Distributions and Operators for Theoretical Physicists

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CHAPTER 1

Integration in Euclidean spaces

1. Riemann integral in Euclidean spaces

1.1. Euclidean spaces. Elements (vectors) of a real (or complex) Euclidean space $\mathbb{R}^N$ (or $\mathbb{C}^N$) are denoted by Roman letters, e.g., x, y, etc. They are ordered $N-$tuples of real (or complex) numbers, $x = (x_1, x_2, ..., x_N)$. Components of x are labeled by i or j. The inner (dot) product and the norm (length) are defined, respectively, by

$$(x, y) = \sum_{i=1}^{N} x_i \overline{y_i}\,, \quad |x| = \sqrt{(x, x)}$$

where $\bar{z}$ denotes the complex conjugation of z. Unless stated otherwise, Einstein's summation rule over repeated indices will be used throughout the text:

$$\sum_{j=1}^{N} \sum_{i=1}^{N} A_{ij} x_i y_j \stackrel{\text{def}}{=} A_{ij} x_i y_j$$

1.1.1. Cauchy sequences in $\mathbb{R}^N$. Indices n or m are used to label elements in a Euclidean space. In particular, a sequence of points is denoted by $\{x_n\}_1^\infty$ or simply $\{x_n\}$ (by default, the index enumerating elements of a point sequence ranges over all positive integers). A sequence $\{x_n\}$ is said to converge to x if

$$\lim_{n\to\infty} |x_n - x| = 0$$

and in this case, one also writes $x_n \to x$. It follows from the inequality $|x_{nj} - x_j| \le |x_n - x|$, where x_{nj} and x_j are the j^{th} components of vectors x_n and x, respectively, that $x_n \to x$ in $\mathbb{R}^N$ if and only if the sequences of components converge to the corresponding components of the limit point, $x_{nj} \to x_j$ in $\mathbb{R}$ for every $j = 1, 2, ..., N$.

A sequence $\{x_n\}_1^\infty$ is called a *Cauchy sequence* if

$$|x_n - x_m| \to 0 \quad \text{for} \quad n, m \to \infty$$

In other words, the distance $|x_n - x_m|$ can be made arbitrary small for all sufficiently large n and m. The *Cauchy criterion* states that *a sequence in a Euclidean space converges to some point if and only if it is a Cauchy sequence.*

1.1.2. Basic sets in a Euclidean space. A collection of all points whose distance from x is less than $a > 0$,

$$B_a(x) = \{y \in \mathbb{R}^N \,|\, |x - y| < a\}\,,$$

is called an open ball of radius a centered at x. For brevity, $B_a(0) = B_a$. A set Ω is *bounded* if it lies in a ball of sufficiently large radius, $\Omega \subset B_a$.

A *neighborhood* of a point x is $B_a(x)$ for some $a > 0$. A point x in a set Ω is called an *interior point* if there exists a neighborhood of x that lies in Ω, $B_a(x) \subset \Omega$ for small enough a. A point x is called a *limit point* of Ω if any neighborhood of x has a point of Ω distinct from x. Clearly, a limit point of Ω may or may not be in Ω. For example, the limit points of an open interval (a, b) form the closed interval $[a, b]$. A set that contains all its limit points is called *closed*. The set obtained from Ω by adding all its limit points is called the *closure* of Ω and will be denoted by $\bar{\Omega}$. The reader is advised to show that the closure is closed. The closure $\bar{\Omega}$ is the smallest closed set that contains Ω.

An *open box* R in $\mathbb{R}^N$ is a collection of points whose coordinates span open bounded intervals, $a_j < x_j < b_j$, $j = 1, 2, ..., N$. For brevity,

$$R = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)\,.$$

A box is closed if all intervals are closed.

A collection of all interior points is called the *interior* of Ω and will be denoted by Ω^o. The interior of Ω is the largest open set that lies in Ω. By the Cauchy criterion, any Cauchy sequence in Ω converges to a limit point of Ω. So, the closure of Ω consists of limit points of all Cauchy sequences in Ω.

The set $\Omega^c = \mathbb{R}^N \backslash \Omega$ is called the *complement* of Ω. The complement of a closed set is open (a proof is left to the reader as an exercise).

1.1.3. Heine-Borel theorem. Let Ω be a bounded and closed set in $\mathbb{R}^N$. Suppose that a collection of open sets $\{U_\alpha\}$ labeled by an index a is such that their union contains Ω

$$\Omega \subset \bigcup_\alpha U_\alpha$$

Then *there exists a finite sub-collection such that the union of open sets from it also contains* Ω. The statement is known as the Heine-Borel theorem[1].

[1] see, e.g., W. Rudin, Principles of mathematical analysis, Chapter 2

In particular, any set Ω is covered by the union of open balls of radius $a > 0$ centered at every point of Ω:

$$\Omega \subset \bigcup_{x\in\Omega} B_a(x)$$

If Ω is closed and bounded in $\mathbb{R}^N$, then, by the Heine-Borel theorem, one can find a finitely many points in Ω such that the union of open balls centered at these points contains Ω:

$$\bar{\Omega} = \Omega\,, \quad \Omega \subset B_R \quad \Rightarrow \quad \Omega \subset \bigcup_{j=1}^{n} B_a(x_j)$$

for any $a > 0$ and some $\{x_j\}_{j=1}^n$.

Let us show first that the theorem holds for a box. Without loss of generality, let $\Omega = I_0$ be a closed cube with side of length L and let the origin be its center so that the coordinates of points in the cube range over the intervals, $-\frac{L}{2} \leq x_j \leq \frac{L}{2}$. Let $\{U_a\}$ be an open cover of I_0. Suppose there exists no finite subcover of I_0. The cube I_0 is the union of 2^N cubes with edges of length $\frac{L}{2}$. At least one of these cubes, denoted I_1, must be covered by uncountably many U_α. The argument can be repeated. The cube I_1 is the union of 2^N cubes with edges of length $\frac{L}{4}$ and at least one of them, denoted I_2, is not covered by the union of finitely many U_α. By repeating the argument over and over, a sequence of cubes is generated such that $I_{n+1} \subset I_n$, $n = 0, 1, ...$, each I_n is not covered by the union of finitely many open sets U_α, and $|x - y| \leq L\sqrt{N}2^{-n}$ for any two points x and y in I_n.

Next note that there exists a point z that lies in all I_n. If $N = 1$, then $I_n = [a_n, b_n]$ where $\{a_n\}$ and $\{b_n\}$ are monotonically increasing and decreasing sequences, respectively. Since the sequence $\{a\}$ is bounded, say, by b_0, it has a limit equal to $z = \sup\{a_n\}$. On the other hand, $z \leq b_m$ for any m because $a_m \leq b_n$ for any n and m. This implies that z lies in any interval I_n. If $N > 1$, then the same argument can be applied to every coordinate of any point in I_n. If the j^{th} coordinate of points in I_n takes values in the closed interval $[a_{jn}, b_{jn}]$, then $z_j = \sup_n\{a_{jn}\}$, and z lies in every I_n.

Now it is easy to see a contradiction. Take any U_α that contains z. Since U_α is open, one can find an open ball $B_\epsilon(z)$ that lies in U_α. On the other hand, the diameter of I_n can be made arbitrary small with increasing n. Therefore for all n large enough, $I_n \subset B_\epsilon(z) \subset U_\alpha$ (if $\sqrt{N}L2^{-n} < 2\epsilon$), which means that I_n is covered by a single U_α and, hence, a contradiction.

Finally, let Ω be bounded and closed in $\mathbb{R}^N$. Then it lies in a cube I with a large enough edge. Let $\{U_\alpha\}$ be an open cover of Ω. The complement Ω^c is an open set. Then the collection $\{U_\alpha\}$ and Ω^c is an open cover of I from which one can select a finite subcover of I and, hence, of Ω. If Ω^c belongs to the selected subcover, then one can remove it from the subcover and still get a finite subcover of Ω because Ω and Ω^c have no common points. This completes the proof of the Heine-Borel theorem.

1.1.4. The boundary of a set. The boundary of Ω is the difference between the closure and interior of Ω:

$$\partial\Omega = \bar{\Omega} \setminus \Omega^o$$

1.1.5. A region in $\mathbb{R}^N$. A *vector function* is a vector-valued function on an interval, $x_i = x_i(t)$, $a \le t \le b$, $i = 1, 2, ..., N$, or, for brevity, $x = x(t)$. A vector function is continuous if every component of $x(t)$ is continuous. A continuous vector function is also called a *parametric curve* in $\mathbb{R}^N$ (think of a trajectory of a point-like particle if t is a physical time). A set Ω is *connected* if any two points in it can be connected by a parametric curve that lies in Ω. A connected open set will be called a *region* and the closure of a region will be called a *closed region*.

1.1.6. A neighborhood of a set. The union of open balls centered at all points of a set Ω is called a *neighborhood* of Ω. By construction, a neighborhood of Ω is open. In what follows, if all balls have the same radius a, then the corresponding neighborhood is said to have radius a. For example, a neighborhood of a closed ball $|x| \le R$ is an open ball $|x| < R + a$.

1.1.7. Distance between sets. A distance between sets A and B is defined by

$$d(A, B) = \inf_{x \in A,\, y \in B} |x - y| \,.$$

Let Ω be a region and let x is not in Ω. Suppose that the distance between x and Ω vanishes

$$d(\Omega, x) = \inf_{y \in \Omega} |y - x| = 0\,.$$

Then x belongs to the boundary of Ω. Indeed, take a point y_1 in Ω and put $a = |y_1 - x|$. Then one can find a point y_2 in Ω such that $|y_2 - x| \le \frac{a}{2}$. By repeating this procedure a sequence of points y_n in Ω can be obtained such that $|y_n - x| \le a2^{-n}$. This means that x is a limit point of Ω. Since Ω is open, x must be in the boundary $\partial\Omega = \bar{\Omega} \setminus \Omega$. Furthermore, $d(\Omega, x) > 0$ if and only if x does not belong to the closure

$\bar{\Omega}$, or, in other words, x lies in the complement of $\bar{\Omega}$. Note that $d(\Omega, x)$ is a continuous function of x. By the extreme value theorem, if A is a closed and bounded set, then there exists a point x_* in A such that

$$d(\Omega, A) = d(\Omega, x_*)\,.$$

1.1.8. A proper subset of a region. A bounded set Ω' is said to be *a proper subset* of a region Ω if its closure lies in Ω, $\overline{\Omega'} \subset \Omega$. A proper subset has a characteristic property that the distance between it and the boundary $\partial\Omega$ does not vanish:

$$d(\Omega', \partial\Omega) > 0\,.$$

The boundary $\partial\Omega$ is a closed set (it can be viewed as the intersection of two closed sets, $\bar{\Omega}$ and the complement of Ω, which is closed because Ω is open). If Ω is bounded, then its boundary is also bounded and, in this case, there exists $x_* \in \partial\Omega$ such that

$$d(\Omega', \partial\Omega) = d(\Omega', x_*) > 0$$

because x_* is not in Ω and, hence, cannot be in $\overline{\Omega'} \subset \Omega$. If Ω is not bounded, then its boundary can be unbounded too. In this case, consider the part of the boundary $\partial\Omega$ that lies in the closed ball of radius R, $\partial\Omega_R = \partial\Omega \cap \overline{B_R}$. Then $\partial\Omega_R$ is closed and bounded for any $R > 0$. Since Ω' is bounded, one can take R large enough so that B_R contains Ω' and

$$d(\Omega', \partial\Omega) = d(\Omega', \partial\Omega_R) = d(\Omega', x_*) > 0\,, \quad x_* \in \partial\Omega_R\,.$$

It also follows that *for any proper subset Ω' of a region Ω there exists a neighborhood of Ω' of radius $\delta > 0$ that is also a proper subset of Ω.* Indeed, by the above reasoning, one can take $\delta = \frac{1}{2}d(\Omega', \partial\Omega) > 0$.

1.2. Functions on a Euclidean space. A function $f:\ \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ is a rule that assigns a unique number $f(x)$ to every point $x \in \Omega$. The sets Ω and $f(\Omega) \subset \mathbb{R}$ are called the domain and the range of f. If $f(x)$ is a complex number (the range lies in the complex plane), then f is called complex-valued. Let y be a limit point of Ω. A function f is said to have a limit A at y if for any sequence $\{x_n\} \subset \Omega$, the sequence $\{f(x_n)\}$ converges to the number A. In this case, one writes

$$\lim_{x\to y} f(x) = A \quad \text{or} \quad f(x) \to A \text{ as } x \to y\,.$$

1.2.1. The characteristic function of a set. For any set Ω, the function defined by

$$\chi_\Omega(x) = \begin{cases} 1\,, & x \in \Omega \\ 0\,, & x \notin \Omega \end{cases}$$

is called the *characteristic function* of Ω.

1.2.2. The classes $C^p(\Omega)$ and $C^p(\bar{\Omega})$. Let Ω be open. A function f is *continuous* at a point $x \in \Omega$ if for any sequence $\{x_n\}$ in Ω that converges to x, the image sequence $\{f(x_n)\}$ converges to $f(x)$, and f is said to be *continuous on* Ω if it is continuous at every point of Ω. The class of all functions that are continuous on Ω will be denoted by $C^0(\Omega)$. The class of functions whose partial derivatives up to order p are continuous on Ω will be denoted by $C^p(\Omega)$.

Let $y \in \partial\Omega$. For an open Ω, f is not defined at any point of the boundary. Suppose that for any sequence $\{x_n\}$ in Ω that converges to a boundary point y, the sequence $\{f(x_n)\}$ has a limit. In this case, f is said to have a *continuous extension to a boundary point* y by the rule

$$f(y) = \lim_{x \to y} f(x)$$

The class of continuous functions on an open set Ω that have a continuous extension to every point of the boundary of Ω will be denoted by $C^0(\bar{\Omega})$. Similarly, the class of functions whose partial derivatives are continuous up to order p on an open set Ω and have continuous extensions to every point of the boundary of Ω will be denoted by $C^p(\bar{\Omega})$. If $\Omega = \mathbb{R}^N$ or an explicit form of Ω is irrelevant, it will be said that f is from class C^p.

1.2.3. Uniform continuity. A function f is said to be *uniformly continuous* on a set Ω if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) >$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta$$

for all x and y in Ω. In other words, values of the function differ from each other by no more than ε for all points in Ω which lie in a ball of radius δ. Clearly, every uniformly continuous function on Ω is continuous on Ω. The converse is not true. The key difference between uniform continuity and continuity is that in the latter δ depends on ε and a point at which the function is continuous, $\delta = \delta(\varepsilon, y)$ if $f(x) \to f(y)$ as $x \to y$. It is always possible to find the same δ for all points in Ω. For example, $f(x) = \frac{1}{x}$ is continuous on $\Omega = (0, 1)$. Fix $\varepsilon > 0$. Then $|f(x) - f(y)| = |x - y|/(xy) < \varepsilon$ always fails for any $|x - y| < \delta$ and any δ if x and y a close enough to zero.

The following assertion provides sufficient conditions for uniform continuity[2]. *Let f be continuous on Ω. If Ω is bounded and closed in $\mathbb{R}^N$, then f is uniformly continuous on Ω.*

1.2.4. Support of a function. The closure of the set on which a function f does not vanish is called the *support* of f and denoted $\operatorname{supp} f$:

$$\operatorname{supp} f = \overline{D}\,, \quad D = \{\, x \,|\, f(x) \neq 0\}\,.$$

For example, the support of $f(x) = \sin(x)$ is $\mathbb{R}$. The support of the characteristic function of a set Ω is the closure $\bar{\Omega}$.

1.2.5. Sequences of functions. A sequence of functions $\{f_n\}$ is said to converge *pointwise* to a function f on a set Ω if for all $x \in \Omega$

$$\lim_{n\to\infty} f_n(x) = f(x)\,.$$

In general, the limit function does not inherit properties of terms of the sequence. For example, f may not be continuous on Ω even if the terms of the sequence are continuous on Ω. It is easy to construct an example. Let $f_n(x) = 0$ if $x < 0$, $f_n(x) = nx$ if $0 \leq x \leq \frac{1}{n}$, and $f_n(x) = 1$ if $x > \frac{1}{n}$. The limit function is the step function $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. It is not continuous at $x = 0$. A stronger condition than just a pointwise convergence is required in order for the limit function to be continuous.

Definition 1.1. (Uniform convergence)
A sequence of functions $\{f_n\}$ is said to converge uniformly to a function f on a set Ω if

$$\lim_{n\to\infty} \sup_{\Omega} |f_n(x) - f(x)| = 0\,.$$

Clearly, every uniformly convergent sequence converges pointwise. The converse is false. For example, if $f_n(x)$ is the sequence defined above. Then

$$\sup |f_n(x) - f(x)| = 1\,.$$

for all n and, hence, f_n does not converge to f uniformly on $\mathbb{R}$. Note also that the uniform convergence depends on the set. For example, f_n converges to the step function f uniformly on $\Omega = (-\infty, -a) \cup (a, \infty)$ for any $a > 0$. If one thinks about terms of a pointwise convergent sequence as an approximation to the limit function, then $|f_n(x) - f(x)|$ is an absolute error of the approximation at a point x. The pointwise convergence means that the error can be made smaller than any positive number ε for all large enough $n > m$ where m naturally depends on ε

[2] W. Rudin, Principles of mathematical analysis, Chapter 4

and the point x. The uniform convergence means that the integer m is independent of x so that the error of the approximation is uniformly bounded by ε for all points in Ω.

THEOREM **1.1**. (Cauchy criterion for uniform convergence) *A sequence of functions $\{f_n\}$ converges uniformly on a set Ω if and only*

$$\sup_{\Omega} |f_n(x) - f_m(x)| \to 0$$

as $n, m \to \infty$.

If $\{f_n\}$ converges to a function f uniformly on Ω, then

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |f(x) - f_n(x)| + |f(x) - f_m(x)| \\ &\le \sup_{\Omega} |f(x) - f_n(x)| + \sup_{\Omega} |f(x) - f_m(x)| \end{aligned}$$

for all $x \in \Omega$. Therefore

$$\sup_{\Omega} |f_n(x) - f_m(x)| \le \sup_{\Omega} |f(x) - f_n(x)| + \sup_{\Omega} |f(x) - f_m(x)|$$

and the sequence satisfies the Cauchy criterion for uniform convergence.

Conversely, suppose that the sequence obeys the Cauchy criterion. Since

$$|f_n(x) - f_m(x)| \le \sup_{\Omega} |f_n(x) - f_m(x)|$$

the numerical sequence $\{f_n(x)\}$ is a Cauchy sequence for any x in Ω and, hence, $\{f_n\}$ converges pointwise to a function $f(x)$ in Ω. By taking the limit $m \to \infty$ in the Cauchy criterion first, it is concluded that the sequence converges uniformly to f on Ω:

$$\lim_{n\to\infty} \lim_{m\to\infty} \sup_{\Omega} |f_n(x) - f_m(x)| = \lim_{n\to\infty} \sup_{\Omega} |f_n(x) - f(x)| = 0\,.$$

This completes the proof.

It turns out that the uniform convergence is sufficient for the continuity of the limit function.

THEOREM **1.2**. (Continuity and uniform convergence)[3] *Suppose that a sequence of continuous functions converges to a function f uniformly on a set Ω. Then the function f is continuous on Ω.*

Suppose a sequence of continuously differentiable functions converges pointwise to a function f on Ω. Two essential questions arise:

(i) Is the limit function continuously differentiable?
(ii) If so, can the derivative of the limit function be obtained as the limit of the sequence of the derivatives of terms?

[3] W. Rudin, Principles of Mathematical Analysis, Chapter 7

The answer is negative to both questions. As an example, put

$$f_n(x) = \begin{cases} 0 & , \quad x < 0 \\ (nx)^2[1-(1-nx)^2] & , \quad 0 \le x \le \frac{1}{n} \\ 1 & , \quad x > \frac{1}{n} \end{cases}$$

Then $f_n \in C^1(\mathbb{R})$. The sequence f_n converges to the step function which is not differentiable at $x = 0$, whereas $f_n'(0) = 0$. The sequence $f_n(x) = \sin(nx)/\sqrt{n}$ converges pointwise to $f(x) = 0$ in $\mathbb{R}$. The terms of the sequence are continuously differentiable infinitely many times and so is the limit function. However, the sequence of the derivatives $f_n'(x) = \sqrt{n}\cos(nx)$ does not converge to $f'(x) = 0$.

THEOREM **1.3**. (Differentiation and uniform continuity)
Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ such that $\{f_n(c)\}$ converges for some $c \in [a, b]$. If $\{f_n'\}$ converges uniformly to a function g on $[a, b]$, then $\{f_n\}$ converges uniformly to a function f on $[a, b]$ and $f' = g$.

1.2.6. Functional series. *Let $\{f_n\}$ be a sequence of bounded functions on a set Ω such that the series of bounds converges:*

$$\sum_n M_n < \infty\,, \quad M_n = \sup_\Omega |f_n(x)|\,.$$

Then the series $\sum_n f_n(x)$ converges uniformly on Ω. Indeed, consider a sequence of partial sums

$$s_n(x) = \sum_{k=1}^n f_k(x)\,.$$

Then by the Cauchy criterion for uniform convergence **1.1**, it converges uniformly on Ω to a function f because

$$\sup_\Omega |s_n(x) - s_m(x)| = \sup_\Omega \Big| \sum_{k=m+1}^n f_k(x) \Big| \le \sum_{k=m+1}^n M_k \to 0$$

as $n > m \to \infty$ because $\sum_k M_k < \infty$ and, hence, its partial sums form a Cauchy sequence.

By combining this assertion with Theorems **1.2** and **1.3**, the following useful criterion for continuity and differentiability of functional series can be established.

PROPOSITION **1.1**. (Differentiation of a series)
Let $\{f_n\}$ be a sequence of continuous and bounded functions on a set

$\Omega \subset \mathbb{R}^N$ *such that the series of the bounds converge:*

$$\sum_n M_n < \infty\,, \quad M_n = \sup_\Omega |f_n(x)|\,.$$

Then the following series converges to a continuous function on Ω*:*

$$f(x) = \sum_n f_n(x)\,.$$

If, in addition, $f_n \in C^1(\Omega)$ *and partial derivatives are bounded on* Ω *so that the series of bounds also converges,*

$$\sum_n M_n^{(j)} < \infty\,, \quad M_n^{(j)} = \sup_\Omega \left|\frac{\partial f_n(x)}{\partial x_j}\right|,$$

then the function f *is from class* $C^1(\Omega)$ *and*

$$\frac{\partial f(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\sum_n f_n(x) = \sum_n \frac{\partial f_n(x)}{\partial x_j}\,.$$

This offers a simple criterion for interchanging the order of differentiation and summation in a functional series.

1.3. Smooth boundary of a set in $\mathbb{R}^N$. The boundary $\partial\Omega$ of a set Ω is called *smooth* if in a neighborhood of any point, $\partial\Omega$ is a level set of a function from class C^1

$$\partial\Omega = \{x \in \mathbb{R}^N \,|\, g(x) = 0\}$$

and the gradient of g does not vanish

$$\nabla g = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, ..., \frac{\partial g}{\partial x_N}\right) \neq 0\,.$$

The boundary $\partial\Omega$ is said to be from class C^p if, in addition, $g \in C^p$, $p \geq 2$.

Let $x_j = x_j(t)$ be parametric equation of a curve that lies in the level set $g(x) = 0$. Then the vector $v(t) = x'(t)$ is tangent to the curve. Since the equation $g(x(t)) = 0$ holds for all t, its differentiation shows that the gradient of g is orthogonal to v at any point of the curve:

$$0 = \frac{d}{dt}g(x(t)) = \Big(\nabla g(x(t)), v(t)\Big)$$

Tangent vectors to all curves through any point x on the level set form a tangent plane to the level set through the point x, and, hence, the gradient $\nabla g(x)$ is orthogonal to all vectors in this plane. Thus, *the gradient* ∇g *is normal to the boundary* $\partial\Omega$. In particular, a *unit normal* to the boundary $\partial\Omega$ can be defined by

$$\hat{n}(x) = \frac{\nabla g(x)}{|\nabla g(x)|}\,, \quad x \in \partial\Omega$$

The vector $-\hat{n}(x)$ is also a unit normal to $\partial\Omega$. If f is from class $C^1(\bar{\Omega})$, then

$$\frac{\partial f}{\partial n} \stackrel{\text{def}}{=} \Big(\hat{n}(x), \nabla f(x)\Big)\,, \quad x \in \partial\Omega$$

is called a *normal derivative* of f on the boundary of Ω.

1.3.1. Notations for partial derivatives. In what follows, partial derivatives are often denoted by

$$\partial_j g = \frac{\partial g(x)}{\partial x_j}\,, \quad \partial_i\partial_j^2 g(x) = \frac{\partial^3 g}{\partial x_i \partial^2 x_j}\,, \quad \text{etc.}$$

A collection of all partial derivatives of a function g of order α will be denoted by $D^\alpha g$. For example,

$$D^2 g = \{\partial_i\partial_j g\}\,, \quad i, j = 1, 2, ..., N\,.$$

The symbol $D^\alpha g$ in equations stands for *any* partial derivative of order α. For example, the product rule has the form

$$D(fg) = gDf + fDg\,.$$

1.3.2. Smooth boundary of a set in $\mathbb{R}^3$. Let g be from the class $C^1(\mathbb{R}^3)$ and $\nabla g \neq 0$. Without loss of generality $\partial_3 g \neq 0$ at some point $x = y$ of the level set. Then by the *implicit function theorem* the equation $g(x_1, x_2, x_3) = 0$ can be solved in a neighborhood of y with respect to x_3, that is, there exists a function $f(x_1, x_2)$ such that $g(x_1, x_2, f(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of (y_1, y_2). Moreover, the function f is from class C^1 and

$$\partial_1 f = -\frac{\partial_1 g}{\partial_3 g}\Big|_{x_3=f}\,, \quad \partial_2 f = -\frac{\partial_2 g}{\partial_3 g}\Big|_{x_3=f}\,.$$

The latter equations are known are *implicit differentiation* equations. So, *a smooth boundary of a set* $\mathbb{R}^3$ *locally looks like a graph of a* C^1 *function of two variables*, which is a two-dimensional *surface* in space.

For example, let $\Omega = B_a$. Then its boundary is a sphere which is a level set $g(x) = |x|^2 = a^2$. The gradient $\nabla g = 2x$ is continuous and does not vanish on the sphere. It is also normal to the sphere. The derivative $\partial_3 g = 2x_3$ does not vanish if $x_3 > 0$ or $x_3 < 0$. So, in a neighborhood of any point in the upper hemisphere the sphere is a graph $x_3 = \sqrt{a^2 - x_1^2 - x_2^2}$ while it is the graph $x_3 = -\sqrt{a^2 - x_1^2 - x_2^2}$

near any point in the lower hemisphere. Near any point at which $\partial_3 g = 0$, the equation cannot be solved for x_3 and should be solved either with respect to x_1 (if $\partial_1 g \neq 0$) or x_2 (if $\partial_2 g \neq 0$).

This picture has a natural generalization to higher dimensional spaces. A smooth boundary of a region in $\mathbb{R}^N$ is locally a graph of a C^1 function of $N-1$ variables obtained by solving the equation $g(x) = 0$ with respect one of the variables. It defines an $N-1$ dimensional surface in $\mathbb{R}^N$.

1.4. A Riemann integral. Suppose a function f of a real variable x is bounded on $[a, b]$, that is, $m \leq f(x) \leq M$ for all $x \in [a, b]$. A finite collection of points $P = \{x_s\}$ in $[a, b]$ that contains a and b is called a *partition* of $[a, b]$. The points from P are endpoints of *partition intervals* $R_s = [x_{s-1}, x_s]$. The smallest partition of $[a, b]$ contains two points $P = \{a, b\}$. Put

$$M_s = \sup_{R_s} f(x)\,, \qquad m_s = \inf_{R_s} f(x)$$

By boundedness of f, the supremum and infimum exist on any subset of $[a, b]$. Define the *upper and lower sums* of f, respectively, by

$$U(P, f) = \sum_s M_s \Delta x_s\,, \quad L(P, f) = \sum_s m_s \Delta x_s\,.$$

where $\Delta x_s = x_s - x_{s-1}$ and the summation is carried over all partition intervals.

A partition P' is called a *refinement* of P if $P \subset P'$. In other words, every point of P also belongs to P' but P' has points which are not in P. Clearly, under a refinement of a partition P some of its partition intervals are split into smaller intervals. Recall that

$$\sup_A f(x) \leq \sup_B f(x)\,, \quad \inf_A f(x) \geq \inf_B f(x)\,, \quad A \subset B$$

These relations imply that the lower sum is increasing upon a refinement whereas the upper sum is decreasing:

$$\begin{aligned} m(b-a) \leq L(P, f) &\leq L(P', f) \\ &\leq U(P', f) \leq U(P, f) \leq M(b-a)\,, \quad P \subset P'\,. \end{aligned}$$

The values of $L(P, f)$ for all partitions form a set of reals that is bounded from above by $M(b-a)$ and, hence, it has the supremum (the least upper bound). Similarly, the values of $U(P, f)$ for all partitions form a set bounded from below by $m(b-a)$ and therefore this set has the infimum (the greatest lower bound).

