15. Properties of the Lebesgue integral

Properties of the Lebesgue integral are analogous to the properties of the Riemann integral because the Lebesgue integral coincides with the absolutely convergent Riemann integral for continuous functions. In fact, one can prove that

*If a function is Riemann integrable, then it is also Lebesgue integrable and the Riemann and Lebesgue integral coincide.*

The converse is not true as noted earlier. The differences between Riemann and Lebesgue integrals are related to the behavior of functions on sets of measure zero.

Recall that Riemann integrable functions are continuous almost everywhere. Let \( f \in C^0(\Omega) \). If the region \( \Omega \) is bounded and has a smooth boundary, then \( f \) is Riemann integrable. Suppose \( g \) coincides with \( f \) almost everywhere on \( \Omega \). In other words, \( g \) can be obtained from \( f \) by altering the values of \( f \) on a set of measure zero. Then \( g \) is Lebesgue integrable and

\[
\int_{\Omega} g \, dN = \mathcal{R} \int_{\Omega} f \, dN
\]

Indeed, if \( f(x) \geq 0 \), then a sequence \( f_n = \chi_{\Omega} f \) of identical piecewise continuous functions converges to \( f \) and satisfies the characteristic properties of the sequence of Riemann integrable functions in the definition of the Lebesgue integral. So, the Lebesgue and Riemann integrals of \( f \) coincide. But this sequence also converges to \( g \) almost everywhere because \( f = g \) a.e. Therefore the Lebesgue integral of \( g \) exists and coincides with the Riemann integral of \( f \). For a general \( f \), the above argument can be applied to the positive and negative parts of \( f \): \( f_{\pm} = \frac{1}{2}(|f| - f) \geq 0 \) which are continuous and, hence, Riemann integrable, so that \( g_{\pm} \in \mathcal{L}^+(\Omega) \) and the Lebesgue integrals of \( g_{\pm} \) coincide with the Riemann integrals of \( f_{\pm} \). Therefore the Lebesgue integral of \( g \) is equal to the Riemann integral of \( f \). Thus, the value of the Lebesgue integral of a continuous function does not change upon altering the function on a set of measure zero.

The function \( g \) may or may not be continuous almost everywhere. For example, let \( f \in C^0(\mathbb{R}) \) and \( g(x) = f(x) \) if \( x \) is not an integer, and \( g(x) \neq f(x) \) if \( x = 0, \pm 1, \pm 2, \ldots \). So, the function \( f \) was altered on a set of measure zero, but the resulting function \( g \) is continuous almost everywhere and, hence, Riemann integrable. The Riemann integrals of \( f \) and \( g \) are equal. However, the Riemann integrability can also be lost upon alterations of values of a continuous function on a set of
measure zero. An example is provided by the Dirichlet function that is obtained from the zero function by alterations of its values at rational values of the argument. The Dirichlet function is nowhere continuous and, hence, cannot be Riemann integrable. More generally, alterations of values of a continuous function on a set measure zero can make the function non-continuous on a set of non-zero measure (or volume) and, hence, non-integrable in the Riemann sense. This is the reason why the set of Lebesgue integrable functions is wider than the set of Riemann integrable functions.

The noted insensitivity of the Lebesgue integral to values of the function on sets of measure zero is the key difference between the Lebesgue and Riemann integrals and it also leads to simplifications of theorems about integrability of the limit function of a functional sequence. In particular, the hypotheses of the uniform convergence can be weakened and simplified.

In what follows, a Lebesgue integrable function will be called just integrable, unless stated otherwise. Proofs of the properties stated below can be found, e.g., in

15.1. The set $\mathcal{L}$ is a linear space. If $f$ and $g$ are integrable, then their linear combination is also integrable and

$$\int [c_1 f(x) + c_2 g(x)] \, d^N x = c_1 \int f(x) \, d^N x + c_2 \int g(x) \, d^N x.$$ 

So, the set $\mathcal{L}(\Omega)$ of Lebesgue integrable functions on $\Omega \subset \mathbb{R}^N$ is a linear space. Recall that the Riemann integral has the same property. This property follows from the limit laws. If $\{f_n\}$ and $\{g_n\}$ are sequences of piecewise continuous functions that define the integrals of $f$ and $g$, then by linearity of the Riemann integral the sequence $c_1 f_n + c_2 g_n$ defines the integral of the linear combination $c_1 f + c_2 g$.

