Assignment 2 with solutions, MAP 6505, Fall 2023

13.10.2 (i) Find a sequence of locally integrable function $f_n(x)$ in $\mathbb{R}^3$ that converges to the spherical delta-function:

$$f_n \to \delta_{S_a} \text{ in } \mathcal{D}', \quad (\delta_{S_a}, \varphi) = \oint_{|x|=a} \varphi(x)\, dS$$

(ii) Find a sequence of test functions $\varphi_n \in \mathcal{D}(\mathbb{R}^3)$ that converges to the spherical delta function in the distributional sense.

SOLUTION: (i) By analogy with a mass density of a point particle, consider a thin sphere of unit mass and radius a. An infinitely thin sphere is an idealization of a spherical layer, $\Omega_n : \ a - \frac{a}{n} \le |x| \le a + \frac{a}{n}$, $n = 2, 3, \ldots$. For simplicity, one can assume that the mass is uniformly distributed within the layer so that the mass density is constant in the layer and zero outside of it:

$$\rho_n(x) = \frac{1}{V_n}, \quad x \in \Omega_n, \quad \rho_n(x) = 0, \quad x \notin \Omega_n, \quad V_n = \frac{8\pi a^3}{3n}\Big(3 + \frac{1}{n^2}\Big),$$

where V_n is the volume of Ω_n. The total mass is $\int \rho_n\, d^3x = 1$ for any n. In the limit $n \to \infty$, the layer tends to a sphere of radius a.

Let us find the distributional limit of ρ_n. In spherical coordinates, $x = \hat{z}r$ where $\hat{z}$ is the unit outward normal on the unit sphere,

$$d^3x = r^2 dr dS_z, \quad r = |x|, \quad \int_{|z|=1} dS_z = 4\pi,$$

and dS_z is the area element on the unit sphere (written via spherical angles). For any test function φ, by the integral mean value theorem there exists $|r_n - a| \le \frac{a}{n}$ such that

$$\begin{aligned}(\rho_n, \varphi) &= \frac{1}{V_n}\int_{\Omega_n} \varphi(x)\, d^3x = \frac{1}{V_n}\int_{|z|=1}\int_{a-\frac{a}{n}}^{a+\frac{a}{n}} \varphi(zr) r^2 dr\, dS_z \\ &= \frac{2a}{nV_n}\int_{|z|=1} \varphi(\hat{z}r_n) r_n^2\, dS_z\end{aligned}$$

Note that $r_n = r_n(\hat{z})$ in general. In the limit $n \to \infty$, $r_n \to a$, $nV_n \to 8\pi a^3$, and by continuity of the test function $\varphi(\hat{z}r_n) \to \varphi(\hat{z}a)$. The order of taking the limit and integration can be interchanged by the Lebesgue dominated convergence theorem because the integrand is bounded by a constant $|\varphi(\hat{z}r_n)r_n^2| \le 2a^2 \sup|\varphi|$ that is integrable on a unit sphere. Thus

$$\lim_{n\to\infty} (\rho_n, \varphi) = \frac{1}{4\pi}\int_{|z|=1} \varphi(\hat{z}a)\, dS_z = \frac{1}{4\pi a^2}\int_{|x|=a} \varphi(x)\, dS \quad \Rightarrow \quad 4\pi a^2 \rho_n \to \delta_{S_a} \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

where $a^2 dS_z = dS$ is the area element on the sphere $|x| = a$.

(ii) Let $r_n \to 0$, $r_n > 0$, as $n \to \infty$. Then by the theorem about bump functions,

$$\rho_n(x) = \int_{|y|=a} \omega_{r_n}(x - y)\, dS,$$

where ω_{r_n} is a hat function, is a test function with support in a spherical layer $a - r_n \leq |x| \leq a + r_n$, and by Fubini's theorem and the normalization property of the hat function,

$$\int \rho_n(x)\, d^3x = \int_{|y|=a} \int \omega_{r_n}(x-y)\, d^3x\, dS = \int_{|y|=a} dS = 4\pi a^2$$

By Part (i), $\rho_n \to \delta_{S_a}$ in $\mathcal{D}'$. Alternatively, $\rho_n = \omega_{r_n} * \delta_{S_a}$ is a regularization of δ_{S_a} that converges to δ_{S_a} in $\mathcal{D}'$ (see properties of the regularization of distributions). Since support of δ_{S_a} is bounded, its regularization is a test function.

13.10.4. Let n be a positive integer and $\theta(x)$ is the step function. Find the following limits in the distributional sense or show that the limit does not exist:

$$\begin{aligned}
&\text{(i)} && \lim_{t\to\infty} t^n e^{itx}\,,\\
&\text{(ii)} && \lim_{t\to\infty} x^n e^{itx}\,,\\
&\text{(iii)} && \lim_{t\to\infty} \sin^n(tx)\,,\\
&\text{(iv)} && \lim_{t\to\infty} e^{itx}\theta(x)\,,\\
&\text{(v)} && \lim_{t\to\infty} t^n e^{itx}\theta(x)\,.
\end{aligned}$$

that is, if the limit exists, then give an explicit rule how to compute the value of the limit distribution for a test function.