A bounded function f is said to be *Riemann integrable* on $[a, b]$ if the supremum of lower sums is equal to the infimum of upper sums and, in this case, their value is called a Riemann integral of f over $[a, b]$:

$$\sup_P L(P, f) = \inf_P U(P, f) = \int_a^b f(x)\, dx\,.$$

1.4.1. The fundamental theorem of calculus. Let f be continuous on $[a, b]$ and F be an antiderivative of f, that is, $F' = f$. Then

$$\int_a^b f(x)\, dx = F(b) - F(b)\,.$$

1.4.2. Riemann sums. A *Riemann sum* for a function f and a partition P is defined by

$$R(P, f) = \sum_s f(x_s^*)\Delta x_s$$

For any partition P, the inequality $m_s \leq f(x_s^*) \leq M_s$ implies that

$$L(P, f) \leq R(P, f) \leq U(P, f)$$

for any choice of sample points in the Riemann sum. Riemann sums can be used for approximations of the Riemann integral.

PROPOSITION **1.2**. *Let f be a Riemann integrable function on $[a, b]$. For any positive number $\varepsilon > 0$, there exist a partition P_ε such that*

$$\Big|\int_a^b f(x)\, dx - R(P, f)\Big| \leq \varepsilon\,, \quad P_\varepsilon \subset P$$

for any choice of sample points in the Riemann sum and any refinement P of P_ε.

In other words, the Riemann integral can be approximated by a Riemann sum with an arbitrary small error ε. Indeed, since f is integrable, for any partition P

$$L(P, f) \leq \int_a^b f(x)\, dx \leq U(P, f)$$

Therefore

$$\Big|\int_a^b f(x)\, dx - R(P, f)\Big| \leq U(P, f) - L(P, f)$$

Fix ε. Then the integrability of f implies that there exists a partition P_ε such that

$$U(P, f) - L(P, f) \leq \varepsilon\,, \quad P_\varepsilon \subset P$$

and the conclusion of the proposition follows.

Let f be integrable on $[a, b]$. Then it follows from the integrability of f that one can find a sequence of partitions $P_n \subset P_{n+1}$ such that

$$\lim_{n\to\infty} R(P_n, f) = \int_a^b f(x)\,dx$$

The sequences of the lower and upper sums, $L(P_n, f)$ and $U(P_n, f)$, also have the same limit. The maximal length of partition intervals associated with P_n must be decreasing to zero, $\max_s \Delta x_s \to 0$ as $n \to \infty$, otherwise it would be impossible to make the difference $U(P_n, f) - L(P_n, f)$ arbitrary small. For example, one can take a sequence of uniform partitions P_n (all partition intervals have equal lengths) defined by $x_0 = a$, $x_j = x_{j-1} + \Delta x$, $j = 1, 2, ..., 2^n$, and $\Delta x = (b - a)/2^n$.

1.4.3. Area under the graph of a continuous function. If f is continuous on interval $[a, b]$ and $f(x) \geq 0$. The region Ω in a plane defined by $a \leq x \leq b$ and $0 \leq y \leq f(x)$ contains the union of rectangles $R_s \times [0, m_s]$ with the total area being equal to $L(P, f)$, whereas the union of rectangles $R_s \times [0, M_s]$, with the total area being equal to $U(P, f)$, contains Ω. The Riemann integral of f defines the area of Ω. The lower sum is an estimate of the area from below and the upper sum is its estimate from above.

1.4.4. Generalization to a Euclidean space. Let a function f be bounded on a box $R = [a_1, b_1] \times \cdots \times [a_N, b_N]$ in $\mathbb{R}^N$. Its volume is $V = (b_1 - a_1) \cdots (b_N - a_N)$. Each coordinate interval can be partitioned so that R is partitioned by boxes R_s of smaller volumes ΔV_s where s enumerates all partition boxes so that $V = \sum_s \Delta V_s$. A partition P of R is a collection of all vertices of partition boxes. A partition P' is a refinement of P if $P \subset P'$. The lower and upper sums of a bounded function f are defined in the same way as in the one-dimensional case

$$\begin{array}{ll} L(P, f) = \sum_s m_s \Delta V_s\ , & m_s = \inf_{R_s} f(x) \\ U(P, f) = \sum_s M_s \Delta V_s\ , & m_s = \sup_{R_s} f(x) \end{array}$$

They have the same properties as in the one-dimensional case.

A bounded function f is said to be *Riemann integrable* on R if the greatest lower bound of upper sums is equal to the least upper bound of lower sums and, in this case, their value is called the Riemann integral of f over R:

$$\int_R f(x)\,d^N x = \inf_P U(P, f) = \sup_P U(P, f)$$

THEOREM 1.4. (Fubini's theorem)
Let $f(x)$ be continuous function on a closed rectangle R. Then it is

Riemann integrable on R and

$$\int_\Omega f(x)\,d^N x = \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} f(x_1, ..., x_N)\,dx_N \cdots dx_1$$

Here x_j, $j = 1, 2, ..., N$, are coordinates of a point x in $\mathbb{R}^N$, and the iterated integral can be computed in any order.

The Fubini allows one to calculate the integral by means of the fundamental theorem of calculus.

1.4.5. Volume under the graph of a continuous function. Let $f(x, y)$ be a a non-negative continuous function of two real variables on a rectangle $R = [a_1, b_1] \times [a_2, b_2]$. Let Ω be the (closed) solid above the rectangle and below the graph $z = f(x, y)$. Then the upper and lower sums are estimates of the volume of Ω from above and below, respectively, because the union of three-dimensional boxes $R_s \times [0, M_s]$ contains Ω, where R_s are partition rectangles, while Ω contains the union of boxes $R_s \times [0, m_s]$. Upon refinement the estimates tends to one another so that the integral of f over the rectangle gives, by definition, the volume of Ω. This geometrical interpretation of the Riemann integral can readily be extended to any Euclidean space.

1.4.6. An example of a non-integrable function. Continuity is not necessary for Riemann integrability. Suppose that a function g coincides with a continuous function f on interval $[a, b]$ everywhere but a point c, $g(c) \neq f(x)$. Then g is still integrable on $[a, b]$. The lower and upper sums for f and g only differs by the term corresponding to a partition interval containing c, but this term can be made arbitrary small (cf. Sec. **1.4.2**). Therefore, g is integrable and the integrals of f and g are equal. Clearly, a continuous function can be altered at finitely many points without destroying its integrability.

The Riemann integrability can be lost if a continuous function is altered at infinitely many points. A simple example is provided by the Dirichlet function defined by

$$f_D(x) = \begin{cases} 1 \ , \ x \in \mathbb{Q} \\ 0 \ , \ x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers. It is continuous nowhere and not Riemann integrable on any interval. Indeed, since any interval contains rational and irrational numbers, for any partition the lower sum is equal to zero, whereas the upper sum is equal to the length of the interval. So, the limits of the lower and upper sums cannot be equal.

1.4.7. Riemann integral over a region. Let Ω be a bounded set in $\mathbb{R}^N$ and f be a bounded function on Ω. Let us extend f to $\mathbb{R}^N$ by zero, that is, $f(x) = 0$ if $x \neq \Omega$. The extension can also be written as $\chi_\Omega(x)f(x)$ where χ_Ω is the characteristic function of Ω. The Riemann integral of f over Ω is defined by

$$\int_\Omega f(x)\,d^N x = \int_R \chi_\Omega(x)f(x)\,d^N x$$

where R is any rectangle that contains Ω, provided $\chi_\Omega f$ is integrable on R. Clearly, $\chi_\Omega f$ is generally not continuous and, hence, Fubini's theorem does not apply.

PROPOSITION 1.3. *Let Ω be a bounded region with piecewise smooth boundary and f be from class $C^0(\bar{\Omega})$. Then f is Riemann integrable on Ω.*

The integral can be evaluated by reducing it to an iterated integral (similarly to Fubini's theorem). Let $\Omega \subset \mathbb{R}^3$ be such that any line parallel to the x_3 coordinate axis intersects Ω at most along one line segment (an interval on the line). Then let D_Ω be the projection of Ω onto the plane $x_3 = 0$, that is, $(x_1, x_2, x_3) \to (x_1, x_2, 0) \in D_\Omega$. Then for any point $(x_1, x_2, 0)$ in D_Ω, $h_1(x_1, x_2) \le x_3 \le h_2(x_1, x_2)$ for all points in Ω. Here the graphs $x_3 = h_i(x_1, x_2)$ are boundaries of Ω from above $(i = 2)$ and from below $(i = 1)$. Then

$$\int_\Omega f(x_1, x_2, x_3)\,d^3x = \int_{D_\Omega}\left(\int_{h_1(x_1,x_2)}^{h_2(x_1,x_2)} f(x_1, x_2, x_3)\,dx_3\right) d^2x$$

The integral over D_Ω can be further reduced to a double iterated integral in a similar fashion and the resulting triple iterated integral can be evaluated by the fundamental theorem of calculus. For example, the integral over a ball of radius a centered at the origin in $\mathbb{R}^3$ can be reduced to the following triple iterated integral:

$$\int_{B_a} f(x)\,d^3x = \int_{-a}^{a}\int_{-\sqrt{a^2-x_1^2}}^{\sqrt{a^2-x_1^2}}\int_{-\sqrt{a^2-x_1^2-x_2^2}}^{\sqrt{a^2-x_1^2-x_2^2}} f(x_1, x_2, x_3)\,dx_3\,dx_2\,dx_1$$

Note that the projection D of B_a is the disk, $x_1^2 + x_2^2 \le a^2$. This procedure can be extended to an integral of any dimensions.

1.5. Properties of the Riemann integrals. A complex-valued function f is Riemann integrable on Ω if its real and imaginary parts are integrable and

$$\int_\Omega f(x)\, d^N x = \int_\Omega \operatorname{Re} f(x)\, d^N x + i \int_\Omega \operatorname{Im} f(x)\, d^N x$$

1.5.1. Linearity. If f and g are Riemann integrable on Ω, then their linear combination is integrable and

$$\int_\Omega \Big(\alpha f(x) + \beta g(x)\Big)\, d^N x = \alpha \int_\Omega f(x)\, d^N x + \beta \int_\Omega g(x)\, d^N x$$

for any (real or complex) numbers α and β.

1.5.2. Positivity. If $f(x) \geq 0$ and f is integrable on Ω, then

$$\int_\Omega f(x)\, d^N x \geq 0\,.$$

1.5.3. Integrability of the absolute value. If f is Riemann integrable on Ω, then its absolute value is also integrable on Ω and

$$\left|\int_\Omega f(x)\, d^N x\right| \leq \int_\Omega |f(x)|\, d^N x$$

The converse is false. For example, put $f(x) = 1$ if x is rational, and $f(x) = -1$ otherwise. This function is not integrable on any interval $[a, b]$ because its lower sum is equal to $-(b - a)$ and the upper sum is equal to $b - a$ for any partition. However, the absolute value $|f(x)| = 1$ is continuous and, hence, integrable on $[a, b]$.

1.5.4. Additivity. Let subsets $\Omega_{1,2} \subset \Omega$ be closed and bounded, and $\Omega_1 \cup \Omega_2 = \Omega$ but the interiors of $\Omega_{1,2}$ do not intersect. If f is integrable on $\Omega_{1,2}$, then it is integrable on Ω and

$$\int_\Omega f(x)\, d^N x = \int_{\Omega_1} f(x)\, d^N x + \int_{\Omega_2} f(x)\, d^N x$$

In particular, if f is continuous on a bounded closed region Ω with a piecewise smooth boundary and the regions $\Omega_{1,2}$ are obtained by cutting Ω into two pieces by a smooth surface, then the above equation holds.

1.5.5. Continuity. Let Ω_n be a family of subsets of a bounded set Ω labeled by a positive integer n such that

$$\Omega_n \subset \Omega_{n+1}\,, \quad \bigcup_n \Omega_n = \Omega$$

In other words, subsets Ω_n becomes larger with increasing n and in the limit $n \to \infty$, Ω_n becomes Ω. If a function f is Riemann integrable on

Ω and on each Ω_n, then

$$\lim_{n\to\infty}\int_{\Omega_n} f(x)\,d^Nx = \int_\Omega f(x)\,d^Nx$$

For example, let Ω be an open bounded set, and its boundary be smooth. Then any f from the class $C^0(\bar{\Omega})$ is integrable on Ω. The subsets Ω_n can be obtained by removing closed balls of radius $1/n$ centered at every point of the boundary of Ω. The boundaries $\partial\Omega_n$ are also smooth if $\partial\Omega$ is smooth enough. Then f is integrable on Ω_n, and the sequence of integrals of f over Ω_n converges to the integral of f over Ω.

1.5.6. More on Riemann integrability. The sets Ω_1 and Ω_2 in the additivity property can share a common boundary, that is, their boundaries can intersect, $\partial\Omega_1 \cap \partial\Omega_1 \neq \emptyset$. Suppose that the boundaries $\partial\Omega_{1,2}$ are smooth and their intersection is also a smooth surface. If $f \in C^0(\bar{\Omega}_{1,2})$, then by Proposition **1.3** the function f is Riemann integrable on $\Omega_{1,2}$ and the additivity property holds. It is worth noting that f is generally not continuous on Ω because it can have jump discontinuities at the common points of the boundaries $\partial\Omega_{1,2}$. If x is a point of $\partial\Omega_1 \cap \partial\Omega_1$, then the value $f(x)$ obtained by a continuous extension from Ω_1 is generally not the same as that obtained by the extension from Ω_2. In other words, the Riemann integrability holds for some non-continuous functions.

PROPOSITION **1.4**. *Let Ω be a bounded region and χ_Ω be its characteristic function. Then a function f is integrable on Ω if the extension $\chi_\Omega f$ is not continuous on finitely many smooth surfaces.*

An interesting question arises: How to characterize all Riemann integrable functions. To answer this question the concept of sets of zero measure is needed.

1.6. Volume (or measure) of a set. The area of a rectangle with sides of length a_1 and a_2 is defined as a_1a_2. Similarly, the volume of a rectangular box in a Euclidean space is defined by

$$V_N(R) = a_1a_2\cdots a_N = \int_R d^Nx$$

where a_j, $j = 1, 2, ..., N$, are lengths of the adjacent edges of the box. Similarly, the volume of any bounded set Ω is defined by

$$V(\Omega) = \int_\Omega d^Nx$$

provided the unit function is Riemann integrable on Ω. In particular, a bounded region with a piecewise smooth boundary always has a volume.

Clearly, the volume is additive, that is, the volume of the union of non-overlapping sets is the sum of the volumes

$$V(\Omega_1 \cup \Omega_2) = V(\Omega_1) + V(\Omega_2)\,, \quad \Omega_1 \cap \Omega_2 = \emptyset$$

In particular, if Ω is divided by smooth surfaces into finitely many pieces, then by Proposition **1.4**, the volume of Ω is the sum of volumes of the pieces. The volume is non-negative and increasing with increasing Ω:

$$V(\Omega_1) \leq V(\Omega_2)\,, \quad \Omega_1 \subset \Omega_2$$

These basic properties of the volume follow from the properties of the Riemann integral.

1.6.1. Volume of a ball in $\mathbb{R}^N$. Let $V_N(a)$ be the volume of a ball $B_a \subset \mathbb{R}^N$ (it exists because the boundary of the ball is a smooth surface). By the scaling property, $V_N(a) = a^N V_N(1)$. To find V_N, let r a coordinate along a diameter of the ball so that $-a \leq r \leq a$ for the whole diameter. The cross-section of the ball by the plane perpendicular to the diameter at a point r is a ball centered at r of radius $\sqrt{a^2 - r^2}$. The volume of a portion of the ball between two such planes at a distance dr is therefore $dV_N = V_{N-1}(\sqrt{a^2 - r^2})dr$ and, hence,

$$V_N(a) = \int_{-a}^{a} V_{N-1}(\sqrt{a^2 - r^2})\, dr\,, \qquad V_1(a) = 2a\,.$$

Put $C_N = V_N(1)$. Using the scaling property it is concluded that

$$C_N = C_{N-1} \int_{-1}^{1} \left(1 - s^2\right)^{\frac{N}{2} - \frac{1}{2}} ds\,, \qquad C_1 = 2$$

Evaluating the integral, one infers that

$$V_N(a) = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})}\, a^N$$

where Γ is Euler's gamma function:

$$\Gamma(z) = \lim_{b \to \infty} \int_0^b e^{-t} t^{z-1}\, dt$$

It has the following properties:

$$\Gamma(z+1) = z\Gamma(z)\,, \quad \Gamma(1) = 1\,, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

The first one is established by integration by parts, while the other two are proved by a direct evaluation of the integral.

If $\sigma_{_N}(a)$ is the surface area of the sphere $|x| = a$, then $dV_N(a) = \sigma_{_N}(a)da$. It follows from this relation that the surface area of the unit sphere in $\mathbb{R}^N$ reads

$$\sigma_{_N} = \sigma_{_N}(1) = \frac{N\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} . \tag{1.1}$$

1.7. Sets of measure zero in $\mathbb{R}^N$. The volume of a point in a Euclidean space is equal to zero because it is contained in a ball of arbitrary small radius. Similarly, any finite collection of points has the zero volume because it is contained in a the union of balls whose total volume can be arbitrary small. This observation is extended to all sets.

DEFINITION 1.2. (Sets of measure zero in $\mathbb{R}^N$)
A set in $\mathbb{R}^N$ is said to be of measure zero if it can be covered by a union of open balls whose total volume can be made smaller than any preassigned positive number.

For brevity, one writes $\mu(\Omega) = 0$ where μ stands for "measure".

1.7.1. Examples of sets of measure zero.

- A finite collection of points in space is a set of measure zero.
- A segment of a straight line of finite length L is a set of measure zero. Indeed, let us split it into n pieces of length L/n. Each such segment can be covered by a ball of radius L/n centered at the midpoint of the segment. The total volume is
$$V_n = nV_N(L/n) = C_N n(L/n)^N \to 0$$
as $n \to \infty$ for any dimension $N \geq 2$.
- Generalizing the previous example, a Euclidean space $\mathbb{R}^M$ can be viewed a hyper-plane in a higher dimensional Euclidean space $\mathbb{R}^N$, $N > M$. Any rectangular box in $\mathbb{R}^M$ is a set of measure zero in $\mathbb{R}^M$. For example, a rectangle in a plane in a three-dimensional space is a set of measure zero. A proof of this assertion is left to the reader as an exercise.
- Any subset of a set of measure zero is also a set of measure zero.

THEOREM 1.5. *A countable union of sets of measure zero is also a set of measure zero.*

Let

$$G = \bigcup_{n=1}^{\infty} G_n$$

where all G_n are sets of measure zero. Fix ε. Then G_n is contained in a union of open balls with the total volume $\varepsilon/2^n$. Therefore G is contained in the union of all such balls with the total volume being

$$V = \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \frac{\varepsilon}{2}\Big(1 + \frac{1}{2} + \Big(\frac{1}{2}\Big)^2 + \Big(\frac{1}{2}\Big)^3 + \cdots\Big) = \frac{\varepsilon}{2}\cdot\frac{1}{1-\frac{1}{2}} = \varepsilon$$

Since ε is arbitrary, G is a set of measure zero.

An immediate consequence of this theorem is that a countable collection of points is a set of measure zero. For example, all rational numbers in the interval $[0,1]$ are countable. So, they form a set measure zero (a set of zero length). Furthermore, the whole real line is a countable union of unit intervals. This implies that all rational numbers form a set of measure zero in $\mathbb{R}$. Points with rational coordinates in the rectangle $[0,1]\times[0,1]$ are also countable. Indeed, pairs (a_n, b_n) with a_n and b_n from countable sets ($n = 1, 2, ...$) can be counted in the order (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), etc. So, all points with rational coordinates in $\mathbb{R}^2$ form a set of measure zero. This conclusion is readily extended to a Euclidean space of any dimension.

Remark. Are there sets of measure zero in $\mathbb{R}$ that are not countable? The answer is affirmative. There are uncountable collections of numbers which contain no interval. One of the most famous examples is the *Cantor set*.

Other uncountable sets of measure zero in $\mathbb{R}^N$ include Euclidean subspaces. A line in space is also a set measure zero because it is a union of countably many line segments of a finite length. Similarly, any subspace $\mathbb{R}^M$ of $\mathbb{R}^N$ is a set of measure zero if $M < N$ because it is a union of countably many boxes, and any box in $\mathbb{R}^M$ is a set of measure zero in $\mathbb{R}^N$, $N > M$. What about measures of curves and surfaces in space?

1.7.2. Curves in a Euclidean space. Intuitively, a curve as a point set in a Euclidean space can be obtained by a continuous deformation (without breaking) of a line segment that has a continuous inverse. The existence of the inverse is needed to avoid gluing parts of the segment together upon deformation. Any such deformation can be described by a continuous one-to-one mapping of an interval into $\mathbb{R}$. So, by definition, *a curve is the range of a continuous vector function $x(t)$ on $[a,b]$ that is one-to-one except possibly at the boundary points of the interval*. If $x(a) = x(b)$, then the curve is called *closed*.

The equations $x = x(t)$ are called *parametric equations* of the curve and t is a parameter on the curve (it labels points of the curve). Let

$t = t(\tau)$ where $t(\tau)$ a continuous one-to-one function $[\alpha, \beta] \to [a, b]$. Then the composition $X(\tau) = x(t(\tau))$ has the same range and $x = X(\tau)$ are also parametric equation of the same curve. So, there are many mappings of an interval into $\mathbb{R}^N$ that define the same curve. A particular choice of the mapping is called a *parameterization* of the curve.

A curve has a self-intersection point if two points of an interval are glued together upon deformation. This implies that any mapping whose range in the curve is not one-to-one at $t = t_1$ and $t = t_2 > t_1$ because $x(t_1) = x(t_2)$ and it is one-to-one on (a, t_1), (t_1, t_2), and (t_2, b). Clearly, in this way one can define a curve with any number of self-intersections by a mapping (modulo a reparameterization).

The curve is said to be from class C^1 if there exists a continuously differentiable parameterization $x = x(t)$ on $[a, b]$. A curve is piecewise C^1 if it consists of finitely many C^1 pieces.

1.7.3. M-surfaces in $\mathbb{R}^N$. By analogy with curves, an M dimensional surface (or simply M-surface) in $\mathbb{R}^N$, $M < N$, can be defined as a continuous deformation of an M dimensional box that has a continuous inverse. *Let D be a closed and bounded set in $\mathbb{R}^M$ and F be a continuous mapping of a neighborhood of D into $\mathbb{R}^N$ that is one-to-one on the interior of D. Then the range $F(D) \subset \mathbb{R}^N$ is called an M-surface.* The equations $x = F(y)$, $y \in D$, are called *parametric equations* of the M-surface, and D is called *the parameter domain*. There are infinitely many mappings of a closed bounded set in $\mathbb{R}^M$ that have the same range in $\mathbb{R}^N$. If $g: D' \to D$, where D' is a closed bounded set in $\mathbb{R}^M$, is a continuous and one-to-one, then the composition $G(z) = F(g(z))$ has the same range $G(D') = F(D)$. Self-intersecting M-surfaces can also be described by a continuous mapping of D that is one-to-one in the interior of D except some of its points. *A surface is said to be from class C^1 if all components of F have continuous partial derivatives.*

Let y_0 be an interior point of D. Consider a coordinate line through y_0. The image of this line $x = F(y)$ is the curve in S_M. The curve is from class C^1 if $F \in C^1$. The span of all unit tangent vectors $T_a = w_a/|w_a|$ where $w_a = \partial F/\partial y_a$, $a = 1, 2, ..., M$, computed at $y = y_0$, is called a tangent space of S_M at a point $x_0 = F(y_0)$. Since F is one-to-one near y_0, the rank of the matrix of partial derivatives $\partial F/\partial y$ is equal to M. So, the tangent space is an M-plane through x_0.

The mapping F is not required to be one-to-one on the boundary ∂D. In this way surfaces with different topologies can be obtained through identification of certain points of the boundary of D upon deformation. For example, a unit sphere in $\mathbb{R}^3$ has the following parametric equations:

$$x_1 = \cos(\xi_1)\,, \quad x_2 = \sin(\xi_1)\cos(\xi_2)\,, \quad x_3 = \sin(\xi_1)\sin(\xi_2)\,,$$

where $D = [0,\pi]\times[0,2\pi]$ is the parameter domain. Here ξ_1 and ξ_2 are the zenith and polar angles of the spherical coordinates. The boundary lines $\xi_2 = 0$ and $\xi_2 = 2\pi$ have the same image curve (the semi-circle from the north to south pole of the sphere), whereas the boundary lines $\xi_1 = 0$ and $\xi_1 = \pi$ are mapped to single points, the north and south poles, respectively. With these identification, the rectangle D is topologically equivalent to a sphere (it can be continuously deformed to a sphere).

Similarly, the parametric equations

$$x_1 = \Big(a + b\cos(\xi_1)\Big)\cos(\xi_2)\,, \quad x_1 = \Big(a + b\cos(\xi_1)\Big)\sin(\xi_2)\,,$$
$$x_3 = b\sin(\xi_1)\,, \quad D = [0,2\pi]\times[0,2\pi]\,,$$

describe a torus with radii $a > b$. The cross section of the torus by the plane $x_3 = 0$ is the union of two circles of radii $a \pm b$. The cross section of the torus by a half-plane bounded by the x_3 coordinate axis is a circle of radius b. The position of the half-plane is defined by the polar angle ξ_2 in the plane $x_3 = 0$, whereas the angle ξ_1 defines a position of the point on the circle of intersection. The map identifies the opposite boundary lines of the rectangle D. After this identification D becomes topologically equivalent to a torus.

1.7.4. Smooth transformations of sets of measure zero. A *transformation* of $\mathbb{R}^N$ is a function $F:\ \mathbb{R}^N \to \mathbb{R}^N$. If all components of F are continuously differentiable, then the transformation is said to be from class C^1. If, in addition, its Jacobian does not vanish, then the transformation is called non-singular:

$$F\ :\quad \mathbb{R}^N \to \mathbb{R}^N\,, \qquad \det\left(\frac{\partial F_j}{\partial y_i}\right) \neq 0$$

Since the Jacobian does not vanish, the transformation is invertible in a neighborhood of each point by the inverse function theorem[4].

Therefore, any straight line passing through a point y_0 becomes a curve passing through the point $x_0 = F(y_0)$ in a neighborhood of x_0. Parametric equations of a line passing through y_0 and parallel to a vector v read $y = y_0 + vt$ where t is a real parameter. Then the image curve is $x = x(t) = F(y_0 + vt)$ so that its tangent vector has

[4]W. Rudin, Principles of Mathematical Analysis, Chapter 9.

components $x_i'(t) = (v, \nabla)F_i$ where ∇ is the gradient operator. Since the Jacobian matrix $\partial_j F_i$ is continuous and not singular, the tangent vector $x'(t)$ is continuous and does not vanish anywhere. So, the curve is also from class C^1.

Similarly, F maps a 2-plane through y_0 into a C^1 2-surface in a neighborhood of $x_0 = F(y_0)$. Parametric equations of a 2-plane through y_0 that is parallel to two linearly independent vectors u and v are $y = y(t,s) = y_0 + su + tv$, where s and t are real parameters. Parametric equations of the image 2-surface are $x = x(t,s) = F(y(t,s))$. At every point, the surface has two non-vanishing continuous tangent vectors $\partial_t x(t,s)$ and $\partial_s x(t,s)$ that are linearly independent because the matrix $\partial_j F_i$ is not singular and continuous. It is not difficult to see that the image of an M-plane through y_0 is an M-surface through $x_0 = F(y_0)$ from class C^1.

This suggests that the question about measure of smooth surfaces can be studied by investigating images of sets of measure zero under transformations from class C^1. The following two theorems answer the posed question [5].

THEOREM **1.6**. *The image of a set of measure zero Ω in $\mathbb{R}^N$ under a transformation F of $\mathbb{R}^N$ from class C^1 is a set of measure zero:*

$$\mu(\Omega) = 0 \quad \Rightarrow \quad \mu(F(\Omega)) = 0\,.$$

For example, a transformation of $\mathbb{R}^3$ can be written in the form

$$x_1 = F_1(y)\,, \quad x_2 = F_2(y)\,, \quad z = F_3(y)\,, \quad y = (y_1, y_2, y_3)\,.$$

Then the coordinate plane $y_3 = 0$ (or its portion) is a 2-surface

$$x_1 = F_1(y_1, y_2, 0)\,, \quad x_2 = F_2(y_1, y_2, 0)\,, \quad x_3 = F_3(y_1, y_2, 0)$$

Since a coordinate plane or any its portion is a set of measure zero in $\mathbb{R}^3$, any parametric surface in $\mathbb{R}^3$ defined by continuously differentiable functions is a set of measure zero.

THEOREM **1.7**. *Let D be an open set in $\mathbb{R}^M$ and the mapping $F : D \to \mathbb{R}^N$ is from class $C^1(D)$ and the the rank of the Jacobian matrix $\partial F_j/\partial y_a$, $j = 1, 2, ..., N$, $a = 1, 2, ..., M$, $y \in D$, is equal to $M < N$. Then the image of D is a set of measure zero in $\mathbb{R}^N$,*

$$\mu(F(D)) = 0\,.$$

By this theorem any M-surface from class C^1 is a set of measure zero in $\mathbb{R}^N$ and, by Theorem **1.6** any image of a C^1 surface under a transformation from class C^1 is also a set of measure zero. For example,

[5] J.M. Lee, Introduction to smooth manifolds

a 2-sphere and a torus and their C^1 transformations are sets of measure zero in $\mathbb{R}^3$.