15.2. Monotonicity. Suppose that $f$ and $g$ are integrable. Then

$$f(x) \geq 0 \quad \Rightarrow \quad \int f(x) \, d^N x \geq 0$$

and, as a consequence,

$$f(x) \geq g(x) \quad \Rightarrow \quad \int f(x) \, d^N x \geq \int g(x) \, d^N x$$

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$^4$F. Riesz and B. Sz.-Nagy, Functional Analysis
15.3. Lebesgue measure of a set. If the characteristic function of a set \( \Omega \) is integrable, then the number

\[
\int \chi_\Omega(x) d^N x = \mu(\Omega)
\]

is called the Lebesgue measure of \( \Omega \). If \( \Omega \) is a region with a smooth boundary, then \( \mu(\Omega) \) is the volume of \( \Omega \) because \( \chi_\Omega \) is continuous and, hence, Riemann integrable

\[
\mu(\Omega) = \mathcal{R} \int_\Omega d^N x
\]

Since a set of measure zero can be covered by a union of balls with an arbitrary small total volume, the Lebesgue measure of such a set is zero. The converse is also proved to be true. If \( \mu(\Omega) = 0 \), then \( \Omega \) is a set of measure zero. Thus, a set of measure zero can also be defined as a set with zero Lebesgue measure.

The Lebesgue measure of any region (an open connected set) is not zero because any such region contains a ball of non-zero radius. Since the Lebesgue integral of a continuous function does not change its value when values of the function are changed on a set of measure zero,

\[
\mu(\Omega \setminus \Omega') = \mu(\Omega) \quad \text{if} \quad \mu(\Omega') = 0
\]

In other words, the Lebesgue measure of \( \Omega \) does not change when a set of measure zero is removed from \( \Omega \). In particular, for any region \( \Omega \) with piecewise smooth boundaries

\[
\mu(\overline{\Omega}) = \mu(\Omega)
\]

Furthermore, it follows from the positivity property that, if \( \Omega \) is a bounded (and measurable) set, then the integral

\[
\int_\Omega d^N x = \mu(\Omega) < \infty
\]

is always finite. Note that any bounded set \( \Omega \) is a subset of a ball \( B \) of a sufficiently large radius so that

\[
\mu(\Omega) = \int \chi_\Omega d^N x \leq \int \chi_B d^N x = \mu(B) < \infty
\]

where \( \mu(B) \) is the volume of the ball.
15.4. Upper and lower bounds. Suppose that \( f \in L(\Omega) \) and \( f \) is bounded almost everywhere in \( \Omega \), then

\[
m \leq f(x) \leq M \quad \text{a.e.} \quad \Rightarrow \quad m\mu(\Omega) \leq \int_{\Omega} f(x) \, d^N x \leq M\mu(\Omega)
\]

A similar property also holds for the Riemann integral over an interval (without a.e.).

15.5. Integrals on sets of measure zero. The following property is a generalization of the property established for continuous functions:

\[
f(x) = g(x) \quad \text{a.e.} \quad \Rightarrow \quad \int f(x) \, d^N x = \int g(x) \, d^N x
\]

and both the integrals either exist or do not exist simultaneously. This property shows that the Lebesgue integral is insensitive to the behavior of a function on sets of measure zero. If \( f \in L \), then every function that coincides with \( f \) almost everywhere (or differs from \( f \) on a set of measure zero) is also integrable and its Lebesgue integral has the same value. Similarly, if \( f \) is not Lebesgue integrable, then any other function that differs from \( f \) on a set of measure zero is also non-integrable. This is one of the main advantages of the Lebesgue integration theory. As noted earlier, this property is not true for the Riemann integral.

In particular, if \( \mu(\Omega) = 0 \), then

\[
\chi_\Omega(x) f(x) = 0 \quad \text{a.e.} \quad \Rightarrow \quad \int_{\Omega} f(x) \, d^N x = \int_{\Omega} \chi_\Omega(x) f(x) \, d^N x = 0
\]

So, the Lebesgue integral of a measurable function over a set measure zero vanishes. Furthermore, if \( \mu(\Omega \cap \Omega^\prime) = 0 \), then

\[
\int_{\Omega \cup \Omega^\prime} f(x) \, d^N x = \int_{\Omega} f(x) \, d^N x + \int_{\Omega^\prime} f(x) \, d^N x
\]

This is called the additivity of the Lebesgue integral.