Solution: (iv) For any test function φ, by integration by parts

$$\Big(e^{itx}\theta(x), \varphi(x)\Big) = \int_0^\infty e^{itx}\varphi(x)\, dx = \frac{e^{itx}}{it}\varphi(x)\Big|_0^\infty - \frac{1}{it}\int_0^\infty e^{itx}\varphi'(x)\, dx$$

The support of φ lies in a bounded interval $|x| \leq R$ for some $R > 0$. Therefore

$$\Big|\Big(e^{itx}\theta(x), \varphi(x)\Big)\Big| \leq \frac{|\varphi(0)|}{t} + \frac{R \sup|\varphi'|}{t} \quad \to \quad 0 \quad \text{as } t \to \infty\,.$$

Thus, the limit exists and the limit distribution is the zero distribution.

(v) Let $n = 1$. Then the integration by parts yields (as in Part (iv))

$$\Big(te^{itx}\theta(x), \varphi(x)\Big) = i\varphi(0) + i\int_0^\infty e^{itx}\varphi'(x)\, dx$$

The last integral vanishes in the limit $t \to \infty$ by the same argument as in Part (iv) if φ is replaced by φ' in it. Thus

$$te^{itx}\theta(x) \to i\delta(x) \quad \text{in } \mathcal{D}' \quad \text{as } t \to \infty$$

Let $n = 2$. Then integrating by parts twice

$$\Big(t^2e^{itx}\theta(x), \varphi(x)\Big) = it\varphi(0) + it\int_0^\infty e^{itx}\varphi'(x)\, dx = it\varphi(0) - \varphi'(0) + \int_0^\infty e^{itx}\varphi''(x)\, dx$$

Therefore the limit does not exist because the integral vanishes in the limit (replace φ by φ'' in Part (iv)) whereas the first term is not unbounded as $t \to \infty$. For any $n > 2$, one can integrate by parts n times to reduce the integral to the sum a polynomial of degree $n-1$ in the parameter t with coefficients being proportional to $\varphi^{(k)}(0)$, $k = 0, 1, ..., n-1$, stemming from the boundary terms, and the integral of $\varphi^{(n)}(x)e^{itx}$ over $(0, \infty)$ that vanishes in the limit $t \to \infty$. Thus, the limit does not exist for any $n \geq 2$.

15.5.7. Let $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Find the support of the distribution defined by the rule

$$(f, \varphi) = \int_0^\infty \int_{|x|=ct} \varphi(x, t)\, dS\, dt$$

where dS stands for the line integral over the circle $|x| = ct$, and $c > 0$ is a constant.

Solution: The integration is curried out over the cone $\Gamma^+ :\ |x| = ct\,,\ t \geq 0$ in $\mathbb{R}^3$ spanned by $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. The distance between $(x_0, t_0) \in \mathbb{R}^3$ and the cone Γ^+ is not zero if the point (x_0, t_0) is not in Γ^+. Therefore there exists an open ball B_a of a small enough radius a and centered at (x_0, t_0) that has no intersection with Γ^+. Then $f(x, t) = 0$ if $(x, t) \in B_a$ because for any test function $\varphi \in \mathcal{D}(B_a)$, $(f, \varphi) = 0$. Thus, $f(x, t) = 0$ in the complement of Γ^+ and, hence, $\operatorname{supp} f = \Gamma^+$.

15.5.15. Prove each of the following distributional limits

$$\text{(i)} \qquad \lim_{t\to+\infty} \frac{e^{itx}}{x + i0^+} = 0$$

$$\text{(ii)} \qquad \lim_{t\to+\infty} \frac{e^{-itx}}{x - i0^+} = 0$$

$$\text{(iii)} \qquad \lim_{t\to+\infty} \frac{e^{-itx}}{x + i0^+} = -2\pi i\delta(x)$$

$$\text{(iv)} \qquad \lim_{t\to+\infty} \cos(tx)\, \mathcal{P}\frac{1}{x} = 0$$

Solution: (iii) It has been shown in the notes that

$$e^{itx}\, \mathcal{P}\frac{1}{x} \to \pi i\delta(x) \quad \text{as } t \to +\infty$$

Put $\varphi_-(x) = \varphi(-x)$ for any test function φ. Then φ_- is also a test function. The above limit means that

$$\lim_{t\to\infty} \left(e^{itx}\, \mathcal{P}\frac{1}{x}, \varphi_-(x)\right) = i\pi\varphi_-(0) = i\pi\varphi(0)\,.$$

On the other hand, by changing variables $x = -y$

$$\left(e^{itx}\, \mathcal{P}\frac{1}{x}, \varphi_-(x)\right) = p.v. \int \frac{e^{itx}\varphi(-x)}{x}\, dx = -p.v. \int \frac{e^{-ity}\varphi(y)}{y}\, dy = -\left(e^{-itx}\, \mathcal{P}\frac{1}{x}, \varphi(x)\right)$$

Therefore

$$e^{-itx}\, \mathcal{P}\frac{1}{x} \to -i\pi\delta(x) \quad \text{as } t \to +\infty\,,$$

and by the Sokhotsky equation and properties of the delta function

$$\frac{e^{-itx}}{x+i0^+} = e^{-itx}\Big(-i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big) = -i\pi\delta(x) + e^{-itx}\mathcal{P}\frac{1}{x} \to -2\pi i\delta(x) \quad \text{as } t \to +\infty$$

as required.

