vector field $\vec{F} = \langle \ln(xyz), \ln(yz), \ln(z)\rangle$:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln(xyz) & \ln(yz) & \ln(z) \end{pmatrix} = \left\langle -\frac{1}{z}, \frac{1}{z}, -\frac{1}{y}\right\rangle$$

42.6.5

The cross product between **a** and **r** can be represented as:

$$\vec{F} = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x\rangle$$

The curl of a vector field F can be represented by $\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\rangle$. Applying this to $\vec{F}$ makes it:

$$\nabla \times \vec{F} = \langle a_1 + a_1, a_2 + a_2, a_3 + a_3\rangle$$

$$2\mathbf{a}$$

42.6.9

$$\mathbf{F} = \langle 2xy, x^2 + 2yz^3, 3z^2y^2 + 1\rangle$$

Let f be the potential of F. That is, $\nabla f = \mathbf{F}$. f can be found by "anti-partial differentiation" of each of the components of $\mathbf{F}$.

$$\frac{\partial f}{\partial x} = 2xy \to f = x^2y + c_1(y, z)$$

$$\frac{\partial f}{\partial y} = x^2y + y^2z^3 + c_2(x, z)$$

$$\frac{\partial f}{\partial z} = 2xy \to f = y^2z^3 + z + c_3(x, y)$$

for some functions $c_1(y, z)$, $c_2(x, z)$, $c_3(x, y)$. These equations can be made to agree when

$$\boxed{f = x^2y + y^2z^3 + z + c}$$

for some constant c.

42.6.10

This vector field does not have any domain restriction. The curl of the vector field must be identically zero in order for the field to be conservative.

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz + 2y\cos(z) & xy - y^2\sin(z) \end{pmatrix}$$

$$= \langle x - 2y\sin(z) - (x - 2y\sin(z)), y - y, z - z\rangle = \vec{0}$$

Evidently the vector field is conservative. Then to construct the potential function the following process involving taking integrals and partial derivatives and comparing with the given components of the vector field take place.

Take the integral with respect to x of the first component to find one representation of f:

$$\frac{\partial f}{\partial x} = yz \implies f = xyz + a(y, z)$$

Take the partial derivative of f given above with respect to y and compare with the second component of the vector field similar to find $a(y, z)$. Then integrate this known partial derivative with respect to y:

$$xz + 2y\cos(z) = xz + a'_y$$

$$\implies f'_y = xz + 2y\cos(z) \implies f = xyz + y^2\cos(z) + b(x, z)$$

Repeat like above with the last component:

$$xy - y^2\sin(z) = xy - y^2\sin(z) + b'_z \implies f'_z = xy - y^2\sin(z)$$

$$\implies f = xyz + y^2\cos(z) + c$$

42.6.11

A requirement for a vector field to be conservative is that its curl is $\vec{0}$. Checking this shows this is the case.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = e^y$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} = 0$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = -2z$$

We can now integrate F_1 with respect to x to obtain part of the potential function.

$$f(x,y,z) = \int e^y dx = xe^y + h(y,z)$$

Taking the partial of f with respect to y should be the same as F_2, and setting this equal yields:

$$xe^y + h'_y(y,z) = xe^y - z^2 \implies h'_y(y,z) = -z^2$$

$$h(y,z) = -z^2 y + k(z) \implies f(x,y,z) = xe^y - z^2 y + k(z)$$

Repeat for F_3 and z:

$$-2zy + k'(z) = -2zy \implies k'(z) = 0$$

$$f(x,y,z) = xe^y - z^2 y + c$$

42.6.17

To determine if the vector field is conservative we can check to see if on its domain the curl of the vector field is identically the zero vector, which it is:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^2 + 2x + 2y & 2xyz^2 + 2x & 2xy^2z + 1 \end{pmatrix}$$

$$= \langle 4xyz - 4xyz, 2y^2z - 2y^2z, (2yz^2 + 2) - (2yz^2 + 2) \rangle = \vec{0}$$

Then reconstruct the potential function (which is in a sense like an antiderivative) $f(x,y,z)$ like so:

$$\frac{\partial f}{\partial x} = y^2z^2 + 2x + 2y \to f(x,y,z) = xy^2z^2 + x^2 + 2xy + a(y,z)$$

$$\frac{\partial f}{\partial y} = 2xyz^2 + 2x = 2xyz^2 + 2x + a'_y(y,z) \to a'_y(y,z) = 0$$

The last result means that $a(y, z)$ is constant with respect to y so we may write $a(y, z)$ instead as $a(z)$. Then:

$$\frac{\partial f}{\partial z} = 2xy^2z + 1 = 2xy^2z + a'_z(z) \to a'_z(z) = 1 \to a(z) = z + C$$

$$\to f(x, y, z) = xy^2z^2 + x^2 + 2xy + z + C$$

Using the fundamental theorem of line integrals we can use this potential function and evaluate it at the endpoints of the curve C. The initial point of C is $(1, 1, 1)$, and the terminal point is $(1, 2, 3)$. Find that:

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 2, 3) - f(1, 1, 1)$$

$$= \big((1)(2)^2(3)^2 + (1)^2 + 2(1)(2) + (3) + C\big)$$

$$- \big((1)(1)^2(1)^2 + (1)^2 + 2(1)(1) + (1) + C\big)$$

$$= 44 - 5 = 39$$

42.6.18

All components of the vector field are polynomials, so the domain is all real numbers for all variables. Knowing this it is sufficient to show that the curl of the vector field is identically zero to determine if the field is conservative.

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx & yz & z^2 \end{pmatrix} = \langle -y, x, 0 \rangle \neq \vec{0}$$

Evidently the field is not conservative so the path given in the problem is the one we must use in the line integral.

The path given is part of the helix that lies in the ellipsoid, but no bounds on t were given. To find them substitute the parameterization into the ellipsoid (make it an inequality because the path should be within the ellipsoid) to find bounds of t:

$$(2\sin(t))^2 + (-2\cos(t))^2 + 2(t)^2 \leq 6 \to -1 \leq t \leq 1$$

The integral becomes

$$\int_C \vec{F} \cdot d\vec{r} \to \int_{-1}^{1} \langle 2t\sin(t), -2t\cos(t), t^2 \rangle \cdot \langle 2\cos(t), 2\sin(t), 1 \rangle dt$$

$$\to \int_{-1}^{1} t^2 dt = \frac{2}{3}$$

42.6.19

One can verify that the vector field $\vec{F}$ is conservative by checking that $\nabla \times \vec{F} = \vec{0}$.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = 1$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} = -2z$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = \cos z$$

As a result, the integral $\int_C \vec{F} d\vec{r}$ is path independent, so we can take the path to be $\vec{r}(t) = \langle a, 0, bt \rangle$ instead of the helix $\langle a\cos t, a\sin t, \frac{b}{2\pi}t\rangle$. The integral then becomes:

$$\int_0^1 \langle -b^2t^2, a + \sin(bt), -2ab^2t\rangle \cdot \langle 0, 0, b\rangle dt$$

$$\int_0^1 -2ab^2 t dt = -ab^2|_0^1$$

$$-ab^2$$

43.5 Exercises

43.5.3

The region is a multiply connected region, so we could take the line integral in the positive sense around the circle of radius 2 and add to it the line integral taken in the negative sense around the circle of radius 1. But we want to use Green's theorem, so it is not necessary to do that.

Our region is an annulus (call it D), which we can easily use polar coordinates to give a rectangular region of integration: $(r, \theta) \in [1, 2] \times [0, 2\pi]$

The integral is rewritten in this manner:

$$\int_C x\sin(x^2)dx + (xy^2 - x^8)dy \to \iint_D \frac{\partial}{\partial x}\left(xy^2 - x^8\right) - \frac{\partial}{\partial y}\left(x\sin(x^2)\right) dA$$

$$\to \iint_D y^2 - 8x^7 dA \to \int_0^{2\pi}\int_1^2 \left(r^3\sin^2(\theta) - 8r^8\cos^7(\theta)\right) dr d\theta$$

$$\to \int_0^{2\pi}\left(\frac{15}{4}\sin^2(\theta)\right) d\theta - \int_0^{2\pi}\left(\frac{8(2^9-1)}{9}\cos^7(\theta)\right) d\theta$$

$$= \frac{15}{4}\pi$$

It is useful to use trigonometric identities and substitutions to help make part of (or all of) some of the integrals vanish.

