Mathematical Methods for Physics and Engineering

Lecture notes

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CHAPTER 1

The theory of convergence

1. Numerical sequences

1.1. Definition of a convergent sequence. An infinite collection of real or complex numbers \( \{z_n\}^\infty_1 \), where a positive integer \( n \) enumerates the elements, is called a sequence. One can also say that a sequence is a real or complex valued function on a set of positive integers, that is, it is a rule that assigns a unique number \( z_n \) to every positive integer \( n \).

**Definition 1.1.** If there exists a number \( L \) such that for any positive number \( \varepsilon > 0 \) one can find a positive integer \( N \) such that

\[
|z_n - L| < \varepsilon, \quad \forall n > N, 
\]

then the sequence \( \{z_n\} \) is said to converge and the number \( L \) is called the limit of the sequence. In this case, one writes

\[
\lim_{n \to \infty} z_n = L \quad \text{or} \quad z_n \to L \quad \text{as} \quad n \to \infty
\]

The definition implies that:
- there are finitely many terms of the sequence outside any neighborhood of the limit \( L \);
- any convergent sequence is bounded:

\[
\lim_{n \to \infty} z_n = L \quad \Rightarrow \quad |z_n| \leq M, \quad \forall n
\]

for some number \( M \) independent of \( n \);
- if the limit of a sequence exists, then it is unique.

The reader is advised to prove the above assertions.

1.2. Basic convergent sequences. It can be proved that for any positive \( p > 0 \) and any real \( \alpha \)

\[
\begin{align*}
\lim_{n \to \infty} \sqrt[n]{n} &= 1 \\
\lim_{n \to \infty} \sqrt[n]{p} &= 1 \\
\lim_{n \to \infty} \frac{1}{n^p} &= 0 \\
\lim_{n \to \infty} \frac{n^\alpha}{(1 + p)^n} &= 0
\end{align*}
\]
Convergence to infinity. If for any positive number $M > 0$, one can find an integer $N$ such that
\[ |z_n| > M, \quad \forall n > N \]
then the sequence \( \{z_n\} \) is said to tend to infinity as \( n \to \infty \), and one writes
\[ \lim_{n \to \infty} |z_n| = \infty \quad \text{or} \quad |z_n| \to \infty \quad \text{as} \quad n \to \infty. \]
Let \( \{x_n\} \) be a sequence of real numbers. If for any positive number $M$ one can find an integer $N$ such that $x_n < -M$ for all $n > N$, then the sequence is said to tend to negative infinity, and one writes
\[ \lim_{n \to \infty} x_n = -\infty \quad \text{or} \quad x_n \to -\infty \quad \text{as} \quad n \to \infty. \]

1.3. Order of magnitude. If two sequences \( \{z_n\} \) and \( \{\xi_n\} \) are such that
\[ \left| \frac{\xi_n}{z_n} \right| < M, \quad \forall n > N \]
for some number $M$ independent of $n$ and some integer $N$, then the sequence are said to be of the same order of magnitude and one writes
\[ \xi_n = O(z_n). \]
If $\xi_n/z_n \to 0$ as $n \to \infty$, then one writes
\[ \xi_n = o(z_n) \]
For example,
\[ \frac{5n+3}{n^4+2n+1} = O\left(\frac{1}{n^3}\right), \quad \frac{3}{2n^2+1} = o\left(\frac{1}{n}\right). \]

1.4. Basic limit laws. Suppose that
\[ \lim_{n \to \infty} z_n = A \quad \text{and} \quad \lim_{n \to \infty} \xi_n = B \]
Then
\[ \lim_{n \to \infty} (z_n + \xi_n) = \lim_{n \to \infty} z_n + \lim_{n \to \infty} \xi_n = A + B \]
\[ \lim_{n \to \infty} (z_n \xi_n) = (\lim_{n \to \infty} z_n)(\lim_{n \to \infty} \xi_n) = AB \]
\[ \lim_{n \to \infty} \frac{z_n}{\xi_n} = \frac{\lim_{n \to \infty} z_n}{\lim_{n \to \infty} \xi_n} = \frac{A}{B}; \]
in the latter equality it is assumed, in addition, that $B \neq 0$ and $\xi_n \neq 0$. Note that the converse is false. The reader is advised to give examples when the limits of the sum or product or ratio of two sequences exists, but the limits of each sequences do not.
1.5. Upper and lower limits. Let $E$ be a set of real numbers. A set $E$ is bounded from above if there is a number $b$ such that $x \leq b, \forall x \in E$, and the number $b$ is called an upper bound of $E$. A set $E$ is bounded from below if there is a number $a$ such that $a \leq x, \forall x \in E$, and the number $a$ is called a lower bound of $E$. A set $E$ is said to be bounded if it has lower and upper bounds.

Definition 1.2. (supremum and infimum of a set of real numbers)
Let $E$ be a bounded set. Then the least upper bound and the greatest lower bound of $E$ are called, respectively, the supremum and infimum of $E$ and denoted as $\text{sup} E$ and $\text{inf} E$. If $E$ is not bounded from above, then $\text{sup} E = \infty$ and, if $E$ is not bounded from below, then $\text{inf} E = -\infty$.

In other words, $\text{sup} E$ is an upper bound of $E$, while $\text{sup} E - \epsilon$ is not an upper bound for any $\epsilon > 0$. Similarly, $\text{inf} E$ is a lower bound of $E$, but $\text{inf} E + \epsilon$ is not a lower bound of $E$ for any $\epsilon > 0$. The numbers $\text{sup} E$ and $\text{inf} E$ are unique. They may or may not be in $E$. For example, if $E = [0, 1]$, then $\text{sup} E = 1$ and $\text{inf} E = 0$ are in $E$, while for $E = (0, 1)$ (an open interval), $\text{sup} E = 1$ and $\text{inf} E = 0$ are not in $E$.

Definition 1.3. (a limit point of a sequence)
A number $x$ is a limit point of real sequence $\{x_n\}$ if any neighborhood of $x$ contains infinitely many terms of $\{x_n\}$:

$$|x_n - x| < \varepsilon, \quad \forall \varepsilon > 0,$$

for infinitely many $n$’s.

For example, the sequence $x_n = (-1)^n$ has two limit points, $\pm 1$.

Definition 1.4. (upper and lower limits of a sequence)
Let $E$ be the set of all limit points of a real sequence $\{x_n\}$. Then $\text{sup} E$ and $\text{inf} E$ are called the upper and lower limits of the sequence, respectively, and denoted as

$$\text{sup} E = \limsup_{n \to \infty} x_n, \quad \text{inf} E = \liminf_{n \to \infty} x_n.$$

For example,

$$\limsup_{n \to \infty} (-1)^n \left(1 + \frac{1}{n}\right) = 1, \quad \limsup_{n \to \infty} (-1)^n \left(1 + \frac{1}{n}\right) = -1.$$

- A real sequence converges if and only if its upper and lower limits exists and coincide.

Theorem 1.1. (Bolzano)
A bounded real sequence has at least one limit point.
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1.6. Increasing sequences. A sequence of real numbers \{x_n\} is called an increasing sequence if \(x_{n+1} \geq x_n\).

**Theorem 1.2.** An increasing sequence either converges or tend to infinity. In particular, if \(x_n < M\) for some \(M\) independent on \(n\), then \{\(x_n\}\} converges, that is, any bounded increasing sequence converges.

Indeed, if \(x_n < M\), then there exists \(\sup\{x_n\} = L\). Since \(x_n\) is an increasing sequence, only finitely many terms of it are not in an interval \((L - \varepsilon, L]\). This implies that \(\lim_{n \to \infty} x_n = L\). If \{\(x_n\)\} is not bounded from above, then \(\lim_{n \to \infty} = \sup\{x_n\} = \infty\).