17.7.2. (i) Let $\{x_n\}$ be any sequence or real numbers, and $\{x_n\}$ be a sequence that has no limit points. Show the series

$$\sum_n a_n\delta(x - x_n)$$

converges in the sense of distribution (converges in $\mathcal{D}'$).
(ii) In part (i), assume that $x_n \to x_0$. Does the series converge in the sense of distributions? If not, construct an explicit example of the sequence $\{a_n\}$ for which the series does not converge.

Solution: (i). If $\{x_n\}$ has no limit point, then any interval $|x| \le R$ can contain only finitely many terms of the sequence for any $R > 0$. Suppose that there are infinitely many distinct terms in a bounded interval. Divide the interval in two intervals of equal length. Then at least one of the two intervals must also contain infinitely many terms. Divide that interval into two intervals of equal length again. Then one of the two intervals must contain infinitely many terms and so on. By repeating this procedure, a sequence of nested intervals $I_{k-1} \subset I_k$ with infinitely many terms of the sequence is obtained. The length of the intervals tends to zero. Since the intervals are nested, one can pick one element of the sequence in each of them, $x_{n_k} \in I_k$, $n_{k-1} < n_k$, to obtain a Cauchy subsequence $\{x_{n_k}\}$. By the Cauchy criterion, it has a limit point that lies in the original interval, thus leading to a contradiction. This is known as the Bolzano-Weierstrass theorem: Every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence (or a limit point). Thus, for any test function with support in $[-R, R]$, only finitely many terms contribute to the value of the series:

$$\Big(\sum_n a_n\delta(x - x_n), \varphi(x)\Big) = \sum_n a_n\varphi(x_n) = \sum_{|x_n|\le R} a_n\varphi(x_n)$$

and hence the series converges for any φ and any choice of a_n.

(ii) Take a bump function $\eta(x)$ for an interval $(x_0 - 1, x_0 + 1)$. It is a test function such that $\eta(x) = 1$ if $|x - x_0| \le 1$. Since $x_n \to x_0$, there are only finitely many terms outside $(x_0 - 1, x_0 + 1)$, say, for $|n| < n_0$, and infinitely many terms in the interval $(x_0 - 1, x_0 + 1)$, say, for $|n| \ge n_0$. Then

$$\Big(\sum_n a_n\delta(x - x_n), \eta(x)\Big) = \sum_{|n|<n_0} a_n\eta(x_n) + \sum_{|n|\ge n_0} a_n$$

So, the series diverges if $\sum_n a_n$ diverges. For example, put $a_n = 1$.

16.7.9. Put $f_k(x) = a(x)\delta^{(k)}(\sin(x))$, $k = 0, 1, 2, \ldots$, where $a \in C^\infty$. Here $\delta^{(k)}(\sin(x))$ is understood as $\delta^{(k)}(z)$ where the substitution $z = \sin(x)$ is made. Express f_k in terms of shifted delta functions and its derivatives or show that f_k is not a distribution.

SOLUTION: The solution is given in detail for $k = 1$. The other cases are solved along the same line of reasoning. Let us find first $\delta'(\sin(x))$ and then multiply it by a smooth function. The change of variables $y = \sin(x) = F(x)$ is studied in the notes. The function $F(x)$ defines a C^∞ transformation $\mathbb{R} \to [-1, 1]$. It is a diffeomorphism $F = F_n : \ \Omega_n \to (-1, 1)$ for every $\Omega_n = (-\frac{\pi}{2} + \pi n, \frac{\pi}{2} + \pi n)$, n is any integer. If $\sin^{-1}(y)$ is the inverse of F_0, then $F_n^{-1}(y) = \pi n + \sin^{-1}(y)$. This change of variable in a distribution $f \in \mathcal{D}'$ was shown to define a distribution only if the support of f lies in $(-1, 1)$ (because the Jacobian vanishes $F'(F_n^{-1}(\pm 1)) = 0$). Since the support of $\delta'(x)$ is just $x = 0$, $\delta'(\sin(x))$ exists as a distribution.

Let $K_f = \operatorname{supp} f$ (a bounded closed interval in $(-1, 1)$). Then there exists is a bump function $\eta \in \mathcal{D}(-1, 1)$ such that $\eta(y) = 1$ in a neighborhood of K_f that is a proper subset in $(-1, 1)$. Moreover, $\eta(y)f(y) = f(y)$ by the hypothesis. For any $\varphi \in \mathcal{D}(\mathbb{R})$, put

$$T_F(\varphi)(y) = \eta(y)\sum_n \frac{\varphi(F_n^{-1}(y))}{|F'(F_n^{-1}(y)|} = \frac{\eta(y)}{\sqrt{1-y^2}}\sum_n \varphi(\pi n + (-1)^n \sin^{-1}(y)) \ \in \mathcal{D}(-1, 1)$$