43.5.4

$$\oint_C y^3 dx - x^3 dy$$

Since the line C bounds the circle of radius a, by Green's theorem the integral can transform to:

$$\iint_D -3x^2 - 3y^2 dA$$

Transforming the integral into polar coordinates yields:

$$-3 \int_0^{2\pi} \int_0^a r^2 r dr d\theta$$

$$-6\pi \int_0^a r^3 dr = \frac{-3\pi}{2} r^4 |_0^a$$

$$\frac{-3\pi}{2} a^4$$

43.5.5

By Green's Theorem:

$$\oint_C (\sqrt{x} + y^3)dx + (x^2 + \sqrt{y})dy$$

$$= \iint_D \frac{\partial(x^2 + \sqrt{y})}{\partial x} - \frac{\partial(\sqrt{x} + y^3)}{\partial y} dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} (2x - 3y^2) dy dx$$

$$= \int_{-\pi/2}^{\pi/2} (2x \cos x - \cos^3 x) dx$$

The first component of the integral cancels by symmetry, leaving:

$$- \int_{-\pi/2}^{\pi/2} \cos^3 x dx$$

$$- \int_{-\pi/2}^{\pi/2} \cos x (1 - \sin^2 x) dx$$

$$-[\sin x - \sin^3 x/3]_{-\pi/2}^{\pi/2}$$

$$= \boxed{-4/3}$$

43.5.6

Call the region within the circle D. Then apply Green's theorem directly to the line integral to find that the integral becomes:

$$\int_C (y^4 - \ln(x^2+y^2))dx + 2\arctan\left(\frac{y}{x}\right) dy$$

$$= \iint_D \frac{\partial}{\partial x}\left(2\arctan\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(y^4 - \ln(x^2+y^2)\right) dA$$

$$= \iint_D \left(-\frac{2y}{x^2+y^2}\right) - \left(4y^3 - \frac{2y}{x^2+y^2}\right) dA$$

$$= -4\iint_D y^3 dA$$

From here it may be useful to apply the transformation $(x,y) \to (x-x_0, y-y_0)$ in order to shift the disk D (so $D \to D'$ and $dA \to dA'$) so that it is centered at the origin. Then we have:

$$-4\iint_{D'} (y-y_0)^3 dA' \to -4\iint_{D'} (y^3 - 3y_0y^2 + 3y_0^2y - y_0^3)dA'$$

Notice that the odd terms in y vanish due to skew symmetry over the new disk centered at the origin. Then the integral becomes

$$-4\iint_{D'} (-3y_0y^2 - y_0^3)dA' \to 4y_0^3\pi a^2 + 12y_0\iint_{D'} y^2 dA'$$

which for the remaining integral we will convert to polar coordinates, knowing that by construction of the disk $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$:

$$12y_0\int_0^{2\pi}\int_0^a r\left(r^2\sin^2(\theta)\right) drd\theta \to 3y_0a^4\int_0^{2\pi}\left(\frac{1-\cos(2\theta)}{2}\right) d\theta = 3y_0\pi a^4$$

Adding the result from before the final answer is $4y_0^3\pi a^2 + 3y_0\pi a^4$.

43.5.9

$$\oint_C (x+y)dx + (y-x)dy$$

By Green's theorem, this becomes:

$$\iint_D (-1) - 1dA = -2\iint_D dA$$

D is the region bounded by the ellipse $(x/a)^2 + (y/b)^2 = 1$. The integral is simply -2 times the area of the ellipse, or

$$-2\pi ab$$

To verify this, one can first change the coordinates to $x/a = u, y/b = v$, which has a Jacobian of ab, to make the integral:

$$-2ab \iint_{D'} dA$$

where D' is the new region in the coordinate system, which is just the unit circle, which has an area of π.

43.5.11

Let D be the disk whose boundary is C. Then apply Green's theorem directly and change the integral as follows:

$$\int_C e^{-x^2+y^2} \left[\cos(2xy)dx - \sin(2xy)dy\right]$$

$$= \iint_D \frac{\partial}{\partial x}\left(-e^{-x^2+y^2}\sin(2xy)\right) - \frac{\partial}{\partial y}\left(e^{-x^2+y^2}\cos(2xy)\right) dA$$

$$= \iint_D \left(-4e^{-x^2+y^2} y\cos(2xy) + 4e^{-x^2+y^2} x\sin(2xy)\right) dA$$

The region of integration is a disk that is symmetric across the coordinate axes. We will consider quarters of the disk lying in each of the quadrants of the coordinate plane, because this integral is not easy to do without a symmetry argument.

Consider evaluating the integrand over the parts of the disk that lie on quadrants 1 and 3. The signs of the integrand evaluated on these regions are opposite, so they nullify each other (represents the transformation $(x, y) \to (-x, -y)$). Similarl y the integrand evaluated on the part of the disk lying on quadrants 2 and 4 are opposite in sign, so the integrals over those two regions nullify each other. Thus the integral vanishes.

43.5.12

$$\mathbf{F} = \langle (x+y)^2, -(x-y)^2 \rangle$$

$y = 5x - 4$ is the line passing through $(1, 1)$ and $(2, 6)$. $y = 2x^2 - x$ is the parabola passing through $(1, 1)$ and $(2, 6)$ and the origin (this can be found by setting up a system of equations using $y = ax^2 + bx + c$ and plugging in the three points). By Green's Theorem:

$$\oint_C \langle (x+y)^2, -(x-y)^2 \rangle \cdot d\mathbf{r} = \iint_D (x+y)^2 dx + (-(x-y)^2) dy$$

$$\iint_D \frac{\partial(-(x-y)^2)}{\partial x} - \frac{\partial(x+y)^2}{\partial y} dA$$

$$-\int_1^2 \int_{2x^2-x}^{5x-4} (2(x-y)+2(x+y))dydx$$
$$-4\int_1^2 \int_{2x^2-x}^{5x-4} xdydx$$
$$-4\int_1^2 (-4x+6x^2-2x^3)dx$$
$$-4[-2x^2+2x^3-x^4/2]_1^2$$
$$-4((-8+16-8)-(-2+2-1/2))$$
$$\boxed{=-2}$$

43.5.13

$$\int_C \langle e^x \sin y - qx, e^x \cos y - q\rangle$$

C is the positively oriented boundary of the region of the top half of the circle of radius a/2 and centered at (a,0). If we suppose C_2 is the x-axis from x=0 to x=a, the union of C and C_2 can create a closed loop ∂D. With Green's theorem, we can relate the initial integral to:

$$\int_C \mathbf{F}\cdot d\vec{r} + \int_{C_2} \mathbf{F}\cdot d\vec{r} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA$$

However, since $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, we can equate the first integral to:

$$\int_C \mathbf{F}\cdot d\vec{r} = -\int_{C_2} \mathbf{F}\cdot d\vec{r}$$

We can write the line C_2 as $\vec{r}(t) = \langle t, 0\rangle$, making the initial integral equivalent to:

$$-\int_0^a \langle 0 - qt, e^t - q\rangle \cdot \langle 1, 0\rangle dt$$
$$\int_0^a qtdt$$
$$1/2qa^2$$

43.5.16

One way to solve this is by simply taking this line integral:

$$\oint_{\partial D} xdy$$

The problem gives us the parametric equations for the path, which we can use to find x and dy. Those quantities are $x = a\cos(t)$ and $dy = b\cos(t)dt$, for $0 \le t \le 2\pi$. Then the integral becomes

$$ab\int_0^{2\pi} \cos^2(t)dt = \pi ab$$

43.5.17

One corollary of Green's Theorem is:

$$A(D) = \oint_{\partial D} ydx$$

For the cycloid, we know:

$$x = a(t - \sin t)\ ,\ dx = a(1 - \cos t)dt$$

$$y = a(1 - \cos t)\ ,\ dy = a \sin t dt$$

Plug in x and dy to the integral. Note that the bound on t is $0 \leq t \leq 2\pi$ because it is one arc of the cycloid.

$$A(D) = a^2 \int_0^{2\pi} (1 - \cos t)^2 dt$$

$$= a^2 \int_0^{2\pi} (\cos^2 t - 2\cos t + 1)dt$$

Use the identity $\cos^2 x = (1 + \cos(2x))/2$ and basic integration techniques to find the anti-derivative.

$$= a^2 [\frac{3}{2}t - 2\sin t + \sin(2x)/4]_0^{2\pi}$$

$$\boxed{= 3\pi a^2}$$

43.5.18

The area of the region can be represented by the closed line integral $\oint_{\partial D} xdy$ by Green's theorem. Substituting in what we have for x and dy gives us:

$$\int_0^{2\pi} a\cos^3(t)(3a\sin^2(t)\cos(t)dt)$$

with the bounds 0 to 2π as that is the period of the curve. This becomes:

$$3a^2 \int_0^{2\pi} \cos^2(t)(\sin^2(t)\cos^2(t))dt$$

$$3a^2 \int_0^{2\pi} \frac{1}{2}(1 + \cos(2t))\frac{1}{4}\sin^2(2t)dt$$

$$\frac{1}{8}3a^2 \int_0^{2\pi} (1 + \cos(2t)\frac{1}{2}(1 - \cos(2t))dt$$

And since any cosine function integrated over a multiple of its period goes to 0, the integral simplifies to:

$$\frac{1}{16}3a^2 \int_0^{2\pi} 1dt = \frac{3\pi a^2}{8}$$

43.5.21

Using the hint, put $y = tx$. Then find that the equation becomes

$$x^3 + t^3x^3 = 3ax^2t$$

which implies that

$$x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$$

and so bounds in t are found by finding two values of t that produce the same coordinates (x, y), and taking all values of t between (possibly even including these endpoints).

Notice that when $t = 0$ the curve starts at the origin. then as t tends to ∞, it is also found that both x and y tend to 0. Thus $0 \leq t\infty$.