Recall that a boundary of a region is smooth if it is a level set of a function from class C^1 whose gradient does not vanish. By the implicit function theorem, the equation $g(x) = 0$ can be solved with respect to one of the components of x, say, with respect to x_N so that $x_N = f(y)$ where $x_j = y_j$, $j = 1, 2, ..., N-1$. Since f is continuously differentiable, these equations can be viewed as parametric equations of an $N-1$ dimensional smooth surface in $\mathbb{R}^N$. Therefore, a piecewise smooth boundary of a region in $\mathbb{R}^N$ is a set of measure zero.

Remark. The condition that a transformation in Theorems **1.6** and **1.7** is from class C^1 is essential. If one takes a merely continuous transformation C^0 (so that the image of a line is a curve that does not necessarily have a tangent vector everywhere), then the theorems are false. There are so-called space-filling curves or surfaces. For example, if $\Omega = [0, 1] \times \{0\} \subset \mathbb{R}^2$ (a unit interval on the first coordinate axis), then one can construct a continuous transformation $F(t) = (x(t), y(t))$ such that it maps this interval, $t \in [0, 1]$, onto a square $[0, 1] \times [0, 1]$. Geometrically, it looks like a curve filling the square (a set of non-zero measure). In other words, the parametric curve $(x(t), y(t))$ passes through every point of the square as t spans the interval [6].

1.8. Riemann integrable functions. The class of Riemann integrable functions is described in the following theorem [7].

Theorem 1.8. (Lebesgue's criterion for Riemann integrability) *A bounded function is Riemann integrable on a rectangular box in a Euclidean space if and only if it is not continuous at most on a set of measure zero.*

It is worth mentioning that it is possible to construct a function on $\mathbb{R}$ that is not continuous at rational numbers but continuous otherwise (e.g., *Thomae's function*). The rational numbers form a countable set of measure zero. This function is Riemann integrable on any bounded interval. The characteristic function of the Cantor set (which is not continuous on an uncountable set of measure zero) is also Riemann integrable on any bounded interval.

1.9. Change of variables. Let $\Omega' \subset \mathbb{R}^N$ be a closed and bounded region with a piecewise smooth boundary. Let $x = F(y)$ be a transformation

[6] W. Rudin, Principles of mathematical analysis, Exercises in Chapter 7
[7] see, e.g., S. Abbott, Understanding Analysis, Springer, 2010

in $\mathbb{R}^N$ from class C^1 such that it is one-to-one on the interior of Ω' and its Jacobian does not vanish in Ω' except possibly on the boundary of Ω'. Let f be an integrable function on $\Omega = F(\Omega')$. Then

$$\int_\Omega f(x)\, d^N x = \int_{\Omega'} f(F(y))\, J(y)\, d^N y\,, \quad J(y) = \Big|\det\Big(\frac{\partial F_j}{\partial y_k}\Big)\Big|$$

1.10. Riemann integrability and uniform convergence. Suppose that a sequence $\{f_n\}$ of Riemann integrable functions on Ω converges pointwise to a function f. Then the function f is not generally Riemann integrable. Even if f happens to be Riemann integrable of Ω, then the integral of f is not generally equal to the limit of the integrals of f_n. The first assertion can be illustrated by the function $f(x)$ defined by

$$f(x) = \lim_{n\to\infty} \lim_{m\to\infty} \Big(\cos(\pi x n!)\Big)^{2m}$$

The first limit is equal to zero if $xn!$ is not an integer and to 1 if $xn!$ is an integer. Therefore $f(x) = 0$ if x is not rational and $f(x) = 1$ if x is rational because any rational number can be written as a ratio of integers $x = p/q$ and $n!/q$ is an integer if $n \geq q$. The limit function is the Dirichlet function that is not Riemann integrable on any interval. Clearly, the terms of the sequence are continuous and, hence, integrable on any bounded interval.

To illustrate the second assertion, put $f_n(x) = 2nx(1-x^2)^n$ where $x \in [0,1]$ and $n = 1, 2, ...$ It is not difficult to verify that the sequence converges pointwise

$$f(x) = \lim_{n\to\infty} f_n(x) = 0\,, \quad 0 \leq x \leq 1$$

However,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 = \int_0^1 f(x)\, dx$$

where the second equality has been obtained by evaluating the integral. So, *the order of neither differentiation nor integration of a convergent functional sequence can be interchanged with taking the limit*, unless the functional sequence satisfies additional conditions.

THEOREM **1.9**. [8] *Let $\{f_n\}$ be a sequence of Riemann integrable functions on a bounded region Ω that converges uniformly to a function f on Ω. Then f is Riemann integrable on Ω and*

$$\lim_{n\to\infty} \int_\Omega f_n(x)\, d^N x = \int_\Omega \lim_{n\to\infty} f_n(x)\, d^N x = \int_\Omega f(x)\, d^N x\,.$$

[8] see, e.g., W. Rudin, Principles of Mathematical Analysis

This theorem offers a sufficient condition for interchanging the order of Riemann integration and taking the limit with respect to a parameter of the integrand.

1.11. Exercises.

1. Show that a plane in a there-dimensional space is a set of measure zero.

2. Can a set in $\mathbb{R}^N$ be a set of measure zero in $\mathbb{R}^N$ if it has an interior point? Give an example or show that the answer is negative.

3. Let f be a function from class $C^1(\mathbb{R})$. Show that $f(\mathbb{Q})$ is a set of measure zero where $\mathbb{Q}$ denotes all rational numbers.

4. Use spherical coordinates in $\mathbb{R}^N$ to calculate the volume of an N dimensional ball.

5. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n , \qquad |x - x_0| < R .$$

(i) Show that the convergence of the series implies that $|c_n|\delta^n \to 0$ as $n \to \infty$ for any $0 < \delta < R$.
(ii) Show that

$$|c_n(x - x_0)^n| \le Mq^n , \quad |x - x_0| \le \delta$$

for some constants $M > 0$ and $0 < q < 1$ and any $\delta < R$. Use this inequality to show that the power series converges uniformly in the interval $|x - x_| \le \delta < R$.
(iii) Show that f is from class C^∞ by investigating uniform convergence of series of derivatives of the terms.
(iv) Prove that $c_n = f^{(n)}(x_0)/n!$.

6. (i) Use the power series representation of the exponential function to show that

$$\left| e^{i\alpha e^{ix}} - 1 \right| \le e^{|\alpha|} - 1 , \quad x \in \mathbb{R}$$

(ii) Use this inequality to show that $e^{i\alpha e^{ix}}$ converges to 1 uniformly on $\mathbb{R}$ as $\alpha \to 0$ and prove that

$$\lim_{\alpha \to 0} \int_a^b e^{i\alpha e^{ix}} dx = b - a\,.$$

2. Improper Riemann integrals

2.1. Preliminaries. The Riemann integral is defined for a bounded function f and a bounded region Ω. Intuitively, a Riemann integral over an unbounded region can be defined as the limit of integrals over bounded subregions. For example, one can take subregions that are intersections of an unbounded region with a ball of radius a, compute the integrals over these subregions, and then investigate the limit $a \to \infty$. Similarly, if a function is not bounded in a neighborhood of a point, one can reduce the region of integration by removing a ball of radius a centered at this point, compute the integral, and then investigate the limit $a \to 0^+$. If there are more then one of such points, the reduced region is obtained by removing the union of such balls centered at all singular points of the function. By combining the two ideas, one can define the integral of an unbounded function over an unbounded region.

For example, a continuous function $f(x) = e^{-x}$ can be integrated over an unbounded interval $[0, \infty)$ using the rule

$$\int_0^\infty e^{-x}\,dx \stackrel{\text{def}}{=} \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty}(1 - e^{-b}) = 1\,.$$

Note f is integrable on every $[0, b]$ because f is continuous. So, the rule makes sense. The function $f(x) = x^{-1/2}$ is not bounded on $[0, 1]$, but it is continuous on every $[a, 1]$ so it makes sense to define

$$\int_0^1 \frac{dx}{\sqrt{x}} \stackrel{\text{def}}{=} \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}(2 - 2\sqrt{a}) = 2\,.$$

Similarly, the function $f(x) = e^{-x}x^{-1/2}$ is not bounded on $[0, \infty)$, but it is continuous on any interval $[\frac{1}{a^2}, a^2]$. So, the integral can be defined by

$$\begin{aligned}\int_0^\infty e^{-x}\,\frac{dx}{\sqrt{x}} &\stackrel{\text{def}}{=} \lim_{a\to\infty} \int_{\frac{1}{a^2}}^{a^2} e^{-x}\frac{dx}{\sqrt{x}} = 2\lim_{a\to\infty}\int_{\frac{1}{a}}^{a} e^{-y^2}\,dy \\ &= 2\lim_{a\to\infty}\left(\int_{\frac{1}{a}}^{1} + \int_1^a\right) e^{-y^2}dy = 2\lim_{a\to\infty}\int_0^a e^{-y^2}dy = \sqrt{\pi}\,,\end{aligned}$$

where $x = y^2$ and the continuity of the integral was used to take the limit in the integral over $[\frac{1}{a}, 1]$.

A Riemann integral in which the integrand or region of integration or both are not bounded are referred to as an *improper Riemann integral*. A limiting procedure used to define the improper Riemann integral is called a *regularization*. A consistency of this definition requires answering the key question: Does the value of the improper integral depend on the regularization? The answer is not straightforward, especially for higher dimensional integrals.

2.1.1. An example. Consider the following function of two real variables:

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

Evidently it is defined everywhere except the origin. One can choose $f(0,0)$ to be any number. Regardless of the choice, f is not bounded in any neighborhood of the origin. Suppose one wants to integrate this function over a bounded closed region

$$\Omega = \{(x,y)\,|\,x^2+y^2 \le 1\,,\ x \ge 0\,,\ y \ge 0\}$$

which is the part of the unit disk that lies in the positive quadrant. So, f is not bounded on Ω. In attempt to mimic a one-dimensional improper integral, let us take a subregion $\Omega_a \subset \Omega$

$$\Omega_a = \{(x,y)\,|\,a^2 \le x^2+y^2 \le 1\,,\ x \ge 0\,,\ y \ge 0\}$$

so that Ω_a gets larger with decreasing a and becomes Ω when $a = 0$. Using the polar coordinates it is not difficult to show that

$$\iint_{\Omega_a} f(x,y)\,dxdy = 0$$

Alternatively, the result follows from the symmetry argument. The region Ω_a is symmetric under the reflection about the line $y = x$: $(x,y) \to (y,x)$, whereas the integrand is skew-symmetric, $f(x,y) = -f(y,x)$. Can one conclude that the improper Riemann integral of f over Ω exists and is equal to zero?

It is obvious that Ω can be obtained in many ways as the limit of subregions. For example, consider a collection of subregions which are defined in polar coordinates

$$x = r\cos(\theta)\,, \quad y = r\sin(\theta)$$

by the conditions

$$\Omega_k = \{(x,y)\,|\,a_k \le r \le 1\,, \quad \beta_k \le \theta \le \pi/2\}\,, \quad k = 1,2,...$$

where $\{a_k\}$ and $\{\beta_k\}$ are positive sequences that converge to 0 monotonically. These regions coincide with Ω_a with $a = a_k$ in which a small sector with the angle β_k is removed. So, with increasing k, the region Ω_k gets larger and eventually becomes Ω in the limit $k \to \infty$:

$$\Omega_k \subset \Omega_{k+1} \subset \Omega\,, \quad \bigcup_{k=1}^{\infty} \Omega_k = \Omega$$

The latter union is a proper mathematical way of saying that Ω_k "approaches Ω and coincides with Ω in the limit $k \to \infty$". Using polar coordinates

$$\iint_{\Omega_k} f(x,y)\,dxdy = \int_{a_k}^{1} \int_{\beta_k}^{\pi/2} \frac{r^2\cos(2\theta)}{r^4}\, rdrd\theta = \frac{1}{2}\sin(2\beta_k)\ln(a_k)$$

The right-hand side is an indeterminate form "$0 \times \infty$" in the limit $k \to \infty$. The limit may or may not exist and, even if it exists, it can have any value! Indeed, take $a_k = e^{-c/\beta_k}$ where $c > 0$ so that $a_k \to 0$ monotonically if $\beta_k \to 0$ monotonically. Then

$$\lim_{k\to\infty} \iint_{\Omega_k} f(x,y)\,dxdy = -\lim_{k\to\infty} \frac{c\sin(2\beta_k)}{2\beta_k} = -c$$

If $a_k = \beta_k$, then the limit is 0 and, if $a_k = e^{-c/\beta_k^2}$, $c > 0$, then the limit is $-\infty$. The reader is asked to verify that if the range of the polar angle in Ω_k is restricted to the interval $0 \le \theta \le \pi/2 - \beta_k$, then the limit can be made arbitrary positive number or $+\infty$ by a suitable choice of a_k.

So, the value of the improper integral really depends on its regularization, that is, on its very definition! A similar result can be established for the integral of f over an unbounded region $x^2 + y^2 \ge 1$, $x \ge 0$, $y \ge 0$ (see Exercises). Naturally, one wants a definitive (or unique) value of an improper integral, and, for this reason, our naive attempt to define improper Riemann integrals should be amended in some way to eliminate the noted deficiency.

2.2. Improper Riemann integrals. Let $\Omega \subset \mathbb{R}^N$ be bounded or unbounded. An *exhaustion* of Ω is a sequence of subsets $\{\Omega_k\}_1^\infty$ such that

- each Ω_k is bounded, closed, and contained in Ω;
- Ω_{k+1} contains Ω_k;
- the union of all Ω_k coincides with Ω except possibly a set of measure zero.

Examples of exhaustions were given in the previous section. If a bounded function f is Riemann integrable on a closed bounded set Ω and on each Ω_k, then by continuity of the Riemann integral

$$\lim_{k\to\infty} \int_{\Omega_k} f(x)\,d^N x = \int_{\Omega} f(x)\,d^N x\,.$$

If f is not bounded and/or Ω is not bounded, the improper integral is defined by demanding that the continuity property still holds.

DEFINITION 2.1. *Let $\{\Omega_k\}_1^\infty$ be an exhaustion of Ω. Suppose that a function f on Ω is Riemann integrable on each Ω_k. Then the function f is said to be Riemann integrable on Ω if the limit*

$$\lim_{k\to\infty}\int_{\Omega_k} f(x)\, d^N x = \int_\Omega f(x)\, d^N x$$

exists and is independent of the choice of Ω_k. In this case, the value of the limit is called an improper Riemann integral of f over Ω.

So, by this definition, the function of two variables considered in the previous section is not integrable on the part of a disk that lies in the positive quadrant or on its complement because the value of the limit depends on the choice of exhaustions (or regularization). To show that an improper integral does not exist, it is sufficient to find two regularizations in which the limits are not equal. However, it is impossible to check the independence of the limit by *computing* the improper integral in every possible regularization. It is therefore important to establish criteria for the existence of improper integrals so that if the limit exists in a particular regularization, then it exists in any other one and has the same value.

2.3. Improper integrals of non-negative functions. Suppose that $f(x)$ is non-negative on Ω. For any exhaustion, the sequence of integrals is monotonically increasing

$$0 \le \int_{\Omega_k} f(x)\, d^N x \le \int_{\Omega_{k+1}} f(x)\, d^N x$$

by the positivity property of the Riemann integral and that $\Omega_k \subset \Omega_{k+1}$. Any monotonic sequence converges if only if it is bounded. So, there are only two possibilities: either the limit is a number

$$\lim_{k\to\infty}\int_{\Omega_k} f(x)\, d^N x = \sup_k \int_{\Omega_k} f(x)\, d^N x = I_f$$

or it is infinite, $I_f = \infty$. Suppose that $I_f < \infty$. Let $\{\Omega'_k\}$ be another exhaustion of Ω. Then the sequence of the integrals is bounded:

$$\int_{\Omega'_k} f(x)\, d^N x \le I_f$$

because $f(x) \ge 0$ and $\Omega'_k \subset \Omega$ for any k'. Since the sequence is also increasing monotonically, it converges

$$\lim_{k\to\infty}\int_{\Omega'_k} f(x)\, d^N x = \sup_{k'} \int_{\Omega'_k} f(x)\, d^N x = I'_f \le I_f$$

and its limit cannot exceed I_f. On the other hand, one can swap the roles of the exhaustions and use the same argument show that

$$\int_{\Omega_k} f(x)\, d^N x \le I'_f \quad \Rightarrow \quad I_f \le I'_f$$

because $\Omega_k \subset \Omega$ and $f(x) \ge 0$. Therefore $I_f = I'_f$ and the value of the limit does not depend on the choice of the exhaustion.

Suppose now that $I_f = \infty$. Then $I'_f = \infty$. Indeed, if $I'_f < \infty$, then the sequence of integrals of f over $\{\Omega_k\}$ is bounded by $I'_f < \infty$ by the above argument (as $\Omega_k \subset \Omega$ for any k) so that, by taking the supremum over k, $I_f \le I'_f < \infty$, which is a contradiction.

THEOREM **2.1**. (Improper integral for non-negative functions) *Suppose that*

(i) $f(x) \ge 0\,, \quad \forall x \in \Omega$;
(ii) $\{\Omega_n\}$ *and* $\{\Omega'_n\}$ *are exhaustions of* Ω;
(iii) f *is Riemann integrable on each* Ω_n *and* Ω'_n

Then

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \lim_{n\to\infty} \int_{\Omega'_n} f(x)\, d^N x$$

where the limit can also be $+\infty$.

By the integrability of the absolute value, the functions

$$f_\pm(x) = \frac{1}{2}\Big(|f(x)| \pm f(x)\Big) \ge 0$$

are Riemann integrable if f is Riemann integrable. The function $f_+(x)$ coincides with $f(x)$ whenever $f(x) \ge 0$ and vanishes otherwise, whereas $f_-(x)$ coincides with $-f(x)$ whenever $f(x) \le 0$ and vanishes otherwise. Thus, any Riemann integrable function can be written as the difference of two non-negative integrable functions:

$$f(x) = f_+(x) - f_-(x)\,,$$
$$\int_\Omega f(x)\, d^N x = \int_\Omega f_+(x)\, d^N x - \int_\Omega f_-(x)\, d^N x$$

and vice versa (integrability of $f_\pm$ implies integrability of f and $|f| = f_+ + f_-$ by the linearity of the integral).

Using the limit laws, the following theorem can be established from the above representation.

THEOREM **2.2**. *Suppose that the improper integrals of* $f_\pm$ *over* Ω *exist. Then the improper integral of* f *over* Ω *exists and can be computed in any exhaustion* $\{\Omega_n\}$ *of* Ω.

Indeed, by the limit laws and the existence of the improper integral of $f_\pm$,

$$\begin{aligned}\int_\Omega f(x)\,d^Nx &= \lim_{n\to\infty}\int_{\Omega_n}\Big(f_+(x)-f_-(x)\Big)\,d^Nx\\ &= \lim_{n\to\infty}\int_{\Omega_n} f_+(x)\,d^Nx - \lim_{n\to\infty}\int_{\Omega_n} f_-(x)\,d^Nx\\ &= \int_\Omega f_+(x)\,d^Nx - \int_\Omega f_-(x)\,d^Nx\end{aligned}$$

and, by Theorem **2.1** the values of the limits in the right side of the equation do not depend on the choice of the exhaustion (or regularization) of the integrals.

COROLLARY **2.1**. *Let $\{\Omega_n\}$ be an exhaustion of Ω. Suppose that f and its absolute $|f|$ are integrable on each Ω_n and*

$$\lim_{n\to\infty}\int_{\Omega_n}|f(x)|\,d^Nx = \int_\Omega |f(x)|\,d^Nx < \infty$$

Then the improper integral of f over Ω exists and

$$\int_\Omega f(x)\,d^Nx = \lim_{n\to\infty}\int_{\Omega_n} f(x)\,d^Nx$$

In other words, if the improper integral of the absolute value of f converges in any particular regularization, then the improper integral of f exists and can be computed in any suitable regularization. Indeed, since

$$0 \le f_\pm(x) \le |f(x)|$$

It is concluded that monotonic sequences of integrals of $f_\pm$ over Ω_n are bounded:

$$0 \le \int_{\Omega_n} f_\pm(x)\,d^Nx \le \int_{\Omega_n}|f(x)|\,d^Nx \le \int_\Omega |f(x)|\,d^Nx < \infty$$

and, hence, converge. By Theorem **2.1** the limits are independent of the choice of Ω_n. By Theorem **2.2**, the improper integral of f over Ω exists (it is independent of regularization).

2.4. Absolutely and conditionally convergent integrals. Am improper Riemann integral of a function f over a region Ω is called *absolutely convergent* if

$$\lim_{n\to\infty}\int_{\Omega_n}|f(x)|\,d^Nx = I_f < \infty$$

The absolute convergence of the Riemann integral implies the existence of the improper Riemann integral.

If the limit

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x$$

exists for some exhaustion (regularization) $\{\Omega_n\}$ but the integral does not converge absolutely, then the integral of f is said to be *conditionally convergent* in the exhaustion $\{\Omega_n\}$. Absolutely and conditionally convergent integrals are analogous to absolutely and conditionally convergent series as illustrated below.

2.4.1. Conditionally convergent integrals. Let the integral of f over Ω be conditionally convergent. In this case, the integrals of $f_\pm$ must diverge, and the value of a conditionally convergent integral is an indeterminate form "$\infty - \infty$" which can happen to be a number in a particular regularization:

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \lim_{n\to\infty} \left(\int_{\Omega_n} f_+(x)\, d^N x - \int_{\Omega_n} f_-(x)\, d^N x \right)$$

Indeed, the divergence of the integral of $|f| = f_+ + f_-$ implies that either the integral of f_+, or f_-, or both diverge because $f_\pm \geq 0$. The conditional convergence of the integral of f (the existence of the limit in the left side) is only possible when the integrals of $f_\pm$ diverge.

The integrals of $f_\pm$ resemble the (divergent) series of positive and negative terms of a conditionally convergent series. The sum of such a series depends on the *arrangement* of terms (the order in which the terms are added). In the case of conditionally convergent integrals, the value depends on the choice of the exhaustion (or regularization). In other words, by choosing a suitable exhaustion one can always make the difference of the integrals of f_+ and f_- over Ω_n to be convergent to any desired number even though both the sequences diverges to $+\infty$, similarly to that the sum of a conditionally convergent numerical series can be made any number or infinity by a suitable rearrangement of terms [9]. This is illustrated with the following example.

Consider the improper integral

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \lim_{n\to\infty} \int_0^{b_n} \frac{\sin(x)}{x}\, dx$$

where $\{b_n\}$ is positive, monotonically increasing, unbounded sequence. Here the integrand extended to $x = 0$ by continuity (the integrand approaches 1 as $x \to 0^+$). In particular, let us take

$$b_n = \pi n\,, \quad n = 1, 2, ...$$

[9] This is known as the Riemann theorem about rearrangements (see, e.g., W. Rudin, Principles of mathematical analysis, Chapter 3).

This regularization corresponds to the exhaustion:

$$\Omega_n = [0, \pi n]$$

so that

$$\Omega_n = \Omega_{n-1} \cup S_n\,, \quad S_n = [\pi(n-1), \pi n]\,.$$

If the limit exists, then is equal to the sum of the series

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \sum_{n=1}^\infty \int_{S_n} \frac{\sin(x)}{x}\,dx = \sum_{n=1}^\infty \int_{\pi(n-1)}^{\pi n} \frac{\sin(x)}{x}\,dx$$

This is an alternating series because the integrand is positive on S_{2k-1} and negative on S_{2k}. It follows from the inequality

$$\frac{|\sin(x)|}{\pi n} \le \frac{|\sin(x)|}{x} \le \frac{|\sin(x)|}{\pi(n-1)}\,, \quad n > 1$$

that

$$\frac{2}{\pi n} \le a_n \le \frac{2}{\pi(n-1)}\,, \quad a_n = \int_{\pi(n-1)}^{\pi n} \frac{|\sin(x)|}{x}\,dx > 0$$

and

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \sum_{n=1}^\infty (-1)^{n+1} a_n$$

The sequence $\{a_n\}$ is positive and converges to 0 monotonically because

$$a_{n+1} \le \frac{2}{\pi n} \le a_n$$

By the alternating series test, the series converges.

However, by the comparison test:

$$\frac{2}{\pi}\sum_{k=1}^n \frac{1}{k} \le \int_0^{\pi n} \frac{|\sin(x)|}{x} = \sum_{k=1}^n a_k \quad \Rightarrow \quad \int_0^\infty \frac{|\sin(x)|}{x}\,dx = \infty$$

the series and the integral do not converge absolutely because $\sum \frac{1}{n} = \infty$. So, the integral is only conditionally convergent and does not exist in the sense of Definition **2.1**. Its value depends on the choice of regularization. In particular, the sum can be made equal to any desired number by a suitable rearrangement in the series. A rearrangement corresponds to a different exhaustion made of unions of the intervals S_n.

Consider a rearrangement $\{S'_n\}$ of the sequence of intervals $\{S_n\}$ and put

$$\Omega'_1 = S'_1\,, \quad \Omega'_{n+1} = \Omega'_n \cup S'_{n+1}$$

So, Ω'_n is a collection of any n intervals from $\{S_n\}$, and Ω'_{n+1} is obtained by adding *any* of remaining intervals in the collection $\{S_n\}$. In other words, the order in which the intervals from $\{S_n\}$ are added to obtain an exhaustion is changed, but

$$\bigcup_{n=1}^{\infty} \Omega_n = \bigcup_{n=1}^{\infty} \Omega'_n = [0,\infty)\,.$$

The function is still integrable on any finite collection of intervals Ω'_n. Therefore in this exhaustion (regularization)

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx = \sum_{n=1}^{\infty} \int_{S'_n} \frac{\sin(x)}{x}\,dx$$

The series in the right-hand side is a rearrangement of the alternating series $\sum_n (-1)^{n+1} a_n$.

Let us show that a rearrangement can converge to any number or $\pm\infty$. Fix a number $I_f > 0$. Make Ω'_1 to be the union of odd intervals, S_1, S_3 and so on until S_{2n-1} where n is the smallest integer for which the integral over the union becomes greater than I_f. Then start adding even intervals, S_2, S_4, and so on until the integral becomes less than I_f. The union of Ω'_1 and the added even intervals is Ω'_2. Then begin to add remaining odd intervals until the integral becomes greater than I_f again. The union of Ω'_2 and the added shells makes Ω'_3. In each step, the overshot or undershot necessarily occurs because the integrals over all odd and all even intervals diverge to $+\infty$ and $-\infty$, respectively. In this way, the sequence of integrals

$$\int_{\Omega'_n} f(x)\,d^N x\,, \quad \Omega'_n \subset \Omega'_{n+1}\,,$$

oscillates about I_f and converges to I_f because the overshot or undershot of the integral is decreasing with increasing the number of iterations:

$$\left| I_f - \int_{\Omega'_n} f(x)\,d^N x \right| \le \int_{S_{k_n}} |f(x)|\,d^N x \to 0$$

for some $k_n \ge n$ so that $k_n \to \infty$ as $n \to \infty$. Since $\{\Omega'_n\}$ is an exhaustion of Ω by construction, the integral conditionally converges to a preassigned positive number I_f. A similar exhaustion can be constructed to make the integral converging to any negative number. The reader is asked to construct exhaustions in which the integral converges to either $+\infty$ or $-\infty$, or does not converge at all (e.g., oscillates between any two numbers).

2.5. Absolutely convergent integrals. There are tests for absolute convergence of improper integrals that are analogous to the corresponding tests for absolute convergence of series.

2.5.1. The comparison test. Let f and g be Riemann integrable on Ω, and

$$|f(x)| \le g(x)\,, \quad \forall x \in \Omega$$

Then

$$\int_\Omega |f(x)|\,d^N x \le \int_\Omega g(x)\,d^N x$$

If now f and g are not integrable in the proper sense, then for any exhaustion $\{\Omega_n\}$

$$\int_{\Omega_n} |f(x)|\,d^N x \le \int_{\Omega_n} g(x)\,d^N x$$

If the improper integral of g converges, then the integral f converges absolutely because

$$\int_{\Omega_n} g(x)\,d^N x \le \int_\Omega g(x)\,d^N x < \infty$$
$$\Rightarrow \lim_{n\to\infty} \int_{\Omega_n} |f(x)|\,d^N x \le \int_\Omega g(x)\,d^N x$$

By Theorem **2.2** the limit does not depend on the choice of the exhaustion and the improper integral of f exists.

THEOREM **2.3**. (Comparison test for absolute convergence) *Let $\{\Omega_n\}$ be an exhaustion of a region Ω and a function f be integrable on any Ω_n. If the absolute value $|f(x)|$ is bounded on Ω by a function whose improper Riemann integral over Ω exists,*

$$|f(x)| \le g(x)\,, \quad x \in \Omega\,, \quad \lim_{n\to\infty} \int_{\Omega_n} g(x)\,d^N x < \infty\,,$$

then the improper integral of f over Ω also exists and converges absolutely.

