Assignment 3 (Final) with solutions, MAP 6505, Fall'23

1. Generalized initial value problem for an oscillator. Consider the initial value problem for an oscillator (e.g., a pendulum):

$$u''(t) + 2\gamma u'(t) + \omega^2 u(t) = f(t)\,, \quad t > 0\,; \quad u\Big|_{t=0} = u_0\,, \quad u'\Big|_{t=0} = u_1$$

where $\gamma \geq 0$ is a damping coefficient. A classical problem is to find $u \in C^2(t > 0) \cap C^1(t \geq 0)$ that satisfy the equation and the initial condition (under some restrictions on smoothness of the inhomogeneity (external force) $f(t)$ that is defined on $[0, \infty)$).

(i) Convert the problem to a generalized Cauchy problem in the algebra $\mathcal{D}'_+$, that is,

$$v''(t) + 2\gamma v'(t) + \omega^2 v(t) = g(t)\,, \quad v \in \mathcal{D}'_+\,, \quad g \in \mathcal{D}'_+\,.$$

Find the explicit form of the distribution g.

(ii) Solve the generalized Cauchy problem by finding a fundamental solution $G(t)$ (a causal Green's function):

$$G''(t) + 2\gamma G'(t) + \omega^2 G(t) = \delta(t)\,, \quad G \in \mathcal{D}'_+$$

Show that the solution to the generalized Cauchy problem is unique in $\mathcal{D}'_+$.

(iii) Assume that f is locally integrable and find an integral representation of $v(t)$. Find the most general condition on $f(t)$ under which $v(t)$ is the classical solution.

(iv) Suppose $u_0 = u_1 = 0$ (the oscillator is at rest for $t < 0$). One wants to model a situation in which the oscillator gets a kick at a time $T > 0$. The kick is defined as an idealization of the process when the momentum (velocity) of the oscillator suddenly changes under the action of an external force f at a time moment $t = T$. For example, $f(t) = f_0 \omega_a(t - T)$ where ω_a is a hat function with $a \to 0^+$. In the limit, such a force can be modeled by $f(t) = f_0 \delta(t - T)$. Show that the solution is smooth (classical) in the intervals $(0, T)$ and (T, ∞) and bounded for all t in this case, $|v(t)| = M < \infty$. Find the distributional solution if the oscillator gets the kicks of the same amplitude periodically

$$f(t) = f_0 \sum_{n=0}^{\infty} \delta(t - nT)$$

(v) Investigate if the solution in this case remains bounded for all values of parameters $\gamma \geq 0$ and $T > 0$. If this is not so, then find a particular set of parameters γ and T for which the solution is not bounded for all t.

Hint: Recall the resonance phenomenon for a harmonic oscillator.

SOLUTION: (i) Suppose $u(t)$ is a classical solution. Put $v(t) = \theta(t)u(t)$ which is a regular distribution. Its derivatives are

$$\begin{aligned} v'(t) &= \theta(t)\{u'(t)\} + \mathrm{disc}_{t=0} u\, \delta(t) = \theta(t)\{u'(t)\} + u_0\, \delta(t) \\ v''(t) &= \theta(t)\{u''(t)\} + \mathrm{disc}_{t=0}\{u'\}\, \delta(t) + u_0 \delta'(t) = \theta(t)\{u''(t)\} + u_1\, \delta(t) + u_0 \delta'(t) \end{aligned}$$

Therefore any classical solution is a distributional solution to the generalized Cauchy problem

$$v''(t) + 2\gamma v'(t) + \omega^2 v(t) = g(t)\,, \quad g \in \mathcal{D}'_+$$

where a distributional solution v is sought in $\mathcal{D}'_+$ (distributions with support in $[0,\infty)$). A classical solution is obtained from the distributional one if

$$g(t) = \theta(t)f(t) + (2\gamma u_0 + u_1)\delta(t) + u_0\delta'(t)$$

for a smooth enough $f(t)$.

(ii) A particular fundamental solution for the stated differential operator can be found in the form $\mathcal{E}(t) = \theta(t)Z(t)$ where

$$Z'' + 2\gamma Z' + \omega^2 Z = 0\,, \quad Z(0) = 0\,, \quad Z'(1) = 1$$

The characteristic equation for this differential equation reads $r^2 + 2\gamma r + \omega^2 = 0$. Its roots are $r_\pm = -\gamma \pm i\omega_\gamma$ where $\omega_\gamma = (\omega^2 - \gamma^2)^{1/2}$. In what follows it is assumed that $\omega^2 > \gamma^2$ (small damping regime). Other cases are treated in the same way (or just replace ω_γ by $i|\omega_\gamma|$ in what follows). Therefore

$$Z(t) = e^{-\gamma t}\Big(A\cos(\omega_\gamma t) + B\sin(\omega_\gamma t)\Big)$$

Using the initial data,

$$\mathcal{E}(t) = \frac{\theta(t)}{\omega_\gamma}\, e^{-\gamma t}\sin(\omega_\gamma t)$$

A general solution reads

$$G(t) = \mathcal{E}(t) + e^{-\gamma t}\Big(C_1\cos(\omega_\gamma t) + C_2\sin(\omega_\gamma t)\Big)$$

Since the causal Green's function must vanish for all $t < 0$, one has to take $C_1 = C_2 = 0$. The causal Green's function reads $G(t) = \mathcal{E}(t)$ and it is unique.

