Assignment 1 with solutions, MAP 6505, Fall 2023

3.3 (1 pt). Which of the following functions are Lebesgue integrable on $\mathbb{R}$:

$$\frac{\sin(x)}{x}, \quad \frac{e^{ikx}}{x}, \quad \frac{\cos(x)}{\sqrt{|x|}}, \quad e^{-x}, \quad x^{100}e^{-x^2}$$

SOLUTION: All of these functions are continuous almost everywhere. Therefore the Lebesgue integrability means the absolute integrability. If the integral does not converge (converge) absolutely in a particular regularization, then it does not converge (converges) in any regularization. It follows

$$\begin{aligned}\int_{\pi}^{\infty} \frac{|\sin(x)|}{x}\,dx &= \sum_{n=1}^{\infty}\int_{\pi n}^{\pi(n+1)} \frac{|\sin(x)|}{x}\,dx \\ &\geq \sum_{n=1}^{\infty}\frac{1}{\pi n}\int_{\pi n}^{\pi(n+1)} |\sin(x)|\,dx = \sum_{n=1}^{\infty}\frac{2}{\pi n} = \infty\end{aligned}$$

So, $\frac{\sin(x)}{x}$ is not integrable on $\mathbb{R}$. The absolute value $|\frac{e^{ikx}}{x}| = \frac{1}{|x|}$ is not integrable,

$$\int \frac{dx}{|x|} = 2\lim_{a\to 0^+} = \int_a^{\frac{1}{a}} \frac{dx}{x} = -4\lim_{a\to 0^+}\ln(a) = \infty$$

and so is $\frac{e^{ikx}}{x}$. The function $\frac{\cos(x)}{\sqrt{|x|}}$ is integrable on any bounded interval containing $x = 0$:

$$\int_0^a \frac{|\cos(x)|}{\sqrt{|x|}}\,dx \leq \int_0^a \frac{dx}{\sqrt{x}} = 2\sqrt{a} < \infty$$

However it is not absolutely integrable on any unbounded interval:

$$\int_{\pi}^{\infty} \frac{|\cos(x)|}{\sqrt{|x|}}\,dx = \sum_{n=1}^{\infty}\int_{\pi n}^{\pi(n+1)} \frac{|\cos(x)|}{\sqrt{|x|}}\,dx \geq \sum_{n=1}^{\infty}\frac{2}{\sqrt{\pi n}} = \infty$$

similarly to the first function in question. The fourth function is also not integrable because

$$\lim_{a\to\infty}\int_{-a}^{a} e^{-x}dx = \lim_{a\to\infty}(e^a - e^{-a}) = \infty$$

The last function is integrable on $\mathbb{R}$ because

$$\begin{aligned}& x^{100} \leq 100!e^{|x|} \\ \Rightarrow \quad & \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2+|x|}dx = 2e^{\frac{1}{4}}\lim_{a\to\infty}\int_{-\frac{1}{2}}^{a-\frac{1}{2}} e^{-y^2 dy} < \sqrt{\pi}e^{\frac{1}{4}} < \infty\end{aligned}$$

where $y = x - \frac{1}{2}$ and after the change of variables the integration region was enlarged to the whole $\mathbb{R}$ to get the last inequality.

4.8 (1 pt). Let $f \in \mathcal{L}(\mathbb{R})$ such that $\int f(x)\,dx = 1$ and φ be a continuous function with bounded support. Put $f_n(x) = nf(nx)$, $n = 1, 2, \ldots$. Show that

$$\lim_{n\to\infty} \int f_n(x)\varphi(x)\,dx = \varphi(0)$$

Hint: Use the Lebesgue dominated convergence theorem and that any continuous function with bounded support is bounded.

SOLUTION: One infers that

$$\begin{aligned} \lim_{n\to\infty} \int f_n(x)\varphi(x)\,dx &\overset{(1)}{=} \lim_{n\to\infty} \int f(y)\varphi(y/n)\,dy \\ &\overset{(2)}{=} \int f(y) \lim_{n\to\infty} f(y)\varphi(y/n)\,dy \\ &\overset{(3)}{=} \varphi(0) \int f(y)\,dy \overset{(4)}{=} \varphi(0) \end{aligned}$$

Here (1) is obtained by changing variables, $y = nx$ so that $dy = ndx$, (2) follows from the Lebesgue dominated convergence theorem. Any continuous function on $\mathbb{R}$ with a bounded support must be bounded $\sup|\varphi| = M < \infty$. Therefore the integrand has an integrable bound independent of n:

$$|f(y)\varphi(y/n)| \le M|f(y)| \in \mathcal{L}$$

and the absolute value of an integrable function is integrable. The equality (3) follows from continuity of φ and (4) holds by the hypothesis.