The form of the line integral we want to use is

$$-\oint_{\partial D} ydx$$

since:

$$y = \frac{3at^2}{1+t^3},\ dx = -\frac{3a(-1+2t^3)}{(1+t^3)^2}$$

Then the integral becomes:

$$(3a)^2 \int_0^\infty \frac{t^2(2t^3-1)}{(1+t^3)^3} dt$$

Using the substitution $u = t^3 + 1$, the integral changes into:

$$\frac{(3a)^2}{3} \int_1^\infty \frac{2(u-1)-1}{u^3} du \to \frac{(3a)^2}{3} \int_1^\infty \left(2u^{-2} - 3u^{-3}\right) du = \frac{3a^2}{2}$$

43.5.24

$$\int_C g(x,y)(ydx + xdy)$$

This line integral is independent of path iff $\oint_K g(x,y)(ydx + xdy) = 0$ for a simple closed curve K.

$$\oint_K g(x,y)(ydx + xdy) = 0$$

By Green's Theorem, this is equivalent to:

$$\iint_D (\frac{\partial(xg(x,y))}{\partial x} - \frac{\partial(yg(x,y))}{\partial y})dA = 0$$

$$\iint_D (g(x,y) + x\frac{\partial g}{\partial x}) - (g(x,y) + y\frac{\partial g}{\partial y}))dA = 0$$

$$\iint_D (x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y})dA = 0$$

The double integral will be zero if the integrand is uniformly zero.

$$\boxed{x\frac{\partial g}{\partial x} = y\frac{\partial g}{\partial y}}$$

44.5 Exercises

44.5.1

There are three cases to consider, depending on which coordinate plane the rectangle lies in. Suppose we take the case where the rectangle lies on the xy plane. Then by the geometric interpretation of the flux, only the component of the constant vector field $\vec{F}$ normal to the xy plane contributes to the flux. The component of the vector field that is normal is indeed the component parallel to $\hat{e}_3$, which is c. Since the rectangle maps onto itself (this is a technicality that means nothing but that the rectangle is just a rectangle), the flux can be found by taking the product of the area A and c, so the flux is cA.

Similarly we find the flux for the other two cases. When the rectangle lies in the xz plane, then the component normal to the rectangle is b, so we have bA. Then for the case when the rectangle lies in the yz plane, the flux is aA.

44.5.2

Please refer to the temporary section for this for now. Sorry for low quality.

44.5.3

We are given that $F = \langle a, b, c\rangle$ and S is the boundary of the pyramid with base $[-q, q]\times[-q, q]$ in the xy plane and vertex $(0, 0, h)$. Recall that $\Phi = \iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})dS$. For this problem, the integral can be split up into five separate integrals, one for each face of the pyramid. The jth face of the pyramid S_j with unit normal vector to the face $\hat{\mathbf{n}}_j$, the dot product $\mathbf{F}\cdot\hat{\mathbf{n}}_j$ will be a constant function. Therefore, the integral becomes:

$$\Phi = \sum_{j=1}^{5}(\mathbf{F}\cdot\hat{\mathbf{n}}_\mathbf{j})\iint_{S_j} dS_j = \mathbf{F}\cdot(\sum_{j=1}^{5} A_j\hat{\mathbf{n}}_j)$$

where A_j is the area of the jth face of the pyramid.
Take S_5 to be the bottom face of the pyramid with unit normal vector $\langle 0, 0, -1\rangle$. It has area $A_5 = (2q)(2q) = 4q^2$. Therefore, its contribution to the sum in the above formula is $\langle 0, 0, -4q^2\rangle$.
The remaining faces have the same area, which is $A_1 = A_2 = A_3 = A_4 =$

$q\sqrt{q^2+h^2}$ by geometry. Therefore, we can simplify the flux formula to:

$$\Phi = \mathbf{F} \cdot (A_5 \hat{\mathbf{n}}_5 + A_1 \sum_{j=1}^{4} \hat{\mathbf{n}}_j)$$

We can find $\hat{\mathbf{n}}_1$ the "front face" of the pyramid, the triangular face with vertices $(-q, q, 0)$, $(q, q, 0)$, and $(0, 0, h)$, using the vector geometry. Taking the cross product of any two vectors between the vertices gives $\mathbf{n}_1 = \langle 0, 2qh, 2q^2 \rangle$.
(Note: There are two vectors perpendicular to the vectors defined by the vertices of the triangular face. We use $\langle 0, 2qh, 2q^2 \rangle$ instead of $\langle 0, -2qh, -2q^2$ because we are given that the normal vectors are oriented outward).
It is now easy to calculate $\hat{\mathbf{n}}_1$ from $\mathbf{n}_1$: we simply multiply $\mathbf{n}_1$ by the reciprocal of its magnitude.

$$\hat{\mathbf{n}}_1 = \frac{1}{\sqrt{q^2+h^2}} \langle 0, h, q \rangle$$

It follows by similar methods that:

$$\hat{\mathbf{n}}_2 = \frac{1}{\sqrt{q^2+h^2}} \langle 0, -h, q \rangle$$

$$\hat{\mathbf{n}}_3 = \frac{1}{\sqrt{q^2+h^2}} \langle h, 0, q \rangle$$

$$\hat{\mathbf{n}}_4 = \frac{1}{\sqrt{q^2+h^2}} \langle -h, 0, q \rangle$$

Plugging the derived values into our simplified equation for the flux of the pyramid yields:

$$\Phi = \mathbf{F} \cdot (\langle 0, 0, -4q^2 \rangle + (q\sqrt{q^2+h^2}) \frac{1}{\sqrt{h^2+q^2}} \langle 0, 0, 4q \rangle)$$

$$= \mathbf{F} \cdot \mathbf{0}$$

$$\boxed{= 0}$$

44.5.4

Consider the symmetry of the cylinder. To start we may consider taking a vertical strip of the cylinder and computing the flux there, and then comparing it with the flux computed on the opposite end (opposite meaning that the x and y coordinates of the strip were both negated). The flux computed over the first strip comes out to be some value, but whatever it is, it is immediately nullified by the flux found on the opposite strip since the normal vector is flipped around while locally the geometry is the same. Thus if we took a sum of the flux on all opposite strips in this manner they would all nullify each other and so the flux is 0.

44.5.5

Please refer to the temporary section for this for now. Sorry for low quality.

44.5.8

We are given $z = g(x,y) = 1 - x^2 - y^2$ and $\mathbf{F} = \langle xy, zx, xy\rangle = \langle xy, x(1 - x^2 - y^2), xy\rangle$. The normal vector (oriented upward) can be calculating $\mathbf{n} = \langle -g'_x, -g'_y, 1\rangle = \langle 2x, 2y, 1\rangle$. Recall the formula:

$$\Phi = \iint_D (\mathbf{F} \cdot \mathbf{n}) dA$$

Plug in the particular $\mathbf{F}$,$\mathbf{n}$.

$$= \iint_D (\langle xy, x(1 - x^2 - y^2), xy\rangle \cdot \langle 2x, 2y, 1\rangle) dA$$

The region D is given to be $[0,1] \times [0,1]$.

$$= \int_0^1 \int_0^1 (2x^2y + 2xy(1 - x^2 - y^2) + xy) dxdy$$

Evaluating the integral gives:

$$\boxed{7/12}$$

44.5.9

We first need to find the unit normal vector $\hat{n}$ by taking some partial derivatives and noting the given orientation of the surface. Because the paraboloid is oriented downward, we need the third (vertical) component of the unit normal to be negative. Then $\hat{n} = \frac{\vec{n}}{||\vec{n}||}$ where $\vec{n} = \langle z'_x, z'_y, -1\rangle$.

We also know that $||\vec{n}|| = J$ where J is the Jacobian of transformation satisfying $dS = JdA$. So it is sufficient to compute $\vec{n}$:

$$\vec{n} = \langle -2x, -2y, -1\rangle$$

In trying to go from a surface integral to a double integral we also want to find a region of integration. Here the region of integration will be a unit disk D centered at the origin since the boundary is where the paraboloid intersects the xy plane. Making sure to substitute the equation of the paraboloid in for z, the integral becomes:

$$\iint_S \vec{F} \cdot \hat{n} dS \to \iint_D \vec{F} \cdot \vec{n} dA \to \iint_D \langle y, -x, (1 - x^2 - y^2)^2\rangle \cdot \langle -2x, -2y, -1\rangle dA$$

$$\to -\iint_D (x^2 + y^2 - 1)^2 dA \to -\int_0^{2\pi} \int_0^1 r(r^2 - 1)^2 dr d\theta = -\frac{\pi}{3}$$

44.5.10

Please refer to the temporary section for this for now. Sorry for low quality.