A solution to the generalized initial value problem

$$v = G * g$$

provided the convolution exists. Since $\mathcal{D}'_+$ is the convolution algebra and $G \in \mathcal{D}'_+$, G has a convolution with any $g \in \mathcal{D}'_+$ and $G * g \in \mathcal{D}'_+$. So, a solution always exists. A general solution to the homogeneous problem has only trivial solution in $\mathcal{D}'_+$ (its convolution with G exists only if $C_1 = C_2 = 0$ as above). Thus, the solution to the generalized initial value problem has a unique solution and, hence, if a classical solution exists (among distributional solutions), it is also unique.

(iii) Since $G * \delta = G$, $G * \delta' = G'$, and $G(t)$ has no jump-discontinuity at $t = 0$ so that $G'(t) = \theta(t)Z'(t)$ and f is locally integrable,

$$v(t) = \theta(t)u(t)\,, \quad u(t) = \int_0^t Z(t-\tau)f(\tau)\,d\tau + (u_1 + 2\gamma u_0)Z(t) + u_0 Z'(t)$$

(see a derivation of the first term in the textbook). Therefore a classical solution (if it exists) must have the form for $t > 0$:

$$u(t) = \frac{1}{\omega_\gamma}\int_0^t e^{-\gamma(t-\tau)}\sin\Big(\omega_\gamma(t-\tau)\Big)f(\tau)\,d\tau + e^{-\gamma t}\Big((u_1 + \gamma u_0)\frac{\sin(\omega_\gamma t)}{\omega_\gamma} + u_0\cos(\omega_\gamma t)\Big)$$

It follows from this relation that $u \in C^0(t \geq 0)$ because it is continuous for $t > 0$ for any bounded f (by continuity of Z) and

$$\lim_{t\to 0^+} u(t) = u_0$$

In order to have $u \in C^1(t \geq 0)$, f is required to be from class $C^0(t \geq 0)$ so that the fundamental theorem of calculus applies in order to differentiate the integral. In this case

$$u'(t) = \int_0^t e^{-\gamma(t-\tau)} \cos\Big(\omega_\gamma(t-\tau)\Big) f(\tau)\, d\tau - -\frac{\gamma}{\omega_\gamma}\int_0^t e^{-\gamma(t-\tau)} \sin\Big(\omega_\gamma(t-\tau)\Big)(t-\tau) f(\tau)\, d\tau$$
$$+e^{-\gamma t}\Big((u_1+\gamma u_0)\cos(\omega_\gamma t) - u_0\omega_\gamma \sin(\omega_\gamma t)\Big) - \gamma e^{-\gamma t}\Big((u_1+\gamma u_0)\frac{\sin(\omega_\gamma t)}{\omega_\gamma} + u_0\cos(\omega_\gamma t)\Big)$$

and by continuity of the integral

$$\lim_{t\to 0^+} u'(t) = u_1 + \gamma u_0 - \gamma u_0 = u_1$$

So, the initial conditions are fulfilled. It follows from the integral representation of $u'(t)$ that the second derivative $u''(t)$ exists and is continuous for all $t > 0$ if f from class $C^0(t \geq 0)$ by the same reasoning as for the existence and continuity of u'. Thus, a classical solution from class $C^2(t > 0) \cap C^1(t \geq 0)$ exists and is unique if $f \in C^0(t \geq 0)$.

(iv) Let $g(t) = f_0\delta(t-T)$. Using the shift property of the convolution $G(t) * g(t-T) = (G * g)(t-T)$, a distributional solution to the initial value problem reads

$$v(t) = (G * g)(t-T) = f_0 G(t-T) = f_0\theta(t-T)Z(t-T)$$

which is a continuous function. Note that $G(t)$ is continuous at $t = 0$ and $G'(t) = \theta(t)Z'(t)$ has a jump discontinuity at $t = 0$. So, the solution is a smooth function everywhere except $t = T$ where its derivative has a jump discontinuity, and it is bounded

$$|v(t)| \leq \frac{f_0}{\omega_\gamma}$$

Owing to continuity of the convolution in the algebra $\mathcal{D}'_+$, if a series converges in $\mathcal{D}'_+$, its convolution with any $G \in \mathcal{D}'_+$ exists in $\mathcal{D}'_+$ and is given by the series of convolutions of terms with G (this series converges in $\mathcal{D}'_+$). Thus the solution to the stated generalized initial value problem exists and is given by the series

$$v(t) = f_0 G(t) * \sum_{n\geq 0}\delta(t-nT) = f_0\sum_{n\geq 0} G(t) * \delta(t-nT) = f_0\sum_{n\geq 0}\theta(t-nT)Z(t-nT)$$