It defines a linear continuous transformation of $\mathcal{D}(\mathbb{R})$ into $\mathcal{D}(-1, 1)$ (proved in the notes). By definition, $f(F(y)) = T_F^*(f)(y)$ is the adjoint transformation of $\mathcal{D}'(-1, 1)$ to $\mathcal{D}'(\mathbb{R})$:

$$\Big(\delta'(\sin(x), \varphi(x)\Big) = \Big(\delta'(y), T_F(\varphi)(y)\Big) = -\frac{d}{dy}T_F(\varphi)(y)\Big|_{y=0}$$

where the last equality is by definition of δ'. All derivatives of η vanish in a neighborhood of $y = 0$. The derivative of $\eta(y)/\sqrt{1-y^2}$ vanishes at $y = 0$ because $\eta'(0) = 0$. So, only the derivative of $\varphi(\pi n + (-1)^n \sin^{-1}(y))$ contributes. Since $\eta(0) = 1$ this contribution reads

$$\Big(\delta'(\sin(x)), \varphi(x)\Big) = -\sum_n (-1)^n \varphi'(\pi n) \quad \Rightarrow \quad \delta'(\sin(x)) = \sum_n (-1)^n \delta'(x - \pi n)$$

For any smooth function a and any x_0, one has

$$\begin{aligned} a(x_0)\delta'(x - x_0) &= [a(x_0)\delta(x - x_0)]' = [a(x)\delta(x - x_0)]' = a'(x)\delta(x - x_0) + a(x)\delta'(x - x_0) \\ &= a'(x_0)\delta(x - x_0) + a(x)\delta'(x - x_0) \end{aligned}$$

It follows from this relation that

$$a(x)\delta'(\sin(x)) = \sum_n (-1)^n \Big[a(\pi n)\delta'(x - \pi n) - a'(\pi n)\delta(x - \pi n)\Big]$$

17.9.3. Let $f(x) = \ln(|x|)$ where $x \in \mathbb{R}^2$.
(i) Show that $f(x)$ is a harmonic function wherever it is twice continuously differentiable (that is, outside of any neighborhood of $x = 0$):

$$\Big\{\Delta \ln(|x|)\Big\} = 0$$

(ii) Use the same method as for the Laplacian of the Coulomb potential in $\mathbb{R}^3$ to find $\Delta \ln(|x|)$ in $\mathbb{R}^2$ in the distributional sense.

SOLUTION: (i) For $x \neq 0$, $\ln(|x|)$ is a smooth function whose first and second (classical) derivatives are

$$\{\partial_j \ln(|x|)\} = \frac{x_j}{|x|^2}\,, \quad \{\partial_k\partial_j \ln(|x|)\} = \frac{\delta_{jk}}{|x|^2} - \frac{2x_j x_k}{|x|^4}\,,$$
$$\Rightarrow \qquad \{\Delta \ln(|x|)\} = \frac{\delta_{jj}}{|x|^2} - \frac{2x_j x_j}{|x|^4} = \frac{2}{|x|^2} - \frac{2|x|^2}{|x|^4} = 0\,, \quad x \neq 0\,.$$

where Einstein's summation rule was used (e.g., $\delta_{jj} = 2$ in $\mathbb{R}^2$).

(ii) For any test function φ, the following chain of equalities holds

$$\begin{aligned}
\Big(\Delta \ln(|x|), \varphi(x)\Big) &\overset{(1)}{=} \Big(\ln(|x|), \Delta\varphi(x)\Big) \overset{(2)}{=} \int \ln(|x|)\Delta\varphi(x)\, d^2x \\
&\overset{(3)}{=} \lim_{a\to 0^+} \int_{|x|>a} \ln(|x|)\Delta\varphi(x)\, d^2x \\
&\overset{(4)}{=} \lim_{a\to 0^+} \Big(\ln(a) \int_{|x|=a} \frac{\partial\varphi}{\partial n}\, dS + \frac{1}{a}\int_{|x|=a} \varphi\, dS\Big) \\
&\overset{(5)}{=} \lim_{a\to 0^+} \int_{|z|=1} \varphi(az)\, dS_z \overset{(6)}{=} \varphi(0)\int_{|z|=1} dS_z = 2\pi\varphi(0) \\
\Rightarrow \quad & \Delta \ln(|x|) = 2\pi\delta(x)\,, \quad x \in \mathbb{R}^2\,.
\end{aligned}$$

Here (1) is by definition of derivatives of distributions, (2) holds because $\ln(|x|)$ is locally integrable (a regular distribution), (3) holds because the integral converges absolutely and does not depend on a regularization (continuity of the Lebesgue integral), (4) is obtained by Green's identity combined with that $\Delta\ln(|x|) = 0$ for $x \neq 0$ and the normal derivative $\partial/\partial n = (\hat{n}, \nabla)$ coincides with $-\partial/\partial r$ where $r = |x|$ because the unit outward normal for the boundary $|x| = a$ of the integration region is $\hat{n} = -x/a$ (no boundary contribution comes from the boundary $|x| = R \to \infty$ because φ has a bounded support), (5) follows from the estimate

$$\Big|\int_{|x|=a} \frac{\partial\varphi}{\partial n}\, dS\Big| \leq \sup|D\varphi| \int_{|x|=a} dS = 2\pi a \sup|D\varphi|$$

and that $a\ln(a) \to 0$ as $a \to 0^+$ (note that $dS = adS_z$), and (6) is justified by the Lebesgue dominated convergence theorem because $|\varphi(az)| \leq \sup|\varphi| < \infty$ and a constant is integrable on the unit circle.