2.5.2. Integrals over unbounded regions. Suppose $\Omega = \mathbb{R}^N$ and f is a continuous function. Clearly it is integrable on any ball $\Omega_n = B_n$ (that is, $|x| \le n$, $n = 1, 2, ...$). So the existence of the improper integral would depend on how fast f falls off as $|x| \to \infty$.

PROPOSITION **2.1**. *Let f be integrable on any ball and*

$$|f(x)| \le \frac{M}{|x|^p}\,, \quad |x| \ge R$$

for some positive constants M and R, and $p > N$, then the improper integral of f over the whole space exists and

$$\int_{\mathbb{R}^N} f(x)\, d^N x = \lim_{n\to\infty} \int_{|x|\le n} f(x)\, d^N x < \infty\,.$$

Consider the case $N = 2$. The integral over the whole plane is split into the integral over the disk B_R and the rest of the plane $\mathbb{R}^2 \setminus B_R$. Since the integral over B_R is a regular integral, one has to investigate the convergence of the improper integral over the rest of the plane. Since $|f(x)| \ge 0$, if it converges in a particular regularization, then it converges in any other regularization to the same value. Let Ω_n be an annulus $R \le |x| \le n$. Then

$$\int_{\Omega_n} |f(x)|\, d^2x \le \int_{\Omega_n} \frac{M}{|x|^p}\, d^2x = \int_0^{2\pi} \int_R^n \frac{M}{r^p}\, r dr d\theta$$
$$= \frac{2\pi M}{p-2}\left(\frac{1}{R^{p-2}} - \frac{1}{n^{p-2}}\right)$$

The right side converges if $p > N = 2$ when $n \to \infty$. Therefore the integral of f converges absolutely and, hence, the improper integral of f exists by the comparison test (it can be computed in any suitable regularization).

For $N > 2$ note that the volume of a spherical shell of thickness dr and radius r is the differential of the volume of the ball of radius r:

$$dV_N(r) = \sigma_N r^{N-1} dr$$

where $\sigma_N = NV_N(1)$ is the area of the unit sphere in $\mathbb{R}^N$. Then using spherical coordinates

$$\int_{\Omega_n} |f(x)|\, d^2x \le \int_{\Omega_n} \frac{M}{|x|^p}\, d^N x = \sigma_N \int_R^n \frac{M}{r^p}\, r^{N-1} dr$$

The integral converges in the limit $n \to \infty$ if $p > N$.

2.5.3. Integrals of unbounded functions. Suppose f is not bounded in any neighborhood of a particular point, and it is continuous otherwise. Without loss of generality, the singular point can be chosen to be the origin $x = 0$ (values of $|f(x)|$ becomes infinitely large as x approaches 0). Then the absolute integrability depends on how fast $|f(x)|$ diverges as $x \to 0$.

PROPOSITION **2.2**. *Suppose that f is not bounded in any ball B_a and integrable on $\Omega \setminus B_a$ where Ω contains $x = 0$. If*

$$|f(x)| \le \frac{M}{|x|^p}\,, \quad |x| \le a$$

for some constants M and R, and $p < N$, then the improper integral of f exists and

$$\int_\Omega f(x)\, d^N x = \lim_{a\to 0^+} \int_{\Omega\setminus B_a} f(x)\, d^N x\,.$$

A proof of this assertion can also be done by using spherical coordinates in $\mathbb{R}^N$. Let $\Omega_{a,R}$ be the intersection of Ω with the spherical shell $a^2 \le |x|^2 \le R^2$. The integral of f over $\Omega \setminus B_R$ exists by continuity of f. Then

$$\begin{aligned}\int\limits_{\Omega_{a,R}} |f(x)|\, d^N x &\le \int\limits_{a\le|x|\le R} \frac{M}{|x|^p}\, d^N x = \sigma_N \int_a^R \frac{M}{r^p}\, r^{N-1} dr\\ &= \frac{\sigma_N M}{N-p}\Big(R^{N-p} - a^{N-p}\Big)\end{aligned}$$

So the integral converges in the limit $a \to 0^+$ if $p < N$. Therefore the integral of f converges absolutely by the comparison test, and, hence, the improper integral of f exists.

2.6. Improper integrals of complex-valued functions. Let f be a complex-valued function of N real variables. If $\{\Omega_n\}$ is an exhaustion of Ω, then the integral of f over Ω is said to converge in this exhaustion if the integrals of the real and imaginary parts of f converge, and in this case

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \lim_{n\to\infty} \int_{\Omega_n} \operatorname{Re} f(x)\, d^N x + i \lim_{n\to\infty} \int_{\Omega_n} \operatorname{Im} f(x)\, d^N x$$

It follows from the inequalities

$$|\operatorname{Re} f| \le |f|\,, \quad |\operatorname{Im} f| \le |f|$$

that the integrals of the real and imaginary parts of f converge absolutely if the integral of the absolute value converges. The converse follows from the inequality

$$|f| \le |\operatorname{Re} f| + |\operatorname{Im} f|$$

that is,

- *the integral of a complex-valued function converges absolutely if and only if the integral of the absolute value converges.*

2.7. Gaussian integrals. The objective is to prove that

$$I_N(A,b) = \int_{\mathbb{R}^N} e^{-(x,Ax)+(b,x)}\, d^N x = \frac{\pi^{N/2}}{\det(A)}\, e^{\frac{1}{4}(b,A^{-1}b)} \tag{2.1}$$

where the quadratic form

$$(x, Ax) = \sum_{k,n=1}^{N} A_{kn} x_k x_n > 0\,, \quad \forall x \neq 0$$

is strictly positive if $x \neq 0$. The integrals of this type are known as *Gaussian integrals*. They are routinely used in various applications. Note that the integrand is positive and, hence, if the integral converges, then it converges absolutely. Therefore it can be computed in any convenient regularization.

2.7.1. A special case. Consider a two-dimensional Gaussian integral

$$I_2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dxdy$$

Let Ω_n be a disk of radius n, $x^2+y^2 \leq n^2$. Then using polar coordinates

$$\begin{aligned} I_2 &= \lim_{n\to\infty} \iint_{\Omega_n} e^{-x^2-y^2}\, dxdy = \lim_{n\to\infty} \int_0^{2\pi}\int_0^n e^{-r^2} r dr d\theta \\ &= \pi \lim_{n\to\infty} \int_0^{n^2} e^{-s}\, ds = \pi \end{aligned}$$

Since the value of the absolutely convergent integral does not depend on the regularization, put $\Omega_n' = [-n, n] \times [-n, n]$ so that by Fubini's theorem

$$\pi = \lim_{n\to\infty} \iint_{\Omega_n'} e^{-x^2-y^2}\, dxdy = \lim_{n\to\infty} \int_{-n}^{n} e^{-x^2} dx \int_{-n}^{n} e^{-y^2} dy$$

Therefore, by the limit laws,

$$I_1 = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

because $I_1^2 = \pi$. Using a scaling transformation, $y = \sqrt{a}x$

$$I_1(a) = \int_{-\infty}^{\infty} e^{-ax^2}\, dx = \frac{I_1(1)}{\sqrt{a}} = \frac{\sqrt{\pi}}{\sqrt{a}}$$

Furthermore using the scaling and shift transformation

$$y = \sqrt{a}\, x\,, \quad s = y - \frac{b}{2\sqrt{a}}$$

one infers that

$$\begin{aligned}
I_1(a,b) &= \int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = \lim_{n\to\infty}\int_{-n}^{n} e^{-ax^2+bx}\,dx \\
&= \frac{1}{\sqrt{a}}\lim_{n\to\infty}\int_{-n\sqrt{a}}^{n\sqrt{a}} e^{-y^2+\frac{b}{\sqrt{a}}y}\,dy \\
&= \frac{e^{\frac{b^2}{4a}}}{\sqrt{a}}\lim_{n\to\infty}\int_{-n\sqrt{a}-\frac{b}{2\sqrt{a}}}^{n\sqrt{a}+\frac{b}{2\sqrt{a}}} e^{-s^2}\,ds \\
&\overset{(1)}{=} \frac{I_1(1,0)}{\sqrt{a}}\,e^{\frac{b^2}{4a}} = \frac{\sqrt{\pi}}{\sqrt{a}}\,e^{\frac{b^2}{4a}}
\end{aligned}$$

Since the value of $I_1(1,0) = I_1$ does not depend on the choice of the exhaustion, the final equality (1) holds. The result (**2.1**) is established for $N = 1$.

2.7.2. General case. Let an exhaustion $\{\Omega_n\}$ of $\mathbb{R}^N$ be rectangular boxes, $|x_j| \leq n$, $j = 1, 2, ..., N$. Let A be a diagonal matrix with diagonal elements a_j. The condition $(x, Ax) > 0$ implies that the diagonal elements are strictly positive, $a_j > 0$. By Fubini's theorem one infers that

$$\begin{aligned}
I_N(A,b) &= \int_{\mathbb{R}^N} \exp\Big(-\sum_{j=1}^{N}(a_j x_j^2 - b_j x_j)\Big)\,d^N x \\
&= \lim_{n\to\infty}\prod_{j=1}^{N}\int_{-n}^{n} e^{-a_j x_j^2 + b_j x_j}\,dx_j \\
&= \prod_{j=1}^{N} I_1(a_j, b_j) = \frac{\pi^{N/2}}{\sqrt{a_1 a_2 \cdots a_N}}\,\exp\left(\frac{1}{4}\sum_{j=1}^{N}\frac{b_j^2}{a_j}\right)
\end{aligned}$$

Any matrix A can be written as a sum of symmetric and skew-symmetric matrix:

$$A = \frac{1}{2}\Big(A + A^T\Big) + \frac{1}{2}\Big(A - A^T\Big) \equiv B + C$$

where B is symmetric, $B^T = B$ (here B^T denotes the transposed matrix B), and C is skew-symmetric, $C^T = -C$. A quadratic form vanishes identically for a skew-symmetric matrix because

$$(x, Cx) = (C^T x, x) = -(Cx, x) = -(x, Cx) \quad \Rightarrow \quad (x, Cx) = 0$$

Therefore without loss of generality $A = A^T$ (a symmetric matrix). Any symmetric matrix A is diagonalizable, and there exists an orthogonal matrix U,

$$U^T = U^{-1}$$

such that

$$A = U^T a U\,, \quad a_{ij} = a_j \delta_{ij}$$

where a is a real diagonal matrix. The diagonal elements a_j are eigenvalues of A. The positivity of a quadratic form requires that all eigenvalues of A are strictly positive because

$$(x, Ax) = (x, U^T a U x) = (Ux, aUx) = (y, ay) = \sum_{j=1}^{N} a_j y_j^2$$

and $Ux = 0$ if and only if $x = 0$ since U is the invertible matrix.

The transformation $y = Ux$ preserves the distance in $\mathbb{R}^N$ because

$$|y|^2 = (y, y) = (Ux, Ux) = (x, U^T U x) = (x, x) = |x|^2$$

because $U^T U = U U^T = I$ is the unit matrix $I_{ij} = \delta_{ij}$. An orthogonal transformations is a composition of rotations and reflections ($x_j \to p_j x_j$ where $p_j = \pm 1$). Owing to the absolute convergence of the Gaussian integral with a diagonal matrix A and that the Jacobian of an orthogonal transformation is equal to one,

$$d^N x = \left| \det \left(\frac{\partial x_j}{\partial y_i} \right) \right| d^N y = |\det U^T|\, d^N y = d^N y$$

one infers that

$$\begin{aligned} I_N(A, b) &= \lim_{n\to\infty} \int_{\Omega_n} e^{-(x,Ax)+(b,x)} d^N x = \lim_{n\to\infty} \int_{U(\Omega_n)} e^{-(y,ay)+(Ub,y)} d^N y \\ &= \int_{\mathbb{R}^N} e^{-(y,ay)+(Ub,y)} d^N y \\ &= \frac{\pi^{N/2}}{\sqrt{a_1 a_2 \cdots a_N}} \exp\left(\frac{1}{4} \sum_{j=1}^{N} \frac{c_j^2}{a_j} \right) \end{aligned}$$

where $c = Ub$. Since the transformation U preserves the distances between points, for any exhaustion $\{\Omega_n\}$, the image $\{U(\Omega_n\}$ is also an exhaustion of $\mathbb{R}^N$.

Next, note that

$$a_1 a_2 \cdots a_N = \det a = \det(UAU^T) = (\det U)^2 \det A = \det A$$

Therefore $\det A \neq 0$ and the inverse A^{-1} exists and

$$A^{-1} = (U^T a U)^{-1} = U^{-1} a^{-1} (U^T)^{-1} = U^T a^{-1} U$$

It is then concluded that

$$\sum_{j=1}^{N} \frac{c_j^2}{a_j} = (c, a^{-1}c) = (Ub, a^{-1}Ub) = (b, U^T a^{-1}Ub) = (b, A^{-1}b)$$

and the relation (**2.1**) follows.

2.8. Exercises.

1. Consider the improper integral

$$\iint_\Omega \frac{x^2 - y^2}{(x^2+y^2)^2}\, dxdy$$

where

$$\Omega = \{(x,y)\,|\, x^2+y^2 \geq 1\,,\ x \geq 0\,,\ y \geq 0\}$$

Take an exhaustion $\{\Omega_n\}$ which is a rectangle in the polar coordinates $(r,\theta) \in [1, a_n] \times [\alpha_n, \pi/2 - \beta_n]$ where α_n and β_n are positive and tend to 0 monotonically, while $1 < a_n$ increases monotonically to infinity, as $n \to \infty$. Show that there is a choice of α_n, β_n, and a_n such that the sequence of integrals over Ω_n can converge to any real number or $\pm\infty$.

2. For the integrand in Problem 1, find $f_\pm(x,y)$ and show that for any exhaustion the improper integrals of $f_\pm$ diverge

$$\iint_\Omega f_\pm(x,y)\, dxdy = \infty$$

3. Prove Proposition **2.1** for $N = 3$ using spherical coordinates.

4. Prove Proposition **2.2** for $N = 3$ using spherical coordinates.

5. Let $f(t)$ be a continuous function and $t \geq 0$. Then for any ball $B_a \subset \mathbb{R}^N$

$$\int_{B_a} f(|x|)\, d^N x = \sigma_N \int_0^a f(r)\, r^{N-1} dr\,, \quad \sigma_N = \int_{|x|=1} dS = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,,$$

where σ_N is the surface area of the unit sphere $|x| = 1$. The assertion is proved by converting the integral to spherical coordinates in $\mathbb{R}^N$. Use this result to prove Propositions **2.1** and **2.2**.

6. Let p and q be positive integers. Do the following improper integrals exist in the sense of Definition **2.1**?

$$\text{(i)} \quad \int_0^\infty \frac{\sin^2(x)}{x^p}\,dx$$

$$\text{(ii)} \quad \int_1^\infty \frac{\cos^q(x)}{x^p}\,dx\,,$$

$$\text{(iii)} \quad \int_0^1 \sin^p\left(\frac{1}{x}\right)\,dx$$

7. Put

$$I_n(a) = \int_0^\infty x^n e^{-ax^2}\,dx\,, \quad n = 0, 1, ..$$

(i) Show that the integral converges absolutely.
(ii) Use integration by parts to prove the recurrence relation

$$I_{n+2} = \frac{n+1}{2a}\,I_n$$

(iii) Find $I_0(a)$ and $I_1(a)$. Use the above recurrence relation to find $I_n(a)$.

8. Let σ_N be the surface area of a unit sphere $|x| = 1$ in $\mathbb{R}^N$.
(i) Let Ω_n be an exhaustion of $\mathbb{R}^N$ made of balls $|x| \leq n$. Show that

$$\int_{\Omega_n} e^{-(x,x)}\,d^N x = \sigma_N \int_0^n e^{-r^2} r^{N-1}\,dr$$

(ii) Use this result and the result of Problem 1 to find σ_N and the volume $V_N(a)$ of a ball of radius a in terms of Euler's gamma function.

3. Lebesgue integral

3.1. Piecewise continuous functions on $\mathbb{R}$. Suppose a function f is not continuous at a point $x = c$ and has a *jump discontinuity* at $x = c$. The latter means that the right and left limits of $f(x)$ at $x = c$ exist but are not equal:

$$\lim_{x\to c^+} f(x) = f_+(c)\,, \quad \lim_{x\to c^-} f(x) = f_-(c)\,, \quad f_+(c) \neq f_-(c)$$

A piecewise continuous function is a function that is not continuous at finitely many points in any bounded interval and has jump discontinuities at these points.

First note that points at which a piecewise continuous function has jump discontinuities form a countable set. Indeed a real line can be viewed as the union of countable many intervals and in each such interval the function has finitely many jump discontinuities. So, a collection $\{c_n\}$ of all such points is either finite or form a sequence. The sequence cannot have any limit point because otherwise the function would have infinitely many jump discontinuities in any open interval containing the limit point. In each interval (c_n, c_{n+1}), the function is continuous and has a *continuous extension* to $[c_n, c_{n+1}]$.

Put

$$m = \inf\{c_n\}\,, \qquad M = \sup\{c_n\}$$

If the sequence $\{c_n\}$ is not bounded from below, then $m = -\infty$ and otherwise m is the smallest number in $\{c_n\}$. If the sequence $\{c_n\}$ has no upper bound, then $M = \infty$ and otherwise M is the largest number in the collection $\{c_n\}$. Clearly, if $-\infty < m \leq M < \infty$, that is, the collection $\{c_n\}$ has the smallest and largest number, then the collection must be finite. Let $\{\Omega_n\}$ denote a collection of open intervals (c_n, c_{n+1}) together with $(-\infty, m)$ and (M, ∞) (if these intervals are not empty). This collection of intervals has the following characteristic properties:

(i) the intervals do not overlap:

$$\Omega_n \cap \Omega_{n'} = \emptyset\,, \quad n \neq n'\,,$$

(ii) any bounded interval (a, b) is covered by finitely many closed intervals $\overline{\Omega}_n$:

$$(a,b) \subset \bigcup_{n=j}^{k} \overline{\Omega}_n\,,$$

(iii) the union of closures of the intervals coincides with whole real line:

$$\bigcup_n \overline{\Omega}_n = \mathbb{R}\,.$$

This observation allows us to give an alternative definition of a piecewise continuous function which can be extended to the multivariable case.

Definition **3.1**. (A piecewise continuous function)
A function $f : \mathbb{R} \to \mathbb{R}$ *is said to be* piecewise continuous *on* $\mathbb{R}$ *if there exists an at most countable collection of open intervals* Ω_n *with no common points such that any bounded interval is covered by finitely many closed intervals* $\overline{\Omega_n}$, *and* $f \in C^0(\overline{\Omega_n})$.

A piecewise continuous function is not continuous on $\{c_n\}$ which is a set measure zero. One can also say that *a piecewise continuous function is continuous almost everywhere*. Therefore any piecewise continuous function is Riemann integrable on any $[a, b]$. The value of the Riemann integral does not depend on the values of a piecewise continuous function at the points where it is not continuous.

3.2. Measurable functions on $\mathbb{R}$. Let $\mathcal{A}$ be a set of functions that is defined by some characteristic property (e.g., continuity, or integrability, etc.). Then the limit function of a pointwise convergent sequence $\{f_n\} \subset \mathcal{A}$ does not in general belong to $\mathcal{A}$. One can ask how large the set $\mathcal{A}$ should be in order to be *complete* in the sense that the limit function of every pointwise convergent sequence in $\mathcal{A}$ belongs to $\mathcal{A}$. It turns out that such a set of functions exists and is known as a set of *measurable functions*.

Suppose that a sequence $\{f_n\}$ of functions on $\mathbb{R}$ converges pointwise almost everywhere. In other words, a numerical sequence $\{f_n(x)\}$ can have no limit for some points x that form a set of measure zero. In this case, one writes

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e.$$

For example,

$$\lim_{n\to\infty} [\cos(\pi x)]^n = 0 \quad a.e.$$

Note that the limit does not exist if x is an integer. If x is not an integer, then $|\cos(\pi x)| < 1$ and the limit is equal to zero. But the integers form a set of measure zero.

Definition **3.2**. (A measurable function)
A function f *is called* measurable *if it coincides almost everywhere with the limit of an almost everywhere convergent sequence of piecewise continuous functions.*

3.2.1. Measurable sets. A set of real numbers is called *measurable* if its characteristic function is measurable. Clearly, any interval (bounded or unbounded, closed or open or semi-open) is measurable. Any set of measure zero is measurable. The following properties of measurable sets can also be established:

- *The complement of a measurable set is measurable.*
- *The union or intersection of countably many measurable sets is measurable.*
- *Every open or closed set is measurable.*

3.3. Properties of measurable functions. [10] Evidently, every piecewise continuous function f is measurable because one can take a sequence of piecewise continuous functions $f_n(x) = f(x)$ of identical terms which obviously converges to $f(x)$. Suppose that f is a measurable function and g coincides with f almost everywhere. Then g is also measurable. Indeed, Let f_n be a sequence of piecewise continuous functions that converges to f almost everywhere. Since f and g differ only on a set of measure zero, f_n converges to g almost everywhere, too:

$$\left.\begin{array}{l} f(x) \text{ is measurable} \\ f(x) = g(x) \ a.e. \end{array}\right\} \quad \Rightarrow \quad g(x) \text{ is measurable}$$

3.3.1. Algebraic operations with measurable functions. Using the basic limit laws, it is not difficult to see that the set of measurable functions is *closed* relative to algebraic operations of addition, multiplication, and division:

$$\left.\begin{array}{l} f(x) \text{ is measurable} \\ g(x) \text{ is measurable} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} f(x)+g(x) \text{ is measurable} \\ f(x)g(x) \text{ is measurable} \\ f(x)/g(x),\ g(x)\neq 0, \text{ is measurable} \end{array}\right.$$

Indeed, if $f_n(x)$ and $g_n(x)$ are sequences of piecewise continuous functions, then the functions $f_n(x) + g_n(x)$, $f_n(x)g_n(x)$, and $f_n(x)/g_n(x)$, $g_n(x) \neq 0$, also form sequences of piecewise continuous functions, and the above assertion follows from the basic laws of limits. This also implies that linear combinations of measurable functions are measurable. Sets that are complete relative to additions and multiplications by a number are called a linear space. Thus, *the set of measurable functions is a linear space*.

[10] Proofs of the listed properties of measurable functions can be found in: A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis

3.3.2. Absolute value of a measurable function. Given two functions f and g, define the following functions

$$\max(f,g)(x) = \begin{cases} f(x)\ , & f(x) > g(x) \\ g(x)\ , & f(x) \le g(x) \end{cases}$$
$$\min(f,g)(x) = \begin{cases} g(x)\ , & f(x) > g(x) \\ f(x)\ , & f(x) \le g(x) \end{cases}$$

One can prove that *the functions* $\max(f,g)$ *and* $\min(f,g)$ *are measurable, if* f *and* g *are measurable*. It follows that *the absolute value*

$$|f(x)| = \max(f,0)(x) - \min(f,0)(x)$$

of a measurable function f *is measurable.*

3.3.3. Measurable and Riemann integrable functions. One can prove the following property

PROPOSITION **3.1**. *A function that is not continuous on a set of measure zero is measurable.*

Therefore every Riemann integrable function is measurable by Theorem **1.8**. Furthermore, every function for which the improper Riemann integral exists is also measurable. So, *the set of measurable functions contains all Riemann integrable functions (either in the proper or improper sense).*

There are measurable functions that are not Riemann integrable. For example, the Dirichlet function introduced in Section **1.4.6** is measurable but not Riemann integrable on any interval. The set $\mathbb{Q}$ of rational numbers has measure zero in $\mathbb{R}$. Therefore $f_D(x) = 0$ a.e., but any constant function and, in particular, $g(x) = 0$ is measurable and, hence, so is the Dirichlet function.

3.3.4. Composition of measurable functions. A composition of measurable functions is measurable

3.3.5. Completeness of the set of measurable functions.

THEOREM **3.1**. *A function that coincides almost everywhere with the limit of an almost everywhere convergent sequence of measurable functions is measurable.*

3.3.6. Non-measurable sets and functions. Thus, the set of measurable functions is quite large. Are there non-measurable functions and sets?

It appears that one can prove that they exist[11] using the so called *axiom of choice*:

- Let $\{E_a\}$ be a collection of subsets of a set E (the indexing set a is of arbitrary nature). Then there exists a *choice function*, $a \to x(a)$ where $x(a) \in E_a$ for all a.

No example of an explicit non-measurable functions has been constructed so far. This suggests that all functions and sets that can possibly be used in applications or otherwise are measurable. For this reason, in what follows all sets are assumed to be measurable and all functions are assumed to be measurable and bounded almost everywhere.

3.4. Definition of the Lebesgue integral. To avoid any confusion between Riemann and Lebesgue integrals, the Riemann integral (proper or improper) will be denoted as

$$\mathcal{R}\int_\Omega f(x)\,dx\,, \quad \Omega \subseteq \mathbb{R}\,,$$

in what follows.

Definition 3.3. (The space $\mathcal{L}_+$)
Let a real function $f(x)$ be the limit of a non-decreasing sequence of piecewise continuous functions $f_n(x)$ such that the sequence of Riemann integrals is bounded:

$$f_n(x) \le f_{n+1}(x)\,, \quad n = 1, 2, ..., \,, \quad \forall x \in \mathbb{R}\,,$$
$$\mathcal{R}\int f_n(x)\,dx \le M\,, \quad n = 1, 2, \ldots\,,$$

for some number M. The limit of the non-decreasing sequence of Riemann integrals is called the Lebesgue integral *of f and is denoted by the symbol $\int f(x)dx$ so that*

$$\int f(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int f_n(x)\,dx\,.$$

The set of all such functions is denoted by $\mathcal{L}_+$.

Definition 3.4. (Lebesgue integral)
A function f is called Lebesgue integrable *if it can be represented as the difference of two functions from the set $\mathcal{L}_+$:*

$$f(x) = f_1(x) - f_2(x)\,, \quad f_1 \in \mathcal{L}_+\,,\ f_2 \in \mathcal{L}_+$$

[11] see, e.g., A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5, Sec. 1.3

The number

$$\int f_1(x)\,dx - \int f_2(x)\,dx = \int f(x)\,dx$$

is called the Lebesgue integral *of the function f. The set of all Lebesgue integrable functions is denoted by $\mathcal{L}$.*

3.4.1. The Lebesgue integral over a set. A function f is said to be Lebesgue integrable on a measurable set $\Omega \subset \mathbb{R}$, if $f\chi_\Omega \in \mathcal{L}$, where χ_Ω is the characteristic function of Ω, and the number

$$\int f(x)\chi_\Omega(x)\,dx = \int_\Omega f(x)\,dx$$

is called the Lebesgue integral of f over Ω. The class of all Lebesgue integrable functions is denoted by $\mathcal{L}(\Omega)$.

3.4.2. Consistency of the definition. Definition **3.3** makes sense only if the Lebesgue integral does not depend on the choice of the sequence $\{f_n\}$. Similarly, Definition **3.4** is consistent if the Lebesgue integral is independent of the choice of f_1 and f_2. To show the consistency, the following property of the Lebesgue integral has to be established.

PROPOSITION **3.2**. *Suppose that $f \in \mathcal{L}_+$ and $f(x) \geq 0$ a.e. Let $\{f_n\}$ be a sequence satisfying the hypotheses of Definition* **3.3**. *Then the Lebesgue integral is non-negative,*

$$\int f(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int f_n(x)\,dx \geq 0$$

Let $f_1(x) = 0$ if $|x| > R$ and $f_1(x) \geq -M$ if $|x| \leq R$ for some positive numbers R and M. Since $\{f_n\}$ is monotonically increasing,

$$f_n(x) \geq 0\,, \quad |x| > R\,; \quad f_n(x) \geq -M\,, \quad |x| \leq R\,.$$

Let $S \subset [-R, R]$ be a set of points at which either one of the terms of the sequence has a jump discontinuity or the sequence $\{f_n(x)\}$ does not converge to $f(x)$. Then S is a set of measure zero. It can be covered by the union I_ε of open intervals whose total length does not exceed an arbitrary small positive number $\varepsilon > 0$. The set $\Omega = [-R, R] \setminus I_\varepsilon$ is bounded and closed. The functions f_n are continuous on Ω and $f_n(x) \to f(x) \geq 0$ for any x in Ω. Therefore for any $x \in \Omega$, one can find an integer n_x (that depends on x) and a positive number $\delta(x) > 0$ such that

$$f_{n_x}(y) \geq -\varepsilon\,, \quad y \in B(x, \delta_x) = (x - \delta_x, x + \delta_x)\,.$$

The neighborhoods $B(x, \delta_x)$, $x \in \Omega$, form an open cover of the closed and bounded set $\Omega \subset \mathbb{R}$. By the Heine-Borel theorem (Sec.

1.1.3), this cover has a finite subcover $B(x_j, \delta_j)$. Put $n_0 = \max_j\{n_{x_j}\}$. Since $\{f_n\}$ is non-decreasing,

$$f_{n_0}(x) \geq f_{n_x}(x) \geq -\varepsilon\,, \quad x \in \Omega \subseteq B_R = [-R, R]\,.$$

and outside of Ω

$$f_{n_0}(x) \geq -M\,, \quad x \in I_\varepsilon$$

Then for all $n > n_0$,

$$\begin{aligned} \mathcal{R}\int f_n(x)\,dx &\geq \mathcal{R}\int f_{n_0}(x)\,dx \geq \mathcal{R}\int_{-R}^{R} f_{n_0}(x)\,dx \\ &\geq -\varepsilon\mathcal{R}\int_{B_R} dx - M\mathcal{R}\int_{I_\varepsilon} dx \geq -\varepsilon(2R + M) \end{aligned}$$

because the total length of intervals in the union I_ε does not exceed ε. This inequality implies that

$$\int f(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int f_n(x)\,dx \geq -\varepsilon(2R + M)$$

Since $\varepsilon > 0$ can be chosen less than any preassigned positive number, the Lebesgue integral of f is non-negative.