2. Series

For a real or complex sequence \{\(z_n\)\}, a formal infinite sum
\[
\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots
\]
is called a series, and a sequence
\[
\{S_n\}^\infty_1 : S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \cdots z_n.
\]
is called a sequence of partial sums of the series \(\sum z_n\).

**Definition 2.1.** (the sum of a series)
A series \(\sum z_n\) is said to converge to a number \(S\) if the sequence of partial sums converges to \(S\), and, in this case, the number \(S\) is called the sum of the series:
\[
\sum_{n=1}^{\infty} z_n = \lim_{n \to \infty} S_n = S.
\]

If the sequence of partial sums has no limit, the series is said to diverge.

2.1. Necessary condition for convergence. Suppose that a series \(\sum z_n\) converges. Then \(z_n \to 0\) as \(n \to \infty\). Indeed, let
\[
\sum_{n=1}^{\infty} z_n = \lim_{n \to \infty} S_n = S.
\]
Then by the Cauchy criterion for convergence of a sequence, for any \(\varepsilon > 0\) one can find an integer \(N\) such that
\[
|z_{n+1}| = |S_{n+1} - S_n| < \varepsilon, \quad \forall n > N \quad \Rightarrow \quad \lim_{n \to \infty} z_n = 0.
\]
For example, the following series diverge:
\[
\sum_{n=1}^{\infty} \frac{n^q + 3n - 2}{(2n + 1)^q}, \quad \sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{(\log n)^p}
\]
for any positive integers \(p\) and \(q\). Explain why!

2.2. Geometric series. Let us investigate the convergence of the complex series
\[
\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots,
\]
where \(z\) is a complex number. It is called a geometric series. Clearly, the series diverges if \(z = 1\). If \(z \neq 1\), by the identity
\[
1 - z^n = (1 - z)(1 + z + \cdots + z^{n-1})
\]
\[
\Rightarrow S_n = 1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}
\]
the sequence of partial sums \(\{S_n\}\) converges
\[
\sum_{n=0}^{\infty} z^n = \lim_{n \to \infty} \frac{1 - z^n}{1 - z} = \frac{1}{1 - z}, \quad |z| < 1,
\]
because \(|z|^n \to 0\) as \(n \to \infty\) if and only if \(|z| < 1\).

2.3. Absolute and conditional convergence. Let \(\sum z_n\) be convergent. If the series of absolute values \(\sum |z_n|\) converges, then the series \(\sum z_n\) is called absolutely convergent, otherwise \(\sum z_n\) is said to converge conditionally. The geometric series converges absolutely for all \(|z| < 1\), while for \(z = -1\), it converges conditionally.

2.4. Series of non-negative terms. If \(a_n \geq 0\) for all \(n\), then the series \(\sum a_n\) is called a series of non-negative terms.

**Theorem 2.1.** (Necessary and sufficient condition for convergence)
Let \(0 \leq a_{n+1} \leq a_n\) for all \(n\) (the sequence \(\{a_n\}\) is monotonically decreasing). Then the series \(\sum a_n\) converges if and only if the series \(\sum 2^k a_{2^k}\) converges:
\[
\sum_{n=1}^{\infty} a_n < \infty \iff \sum_{k=0}^{\infty} 2^k a_{2^k} < \infty.
\]
By this criterion, the following series converge

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty, \quad p > 1, \]
\[ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} < \infty, \quad p > 1, \]
\[ \sum_{n=3}^{\infty} \frac{1}{n \log(n) \log(\log(n))} < \infty, \]
\[ \sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^2} < \infty. \]

For example,

\[ \sum_{k=0}^{\infty} 2^k \left(\frac{1}{2^k}\right)^p = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k < \infty \]

if and only if \(2^{1-p} < 1\) or \(1 - p < 0\) or \(p > 1\). Therefore the series \(\sum 1/n^p\) converges for \(p > 1\) and diverges for \(p \leq 1\).

2.5. Comparison test. If \(|z_n| \leq x_n\) for all \(n > N\) for some \(N\) and the series \(\sum x_n\) converges, then the complex series \(\sum z_n\) converges, too. If \(u_n \geq x_n \geq 0\) and the series \(\sum x_n\) diverges, then the real series \(\sum u_n\) diverges, too.

Indeed, consider the sequence of partial sums for \(\sum z_n\):
\[ |S_n| = |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| \]
\[ \leq x_1 + x_2 + \cdots + x_n \leq \sum_{n=1}^{\infty} x_n = A. \]

The sequence \(\{\sum_{k=1}^{n} |z_k|\}\) is bounded by \(A\) and increasing monotonically. Therefore it converges, which means that the series \(\sum z_n\) converges absolutely (and the absolute convergence implies convergence).

For example, the series
\[ \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^p} \]
converges absolutely for all real \(x\) and \(p > 1\) because
\[ \left| \frac{\cos(nx)}{n^p} \right| \leq \frac{1}{n^p} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty. \]
The series
\[ \sum_{n=0}^{\infty} 2^n \sin \left( \frac{z}{3^n} \right) \]
converges absolutely for all complex \( z \). Indeed, since \( \sin(u)/u \to 1 \) as \( u \to 0 \),
\[ \lim_{n \to \infty} 3^n \sin \left( \frac{z}{3^n} \right) = z \Rightarrow |3^n \sin \left( \frac{z}{3^n} \right)| \leq M, \]
for some \( M \) independent of \( n \) because every convergent sequence is bounded. Therefore
\[ |2^n \sin \left( \frac{z}{3^n} \right)| = \left( \frac{2}{3} \right)^n |3^n \sin \left( \frac{z}{3^n} \right)| \leq M \left( \frac{2}{3} \right)^n \]
and \( \sum_{n=0}^{\infty} (2/3)^n = 3 < \infty \) as the sum of a geometric series.

2.6. Root and ratio tests.

**Theorem 2.2.** (Root test)

Let
\[ \alpha = \limsup_{n \to \infty} \sqrt[n]{|z_n|}. \]

Then

(i) \( \alpha < 1 \) \( \Rightarrow \) \( \sum z_n \) converges absolutely

(ii) \( \alpha > 1 \) \( \Rightarrow \) \( \sum z_n \) diverges

(iii) \( \alpha = 1 \) the test gives no information

For example, the geometric series \( \sum z^n \) converges for all \( |z| < 1 \) by the root test. However, the root test gives no information about the series \( \sum 1/n^p \) because \( \sqrt[n]{n} \to 1 \) as \( n \to \infty \).

**Theorem 2.3.** (Ratio test)

\[ \limsup_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| < 1 \Rightarrow \sum z_n \text{ converges absolutely} \]
\[ \left| \frac{z_{n+1}}{z_n} \right| \geq 1, \quad \forall n > N \Rightarrow \sum z_n \text{ diverges} \]
\[ \lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = 1 \quad \text{the test gives no information} \]
The ratio test is convenient to use if terms of the series contain factorials. For example, the series
\[ \sum_{n=1}^{\infty} \frac{n^2 z^n}{n!} \]
converges absolutely for all complex \( z \). Indeed,
\[ \left| \frac{z_{n+1}}{z_n} \right| = \frac{(n+1)^2}{n^2} \frac{n!}{(n+1)!} |z| = \frac{n+1}{n^2} |z| \to 0 \quad \text{as} \quad n \to \infty \]
for any \( |z| \). The use of the root test would require to study the asymptotic behavior of \( \sqrt[n]{n!} \). This can be done with the help of Stirling’s formula:
\[ \lim_{n \to \infty} \frac{e^n n!}{n^n \sqrt{2\pi n}} = 1. \]
It is not difficult to show using Stirling’s formula that
\[ \lim_{n \to \infty} \frac{n^{\sqrt[n]{n!}}}{n} = e. \]