18.9.7. Let $\Omega \subset \mathbb{R}^N$ be an open set, and $\Omega_c = \mathbb{R}^N \setminus \overline{\Omega}$ be its complement. Supposed that f is a piecewise continuous function such that $f \in C^1(\overline{\Omega})$ and $f \in C^1(\overline{\Omega}_c)$. So, f is generally not continuous at any point of the boundary $\partial\Omega$. Let the boundary $\partial\Omega$ be piecewise smooth and the jump discontinuity of f at a point $x \in \partial\Omega$ defined by

$$[f]_{\partial\Omega}(x) = \lim_{y\to x} f(y) - \lim_{z\to x} f(z)\,, \quad y \in \Omega\,,\ z \in \Omega_c\,,\ x \in \partial\Omega$$

be continuous on $\partial\Omega$. Show that

$$\frac{\partial f}{\partial x_j} = \Big\{\frac{\partial f}{\partial x_j}\Big\} - n_j[f]_{\partial\Omega}\delta_{\partial\Omega}$$

where $n(x)$ is a unit outward normal at $x \in \partial\Omega$.

Solution: To simplify notations, for any $x \in \partial\Omega$, put

$$f_+(x) = \lim_{y\to x} f(y)\,, \quad y \in \Omega\,, \quad f_-(x) = \lim_{z\to x} f(z)\,,\ z \in \Omega_c\,, \quad [f]_{\partial\Omega} = f_+ - f_-\,.$$

Define

$$\Omega^a = \{x \in \Omega \,|\, d(x, \partial\Omega) < a\} \subset \Omega$$

and similarly for Ω_c^a. So, Ω^a is obtained from Ω by removing closed balls of radius a that are centered at all points of $\partial\Omega$. For any test function φ, the following chain of equalities holds

$$\begin{aligned}
(\partial_j f, \varphi) &\overset{(1)}{=} -(f, \partial_j\varphi) \overset{(2)}{=} -\int_\Omega f(x)\partial_j\varphi(x)\, d^N x - \int_{\Omega_c} f(x)\partial_j\varphi(x)\, d^N x \\
&\overset{(3)}{=} -\lim_{a\to 0^+}\int_{\Omega^a} f(x)\partial_j\varphi(x)\, d^N x - \lim_{a\to 0^+}\int_{\Omega_c^a} f(x)\partial_j\varphi(x)\, d^N x \\
&\overset{(4)}{=} \lim_{a\to 0^+}\int_{\Omega^a} \{\partial_j f(x)\}\varphi(x)\, d^N x - \int_{\partial\Omega} f_+(x)\varphi(x)\, d\Sigma_j \\
&\quad + \lim_{a\to 0^+}\int_{\Omega_c^a} \{\partial_j f(x)\}\varphi(x)\, d^N x + \int_{\partial\Omega} f_-(x)\varphi(x)\, d\Sigma_j \\
&\overset{(5)}{=} \int \{\partial_j f(x)\}\varphi(x)\, d^N x - \int_{\partial\Omega} [f]_{\partial\Omega}(x)\varphi(x)\, d\Sigma_j \\
&\overset{(6)}{=} \Big(\{\partial_j f\} - n_j [f]_{\partial\Omega}\delta_{\partial\Omega}, \varphi\Big)\,,
\end{aligned}$$

as required. Here (1) is by definition of a distributional derivative, (2) holds because f is locally integrable, (3) is by the continuity of the Lebesgue integral, (4) is by integration by parts and that $d\Sigma_j^c = -d\Sigma_j$ is $d\Sigma_j^c$ is the oriented outward area element for the boundary of Ω_c (note that $\operatorname{supp}\varphi \subset B_R$ so that any part of the sphere $|x| = R$ in Ω or Ω_c does not contribute to the boundary surface integrals), (5) is again by continuity of the Lebesgue integral and by that $\{\partial_j f\}$ has continuous extensions from Ω and from Ω_c to $\partial\Omega$. Finally, (6) follows from the definition of the simple layer distribution and that $d\Sigma_j = n_j dS$ where n_j are components of the unit outward normal on $\partial\Omega$.

19.9.6. Let $f(x) = 1 - |x|$ if $|x| < 1$ and $f(x+2) = f(x)$. Show that

$$f''(x) = -2\sum_n e^{i\pi(2n+1)x}$$

in the sense of distributions.

Solution: The function has a period 2. So it can be expanded into a Fourier series over $e^{i\pi nx}$, where n is an integer. Since $f'(x)$ is locally integrable, the Fourier series converges uniformly to f. This guarantees that the Fourier coefficients a_n are bounded $|a_n| < M$ because $a_n \to 0$ as $|n| \to \infty$. By the theorem about differentiation of Fourier series in the distributional sense, the distributional derivative of a regular distribution f can be obtained by term-by-term differentiation of its Fourier series:

$$f(x) \;=\; 1 + \sum_{n\neq 0} a_n e^{i\pi nx} \quad \Rightarrow \quad f''(x) = -\pi^2 \sum_{n\neq 0} n^2 a_n e^{i\pi nx} \;\in \mathcal{D}'\,.$$

It remains to calculate the Fourier coefficients:

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx = \frac{1}{2}\int_{-1}^{0}(x+1)e^{-i\pi nx}\,dx + \frac{1}{2}\int_{0}^{1}(1-x)e^{-i\pi nx}\,dx \\
&= \frac{i}{2\pi n}\int_{-1}^{0}(x+1)de^{-i\pi nx} + \frac{i}{2\pi n}\int_{0}^{1}(1-x)de^{-i\pi nx} \\
&= \frac{i}{2\pi n}\Big(1 - \int_{-1}^{0} e^{-i\pi nx}dx - 1 + \int_{0}^{1} e^{-i\pi nx}dx\Big) = \frac{1-(-1)^n}{\pi^2 n^2}
\end{aligned}$$

So, $a_{2n} = 0$ and substituting a_{2n+1} into the series for f'', it is concluded that

$$f''(x) = -2\sum_n e^{i\pi(2n+1)x}$$

as required.