Proposition **3.3**. *Let $f \in \mathcal{L}_+$. Then the Lebesgue integral of f is independent of the choice of a sequence $\{f_n\}$ in Definition* **3.3**.

Let $\{f_n\}$ and $\{g_n\}$ be two sequences that define the Lebesgue integral of a function f from $\mathcal{L}_+$. Put

$$A = \lim_{n\to\infty} \mathcal{R}\int f_n(x)\,dx\,, \quad B = \lim_{n\to\infty} \mathcal{R}\int g_n(x)\,dx\,.$$

One has to show that the limits are equal, $A = B$. For any k,

$$\lim_{n\to\infty}\Big(f_n(x) - g_k(x)\Big) = f(x) - g_k(x) \geq 0\,, \quad \text{a.e.}$$

because $\{g_k\}$ is monotonically increasing and converging to f. By Proposition **3.2** applied to the sequence $\{f_n - g_k\}$ (k is fixed) that converges to $f - g_k$, it is concluded that

$$\lim_{n\to\infty} \mathcal{R}\int \Big(f_n(x) - g_k(x)\Big)\,dx = A - \mathcal{R}\int g_k(x)\,dx \geq 0\,.$$

This inequality holds for any k. Therefore, by taking the limit $k \to \infty$, it is found that $A \geq B$. Swapping the roles of the sequences $\{f_n\}$ and $\{g_n\}$ in this reasoning, it is inferred that $B \geq A$, and, hence, $A = B$.

Proposition **3.4**. *Let $f \in \mathcal{L}$. Then the Lebesgue integral of f is independent of the choice of functions f_1 and f_2 in Definition* **3.4**.

Suppose that there is another pair of functions $g_{1,2} \in \mathcal{L}_+$ such that

$$f_1(x) - f_2(x) = f(x) = g_1(x) - g_2(x)\,.$$

It follows from the basic laws for limits and Definition **3.3** that $f_1 + g_2$ and $g_1 + f_2$ are also from $\mathcal{L}_+$ and

$$\int \Big(f_1(x) + g_2(x)\Big)\,dx = \int f_1(x)\,dx + \int g_2(x)\,dx$$

and similarly for $g_1 + f_2$. Therefore integrating the equality

$$f_1(x) + g_2(x) = g_1(x) + f_2(x)$$

and using the above integral relation it is concluded that

$$\int f(x)\,dx = \int f_1(x)dx - \int f_2(x)dx = \int g_1(x)dx - \int g_2(x)dx\,.$$

This completes the proof of consistency of the definition of the Lebesgue integral.

3.5. Riemann and Lebesgue integrals in $\mathbb{R}$. If the Lebesgue integral of a piecewise continuous function f over any bounded interval coincides with the Riemann integral because any such function is from class $\mathcal{L}_+$:

$$f \in C^0[a,b] \quad \Rightarrow \quad \int_a^b f(x)\,dx = \mathcal{R}\int_a^b f(x)\,dx$$

Note that one can take $f_n(x) = f(x)\chi_{[a,b]}(x)$ in Definition **3.3**.

3.5.1. Linearity. By the limit laws and linearity of the Riemann integral, the Lebesgue integral is also linear, that is, if f and g are integrable, then their linear combination is integrable and

$$\int \Big(\alpha f(x) + \beta g(x)\Big)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx\,.$$

3.5.2. Lebesgue integrability and set of measure zero. One of the key differences between the Lebesgue and Riemann integrals is that alterations of an integrable function on a set measure zero does not affect integrability and the value of the integral does not change. Let $f(x) = 0$ a.e. Put $f_n(x) = 0$ in Definition **3.3**. Clearly f_n converges to f almost everywhere. Therefore

$$f(x) = 0 \ \text{a.e.} \quad \Rightarrow \quad \int_\Omega f(x)\,dx = 0$$

In particular, the Lebesgue integral of the Dirichlet function vanishes over any (measurable) set

$$\int_\Omega f_D(x)\,dx = 0$$

because $f_D(x) = 0$ a.e.

One can show that the converse is also true if f is a non-negative function[12]

Proposition 3.5. *Let $f(x) \geq 0$. Then its Lebesgue integral vanishes if and only if $f(x) = 0$ almost everywhere.*

It follow from linearity of Lebesgue integral that if $f \in \mathcal{L}$ and g differs from f only on a set of measure zero, then g is also integrable and its integral is equal to the integral of f:

$$f \in \mathcal{L}\,, \quad g(x) = f(x) \text{ a.e.} \quad \Rightarrow \quad g \in \mathcal{L}\,, \int g(x)\,dx = \int f(x)\,dx$$

Thus, in full contrast to the Riemann integral, the Lebesgue integral is insensitive to alterations of an integrable function on sets of measure zero. Note that if f is continuous and $g(x) = f(x)$ a.e., then g can be continuous nowhere, just like the Dirichlet function, and hence g may not even be Riemann integrable.

3.5.3. Lebesgue integrability of Riemann integrable functions. Let us show that *any function f that is Riemann integrable on $[a, b]$ is Lebesgue integrable and*

$$\mathcal{R}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

As noted earlier, for any Riemann integrable function f there exists a sequence of partitions P_n such that P_{n+1} is a refinement of P_n and

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = \mathcal{R}\int_a^b f(x)\,dx$$

Define two sequences of piecewise constant functions

$$L_n(x) = m_s\,, \quad x \in R_s\,, \qquad U_n(x) = M_s\,, \quad x \in R_s$$

where R_s are partition intervals for P_n. Then

$$L_n(x) \leq L_{n+1}(x) \leq f(x) \leq U_{n+1}(x) \leq U_n(x)$$

[12] see, e.g., A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5.

The sequence $\{L_n(x)\}$ is monotonically increasing and bounded from above, and the $\{U_n(x)\}$ is monotonically decreasing and bounded from below. Therefore they converge for all x:

$$\lim_{n\to\infty} L_n(x) = L(x)\,, \quad \lim_{n\to\infty} U_n(x) = U(x)\,.$$

and

$$L(x) \le f(x) \le U(x)\,, \quad a \le x \le b\,.$$

The limit function L is Lebesgue integrable because the sequence of its Riemann integrals is nothing but the sequence of lower sums for f:

$$\int_a^b L(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int_a^b L_n(x) = \lim_{n\to\infty} L(P_n, f)$$

Similarly, the function $-U(x)$ is also Lebesgue integrable because the sequence $\{-U_n\}$ satisfies the conditions in Definition **3.3**. Therefore U is Lebesgue integrable and

$$\int_a^b U(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int_a^b U_n(x)\,dx = \lim_{n\to\infty} U(P_n, f)$$

Thus, the Lebesgue integrals of U and L are equal, and therefore the integral of a non-negative function $U(x) - L(x) \ge 0$ vanishes. By Proposition **3.5**, this implies that $U(x) = L(x)$ a.e. and, hence,

$$f(x) = L(x) \quad \text{a.e.}$$

from which it follows that f is Lebesgue integrable and

$$\int_a^b f(x)\,dx = \mathcal{R}\int_a^b f(x)\,dx\,.$$

3.5.4. Lebesgue and improper Riemann integrals. Suppose that f is not bounded on (a, b) but $f \in C^0(a, b)$ (f is singular at one or both endpoints of the interval). Suppose that the improper Riemann integral of f over (a, b). This implies that

$$\mathcal{R}\lim_{n\to\infty} \int_{a_n}^{b_n} f_\pm(x)\,dx = \mathcal{R}\int_a^b f_\pm(x)\,dx < \infty$$

for an exhaustion $[a_n, b_n] \subset (a, b)$ where a_n and b_n converge monotonically to a and b, respectively. It follows that $f_\pm$ are from class $\mathcal{L}_+$ because they are limits of monotonically increasing sequences of piecewise continuous functions $\chi_n(x)f_\pm(x)$ where χ_n is the characteristic function of $[a_n, b_n]$ whose Riemann integrals are bounded. Since $f(x) = f_+(x) - f_-(x)$, it is concluded that f is Lebesgue integrable on (a, b) and its Lebesgue integral is equal to the improper Riemann integral. Conversely, if a continuous function is Lebesgue integrable, then its Riemann integral converges absolutely is equal to the Lebesgue integral. Clearly, the argument can readily be extended to a continuous (or piecewise continuous) function on an unbounded interval. So, for *continuous (or piecewise continuous) functions, the Lebesgue and absolutely convergent Riemann integrals are equivalent.*

In fact, a more general assertion is true (see Sec. **4.10**).

PROPOSITION **3.6**. *If the function $f(x)$ and $|f(x)|$ are Riemann integrable on a set Ω (possibly in the improper sense), then they are Lebesgue integrable on Ω, and their Lebesgue and Riemann integrals are equal:*

$$\mathcal{R}\int_\Omega f_\pm(x) < \infty\, dx \quad \Rightarrow \quad \int_\Omega f(x)\, dx = \mathcal{R}\int_\Omega f(x)\, dx$$

where $f_\pm(x) = \frac{1}{2}(|f(x)| \pm f(x))$.

Thus, any function g that coincides almost everywhere with an absolutely Riemann integrable function f is Lebesgue integrable and, in this case, the Lebesgue integral of g is equal to the Riemann integral of f.

3.5.5. The Lebesgue integral of a complex-valued function. A complex-valued function $f(x)$, $x \in \mathbb{R}$, is said to be integrable if its real and imaginary parts are integrable, and in this case

$$\int f(x)dx = \int \operatorname{Re} f(x)\, dx + i\int \operatorname{Im} f(x)\, dx\,,$$

It follows from Proposition **3.6** that if the Riemann integral of a complex-valued function converges absolutely, then the function is Lebesgue integrable and its Lebesgue and Riemann integrals are equal.

3.6. Lebesgue integral in $\mathbb{R}^N$. The Lebesgue integral in any Euclidean space is defined in the same way, that is, as a limit of Riemann integrals of piecewise continuous functions.

3.6.1. Piecewise continuous functions on $\mathbb{R}^N$. Recall that a region is an open connected set $\mathbb{R}^N$. A function f is called *piecewise continuous* in $\mathbb{R}^N$ if

(i) there is at most countably many non-intersecting regions Ω_n, $n = 1, 2, ...,$
(ii) with piecewise smooth boundaries $\partial\Omega_n$,
(iii) any ball is contained in the union of finitely many closed regions $\overline{\Omega}_n$,
(iv) the union of $\overline{\Omega}_n$ coincides with $\mathbb{R}^N$, and

(v) $f \in C^0(\overline{\Omega}_n)$

This definition is to be compared with the definition of a piecewise continuous function on $\mathbb{R}$. Regions Ω_n are analogs of open intervals.

A piecewise continuous function is continuous almost everywhere and at any point where it is not continuous the function can only have a jump discontinuity. A piecewise continuous function is bounded on any ball. Therefore a piecewise continuous function with a bounded support is Riemann integrable on $\mathbb{R}^N$.

3.6.2. Definition of the Lebesgue integral in $\mathbb{R}^N$. Let a real-valued function f coincide almost everywhere with the limit of a non-decreasing sequence of piecewise continuous functions $f_n(x)$,

$$f_n(x) \le f_{n+1}(x)\,, \quad \forall x \in \mathbb{R}^N\,, \quad n = 1, 2, ..$$

such that the sequence of the Riemann integrals is bounded:

$$\mathcal{R}\int f_n(x)\, d^N x \le M\,,$$

for all n, where the Riemann integral is understood in the improper sense if supports of f_n are not finite. The limit

$$\lim_{n\to\infty} \mathcal{R}\int f_n(x) d^N x = \int f(x) d^N x < \infty$$

of this non-decreasing bounded sequence is called the Lebesgue integral of f. The set of such functions is denoted by $\mathcal{L}_+$. A real function f is called Lebesgue integrable if it can be represented as the difference of two functions from $\mathcal{L}_+$, $f = f_1 - f_2$, $f_{1,2} \in \mathcal{L}_+$ and

$$\int f(x) d^N x = \int f_1(x) d^N x - \int f_2(x) d^N x\,.$$

The set of Lebesgue integrable functions is denoted by $\mathcal{L}$. The proof of consistency of the Lebesgue integral over $\mathbb{R}$ given in Sec.**3.4.2** is easily extended to $\mathbb{R}^N$ by replacing all intervals in $\mathbb{R}$ in Sec.**3.4.2** by the corresponding balls in $\mathbb{R}^N$.

Similarly to the one dimensional case, a function f is said to be from $\mathcal{L}(\Omega)$ if the function $\chi_\Omega f \in \mathcal{L}$, where χ_Ω is the characteristic function of the set Ω and, in this case,

$$\int_\Omega f\, d^N x = \int \chi_\Omega f\, d^N x$$

3.6.3. Lebesgue and Riemann integrability in $\mathbb{R}^N$. Let Ω be a region in $\mathbb{R}^N$ and $f \in C^0(\Omega)$. *Then $f \in \mathcal{L}(\Omega)$ if and only if its Riemann integral over Ω converges absolutely, that is, if and only if*

$$\lim_{n\to\infty} \mathcal{R}\int_{\Omega_n} |f(x)|\, d^N x < \infty$$

for some exhaustion (or regularization) $\{\Omega_n\}$ of Ω, and, in this case,

$$\int_{\Omega} f(x)\, d^N x = \lim_{n\to\infty} \mathcal{R}\int_{\Omega_n} f(x)\, d^N x\,.$$

A proof of this assertion is left to the reader (cf. Sec. **3.5.4**).

Proposition **3.6** can be extended to integrals in $\mathbb{R}^N$. Let f an absolutely Riemann integrable function and $g(x) = f(x)$ a.e.. Then g is Lebesgue integrable and its integral is equal to the Riemann (improper) integral of f.

3.7. Exercises.

1. Let $f(x) = 0$ if x is rational and $f(x) = e^{-x}$ otherwise Find the Lebesgue integral

$$\int_0^\infty f(x)\, dx$$

or show that it does not exist.

2. Let L_Q be a collection of lines through the origin in $\mathbb{R}^2$ such that the angle between any two lines is equal πq where q is a rational number. Let $f(x) = 0$ if $x \in L_Q$ and $f(x) = e^{-|x|^2}$ otherwise. Investigate the existence of the integrals

$$\int f(x)\, d^2x\,, \quad \mathcal{R}\int f(x)\, d^2x$$

and, if an integral exists, find its value.

3. Which of the following functions are Lebesgue integrable on $\mathbb{R}$:

$$\frac{\sin(x)}{x}\,, \quad \frac{e^{ikx}}{x}\,, \quad \frac{\cos(x)}{\sqrt{|x|}}\,, \quad e^{-x}\,, \quad x^{100}e^{-x^2}$$

4. A function is said to be Lebesgue square integrable on Ω, or from the space $\mathcal{L}_2(\Omega)$, if $|f|^2 \in \mathcal{L}(\Omega)$. Which of the functions from Problem 3 are square integrable?

5. Let f be continuous and Lebesgue integrable on $\mathbb{R}^N$. Show that its Fourier transform

$$\mathcal{F}[f](k) = \int e^{i(k,x)}\, f(x)\, d^N x$$

exists for any $k \in \mathbb{R}^N$.

4. Properties of the Lebesgue integral in $\mathbb{R}^N$

Properties of the Lebesgue integral are analogous to the properties of the Riemann integral (cf. Sec. **1.5**) because the Lebesgue integral coincides with the absolutely convergent Riemann integral whenever the latter exists (Proposition **3.6**).

The key difference between the Lebesgue and Riemann integrals is that the Lebesgue integral is insensitive to alterations of the integrand on sets of measure zero, whereas the Riemann integrability can be lost after such alterations. This leads to simplifications of theorems about integrability of the limit function of a functional sequence. In particular, the hypotheses of the uniform convergence can be weakened and simplified, which is a major advantage of the Lebesgue integral.

In what follows, a Lebesgue integrable function will be called just integrable and integrals are always understood in the Lebesgue sense (unless stated otherwise) and the term integrability means integrability in the Lebesgue sense. In mathematical literature, Lebesgue integrable functions are often called *summable* to distinguish them from integrable functions in the Riemann sense.

4.1. The set $\mathcal{L}$ is a linear space. If f and g are integrable, then their linear combination is also integrable and

$$\int \Big(c_1 f(x) + c_2 g(x)\Big)\, d^N x = c_1 \int f(x)\, d^N x + c_2 \int g(x)\, d^N x\,.$$

So, the set $\mathcal{L}(\Omega)$ of Lebesgue integrable functions on $\Omega \subset \mathbb{R}^N$ is a linear space. Recall that the Riemann integral has the same property. This property follows from the limit laws. If $\{f_n\}$ and $\{g_n\}$ are sequences of piecewise continuous functions that define the integrals of f and g, then by linearity of the Riemann integral the sequence $c_1 f_n + c_2 g_n$ defines the integral of the linear combination $c_1 f + c_2 g$.

4.2. Monotonicity. Suppose that f and g are integrable. Then[13]

$$f(x) \geq 0 \quad \Rightarrow \quad \int f(x)\, d^N x \geq 0$$

and, as a consequence,

$$f(x) \geq g(x) \quad \Rightarrow \quad \int f(x)\, d^N x \geq \int g(x)\, d^N x$$

[13]see Proposition **3.2**

4.3. Integrals on sets of measure zero. The Lebesgue integral is insensitive to alterations of a function on sets of measure zero. If $f \in \mathcal{L}$, then every function that coincides with f almost everywhere is also integrable and its Lebesgue integral has the same value. Similarly, if f is not Lebesgue integrable, then any other function that differs from f on a set of measure zero is also non-integrable. In other words,

$$f(x) = g(x) \;\; a.e. \quad \Rightarrow \quad \int f(x)\, d^N x = \int g(x)\, d^N x$$

and both the integrals either exist or do not exist simultaneously. As noted earlier, *this property is not true for the Riemann integral*.

In particular, if the integral of any (measurable) function over a set of measure zero vanishes:

$$\chi_\Omega(x) f(x) = 0 \; a.e. \quad \Rightarrow \quad \int_\Omega f(x)\, d^N x = \int \chi_\Omega(x) f(x)\, d^N x = 0\,.$$

4.4. Additivity of the Lebesgue integral. Suppose that f is integrable on Ω and Ω' and the intersection $\Omega \cap \Omega')$ is a set of measure zero. Then f is integrable on the union $\Omega \cup \Omega'$ and

$$\int_{\Omega \cup \Omega'} f(x)\, d^N x = \int_\Omega f(x)\, d^N x + \int_{\Omega'} f(x)\, d^N x\,.$$

This follows from that

$$\chi_{\Omega \cup \Omega'}(x) = \chi_\Omega(x) + \chi_{\Omega'}(x) \quad \text{a.e.}$$

and the linearity of the Lebesgue integral.

4.5. Lebesgue measure of a set. If the characteristic function of a set $\Omega \subset \mathbb{R}^N$ is integrable, then the number

$$\mu(\Omega) = \int \chi_\Omega(x) d^N x$$

is called the *Lebesgue measure* of Ω. For example, if Ω is a bounded region with a smooth boundary, then its characteristic function is piecewise continuous and the Lebesgue measure is equal to the volume of Ω defined by the Riemann integral. If Ω is not bounded, then the volume is defined by the improper Riemann integral. The volume can be infinite if the improper Riemann integral diverges.

In general, a characteristic function of a measurable set is measurable. So, every bounded measurable set Ω has the Lebesgue measure. If a measurable set is not bounded and its characteristic function is not integrable, then the set is said to have infinite measure $\mu(\Omega) = \infty$ (similarly to sets of infinite volume). Thus, in contrast to the volume, the Lebesgue measure is defined on all measurable sets if it is allowed to have the infinite value. The Lebesgue measure has the following properties similarly to the volume[14].

4.5.1. Positivity. The Lebesgue measure is non-negative function of a set:

$$\mu(\Omega) \geq 0$$

and it vanishes if and only if Ω is a set of measure zero.

4.5.2. Monotonicity. The Lebesgue measure is increasing with enlarging the set:

$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \mu(\Omega_1) \leq \mu(\Omega_2)\,.$$

In particular,

$$\mu(\Omega) \leq \mu(\bar{\Omega})$$

and the strict inequality is also possible. Let $\Omega \subset \mathbb{R}$ consists of all rational numbers in an interval $[a, b]$. Evidently $\chi_\Omega(x) = 0$ a.e. and $\mu(\Omega) = 0$. However, $\bar{\Omega} = [a, b]$ and $\mu(\bar{\Omega}) = b - a > 0$.

4.5.3. Countable additivity. If a set is the union of countably many non-intersecting sets, then its measure is the sum of measures of sets in the union:

$$\Omega = \bigcup_n \Omega_n\,, \quad \Omega_n \cap \Omega_m = \emptyset\,,\ n \neq m \quad \Rightarrow \quad \mu(\Omega) = \sum_n \mu(\Omega_n)$$

In particular, the Lebesgue measure of Ω does not change when a set of measure zero is removed from Ω:

$$\mu(\Omega \setminus \Omega') = \mu(\Omega) \quad \text{if} \quad \mu(\Omega') = 0$$

For example, if Ω is a bounded region with a piecewise smooth boundary, then $\bar{\Omega} = \Omega \cup \partial\Omega$ has the same Lebesgue measure.

4.5.4. Measure of an unbounded set. Let Ω be an unbounded set. Let Ω_n a sequence of subsets of a finite measure such that $\Omega_n \subset \Omega_{n+1}$ for any n and the union of all Ω_n coincides with Ω up to a set of measure zero. Then it follows from the countable additivity that

$$\Omega = \bigcup_n \Omega_n\,,\ \Omega_n \subset \Omega_{n+1} \quad \Rightarrow \quad \mu(\Omega) = \lim_{n\to\infty} \mu(\Omega_n)\,.$$

This procedure can be used to evaluate the measure of unbounded sets. The limit either exists or is infinite and does not depend on the choice

[14] Proofs of the properties of the Lebesgue measure can be found in: A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5, Sec. 1

of exhaustion of Ω (similarly to the absolutely convergent Riemann integrals). For example, $\Omega_n = \Omega \cap B_n$ where B_n is a ball of radius n.

4.5.5. Continuity. Let $\{\Omega_n\}$ be a sequence of set embedded into one another $\Omega_{n+1} \subset \Omega_n$ and Ω is the intersection of all Ω_n. Then

$$\Omega = \bigcap_n \Omega_n\,,\ \Omega_{n+1} \subset \Omega_n \quad \Rightarrow \quad \mu(\Omega) = \lim_{n\to\infty} \mu(\Omega_n)$$

For example, if Ω is a bounded set, then one can take Ω_a to be the union of open ball of radius a that are centered at every point of Ω. Then $\mu(\Omega_a) \to \mu(\Omega)$ as $a \to 0^+$.

4.5.6. Geometrical properties of measurable sets. The *symmetric difference* of two sets A and B is defined by

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

So, it consists of elements that are either in A or B but not in their intersection. Therefore, if $\mu(A \triangle B) = 0$, then A and B differs at most by a set of measure zero.

Two rectangular boxes are called *almost disjoint* if their interiors do not intersect. In other words, two almost disjoint boxes can have a non-empty intersection of their boundaries. A set is called *elementary* if it can be represented as a finite union of almost disjoint rectangular boxes. The measure of an elementary set is just its volume. One can show that

- the union, intersection, difference, and symmetric difference of two elementary sets is elementary;
- the union, intersection, set difference, and symmetric difference of two elementary sets is elementary.

Any measurable set has the following characteristic property. For any measurable set A and any $\varepsilon > 0$, there exists an elementary set B such that

$$\mu(A \triangle B) < \varepsilon$$

In other words, any measurable set in $\mathbb{R}^N$ can be "approximated with any desired accuracy" by a finite collection of almost disjoint boxes.

4.6. Upper and lower bounds. Suppose that $f \in \mathcal{L}(\Omega)$ and f is bounded almost everywhere in Ω, then

$$m \le f(x) \le M \ \ a.e. \quad \Rightarrow \quad m\mu(\Omega) \le \int_\Omega f(x)\, d^N x \le M\mu(\Omega)$$

A similar property also holds for the Riemann integral over an interval (without a.e.).

4.7. Integrability of the absolute value. If $f \in \mathcal{L}$, then $|f| \in \mathcal{L}$. If f is measurable and $|f| \in \mathcal{L}$, then $f \in \mathcal{L}$ and

$$\left|\int f(x) d^N x\right| \leq \int |f(x)| d^N x\,.$$

In view of the early remark about non-measurable functions, the integrability of f and $|f|$ is practically equivalent in the Lebesgue theory. So, *if f is measurable, then the integrals*

$$\int f(x)\,dx \quad \text{and} \quad \int |f(x)|\,dx$$

exist or do not exist simultaneously.

This property does not hold for the Riemann integral. For example, let $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ otherwise so that $|f(x)| = 1$ for all x. Clearly, $|f(x)|$ is Riemann integrable on any bounded interval, while this is not so for $f(x)$. In the Lebesgue theory, $f(x) = -1$ a.e. (because $\mathbb{Q}$ has measure zero) and therefore it is Lebesgue integrable on any bounded interval.

4.8. Vanishing integral of the absolute value. Recall that, if f is continuous and the Riemann integral of the absolute value $|f|$ vanishes, then $f(x) = 0$. The converse is obviously true. The Lebesgue integral has a similar property: *if $f \in \mathcal{L}$ and the integral of $|f|$ vanishes, then $f(x) = 0$ almost everywhere (and the converse is obviously true)*:

$$f \in \mathcal{L}\,, \quad \int |f(x)| dx = 0 \qquad \Leftrightarrow \qquad f(x) = 0\,, \quad a.e.$$

4.9. Comparison test for integrability. If a function g is integrable on Ω and $|f(x)| \leq g(x)$ a.e., then f is also integrable on Ω:

$$|f(x)| \leq g(x)\ a.e.\,, \quad g \in \mathcal{L}(\Omega) \quad \Rightarrow \quad f \in \mathcal{L}(\Omega)$$

This implies that *any bounded (and measurable) function is Lebesgue integrable on any bounded (and measurable) set*. Indeed,

$$|f(x)| \leq M\ a.e. \quad \Rightarrow \quad \int_\Omega |f(x)|\, d^N x \leq M\mu(\Omega) < \infty$$

because Ω is bounded. In particular, all Riemann integrable functions (in the proper sense) are bounded. Therefore, every Riemann integrable function is Lebesgue integrable.

For example,

$$\left|\sin\left(\frac{1}{|x|^p}\right)\right| \leq 1 \quad \text{a.e.}$$

for any real p. Therefore $\sin(|x|^{-p})$ in integrable on any bounded interval. If $p > 0$, the function is not defined at $x = 0$. One can assign any value to the function at $x = 0$. The Lebesgue integral does not change.

4.9.1. Comparison tests. Let f be integrable on $\Omega \setminus B_R(x_0)$ and f is not bounded in a ball $B_R(x_0)$, then f is integrable on $\Omega \subset \mathbb{R}^N$ if

$$|f(x)| \le \frac{M}{|x - x_0|^p} \quad a.e.\,, \quad p < N\,,\ x \in B_R(x_0)$$

because the Riemann integral of the right side of this inequality was shown to converge absolutely. Similarly, if Ω is not bounded and f is integrable on $\Omega \cap B_R$ for some ball B_R, then f is integrable on Ω if

$$|f(x)| \le \frac{M}{|x|^p} \quad a.e., \quad |x| > R\,,\ p > N$$

for some M.

4.10. Absolute continuity of the Lebesgue integral. Consider the Lebesgue integral as a function of the integration set:

$$F(\Omega) = \int_\Omega f(x)\, d^N x\,. \tag{4.1}$$

The function F has the following properties[15].

THEOREM 4.1. *Suppose that*

$$\Omega = \bigcup_n \Omega_n\,, \quad \Omega_k \cap \Omega_n = \emptyset\,,\ k \ne n$$

and f is integrable on Ω. Then f is integrable on any Ω_n and

$$\int_\Omega f(x)\, d^N x = \sum_n \int_{\Omega_n} f(x)\, d^N x \tag{4.2}$$

where there the series converges absolutely. Conversely, if f is integrable on every Ω_n and the series

$$\sum_n \int_{\Omega_n} |f(x)|\, d^N < \infty$$

*converges, then f is integrable on Ω and relation (**4.2**) holds.*

[15] A proof can be found in: A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5, Sec. 5

There are a few consequences that can be deduced from this theorem. A measurable set Ω in Theorem **4.1** is represented as the union of arbitrary non-intersecting measurable sets. Therefore *the Lebesgue integrability on* Ω *implies the Lebesgue integrability on any measurable subset of* Ω:

$$f \in \mathcal{L}(\Omega)\,, \quad \Omega' \subset \Omega \quad \Rightarrow \quad f \in \mathcal{L}(\Omega')$$

Note that if f is bounded almost everywhere and Ω is bounded, then this conclusion follows from the comparison test $|f(x)| \leq M\chi_\Omega(x)$ a.e. and that $\mu(\Omega') \leq \mu(\Omega) < \infty$.