15.6. Integrability of the absolute value. If \( f \in L \), then \( |f| \in L \). If \( f \) is measurable and \( |f| \in L \), then \( f \in L \) and

\[
\left| \int f(x) \, d^N x \right| \leq \int |f(x)| \, d^N x.
\]

In view of the early remark about non-measurable functions, the integrability of \( f \) and \( |f| \) is practically equivalent in the Lebesgue theory. So, if \( f \) is measurable, then the integrals

\[
\int f(x) \, dx \quad \text{and} \quad \int |f(x)| \, dx
\]
exist or do not exist simultaneously. *This property does not hold for the Riemann integral.* For example, let $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ otherwise so that $|f(x)| = 1$ for all $x$. Clearly, $|f(x)|$ is Riemann integrable on any bounded interval, while this is not so for $f(x)$. In the Lebesgue theory, $f(x) = -1$ a.e. (because $\mathbb{Q}$ has measure zero) and therefore it is Lebesgue integrable on any bounded interval.

Recall that it was proved earlier that the simultaneous integrability or non-integrability of $f$ and $|f|$ holds for the absolutely convergent Riemann integral of continuous functions. So, the property stated above is an extension from continuous to measurable functions in the Lebesgue integration theory.

**15.7. Vanishing integral of the absolute value.** Recall that, if $f$ is continuous and the Riemann integral of the absolute value $|f|$ vanishes, then $f(x) = 0$. The converse is obviously true. The Lebesgue integral has a similar property: if $f \in \mathcal{L}$ and the integral of $|f|$ vanishes, then $f(x) = 0$ almost everywhere (and the converse is obviously true):

$$f \in \mathcal{L}, \quad \int |f(x)|\,dx = 0 \quad \Leftrightarrow \quad f(x) = 0, \quad \text{a.e.}$$

**15.8. Sufficient condition for integrability.** If a function $g$ is integrable on $\Omega$ and $|f(x)| \leq g(x)$ a.e., then $f$ is also integrable on $\Omega$:

$$|f(x)| \leq g(x) \text{ a.e., } g \in \mathcal{L}(S) \quad \Rightarrow \quad f \in \mathcal{L}(\Omega)$$

This implies that *any bounded (and measurable) function is Lebesgue integrable on any bounded (and measurable) set.* Indeed,

$$|f(x)| \leq M \text{ a.e. } \Rightarrow \int_\Omega |f(x)|\,d^N x \leq M \mu(\Omega) < \infty$$

because $\Omega$ is bounded.

If $f$ is not bounded in a neighborhood of a point $x_0$, then $f$ is integrable on any bounded $\Omega$ if

$$|f(x)| \leq \frac{M}{|x-x_0|^p} \text{ a.e., } p < N,$$

because the Riemann integral of the right side of this inequality was shown to converge absolutely. Similarly, if $\Omega$ is not bounded, then a bounded $f$ is integrable on $\Omega$ if

$$|f(x)| \leq \frac{M}{|x|^p} \text{ a.e., } |x| > R, \quad p > N$$

for some $R$ and $M$. 
15.9. **Absolute continuity of the Lebesgue integral.** Let \( f \in L(\Omega) \). Then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\left| \int_{\Omega'} f(x) \, dx \right| < \varepsilon
\]

for any measurable subset \( \Omega' \subset \Omega \) such that \( \mu(\Omega') < \delta \). In other words, the value of the integral can be made arbitrarily small if the set of integration has a sufficiently small measure. This property is obvious for any \( f \) that is bounded almost everywhere on \( \Omega \) because

\[
\left| \int_{\Omega'} f(x) \, dx \right| \leq \int_{\Omega'} |f(x)| \, dx \leq M \mu(\Omega')
\]

if \( |f(x)| \leq M \) a.e.. In this case, \( \delta = \varepsilon/M \). Let \( \Omega \) be a region (an open connected set). A set \( \Omega' \) is called a *proper* subset of \( \Omega \) if its closure lies in \( \Omega \), \( \overline{\Omega'} \subset \Omega \). In particular, \( \Omega' \) is a proper subregion of a region \( \Omega \) if \( \Omega' \) and its boundary \( \partial \Omega' \) lie in \( \Omega \). By the absolute continuity of the Lebesgue integral, if \( f \in L(\Omega) \), then for any \( \varepsilon > 0 \) one can find a proper subregion \( \Omega' \subset \Omega \) such that

\[
\int_{\Omega \setminus \Omega'} |f(x)| \, d^N x < \varepsilon
\]