44.5.11

We are given that S is the part of the sphere in the first octant with radius R centered at the origin and $\mathbf{F} = \langle x, -z, y \rangle$. We seek to evaluate the integral:

$$\iint_D (\mathbf{F} \cdot \mathbf{n}) dA$$

where $\mathbf{n}$ is the normal vector to the sphere pointed in towards the origin. A spherical integral transforms $\mathbf{F} = \langle R\cos\theta\sin\phi, -R\cos\phi, R\sin\theta\sin\phi \rangle$. The inward normal vector in spherical coordinates is $\mathbf{n} = -R^2 \sin\phi \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$ (this is derived in Example 39.7 in the textbook).

$$-R^3 \int_0^{\pi/2} \int_0^{\pi/2} (\cos^2\theta \sin^3\phi - \sin\theta\sin^2\phi\cos\phi + \sin\theta\sin^2\phi\cos\phi) d\phi d\theta$$

$$= -R^3 (\int_0^{\pi/2} \cos^2\theta d\theta)(\int_0^{\pi/2} \sin^3\phi d\phi)$$

The first integral can be calculated using the identity $\cos^2 u = (1 + \cos(2u))/2$. The second integral can be solved by rearranging the integral to $\sin\phi(1 - \cos^2\phi)$ and solving using standard methods.

$$= -R^3(\pi/2)(2/3)$$

$$\boxed{= -\pi R^3/6}$$

44.5.12

This integral may be computed without actually taking an integral. In trying to find a unit normal vector it becomes apparent that the unit normal vector $\hat{n}$ is in fact parallel to $\vec{r}$.

To illustrate the significance of this, give $\hat{n} = k\vec{r}$ for some real k. Then the integral becomes

$$\iint_S (\vec{a} \times \vec{r}) \cdot k\vec{r} dS = 0$$

which vanishes due to the properties of the triple product (use a cyclic transformation of the triple product in the integrand or just know that because two of the vectors are coplanar the triple product vanishes).

Alternatively to see how this integral vanishes, notice that the cross product $\vec{a} \times \vec{r}$ produces a vector that is perpendicular to $\vec{r}$ itself and similarly $\hat{n}$. Thus any dot product (like the one in the integrand) will be zero and the integral is zero.

44.5.13

Please refer to the temporary section for this for now. Sorry for low quality.

44.5.15

$$\mathbf{F} = \langle 2y, x, -z \rangle$$

We are given that S is the surface in the positive octant such that $y = g(x, z) = 1 - x^2 - y^2$. The flux is given by $\Phi = \iint_D (\mathbf{F} \cdot \mathbf{n}) dA$. The normal vector (such that the y-component is always positive) can be calculated using $\mathbf{n} = \langle -\frac{\partial g}{\partial x}, 1, -\frac{\partial g}{\partial z} \rangle = \langle 2x, 1, 2z \rangle$. Plug $\mathbf{F}$ (setting $y = g(x, z)$) and $\mathbf{n}$ into the flux equation.

$$\begin{aligned}\Phi &= \iint_D \langle 2(1 - x^2 - z^2), x, -z \rangle \cdot \langle 2x, 1, 2z \rangle dA \\ &= \iint_D (5x - 2z^2 - 4x^3 - 4xz^2) dA\end{aligned}$$

If the bound on x in the first octant is $0 \le x \le 1$, the z is bounded by $0 \le z \le \sqrt{1 - x^2}$ to ensure that $x, y, z \ge 0$.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (5x - 2z^2 - 4x^3 - 4xz^2) dA$$

Calculation of this integral can be simplified using the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, $J(r, \theta) = r$. The integral becomes:

$$\int_0^1 \int_0^{\pi/2} (5r^2 \cos\theta - 2r^3 \sin^2\theta - 4r^4 \cos^3\theta - 4r^4 \sin\theta \cos^2\theta) d\theta dr$$

We can integrate first w.r.t. θ using basic methods and making use of the identity $\cos^2 u = (1 - \cos(2u))/2$.

$$\int_0^1 (5r^2 - \frac{\pi}{2} r^3 - 4r^4) dr$$

$$\boxed{= \frac{13}{15} - \frac{\pi}{8}}$$

44.5.18

The surface seems to be only the outer sphere and the inner sphere. We will take each of these surfaces individually (due to the additivity of the integral) to simplify computation.

The surface of the outer sphere is the surface of a sphere of radius R. To construct the unit normal vector the easiest way, simply take the position vector $\vec{r} = \langle x, y, z \rangle$ and divide through by the radius of the sphere itself (this is due to the geometry of the sphere). Thus $\hat{n} = R^{-1}\vec{r}$.

We may opt to use a symmetry argument to simplify the integral, because the surface is oriented outward. Notice that we may give the sphere as $z = \pm\sqrt{R^2 - x^2 - y^2}$, and so for the hemisphere that contains negative z values, an extra negative sign must be introduced into the unit normal vector in order to retain the outward orientation. Thus the surface integral over the positive hemisphere is equivalent to the bottom hemisphere. We can double the flux integral for the positive hemisphere and obtain the same result, which is how we will proceed.

To actually go about taking the integral convert the surface integral into a generic double integral. So first find dS in terms of dA:

$$dS = JdA,\ J = \sqrt{1 + (z_x')^2 + \left(z_y'\right)^2} = \sqrt{1 + \left(-\frac{x}{z}\right)^2 + \left(-\frac{y}{z}\right)^2}$$

$$J = z^{-1}\sqrt{z^2 + x^2 + y^2} = Rz^{-1}$$

Now piece together the integral:

$$2\iint_S \vec{F} \cdot \hat{n} dS = 2\iint_D \langle x, y, z\rangle \cdot R^{-1}\langle x, y, z\rangle (z^{-1}R) dA$$

$$\to \iint_D \frac{1}{z}(x^2 + y^2 + y^2) dA \to 2R^2 \iint_D \frac{1}{\sqrt{R^2 - x^2 - y^2}} dA$$

$$2R^2 \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{R^2 - r^2}} dr d\theta = 4\pi R^3$$

To compute the flux over the inner sphere oriented in the opposite way, the steps are identical to the above except that we replace R with a, and due to the normal vector being oriented in the opposite direction, the quantity found is negated. Find that the flux over the inner sphere is:

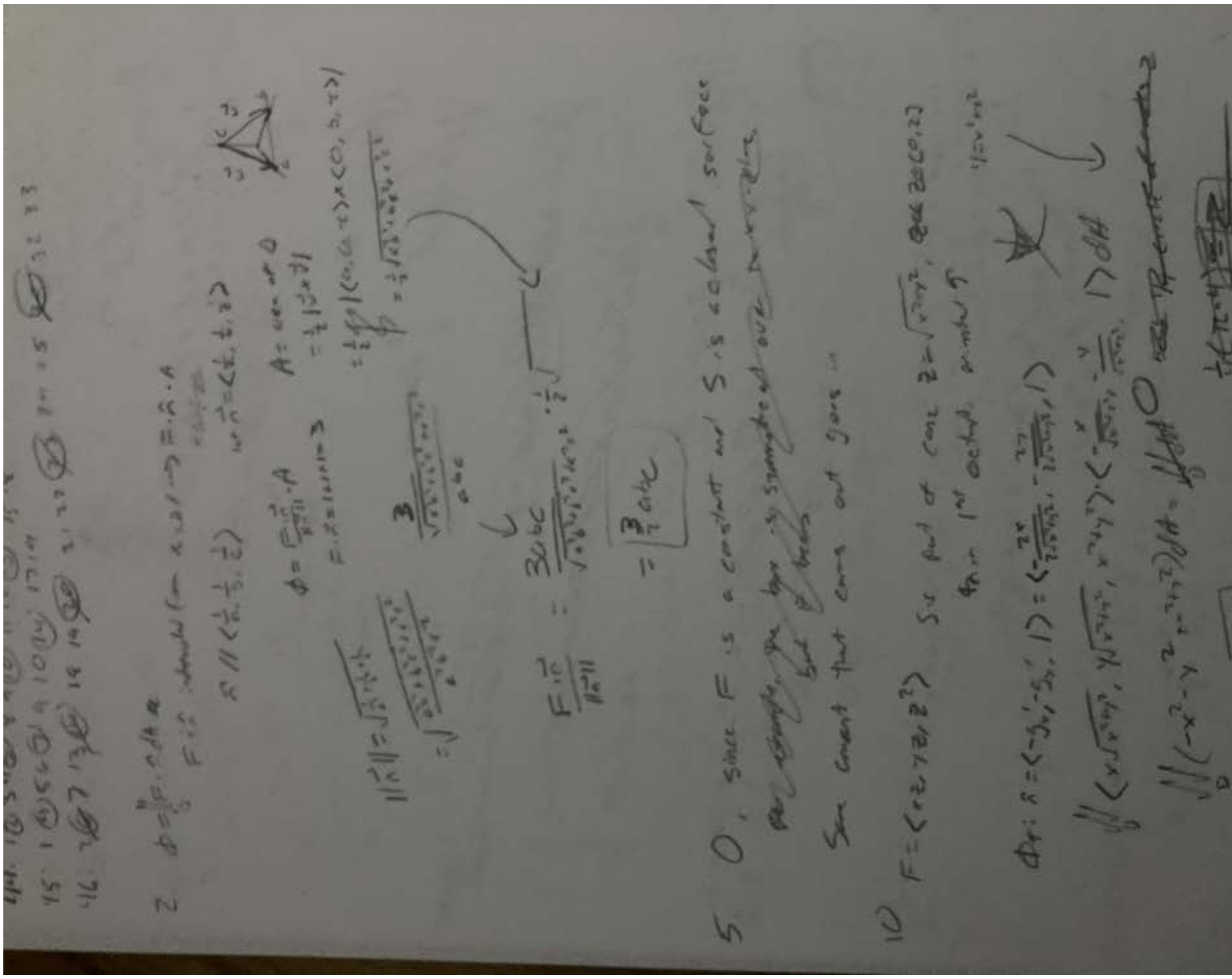
$$2\iint_S \vec{F} \cdot \hat{n} dS = 2\iint_D \langle x, y, z\rangle \cdot -a^{-1}\langle x, y, z\rangle (z^{-1}a) dA$$