For the above series
\[ \lim_{n \to \infty} \sqrt[n]{|z_n|} = \lim_{n \to \infty} \frac{\sqrt[n]{n^2} |z|}{\sqrt[n]{n!}} = |z| \lim_{n \to \infty} \frac{1}{\sqrt[n]{n!}} = |z| \lim_{n \to \infty} \frac{n}{\sqrt[n]{n!}} = \frac{|z|}{e} \lim_{n \to \infty} \frac{1}{n} = 0. \]

2.7. Wider scope of the root test. The ratio test is easier to apply than the root test. However, the root test has a wider scope. Let \( b > a > 1 \).
Consider the series
\[ \sum_{n=1}^{\infty} u_n = \frac{1}{a} + \frac{1}{b} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{a^3} + \frac{1}{b^3} + \cdots \]
A general term of the series can be written as
\[ u_n = \begin{cases} a^{-k}, & n = 2k - 1, \\ b^{-k}, & n = 2k, \end{cases} \quad k = 1, 2, \ldots \]
One has
\[ \liminf_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{k \to \infty} \left( \frac{a}{b} \right)^k = 0 \]
\[ \limsup_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{k \to \infty} \left( \frac{b}{a} \right)^k = \infty \]
So, the ratio test does not even apply. However, the root test indicates convergence:
\[ \limsup_{n \to \infty} \sqrt[n]{u_n} = \lim_{k \to \infty} \sqrt[k]{ \frac{1}{a^k} } = \frac{1}{\sqrt[a]{a}} < 1. \]
The following results make a description of a wider scope of the root test more precise.

**Theorem 2.4. (Wider scope of the root test)**

Let \( \{x_n\} \) be a sequence of positive numbers, \( c_n > 0 \). Then

\[
\limsup_{n \to \infty} \sqrt[n]{x_n} \leq \limsup_{n \to \infty} \frac{x_{n+1}}{x_n}
\]

\[
\liminf_{n \to \infty} \sqrt[n]{x_n} \geq \liminf_{n \to \infty} \frac{x_{n+1}}{x_n}
\]

The first inequality shows that the root test may indicate convergence when the ratio test does not do so.

**Corollary 2.1.** Suppose that the limit of \( \left| \frac{x_{n+1}}{x_n} \right| \) exists and is equal to \( \alpha \). Then the limit of the sequence \( \sqrt[n]{|x_n|} \) also exists and is equal to \( \alpha \):

\[
\lim_{n \to \infty} \sqrt[n]{|x_n|} = \lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = \alpha
\]

Indeed, under the hypothesis, the upper and lower limits of the sequence \( \left| \frac{x_{n+1}}{x_n} \right| \) are equal. It then follows from Theorem 2.4 that

\[
\liminf_{n \to \infty} \frac{x_{n+1}}{x_n} \leq \limsup_{n \to \infty} \sqrt[n]{x_n} \leq \limsup_{n \to \infty} \sqrt[n]{x_{n+1}} \leq \limsup_{n \to \infty} \frac{x_{n+1}}{x_n}
\]

\[
\Rightarrow \liminf_{n \to \infty} \sqrt[n]{x_n} = \limsup_{n \to \infty} \sqrt[n]{x_n}
\]

\[
\Rightarrow \lim_{n \to \infty} \sqrt[n]{|x_n|} = \lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|}
\]

In practical applications, the existence of the limit of \( \left| \frac{x_{n+1}}{x_n} \right| \) is easier to verify than to compute the upper limit. If the said limit exists, then the root test would have the same scope as the ratio test for any such series, that is, the series diverges if \( \alpha > 1 \), or converges if \( \alpha < 1 \), or both tests give no information.

2.8. **De Morgan test.** The ratio test gives no information if \( \left| \frac{z_{n+1}}{z_n} \right| \to 1 \) as \( n \to \infty \). It turns out that in this case the convergence depends on how fast the ratio approaches 1.

**Theorem 2.5. (De Morgan test)**

Suppose that

\[
\lim_{n \to \infty} \frac{|z_{n+1}|}{|z_n|} = 1,
\]

\[
\limsup_{n \to \infty} n \left( \frac{|z_{n+1}|}{|z_n|} - 1 \right) = -1 - c, \quad \text{for some } c > 0,
\]
Then the series \( \sum z_n \) converges absolutely.

**Corollary 2.2.** If

\[
\left| \frac{z_{n+1}}{z_n} \right| = 1 + \frac{A}{n} + O \left( \frac{1}{n^2} \right), \quad A < -1
\]

then the series \( \sum z_n \) converges absolutely.

For example, the series \( \sum n^{-p} \) converges for \( p > 1 \). However, the ratio or root test fails to detect convergence. The De Morgan test shows convergence:

\[
\frac{n^p}{(n+1)^p} = \frac{1}{(1 + \frac{1}{n})^p} = 1 - \frac{p}{n} + O \left( \frac{1}{n^2} \right)
\]

and, hence, the series converges if \( A = -p < -1 \) or \( p > 1 \). The reader is advised to investigate the series

\[
\sum_{n=1}^{\infty} n^r \exp \left( -q \sum_{k=1}^{n} \frac{1}{k} \right)
\]

when \( r > q \) and \( r < q \).

### 2.9. Conditionally convergent series.

**Theorem 2.6.** (Abel test)

Let the sequence \( \{A_n\}, A_n = \sum_{k=1}^{n} a_k \), be bounded, and the sequence \( \{b_n\} \) is decreasing monotonically, \( b_1 \geq b_2 \geq b_3 \geq \cdots \), so that \( \lim_{n \to \infty} b_n = 0 \). Then the series \( \sum a_n b_n \) converges.

By the root test, the series

\[
\sum_{n=1}^{\infty} \frac{z^n}{n}
\]

converges absolutely in the disk \( |z| < 1 \) in the complex plane. Let us investigate the convergence on the boundary of the disk. Put \( z = e^{i\theta} \) so that \( a_k = e^{ik\theta} \) and \( b_n = 1/n \to 0 \) monotonically as \( n \to \infty \). Then

\[
A_n = e^{i\theta} + e^{2i\theta} + \cdots + e^{in\theta} = e^{i\theta} \frac{1 - e^{in\theta}}{1 - e^{i\theta}}
\]

\[
|A_n| = \left| \frac{1 - e^{in\theta}}{1 - e^{i\theta}} \right| = \sqrt{\frac{1 - \cos(n\theta)}{1 - \cos \theta}} \leq \sqrt{\frac{1}{1 - \cos \theta}} \leq \frac{1}{|\sin(\theta/2)|}
\]

Thus, the sequence \( \{A_n\} \) is bounded for all \( 0 < \theta < 2\pi \), and the series in question converges at all points of the closed disk \( |z| \leq 1 \) except \( z = 1 \). The convergence on the boundary is not absolute.
**Corollary 2.3. (Alternating series)**
Suppose that $|c_n|$ is decreasing monotonically, $|c_{n+1}| \leq |c_n|$, so that $c_n \to 0$ as $n \to \infty$, and $c_{2m-1} \geq 0$, $c_{2m} \leq 0$. Then the series $\sum c_n$ converges.

This is a simple consequence of the previous theorem. Indeed, put $a_n = (-1)^n$ and $b_n = |c_n|$. Then $|A_n| \leq 1$ and $b_n = 1/n \to 0$ monotonically. By the alternating series test, the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
converges, but not absolutely.

**2.10. Rearrangements.** Consider a one-to-one map of the set of all positive integer onto itself so that \{1, 2, 3, \ldots\} is mapped to \{k_1, k_2, k_3, \ldots\}. The series
$$\sum_n a_{k_n} = a_{k_1} + a_{k_2} + a_{k_3} + \cdots = a_1' + a_2' + a_3' + \cdots = \sum_n a_n'$$
is called a rearrangement of the series $\sum_n a_n = a_1 + a_2 + a_3 + \cdots$.