The convergence of the series in (**4.2**) implies that the terms of the series must tend to zero. Therefore for any function $f \in \mathcal{L}(\Omega)$ one can find a measurable subset Ω' such that the integral of f over Ω' is arbitrary small. This property is known as the *absolute continuity of the Lebesgue integral*.

Theorem **4.2**. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left|\int_{\Omega'} f(x)\, d^N x\right| < \varepsilon \quad \text{whenever} \quad \mu(\Omega') < \delta\,,\ \Omega' \subset \Omega\,.$$

The assertion is obvious if f is *bounded almost everywhere* on Ω because

$$\left|\int_{\Omega'} f(x)\, d^N x\right| \leq \int_{\Omega'} |f(x)|\, d^N x \leq M\mu(\Omega')$$

if $|f(x)| \leq M$ *a.e.*. In this case, $\delta = \varepsilon/M$.

By the absolute continuity of the Lebesgue integral, if $f \in \mathcal{L}(\Omega)$, then for any $\varepsilon > 0$ one can find a proper subregion Ω' or a region Ω (cf. Sec. **1.1.8**) such that

$$\int_{\Omega\setminus\Omega'} |f(x)|\, d^N x < \varepsilon$$

For example, if Ω is bounded, then Ω' can be constructed by removing closed balls of sufficiently small radius from Ω whose centers are on the boundary $\partial\Omega$.

4.10.1. Lebesgue integral over unbounded regions. Let $\{\Omega_n\}$ be an exhaustion of a region Ω. If f is integrable on Ω, then

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \int_\Omega f(x)\, d^N x\,. \tag{4.3}$$

In contrast to the continuity of the Riemann integral, the integrability of f on every Ω_n is redundant because f is integrable on any measurable subset of Ω. Conversely, suppose one wants to investigate integrability of f on Ω. If $f \in \mathcal{L}(\Omega_n)$ for all n, then its absolute value is also integrable on every Ω_n. By the second part of Theorem **4.1**, if the limit

$$\lim_{n\to\infty} \int_{\Omega_n} |f(x)|\, d^N x < \infty$$

exists (not infinite), then f is integrable on Ω and (**4.3**) holds. In what follows, this continuity property will often be used to investigate integrability in combination with comparison tests.

4.10.2. The space $\mathcal{L}_{\rm loc}(\Omega)$ of locally integrable functions. Let $\Omega \subset \mathbb{R}^N$ be an open set and f is continuous on Ω. Then f is integrable on any proper bounded subset Ω' of Ω, $\overline{\Omega'} \subset \Omega$ and $\Omega' \subset B_R$ for some R. However, f is not necessarily integrable on Ω. Recall that in this case, the integrability means that the Riemann integral of f converges absolutely on Ω if f has singular points on the boundary of Ω, or Ω is not bounded, or both. A function f is called *locally integrable on an open set* $\Omega \subseteq \mathbb{R}^N$ if it is integrable on any proper bounded subset of Ω. The class of such function is denoted by $\mathcal{L}_{\rm loc}(\Omega)$ or simply by $\mathcal{L}_{\rm loc}$ if $\Omega = \mathbb{R}^N$:

$$f \in \mathcal{L}_{\rm loc}(\Omega) \ : \quad \int_{\Omega'} |f(x)|\, d^N x < \infty$$

for any proper bounded subset Ω' of Ω.

4.10.3. The space $\mathcal{L}(\Omega;\sigma)$. Let σ be a non-negative integrable function on Ω. Then the function

$$\mu_\sigma(\Omega) = \int_\Omega \sigma(x)\, d^N x$$

has the same properties at the Lebesgue measure $\mu(\Omega)$. It is defined on all measurable sets, it is non-negative, monotonic, and countably additive, and the condition $\mu(\Omega) = 0$ implies $\mu_\sigma(\Omega) = 0$

Let $\sigma(x) \geq 0$. A function f is called *Lebesgue integrable on* Ω *with weight (or measure)* σ if the product $f\sigma$ is integrable on Ω. The space of all integrable functions on Ω with weight σ is denoted by $\mathcal{L}(\Omega;\sigma)$.

4.11. Taking limits under the integral sign. It was shown that the limit of a pointwise convergent sequence of Riemann integrable functions is not generally Riemann integrable. A uniform convergence of the sequence is sufficient for the Riemann integrability of the limit function (cf. Theorem **1.9**). In the Lebesgue theory, taking limits under the integral sign is simpler (requires weaker conditions). This stems from insensitivity of the Lebesgue integral to alterations of the integrand on sets of measure zero. Here a few theorems stating sufficient conditions for interchanging the order of taking the limit and integral are discussed (their proofs can be found in[16])

4.11.1. The Lebesgue dominated convergence theorem. *Let a sequence of (measurable) functions* $\{f_n\}_1^\infty$ *converge to* f *a.e.,*

$$\lim_{n\to\infty} f_n(x) = f(x) \ a.e.$$

If there exists an integrable function g *independent of* n *such that*

$$|f_n(x)| \le g(x) \ a.e.\,, \quad g \in \mathcal{L}\,,$$

then $f \in \mathcal{L}$ *and*

$$\lim_{n\to\infty} \int f_n(x)dx = \int \lim_{n\to\infty} f_n(x)dx = \int f(x)dx\,. \tag{4.4}$$

This theorem is perhaps one of the most useful theorems from analysis. To illustrate it, recall the first example in Section **1.10**. The sequence is bounded by $g(x) = 1$ that is integrable on any bounded interval and, hence, (**4.4**) holds for any such interval. The limit function is the Dirichlet function that is nowhere continuous and, hence, not Riemann integrable, but it is Lebesgue integrable because it is zero almost everywhere.

4.11.2. Example. Let

$$f_n(x) = \frac{n\sin(x^2/n)}{x^2(x^2+a_n^2)}\,, \quad x \ne 0\,,$$

where $a_n > 0$ and $a_n \to a > 0$ as $n \to \infty$. The functions f_n are not defined at $x = 0$. For example, they can be extended by continuity $f_n(x) \to 1/a_n^2$ as $x \to 0$, or one can set $f_n(0) = b_n$ for some sequence $\{b_n\}$. Then

$$\lim_{n\to\infty} f_n(x) = \frac{1}{x^2+a^2} \quad \text{a.e.}$$

Indeed, the limit may or may not exist at $x = 0$, and for $x \ne 0$, the limit follows from that

$$\lim_{y\to 0} \frac{\sin(y)}{y} = 1$$

where $y = x^2/n \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$$

[16] A.N. Kolmogorov nad S.V. Fomin, Elements of the theory functions and functional analysis

provided there exists a Lebesgue integrable bound g, $|f_n(x)| \leq g(x)$ a.e., that is *independent* of the parameter n. To find $g(x)$, note first that $|\sin(y)| \leq |y|$, and it follows that

$$|f_n(x)| \leq \frac{1}{x^2 + a_n^2} \quad \text{a.e.}$$

A positive sequence a_n converges to $a > 0$ and, hence, its greatest lower bound cannot be equal to zero $a_0 = \inf_n\{a_n\} > 0$. Indeed, any interval $|x - a| < \delta < a$ contains all but finitely many terms of the sequence $\{a_n\}$. Since $a_n > 0$, a_0 is the smallest among finitely many terms outside the interval. If $a_n \neq a$, then for small enough δ there will always be terms outside the interval. Therefore

$$|f_n(x)| \leq \frac{1}{x^2 + a_n^2} \leq \frac{1}{x^2 + a_0^2} = g(x) \quad \text{a.e.}, \quad a_0 = \inf_n\{a_n\} > 0\,.$$

4.11.3. An example of a convergent sequence with no integrable bound. If there exists no integrable bound, then (**4.4**) can be false. Consider the sequence

$$f_n(x) = \frac{n}{1 + n^2x^2}\,, \quad x \in \mathbb{R}$$

Then for any n,

$$\int f_n(x)\,dx = \lim_{b\to\infty} \int_{-b}^{b} \frac{ndx}{1 + n^2x^2} = \lim_{b\to\infty} \int_{-bn}^{bn} \frac{dy}{1 + y^2} = \pi$$

However the integral of the limit function is zero. Indeed, the sequence converges to zero if $x \neq 0$ and to infinity if $x = 0$. Therefore

$$\lim_{n\to\infty} f_n(x) = 0 \quad a.e.$$

and

$$\lim_{n\to\infty} \int f_n(x)\,dx = \pi \neq 0 = \int \lim_{n\to\infty} f_n(x)\,dx$$

Note that $\frac{2}{3}n \leq f_n(x) \leq n$ if $|x| \leq \frac{1}{n}$. This implies that if $f_n(x) \leq g(x)$ for all x and all n, then $g(x) \sim \frac{1}{|x|}$ near $x = 0$ which is not integrable.

4.11.4. Levi's theorem. If the sequence has no integrable bound, then the integrability of the limit function can be established by means of Levi's theorem: *Let $\{f_n\}$ be an almost everywhere non-decreasing sequence of integrable functions, $f_n \in \mathcal{L}(\Omega)$, and the sequence of the integrals of f_n is bounded,*

$$f_n(x) \leq f_{n+1}(x) \ a.e.$$
$$\left|\int f_n(x)\,d^Nx\right| \leq M\,,$$

for all n. Then there exists $f \in \mathcal{L}(\Omega)$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e.}$$

*and the relation (**4.4**) holds.*

As an example, put

$$f_n(x) = \left(1 + \frac{x f_D(x)}{n}\right)^n$$

where f_D is the Dirichlet function. Recall the sequence $(1 + p/n)^n$ converges to e^p and it is monotonically increasing if $p > 0$. Therefore $f_n(x) \le f_{n+1}(x)$ if $x \ge 0$ and

$$0 \le \int_a^b f_n(x)\,dx \le \int_a^b e^x\,dx < \infty\,, \qquad 0 \le a < b < \infty\,.$$

The limit function is

$$f(x) = \lim_{n\to\infty} f_n(x) = 1 \quad \text{a.e.}$$

because $f(x) = e^x$ if x is rational and $f(x) = 1$ otherwise so that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx = b - a\,.$$

In Levi's theorem the hypothesis of the boundedness of a sequence by an integrable function is replaced by the hypothesis of monotonicity of the sequence and boundedness of the sequence of integrals. The monotonicity hypothesis is essential. For example, the sequence of functions in Sec. **4.11.3** has a bounded sequence of integrals. But, by graphing $f_n(x)$, it is not difficult to see that the sequence is not monotonic: if $n > m$, then $f_n(x) > f_m(x)$ near $x = 0$ and $f_n(x) < f_m(x)$ for all large enough $|x|$.

There is a simple consequence of Levi's theorem for functions defined by functional series of non-negative terms that allows one to interchange the summation and integration signs.

Corollary **4.1**. *If $f_n(x) \ge 0$ and*

$$\sum_{n=1}^{\infty} \int_{\Omega} f_n(x)\,d^N x < \infty$$

then there exists $f \in \mathcal{L}(\Omega)$ such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{a.e.}$$

and

$$\int_\Omega \Big(\sum_{n=1}^\infty f_n(x)\Big)\, d^N x = \sum_{n=1}^\infty \int_\Omega f_n(x)\, d^N x$$

Note that partial sums of the series $\sum_n f_n(x)$ form a sequence satisfying the hypotheses of Levi's theorem.

For example, the series

$$\sum_{n=1}^\infty \frac{1}{n(1+n^2x^2)}$$

converges almost everywhere to $f(x)$ that is integrable on $\mathbb{R}$ and

$$\int_{-\infty}^\infty f(x)\, dx = \frac{\pi}{2}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^3}{12}$$

The series converges for any $x \neq 0$ and diverges for $x = 0$. So, f exists almost everywhere. Its integrability follows from that $f_n(x) > 0$ and

$$\int_{-\infty}^\infty f_n(x)\, dx = \frac{\pi}{2}\cdot\frac{1}{n^2}\,, \quad \sum_{n=1}^\infty \frac{1}{n^2} < \infty\,.$$

4.12. Exercises.

1. Can the Lebesgue measure of an unbounded region be finite? If so, construct an example. *Hint*: Think of the area under the graph of a non-negative continuous function on $\mathbb{R}$.

2. Construct an example of set in $\mathbb{R}^N$ that contains an open ball $|x| < R$ whose measure is equal to the volume of the ball but the closure of the set has measure that twice as much as the volume of the ball.

3. Are there any values of p for which the function

$$f(x) = \frac{\sin^2(|x|)}{|x|^p}\,, \quad x \in \mathbb{R}^N$$

is integrable on
(i) a bounded set that contains $x = 0$;
(ii) $\mathbb{R}^N$;
(iii) on the complement of a region containing $x = 0$

4. Suppose that

$$|f(x)| \leq \frac{M}{1+|x|^p}$$

For what values of p does f have a Fourier transform

$$\mathcal{F}[f](k) = \int e^{i(k,x)} f(x)\, d^N x\,, \quad k \in \mathbb{R}^N$$

5. Suppose $|f(x)| \leq M|x|^p$ a.e., where $p > 0$. For what values of p is the function $e^{-|x|}f(x)$ is integrable on $\mathbb{R}^N$? Give an upper bound of the value of the integral.

6. Let $f_n(x) = \Big(1 - x/n\Big)^n$, $n = 1, 2, \ldots$.
(i) Show that $f_n(x)$ converges to e^{-x} uniformly on $[0,1]$, that is,

$$\lim_{n\to\infty} \sup_{[0,1]} \Big| f_n(x) - e^{-x} \Big| = 0$$

Note that $f_n(x) - e^{-x}$ is continuous on $[0,1]$ and, hence, attains its extreme values on $[0,1]$. Find them and compute the limit. Conclude that

$$\lim_{n\to\infty} \int_0^1 f_n(x) = \int_0^1 e^{-x}\, dx$$

(ii) Show that $|f_n(x)| \leq M$ for all $x \in [0,1]$, where M is some constant independent of n. Use the Lebesgue dominated convergence theorem to established the same result.

7. Let $\varphi \in C^1(\mathbb{R})$ and the support of φ is bounded. Show that

$$\lim_{n\to\infty} \int e^{inx} \varphi(x)\, dx = 0$$

Hint: Use integration by parts in combination with the Lebesgue dominated convergence theorem (or with the theorem about the uniform convergence and integrability).

8. Let $f \in \mathcal{L}(\mathbb{R})$ such that $\int f(x)\, dx = 1$ and φ be a continuous function with bounded support. Put $f_n(x) = nf(nx)$, $n = 1, 2, \ldots$. Show that

$$\lim_{n\to\infty} \int f_n(x)\varphi(x)\, dx = \varphi(0)$$

Hint: Use the Lebesgue dominated convergence theorem and that any continuous function with bounded support is bounded.

9. Use the Lebesgue dominated convergence theorem to find the following limit

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{4}} e^{-n^2 \sin(2t)}\, dt$$

5. Functions defined by Lebesgue integrals

Let $f(x, y)$ be a function of two variables $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. Suppose that f is Lebesgue integrable with respect to y for any x. Then the integral defines a function

$$u(x) = \int f(x, y)\, d^M y\,.$$

Under what conditions on the function f is the function u integrable, or continuous, or differentiable? These questions will be answered in this section.

5.1. Fubini's theorem. *Suppose that the iterated integral of $|f(x, y)|$ exists, then f is Lebesgue integrable on $\mathbb{R}^{N+M}$:*

$$\int \left(\int |f(x, y)|\, d^N x \right) d^M y < \infty \quad \Rightarrow \quad f(x, y) \in \mathcal{L}(\mathbb{R}^{N+M})$$

Conversely, if f is Lebesgue integrable, then the function defined by the integrals of f either with respect to x or y

$$h(x) = \int f(x, y)\, d^M y\,, \quad g(y) = \int f(x, y)\, d^N x$$

exist almost everywhere and are Lebesgue integrable:

$$f \in \mathcal{L}(\mathbb{R}^{N+M}) \quad \Rightarrow \quad h \in \mathcal{L}(\mathbb{R}^N)\,, \quad g \in \mathcal{L}(\mathbb{R}^M)$$

and, in this case, the integral of f is equal to the iterated integrals:

$$\iint f(x, y)\, d^N x d^M y = \int \left(\int f(x, y)\, d^N x \right) d^M y$$
$$= \int \left(\int f(x, y)\, d^M y \right) d^N x$$

Funini's theorem also holds if f is defined on $\Omega \times \Omega'$, that is, $x \in \Omega \subset \mathbb{R}^N$ and $y \in \Omega' \subset R^M$. Indeed, one can replace $f(x, y)$ by $\chi_\Omega(x)\chi_{\Omega'}(y) f(x, y)$ in the above formulation and use the definition of the Lebesgue integral over a region.

It should be noted that if f is not integrable, then its iterated integrals either do not exist or, if they exist, they are not equal. The latter can happens if f has a conditionally convergent Riemann integral. For example, consider

$$h(x) = \lim_{b \to 0^+} \int_b^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy = \lim_{b \to 0^+} \int_b^1 \frac{\partial}{\partial y} \frac{y}{x^2 + y^2}\, dy$$
$$= \lim_{b \to 0^+} \frac{y}{x^2 + y^2}\Big|_b^1 = \frac{1}{1 + x^2}\,, \quad x \neq 0$$

Similarly,

$$g(y) = \lim_{a\to 0^+} \int_a^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx = - \lim_{a\to 0^+} \int_b^1 \frac{\partial}{\partial x} \frac{x}{x^2+y^2}\, dx$$
$$= - \lim_{a\to 0^+} \frac{x}{x^2+y^2}\Big|_a^1 = -\frac{1}{y^2+1}\,, \quad y \neq 0$$

Therefore, the functions

$$h(x) = \int_0^1 f(x,y)\, dy = \frac{1}{1+x^2} \quad a.e.$$
$$g(y) = \int_0^1 f(x,y)\, dx = -\frac{1}{1+y^2} \quad a.e.$$

are integrable on $(0,1)$ and

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy\right) dx = \int_0^1 h(x)\, dx = \frac{\pi}{4}\,,$$
$$\int_0^1 \left(\int_0^1 f(x,y)\, dx\right) dy = \int_0^1 g(y)\, dy = -\frac{\pi}{4}$$

It was shown earlier that the improper Riemann integral of f over any bounded closed region that contains the origin does not converge absolutely so that f is not Lebesgue integrable.

The first part of Fubini's theorem is a criterion for Lebesgue integrability of a function of two variables, while the second part gives a criterion for changing the order of integration. If

$$\int_\Omega \int_{\Omega'} |f(x,y)|\, d^N x d^M y < \infty$$

in any particular order, then

$$\int_\Omega \int_{\Omega'} f(x,y)\, d^M y d^N x = \int_{\Omega'} \int_\Omega f(x,y)\, d^N x d^M y$$

In the above example

$$\int_0^1 \left(\int_0^1 |f(x,y)| dx\right) dy = \infty$$

This is left to the reader as an exercise.

5.2. Continuity.

THEOREM 5.1. (Continuity of a function defined by an integral) *Let $f(x,y)$ be defined on $\mathbb{R}^N \times \Omega$, $\Omega \subset \mathbb{R}^M$. Suppose f is continuous in $y \in \Omega$ for almost all $x \in \mathbb{R}^N$, and there exists an integrable function*

$F(x)$ such that $|f(x,y)| \leq F(x)$ a.e. for every $y \in \Omega$. Then the function

$$g(y) = \int f(x,y)d^N x$$

is continuous on Ω, that is,

$$\lim_{z\to y} \int f(x,z)d^N x = \int_\Omega \lim_{z\to y} f(x,z)d^N x = \int f(x,y)d^N x$$

for any $y \in \Omega$

Recall that g is continuous at a point y if and only if for any sequence $\{y_n\}$ converging to y, the sequence $\{g(y_n)\}$ converges to $g(y)$. Consider the sequence of functions $f_n(x) = f(x, y_n)$. Then

$$\lim_{n\to\infty} f_n(x) = f(x,y) \quad a.e.$$

because $f(x,y)$ is continuous in y for almost every $x \in \mathbb{R}^N$. The sequence $\{f_n\}$ is bounded for all n by a Lebesgue integrable function

$$|f_n(x)| \leq F(x)$$

for any choice of $\{y_n\}$. By the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} \int f_n(x)\, d^N x = \int \lim_{n\to\infty} f_n(x) d^N x = g(y)$$

This proves the theorem.

5.2.1. Continuity of the Fourier transform. As an example, let us show that the Fourier transform of a Lebesgue integrable function is a continuous function:

$$\mathcal{F}[f](k) = \int e^{i(k,x)} f(x)\, d^N x\,, \quad k \in \mathbb{R}^N$$

First note that the Fourier transform exists for any $k \in \mathbb{R}^N$ because $|e^{i(k,x)}f(x)| = |f(x)|$ and the absolute value is integrable if $f \in \mathcal{L}$. Let $g(x,k) = e^{i(k,x)}f(x)$. The exponential $e^{i(k,x)}$ is continuous with respect to k for any x and so is $g(x,k)$. So,

$$g(x,k) \in C^0\,, \quad \forall x\,; \qquad |g(x,k)| = |f(x)| \in \mathcal{L}$$

By the stated theorem $\mathcal{F}[f](k) = \int g(x,k)\, d^N x$ is a continuous function.

5.3. Differentiability. In what follows the following notations for partial derivatives are adopted

$$D_x^p f = \frac{\partial^p f(x,y)}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_p}}$$

for any choice of i_1, i_2,..., i_p from 1 to N (here $x \in \mathbb{R}^N$). So, $D_x^p f$ stands for any partial derivative of f of order p with respect to x.

THEOREM 5.2. (Differentiation of an integral)
Let $f(x,y)$ be defined on $\mathbb{R}^N \times (a,b)$. Suppose that the partial derivative $D_y f(x,y)$ is continuous in $y \in (a,b)$ for almost all $x \in \mathbb{R}^N$. Furthermore, there exists an integrable function $F(x)$ such that for every $y \in (a,b)$, $|D_y f(x,y)| \leq F(x)$ almost everywhere, and the integral of $f(x,y)$ with respect to x exists for some particular $y_0 \in (a,b)$. Then the function

$$g(y) = \int f(x,y) d^N x \in C^1(a,b)$$

has the derivative continuous in (a,b) and the following equality holds

$$h'(y) = \frac{d}{dy} \int f(x,y) d^N x = \int D_y f(x,y)\, d^N x\,. \qquad (5.1)$$

Put

$$\phi(y) = \int D_y f(x,y)\, d^N x$$

Since $D_y f(x,y)$ is continuous in y for almost every x and is bounded by an integrable function:

$$D_y f(x,y) \in C^0(a,b) \;\; \forall x\,; \qquad |D_y f(x,y)| \leq F(x) \in \mathcal{L}$$

by the theorem about continuity of a function defined by the Lebesgue integral, the function $\phi(y)$ is continuous on (a,b). Therefore for any y and y_0 in (a,b), its integral

$$\Phi(y) = \int_{y_0}^{y} \phi(t)\, dt \in C^1(a,b)$$

is continuously differentiable in (a,b) and, by the Fundamental theorem of calculus,

$$\Phi'(y) = \phi(y)$$

Since $F \in \mathcal{L}$, one infers that

$$\int_a^b \int |D_y f(x,y)|\, d^N x\, dy \leq \int_a^b \int F(x)\, d^N x\, dy$$
$$= (b-a) \int F(x)\, d^N x < \infty$$

Therefore the function $D_y f(x,y)$ is Lebesgue integrable on $\mathbb{R}^N \times (a,b)$ by the first part of Fubini's theorem. By the second part of Fubini's theorem, the order of integration can be changed:

$$\begin{aligned}\Phi(y) &= \int_{y_0}^{y}\int D_t f(x,t)\, d^N x dt = \int\int_{y_0}^{y} D_t f(x,t)\, dt\, d^N x \\ &= \int [f(x,y) - f(x,y_0)]\, d^N x = g(y) - g(y_0)\end{aligned}$$

This shows that $g(y)$ is continuously differentiable and $g'(y) = \Phi'(y) = \phi(y)$ as required. This completes the proof of the theorem.

It is clear from the proof that the same result holds if $y \in \Omega \subset \mathbb{R}^M$. If Ω is open, then each coordinate y_i ranges over some open interval for given values of the other coordinates. Similarly, $g(y)$ is from class C^p if partial derivatives $D_y^\beta f(x,y)$, $\beta = 1, 2, ..., p$, are continuous with respect to y for almost every x and are bounded by Lebesgue integrable functions, $|D_y^\beta f(x,y)| \le F_\beta(x) \in \mathcal{L}$:

$$\begin{aligned}& D_y^\beta f(x,y) \in C^0(\Omega) \quad \forall x\,; \quad |D_y^\beta f(x,y)| \le F_\beta(x) \in \mathcal{L}\,,\ \beta \le p\,, \\ \Rightarrow \quad & g(y) = \int f(x,y)\, d^N x \in C^p(\Omega) \\ \Rightarrow \quad & D_y^\beta g(y) = \int D_y^\beta f(x,y)\, d^N x\,, \quad \beta \le p\end{aligned}$$

So, *the order of differentiation with respect to parameters and the integration with respect to other variables can be interchanged if partial derivatives of the integrand with respect to parameters are bounded by a Lebesgue integrable function that is independent of the parameters.*

5.3.1. Interchanging the order of integration and differentiation. Theorem **5.2** states sufficient but not necessary conditions for differentiation of the integral with respect to a parameter. In fact, the integral can be differentiable infinitely many times while partial derivatives with respect to parameters do not have integrable bounds independent of parameters. This implies that *in general the order of differentiation and integration cannot be interchanged.* For example, one can show that[17],

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\, dx = \pi\, e^{-|k|}$$

This function is infinitely many times differentiable on any interval that does not contain $k = 0$. However, the derivatives of the integrand with

[17] Example 1 in Sec. **7.3**

respect to k are not even integrable:

$$\left|\frac{d^n}{dk^n}\frac{e^{ikx}}{1+x^2}\right| = \frac{|x|^n}{1+x^2} \notin \mathcal{L}\,, \quad n>0\,.$$

5.3.2. Differentiability of the Fourier transform. Let us investigate differentiability of the Fourier transform in $\mathbb{R}^N$. It follows from integrability of f that

$$\int |f(x)|\,d^N x = \lim_{R\to\infty}\int_{B_R}|f(x)|\,d^N x < \infty$$

and, owing to non-negativity of the absolute value,

$$|f(x)| \to 0 \quad \text{as} \quad |x|\to\infty$$

For $x \in \mathbb{R}^N$, define

$$x^\beta = x_1^{\beta_1}x_2^{\beta_2}\cdots x_N^{\beta_N}\,, \quad \beta_1+\beta_2+\cdots+\beta_N=\beta\,, \quad \beta_j \ge 0$$

where β_j are non-negative integers. In other words, x^β stands for any monomial of order β in coordinates of x. In particular, it follows from $|x_i| \le |x|$ that

$$|x^\beta| \le |x|^\beta.$$

Then

$$\left|D_k^\beta e^{i(k,x)f(x)}\right| = \left|i^\beta x^\beta e^{i(k,x)}f(x)\right| \le |x|^\beta |f(x)|$$

The Fourier transform is from class C^p if $|x|^p|f(x)| \in \mathcal{L}$ and

$$\begin{aligned} D_k^\beta \mathcal{F}[f](k) &= D_k^\beta \int e^{i(k,x)}f(x)\,d^N x = i^\beta \int e^{i(k,x)}x^\beta f(x)\,d^N x \\ &= \mathcal{F}[(ix)^\beta f(x)](k)\,, \quad \beta = 1,2,...,p \end{aligned}$$

So, differentiability of the Fourier transform depends on how fast the function decreases in the asymptotic region $|x|\to\infty$.

Recall that if f is integrable on any ball $|x|<R$ and $|f(x)| = O(|x|^{-m})$, $m>N$, in the asymptotic region $|x|\to\infty$, then $f\in\mathcal{L}$. Therefore the Fourier transform $\mathcal{F}[f]$ is p times continuously differentiable if

$$|f(x)| = O(|x|^{-n})\,, \quad n > N+p$$

in the asymptotic region $|x|\to\infty$. In particular, if f decreases faster than any power function, its Fourier transform is from class C^∞:

$$\lim_{|x|\to\infty}|x|^p|f(x)| = 0 \quad \text{for all } p \quad \Rightarrow \quad \mathcal{F}[f](k)\in C^\infty$$

5.3.3. Differentiability of a Gaussian integral. Let us prove that for $n = 1, 2, \ldots$, and $t > 0$

$$\int_{-\infty}^{\infty} x^{2n} e^{-tx^2} dx = \sqrt{\pi}\,(-1)^n \frac{d^n}{dt^n}\, t^{-1/2}\,.$$

For $n = 0$, the Gaussian integral can easily be computed

$$h(t) = \int_{-\infty}^{\infty} e^{-tx^2} dx = \sqrt{\pi}\, t^{-1/2}\,.$$

Let $f(x,t) = e^{-tx^2}$. Fix $t_0 > 0$ (an arbitrary positive number). Using the power series for e^x about $x = 0$ is not difficult to establish the inequality

$$|x|^m \le m! e^{|x|}\,, \qquad x \in \mathbb{R}\,.$$

Therefore for every positive integer n

$$x^{2n} e^{-tx^2} \le (2n)! e^{-tx^2+|x|} \le (2n)! e^{-t_0x^2+|x|} = g_n(x)$$

for all $0 < t_0 \le t < \infty$ and $x \in \mathbb{R}$. The function $g_n(x)$ is integrable. In particular

$$|f_t'(x,t)| = x^2 e^{-tx^2} \le g_1(x)\,, \quad 0 < t_0 \le t < \infty$$

and all $x \in \mathbb{R}$. Therefore

$$\frac{d}{dt}\int_{-\infty}^{\infty} e^{-tx^2} dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} e^{-tx^2} dx = -\int_{-\infty}^{\infty} x^2 e^{-tx^2} dx = -\sqrt{\pi}(t^{-1/2})'$$

which holds for all $t_0 \le t < 0$. Since $t_0 > 0$ is arbitrary, the above relation is true for all $t > 0$. Repeating this argument successively for $f(x,t) = x^{2n-2}e^{-tx^2}$, $n = 1, 2, \ldots$ it is concluded that

$$\begin{aligned}(-1)^n h^{(n)}(t) &= (-1)^n \sqrt{\pi}\, \frac{d^n}{dt^n}\, t^{-1/2} = (-1)^n \frac{d^n}{dt^n} \int_{-\infty}^{\infty} e^{-tx^2} dx \\ &= (-1)^n \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n} e^{-tx^2} dx = \int_{-\infty}^{\infty} x^{2n} e^{-tx^2} dx\,.\end{aligned}$$

for all $t > 0$.