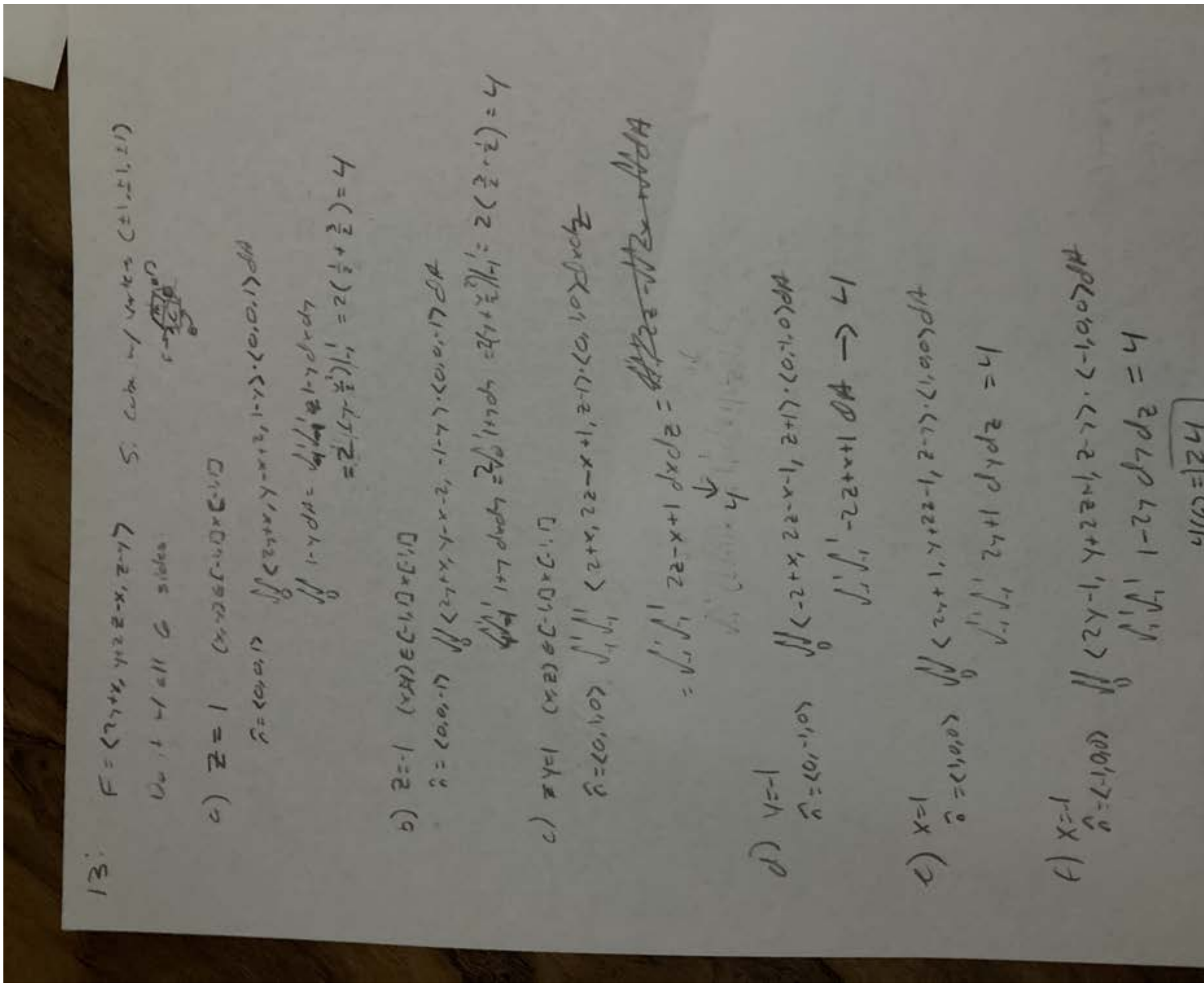
$$\to -\iint_D \frac{1}{z}(x^2 + y^2 + y^2) dA \to -2a^2 \iint_D \frac{1}{\sqrt{a^2 - x^2 - y^2}} dA$$

$$-2a^2 \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr d\theta = -4\pi a^3$$

Add the two results together to find that the flux is $4\pi(R^3 - a^3)$.

44.5.TEMP





45.7 Exercises

45.7.1

To compute the circulation of $\vec{F}$ along ∂S, we must choose an orientation for the boundary curve that is given by our own choice for the orientation of S. For this problem it seems natural to take the outward orientation, so we take the curve given by the intersection of the plane $z = 1$ and the sphere $x^2 + y^2 + z^2 = 2$. The curve itself is the unit circle suspended above where $z = 1$, so the path may be given as $\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$ for $0 \leq t \leq 2\pi$ (counterclockwise since the surface is oriented outwards).

The line integral is then

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} \to \int_0^{2\pi} \langle \sin(t), -\cos(t), 1\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle dt$$

$$\to \int_0^{2\pi} (-1) dt = -2\pi$$

Then to compute the equivalent flux integral the surface S given by the graph $z = \sqrt{1 - x^2 - y^2}$ is again oriented outward. Thus the normal vector $\vec{n}$ (not the unit normal, because computing the norm of $\vec{n}$ is unnecessary) is given by:

$$\vec{n} = \langle -z'_x, -z'_y, 1\rangle = \left\langle \frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1 \right\rangle$$

Then to compute the curl of $\vec{F}$:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{e}_1 & \hat{e}_1 & \hat{e}_1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{pmatrix} = \langle 0, 0, -2\rangle$$

In computing the flux integral we will be making a transformation to integrate over a planar region. The region of integration will be the disk of unit radius D, since that is the projection of the surface itself onto the xy plane. The integral becomes:

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n} dS \to \iint_D \left(\nabla \times \vec{F}\right) \cdot \vec{n} dA$$

$$\to \iint_D \langle 0, 0, -2\rangle \cdot \left\langle \frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1 \right\rangle dA$$

$$\to -2 \iint_D dA = -2\pi$$

The last integral is just the surface area of the disk, which was then scaled by -2 (geometry). Both methods result with the same value, so we seem to be correct.

45.7.4

Please refer to the temporary section for this for now. Sorry for low quality.

45.7.5

$$\mathbf{F} = \langle yz, 2xz, e^{xy}\rangle$$

The closed curve C is the intersection of the cylinder $x^2 + y^2 = 1$ and $z = 3$. The curve is oriented clockwise, so the line integral is:

$$\oint_{-C} \mathbf{F} \cdot d\mathbf{r} = -\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Let S be the surface such that $\partial S = C$. That is, S is the part of the plane $z = 3$ bound by $x^2 + y^2$. Let D be the region on the xy plane that lies beneath S. By Stokes' Theorem:

$$-\oint_C \mathbf{F} \cdot d\mathbf{r} = -\iint_D ((\nabla \times \mathbf{F}) \cdot \mathbf{n}) dA$$

The the curl of $\mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix} = \langle xe^{xy} - 2x, -ye^{xy} + y, z \rangle$$

The normal vector to the plane $z = 3$ (oriented upward) is:

$$\hat{\mathbf{n}} = \langle 0, 0, 1 \rangle$$

Thus, the integral is:

$$-\iint_D (\langle xe^{xy} - 2x, -ye^{xy} + y, z \rangle \cdot \langle 0, 0, 1 \rangle) dA$$

$$= -\iint_D z dA$$

It is already known that $z \equiv 3$.

$$= -3 \iint_D dA$$

$$= -3A(D)$$

D is a circle with radius 1, so its area is $A(D) = \pi r^2 = \pi$.

$$\boxed{= -3\pi}$$

45.7.6

Graphically it may be helpful to imagine that the x axis is the vertically aligned one, since the cylinder is oriented in this manner. As for the line integral, we wish to use Stokes' theorem to evaluate it. So we need to have a surface that the curve C is the boundary of. We shall choose the simplest one, that is, the plane $x + y = 1$ (or $x = 1 - y$) that is bounded by the curve C. Call this surface S, and because the curve C is oriented counterclockwise when viewed from above, the orientation of S is upward.