**Theorem 2.7. (Rearrangement and absolute convergence)**
A rearrangement of any absolutely convergent series converges absolutely to the same sum:
$$\sum_n a_n = \sum_n a_n'.$$

In contrast, a rearrangement of a conditionally convergent series can converge to a different sum and even fail to converge. For example, consider the alternating series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = S$$
and let $S_n$ be the sequence of its partial sums so that $S_n \to S$ as $n \to \infty$. Evidently,
$$S < S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$Consider the rearrangement of the series in which two positive terms are followed by one negative:
$$\sum_{n=1}^{\infty} a_n' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} + \cdots$$
and let \( S'_n \) be the sequence of its partial sums. It follows that
\[
\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0 \quad \Rightarrow \quad S_3 = S'_3 < S'_6 < S'_9 < \cdots
\]
Therefore there are infinitely many terms of the sequence \( S'_n \) greater than \( 5/6 \). This implies that
\[
\limsup_{n \to \infty} S'_n > S'_3 = \frac{5}{6} > S
\]
and, hence, the sequence \( S'_n \) cannot converge to \( S \).

**Theorem 2.8.** (Riemann)

*Let \( \sum a_n \) converges but not absolutely. For any two elements of the extended real number system \( -\infty \leq \alpha \leq \beta \leq \infty \), there exists a rearrangement \( \sum a'_n \) with partial sums \( S'_n \) such that*
\[
\liminf_{n \to \infty} S'_n = \alpha, \quad \limsup_{n \to \infty} S'_n = \beta
\]

The Riemann theorem about rearrangements shows that a rearrangement of a conditionally convergent series may be constructed to converge to any desired number \( \alpha = \beta \), or diverge to \( \pm \infty \), or to have a sequence of partial sums that does not converge and oscillates between any two numbers \( (\alpha < \beta) \) (or has unbounded oscillations, \( \alpha = -\infty \), \( \beta = \infty \), or both). Such a behavior of conditionally convergent series stems from the following property.

**Theorem 2.9.** (Properties of conditionally convergent series)

*Let \( \sum a_n \) be a real series that converges but not absolutely. If \( S^+_p \) is the sum of the first \( p \) positive terms of the series and \( S^-_q \) is the sum of its first \( q \) negative terms, then*
\[
\lim_{p \to \infty} S^+_p = \infty, \quad \lim_{q \to \infty} S^-_q = -\infty.
\]

For example,
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots
\]
\[
\lim_{p \to \infty} S^+_p = \lim_{p \to \infty} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2p-1} \right) = \sum_{p=1}^{\infty} \frac{1}{2p-1} = \infty
\]
\[
\lim_{q \to \infty} S^-_q = \lim_{q \to \infty} \left( -\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2q} \right) = \frac{1}{2} \sum_{q=1}^{\infty} \frac{1}{q} = -\infty
\]
Note that a real conditionally convergent series necessarily contains infinitely many negative terms and infinitely many positive terms (otherwise it were an absolutely convergent series). As the series made of positive and negative terms of a conditionally convergent series diverge to infinity, the sum of the series looks like an undetermined form “\( \infty - \infty \)” whose value depends on how the infinities in this form are “regularized”, or, more precisely, how negative and positive terms are ordered when computing the sequence of partial sums.

For example, let us construct a rearrangement that converges to a given number \( \beta > 0 \) (the case \( \beta \leq 0 \) can be considered along a similar line of arguments). Let \( S'_n \) denote the sequence of partial sums of the sought-after rearrangement. The rearrangement should be such that \( S'_n \to \beta \) as \( n \to \infty \). Let \( \{a^+_n\} \) and \( \{a^-_n\} \) denote the subsequences of positive and negative terms in the sequence \( \{a_n\} \). Clearly, these sequences can be reordered so that they converge to 0 monotonically. Note that \( a_n \to 0 \) as \( n \to \infty \) owing to the conditional convergence of \( \sum a_n \). Let us keep the same notations for the reordered (monotonic) sequences, \( |a^+_{n+1}| \leq |a^+_n| \) for all \( n \). Put \( a'_n = a^+_n \) for \( n = 1, 2, \ldots, m_1^+ \) where \( m_1 \) is determined by the condition \( S'_{m_1^+-1} < \beta \leq S'_{m_1^+} \). It is always possible to fulfill this condition for any \( \beta > 0 \) because \( \sum a^+_n \) diverges. Then put \( a'_{m_1^++n} = a^-_n \) for \( n = 1, 2, \ldots, m_1^- \) where \( m_1^- \) is determined by the condition \( S'_{m_1^-+m_1^+-1} < \beta \leq S'_{m_1^-+m_1^+} \). The integer \( m_1^- \) always exists as \( \sum a^-_n = -\infty \). Next, positive terms \( a^+_n, n > m_1^+ \), are added until the value of \( S'_n \) exceeds \( \beta \), after that the negative terms \( a^-_n, n > m_1^- \), are added until the value of \( S'_n \) drops below \( \beta \) and so on to obtain a sequence \( S_n \) oscillating about \( \beta \): 

\[
\{a'_n\} = \{a^+_1, \ldots, a^+_1, a^-_1, \ldots, a^-_1, a^+_2, \ldots, a^+_2, a^-_2, \ldots, a^-_2, \ldots\}.
\]

The sequence of partial sums of \( \sum a'_n \) oscillates about \( \beta \) with the amplitude of oscillations decreasing to zero, which implies that \( S'_n \to \beta \) as \( n \to \infty \). Indeed, the procedure generates two sequences of integers \( m_k^\pm, k = 1, 2, \ldots \) which determine the rearrangement so that 

\[
m^\pm = m_1^\pm + m_2^\pm + \cdots + m_k^\pm, \quad m = m^+ + m^-,
\]

\[
S'_{m-k^+} - \beta < S'_{m-k^-} \quad \text{or} \quad 0 \leq S'_{m-k^-} - \beta < a^+_m
\]

\[
S'_m - \beta \leq S'_{m-1} \quad \text{or} \quad 0 < \beta - S'_m \leq -a^-_m.
\]

Since \( |a^+_n| \to 0 \) monotonically as \( n \to \infty \), it is concluded that the values of \( S'_n \) overshoot \( \beta \) by no more than \( a^+_m \) and undershoot \( \beta \) by no less than \( |a^-_m| \) for all \( n > m \). In other words, the amplitude of oscillations
of the sequence $S'_{n}$ is monotonically decreasing to 0 with increasing the number of steps $k$ because $m^{\pm} \to \infty$ as $k \to \infty$ so that $|a_{m^{\pm}}| \to 0$ and, hence, $S'_{n} \to \beta$ as $n \to \infty$. The reader is advised to carry out this procedure explicitly for the alternating harmonic series $\sum(-1)^{n+1}/n$.

2.11. Double series. Consider an infinite table of numbers (real or complex)

\[
\begin{array}{cccc}
  u_{11} & u_{12} & u_{13} & \cdots \\
  u_{21} & u_{22} & u_{23} & \cdots \\
  u_{31} & u_{32} & u_{33} & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

The collection of elements $u_{mn}$ is called a double sequence. The formal sum of all elements of the sequence $\{u_{mn}\}$ is called a double series and denoted as

\[
\sum_{m,n=1}^{\infty} u_{mn}.
\]

**Definition 2.2. (the sum of a double series)**

A double series is said to converge to a number $S$ if for any $\varepsilon > 0$ one can find two integers $N$ and $M$ such that for all $n > N$ and $m > M$

\[|S - S_{mn}| < \varepsilon, \quad S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij}.
\]

The number $S$ is called the sum of the double series.

The sequence $\{S_{mn}\}$ is also called a sequence of partial sums. A double series is said to converge absolutely if the series of absolute values converges, $\sum_{n,m} |u_{nm}| < \infty$.