This function is smooth in any interval $(mT, (m+1)T)$, $m = 0, 1, 2, ...$, (from C^∞) and is continuous at $t = mT$ for any m. Thus, $v \in C^0$. Indeed, the characteristic function for the interval $(mT, (m+1)T)$ is

$$\chi_m(t) = \theta(t-mT) - \theta(t-(m+1)T)\,.$$

Then the solution can be written in the form

$$v(t) = \sum_{m=0}^{\infty}\chi_m(t)v_m(t)\,, \qquad v_m(t) = f_0\sum_{n=0}^{m} Z(t-nT)$$

The smooth functions $v_m(t)$ define the solution in the intervals $(mT, (m+1)T)$. By continuity of $Z(t)$, v_m has continuous extensions to the end-points of the interval such that $v_{m-1}(mT) = v_m(mT)$ which implies continuity of $v(t)$ at $t = mT$:

$$\lim_{t\to mT^-} v(t) = v_{m-1}(mT) = v_m(mT) = \lim_{t\to mT^+} v(t)\,.$$

For example, $v_0(T) = Z(T)$ and $v_1(T) = Z(T) + Z(0) = Z(T) = v_0(T)$. This shows that $v(t)$ is continuous at $t = T$. Then $v_1(2T) = Z(2T)+Z(T)$ and $v_2(2T) = Z(2T)+Z(T)+Z(0) = v_1(T)$ because $Z(0) = 0$ so that $v(t)$ is continuous at $t = 2T$. An extension of the argument to any m is obvious.

It is also worth noting that if the bound $\max|v_m(t)| \le M < \infty$ does not depend on m, then the solution $v(t)$ is bounded for all $t \ge 0$:

$$|v(t)| \le \sup_{m\ge 0} \max_{[mT,(m+1)T]} |v_m(t)| \le M < \infty\,.$$

(v) Let us find an upper bound on $v_m(t)$ if $\gamma > 0$. Using $|Z(t)| \le e^{-\gamma t}/\omega_f$, one infers that

$$|v_m(t)| \le \frac{f_0}{\omega_f} e^{-\gamma t} \sum_{n=0}^{m} \left(e^{\gamma T}\right)^n = \frac{f_0}{\omega_f} e^{-\gamma t}\, \frac{e^{(m+1)\gamma T} - 1}{e^{\gamma T} - 1}$$

where the equation for the geometric partial sum has been used. The bound is monotonically decreasing over the interval $(mT, (m+1)T)$ and, hence, attains its largest value at $t = mT$ so that

$$|v_m(t)| \le \frac{f_0}{\omega_f} \frac{e^{\gamma T} - e^{-\gamma T}}{e^{\gamma T} - 1}$$

The bound does not depend on m and, hence, the solution remains bounded for any $T > 0$ if $\gamma > 0$. If $\gamma = 0$, then setting $\gamma = 0$ in the sum yields the bound that increases with increasing m:

$$|v_m(t)| \le \frac{f_0}{\omega} \sum_{n=0}^{m} 1 = \frac{f_0}{\omega}\,(m+1)$$

However this does not imply that the solution is not bounded. Let us calculate $v_m(t)$ in the case $\gamma = 0$. The method would actually work for any γ, but the analysis is somewhat more cumbersome that the one given above to show boundedness of the solution if $\gamma > 0$. Using again the partial geometric sum,

$$v_m(t) = \frac{f_0}{\omega} \sum_{n=0}^{m} \sin[\omega(t-nT)] = \frac{f_0}{\omega}\,\mathrm{Im} \sum_{n=0}^{m} e^{i\omega(t-nT)} = \frac{f_0}{\omega}\,\mathrm{Im}\, e^{i\omega t}\, \frac{1 - e^{-i(m+1)T}}{1 - e^{-i\omega T}}$$

This shows that if $T \ne \frac{2\pi}{\omega} N$ where N is any positive integer, then the solution remains bounded because the bound does not depend on m:

$$|v_m(t)| \le \frac{f_0}{\omega} \left| \frac{\sin[\frac{\omega T}{2}(m+1)]}{\sin(\frac{\omega T}{2})} \right| \le \frac{f_0}{\omega} \frac{1}{|\sin(\frac{\omega T}{2})|}\,, \quad \omega T \ne 2\pi N\,.$$

If $\omega T = 2\pi N$, then the solution is not bounded for $t \ge 0$ because its amplitude grows linearly with increasing m:

$$v_m(t) = \frac{f_0}{\omega} \sum_{n=0}^{m} \sin(\omega(t-nT)) = \frac{f_0}{\omega}\, \sin(\omega t) \sum_{n=0}^{m} 1 = \frac{f_0}{\omega}\, \sin(\omega t)\,(m+1)$$

So, the amplitude of oscillations increases by f_0/ω after each "kick". The "kicks" are in *resonance* with the oscillator frequency.