ALTERNATIVE SOLUTION: In the Poisson summation formula, replace x by πx and by the scaling property of the delta function, the formula has the following form

$$\sum_n \delta(x-2n) = \frac{1}{2}\sum_n e^{i\pi nx}$$

In this equation, replace x by $x-1$ so that

$$\sum_n \delta(x-2n-1) = \frac{1}{2}\sum_n (-1)^n e^{i\pi nx}$$

The classical derivative $\{f'\}$ is a piecewise constant function. So, $\{f''(x)\} = 0$ a.e. It does not exist at $x = n$ where $\{f'\}$ has a jump discontinuity, $\mathrm{disc}_n[\{f'\}] = -2(-1)^n$. By the relation between the classical and distributional derivatives of a piecewise smooth function

$$f''(x) = -2\sum_n \delta(x-2n) + 2\sum_n \delta(x-2n-1) = -\sum_n [1-(-1)^n]e^{i\pi nx} = -2\sum_n e^{i\pi(2n+1)x}\,.$$

20.6.1. Let $x_0 > 0$. Find the product of distributions

(i) $\quad \dfrac{1}{x \pm i0}\,\delta'(x-x_0)\,,$

(ii) $\quad \mathcal{P}\dfrac{1}{|x|}\,\delta(x^2-x_0^2)\,,$

(iii) $\quad \dfrac{1}{x-x_0+i0}\mathcal{P}\dfrac{1}{x^2}\,,$

SOLUTION (ii) The singular support of $\mathcal{P}\frac{1}{|x|}$ is $\{0\}$ and the singular support of $\delta(x^2-x_0^2)$ is $\{\pm x_0\}$. They do not overlap. Therefore the product exists and can be found by the localization method. Let us take three open intervals whose union is $\mathbb{R}$ and each of which contains only one of the three singular points:

$$\Omega_1 = (-\infty, -\delta)\,, \quad \Omega_2 = (-x_0+\delta, x_0-\delta)\,, \quad \Omega_3 = (\delta, \infty)$$

for some $0 < \delta < x_0/2$. Using the property that $\delta(g(x)) = |g'(a)|^{-1}\delta(x-a)$ near any simple isolated root a of g and vanishes otherwise,

$$\delta(x^2 - x_0^2) = \frac{1}{2x_0}\begin{cases} \delta(x+x_0) & , \quad x \in \Omega_1 \\ 0 & , \quad x \in \Omega_1 \\ \delta(x-x_0) & , \quad x \in \Omega_3 \end{cases}$$

and the principal value distribution is a function from class C^∞ outside any neighborhood of $x = 0$:

$$\mathcal{P}\frac{1}{|x|} = -\frac{1}{x}, \quad x \in \Omega_1\,, \quad \mathcal{P}\frac{1}{|x|} = \frac{1}{x}, \quad x \in \Omega_3\,,$$

Let $f \in \mathcal{D}'$ denote the product in question. Then f has the following form on each of Ω's:

$$\begin{aligned} f(x) &= f_1(x) = -\frac{1}{x}\frac{1}{2x_0}\delta(x+x_0) = \frac{1}{2x_0^2}\delta(x+x_0)\,, \quad x \in \Omega_1 \\ f(x) &= f_3(x) = \frac{1}{x}\frac{1}{2x_0}\delta(x-x_0) = \frac{1}{2x_0^2}\delta(x-x_0)\,, \quad x \in \Omega_3 \\ f(x) &= f_2(x) = 0 \cdot \mathcal{P}\frac{1}{|x|} = 0\,, \quad x \in \Omega_2 \end{aligned}$$

Let φ be a test function from $\mathcal{D}(\mathbb{R})$ and $K = \operatorname{supp}\varphi$. Take a partition of unity $\psi_j \in \mathcal{D}(\Omega_j)$, $j = 1, 2, 3$, such that

$$\psi_1(x) + \psi_2(x) + \psi_3(x) = 1\,, \quad x \in K\,.$$

Then by the localization method,

$$\begin{aligned} (f, \varphi) &= (f_1, \psi_1\varphi) + (f_2, \psi_2\varphi) + (f_3, \psi_3\varphi) = \frac{1}{2x_0^2}\Big(\psi_1(-x_0)\varphi(-x_0) + \psi_3(x_0)\varphi(x_0)\Big) \\ &= \frac{1}{2x_0^2}\Big(\varphi(-x_0) + \varphi(x_0)\Big) \end{aligned}$$

Here $\psi_1(-x_0) = 1$ and $\psi_3(x_0) = 1$ because $\psi_{2,3}(-x_0) = 0$ and $\psi_{1,2}(x_0) = 0$ by the construction of partition of unity. Thus,

$$f(x) = \mathcal{P}\frac{1}{|x|}\,\delta(x^2 - x_0^2) = \frac{1}{2x_0^2}\Big(\delta(x+x_0) + \delta(x-x_0)\Big)$$