5.6 (2 pts). Let

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad (x, y) \in \Omega = (1, \infty) \times (1, \infty)$$

(i) Calculate the iterated integral

$$\int_1^\infty \left(\int_1^\infty |f(x, y)|\,dx \right) dy$$

Is it true that $f \in \mathcal{L}(\Omega)$?
(ii) Calculate and compare the iterated integrals

$$\int_1^\infty \left(\int_1^\infty f(x, y)\,dx \right) dy\,, \quad \int_1^\infty \left(\int_1^\infty f(x, y)\,dy \right) dx$$

SOLUTION: (i) To evaluate the integral, let us use the identity

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} = -\frac{\partial}{\partial x} \frac{x}{x^2 + y^2}$$

so that

$$\begin{aligned} \int_1^\infty \left(\int_1^\infty |f(x, y)|\,dx \right) dy &= \int_1^\infty \left(\int_y^\infty - \int_1^y \right) f(x, y)\,dx\,dy \\ &= \int_1^\infty \left(\frac{1}{y} - \frac{1}{1 + y^2} \right) dy = \infty \end{aligned}$$

where the first equality follows from that $f(x,y)\ge 0$ if $x\ge y$, and the second is obtained by the fundamental theorem of calculus and the aforementioned identity. By the first part of Fubini's theorem the function $f(x,y)$ is not integrable on $(1,\infty)\times(1,\infty)$.

(ii) By the identity used in Part (i) and the fundamental theorem of calculus

$$\int_1^\infty\left(\int_1^\infty f(x,y)\,dx\right)dy=\int_1^\infty\frac{dy}{1+y^2}=\frac{\pi}{4}$$

The second integral is evaluated using the identity

$$f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}=\frac{\partial}{\partial y}\frac{y}{x^2+y^2}$$

and the fundamental theorem of calculus

$$\int_1^\infty\left(\int_1^\infty f(x,y)\,dy\right)dx=-\int_1^\infty\frac{dx}{1+x^2}=-\frac{\pi}{4}\,.$$

The second part of Fubini's theorem does not hold because the function is not absolutely (Lebesgue) integrable.

8.4 (6 pts). **Solution to the Poisson equation**. Suppose that $\rho\in C^1(\mathbb{R}^3)$ and has a bounded support Ω. Suppose that the boundary $\partial\Omega$ is smooth (or piecewise smooth). Prove that

$$\Delta u(x)=-4\pi\rho(x)\,,\quad x\in\mathbb{R}^3\,,\quad u(x)=\int\frac{\rho(y)}{|x-y|}\,d^3y$$

by justifying each of the following assertions:

(i) $$u\in C^1(\mathbb{R}^3)\,,\quad u\in C^\infty\Big(\mathbb{R}^3\setminus\Omega\Big)\,,$$

(ii) $$\Delta_x\frac{1}{|x-y|}=0\,,\quad \forall x\ne y$$

(iii) $$x\notin\Omega\quad\Rightarrow\quad \Delta u(x)=0\,,$$

(iv) $$x\in\Omega\quad\Rightarrow\quad \Delta u(x)=-\left(\nabla,\int_\Omega\rho(y)\nabla_y\frac{1}{|x-y|}\,d^3y\right)$$
$$=-\int_\Omega\left(\nabla_y\rho(y),\nabla_y\frac{1}{|x-y|}\right)d^3y$$
$$=-\left(\int_{\Omega\setminus B_\varepsilon(x)}+\int_{B_\varepsilon(x)}\right)\left(\nabla_y\rho(y),\nabla_y\frac{1}{|x-y|}\right)d^3y$$

(v) $$\lim_{\varepsilon\to0}\int_{\Omega\setminus B_\varepsilon(x)}\left(\nabla_y\rho(y),\nabla_y\frac{1}{|x-y|}\right)d^3y=4\pi\rho(x)\,,\quad x\in\Omega$$

(vi) $$\lim_{\varepsilon\to0}\int_{B_\varepsilon(x)}\left(\nabla_y\rho(y),\nabla_y\frac{1}{|x-y|}\right)d^3y=0$$

where ∇ denotes the gradient.