2. Fourier transforms of distributions. Show that

$$\text{(i)} \qquad \mathcal{F}\Big[\mathcal{P}\frac{1}{x^2}\Big](k) = -\pi|k|\,,$$

$$\text{(ii)} \qquad \mathcal{F}[|x|](k) = -2\mathcal{P}\frac{1}{k^2}\,.$$

Find the Fourier transform of the following distributions

$$\text{(iii)} \qquad f(x_0, x) = \theta(x_0 - |x|)\,, \quad x_0 \in \mathbb{R}\,, \quad x \in \mathbb{R}$$

$$\text{(iv)} \qquad \delta_{C_a}(x)\,, \quad x \in \mathbb{R}^2\,, \quad (\delta_{C_a}, \varphi) = \int_{|x|=a} \varphi(x)\, ds$$

Hints: Use the continuity of the Fourier transform and that $f_a(x_0, x) = e^{-ax_0} f(x_0, x) \to f(x_0, x)$ in $\mathcal{S}'$ (prove this!). For (iv), recall an integral representation of the Bessel functions $J_\nu(z)$.

SOLUTION: (i) Recall that

$$\frac{d}{dx}\mathcal{P}\frac{1}{x} = -\mathcal{P}\frac{1}{x^2}$$

By taking the Fourier transform of this relation and using the Fourier transform of a derivative

$$\mathcal{F}\Big[\mathcal{P}\frac{1}{x^2}\Big](k) \;=\; -\mathcal{F}\Big[\frac{d}{dx}\mathcal{P}\frac{1}{x}\Big](k) = ik\mathcal{F}\Big[\mathcal{P}\frac{1}{x}\Big](k) = ik \cdot i\pi\varepsilon(k) = -\pi|k|$$

(ii) Swapping the variables x and k in the above relation and taking the inverse Fourier transform of both sides

$$\mathcal{P}\frac{1}{k^2} = -\pi\mathcal{F}^{-1}[|x|](k) = -\frac{1}{2}\mathcal{F}[|-k|](x) = -\frac{1}{2}\mathcal{F}[|x|](k)$$

from which the desired relation follows.

(iii) Put $f_a(x_0, x) = e^{-ax_0} f(x_0, x)$, $a > 0$, so that $f_a \to f_a$, as $a \to 0^+$ pointwise almost everywhere in $\mathbb{R}^2$. Let us show that $f_a \to f$ in $\mathcal{S}'$ as $a \to 0^+$. If the latter is true, then

$$\mathcal{F}[f](k_0, k) = \lim_{a\to 0^+} \mathcal{F}[f_a](k_0, k) \quad \text{in } \mathcal{S}'$$

by continuity of the Fourier transform. Then $\mathcal{F}[f_a]$ is just the classical Fourier transforms because f_a is integrable on $\mathbb{R}^2$. Its limit in $\mathcal{S}'$ should be calculated.

For any test function $\varphi \in \mathcal{S}$,

$$|f_a(x_0, x)\varphi(x_0, x)| \le |\varphi(x_0, x)| \in \mathcal{L}(\mathbb{R}^2)$$

Therefore by the Lebesgue dominated convergence theorem

$$\lim_{a\to 0^+} (f_a, \varphi) \;=\; \lim_{a\to 0^+} \int\!\!\int f_a(x_0, x)\varphi(x_0, x)\, dx\, dx_0 = \int\!\!\int f(x_0, x)\varphi(x_0, x)\, dx\, dx_0 = (f, \varphi)$$

So, $f_a \to f$ in $\mathcal{S}'$ as required. Next,

$$\begin{aligned}
\mathcal{F}[f_a](k_0,k) &= \int\int e^{ik_0x_0+ikx} f_a(x_0,x)\,dx dx_0 = \int_0^\infty e^{-x_0(a-ik_0)}\int_{-x_0}^{x_0} e^{ikx}dx\,dx_0\\
&= \frac{1}{ik}\int_0^\infty \Big(e^{-x_0[a-i(k_0+k)]} - e^{-x_0[a-i(k_0-k)]}\Big)\,dx_0\\
&= \frac{1}{k}\Big(\frac{1}{k_0+k+ia} - \frac{1}{k_0-k+ia}\Big) = \frac{2}{k^2-(k_0+ia)^2}
\end{aligned}$$

The distributional limit is nothing but a distributional regularization of a singular function

$$\mathcal{F}[f](k_0,k) = \mathrm{Reg}\,\frac{2}{k^2-k_0^2}\,.$$

The regularization prescription is defined by shifting the poles $k_0 = \pm|k|$ in the complex k_0 plane like in the Sokhotsky distributions. The action of this distribution on a test function can be written via the direct product of the principle value and Sokhotsky's distributions

$$\begin{aligned}
\Big(\mathcal{F}[f](k_0,k),\varphi(k_0,k)\Big) &= \lim_{a\to0^+}\int\int \frac{\varphi(k_0,k)}{k}\Big(\frac{1}{k_0+k+ia} - \frac{1}{k_0-k+ia}\Big)\,dk_0\,dk\\
&= \lim_{a\to0^+}\lim_{\varepsilon\to0^+}\int_{|k|>\varepsilon}\int \frac{\varphi(k_0,k)}{k}\Big(\frac{1}{k_0+k+ia} - \frac{1}{k_0-k+ia}\Big)\,dk_0\,dk\\
&= \lim_{a\to0^+}\lim_{\varepsilon\to0^+}\int_{|k|>\varepsilon}\int \frac{1}{k}\cdot\frac{1}{k_0+ia}\Big(\varphi(k_0-k,k) - \varphi(k_0+k,k)\Big)\,dk_0\,dk\\
&= \Big(\mathcal{P}\frac{1}{k}\cdot\frac{1}{k_0+i0},\psi(k_0,k)\Big)\\
\psi(k_0,k) &= \varphi(k_0-k,k) - \varphi(k_0+k,k)
\end{aligned}$$