**Theorem 2.10. (Stolz)** A double series converges if and only if for any $\varepsilon > 0$ there exist two integers $N$ and $M$ such that

\[|S_{M+m,N+n} - S_{MN}| < \varepsilon
\]

for all $n, m$.

Stolz’ theorem is nothing but the analog of the Cauchy criterion for sequences. Its advantage is that the convergence can be established without any knowledge about the value of the sum. Clearly, the terms of a convergent double series must tend to zero as $n \to \infty$ and $m \to \infty$. This comprises the necessary condition for convergence:

\[
\sum_{m,n=1}^{\infty} u_{mn} = S \quad \Rightarrow \quad \lim_{(m,n) \to \infty} u_{mn} = 0
\]
2. SERIES

Put

\[ S_m = \sum_{n=1}^{\infty} u_{mn}, \quad S = \sum_{m=1}^{\infty} S_m, \]
\[ \tilde{S}_n = \sum_{m=1}^{\infty} u_{mn}, \quad \tilde{S} = \sum_{n=1}^{\infty} \tilde{S}_n. \]

If \( S \) and \( \tilde{S} \) exist, then they are called the sum of the double series by rows and by columns, respectively. In general,

\[ S \neq \tilde{S} \quad \text{or} \quad \sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} u_{mn} \right) \neq \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} u_{mn} \right) \]

For example, if

\[ u_{mn} = \frac{m - n (m + n - 1)!}{2^{m+n} m! n!}, \quad n, m > 0; \]
\[ u_{m0} = 2^{-m}; \quad u_{0n} = -2^{-n}; \quad u_{00} = 0. \]

Then

\[ \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} u_{mn} \right) = -1, \quad \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} u_{mn} \right) = 1. \]

**Theorem 2.11.** (Pringsheim)

If a double series converges to \( S \) and the sums by rows and columns exist, then they are equal to \( S \).

**Theorem 2.12.** (Cauchy)

Suppose that series \( \sum a_n = A \) and \( \sum b_m = B \) converge absolutely. Then the double series

\[ \sum a_n b_m = a_1 b_1 + a_2 b_1 + a_1 b_2 + \cdots, \]

consisting of the products \( a_n b_m \) written in any order, converges absolutely and its sum is equal to \( AB \).

For example, if

\[ \sum a_n = 1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots, \quad \sum b_n = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \]

Then the series \( \sum a_n b_m \) represented as a series over powers of \( z \) converges absolutely for all \( 1 \leq |z| \leq 2 \).
2.12. Power series. Let \( \{a_n\}_{0}^{\infty} \) be a sequence of complex numbers. The series
\[
\sum_{n=0}^{\infty} a_n z^n
\]
is called a power series. Put
\[
R = \frac{1}{\alpha}, \quad \alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}
\]
The upper limit may take values in the extended system of real numbers. If \( \alpha = 0 \), then \( R = \infty \) and, if \( \alpha = \infty \), then \( R = 0 \). The quantity \( R \) is called the radius of convergence of the power series. This term is justified by the following property of \( R \).

**Theorem 2.13.** A power series \( \sum a_n z^n \) converges absolutely if \( |z| < R \) and diverges if \( |z| > R \).

Indeed, by the root test
\[
\limsup_{n \to \infty} \sqrt[n]{|a_n||z|^n} = |z| \limsup_{n \to \infty} \sqrt[n]{|a_n|} = |z||\alpha| = \frac{|z|}{R}
\]
the power series converges if \( |z|/R < 1 \) or \( |z| < R \) and diverges if \( |z| > R \). In the complex plane, a power series converges in the interior of the disk of radius \( R \) centered at the origin. For the boundary points \( |z| = R \), the series may converge or diverge. The radius of converges of the series
\[
\sum_{n=1}^{\infty} \frac{z^n}{n}
\]
is
\[
R = \left( \limsup_{n \to \infty} \sqrt[n]{n} \right)^{-1} = \left( \lim_{n \to \infty} \sqrt[n]{n} \right)^{-1} = 1.
\]
On the boundary of the unit disk \( z = e^{i\theta}, 0 \leq \theta \leq 2\pi \), the series was shown above to converge conditionally for all \( 0 < \theta < 2\pi \) (for all points of the boundary except \( z = 1 \)).

It follows from Corollary 2.1 that

**Corollary 2.4.** If
\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \alpha
\]
then the radius of convergence of the power series \( \sum a_n z^n \) is \( R = \frac{1}{\alpha} \), where \( R = 0 \) if \( \alpha = 0 \) and \( R = \infty \) if \( \alpha = 0 \).
2.13. **Infinite products.** Let a sequence \( \{1 + a_n\} \) have no zero terms. Consider a sequence of products

\[
p_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n), \quad n = 1, 2, \ldots
\]

If the limit

\[
\lim_{n \to \infty} p_n = p
\]

exists, then the number \( p \) is called the *value* of the infinite product

\[
(1 + a_1)(1 + a_2)(1 + a_3) \cdots = \prod_{n=1}^{\infty} (1 + a_n)
\]

and, in this case, the infinite product is said to converge, otherwise the product is said to diverge. It follows from the relation \( p_n = (1 + a_n)p_{n-1} \) that

\[
p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} (1 + a_n)p_{n-1} = (1 + \lim_{n \to \infty} a_n)p,
\]

that is, if the infinite product converges, then

\[
\lim_{n \to \infty} a_n = 0,
\]

which, therefore, comprises a *necessary condition* for convergence.

An analysis of convergence of an infinite product can be reduced to analyzing convergence of a series:

\[
p_n = \exp \left[ \sum_{k=1}^{n} \ln(1 + a_k) \right] \equiv \exp S_n
\]

so that from continuity of the exponential function it follows that

\[
p = \lim_{n \to \infty} p_n = \exp \left[ \lim_{n \to \infty} S_n \right]
\]

provided the limit of the sequence of partial sums \( S_n \) exists. Thus, the existence of the limit of \( \{S_n\} \) is a *sufficient condition* for the value \( p \) of the infinite product to exist. The infinite product \( \prod (1 + a_n) \) is said to *converge absolutely* if the series \( \sum \ln(1 + a_n) \) converges absolutely.

**Theorem 2.14. (Absolute convergence of infinite products)**

*In order for an infinite product \( \prod (1 + a_n) \) to converge absolutely, it is necessary and sufficient that the series \( \sum a_n \) converges absolutely.*

One has to prove that the series \( \sum \ln(1 + a_n) \) converges absolutely if and only if the series \( \sum a_n \) converges absolutely. This objective is achieved by means of the comparison test. In other words, the convergence of \( \sum |a_n| \) implies the convergence of \( \sum |\ln(1 + a_n)| \) and the divergence of \( \sum |a_n| \) implies the divergence of \( \sum |\ln(1 + a_n)| \). First,
note that \( a_n \to 0 \) as \( n \to 0 \) means that there are only finitely many terms \( |a_n| \) that are greater than any \( \varepsilon > 0 \). In particular,

\[
|a_n| < \frac{1}{2}, \quad \forall n > N
\]

for some integer \( N \). Using the Taylor expansion of the log function for \( n > N \):

\[
\left| \frac{\ln(1 + a_n)}{a_n} - 1 \right| = \left| -\frac{a_2}{2} + \frac{a_n^2}{3} - \frac{a_n^3}{4} + \cdots \right|
\leq \frac{1}{2} |a_n| + \frac{1}{3} |a_n|^2 + \frac{1}{4} |a_n|^3 + \cdots
\leq \frac{1}{2} \left( |a_n| + |a_n|^2 + |a_n|^3 + \cdots \right)
\leq \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) = \frac{1}{2}
\]

This inequality can rewritten in the form

\[
\frac{1}{2} |a_n| \leq |\ln(1 + a_n)| \leq \frac{3}{2} |a_n|
\]

By the comparison test it follows from the first inequality that the divergence of \( \sum |a_n| \) implies the divergence of \( \sum |\ln(1 + a_n)| \), and, from the second inequality, that the convergence of \( \sum |a_n| \) implies the convergence of \( \sum |\ln(a + a_n)| \), as required.