Since the vertical axis is the x axis, give the unit normal vector $\hat{n}$ as

$$\hat{n} = \frac{\vec{n}}{||\vec{n}||} = \frac{1}{||\langle 1, x'_y, x'_z \rangle||} \langle 1, x'_y, x'_z \rangle = \frac{1}{\sqrt{2}} \langle 1, -1, 0 \rangle$$

The curl of the vector field $\vec{F}$ can be computed like so:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{e}_1 & \hat{e}_1 & \hat{e}_1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 3z & 3y \end{pmatrix} = \langle 0, 0, -x \rangle$$

Then the integral so far is given as follows:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F} \right) \cdot \hat{n} dS \to \iint_S \langle 0, 0, -x \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1, 0 \rangle dS$$

$$\to \iint_S (0) dS = 0$$

The integral vanishes.

45.7.7

Please refer to the temporary section for this for now. Sorry for low quality.

45.7.9

We are given:

$$\mathbf{F} = \langle z^2 y/2, -z^2 x/2, 0 \rangle$$

and C is the boundary of $z = 1 - \sqrt{x^2 + y^2}$ in the first quadrant. The integral is then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Let S be the surface such that $\partial S = C$ and D be the region on the xy plane that lies beneath S. By Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D ((\nabla \times \mathbf{F}) \cdot \mathbf{n}) dA$$

The the curl of $\mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 y/2 & -z^2 x/2 & 0 \end{vmatrix} = \langle xz, yz, -z^2 \rangle$$

We are given that $z = g(x, y) = 1 - \sqrt{x^2 + y^2}$. The normal vector (oriented upward) to S is:

$$\mathbf{n} = \langle -g'_x, -g'_y, 1 \rangle = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 1 \rangle$$

The integral can thus be simplified to:

$$\iint_D ((1 - \sqrt{x^2 + y^2}) \sqrt{x^2 + y^2} - (1 - \sqrt{x^2 + y^2})^2) dA$$

It is easiest to evaluate this integral using a polar substitution:

$$\int_0^{\pi/2}\int_0^1 r((1-r)r-(1-r)^2)drd\theta$$
$$=\frac{1}{2}\int_0^1(-2r^3+3r^2-r)dr$$
$$=\frac{1}{2}(-\frac{1}{2}+1-\frac{1}{2})$$
$$\boxed{=0}$$

45.7.10

The surface S given by the graph $z=x^2+(y-1)^2$ is oriented upwards because its boundary curve is oriented counterclockwise when viewed from above. We would also like to take note of the region that it maps to when we take a vertical projection of this surface onto the xy plane since we will be making a transformation to change the surface integral to a normal double integral. Such a region is the unit disk D, because the cylinder bounds both the surface and the disk in that manner.

We would like to find the normal vector (not the unit normal vector since the normalization factor will be canceled out by the Jacobian of transformation from the disk D to the surface S). The normal vector $\vec{n}$ is given by:

$$\vec{n}=\langle -z'_x,-z'_y,1\rangle=\langle -2x,-2(y-1),1\rangle$$

The curl of the vector field is computed as follows:

$$\nabla\times\vec{F}=\det\begin{pmatrix}\hat{e}_1 & \hat{e}_1 & \hat{e}_1\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y-z & -x & x\end{pmatrix}=\langle 0,-2,-2\rangle$$

Then the integral is computed:

$$\oint_C\vec{F}\cdot d\vec{r}=\iint_S\left(\nabla\times\vec{F}\right)\cdot\hat{n}dS\to\iint_D\left(\nabla\times\vec{F}\right)\cdot\vec{n}dA$$
$$\to\iint_D\langle 0,-2,-2\rangle\cdot\langle -2x,-2(y-1),1\rangle dA\to -2\iint_D(-2y+3)dA$$
$$\to -2\int_0^{2\pi}\int_0^1 r(-2r\sin(\theta)+3)drd\theta=-6\pi$$

45.7.14

Please refer to the temporary section for this for now. Sorry for low quality.

45.7.17

We are given:

$$\mathbf{F} = \langle -yz, xz, z^2 \rangle$$

and C is the boundary of $z = 1 - x^2 - y^2$ in the first octant traversed counter-clockwise. The work done by the force is given by:

$$\oint_{-C} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

Let the surface S be defined by $z = 1 - x^2 - y^2$. That is, $\partial S = C$. Let D be the region of the xy plane that lies beneath S. By Stokes' Theorem:

$$-\oint \mathbf{F} \cdot d\mathbf{r} = -\iint_D ((\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}) dA$$

The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & z^2 \end{vmatrix} = \langle -x, -y, 2z \rangle$$

We are given that $z = g(x, y) = 1 - x^2 - y^2$. The normal vector (oriented upward) to S is:

$$\mathbf{n} = \langle -g'_x, -g'_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$$

The integral can thus be simplified to:

$$-\iint_D (2(1 - x^2 - y^2) - 2x^2 - 2y^2) dA$$

$$-\iint_D (2 - 4x^2 - 4y^2) dA$$

It is easiest to evaluate this integral using a polar substitution:

$$= -\int_0^{\pi/2} \int_0^1 r(2 - 4r^2) dr d\theta$$

$$\boxed{= 0}$$

45.7.19

From the vector field it seems pretty unreasonable to take the line integral normally. So we will use Stokes' theorem to help.

Notice that the field is conservative, which is shown by taking the curl of the vector field like so:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \hat{e}_1 & \hat{e}_1 & \hat{e}_1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2} - yz & e^{y^2} - xz & z^2 - xy \end{pmatrix} = \langle -x + x, -y + y, -z + z \rangle = \vec{0}$$

We can use this to our advantage by making the contour C closed by taking the union of C with a straight line segment BA from $(a, 0, h)$ to $(a, 0, 0)$. From the fundamental theorem of line integrals or Stokes' theorem it follows that this line integral over the closed contour is zero. But what we add to the contour we must also remove. It follows from the additivity of the line integral that:

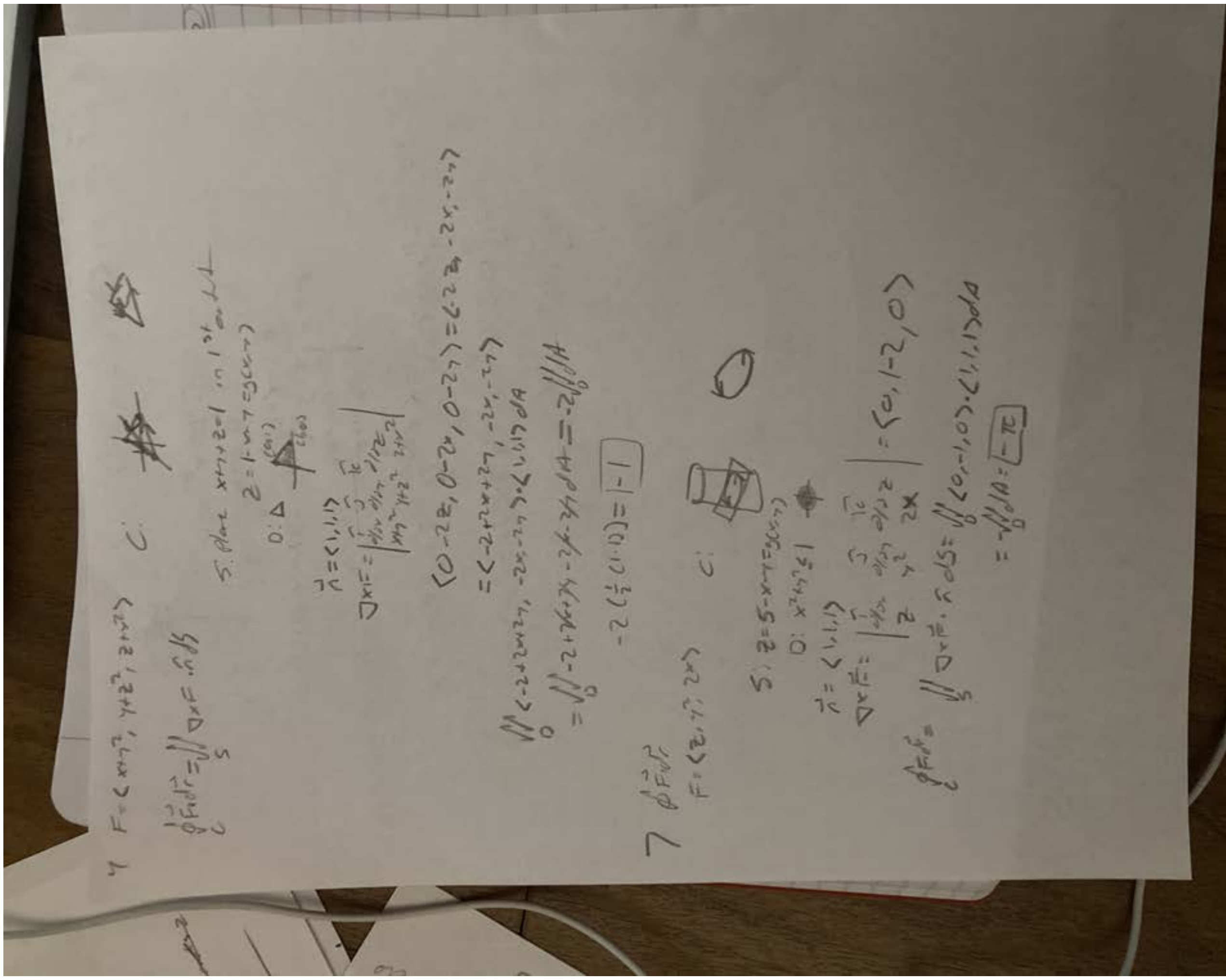
$$\int_C \vec{F} \cdot d\vec{r} = \oint_{C \cup BA} \vec{F} \cdot d\vec{r} - \int_{BA} \vec{F} \cdot d\vec{r} = 0 - \int_{BA} \vec{F} \cdot d\vec{r}$$