The function ψ is a test function of two variables because it is obtained by a non-singular transformation of the argument $(k_0,k)\to(k_0\pm k,k)$ in a test function. Thus, the limit is the direct product:

$$\mathcal{F}[f](k_0,k) = \mathcal{P}\frac{1}{k}\cdot\Big(\frac{1}{k_0+k+i0} - \frac{1}{k_0-k+i0}\Big)$$

(iv) Let $\eta(x)$ be a bump function from $\mathcal{D}$ for a neighborhood of the circle $|x| = a$ (the support of δ_{S_a}. Using the theorem about the Fourier transform of compactly supported distributions,

$$\begin{aligned}
\mathcal{F}[\delta_{S_a}](k) &= \Big(\delta_{S_a}(x),\eta(x)e^{i(k,x)}\Big) = \int_{|x|=a} e^{i(k,x)}\,dS = \int_{-\pi}^{\pi} e^{ia|k|\cos(\phi)}a d\phi\\
&= 2\pi a J_0(a|k|)
\end{aligned}$$

where the integral representation of Bessel functions was used:

$$J_n(z) = \frac{1}{\pi}\int_0^\pi e^{iz\cos(\phi)}\cos(n\phi)\,d\phi\,,\quad z\in\mathbb{C}$$

3. Convolution of distributions. Find each of the following convolutions or show that the convolution does not exist:

(i) $\quad \theta(x) * \Big(x\theta(x)\Big)\,,\qquad \dfrac{d}{dx}\Big[\Big(x\theta(x)\Big) * \Big(x\theta(x)\Big)\Big]$

(ii) $\quad \dfrac{1}{|x|} * \Delta\delta_{S_a}(x)\,,\quad x\in\mathbb{R}^3$

where $\delta_{S_a}(x)$ is the spherical delta function.

SOLUTION: (i) This is a convolution in $\mathcal{D}'_+$. If f and g are bounded functions that vanish for $x<0$, then their classical convolution exists:

$$(f*g)(x)=\theta(x)\int_0^\infty f(x-y)g(y)\,dy=\theta(x)\int_0^x f(x-y)g(y)\,dy\,,$$

Therefore

$$\theta(x)*\Big(x\theta(x)\Big)=\theta(x)\int_0^x y\,dy=\frac{1}{2}\,\theta(x)x^2$$

Since the convolution exists in $\mathcal{D}'_+$ in the second part of the question, it can be differentiated

$$\frac{d}{dx}\Big[\Big(x\theta(x)\Big)*\Big(x\theta(x)\Big)\Big]=\Big[\frac{d}{dx}\Big(x\theta(x)\Big)\Big]*\Big(x\theta(x)\Big)=\theta(x)*\Big(x\theta(x)\Big)=\frac{1}{2}\,\theta(x)x^2$$

because $(x\theta(x))'=\theta(x)+x\delta(x)=\theta(x)$.

(iv) The convolution $\frac{1}{|x|}*\delta_{S_a}(x)$ exists because the spherical delta function has a compact support. Therefore the operator Δ can be moved to act on the other distribution which happens to be proportional to its fundamental solution:

$$\frac{1}{|x|}*\Delta\delta_{S_a}(x)=\Delta\Big(\frac{1}{|x|}*\delta_{S_a}(x)\Big)=\Delta\frac{1}{|x|}*\delta_{S_a}(x)=-4\pi\delta(x)*\delta_{S_a}(x)=-4\pi\delta_{S_a}(x)$$

4. Cauchy problem for the transfer equation. The Cauchy problem for the transfer or flow equation is to find a solution to the following initial value problem:

$$\frac{1}{c}\frac{\partial u(x,t)}{\partial t}+(s,\nabla_x)u(x,t)+\alpha u(x,t)=f(x,t)\,,\quad t>0\,,\quad x\in\mathbb{R}^N$$
$$u\Big|_{t=0}=u_0(x)$$

where $s\in\mathbb{R}^N$, $|s|=1$, $c>0$, and $\alpha>0$. A classical solution must be from class $u\in C^1(t>0)\cap C^0(t\geq 0)$ if it exists (under some smoothness conditions on the inhomogeneity f and the initial data u_0).
(i) Let u be a classical solution. Consider the distributions from $\mathcal{D}'(\mathbb{R}^{N+1})$:

$$v(x,t)=\theta(t)u(x,t)\,,\quad g(x,t)=\theta(t)f(x,t)$$

Using similar arguments as for the Cauchy problem for the heat equation, show that

$$Lv=\Big[\frac{1}{c}\partial_t+(s,\nabla_x)+\alpha\Big]v(x,t)=g(x,t)+\frac{1}{c}\,u_0(x)\cdot\delta(t)\quad\text{in}\quad\mathcal{D}'(\mathbb{R}^{N+1})$$