Solution: (i) It was proved in this section that the function

$$u(x)=\int_\Omega\frac{\rho(y)}{|x-y|^\alpha}\,dy\,,\quad |\rho(y)|\le M\,,$$

is from $C^p(\mathbb{R}^N)$ and $C^\infty\Big(\mathbb{R}^N \setminus \bar\Omega\Big)$. In the case in question, $N=3$, $\alpha=1$, and, hence, $p=1$. The integration region Ω is the support of ρ and, hence, a closed subset in $\mathbb{R}^3$.

(ii) By a direct evaluation of partial derivatives

$$\begin{aligned}\Delta\frac{1}{|x|} &= (\nabla,\nabla)\frac{1}{|x|} = -\Big(\nabla,\frac{x}{|x|^3}\Big) = -\frac{1}{|x|^3}(\nabla,x) - \Big(x,\nabla\frac{1}{|x|^3}\Big)\\ &= -\frac{3}{|x|^3}+\frac{3(x,x)}{|x|^5} = 0\,,\quad \forall x\neq 0\end{aligned}$$

Shifting the variable x by a constant vector y, it is concluded that

$$\Delta_x\frac{1}{|x-y|}=0\,,\quad \forall x\neq y$$

(iii) Any partial derivative D^β of $u(x)$ for $x\notin\Omega$ can be evaluated by rearranging the order of integration and differentiation. In particular,

$$\Delta u(x) = \int_\Omega \rho(y)\Delta_x\frac{1}{|x-y|}\,d^3y = 0\,,\quad \forall x\in\mathbb{R}^3\setminus\Omega\,,$$

by Part (ii) because $y\neq x$ if y spans Ω.

(iv) If $x\in\Omega$, then only the first partials can be evaluated by rearranging the order of integration and differentiation. In this case,

$$\begin{aligned}\nabla u(x) &= \int_\Omega \rho(y)\nabla_x\frac{1}{|x-y|}\,d^3y \overset{(1)}{=} -\int_\Omega\rho(y)\nabla_y\frac{1}{|x-y|}\,d^3y\\ &\overset{(2)}{=} -\oint_{\partial\Omega}\rho(y)\frac{n_y}{|x-y|}\,dS_y + \int_\Omega \nabla_y\rho(y)\,\frac{1}{|x-y|}\,d^3y\\ &\overset{(3)}{=} \int_\Omega \nabla_y\rho(y)\,\frac{1}{|x-y|}\,d^3y\end{aligned}$$

Here: (1) is justified by $\nabla_x f(x-y) = -\nabla_y f(x-y)$ for any C^1 function f; (2) the divergence theorem was applied to integrate by parts, where n_y is the outward unit normal on the boundary $\partial\Omega$ and dS_y is the surface area element; (3) the surface integral vanishes because ρ vanishes on $\partial\Omega$.

Next, note that the components of the gradient $\nabla\rho$ are bounded because $\rho\in C^1$ and has a bounded support (partial derivatives of ρ vanish outside of a ball, and, hence, by continuity must attain their extreme values). Therefore by the aforementioned theorem, the components of the gradient ∇u are from the class C^1 and their partial derivatives can be obtained by changing the order of differentiation and integration. In particular, for $x\in\Omega$,

$$\begin{aligned}\Delta u(x) &= (\nabla,\nabla u) \overset{(1)}{=} -\int_\Omega\Big(\nabla_y\rho(y),\nabla_y\frac{1}{|x-y|}\Big)\,d^3y\\ &\overset{(2)}{=} -\lim_{\varepsilon\to0+}\Big(\int_{\Omega\setminus B_\varepsilon(x)}+\int_{B_\varepsilon(x)}\Big)\Big(\nabla_y\rho(y),\nabla_y\frac{1}{|x-y|}\Big)\,d^3y\end{aligned}$$

Here (1) is again due to $\nabla_x f(x-y) = -\nabla_y f(x-y)$ and (2) is due to the additivity of the integral where $B_\varepsilon(x)$ is the ball of radius ε centered at x (note that $\nabla\rho(y)=0$ if $y\notin\bar\Omega$ for this reason the second integral can be extended to the whole $B_\varepsilon(x)$ if $x \in \partial\Omega$).