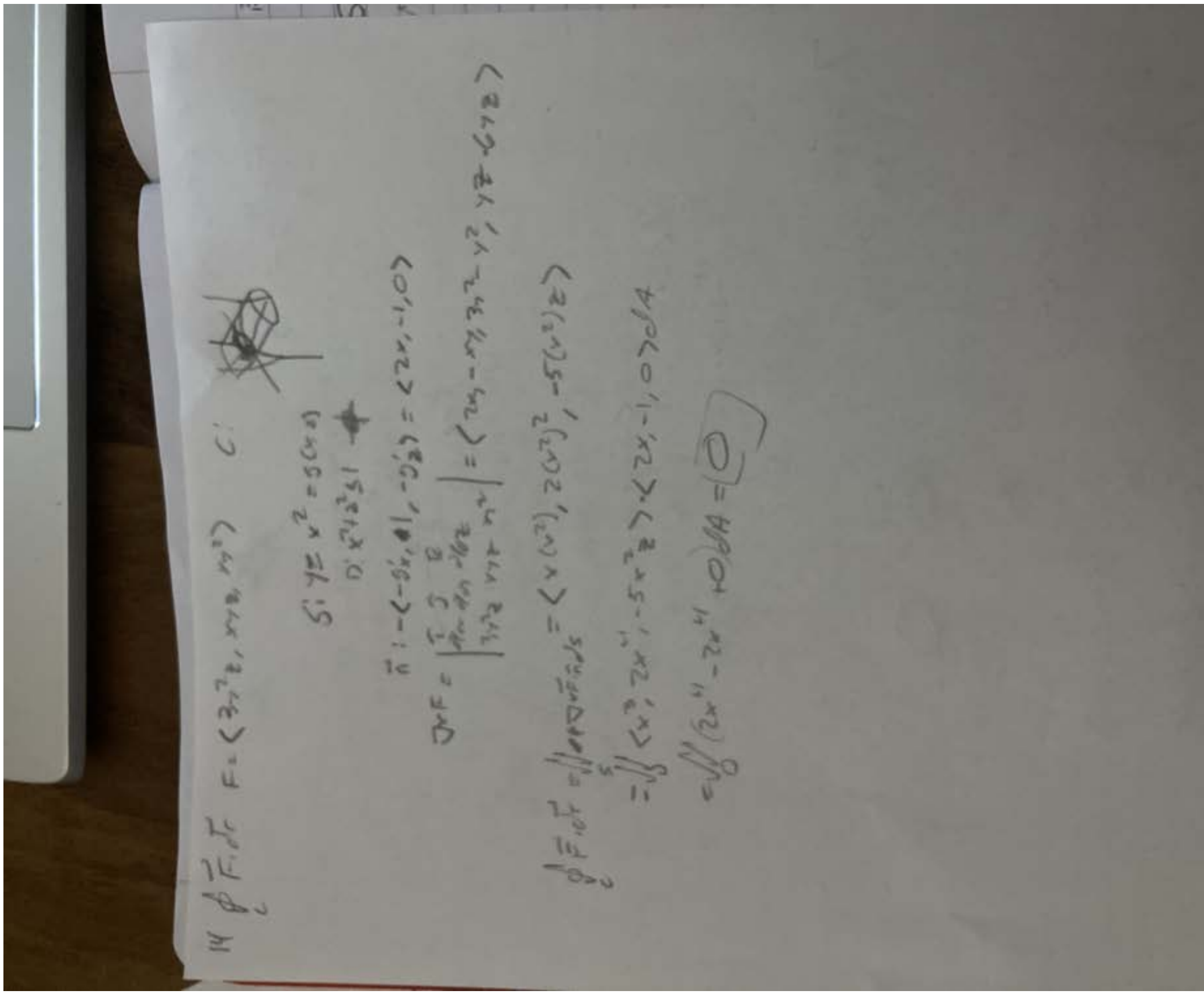
So all we need to compute is the last integral over the line segment. Because the parameterization does not matter we may choose the easiest one, namely $\vec{r}(t) = (1-t)\langle a, 0, h\rangle + t\langle a, 0, 0\rangle = \langle a, 0, h - th\rangle$ where $0 \leq t \leq 1$. Then it follows that the line integral is computed as follows:

$$-\int_{BA} \vec{F} \cdot d\vec{r} \to -\int_0^1 \langle e^{a^2}, 1 - a(h - th), (h - th)^2\rangle \cdot \langle 0, 0, -h\rangle dt$$

$$\to h^3 \int_0^1 (1-t)^2 dt = h^3 \int_0^1 u^2 du = \frac{1}{3}h^3$$

45.7.TEMP





46.6 Exercises

46.6.2

The divergence of a vector field can be found by taking the formal dot product

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \vec{F}$$

Perform this computation for the vector field $\vec{F}$. Rewrite the vector field as

$\vec{F} = \frac{1}{||\vec{r}||}\vec{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\langle x, y, z\rangle$:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$\rightarrow \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\rightarrow \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{r}$$

46.6.6

Please refer to the temporary section for this for now. Sorry for low quality.

46.6.7

$$\mathbf{F} = \mathbf{a} \times \nabla g$$

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. The cross product can now be evaluated algebraically.

$$\mathbf{F} = \langle a_1, a_2, a_3 \rangle \times \langle g'_x, g'_y, g'_z \rangle = \langle a_2 g'_z - a_3 g'_y, a_3 g'_x - a_1 g'_z, a_1 g'_y, a_2 g'_x \rangle$$

Now evaluate the divergence of F:

$$\nabla \cdot \mathbf{F} = (a_2 g''_{zx} - a_3 g'_{yx}) + (a_3 g'_{xy} - g_1 g'_{zy}) + (a_1 g'_{yz} - a_2 g'_{xz})$$

If g is a "nice enough" function such that it satisfies Clairaut's Theorem, then the curl of $\mathbf{F}$ vanishes.

$$\boxed{= 0}$$

46.6.13

Apply the other form of Green's theorem:

$$\oint_{\partial D} \vec{F} \cdot \hat{n} ds = \iint_D \nabla \cdot \vec{F} dA$$

We should find that (give D as the planar region bounded by C):

$$\oint_C \vec{a} \cdot \hat{n} ds = \iint_D \nabla \cdot \vec{a} dA = 0$$

Note that the vector $\vec{a}$ is a constant vector. The divergence of the vector field given by $\vec{F} = \vec{a}$ is zero, since derivatives of constants vanish. Thus the double integral vanishes as shown above.

46.6.14

Please refer to the temporary section for this for now. Sorry for low quality.

46.6.18

For this it may be easier to compute the flux using the divergence theorem, instead of taking the surface integral. Give D as the rectangular region bounded by S. Then compute the divergence of the vector field $\vec{F} = \langle x^2, y^2, z^2 \rangle$:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2, y^2, z^2 \rangle = 2x + 2y + 2z$$

From the divergence theorem we know that:

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_D (2x + 2y + 2z) dV$$

Because the region D is a rectangular prism, apply Fubini's theorem directly to find the flux:

$$\iiint_D (2x + 2y + 2z) dV = \int_0^c \int_0^b \int_0^a (2x + 2y + 2z) dx dy dz = a^2bc + ab^2c + abc^2$$

Because the flux is positive, we have a faucet within the region.

46.6.19

We are given:

$$\mathbf{F} = \langle x^3, y^3, z^3 \rangle$$

and S is the sphere $x^2 + y^2 + z^2 = R^2$ oriented inward. We are asked to find the flux:

$$\iint_{-S} ((\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}) dS = -\iint_S ((\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}) dS$$

Let E be the ball such that $\partial E = S$. By Divergence Theorem, the flux is also given by:

$$-\iiint_E (\nabla \cdot \mathbf{F}) dV$$

The flux of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \langle 3x^2 + 3y^2 + 3z^2 \rangle$$

So the integral becomes:

$$3 \iiint_E (x^2 + y^2 + z^2) dV$$

Make a spherical substitution.

$$3 \int_0^R \int_0^\pi \int_0^{2\pi} \rho^4 \sin \phi \, d\theta d\phi d\rho$$

$$\frac{6\pi R^5}{5} \int_0^\pi \sin \phi \, d\phi$$

$$\boxed{= -\frac{12\pi R^5}{5}}$$

The orientation is inward but the flux is negative, implying that this is a faucet.