(ii) Let G_c be the causal Green's function for the transfer operator L:

$$LG_c(x,t)=\delta^N(x)\cdot\delta(t)\,,\quad G_c\in\mathcal{D}'(\mathbb{R}^{N+1})\,,\quad \operatorname{supp}G_c\subset\{t\geq 0\}$$

Show that its Fourier transform with respect to x satisfies the equation

$$\Big[\frac{1}{c}\partial_t-i(s,k)+\alpha\Big]\mathcal{F}_x[G_c](k,t)=\delta(t)$$

and find its solution.
(iii) Use the inverse Fourier transform with respect to variable $k \in \mathbb{R}^N$ to show that

$$G_c(x,t) = c\theta(t)e^{-\alpha ct} \cdot \delta(x - cts)$$

(iv) Assume that $u_0 \in \mathcal{D}'(\mathbb{R}^N)$ with a bounded support, and $g(x,t) \in \mathcal{D}'(\mathbb{R}^{N+1})$ with a bounded support that lies in $t \geq 0$. Show that the equation

$$Lv(x,t) = g(x,t) + \frac{1}{c}u_0(x) \cdot \delta(t)\,, \quad v(t,x) = 0\,, \quad t < 0\,.$$

has a unique solution and express it using the convolution of distributions (give arguments for the uniqueness and the existence of the convolution).
(v) Assume that $f(x,t)$ is smooth in the half-space $t \geq 0$ and $u_0(x)$ is also smooth. Find an explicit (integral) representation for the solution $v(x,t)$. Find sufficient conditions on on smoothness of f such that $v(x,t) \in C^1(t > 0) \cap C^0(t \geq 0)$

Solution: (i) Using the rule for distributional differentiation of piecewise smooth functions,

$$\begin{aligned} D_x v(t,x) &= \theta(t)\{D_x u(t,x)\}\,, \\ D_t v(t,x) &= \theta(t)\{D_t u(t,x)\} + \mathrm{disc}_{t=0} u\, \delta(t) = \theta(t)\{D_t u(t,x)\} + \delta(t) \cdot u_0(x)\,. \end{aligned}$$

It follows from this relations that

$$Lv(t,x) = \theta(t)\{Lu(t,x)\} + \frac{1}{c}\,\delta(t) \cdot u_0(x) = \theta(t) f(t,x) + \frac{1}{c}\,\delta(t) \cdot u_0(x)\,.$$

(ii) Let us try to find the Green's function of L in the space of temperate distributions. If it exists, then it can be found by the Fourier method. By taking the Fourier transform of both sides of the equation with respect to the variable x, and using the property that $\mathcal{F}[Df](k) = -ik\mathcal{F}[f](k)$, $f \in \mathcal{S}'$, it is concluded that the Fourier transform $\mathcal{F}[G](k,t)$ satisfies the ordinary differential equation

$$\Big[\frac{1}{c}\frac{d}{dt} - i(s,k) + \alpha\Big]\mathcal{F}_x[G_c](k,t) = \delta(t)\,, \quad \mathcal{F}_x[G_c](k,t) = 0\,,\ t < 0$$

This problem has been shown (see the textbook) to have a *unique* distributional solution of the form $\mathcal{F}_x[G_c](k,t) = c\theta(t)Z(t,x)$ where

$$\Big[\frac{1}{c}\frac{d}{dt} - i(s,k) + \alpha\Big]Z(t,k) = 0\,, \quad Z(0,k) = 1\,.$$

Note the factor c in the solution and the factor $\frac{1}{c}$ at the derivative in the equation (to be compared with the unit factor in the textbook). Thus,

$$\mathcal{F}_x[G_c](k,t) = c\theta(t)e^{-\alpha ct + ic(s,k)t} \ \in \mathcal{S}'(\mathbb{R}^{N+1})$$

Since $\alpha \geq 0$, the solution is a smooth function in $t > 0$ that is bounded

$$|\mathcal{F}_x[G_c](k,t)| \leq c$$

and, hence, it is a regular temperate distribution. Therefore G_c is also a temperate distribution.

(iii) To find the inverse Fourier transform, recall that

$$\mathcal{F}[\delta(x-x_0)](k) = e^{i(k,x_0)}\,, \quad \mathcal{F}^{-1}[e^{i(k,x_0)}] = \delta(x-x_0)\,.$$

Therefore for any test function $\varphi(x,t) \in \mathcal{S}$,

$$\begin{aligned}
(G_c,\varphi) &= \Big(\mathcal{F}_x[G_c], \mathcal{F}_x^{-1}[\varphi]\Big) = c\int_0^\infty \int e^{-\alpha t} e^{ic(s,k)t}\mathcal{F}_x^{-1}[\varphi](k,t)\, d^N k dt \\
&\overset{(1)}{=} c\int_0^\infty e^{-\alpha t}\Big(e^{ic(s,k)t}, \mathcal{F}_x^{-1}[\varphi](k,t)\Big)\, dt \\
&= c\int_0^\infty e^{-\alpha t}\Big(\mathcal{F}_k^{-1}[e^{ic(s,k)t}](x), \varphi(x,t)\Big)\, dt \\
&= c\int_0^\infty e^{-\alpha t}\Big(\delta(x-cst), \varphi(x,t)\Big)\, dt \\
&= c\int_0^\infty e^{-\alpha ct}\varphi(cst,t)\, dt
\end{aligned}$$