5.4. Exercises.

1. Let

$$f(x,y) = \frac{xy}{(x^2+y^2)^2}\,, \quad (x,y) \in (-1,1) \times (-1,1) = \Omega \subset \mathbb{R}^2$$

(i) Show that

$$h(x) = \int_{-1}^{1} f(x,y)\, dy = 0\,, \quad g(y) = \int_{-1}^{1} f(x,y)\, dx = 0$$

so that these functions are integrable on $(-1, 1)$ and their integrals vanish.
(ii) Show that the function f is not integrable on the rectangle Ω. Explain why Fubini's theorem does not apply in this case.

2. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $[0, 1]$ that converge to 0 monotonically, $x_1 = 1 > x_2 > \cdots$ and $y_1 = 1 > y_2 > \cdots$. Put $\Delta x_n = x_n - x_{n+1}$ and $\Delta y_n = n_n - y_{n+1}$, and suppose that

$$p = \frac{\Delta x_n}{\Delta x_{n+1}}, \quad q = \frac{\Delta y_n}{\Delta y_{n+1}}$$

for any n. Consider the function of two real variables

$$f(x, y) = \begin{cases} p^n q^n , & (x, y) \in [x_{n+1}, x_n] \times (y_{n+1}, y_n] , \\ -p^n q^{n+1} , & (x, y) \in [x_{n+2}, x_{n+1}) \times (y_{n+1}, y_n] , \\ 0 , & \text{otherwise} \end{cases}$$

(i) Is the function f piecewise continuous? Explain.
(ii) Evaluate the iterated integrals of f:

$$\int\int f(x, y)\, dxdy\,, \quad \int\int f(x, y)\, dydx\,.$$

(iii) Is the function f integrable on $\mathbb{R}^2$?

3. (i) Show that the function defined by the integral

$$h(x) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{k^2 + m^2}\, dk\,,$$

where m is a positive constant, exists and is continuous for all $x \in \mathbb{R}$.
(ii) Find an explicit form of $h(x)$. Is $h(x)$ differentiable for all x? Is it true that

$$h'(x) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \frac{\cos(kx)}{k^2 + m^2}\, dk\,,$$

if $h'(x)$ exists for some x?

4. Let $f(u) = 1 - |u|$ if $|u| \leq 1$ and f is extended periodically to all $u \in \mathbb{R}$, $f(u + 2) = f(u)$. Define a function

$$F(t) = \int_0^{\infty} \frac{f(tx)}{1 + x^p}\, dx\,, \quad p > 2\,, \quad t \in \mathbb{R}$$

(i) Show that $F(t)$ exists and $F(-t) = F(t)$;
(ii) Show that $F \in C^1(a,b)$ for any $0 < a < b$ and

$$F'(t) = \int_0^\infty \frac{xf'(tx)}{1+x^p}\, dx\,, \quad 0 < a \le t \le b$$

Hint: Consider a change of the integration variable $u = tx$. Use the theorem about differentiation of a function defined by a Lebesgue integral.
(iii) Show that

$$\left|\frac{f(xt) - f(0)}{t}\right| \le |x|$$

(iv) Use the above inequality and the Lebesgue dominated convergence theorem to show that the left and right limits

$$\lim_{t\to 0^\pm} \frac{F(t) - F(0)}{t}$$

exist but are not equal. Is F differentiable at $t = 0$?

5. Let A be a positive matrix (all eigenvalues are strictly positive). Define the function

$$J(y) = \int e^{-(x,Ax)+(x,y)}\, d^N x\,, \qquad y \in \mathbb{R}^N$$

(i) Show that $J \in C^\infty$ and

$$D_y^\beta J(y) = \int D_y^\beta e^{-(x,Ax)+(x,y)}\, d^N x\,.$$

(ii) Calculate $J(y)$ and show that for any polynomial $P(x)$

$$\int P(x) e^{-(x,Ax)}\, d^N x = P(D_y) J(y)\Big|_{y=0}$$

6. Let

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}\,, \quad (x,y) \in \Omega = (1,\infty)\times(1,\infty)$$

(i) Calculate the iterated integral

$$\int_1^\infty \left(\int_1^\infty |f(x,y)|\, dx\right) dy$$

Is it true that $f \in \mathcal{L}(\Omega)$?
(ii) Calculate and compare the iterated integrals

$$\int_1^\infty \left(\int_1^\infty f(x,y)\, dx\right) dy\,, \quad \int_1^\infty \left(\int_1^\infty f(x,y)\, dy\right) dx$$

6. Line and surface integrals

6.1. Line integrals in a Euclidean space. Let $x = x(t)$, $a \le t \le b$, be a parameterization of a curve C. Consider a partition $P = \{t_n\}$ of $[a, b]$. Put

$$L(P, C) = \sum_n |x(t_n) - x(t_{n-1})|$$

The number $L(P, C)$ is the length of a polygonal path with vertices at $x(t_n)$ on the curve C. Upon a refinement of P, the polygon path gets closer to the curve C but its length is increasing by the triangle inequality:

$$L(P, C) \le L(P', C)\,, \quad P \subset P'$$

for any refinement P' of P. The quantity

$$L_C = \sup_P L(P, C)$$

is called the *arclength* of C. Note that L_C can be infinite.

One can prove that[18] *if a curve is from class C^1, then*

$$L_C = \int_a^b |x'(t)|\, dt < \infty$$

In physics, this equation has a simple meaning. If $x = x(t)$ is the trajectory of a point-like particle, then $x'(t)$ is the velocity vector, and its magnitude $|x'(t)|$ is the speed. The distance traveled along path C is the integral of the speed with respect to time.

6.1.1. Natural parameterization of a smooth curve. Let $x = x(t)$ be a parameterization of a curve C from class C^1. Define an *arclength* parameter by

$$s = s(t) = \int_a^t |x'(\tau)|\, d\tau\,.$$

Then $s'(t) = |x'(t)| > 0$ and, hence, $s(t)$ is monotonic and maps $[a, b]$ onto $[0, L_C]$. The map is invertible, $t = t(s)$. A parameterization of C in terms of the arclength, $x = X(s) = x(t(s))$, is called a *natural parameterization* of C. Note that $X'(s)$ is a unit tangent vector, $|X'(s)| = 1$, as one infers from the chain rule, $dX(s)/ds = (dx(t)/dt)(ds/dt)^{-1}$.

[18] W. Rudin, Principles of mathematical analysis, Theorem 6.27

6.1.2. Line integral of a scalar function. Let $x(s)$ be a natural parameterization of a curve C and $f(x)$ be continuous in a neighborhood of C. Then the integral

$$\int_C f\,ds = \int_0^{L_C} f(x(s))\,ds$$

exists and is called a *line integral of a function f over a curve C*. A line integral can be defined for any f as long as the function $f(x(s))$ is integrable on $[0, L_C]$. One might think about a thin wire with a non-uniform mass density $f(x)$ so that $dm(x) = f(x)ds$ is the mass in a segment of length ds at a sample point x of the segment. The line integral is equal to the total mass of the wire.

If $x = x(t)$ is any parameterization of a curve from class C^1, then using the change of variables, $ds = |x'(t)|dt$,

$$\int_C f\,ds = \int_a^b f(x(t))\,|x'(t)|\,dt\,.$$

Recall that the center of mass a collection of point-like particles with positions x_p and masses m_p (the index p labels the particles) is

$$x_c = \frac{1}{m}\sum_p m_p x_p\,, \quad m = \sum_p m_p$$

Suppose these particles are assembled into a smooth curve C with a linear mass density $\sigma(x)$ so that $dm(x_p) = \sigma(x_p)ds$ is the mass of a segment of the curve of length ds at a sample point x_p. Then it follows that the coordinates of the center of mass of this wire are given by the following line integrals:

$$x_{0j} = \frac{1}{m}\int_C \sigma(x)x_j\,ds\,, \quad m = \int_C \sigma(x)\,ds$$

6.1.3. Line integral of a vector field. Let T be a unit tangent vector at some point of a smooth curve. Then the vector $-T$ is also a unit tangent vector. A unit tangent vector is continuous for a curve from class C^1. Therefore there are only two ways of choosing the latter. If a curve connects points x_a and x_b, then a natural parameter can be counted either from x_a or from x_b, the derivatives of the corresponding natural parameterizations are opposite unit tangent vectors at any point of the curve. This choice defines an *orientation* of the curve C.

Let C be a curve oriented by a unit tangent vector T. Let $F(x)$ be a continuous vector field (its components $F_j(x)$ are continuous). The dot product

$$F_T(x) = \Big(F(x), T(x)\Big)\,, \quad x \in C$$

is the tangent component of the vector field F at a point x of C. If $x = x(s)$ is a natural parameterization of C such that $x'(s) = T(x(s))$, then $F_T = (F, x')$. The line integral

$$\int_C F_T(x)\, ds = \int_C F_j(x) dx_j$$

where Einstein's summation rule over repeated indices j is assumed, is called the *integral of a vector field F along a curve C*.

If $F(x)$ is a force acting on a point-like particle at a point x, then the work done by F in moving the particle along an infinitesimal straight-line segment from x to $x + dx$ is given by

$$dW(x) = F_j(x) dx_j = |F(x)| \cos(\theta)\, ds\,,$$

where θ is the angle between $F(x)$ and $T(x)$ and $|dx| = ds$. The line integral of F along C is nothing but the total work done by F in moving the particle along the curve C.

Let $x = x(t)$ be a parameterization of a curve C such that $x'(t)$ defines the correct orientation of C. Then by changing variable in the line integral, $dx_j = x'(t)\, dt$, one infers that

$$\int_C F_j(x) dx_j = \int_a^b F_j(x(t)) x_j'(t)\, dt$$

Let $-C$ be the curve C with the opposite orientation, then

$$\int_{-C} F_j(x) dx_j = - \int_C F_j(x) dx_j\,.$$

Consequently, any parameterization can be used to evaluate the line integral over a curve from class C^1 (if a parameterization defines an opposite parameterization, the sign of the integral should be changed after evaluating it).

If $x = x(t)$ is a physical trajectory of a point-particle of mass m, then the trajectory satisfies Newton's Law $mx''(t) = F(x(t))$ (recall that the second derivative $x''(t)$ is the acceleration of the particle). Then the work done by F in moving the particle is a net change of the kinetic energy $\frac{1}{2} mv^2$, where $v(t) = x'(t)$ is the velocity of the particle,

$$W = \int_C F_j(x) dx_j = \int_a^b m v_j'(t) v_j(t) dt = \frac{1}{2}\, mv^2(b) - \frac{1}{2}\, mv^2(a)\,.$$

6.1.4. Fundamental theorem for line integrals. A vector field is said to be *conservative* in an open set Ω if it is the gradient of some function U, that is, $F = \nabla U$ in Ω. The function U is called a *potential* of F. Note that U is not unique as it can be changed by an additive constant.

Let C be a C^1 curve in an open set Ω and a vector field F be conservative in Ω. Then

$$\int_C F_j dx_j = \int_a^b \frac{\partial U}{\partial x_j}\frac{dx_j}{dt}\,dt = \int_a^b dU(x(t)) = U(x(b)) - U(x(a))$$

Thus, *for a conservative vector field, its line integral along a path from point A to point B does not depend on the path and is determined by the difference of its potential at the endpoints of the path.* This comprises the fundamental theorem for line integrals.

In physics, this is a familiar statement that the work done by a conservative force is determined by the net change change of a potential energy V:

$$W = V(A) - V(B)\,, \quad F = -\nabla V\,.$$

Combining this relation with the work being the net change of kinetic energy, it is concluded that the energy of a particle

$$E(t) = \frac{1}{2}\,v^2(t) + V(x(t))$$

remains constant along any trajectory in a conservative force field, $E(b) = E(a)$. It can also be shown that $E'(t) = 0$ by a direct evaluation of the derivative $E'(t)$ and Newton's Law.

6.2. Surface area. Let S be an M-surface from class C^1 and $x = F(y)$ be its parameterization, $y \in D$. Then vectors $w_a = \partial_a F$ are tangent to the surface. Define a matrix W whose columns are the tangent vectors $w_a \in \mathbb{R}^N$:

$$W_M = [w_1\, w_2\, \cdots\, w_M]\,, \qquad w_a = \frac{\partial F}{\partial y_a}$$

Then the area of S is defined by the integral

$$A_S = \int_D J(y)\, d^M y\,, \qquad J = \sqrt{\det(W_M^T W_M)} \tag{6.1}$$

where W_M^T is the transposed matrix W_M. Note that $J(y)$ is continuous on a closed and bounded D and, hence, the integral exists for any C^1 surface in $\mathbb{R}^N$.

The quantity $J(y)$ is the volume on M dimensional parallelepiped with adjacent sides being vectors w_a. It is easy to verify the assertion for $M = 2$. If $0 \leq \theta \leq \pi$ is the angle between w_1 and w_2, then the area of parallelogram with adjacent sides w_1 and w_2 is

$$V_2 = |w_1||w_2|\sin(\theta) = \sqrt{|w_1|^2|w_2|^2 - (w_1, w_2)^2} = \sqrt{\det(W_2^T W_2)}$$

because

$$W_2^T W_2 = \begin{pmatrix} |w_1|^2 & (w_1, w_2) \\ (w_1, w_2) & |w_2|^2 \end{pmatrix}$$

Consider an n dimensional parallelepiped with adjacent sides w_1, w_2,..., w_n. Its volume is denoted by V_n. Its base is the $n-1$ dimensional parallelepiped with adjacent sides w_1, w_2,..., w_{n-1} with volume V_{n-1}. Then

$$V_n = V_{n-1} h \,,$$

where h is the height. If $w_n^{\|}$ is the orthogonal projection of w_n onto $\mathrm{span}\{w_1, ..., w_{n-1}\}$, then by the Pythagorean theorem

$$h^2 = |w_n|^2 - |w_n^{\|}|^2 \,.$$

One has

$$w_n^{\|} = c_1 w_1 + c_2 w_2 + \cdots + c_{n-1} w_{n-1} \,,$$

where the constants c_a are such that $w_n - w_n^{\|}$ is orthogonal to all w_a, $a = 1, 2, ..., n-1$, so that

$$\sum_{b=1}^{n-1} (w_a, w_b) c_b = (w_a, w_n) \,.$$

If $c \in \mathbb{R}^{n-1}$ with components c_a satisfying the above equation, then

$$c = (W_{n-1}^T W_{n-1})^{-1} W_{n-1}^T w_n$$

because (w_a, w_b) are matrix elements of $W_{n-1}^T W_{n-1}$ and this matrix is invertible because w_a are linearly independent. Therefore

$$|w_n^{\|}|^2 = (w_n^{\|}, w_n^{\|}) = c^T W_{n-1}^T W_{n-1} c = w_n^T W_{n-1} (W_{n-1}^T W_{n-1})^{-1} W_{n-1}^T w_n$$

On the other hand, W_n is obtained from W_{n-1} by adding an extra column w_n so that in the block-matrix notation, $W_n = [W_{n-1}\, w_n]$. Using the block-matrix multiplication

$$W_n^T W_n = \begin{bmatrix} W_{n-1}^T \\ w_n^T \end{bmatrix} [W_{n-1}\, w_n] = \begin{bmatrix} W_{n-1}^T W_{n-1} & W_{n-1}^T w_n \\ w_n^T W_{n-1} & |w_n|^2 \end{bmatrix}$$

Suppose that the equation for the volume is correct for $n-1$. It follows from the determinant of a block-matrix

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - C A^{-1} B)$$

that the equation is also correct for n

$$\begin{aligned} \det(W_n^T W_n) &= \det(W_{n-1}^T W_{n-1})(|w_n|^2 - |w_n^{\|}|^2) \\ &= V_{n-1}^2 (|w_n|^2 - |w_n^{\|}|^2) = V_n^2 \end{aligned}$$

By mathematical induction, the equation is true for any $n \le N$.

A *linearization* of F an a point y^* is the linear function $L:\ D \to \mathbb{R}^N$ defined by

$$L(y) = F(y^*) + \sum_{a=1}^{M} w_a(y^*)(y_a - y_a^*)\,, \quad w_a(y^*) = \frac{\partial F}{\partial y_a}\Big|_{y=y^*}\,.$$

It maps D into the tangent space of S at a point $x^* = F(y^*)$. Equation (**6**.**1**) has a simple geometrical meaning. For any partition box R_p in D, the surface area of $F(R_p)$ is approximated by the volume of a parallelepiped that is the image $L(R_p)$ of a partition box R_p in the tangent space taken at a sample point $x_p = F(y_p)$, $y_p \in R_p$. The total volume depends on the choice of sample points. But since it depends continuously on them for a C^1 surface, variations of the volume related to different choice of sample points do not contribute in the limit when dimensions of all partition boxes tend to zero uniformly, just like a Riemann sum converges to the integral of a continuous function for any choice of sample points.

6.3. Surface integrals in $\mathbb{R}^N$ of a scalar function. Let S be a M-surface from class C^1. Let $x = F(y)$ be a parameterization of S. The surface integral of a function f is defined by

$$\int_S f(x)\,dS \stackrel{\text{def}}{=} \int_D f(F(y))\,J(y)\,d^M y$$

if $f(F(y))$ is integrable on D.

6.3.1. Integration over a sphere in $\mathbb{R}^N$. A sphere of radius a in $\mathbb{R}^N$ is defined by

$$|x|^2 = x_1^2 + x_2^2 + \cdots + x_N^2 = a^2$$

Its parameterization

$$\begin{aligned}
x_1 &= a\cos(\xi_1)\,,\\
x_2 &= a\sin(\xi_1)\cos(\xi_2)\,,\\
x_{N-1} &= a\sin(\xi_1)\sin(\xi_2)\cdots\sin(\xi_{N-2})\cos(\xi_{N-1})\,,\\
x_N &= a\sin(\xi_1)\sin(\xi_2)\cdots\sin(\xi_{N-2})\sin(\xi_{N-1})\,,
\end{aligned}$$

where $\xi_p \in [0,\pi]$ for $p < N-1$ and $\xi_{N-1} \in [0, 2\pi]$, can be obtained using spherical coordinates. Here $0 \le \xi_1 \le \pi$ is the angle between the x_1 axis and the vector x. This axis is called the axis of a spherical coordinate system. Let $x_\perp$ be the orthogonal projection of x onto the $N-1$ dimensional plane orthogonal to the first coordinate axis. Then the length of the projection is $|x_\perp| = a\sin(\xi_1)$. With this choice, x_2 is the scalar projection of $x_\perp$ onto the second coordinate axis, where

$0 \le \xi_2 \le \pi$ is the angle between $x_\perp$ and the x_2 coordinate axis. Then the length of the orthogonal projection of $x_\perp$ onto the plane orthogonal to the second coordinate axis is $a\sin(\xi_1)\sin(\xi_2)$, and x_3 is the scalar projection of this vector projection onto the third axis. This procedure is repeated N times to get all x_j as functions of the angles ξ_a. The angle between any two vectors changes from 0 (parallel vectors) to π (anti-parallel vectors), which explains the range of ξ_b for $b < N-1$, and ξ_{N-1} is nothing but a polar angle a 2-plane.

The tangent vectors to curves that are images of the coordinate lines of parameters ξ are orthogonal:

$$
\begin{aligned}
w_b &= \frac{\partial x}{\partial \xi_b} \quad \Rightarrow \quad (w_b, w_{b'}) = |w_b|^2 \delta_{bb'}\,, \\
|w_1| &= a\,, \quad |w_b| = a\sin(\xi_1)\sin(x_2)\cdots\sin(\xi_{b-1})\,, \quad b = 2,3,...,N-1 \\
J(\xi) &= a^{N-1}\sin^{N-2}(\xi_1)\sin^{N-3}(\xi_2)\cdots\sin(\xi_{N-2})\,.
\end{aligned}
$$

Then an integral of a function over the sphere is reduced to the following iterated integral

$$
\int_{|x|=a} f(x)\,dS = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} f(x(\xi))\,J(\xi)\,d\xi_1\cdots d\xi_{N-2}\,d\xi_{N-1}
$$

By construction of the spherical coordinate system, the integral over a unit sphere in $\mathbb{R}^N$ can also be written as an iterated integral over a sphere in $\mathbb{R}^{N-1}$ of radius $|x_\perp| = \sin(\xi_1)$ and an integral over the angle with the axis of the spherical coordinates ξ_1

$$
\int\limits_{|x|=1} f(x)\,dS_N = \int_0^{\pi}\int\limits_{|y|=1} f\Big(\hat{e}_1\cos(\xi_1) + y\sin(\xi_1)\Big)\,\sin^{N-2}(\xi_1)\,dS_{N-1}\,d\xi_1
$$

where dS_M is the surface area element of a unit sphere in $\mathbb{R}^M$, and $\hat{e}_1$ is the unit vector parallel to the axis of the spherical coordinates. If the function $f(x)$ is invariant under rotations about the axis of the spherical coordinate system, then it depends only on $x_1 = \cos(\xi_1)$ and $|x_\perp| = \sin(\xi_1)$, that is, $f(x) = g(\xi_1)$. In this case,

$$
\int_{|x|=1} f(x)\,dS = \sigma_{_{N-1}}\int_0^{\pi} g(\xi_1)\sin^{N-2}(\xi_1)\,d\xi_1\,.
$$

where $\sigma_{_N}$ is the surface are of a unit sphere in $\mathbb{R}^N$.

6.3.2. Levi-Civita symbol. Let the symbol $\varepsilon_{j_1 j_2\cdots j_N}$ be defined so that it is skew-symmetric under permutation of any two indices, and $\varepsilon_{12\cdots N} = 1$. This symbol is called the *Levi-Civita symbol* in $\mathbb{R}^N$. Any symbol with N indices has N^N indexed values. But the Levi-Civita symbol has only one independent value because its indexed values vanish if any two indices are equal and

$$\varepsilon_{j_1 j_2 \cdots j_N} = (-1)^P \varepsilon_{12\cdots N}$$

where P is the number of permutations needed to convert the set $j_1 j_2 \cdots j_N$ to $12 \cdots N$ by permutations.

The product of two symbols can be expressed in terms the Kronecker delta symbol:

$$\varepsilon_{i_1 i_2 \cdots i_N} \varepsilon_{j_1 j_2 \cdots j_N} = \det \begin{pmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \cdots \delta_{i_1 j_N} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots \delta_{i_2 j_N} \\ \vdots & \vdots & \vdots \\ \delta_{i_N j_1} & \delta_{i_N j_2} & \cdots \delta_{i_N j_N} \end{pmatrix} \stackrel{\text{def}}{=} \delta^{j_1 j_2 \cdots j_N}_{i_1 i_2 \cdots i_N}$$

The determinant of Kronecker deltas, denoted by $\delta^{j\cdots}_{i\cdots}$, is called the *generalized Kronecker delta symbol*. This symbol is convenient to write any *contraction* of indices in the product of Levi-Civita symbols:

$$\sum_{i_1, i_2, \ldots, i_n = 1}^{N} \varepsilon_{i_1 i_2 \cdots i_n i_{n+1} \cdots i_N} \varepsilon_{i_1 i_2 \cdots i_n j_{n+1} \cdots j_N} = n! \delta^{j_{n+1} j_{n+2} \cdots j_N}_{i_{n+1} i_{n+2} \cdots i_N}$$

Note that free indices in the contraction take integer values from 1 to N whereas the generalized Kronecker delta symbol in this equation is defined by the determinant of an $(N-n) \times (N-n)$ matrix.

Let A_{ij} be an $N \times N$ matrix. It is proved in linear algebra that

$$\det A = \varepsilon_{j_1 j_2 \cdots j_N} A_{1 j_1} A_{2 j_2} \cdots A_{N j_N}$$

where the Einstein summation rule is used for repeated indices. The absolute value $|\det A|$ is the volume on an $N-$dimensional parallelepiped with adjacent sides being the vectors defined by the columns of the matrix A.

For any $N-1$ vectors w_a, $a = 1, 2, \ldots., N-1$, define a vector n with components

$$n_i = \varepsilon_{i j_1 \cdots j_{N-1}} w_{1 j_1} w_{2 j_2} \cdots w_{N-1 j_{N-1}}$$

This vector is not zero if and only if the non-zero vectors w_a are linearly independent, and n is orthogonal to the span of vectors w_a because $(n, w_a) = 0$. This follows from the skew-symmetry of the Levi-Civita symbol under a permutation of two indices.

6.3.3. Oriented surface area element of the boundary of a region. Let Ω be an open set in $\mathbb{R}^N$ and its boundary be from class C^1. If $x = F(\xi)$ is a parameterization of the boundary $\partial\Omega$, then the span of vectors $w_a = \partial_a F$ is the tangent space to $\partial\Omega$ which is an $N-1$ dimensional plane that is orthogonal to the vector n defined by the Levi-Civita symbol and the tangent vectors w_a. So, the vector n will called a *normal* to the boundary $\partial\Omega$.

Using the contraction formula for one index in the product of Levi-Civita symbols, one infers that

$$\begin{aligned}|n|^2 &= n_i n_i = \delta^{j_1 j_2 \cdots j_{N-1}}_{i_1 i_2 \cdots i_{N-1}} w_{1j_1} w_{2j_2} \cdots w_{N-1 j_{N-1}} w_{1i_1} w_{2i_2} \cdots w_{N-1 i_{N-1}} \\ &= \det(W^T W) = J^2(\xi)\end{aligned}$$

So, the length of n is equal to the volume of the parallelepiped with adjacent sides being vectors w_a. The vector

$$d\Sigma_i = n_i d^{N-1}\xi$$

is called an *oriented surface area element* on the boundary $\partial\Omega$ and

$$|d\Sigma| = |n| d^{N-1}\xi = J(\xi)\, d^{N-1}\xi\,.$$

6.4. Flux of a vector field. Consider a vector field $F:\ \mathbb{R}^N \to \mathbb{R}^N$. Let S be an $N-1$ surface from class C^1 described by parametric equations $x_j = x_j(\xi)$, $\xi \in D$. The surface integral

$$\Phi = \int_S (F, d\Sigma) = \int_D F_j(x(\xi)) n_j(\xi)\, d^{N-1}\xi$$

is called a *flux* of the vector field F across the surface S.

Suppose that F describes a flow of some quantity. For example, consider a moving air with the velocity vector field $v(x)$ and mass density $\rho(x)$. Then $F(x) = \rho(x)v(x)$ is a mass flow. Suppose that $|n(\xi)| \neq 0$ so that $\hat{n} = n/|n|$ is a unit normal vector. By construction

$$d\Phi(x) = \Big(F(x), d\Sigma(x)\Big) = (F, \hat{n})\, dS = \rho(x)\Big(v(x), \hat{n}(x)\Big) dS(x)$$

is the mass carried by the flow per unit time across the surface dS at a sample point x of the surface in the direction of $\hat{n}(x)$. Note that $(v, \hat{n})$ is the normal component of the velocity (the scalar projection of v on $\hat{n}$ at a point x). If the vector field is orthogonal to the normal, the flux vanishes. Therefore Φ is the total mass carried by the flow across S per unit time.