(v) Let us estimate the behavior of the integral over $B_\varepsilon(x)$ as $\varepsilon \to 0^+$.

$$\begin{aligned}
\left|\int_{B_\varepsilon(x)} \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y\right| &\overset{(1)}{\leq} \int_{B_\varepsilon(x)} \left|\left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right)\right| d^3y \\
&\overset{(2)}{\leq} \int_{B_\varepsilon(x)} \left|\nabla_y \rho(y)\right| \left|\nabla_y \frac{1}{|x-y|}\right| d^3y \\
&\overset{(3)}{=} \int_{B_\varepsilon(x)} \left|\nabla_y \rho(y)\right| \frac{1}{|x-y|^2}\, d^3y \\
&\overset{(4)}{\leq} M \int_{B_\varepsilon(x)} \frac{d^3y}{|x-y|^2} = M \int_{B_\varepsilon} \frac{d^3z}{|z|^2} \\
&\overset{(5)}{=} 4\pi M\varepsilon \to 0 \quad \text{as } \varepsilon \to 0
\end{aligned}$$

Here (1) is by the properties of the integral;
(2) is by the Cauchy-Schwartz inequality for the dot product $|(a,b)| \leq |a||b|$ for any $a, b \in \mathbb{R}^N$;
(3) is by a direct evaluation of the gradient, $|\nabla|x|^{-1}| = |-x|x|^{-3}| = |x|^{-2}$;
(4) the gradient $\nabla\rho$ is continuous on a closed and bounded region Ω and, hence, is bounded $|\nabla\rho| \leq \sup|\nabla\rho| = M$ by the extreme value theorem;
(5) after the change of variables $z = y - x$ so that the new integration region is centered at the origin, the integral is evaluated in spherical coordinates; $d^3z = r^2 \sin\phi\, dr\, d\phi\, d\theta$, $r = |z|$, and $(r, \phi, \theta) \in [0, \varepsilon] \times [0, \pi] \times [0, 2\pi]$. It is concluded that the integral over $B_\varepsilon(x)$ tends to 0 as $\varepsilon \to 0^+$.

(vi) Let us analyze the second integral in the limit $\varepsilon \to 0^+$:

$$\begin{aligned}
&\int_{\Omega\setminus B_\varepsilon(x)} \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y \\
&\overset{(1)}{=} \oint_{\partial(\Omega\setminus B_\varepsilon(x))} \rho(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|}\, dS_y - \int_{\Omega\setminus B_\varepsilon(x)} \rho(y) \Delta_y \frac{1}{|x-y|}\, d^3y \\
&\overset{(2)}{=} \oint_{\partial(\Omega\setminus B_\varepsilon(x))} \rho(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|}\, dS_y + 0 \\
&\overset{(3)}{=} \left(\oint_{\partial\Omega} + \oint_{\partial B_\varepsilon(x)}\right) \rho(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|}\, dS_y
\end{aligned}$$

$$\begin{aligned}
&\overset{(4)}{=} 0 - \oint_{|z|=\varepsilon} \rho(z+x) \frac{\partial}{\partial|z|} \frac{1}{|z|}\, dS_z \\
&\overset{(5)}{=} \frac{1}{\varepsilon^2} \oint_{|z|=\varepsilon} \rho(z+x)\, dS_z \\
&\overset{(6)}{=} \frac{4\pi\varepsilon^2}{\varepsilon^2} \rho(z_\varepsilon + x)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \Delta u(x) &= -\lim_{\varepsilon\to 0} \int_{\Omega\setminus B_\varepsilon(x)} \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y \\
&= -4\pi \lim_{\varepsilon\to 0} \rho(z_\varepsilon + x) \overset{(7)}{=} -4\pi\rho(x)
\end{aligned}$$