Here (1) holds by the consistency theorem for the direct product of temperate distributions, that is, $(f(y),\varphi(x,y))$ is a test function from $\mathcal{S}$ for any test function φ of two variables and any temperate distribution f. So, G_c is a line-delta function supported on the half-line in $\mathbb{R}^{N+1}$ that is parallel to the vector s. Its parametric equations are $x = cst$, $t = t$, $t > 0$. Since $\mathcal{S}' \subset \mathcal{D}'$, the causal Green's function exists in $\mathcal{D}'(\mathbb{R}^{N+1})$.

Consider the direct product $c\theta(t)e^{-\alpha ct}\cdot\delta(x)$. Then G_c is obtained from it by a non-singular linear transformation of the argument $(t,x) \to (t, x-cts)$:

$$G_c(x,t) = c\theta(t)e^{-\alpha ct}\cdot\delta(x-cts) \;\in\; \mathcal{D}'(\mathbb{R}^{N+1})\,.$$

Indeed, owing to the definitions of the direct product and a linear transform of the argument in a distribution, the action of G_c on a test function reads

$$\begin{aligned}
(G_c,\varphi) &= \Big(c\theta(t)e^{-\alpha t}\cdot\delta(x-cst), \varphi(x,t)\Big) = \Big(c\theta(t)e^{-\alpha t}\cdot\delta(x), \varphi(x+cts,t)\Big) \\
&= \Big(c\theta(t)e^{-\alpha t}, \Big(\delta(x), \varphi(x+cts,t)\Big)\Big) = \Big(c\theta(t)e^{-\alpha t}, \varphi(cts,t)\Big) = c\int_0^\infty e^{-\alpha ct}\varphi(cts,t)\, dt\,.
\end{aligned}$$

as required.

(iv) Consider first the case when $g = 0$ and the inhomogeneity is $h(t,x) = c^{-1}\delta(t)\cdot u_0(x)$ where $u_0 \in \mathcal{D}'$. Then the solution is given by the convolution $v = G_c * h$ provided the convolution exists. Let us investigate the convolution. Let $\eta_n(t,\tau,x,y)$ be a unit sequence in $\mathbb{R}^{2N+2}$. Then using the definitions of the convolution and the direct product, one infers that for any

$\varphi \in \mathcal{D}$

$$\begin{aligned}
\Big(G_c * h, \varphi\Big) &= \lim_{n\to\infty} \Big(G_c(t,y)\cdot h(\tau,x), \eta_n(t,\tau,x,y)\varphi(t+\tau,x+y)\Big)\\
&= c^{-1}\lim_{n\to\infty} \Big(G_c(t,y), \Big(u_0(x), \eta_n(t,0,x,y)\varphi(t,x+y)\Big)\Big)\\
&= \lim_{n\to\infty} \Big(\theta(t)e^{-\alpha t}, \Big(u_0(x), \eta_n(t,0,x,cts)\varphi(t,x+cts)\Big)\Big)\\
&= \lim_{n\to\infty} \Big(\theta(t)e^{-\alpha t}, \Big(u_0(x-cts), \eta_n(t,0,x-cts,cts)\varphi(t,x)\Big)\Big)\\
&= \lim_{n\to\infty} \Big(\theta(t)e^{-\alpha t}\cdot u_0(x-cts), \eta_n(t,0,x-cts,cts)\varphi(t,x)\Big)\\
&= \Big(\theta(t)e^{-\alpha t}\cdot u_0(x-cts), \varphi(t,x)\Big)
\end{aligned}$$

where the definition of a linear transformation of the argument in distributions has been used. Since $\varphi \in \mathcal{D}(\mathbb{R}^{N+1}$, the sequence $\eta_n(t,0,x-cts,cts)\varphi(t,x)$ converges to $\varphi(t,x)$ in $\mathcal{D}(\mathbb{R}^{N+1})$. The proof is identical to that given in the textbook when discussing the existence of the convolution for distributions with bounded support (note that terms of the sequence coincide with φ for all large enough n). The limit follows from the continuity of the direct product Thus,

$$(G_c * h)(t,x) = \theta(t)e^{-\alpha t}\cdot u_0(x-cts)\,.$$

for any $u_0 \in \mathcal{D}'(\mathbb{R}^N)$.