(iii). The singular support of $\mathcal{P}\frac{1}{x^2}$ is $\{0\}$ and the singular support of $[x - x_0 + i0]^{-1}$ is $\{x_0\}$. They do not overlap. Therefore the product exists and can be found by the localization method. It is convenient to use the Sokhotsky equation

$$\frac{1}{x - x_0 + i0} = -i\pi\delta(x-x_0) + \mathcal{P}\frac{1}{x-x_0}$$

Since $\mathcal{P}\frac{1}{x^2} = \frac{1}{x^2} \in C^\infty$ near $x = x_0 \neq 0$

$$-i\pi\delta(x-x_0)\mathcal{P}\frac{1}{x^2} = -\frac{i\pi}{x_0^2}\delta(x-x_0)$$

To find the product of two principal value distributions, take two open intervals covering $\mathbb{R}$ each of which contains only one of the two singular points

$$\Omega_1 = (-\infty, x_0 - \delta)\,, \quad \Omega_2 = (\delta, \infty)$$

for some $0 < \delta < x_0/2$. Let $f = \mathcal{P}\frac{1}{x-x_0}\mathcal{P}\frac{1}{x^2}$. Then

$$f(x) = f_1(x) = \frac{1}{x-x_0}\mathcal{P}\frac{1}{x^2}\,, \quad x \in \Omega_1\,, \quad f(x) = f_2(x) = \frac{1}{x^2}\mathcal{P}\frac{1}{x-x_0}\,, \quad x \in \Omega_2\,.$$

Let φ be a test function from $\mathcal{D}(\mathbb{R})$ and $K = \operatorname{supp}\varphi$. Take a partition of unity $\psi_j \in \mathcal{D}(\Omega_j)$, $j = 1, 2$, such that

$$\psi_1(x) + \psi_2(x) = 1\,, \quad x \in K\,.$$

Then by the localization method,

$$(f,\varphi) \;=\; (f_1, \psi_1\varphi) + (f_2, \psi_2\varphi) = \Big(\mathcal{P}\frac{1}{x^2}, \frac{\psi_1(x)\varphi(x)}{x-x_0}\Big) + \Big(\mathcal{P}\frac{1}{x-x_0}, \frac{\psi_2(x)\varphi(x)}{x^2}\Big)$$

Put $I_{ab} = (-\infty, -a) \cup (a, x_0 - b) \cup (x_0 + b, \infty)$. Then using the definitions of the principal value distributions and that $\psi_1(0) = 1$ by construction,

$$(f,\varphi) = \lim_{a\to 0^+}\lim_{b\to 0^+}\int_{I_{ab}} \Big(\frac{\psi_1(x)\varphi(x)}{x^2(x-x_0)} + \frac{\varphi(0)}{x_0x^2} + \frac{\psi_2(x)\varphi(x)}{x^2(x-x_0)}\Big)\,dx$$

Using the partial fraction decomposition

$$\frac{1}{x^2(x-x_0)} = \frac{1}{x_0^2}\Big(\frac{1}{x-x_0} - \frac{x_0}{x^2} - \frac{1}{x}\Big)$$

and that $\psi_1 + \psi_2 = 1$ in the support of φ, one infers that

$$\begin{aligned}(f,\varphi) &= \lim_{a\to 0^+}\lim_{b\to 0^+}\int_{I_{ab}} \Big(\frac{1}{x_0^2}\frac{\varphi(x)}{x-x_0} - \frac{1}{x_0}\frac{\varphi(x)-\varphi(0)}{x^2} - \frac{1}{x_0^2}\frac{\varphi(x)}{x}\Big)\,dx \\ &= \frac{1}{x_0^2}\Big(\mathcal{P}\frac{1}{x-x_0}, \varphi\Big) - \frac{1}{x_0}\Big(\mathcal{P}\frac{1}{x^2}, \varphi\Big) - \frac{1}{x_0^2}\Big(\mathcal{P}\frac{1}{x}, \varphi\Big)\,.\end{aligned}$$

Thus, the product in question is given by

$$\begin{aligned}\frac{1}{x-x_0+i0}\mathcal{P}\frac{1}{x^2} &= \frac{1}{x_0^2}\mathcal{P}\frac{1}{x-x_0} - \frac{1}{x_0}\mathcal{P}\frac{1}{x^2} - \frac{1}{x_0^2}\mathcal{P}\frac{1}{x} - \frac{i\pi}{x_0^2}\delta(x-x_0) \\ &= \frac{1}{x_0^2}\frac{1}{x-x_0+i0} - \frac{1}{x_0}\mathcal{P}\frac{1}{x^2} - \frac{1}{x_0^2}\mathcal{P}\frac{1}{x}\end{aligned}$$

where the Sokhotsky equation was used again.