46.6.20

Please refer to the temporary section for this for now. Sorry for low quality.

46.6.21

Give D as the unit ball bounded by the S, and apply the divergence theorem. We seek to find the divergence of the vector field first:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle -xy^2, -yz^2, zx^2 \rangle = -y^2 - z^2 + x^2$$

The statement of the divergence theorem uses an outwardly oriented surface. In this problem the surface S is oriented inwards, so we must adjust our use of the divergence theorem to reflect this. Notice that we may form the outward orientation of S by simply negating the unit normal $\hat{n}$, and so turn the surface integral into one that we can apply the divergence theorem on. The integral becomes:

$$\iint_S \vec{F} \cdot (-1)\hat{n} dS = \iiint_D (-y^2 - z^2 + x^2) dV$$

$$\rightarrow \iint_S \vec{F} \cdot \hat{n} dS = -\iiint_D (-y^2 - z^2 + x^2) dV$$

Make a change of variables into spherical coordinates where the x axis is the vertical axis. This means that $x = \rho\cos(\phi)$, and $y^2 + z^2 = r^2 = \rho^2 \sin^2(\phi)$. Since the whole ball is the region of integration, all parameters take on their natural ranges (and ρ ranges from 0 to 1).

$$-\iiint_D (-y^2 - z^2 + x^2) dV$$

$$\rightarrow -\int_0^{2\pi} \int_0^{\pi} \int_0^1 (-\rho^2 \sin^2(\phi) + \rho^2 \cos^2(\phi))(\rho^2 \sin(\phi)) d\rho d\phi d\theta$$

$$\rightarrow -2\pi \int_0^{\pi} \int_0^1 (\rho^4 (2\cos^2(\phi) - 1)) \sin(\phi)) d\rho d\phi$$

$$\rightarrow -\frac{2\pi}{5} \int_0^{\pi} (2\cos^2(\phi) - 1)) \sin(\phi) d\phi = \frac{4\pi}{15}$$

Because the surface S was oriented inwards and we found a positive flux, the interpretation is that the vector field seems to be converging somewhere within the sphere. Thus we have a sink.

46.6.22

Give D as the cylindrical region within the cylindrical surface S. It is quickly apparent that we can use the divergence theorem directly. First compute the divergence of the vector field $\vec{F} = \langle xy, z^2y, zx\rangle$:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy, z^2y, zx\rangle = y + z^2 + x$$

Then the integral becomes:

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_D \nabla \cdot \vec{F} dV \to \iiint_D (y + z^2 + x) dV$$

Use cylindrical coordinates where $-2 \leq z \leq 2$, $0 \leq r \leq 2$, and θ takes on its natural range:

$$\iiint_D (y + z^2 + x) dV \to \int_0^{2\pi} \int_0^2 \int_{-2}^2 (r\sin(\theta) + z^2 + r\cos(\theta)) dz(r) dr d\theta$$

$$\to \int_0^{2\pi} \int_0^2 \left(4r^2 \sin(\theta) + 4r^2 \cos(\theta) + \frac{16}{3} r\right) dr d\theta$$

$$\to \int_0^{2\pi} \frac{32}{3} (\sin(\theta) + \cos(\theta) + 1) = \frac{64}{3}\pi$$

Since we have a positive flux on the outwardly oriented surface S, we have a faucet.

46.6.23

Give D as the solid region bounded by S. Then compute the divergence of the vector field $\vec{F} = \langle xz^2, \frac{y^3}{3}, zy^2 + xy\rangle$:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xz^2, \frac{y^3}{3}, zy^2 + xy\rangle = z^2 + y^2 + x^2$$

Remembering that the orientation of S was inwards, we know that due to the divergence theorem the integral becomes:

$$\iint_S \vec{F} \cdot (-1)\hat{n} dS = -\iiint_D \nabla \cdot \vec{F} dV \to -\iiint_D (z^2 + y^2 + x^2) dV$$

This is easier to do in spherical coordinates where ρ varies from 0 to 1, and ϕ and θ both vary from 0 to $\frac{\pi}{2}$. Continuing the computation:

$$-\iiint_D (z^2 + y^2 + x^2) dV \to -\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho^2)\rho^2 \sin(\phi) d\rho d\phi d\theta = -\frac{\pi}{10}$$

46.6.24

We will use the divergence theorem. Give D as the solid region bounded by S. First we must find the divergence of the vector field:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle yz, z^2x + y, z - xy \rangle = 2$$

Then we can apply the divergence theorem to find the flux:

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_D (2) dV$$

The integral is fairly simple to compute, especially using spherical coordinates. Since we have a unit sphere and the coefficient of $\sqrt{x^2 + y^2}$ is 1, the bounds are nice. The parameter ρ varies from 0 to 1, θ takes on its natural range, and ϕ ranges from 0 to $\frac{\pi}{4}$. Then the integral becomes:

$$2 \iiint_D dV \to 2 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin(\phi) d\rho d\phi d\theta = \frac{4\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$$

46.6.25

We are given:

$$\mathbf{F} = \langle x + \tan(yz), \cos(xz) - y, \sin(xy) + z \rangle$$

and S is the solid region between the sphere $x^2 + y^2 + z^2 = 2z$ and the cone $z = \sqrt{x^2 + y^2}$ oriented outward. We are asked to find the flux:

$$\iint_S ((\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}})$$

Let E be the region such that $\partial E = S$. By Divergence Theorem, the flux is also given by:

$$\iiint_E (\nabla \cdot \mathbf{F}) dV$$

The flux of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = 1$$

So the integral becomes:

$$\iiint_E dV$$

which is just the volume of the solid E. This integral will be easier to evaluate if we make a spherical substitution. The boundary surfaces becomes $z = \sqrt{x^2 + y^2} \to \phi = \pi/4$ and $x^2 + y^2 + z^2 = 2z \to \rho = 2\cos\phi$. Thus, the integral becomes:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} [\rho^3 \sin\phi]_0^{2\cos\phi} d\phi$$
$$= \frac{16\pi}{3}\int_0^{\pi/4} \cos^3\phi \sin\phi d\phi$$
$$= -\frac{4\pi}{3}[\cos^4\phi]_0^{\pi/4}$$
$$= -\frac{4\pi}{3}(1/4 - 1)$$
$$\boxed{= \pi}$$

The orientation is outward and the flux is positive, implying that this is a faucet.

46.6.26

Please refer to the temporary section for this for now. Sorry for low quality.

46.6.32

We would like to use the divergence theorem, which means we want to form a closed surface somehow. Notice that if we take the union of S with a disk of radius 2 centered at the origin (call this disk S_d, and orient it upwards to mimic the inward orientation of S), we have a surface that is closed. In particular this surface would be the boundary of the upper hemisphere of a ball of radius 2.

Keep in mind that if we want to use the divergence theorem, we must introduce negative signs into the surface integral since the statement involves surfaces that are oriented outward, unlike our surface $S \cup S_d$. Give the region bounded by this surface as D.

This deformation by introducing the disk S_d means that the following equality (by additivity) is true:

$$\iint_S \vec{F} \cdot (-1)\hat{n} dS + \iint_{S_d} \vec{F} \cdot (-1)\hat{n} dS = \iiint_D \left(\nabla \cdot \vec{F}\right) dV$$
$$\to \iint_S \vec{F} \cdot \hat{n} dS = -\iiint_D \left(\nabla \cdot \vec{F}\right) dV - \iint_{S_d} \vec{F} \cdot \hat{n} dS$$

Compute the divergence of the vector field $\vec{F}$:

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy^2, yz^2, zx^2 + x^2 \rangle = y^2 + z^2 + x^2$$

We shall first compute the triple integral using the divergence we just computed. It will be useful to use spherical coordinates once more. Keep in mind that ρ ranges from 0 to 2 and ϕ ranges from 0 to $\frac{\pi}{2}$, where θ takes on its natural range.

$$-\iiint_D \left(\nabla \cdot \vec{F}\right) dV \to -\iiint (y^2 + z^2 + x^2) dV$$

$$\rightarrow -\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (\rho^2)(\rho^2\sin(\phi))d\rho d\phi d\theta = -\frac{64\pi}{5}$$

Then we must compute the surface integral over the disk S_d. To aid in computation, note that it is easiest to give the unit normal vector as $\hat{e}_3$, so that the integrand reduces to a simpler form:

$$-\iint_{S_d} \vec{F}\cdot\hat{e}_3 dS \rightarrow -\iint_{S_d} \langle xy^2, yz^2, zx^2+x^2\rangle\cdot\langle 0,0,1\rangle dS$$

$$\rightarrow -\iint_{S_d} x^2(z+1)dS$$

On all points on the disk S_d, the value of z is 0. The integral becomes much simpler and it is useful to use polar coordinates (remember the radius of the boundary of the disk is 2) to compute the integral:

$$-\iint_{S_d} x^2 dS \rightarrow -\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta)(r)drd\theta = -4\pi$$

Add the two results we found from both of the integrals to find the desired flux over the original surface S, which is $-4\pi - \frac{64\pi}{5} = -\frac{84}{5}\pi$.

46.6.TEMP