2.14. Rearrangements of terms in infinite products. If \( \{a'_n\} \) is a rearrangement of \( \{a_n\} \), then the product \( \prod a'_n \) is called a rearrangement of the product \( \prod a_n \). If the series \( \sum a_n \) converges absolutely, then the series \( \sum \ln(1 + a_n) \) converges absolutely. Therefore the sequences of partial sums of the series \( \sum \ln(1 + a_n) \) and its rearrangement \( \sum \ln(1 + a'_n) \) converge to the same number. In turn, this implies that the value of the product \( \prod(1 + a_n) \) does not depend on the order of terms in the product, provided \( \sum a_n \) converges absolutely.

Consider three infinite products:

\[
\left( 1 - \frac{z^2}{\pi^2} \right) \left( 1 - \frac{z^2}{2^2 \pi^2} \right) \left( 1 - \frac{z^2}{3^2 \pi^2} \right) \cdots
\]
\[
\left( 1 - \frac{z}{\pi} \right) \left( 1 + \frac{z}{\pi} \right) \left( 1 - \frac{z}{2 \pi} \right) \left( 1 + \frac{z}{2 \pi} \right) \cdots
\]
\[
\left[ \left( 1 - \frac{z}{\pi} \right) e^{\frac{z}{\pi}} \right] \left[ \left( 1 + \frac{z}{\pi} \right) e^{-\frac{z}{\pi}} \right] \left[ \left( 1 - \frac{z}{2 \pi} \right) e^{\frac{z}{2 \pi}} \right] \left[ \left( 1 + \frac{z}{2 \pi} \right) e^{-\frac{z}{2 \pi}} \right] \cdots
\]
At a glance they seem identical because

\[ 1 - \frac{z^2}{n^2 \pi^2} = \left(1 - \frac{z}{\pi n}\right) \left(1 + \frac{z}{\pi n}\right) = \left(1 - \frac{z}{\pi n}\right) e^{\frac{z}{\pi n}} \left(1 + \frac{z}{\pi n}\right) e^{-\frac{z}{\pi n}} \]

However, the first product converges absolutely because the series

\[ \sum_{n=1}^{\infty} \frac{z^2}{n^2 \pi^2} = \frac{z^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \]

converges absolutely, while the second product converges only conditionally because the series

\[ -\frac{z}{\pi} + \frac{z}{2\pi} - \frac{z}{2\pi} + \cdots \]

does not converge absolutely (because \( \sum \frac{1}{n} = \infty \)). Therefore the first two products are equal if the order of terms in the second product is as given, and a rearrangement of it can have a different value or even diverge. The third product also converges absolutely because its \((2m - 1)\)th and \(m\)th terms of the corresponding sequence \(\{a_n\}\) have the form

\[ \left(1 \mp \frac{z}{\pi m}\right) e^{\pm \frac{z}{m \pi}} - 1 = \left(1 \mp \frac{z}{\pi m}\right) \left(1 \pm \frac{z}{\pi m} + O\left(\frac{1}{m^2}\right)\right) - 1 \]

\[ = O\left(\frac{1}{m^2}\right) \]

and the series \(\sum \frac{1}{m^2}\) converges. So, for any rearrangements the first and third products have the same value as they converge absolutely.

A similar analysis show that the product

\[ \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{c + n}\right) e^{\frac{z}{n}}\right] \]

converges absolutely for all complex \(z\) for any \(c\) that is not equal to a negative integer, while the product

\[ \prod_{n=2}^{\infty} \left[1 - \left(1 - \frac{1}{n}\right)^{-n} \frac{1}{z^n}\right] \]

converges absolutely for all \(|z| > 1\).

### 2.15. Evaluation of infinite products

In general, an evaluation of infinite products is a difficult task. The following theorem gives an expansion of analytic functions into infinite products.
Theorem 2.15. (Weierstrass)
Let an analytic function \( f(z) \) of a complex variable \( z \) have simple zeros that form a sequence \( \{a_n\}_1^\infty \) of non-zero elements such that \( |a_n| \to \infty \) as \( n \to \infty \). Then

\[
f(z) = f(0) \exp \left( \frac{f'(0)}{f(0)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n} \right) e^{\frac{z}{a_n}} \right)
\]

where the product converges absolutely in any disk \( |z| < K \).

For example, the function \( \frac{\sin(z)}{z} \) is analytic in the entire complex plane (the corresponding power series has infinite radius of convergence) and has simple zeros at \( z = \pi n \), where \( n \) is a non-zero integer, which can be arranged into a sequence \( \{a_n\} = \{\pi, -\pi, 2\pi, -2\pi, \ldots\} \) so that \( |a_n| \to \infty \) as \( n \to \infty \). It follows from the Weierstrass theorem that the third infinite product considered in the previous section converges absolutely to \( \frac{\sin(z)}{z} \) and, hence, so does the first one. The second product converges to \( \frac{\sin(z)}{z} \) in the given order of terms.

3. Sequences and series of functions

In what follows, it is assumed that \( x \in \mathbb{R}^N \), and \( x \to a \) means that the Euclidean distance between \( x \) and \( a \) tends to zero, \( |x - a| \to 0 \).

3.1. Pointwise convergence. Consider a sequence of functions \( u_n(x) \) (real or complex valued), \( n = 1, 2, \ldots \). The sequence \( \{u_n\} \) is said to converge pointwise to a function \( u \) on a set \( D \) if

\[
\lim_{n \to \infty} u_n(x) = u(x), \quad \forall x \in D.
\]

Similarly, the series \( \sum u_n(x) \) is said to converge pointwise to a function \( u(x) \) on \( D \subset \mathbb{R}^N \) if the sequence of partial sums converges pointwise to \( u(x) \):

\[
S_n(x) = \sum_{k=1}^{n} u_k(x), \quad \lim_{n \to \infty} S_n(x) = u(x), \quad \forall x \in D.
\]

The most important question: does the limit function (or the sum) \( u(x) \) inherit some properties of the terms of the sequence (or the series)? In particular,
• If all terms \( u_n \) are continuous functions, is the limit function or the sum \( u(x) \) continuous so that
\[
\lim_{x \to a} \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} \lim_{x \to a} u_n(x),
\]
\[
\lim_{x \to a} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to a} u_n(x) ?
\]

• If all terms \( u_n \) are differentiable functions, is the limit function or the sum \( u(x) \) differentiable, and, if affirmative, are the equations
\[
\frac{d}{dx} \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} \frac{d}{dx} u_n(x),
\]
\[
\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x),
\]
valid?

• If all terms \( u_n \) are (Riemann) integrable on an interval \([a, b]\), is the limit function or the sum \( u(x) \) Riemann integrable so that
\[
\lim_{n \to \infty} \int_{a}^{b} u_n(x) \, dx = \int_{a}^{b} \left( \lim_{n \to \infty} u_n(x) \right) \, dx,
\]
\[
\sum_{n=1}^{\infty} \int_{a}^{b} u_n(x) \, dx = \int_{a}^{b} \left( \sum_{n=1}^{\infty} u_n(x) \right) \, dx?
\]
The answer to all these questions is negative in general. A sequence or series of smooth functions (with infinitely many derivatives) may converge to a function that is not continuous, or not differentiable, or not integrable so that the stated equations are generally false.