Let now the inhomogeneity be any distribution $g(t,x)$ that vanishes in the half-space $t<0$ so that $g(t,x) = \eta(t)g(t,x)$ where η is a bump function for $[0,\infty)$. The same holds for the Green's function, $G_c(t,x) = \eta(t)G_c(t,x)$. Then the solution to the generalized Cauchy problem is given by the convolution $G_c * g$. Let us investigate the convolution using the same line of arguments

$$\begin{aligned}
\Big(G_c * g, \varphi\Big) &= \lim_{n\to\infty} \Big(G_c(t,y)\cdot g(\tau,x), \eta(t)\eta(\tau)\eta_n(t,\tau,x,y)\varphi(t+\tau,x+y)\Big)\\
&= c\lim_{n\to\infty} \Big(\theta(t)e^{-\alpha t}, \Big(g(\tau,x), \eta_n(t,\tau,x,cts)\eta(t)\eta(\tau)\varphi(t+\tau,x+cts)\Big)\\
&= c\int_0^\infty e^{-\alpha t}\Big(g(\tau,x), \psi(t,\tau,x)\Big)\,dt\\
\psi(t,\tau,x) &= \eta(t)\eta(\tau)\varphi(t+\tau,x+cts)\,,
\end{aligned}$$

where ψ is a test function from $\mathcal{D}(\mathbb{R}^{N+2})$, as shown below, and the existence of the limit follows from continuity of the functional g. Therefore the action of g on ψ is a test function of one real variable t so that integral exists for any distribution g vanishing in the lower half-space. Thus, the convolution exists, and is given by the above equation.

It remains to show that $\psi \in \mathcal{D}(\mathbb{R}^{N+2})$ for any $\varphi \in \mathcal{D}(\mathbb{R}^{N+1})$ and the existence of the limit. The support of $\varphi(t,x)$ lies in, say, $|t|<T$ and $|x|<R$. Therefore $\varphi(t+\tau,x+cts) = 0$ if $|t+\tau|>T$ and $|x+cts|>R$. Owing to that the bump function $\eta(t)$ vanishes for all $t<-\delta$ for some $\delta>0$, the support of ψ is bounded in the variables t and τ. It lies in the triangle, $t>-\delta$, $\tau>-\delta$, and $t+\tau<T$. This implies that $\psi(t,\tau,x)=0$ for all $|x|>R+c(T+\delta)$ (note that $|s|=1$) and all t and τ outside the said triangle. Thus, the support of ψ is bounded. Since η and φ are from class C^∞, ψ is a test function. Therefore the sequence $\psi_n = \eta_n(t,\tau,x,cts)\psi(t,\tau,x)$ converges to $\psi(t,\tau,x)$ in $\mathcal{D}(\mathbb{R}^{N+2})$. The proof is identical to that given in the textbook when discussing the existence of the convolution for distributions with bounded support. By continuity of the direct product $f(t)\cdot g(\tau,x)$, $(f\cdot g,\psi_n)\to(f\cdot g,\psi)$ as $n\to\infty$ for any distributions f and g.

Suppose now that $g(t,x)=f(t,x)$ is a regular distribution that vanishes for $t<0$. Then

$$\begin{aligned}(G_c * g,\varphi) &= \int_0^\infty\int_0^\infty\int e^{-\alpha t} f(\tau,x)\varphi(t+\tau,x+cts)\,d^Nx\,d\tau\,dt\\ &\overset{(1)}{=} \int_0^\infty\int_0^\infty\int e^{-\alpha t} f(\tau,x-cts)\varphi(t+\tau,x)\,d^Nx\,dt\,d\tau\\ &\overset{(2)}{=} \int_0^\infty\int_\tau^\infty e^{-\alpha c(t-\tau)}\int f(\tau,x-cs(t-\tau))\varphi(x,t)\,d^Nx\,dt\,d\tau\\ &\overset{(3)}{=} \int_0^\infty\int\int_0^t e^{-\alpha c(t-\tau)} f(\tau,x-cs(t-\tau))\,\varphi(x,t)\,d\tau\,d^Nx\,dt\end{aligned}$$

Here (1) is obtained by the shift of the integration variable x and a subsequent change of order of integration with respect to t and τ (allowed by Fubini's theorem because the support of $\varphi(t+\tau,x)$ is bounded in all three variables t, τ and x in the stated region of integration and f is locally integrable), (2) is obtained by the shift of the integration variable t, and (3) is obtained by reversing the order of integration with respect to t and τ (the integration with respect to x can be carried out in any order by Fubini's theorem). This shows that the solution to the generalized Cauchy problem has the integral representation for $t>0$

$$u(t,x)=\int_0^t e^{-\alpha c(t-\tau)} f(\tau,x-cs(t-\tau))\,d\tau$$

Evidently, $u(t,x)\to 0$ as $t\to 0^+$ (it satisfies the zero initial condition).

(v) If the inhomogeneity and the initial data are continuous functions, $u_0\in C^0$ and $f\in C^0(t\geq 0)$, then the distributional solution

$$u(t,x)=e^{-\alpha t}u_0(x-cts)+\int_0^t e^{-\alpha c(t-\tau)} f(\tau,x-cs(t-\tau))\,d\tau$$

is continuous and satisfies the initial conditions, $u(t,x)\to u_0(x)$ as $t\to 0^+$ by continuity of u_0 and f. Thus, $u\in C^0(t\geq 0)$. The solution belongs to class $C^1(t>0)$ if, in addition, $u_0\in C^1$ and $f\in C^1(t>0)$ by the same line of reasonings as in Problem 1 (iii) for continuity of D_tu (the necessity of these conditions for continuity of D_xu is obvious). In this case, the distributional solution is a solution to the classical Cauchy problem for the transfer equation, and it is unique.