Here (1) the Green's formula was used to integrate by parts, and n_y is the outward unit normal on the boundary of the integration domain (note that the singular point $y = x$ is not in the integration region and, hence, the hypotheses of the Green's formula are met);
(2) $\Delta_y |y - x|^{-1} = 0$ for any $y \neq x$;
(3) The boundary consists of two pieces, $\partial\Omega$ and the sphere $|y - x| = \varepsilon$, if x is an interior point. Since $\rho = 0$ on the boundary $\partial\Omega$, The surface integral over $\partial\Omega$ vanishes. If $x \in \partial\Omega$, then the first integral is taken over the part of $\partial\Omega$ that is not in the ball $B_\varepsilon(x)$, but the latter does not matter as $\rho = 0$ on $\partial\Omega$ anyway. The second surface integral is taken over the part of the sphere $\partial B_\varepsilon(x)$ lies in Ω, but since $\rho = 0$ outside Ω, the integral can be extended to the whole sphere.
(4) For any $x \in \Omega$, the first surface integral is equal to zero, while the second one is taken over the whole sphere $|y - x| = \varepsilon$. In the second integral, put $z = y - x$ so that the normal derivative on the sphere $|z| = \varepsilon$ oriented toward the origin (the outward normal on the boundary of the integration region) is the negative of the radial derivative, and $dS_y = dS_z$ since a sphere remains a sphere under a parallel translation.
(5) The radial derivative on the sphere was evaluated.
(6) The integral mean value theorem was used, where z_ε is a point on the sphere, $|z_\varepsilon| = \varepsilon$, and $4\pi\varepsilon^2$ is the sphere area.
(7) By continuity of ρ and that $z_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

9.2 (4 pts). Consider the function defined by the Fourier integral

$$F(k) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{1 + x^4}\, dx$$

(i) Show that $F \in C^2(\mathbb{R})$
(ii) Show that $F \in C^3(|k| \geq \delta)$ for any $\delta > 0$.
(iii) Use the residue theorem to find an explicit form of $F(k)$. Compute $F'''(k)$. Does it exist for all k?
(iv) Can $F'''(k)$ be obtained by interchanging the order of D_k^3 and integration with respect to x? If so, evaluate the integral after differentiation of the integrand with respect to k.

SOLUTION: (i) One has

$$\left|\frac{\partial^p}{\partial^p k} \frac{\cos(kx)}{1 + x^4}\right| \leq \frac{|x|^p}{1 + x^4} \in \mathcal{L}, \quad p = 1, 2,$$

and the first and second partial derivatives of the integrand with respect to k are continuous for all x. By Theorem **5.2**, $F \in C^2$.

(ii) The integrable bound of the third derivative that is independent of the parameter k does not exist. So, put

$$F_n''(k) = -\int_{-n}^{n} \frac{x^2 \cos(kx)}{1 + x^4}\, dx \to F''(k)$$

as $n \to \infty$ for any $k \in \mathbb{R}$. Then

$$\left|\frac{\partial}{\partial k} \frac{x^2 \cos(kx)}{1 + x^4}\right| \leq \frac{|x|^3}{1 + x^4} \in \mathcal{L}(-n, n)$$

By Theorem **5.2**,

$$F_n'''(k) = -2\int_{0}^{n} \frac{x^3 \sin(kx)}{1 + x^4}\, dx\,.$$

The sequence $F_n'''(k)$ converges by Abel's theorem for conditionally convergent integrals for all $|k|\geq\delta>0$ and any such δ because

$$\begin{aligned}\Big|\int_c^d \sin(kx)\,dx\Big| &= \Big|\frac{\cos(kd)-\cos(kc)}{k}\Big| \leq \frac{2}{|k|}\leq\frac{2}{\delta}\\ \Big(\frac{x^3}{1+x^4}\Big)' &= \frac{x^2(3-x^4)}{(1+x^4)^2}<0\,,\quad x>2\end{aligned}$$

so that the factor at $\sin(kx)$ in the integrand is monotonically decreasing for all $x>2$. Therefore there exists a function $G(k)$ such that $F_n'''(k)\to G(k)$ as $n\to\infty$ for all k (note that $G(0)=0$ because $F_n'''(0)=0$). By the second part of Abel's theorem

$$|F_n'''(k)-G(k)|\leq 2\cdot\frac{2}{\delta}\cdot\frac{n^3}{1+n^4}\,,\quad |k|\geq\delta>0\,,\quad n>2\,.$$

Therefore $F_n'''\to G$ converges uniformly on the set $|k|\geq\delta>0$

$$\sup_{|k|\geq\delta>0}|F_n'''(k)-G(k)|\leq\frac{4}{\delta}\frac{n^3}{1+n^4}\to 0$$

as $n\to\infty$. By Theorem **1.3**, $G(k)=F'''(k)$ and $F\in C^3(k\neq 0)$ because $\delta>0$ is arbitrary.