For example, if \( \Omega \) has a smooth boundary, \( \Omega' \) can be constructed by removing from \( \Omega \) all balls of sufficiently small radius whose centers are on the boundary \( \partial \Omega \).

In particular, if \( \{\Omega_n\} \) is an exhaustion of \( \Omega \), where \( \Omega_n \) are proper subsets of \( \Omega \), then by the continuity property of the Lebesgue integral

\[
\lim_{n \to \infty} \int_{\Omega_n} f(x) \, d^N x = \int_{\Omega} f(x) \, d^N x
\]

because \( \mu(\Omega \setminus \Omega_n) \to 0 \) as \( n \to \infty \). In what follows, the continuity property will often be used in this form.

15.10. **The Lebesgue dominated convergence theorem.** Let a sequence of (measurable) functions \( \{f_n\}_{n=1}^\infty \) converge to \( f \) a.e.,

\[
\lim_{n \to \infty} f_n(x) = f(x) \text{ a.e.}
\]

If there exists an integrable function \( g \) independent of \( n \) such that

\[
|f_n(x)| \leq g(x) \text{ a.e., \quad } g \in L,
\]

then \( f \in L \) and

\[
\lim_{n \to \infty} \int f_n(x) \, dx = \int \lim_{n \to \infty} f_n(x) \, dx = \int f(x) \, dx.
\]
To illustrate the use of the Lebesgue dominated convergence theorem, consider the following limit
\[
\lim_{n \to \infty} \int_0^\infty n e^{-n^p x^q} \, dx, \quad p > q > 0.
\]
Let us verify the hypotheses. The sequence \( f_n(x) = n e^{-n^p x^q} \) converges to zero for any \( x > 0 \) and the limit is infinite if \( x = 0 \). Therefore
\[
\lim_{n \to \infty} f_n(x) = f(x) = 0 \text{ a.e.}
\]
In order to change the order of taking the limit and the integral one has to find an integrable bound \( |f_n(x)| \leq g(x) \) where \( g \) is independent of \( n \). Such \( g(x) \) does exist (it is left to the reader as an exercise). However there is a simpler way. Since \( f_n \) are continuous and positive, their Lebesgue integrals are the improper Riemann (absolutely convergent) integrals. Using the change of variables, the problem can be converted to the following problem:
\[
\int_0^\infty n e^{-n^p x^q} \, dx = \lim_{b \to \infty} \int_0^b n e^{-n^p x^q} \, dx = \lim_{b \to \infty} \int_0^b e^{-n^p y^q} \, dy
\]
\[
= \int_0^\infty e^{-n^p y^q} \, dy
\]
Since
\[
\lim_{n \to \infty} e^{-n^p y^q} = 0, \quad x > 0, \quad p > q
\]
\[
e^{-n^p x^q} \leq e^{-x^q} \in \mathcal{L}(0, \infty), \quad n = 1, 2, ...
\]
by the Lebesgue dominated convergence theorem
\[
\lim_{n \to \infty} \int_0^\infty n e^{-n^p x^q} \, dx = \lim_{n \to \infty} \int_0^\infty e^{-n^p x^q} \, dx = 0
\]

15.11. Levi’s theorem. If the sequence is not bounded, then the integrability of the limit function can be established by means of Levi’s theorem: If an almost everywhere non-decreasing sequence of functions \( f_n \in \mathcal{L}, n = 1, 2, ... \), converges to a function \( f \) a.e., and the sequence of the integrals of \( f_n \) is bounded,
\[
\lim_{n \to \infty} f_n(x) = f(x) \text{ a.e.}
\]
\[
f_n(x) \leq f_{n+1}(x) \text{ a.e.}
\]
\[
\left| \int f_n(x) \, d^N x \right| \leq M, \quad \forall n
\]
then \( f \in \mathcal{L} \) and the relation (15.1) holds.
In this theorem the hypothesis of the boundedness of a sequence by an integrable function is replaced by the hypothesis of monotonicity of the sequence and boundedness of the sequence of integrals. The monotonicity hypothesis is essential. Consider the sequence

\[ f_n(x) = \frac{n}{1 + n^2 x^2}, \quad x \in \mathbb{R} \]