20.6.1. Find a general distributional solution $f \in \mathcal{D}'(\mathbb{R})$ to the equations

$$\begin{aligned}&\text{(i)} && x^2 f(x) = 1\,, \\ &\text{(ii)} && (x-a)^2 f(x) = x\,, \\ &\text{(iii)} && x^2(x-1)f(x) = x^2+1\,, \\ &\text{(iv)} && (x-a)f(x) = \delta'(x)\,,\ a \neq 0\end{aligned}$$

Solution: (iv) A general solution to the associated homogeneous equation is

$$(x-a)h(x) = 0 \quad \Rightarrow \quad h(x) = c\delta(x-a)\,,$$

where c is a constant. A particular solution can be found by the localization method. Since the support of $\delta'(x)$ is $x=0$ and $\frac{1}{x-a}$ is from class C^∞ near $x=0$ (because $a\neq 0$), a particular solution reads

$$f_p(x) = \frac{1}{x-a}\,\delta'(x) = \Big(\frac{1}{x-a}\delta(x)\Big)' + \frac{1}{(x-a)^2}\,\delta(x) = -\frac{1}{a}\delta'(x) + \frac{1}{a^2}\delta(x)$$

Thus,

$$f(x) = \frac{1}{a^2}\delta(x) - \frac{1}{a}\delta'(x) + c\delta(x-a)$$

Alternative solution: Any solution to the problem is also a solution to the equation

$$x^2(x-a)f(x) = x^2\delta'(x) = 0 \quad \Rightarrow \quad f(x) = c_1\delta(x) + c_2\delta'(x) + c\delta(x-a)$$

Substituting this distribution into the original equation, one gets

$$\delta'(x) \;=\; (x-a)\Big(c_1\delta(x) + c_2\delta'(x) + c\delta(x-a)\Big) = -c_1 a\delta(x) - c_2 a\delta'(x) - c_2\delta(x)$$

The original equation is satisfied only if $c_2 = -\frac{1}{a}$ and $c_1 = \frac{1}{a^2}$.

22.8.2. Find a general distributional solution to each of the following equations

$$\begin{array}{ll}
\text{(i)} & (x-a)(x-b)f'(x) = 1 \\
\text{(ii)} & (x-a)(x-b)f''(x) = \delta(x) \\
\text{(iii)} & f'(x) + a(x)f(x) = \delta'(x)\,, \quad a\in C^\infty \\
\text{(iv)} & xf'(x) + xa(x)f(x) = \delta(x)\,, \quad a\in C^\infty
\end{array}$$

Solution: (ii) Let a and b be non-equal and non-zero. Put $g(x) = f''(x)$. Then the associated homogeneous equation has a general solution

$$(x-a)(x-b)g(x) = 0 \quad \Rightarrow \quad g(x) = c_a\delta(x-a) + c_b\delta(x-b)$$

where $c_{a,b}$ are arbitrary constants. A localization method can be used to find a particular solution. Since the support of $\delta(x)$ is $x=0$ and $[(x-a)(x-b)]^{-1}$ is from class C^∞ near $x=0$ (assuming a and b are not equal to 0), a particular solution reads

$$g_p(x) = \frac{1}{(x-a)(x-b)}\,\delta(x) = -\frac{1}{ab}\,\delta(x)$$

Applying the antiderivative twice to g, one infers

$$\begin{aligned}
f'(x) &= D^{-1}g = c_a\theta(x-a) + c_b\theta(x-b) - \frac{1}{ab}\theta(x) + c_1\,, \\
f(x) &= D^{-1}f' = c_a(x-b)\theta(x-a) + c_b(x-b)\theta(x-b) - \frac{x}{ab}\theta(x) + c_1 x + c_0
\end{aligned}$$

where $c_{0,1}$ are integration constants. Here it was used that that $D^{-1}(f(x-x_0)) = (D^{-1}f)(x-x_0)$ (an antiderivative of a shifted distribution is the shifted antiderivative of the distribution).

Let $a \neq 0$ but $b = 0$ (a solution in the case $a = 0$ and $b \neq 0$ is obtained by swapping a and b in what follows). Then any solution to the problem is also a solution to

$$x^2(x-a)g(x) = x\delta(x) = 0 \quad \Rightarrow \quad g(x) = c_a\delta(x-a) + c_0\delta(x) + c_1\delta'(x)$$

A substitution into the original equation yields

$$\delta(x) = x(x-a)g(x) = c_1 x(x-a)\delta'(x) = c_1 a\delta(x) \quad \Rightarrow \quad c_1 = \frac{1}{a}$$

because $x^2\delta'(x) = 0$ and $x\delta'(x) = -\delta(x)$. Therefore

$$f(x) = D^{-2}g(x) = c_a(x-a)\theta(x-a) + c_0 x\theta(x) + \frac{1}{a}\theta(x) + c_1 + c_2 x$$

where $c_{1,2}$ are integration constants.

Let $a = b = 0$. Then any solution to the problem is also a solution to

$$x^3 g(x) = x\delta(x) = 0 \quad \Rightarrow \quad g(x) = c_0\delta(x) + c_1\delta'(x) + c_2\delta''(x)$$

A substitution into the original equation yields

$$\delta(x) = x^2 g(x) = c_2 x^2\delta''(x) = 2c_2\delta(x) \quad \Rightarrow \quad c_2 = \frac{1}{2}$$

because $x^2\delta''(x) = 2\delta(x)$. Therefore

$$f(x) = D^{-2}g(x) = (c_0 x + c_1)\theta(x) + \frac{1}{2}\delta(x) + c_2 + c_3 x$$

where $c_{2,3}$ are integration constants.