(iii) Consider the function $f(z)=e^{ikz}(1+z^4)^{-1}$ which is analytic in the complex plane and has four simple poles

$$z_1=\frac{1+i}{\sqrt{2}}\,,\quad z_2=\frac{i-1}{\sqrt{2}}\,,\quad z_3=-z_1\,,\quad z_4=-z_2$$

Then

$$F(k)=\mathrm{Re}\lim_{R\to\infty}\int_{-R}^{R}f(z)\,dz$$

Let $k\geq 0$. Take the closed contour in the complex plane that consists of the interval $I_R=[-R,R]$ in the real axis and the circular arc C_R^+: $|z|=R$, $\mathrm{Im}\,z\geq 0$. Then the integral of f over the arc vanishes in the limit $R\to\infty$ because for all $R>1$

$$\begin{aligned}\Big|\int_{C_R^+}f(z)dz\Big| &= \Big|\int_0^\pi f(Re^{it})Re^{it}dt\Big|\leq\int_0^\pi\frac{Re^{-kR\sin(t)}}{|1+R^4e^{4it}|}\,dt\\ &\leq \frac{R}{R^4-1}\int_0^\pi dt=\frac{\pi R}{R^4-1}\to 0\end{aligned}$$

as $R\to\infty$. By the residue theorem

$$\int_{-\infty}^{\infty}\frac{e^{ikx}}{1+x^4}\,dx = 2\pi i\Big(\mathrm{Res}_{z_1}f(z)+\mathrm{Res}_{z_1}f(z)\Big)=\frac{\pi}{2}\Big(\frac{e^{ikz_1}}{z_1}-\frac{e^{ikz_2}}{z_2}\Big)$$

If $k\leq 0$, then the residue theorem is applied to the closed contour that consists of the interval I_R and the circular arc C_R^-: $|z|=R$, $\mathrm{Im}\,z\leq 0$. The integral of f over the arc vanishes in the limit $R\to\infty$ by the same argument. Therefore

$$\int_{-\infty}^{\infty}\frac{e^{ikx}}{1+x^4}\,dx=-2\pi i\Big(\mathrm{Res}_{z_3}f(z)+\mathrm{Res}_{z_4}f(z)\Big)=\frac{\pi}{2}\Big(\frac{e^{ikz_4}}{z_4}-\frac{e^{ikz_3}}{z_3}\Big)$$

The extra minus sign is due to the negative orientation of the contour. Therefore

$$F(k) = \frac{\pi}{\sqrt{2}}\, e^{-|k|/\sqrt{2}} \Big(\cos(k/\sqrt{2}) + \sin(|k|/\sqrt{2}) \Big)$$

It follows from the explicit form of $F(k)$ that $F \in C^\infty(k \neq 0)$. Near $k = 0$, using the power series for the exponential and trigonometric functions in the variable $q = k/\sqrt{2}$

$$\begin{aligned} F(k) &= \frac{\pi}{\sqrt{2}} \Big(1 - |q| + \frac{1}{2} q^2 - \frac{1}{6} |q|^3 + O(q^4) \Big) \Big(1 + |q| - \frac{1}{2} q^2 - \frac{1}{3} |q|^3 + O(q^4) \Big) \\ &= \frac{\pi}{\sqrt{2}} \Big(1 - q^2 + \frac{1}{2} |q|^3 + O(q^4) \Big) \end{aligned}$$

This shows that $F \in C^2$ but $F'''(0)$ does not exists.

(iv) No. The said integral does not exist for any $k \neq 0$ because it does not converge absolutely:

$$\sum_{n=2}^{\infty} \int_{\pi n/k}^{\pi(n+1)/k} \frac{x^3 |\sin(kx)|}{1 + x^4}\, dx \geq \sum_{n=2}^{\infty} \frac{2(\frac{\pi n}{k})^3}{1 + (\frac{\pi n}{k})^4} = \infty\,, \qquad k \neq 0$$

Note that the left-hand side is the integral of the absolute value of the third derivative of the integrand with respect to k over the interval $(2\pi/k, \infty)$. If $k = 0$, then the said integral vanishes but the explicit form of F shows that $F'''(0)$ does not exist. So, the order of D_k^3 and the integration cannot be interchanged for any k.