Since the functions are positive and continuous, their Lebesgue integrals are the absolutely convergent Riemann integrals:

\[ \int f_n(x) \, dx = \lim_{b \to \infty} \int_{-b}^{b} \frac{nx}{1 + n^2 x^2} = \lim_{b \to \infty} \int_{-bn}^{bn} \frac{dy}{1 + y^2} = \pi \]

So, the sequence of Riemann integrals is bounded (it is a constant sequence). It converges to \( \pi \). However the integral of the limit function is zero. Indeed, the sequence converges to zero if \( x \neq 0 \) and to infinity if \( x = 0 \). Therefore

\[ \lim_{n \to \infty} f_n(x) = 0 \quad a.e. \]

and

\[ \lim_{n \to \infty} \int f_n(x) \, dx = \pi \neq 0 = \int \lim_{n \to \infty} f_n(x) \, dx \]

By graphing \( f_n(x) \) is not difficult to see that the sequence is not monotonic: if \( n > m \), then \( f_n(x) > f_m(x) \) near \( x = 0 \) and \( f_n(x) < f_m(x) \) for all large enough \( |x| \).


1. Can the Lebesgue measure of an unbounded region be finite? If so, construct an example. *Hint:* Think of the area under the graph of a non-negative continuous function on \( \mathbb{R} \).

2. Are there any values of \( p \) for which the function

\[ f(x) = \frac{\sin^2(|x|)}{|x|^p}, \quad x \in \mathbb{R}^N \]

is integrable on

(i) a bounded set that contains \( x = 0 \);

(ii) \( \mathbb{R}^N \);

(iii) on the complement of a region containing \( x = 0 \)

3. Suppose that

\[ |f(x)| \leq \frac{M}{1 + |x|^p} \]
2. THE LEBESGUE INTEGRATION THEORY

For what values of $p$ does $f$ have a Fourier transform

$$\mathcal{F}[f](k) = \int e^{i(k,x)} f(x) \, d^N x, \quad k \in \mathbb{R}^N$$

4. Suppose $|f(x)| \leq M|x|^p$ a.e., where $p > 0$. For what values of $p$ is the function $e^{-|x|}f(x)$ is integrable on $\mathbb{R}^N$? Give an upper bound of the value of the integral.

5. Let $f_n(x) = \left(1 - \frac{x}{n}\right)^n$, $n = 1, 2, \ldots$

   (i) Show that $f_n(x)$ converges to $e^{-x}$ uniformly on $[0, 1]$, that is,

   $$\lim_{n \to \infty} \sup_{[0, 1]} |f_n(x) - e^{-x}| = 0$$

   Note that $f_n(x) - e^{-x}$ is continuous on $[0, 1]$ and, hence, attains its extreme values on $[0, 1]$. Find them and compute the limit. Conclude that

   $$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 e^{-x} \, dx$$

   (ii) Show that $|f_n(x)| \leq M$ for all $x \in [0, 1]$, where $M$ is some constant independent of $n$. Use the Lebesgue dominated convergence theorem to established the same result.

6. Let $\varphi \in C^1(\mathbb{R})$ and the support of $\varphi$ is bounded. Show that

   $$\lim_{n \to \infty} \int e^{inx} \varphi(x) \, dx = 0$$

   *Hint:* Use integration by parts in combination with the Lebesgue dominated convergence theorem (or with the theorem about the uniform convergence and integrability).

7. Let $f \in L(\mathbb{R})$ such that $\int f(x) \, dx = 1$ and $\varphi$ be a continuous function with bounded support. Put $f_n(x) = nf(nx)$, $n = 1, 2, \ldots$

   Show that

   $$\lim_{n \to \infty} \int f_n(x) \varphi(x) \, dx = \varphi(0)$$

   *Hint:* Use the Lebesgue dominated convergence theorem and that any continuous function with bounded support is bounded.