### 3.2. Examples of convergent functional sequences and series.

Put
\[
u_n(x) = \frac{x^2}{(1 + x^2)^n}, \quad x \in \mathbb{R}, \quad n = 0, 1, 2, \ldots
\]
and consider the series \( \sum u_n(x) \). The sequence of partial sums is easy to find using the geometric sum:
\[
S_n(x) = x^2 \sum_{k=0}^{n-1} \left( \frac{1}{1 + x^2} \right)^k = x^2 + 1 - \frac{1}{(1 + x^2)^{n-1}}.
\]
Therefore
\[
\lim_{n \to \infty} S_n(x) = u(x) = \begin{cases} 
0 & x = 0 \\
1 + x^2 & x \neq 0
\end{cases}
\]
The terms $u_n(x)$ are rational functions defined on $\mathbb{R}$ and, hence, differentiable any number of times, whereas the sum is not even continuous at $x = 0$.

Consider the function $u(x)$, $x \in \mathbb{R}$, that is defined as the limit of the double sequence:

$$u_m(x) = \lim_{n \to \infty} u_{nm}(x) = \lim_{n \to \infty} [\cos(\pi x m!)]^{2n},$$

$$u(x) = \lim_{m \to \infty} u_m(x).$$

If $x = p/q$ is a rational number, then $\cos^2(\pi x m!) = 1$ for $m \geq q$ so that $u_m(x) = 1$ and $u(x) = 1$. If $x$ is irrational, then the number $xm!$ cannot be an integer for any $m$ so that $\cos^2(\pi x m!) < 1$ for any $m$ so that $u_m(x) = 0$ and, hence, $u(x)$. Thus, the limit function is the Dirichlet function:

$$u(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ denotes the set of all rational numbers. The terms $u_{nm}(x)$ are smooth functions, whereas the limit function $u(x)$ is continuous nowhere and not Riemann integrable on any interval. Consider a uniform partition of an interval $[a, b]$: $x_k = a + k\Delta x, k = 0, 1, 2, \ldots, N$, $\Delta x = (b - a)/N$. Put $M_k = \sup_{I_k} u(x)$ and $m_k = \inf_{I_k} u(x)$, where $I_k = [x_{k-1}, x_k]$. Recall that $u(x)$ is Riemann integrable on $[a, b]$ if the limits of the upper and lower sums,

$$U_N = \sum_{k=1}^{N} M_k \Delta x, \quad L_N = \sum_{k=1}^{N} m_k \Delta x$$

exist and are equal:

$$\lim_{N \to \infty} U_N = \int_{a}^{b} u(x) dx = \lim_{N \to \infty} L_N$$

Any partition interval $I_k$ contains rational and irrational numbers so that $M_k = 1$ and $m_k = 0$ for the Dirichlet function. Therefore $U_N = b - a$ and $L_N = 0$, that is, the limits of the upper and lower sums exist but are not equal, which means that the Dirichlet function is not Riemann integrable.

Even if the limit function $u(x)$ of a sequence $\{u_n\}$ happens to be differentiable, the sequence $\{u'_n\}$ of the derivatives does not necessarily converge to the derivative $u'$ (the operations of $\lim$ (or $\sum$) and $d/dx$ are not commutative). Consider a sequence $u_n(x) = \sin(nx)/\sqrt{n}$. It converges pointwise to $u(x) = 0$ for all $x \in \mathbb{R}$. A constant function is
differentiable everywhere, \( u'(x) = 0 \). However, the limit of the sequence of derivatives
\[
u_n'(x) = \sqrt{n} \cos(nx)
\]
does not exist and, hence, \( \lim_{n \to \infty} u_n'(x) \neq u'(x) \).

The next important question is: What are conditions under which the limit function or the sum of a series inherits some properties of the terms of a sequence or series? In particular, under what conditions can the order of operations \( \lim \) (or \( \sum \)) and \( d/dx \) (and/or \( \int_a^b \)) be changed?

### 3.3. Uniform convergence.

**Definition 3.3. (Uniform convergence)**

A sequence of functions \( \{u_n\} \) is said to converge uniformly to a function \( u \) on a set \( D \subset \mathbb{R}^N \) if
\[
\lim_{n \to \infty} \sup_{D} |u_n(x) - u(x)| = 0
\]

Similarly, a series \( \sum u_n(x) \) converges uniformly to \( u(x) \) on \( D \) if the sequence of partial sums converges uniformly to \( u(x) \) on \( D \).

For example, the sequence \( u_n(x) = \sin(nx)/\sqrt{n} \) converges uniformly to \( u(x) = 0 \) on \( \mathbb{R} \) because
\[
|u_n(x) - u(x)| = \frac{\left| \sin(nx) \right|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \to 0
\]
as \( n \to \infty \).

The series \( \sum x^2/(1 + x^2)^n \) does not converge uniformly on any interval whose closure contains \( x = 0 \). Indeed, as shown in the previous section
\[
|S_n(x) - u(x)| = \frac{1}{(1 + x^2)^{n+1}} \Rightarrow \sup_I |S_n(x) - u(x)| = 1
\]
\[
\Rightarrow \lim_{n \to \infty} \sup_I |S_n(x) - u(x)| = 1 \neq 0
\]
for any interval (closed or open) \( I \) for which \( x = 0 \) is an endpoint. However, the series converges uniformly for \( 0 < a \leq x < \infty \) or \( -\infty < x \leq a < 0 \) because
\[
\lim_{n \to \infty} \sup_I |S_n(x) - u(x)| = \lim_{n \to \infty} \frac{1}{(1 + a^2)^{n+1}} = 0, \quad a \neq 0.
\]

**Definition 3.4.** A series
\[
\sum_{n=1}^{\infty} u_n(x) = u(x)
\]
is said to converge uniformly to \( u(x) \) on a set \( D \) if the sequence of partial sums of the series converges uniformly to \( u(x) \) on \( D \).
Theorem 3.16. (Sufficient conditions for uniform convergence)
Suppose that \(|u_n(x)| \leq M_n\) for all \(x \in D\) and \(n = 1, 2, \ldots\). If the series of the upper bounds \(\sum M_n < \infty\) converges, then the series \(\sum u_n(x)\) converges uniformly on \(D\).

For example, a power series \(\sum_{n=0}^{\infty} a_n z^n\) converges uniformly on any disk \(|z| \leq b < R\), where \(R\) is the radius of convergence of the power series. Indeed, put \(u_n(z) = a_n z^n\). Then
\[
|u_n(z)| = |a_n| |z|^n \leq |a_n| b^n = M_n, \quad \forall |z| \leq b.
\]
The series of upper bounds converges \(\sum M_n < \infty\) by the root test:
\[
\limsup_{n \to \infty} \sqrt[n]{M_n} = b \limsup_{n \to \infty} \sqrt[n]{|a_n|} = \frac{b}{R} < 1
\]
where the definition of the radius of convergence has been used. Therefore the power series converges uniformly in any disk of radius less than the radius of convergence.

Theorem 3.17. (Cauchy criterion for uniform convergence)
A sequence of functions \(\{u_n\}\) converges to a function \(u\) uniformly on a set \(D\) if and only if for any \(\varepsilon > 0\) one can find an integer \(N\) such that \(n, m > N\) implies that
\[
|u_n(x) - u_m(x)| \leq \varepsilon \quad \forall x \in D
\]
The difference with the ordinary Cauchy criterion for convergence is that the integer \(N\) here depends only on \(\varepsilon\), that is, for all \(x\) in \(D\) one can find the same \(N\), given \(\varepsilon > 0\).

3.4. Uniform convergence and continuity.

Theorem 3.18. Suppose that a sequence \(\{u_n(x)\}\) converges uniformly to \(u(x)\) on a set \(D \subset \mathbb{R}^N\). Let \(y\) be a limit point of \(D\) so that
\[
\lim_{x \to y} u_n(x) = A_n, \quad n = 1, 2, \ldots.
\]
Then the sequence of limits \(\{A_n\}\) converges and
\[
\lim_{n \to \infty} A_n = \lim_{x \to y} u_n(x)
\]
If \(y \in D\) so that \(u_n(y)\) is defined, then the stated theorem implies that the limit of a uniformly convergent sequence of continuous functions is a continuous function:
\[
\lim_{x \to y} \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} u_n(y).
\]
For example, the function defined by the Fourier series
\[ u(x) = \sum_{n=0}^{\infty} \frac{e^{inx}}{1 + n^2} \]
is continuous everywhere. Indeed,
\[ u_n(x) = \frac{e^{inx}}{n^2} \Rightarrow |u_n(x)| = \frac{1}{1 + n^2} \leq \frac{1}{n^2}, \quad n \geq 1 \]
and the series of upper bounds converges \( \sum \frac{1}{n^2} < \infty \). Therefore the series \( \sum u_n(x) \) converges uniformly for all \( x \). Since \( u_n(x) \) are continuous everywhere, the sum of the Fourier series is also a continuous function.

### 3.5. Uniform convergence and differentiation.

**Theorem 3.19.** Let \( \{u_n\} \) be a sequence of differentiable functions on \( [a, b] \). Suppose that the sequence \( \{u_n(x_0)\} \) converges for some \( x_0 \in [a, b] \) and the sequence of derivatives \( \{u'_n(x)\} \) converges uniformly on \([a, b]\). Then the sequence \( \{u_n\} \) converges uniformly to some \( u(x) \) on \([a, b]\), the limit function \( u(x) \) is differentiable on \([a, b]\) and
\[ u'(x) = \left( \lim_{n \to \infty} u_n(x) \right)' = \lim_{n \to \infty} u'_n(x). \]

For example, the sum of the series
\[ u(x) = \sum_{n=1}^{\infty} \frac{\sin^2(nx)}{2 + n^3} \]
is a differentiable function everywhere. Indeed, each term of the series is differentiable everywhere. If \( x = 0 \), then the sum is \( u(0) = 0 \). The series of the derivatives converges uniformly everywhere because the series of upper bounds
\[ |u'_n(x)| = \frac{n|\sin(2nx)|}{2 + n^3} \leq \frac{n}{2 + n^3} \leq \frac{1}{n^2} \]
converges \( \sum \frac{1}{n^2} < \infty \). Therefore
\[ u'(x) = \sum_{n=1}^{\infty} \frac{n \sin(2nx)}{2 + n^3}. \]

The sum of a power series is infinitely many times differentiable function in the disk \(|z| < R\) where \( R \) is the radius of convergence. Indeed, the series of derivatives converges uniformly in any disk \(|z| \leq b < B\) because the corresponding series of upper bounds
\[ |u'_n(z)| = n|a_n||z|^{n-1} \leq n|a_n|b^{n-1} = M_n \]
converges, $\sum M_n < \infty$, by the root test:

$$\limsup_{n \to \infty} \sqrt[n]{M_n} = b \limsup_{n \to \infty} \sqrt[n]{|a_n|} \cdot \lim_{n \to \infty} \sqrt[n]{|1/b|} = \frac{b}{R} < 1.$$ Therefore

$$u'(z) = \sum_{n=0}^{\infty} u'_n(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$$

Similarly, the series of the $k$th derivatives

$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k) a_n z^{n-k}$$

converges uniformly in $|z| \leq b < R$ by the same reason. Therefore it converges to the $k$th derivative of the sum $u^{(k)}(z)$. The conclusion holds for any $k$. So, $u(z)$ is differentiable infinitely many times and any derivative is obtained by term-by-term differentiation of the power series.

A function defined as limit of a sequence or the sum of series may be continuous everywhere but nowhere differentiable! Let $f(x) = |x|$ for $|x| \leq 1$. Let us extend $f$ to $\mathbb{R}$ by periodicity:

$$f(x + 2) = f(x)$$

Evidently, $f(x)$ is continuous everywhere and bounded:

$$|f(x)| \leq 1, \quad \forall x \in \mathbb{R}.$$ Define the function $u(x)$ on $\mathbb{R}$ by the series

$$u(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n f(4^n x).$$

This series of continuous terms converges uniformly on $\mathbb{R}$ because the series of upper bounds converges:

$$\left|\left(\frac{3}{4}\right)^n f(4^n x)\right| \leq \left(\frac{3}{4}\right)^n \Rightarrow \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 4 < \infty$$

as a geometric series. Therefore the sum $u(x)$ is continuous on $\mathbb{R}$. However $u(x)$ is differentiable nowhere. By definition

$$u'(x) = \lim_{\delta \to 0} \frac{u(x + \delta) - u(x)}{\delta}$$

This limit does not exist for any $x$. If this limit existed, then for any sequence $\delta_m$ converging to 0 as $m \to \infty$, the limit

$$\lim_{m \to \infty} \frac{u(x + \delta_m) - u(x)}{\delta_m}$$
must exist and coincide with \( u'(x) \). Put \( \delta_m = \pm \frac{1}{2} 4^{-m} \) where the sign is chosen so that no integer between lies between \( 4^m x \) and \( 4^m (x + \delta_m) = 4^m x \pm \frac{1}{2} \). Clearly, \( \delta_m \to 0 \) as \( m \to \infty \). By the choice of \( \delta_m \):

\[
|f(4^m (x + \delta_m)) - f(4^m x)| = |f(4^m x \pm \frac{1}{2}) - f(4^m x)| = |\pm \frac{1}{2}| = \frac{1}{2}
\]

because the function \( f \) has a constant slope, 1 or \(-1\), between any two integers, and no integer lies between \( 4^m x \) and \( 4^m (x + \delta_m) \). Put

\[
\gamma_n = \frac{f(4^n (x + \delta_m)) - f(4^n x)}{\delta_m}
\]

Then it follows from the above property of \( f \) that

\[
|\gamma_n| = \frac{\frac{1}{2}}{|\delta_m|} = 4^m.
\]

If \( n > m \), the number \( 4^n \delta_m \) is even, and by periodicity of \( f \)

\[
\gamma_n = 0, \hspace{1em} n > m
\]

Then

\[
\frac{u(x + \delta_m) - u(x)}{\delta_m} = \frac{1}{\delta_m} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n f(4^n (x + \delta_m)) - \frac{1}{\delta_m} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n f(4^n x)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n \frac{f(4^n (x + \delta_m)) - f(4^n x)}{\delta_m}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n \gamma_n = \sum_{n=0}^{m} \left( \frac{3}{4} \right)^n \gamma_n = 3^m - \sum_{n=1}^{m-1} \left( \frac{3}{4} \right)^n \gamma_n.
\]

This sequence diverges as \( m \to \infty \). Indeed, using the inequality

\[
|A - B| \geq |A| - |B|
\]

\[
\left| 3^m - \sum_{n=0}^{m-1} \left( \frac{3}{4} \right)^n \gamma_n \right| \geq \left| 3^m - \sum_{n=0}^{m-1} \left( \frac{3}{4} \right)^n |\gamma_n| \right|
\]

\[
\geq 3^m - \sum_{n=0}^{m-1} 3^m = \frac{1}{2} (3^m - 1) \to \infty
\]

as \( m \to \infty \). So, the lower bound diverges as \( m \to \infty \) and this implies that the limit that defines the derivative \( u'(x) \) does not exist. Thus, \( u'(x) \) does not exist for any \( x \in \mathbb{R} \).
3.6. Uniform convergence and Riemann integration.

**Theorem 3.20.** Let \( \{u_n\} \) be a sequence of functions that are Riemann integrable on \([a, b]\). If the sequence converges to \(u(x)\) uniformly on \([a, b]\), then the limit function \(u(x)\) is Riemann integrable on \([a, b]\) and

\[
\int_a^b u(x) \, dx = \lim_{n \to \infty} \int_a^b u_n(x) \, dx.
\]

In particular, a function defined by a convergent power series can be integrated term-by-term and the resulting series converges to an integral of the sum in any disk whose radius is less than the radius of convergence of the original series.