6.4.1. Flux integral in $\mathbb{R}^3$. The 3-dimensional Levi-Civita symbol defines the components of the cross product of two vectors

$$(x \times y)_i = \varepsilon_{ijk} x_j y_k$$

A parameterization of a 2-surface S in $\mathbb{R}^3$ is defined by a C^1 map of a rectangle in $\mathbb{R}^2$ to $\mathbb{R}^3$

$$x = x(\xi_1, \xi_2)\,, \quad (\xi_1, \xi_2) \in [a, b] \times [c, d] = D$$

such that the normal vector

$$n(\xi) = \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2}$$

is not zero in D except possibly on the boundary of D. The surface area is given by

$$A(S) = \int_S dS = \int_a^b \int_c^d |n(\xi)| \, d\xi_2 d\xi_1$$

A surface integral of a continuous function $f(x)$ over S is given by

$$\int_S f(x) \, dS = \int_a^b \int_c^d f(x(\xi)) \, |n(\xi)| \, d\xi_2 d\xi_1$$

and a flux of a vector field $F(x)$ across S reads

$$\int_S (F, d\Sigma) = \int_a^b \int_c^d \Big(F(x(\xi)), n(\xi) \Big) \, d\xi_2 \, d\xi_1$$

6.4.2. Orientable surfaces. If the vector field is tangent to a surface, then its flux across this surface vanishes. So, the definition of the flux makes sense only if it is not possible to get across the surface at a point x by traveling along the surface and getting back to x but on the "other side" of the surface. To make this concept precise, let $\hat{n}$ be unit normal to the surface at x. Then $-\hat{n}$ is also a unit normal at x and no other unit normal vectors exist. One can define a side of the surface by saying that $\hat{n}$ always points up from the surface. For example, on the outer side of the sphere, the unit normal points from the sphere center, and on the inner side the unit normal points toward the center. So, in a neighborhood of any point of a C^1 surface there are always two sides.

Suppose $\hat{n}$ can be defined continuously on the whole S. This implies that a net variation of $\hat{n}$ along any closed curve in S must be zero. In this case, S must have two sides, one is defined by $\hat{n}$ and the other by $-\hat{n}$, like a sphere or a portion of a plane. It would be impossible to get to the other side of the surface at any point by traveling along a closed curve in the surface because it would contradict the continuity of $\hat{n}$.

A surface is said to be *orientable* if a continuous unit normal vector can be defined on it. In this case, the surface is *oriented* by the unit normal vector. An orientable surface can have two orientations. *The flux of a vector field can only be defined across an orientable surface, and its changes its sign when the orientation is changed.*

If a smooth surface is one-sided or non-orientable, then there should exist a closed curve such that the net change of a unit normal vector along it is not zero. Imagine an ant carrying a flagpole as a unit normal always pointing up from the surface. Since a surface is one-sided, it is possible to find a closed path in the surface such that, when the ant is back to the initial point, the pole points in the direction opposite the initial one. This implies that it is not possible to define a *continuous* unit normal vector on a one-sided surface. For a C^1 surface it is always possible to define a continuous unit normal in a neighborhood of any point (as a unit normal to the tangent space). If however the surface is not orientable, then it is impossible to extend the unit normal vector continuously to the whole surface. It turns out that non-orientable surface do exist. Here is the simplest example.

6.4.3. Möbius strip. Consider a circle in a plane. Take a pole perpendicular to the plane. If the midpoint of the pole is moved along the circle while keeping the pole orthogonal to the plane, the pole sweeps a portion of a cylinder. Let z be a vector perpendicular to the plane, and x_m be the position vector of the pole midpoint relative to the center of the circle. The vectors z and x_m are orthogonal and their span is a plane normal to the circle at any point. At every point of the circle, the pole occupies that same position in this plane. Now image that the pole is rotated in this plane as its midpoint moves around the circle. Suppose that when the midpoint returns to the initial point, the pole net rotation angle is π so that it will occupy the same (staring) position. The surface swept by the pole is smooth and one-sided by construction. If in the beginning of the motion, the swept surface is colored so that one side is red, and the other is blue, then at the end of the motion, the red side is glued to the blue one and vice versa. So, it is impossible to define the either "red" or "blue" normal continuously on this surface because, after making around the circle, the normal becomes the opposite to that at the starting point. This surface is known as a *Möbius strip*.

It is not difficult to find its parametric equations. Let ξ_1 be a parameter that labels points on the pole, so that the straight line segment $x_1 = a$, $x_2 = 0$, and $x_3 = \frac{1}{2}\xi_1$, $-b \le \xi \le b$, is the initial position of the pole of length b. So, the pole is parallel to the x_3 axis and its midpoint at a distance a from the origin on the x_1 axis. The midpoint moves along a circle $x_1 = a\cos(\xi_2)$, $x_2 = a\sin(\xi_2)$, $x_3 = 0$, making one full turn when $0 \le \xi_2 \le 2\pi$. Suppose that the pole is rotated through the angle $\frac{1}{2}\theta$ and the midpoint rotates through the angle ξ_2 so that the pole rotates through the angle π as the midpoint returns to the initial position. Then the projection of the position vector of a point of the pole relative to the midpoint onto the x_3 axis is $\frac{1}{2}\xi_1\cos(\xi_2/2)$, and its projection on the axis from the origin to the midpoint is $\frac{1}{2}\xi_1\sin(\xi_2/2)$.

Therefore the position vector of a point of the pole relative to the origin reads

$$
\begin{aligned}
x_1 &= \Big(a + \frac{\xi_1}{2}\sin(\xi_2/2)\Big)\cos(\xi_2)\,,\\
x_2 &= \Big(a + \frac{\xi_1}{2}\sin(\xi_2/2)\Big)\sin(\xi_2)\,,\\
x_3 &= \frac{\xi_1}{2}\cos(\xi_2/2)\,,
\end{aligned}
$$

where $(\xi_1, \xi_2) \in D = [-b, b] \times [0, 2\pi]$. These are the parametric equations of a Möbius strip.

Let us investigate continuity of a unit normal vector along the circle traversed by the midpoint of the pole, that is, when $\xi_1 = 0$. The vector

$$
n(\xi) = \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2}
$$

is normal to the surface. By evaluating the derivatives, the cross product, and setting $\xi_1 = 0$, the normal is found to be

$$
\begin{aligned}
n_1(0, \xi_2) &= \frac{1}{2}\sin(\xi_2/2)\cos(\xi_2)\,,\\
n_2(0, \xi_2) &= \frac{1}{2}\sin(\xi_2/2)\sin(\xi_2)\,,\\
n_3(0, \xi_2) &= \frac{1}{2}\cos(\xi_2/2)\,.
\end{aligned}
$$

so that $|n(0, \xi_2)| = a/2$. The values $\xi_2 = 0$ and $\xi_2 = 2\pi$ correspond to the same point of the circle. It follows from this equations that

$$
n(0, 2\pi) = -n(0, 0)
$$

Therefore a continuous unit normal cannot be defined. Note that parametric equations define a smooth map of the rectangle $\Omega = [-b, b] \times [0, 2\pi]$ to $\mathbb{R}^3$ that is one-to-one in the interior, but maps the boundaries $\xi = 0$ and $\xi = 2\pi$ onto the same set. This identification is done with a twist which leads to a one-sided smooth surface. One can easily construct a similar map that sends the boundaries $\xi = 0$ and $\xi = 2\pi$ to the same line segment with any numbers of twists. A surface with an odd number of twists is one-sided. Note that the unit normal $\hat{n}(\xi)$ computed for the map $x = x(\xi)$ can be a continuous function on the closed rectangle Ω. But the surface can nonetheless be one-sided because the map is not one-to-one on the boundary and for this reason $\hat{n}$ cannot always be continuously defined on the surface.

It is worth noting that there are surfaces without boundaries (like a sphere) that are one-sided. An example is provided by the famous Klein bottle.

6.5. The divergence (Gauss-Ostrogradsky) theorem. Let Ω be a bounded region with a smooth boundary which is a level set of a C^1 function g with the non-vanishing gradient. The boundary divides $\mathbb{R}^N$ into two non-intersecting regions. Then unit normal $\hat{n} = \nabla g/|\nabla g|$ is continuous on the boundary (cf. Sec. **1.3**). The boundary $\partial\Omega$ is said to be oriented *positively* if the unit normal points outward from Ω. The other orientation is called *negative*. Unless stated otherwise, $\partial\Omega$ will always denote the positively oriented boundary of Ω. The divergence of a vector field F is defined by

$$\operatorname{div} F(x) = \sum_{j=1}^{N} \frac{\partial F_j(x)}{\partial x_j} = (\nabla, F)\,,$$

where ∇ is a formal vector with components being $\partial/\partial x_j$.

THEOREM **6.1**. *Let Ω be an open bounded set in $\mathbb{R}^N$ such that its boundary piecewise smooth. Suppose that a vector field F and a function u are from class $C^1(\overline{\Omega})$. Then*

$$\int_\Omega (\nabla, F)\, u\, d^N x = -\int_\Omega (F, \nabla u)\, d^N x + \int_{\partial\Omega} u\,(F, d\Sigma)\,,$$

where $d\Sigma = \hat{n} dS$ is the surface area element on $\partial\Omega$ oriented positively.

In particular, if $u(x) = 1$, then

$$\int_\Omega \operatorname{div} F\, d^N x = \int_{\partial\Omega} (F, \hat{n})\, dS\,.$$

This statement is known as the divergence or Gauss-Ostrogradsky theorem. Recall that if F describes a flow of some quantity, then the divergence of F is the density of sources of the flow. The divergence theorem states that the net flux of a vector field across the boundary of a bounded region is equal to the sum of all sources of the field in the region. It should be noted that the boundary $\partial\Omega$ can have several disjoint pieces. For example, Ω can have several "cavities" obtained by removing proper open subsets from Ω. All separate parts of $\partial\Omega$ are oriented outward and the surface integral is the sum over all separate parts.

6.5.1. Green's theorem. In a two-dimensional Euclidean space, consider a bounded region Ω whose boundary is a C^1 closed curve without self-intersections. Suppose that the boundary curve is oriented counterclockwise (the x_2 axis is directed upward, while the x_1 is directed to the right). If $x_j = x_j(t)$ are parametric equations of the boundary, then the unit normal to the boundary directed outward is

$$n_j(t) = \varepsilon_{ji} T_i(t)\,, \quad T_j(t) = \frac{x_j'(t)}{|x'(t)|}$$

Indeed, suppose the origin is in the interior of Ω and $T_1 < 0$ and $T_2 > 0$ (for a counterclockwise orientation). Since $n_1 = T_2$ and $n_2 = -T_1$, $n_{1,2} > 0$. This implies that T is obtained by rotating n counterclockwise through the angle $\frac{\pi}{2}$. Any continuous deformation of the boundary preserves this property of T and n. So the equation holds for any shape of Ω that can be continuously deformed to a disk.

The dot product of any two vectors A_j and B_j is equal to the dot product of the (dual) vectors $\varepsilon_{jk} A_k$ and $\varepsilon_{jk} B_k$ so that

$$F_j dx_j = F_j T_j ds = \varepsilon_{jk} F_k n_j ds = \varepsilon_{jk} F_k d\Sigma_j$$

Therefore by the divergence theorem for a vector field $\varepsilon_{jk} F_k$

$$\oint_{\partial\Omega} F_j dx_j = \int_\Omega \varepsilon_{jk} \partial_j F_k \, d^2x = \int_\Omega \left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1} \right) d^2x$$

This statement is known as *Green's theorem.*

It is also valid if Ω has holes, that is, if its boundary contains several closed curves without self-intersections. In this case, the outer boundary must be oriented counterclockwise, while all the inner boundaries must be oriented clockwise. Indeed, let us cut a region Ω without any holes by a curve C into two regions Ω_1 and Ω_2. Then Green's theorem can be applied to both of them. Note that the curve C is a part of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, but it has opposite orientation in them so that for any line integral of a vector field

$$\oint_{\partial\Omega_1} + \oint_{\partial\Omega_2} = \oint_{\partial\Omega}$$

because the line integral over the cut curve C is cancelled. The line integrals in the left side also contain integration over the inner boundary of Ω (over the boundary of the hole) that must be oriented clockwise if $\partial\Omega_1$ and $\partial\Omega_2$ are oriented counterclockwise. Evidently, this argument can be extended to any number of holes.

6.6. Integration by parts in $\mathbb{R}^N$. It follows from the fundamental theorem of calculus that for any two functions from class $C^1[a,b]$,

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx$$

This equation has a multi-variable generalization.

Let b be a non-zero constant vector and v is a function from class $C^1(\overline{\Omega})$. Put $F = bv$ in Theorem **6.1**. Then

$$\int_\Omega u(b,\nabla)v\,d^Nx = \oint_{\partial\Omega} vu(b,d\Sigma) - \int_\Omega v(b,\nabla)u\,d^Nx$$

Since the vector b is arbitrary, the integration by parts can be stated in the form

$$\int_\Omega u(x)D_jv(x)\,d^Nx = \oint_{\partial\Omega} v(x)u(x)d\Sigma_j - \int_\Omega v(x)D_ju(x)\,d^Nx\,.$$

where $D_j = \partial/\partial x_j$, $j = 1,2,...,N$. If Ω is not bounded, the integration by parts can still be used with a suitable regularization. For example, it can be applied to a part of Ω that lies in a ball of radius R and the limit $R \to \infty$ should be taken after evaluation of the integrals. The answer may depend on the regularization if the improper integral does not converge absolutely.

Suppose that u and v are from class C^p and their supports are in Ω. This implies that any partial derivative of u and v up order p vanishes on the boundary $\partial\Omega$. Then by applying the integration by parts several times

$$\int_\Omega uD_x^\beta v\,d^Nx = (-1)^\beta \int_\Omega vD_x^\beta u\,d^Nx\,,\quad 0 \le \beta \le p$$

where $D_x^\beta u$ stands for any partial derivative of order β. The surface integrals arising upon integration by parts vanish because of the said properties of the functions u and v.

6.6.1. Green's identity. Let the boundary of a bounded region Ω be oriented outward by a unit normal $\hat{n}$, and let u and v be functions from class C^2 in a neighborhood of Ω. Integrating the identity

$$u\Delta v - \Delta uv = \partial_j\Big(u\partial_jv - \partial_juv\Big)$$

where $\Delta = \partial_j\partial_j$ is the Laplace operator, over Ω and using the divergence theorem to transform the integral in the right-hand side to a surface integral, one infers that

$$\int_{\Omega} \Big(u\Delta v - \Delta u v\Big)\, d^N x = \int_{\partial\Omega} \Big(u\partial_j v - \partial_j u v\Big)\, d\Sigma_j$$
$$= \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v\Big)\, dS$$

This is known as Green's first identity. Here $\frac{\partial v}{\partial n} = (\hat{n}, \nabla v)$ is the normal derivative of v.

6.7. Exercises.

1. Suppose S is a surface in $\mathbb{R}^3$ obtained by a revolution of the graph $x_3 = f(s)$, $a \le s \le b$, about the x_3 axis.
(i) Show that its parametric equations can be written in the form

$$x_1 = s\cos(\phi)\,,\ x_2 = s\sin(\phi)\,,\quad x_3 = f(s)\,,\quad (s,\phi) \in [a,b]\times[0,2\pi]$$

or

$$x_1(s,t) = \frac{s(1-t^2)}{1+t^2}\,,\ x_2 = \frac{2st}{1+t^2}\,,\ x_3 = f(s)\,,\quad (s,t) = [a,b]\times\mathbb{R}$$

(ii) Find the normal vectors $n(s,\phi)$ and $n(s,t)$ for both parameterizations. Express the surface area in terms of the function f.

2. Let σ_N be the surface are of a unit sphere, $|x| = 1$, in $\mathbb{R}^N$. Suppose f is continuous on $\mathbb{R}^N$. Show that

$$\lim_{a\to 0^+} \frac{1}{\sigma_N a^{N-1}} \int_{|x|=a} f(x)\, dS = f(0)$$

3. Show that the volume of a bounded region in $\mathbb{R}^N$ with a piecewise smooth boundary is given by the surface integral

$$V(\Omega) = \frac{1}{N}\oint_{\partial\Omega} x_j d\Sigma_j$$

Use this relation to show that the volume V_N and the surface area σ_N of an $N-$ball of radius a are related by

$$V_N(a) = \frac{a}{N}\,\sigma_N(a)\,.$$

4. Suppose that u and its partial derivative $\partial_j u$ are continuous and

$$|u(x)| \le \frac{M_0}{|x|^\alpha}\,,\quad \Big|\frac{\partial u}{\partial x_j}\Big| \le \frac{M_1}{|x|^\beta}\,,\quad |x| > R > 0$$

for some constants $M_{0,1}$. Show that if $\alpha > N-1$ and $\beta > N$, then

$$\int \frac{\partial u}{\partial x_j}\, d^N x = 0$$

Hint: Reduce the integration domain to $[-a, a] \times \mathbb{R}^{N-1}$ and use continuity of the Lebesgue integral as $a \to \infty$. Use Fubini's theorem to evaluate the integral with respect to x_j and then investigate the limit.

5. Put

$$u(x, y) = \frac{\arctan(x)}{1+y^2}\,, \quad x, y \in \mathbb{R}$$

Show that the partial derivative $D_x u = \frac{\partial u}{\partial x}$ is integrable in the plane $\mathbb{R}^2$ spanned by real variables x and y, and find the value of the integral of $D_x u(x, y)$ over the plane. Does the answer contradict to the result of Problem 4?

6. Suppose that u and v are from class C^1 and

$$|u(x)| \leq \frac{A_0}{|x|^{\alpha_0}}\,, \quad |Du(x)| \leq \frac{A_1}{|x|^{\alpha_1}}\,,$$
$$|v(x)| \leq \frac{b_0}{|x|^{\beta_0}}\,, \quad |Dv(x)| \leq \frac{B_1}{|x|^{\beta_1}}\,,$$

for all $|x| > R > 0$ and constants $A_{0,1}$ and $B_{0,1}$. Find a condition on parameters $\alpha_{0,1}$ and $\beta_{0,1}$ under which

$$\int u(x) Dv(x)\, d^N x = -\int Du(x) v(x)\, d^N x$$

7. Cauchy line integrals of analytic functions

7.1. Functions of a complex variable. A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is called a function of a complex variable. A function $f(z)$ has a limit $w \in \mathbb{C}$ at z_0 if $|f(z) - w| \to 0$ as $|z - z_0| \to 0$, and in this case one writes

$$\lim_{z\to z_0} f(z) = w\,.$$

A function f is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$, and f is continuous on a set Ω is it is continuous at all point of Ω. The derivative defined by

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided the limit exists. In particular, $(z^n)' = nz^{n-1}$ for integer n. A function f is (complex) differentiable on a set Ω if the derivative exists at every point of Ω.

7.1.1. Analytic functions. A function of a complex variable is said to be *analytic* at a point z_0 if in a neighborhood of z_0 it is given by a power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n\,, \quad |z - z_0| < R\,, \quad R > 0\,.$$

A function is *analytic on a set* Ω if it is analytic at every point of Ω. Using Proposition **1.1** one can show that f is from class C^∞ and its derivatives can be obtained by term-by-term differentiation of the series and $c_n = f^{(n)}(z_0)/n!$ (see Exercises). By the Taylor theorem,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n\,, \quad |z - z_1| < R_1$$

for some $R_1 > 0$ and any z_1 in the disk, $|z_1 - z_0| < R$. Therefore analyticity at a point implies analyticity in a neighborhood of the point.

For example,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is analytic in the whole complex plane because the above series has infinite radius of convergence. The function

$$f(z) = \frac{1}{1-z}$$

is analytic everywhere except the point $z = 1$. Recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\,, \quad |z| < 1$$

This shows that the function is analytic in the open disk $|z| < 1$. Let $z_0 \neq 1$. Then the following identity holds

$$\frac{1}{1-z} = \frac{1}{1-z_0}\frac{1}{1-\frac{z-z_0}{1-z_0}}$$

Therefore near z_0, the function is represented by a power series

$$\frac{1}{1-z} = \frac{1}{1-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{1-z_0}\right)^n\,, \qquad |z-z_0| < |1-z_0|$$

whose radius of convergence coincides with the distance from 1 to z_0.

7.1.2. Holomorphic functions. A function $f(z)$ is said to be *holomorphic* on an open set Ω of the complex plane if it is differentiable in a neighborhood of every point of Ω. In particular, the functions e^z and $\frac{1}{1-z}$ are holomorphic on their domains.

A major theorem in complex analysis states that *every holomorphic function is analytic and vice versa*[19]. Note that every analytic function is differentiable so it is holomorphic. It turns out that a complex differentiability (the existence of the derivative $f'(z)$) implies that all derivatives exist so that the function can be given by a Taylor series (which is a power series) so that the function is analytic.

7.1.3. Cauchy-Riemann equations. Let $f(z)$ be analytic. Put $z = x+iy$ so that

$$f(z) = u(x,y) + iv(x,y)\,, \quad x = \frac{1}{2}(z+\bar{z})\,, \quad y = \frac{1}{2i}(z+\bar{z})$$

Since $f(z)$ is independent of $\bar{z}$, it follows from the chain rules $\partial_{\bar{z}} = \frac{1}{2}\partial_x - \frac{1}{2i}\partial_y$ that

$$\frac{\partial f(z)}{\partial \bar{z}} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,,$$

These relations are known as *the Cauchy-Riemann equations*. It follows from them that real and imaginary parts of an analytic function in Ω are *harmonic functions*, that is, they are solutions to the Laplace equation

$$\Delta u(x,y) = 0\,, \quad \Delta v(x,y) = 0$$

[19] see, e.g.,

The assertion follows from Clairaut's theorem $\partial_x\partial_y u = \partial_y\partial_x u$ (and similarly for v).

The set of analytic functions is closed under basic algebraic operations with functions. The sum and product of two analytic functions on Ω is analytic on Ω. The reciprocal of an analytic function is analytic except points where the functions vanishes. A composition of two analytic functions is analytic.

7.1.4. Poles. A function $f(z)$ is said to have a *pole* at $z = z_0$ of order n if near z_0

$$f(z) = \sum_{k=1}^{n} \frac{a_k}{(z - z_0)^k} + g(z)\,, \tag{7.1}$$

where g is analytic at z_0. If $n = 1$, the pole is called *simple*. The coefficient a_1 is called the *residue* of f at the pole z_0 and is denoted by

$$a_1 = \operatorname*{res}_{z_0} f$$

If the pole is simple, then

$$\operatorname*{res}_{z_0} f = \lim_{z\to z_0} (z - z_0) f(z)$$

For example,

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

So, the function has two simple poles at $z = \pm i$ and

$$\operatorname*{res}_{\pm i} \frac{1}{1+z^2} = \pm\frac{1}{2i}$$

7.2. The line integral in a complex plane. A curve in the complex plane is defined in the same as a curve in $\mathbb{R}^2$, that is, by a continuous mapping of an interval to $\mathbb{C}$ that is one-to-one except possibly finitely many points in the interval. If $z = w(t)$, $a \le t \le b$, are parametric equations, then the curve is closed if $w(a) = w(b)$. If $w(t_1) = w(t_2)$ implies that $t_1 = t_2$ (except possibly for $t_1 = a$ and $t_2 = b$ for a closed curve), then the curve is called *simple* (no self-intersections). If $w'(t)$ is continuous on $[a, b]$, the curve is said to be from class C^1.

For example

$$z = ae^{it}\,, \quad 0 \le t \le 2\pi$$

is a circle of radius a centered at the origin because $|z(t)| = a$. The circle is oriented counterclockwise. Parametric equations $z = ae^{-it}$ describe the same circle oriented clockwise.

Let $z = w(t)$, $a \le t \le b$, be parametric equations of a curve C from class C^1 in the complex plane. Let $f(z)$ be a continuous function of a complex variable z. Then the integral

$$\int_C f(z)\,dz = \int_a^b f(w(t))w'(t)\,dt$$

is called the *Cauchy line integral of f over the curve C*. Note that the integral depends on the orientation of C in full contrast to the line integral of a scalar function. The Cauchy line integral changes its sign if the orientation of the curve is changed. Parametric equations $z = w(\tau(t))$, $a \le t \le b$, $\tau(t) = b + a - t$, describe the same curve but with opposite orientation, denoted by $-C$. Then by changing the integration variable

$$\begin{aligned}\int_{-C} f(z)\,dz &= \int_a^b f(w(\tau(t)))w'(\tau(t))\tau'(t)\,dt \\ &= \int_b^a f(w(\tau))w'(\tau)\,d\tau = -\int_C f(z)\,dz\,.\end{aligned}$$

For example, if C is a circle $|z| = a$ oriented counterclockwise, then for any integer n

$$\begin{aligned}\oint_C z^n\,dz &= \int_0^{2\pi} a^n e^{int}\,iae^{it}\,dt = ia^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt \\ &= \begin{cases} 0\,, & n \ne -1 \\ 2\pi i\,, & n = -1 \end{cases}\end{aligned} \tag{7.2}$$

By the 2π periodicity of the integrand.

A region is called *simply connected* if any closed curve can be continuously contracted to a point in the region without crossing its boundary. In other words, a simply connected region in the complex plane has no holes.

THEOREM **7.1**. (Cauchy's integral theorem)
Let f be analytic in a simply connected region and C be a simple, closed curve in this region from class C^1. Then the line integral of f over C vanishes

$$\oint_C f(z)\,dz = 0$$

This theorem follows from Green's theorem. Recall that

$$\oint_{\partial\Omega} F_1(x,y)dx + F_2(x,y)dy = \iint_\Omega \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right)\,dxdy$$

where the boundary $\partial\Omega$ is oriented counterclockwise. The hypotheses of Green's theorem are met for the Cauchy integral if $C = \partial\Omega$ (the boundary of some simply connected Ω). Therefore by Green's theorem and the Cauchy-Riemann equations

$$\begin{aligned}\oint_C f(z)dz &= \oint_C (udx - vdy) + i\oint_C (vdx + udy)\\ &= -\iint_\Omega \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy\\ &= 0\end{aligned}$$

7.3. The residue theorem.

THEOREM **7.2**. *Let f have finitely many poles at $z = z_k$, $k = 1, 2, ..., n$, in a simply connected region Ω. If the boundary $\partial\Omega$ is from class C^1 and oriented counterclockwise, then*

$$\oint_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z_k} f(z)$$

Take z_k and connect it to some point on the boundary by a smooth curve C_k. Let Ω_a be the region obtained from Ω by removing the curves C_k and the disks $|z - z_k| \le a$. Then the boundary of Ω_a consists of $\partial\Omega$, the circles $|z - z_k| = a$, denoted by S_k, and the curves C_k. If $\partial\Omega$ is oriented counterclockwise, then $\partial\Omega_a$ must also be oriented counterclockwise. This implies that the circles S_k are oriented clockwise, and the curves C_k must be traversed twice (from the boundary toward the pole and back after traversing S_k clockwise). Let C_k^+ denote C_k oriented from the boundary to the pole, and C_k^- from the pole to the boundary. The function f is analytic in Ω_a. By the Cauchy integral theorem

$$\int_{\partial\Omega_a} f(z)\,dz = 0$$

On the other hand,

$$\int_{\partial\Omega_a} f(z)\,dz = \int_{\partial\Omega} f(z)\,dz + \sum_{k=1}^{n}\left(\int_{C_k^+} + \int_{C_k^-} + \oint_{S_k}\right) f(z)\,dz$$

The integrals over $C_k^{\pm}$ are taken along the same curve but with opposite orientations. So, they cancel each other in the sum. Near z_k, f has the form (**7.1**). So, only the term proportional to $(z - z_k)^{-1}$ contributes to the integral over S_k according to (**7.2**). Since the circles S_k are oriented clockwise, there is an extra minus sign as compared to (**7.2**) so that

$$\int_{\partial\Omega_a} f(z)\,dz = \int_{\partial\Omega} f(z)\,dz - 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z_k} f(z)$$

and the conclusion of the residue theorem follows.

Example 1. Let k be real parameter. Put

$$F(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\,dx\,. \tag{7.3}$$

The integral converges absolutely for any k because

$$\left|\frac{e^{ikx}}{1+x^2}\right| = \frac{1}{1+x^2}\,,$$

$$\lim_{n\to\infty} \int_{-n}^{n} \frac{dx}{1+x^2} = \lim_{n\to\infty} \arctan(x)\Big|_{-n}^{n} = \pi < \infty$$

Therefor $F(k)$ can be computed in any suitable regularization. In particular,

$$F(k) = \lim_{n\to\infty} \int_{-n}^{n} \frac{e^{ikx}}{1+x^2}\,dx$$

One has $F(0) = \pi$. Put $f(z) = e^{ikz}(1+z^2)^{-1}$. The function f has two simple poles at $z = \pm i$ and analytic otherwise. If $k > 0$, it decays exponentially with increasing $|z|$ if the upper half-plane, $\operatorname{Im} z > 0$. If $k < 0$, it decays exponentially with increasing $|z|$ in the lower half-plane, $\operatorname{Im} z < 0$.

Let C_n^+ be the closed contour that consists of the interval $[-n, n]$ in the real axis and the circular arc $|z| = n$, $\operatorname{Im} z \geq 0$, denoted S_n^+. If C_n^+ is oriented counterclockwise, then by the residue theorem

$$\begin{aligned}\oint_{C_n^+} f(z)\,dz &= \int_{-n}^{n} f(x)\,dx + \int_{S_n^+} f(z)\,dz \\ &= 2\pi i \operatorname*{res}_{i} f(z) = \pi e^{-k}\,, \quad k > 0\end{aligned}$$

The integral over S_n^+ vanishes in the limit $n \to \infty$:

$$\begin{aligned}\left|\int_{S_n^+} f(z)\,dz\right| &\overset{(1)}{=} \left|\int_0^{\pi} \frac{e^{ikne^{it}}}{1+n^2e^{2it}} ine^{it}\,dt\right| \overset{(2)}{\leq} n \int_0^{\pi} \frac{e^{-kn\sin(t)}}{|1+n^2e^{2it}|}\,dt \\ &\overset{(3)}{\leq} n \int_0^{\pi} \frac{dt}{|1+n^2e^{2it}|} \overset{(4)}{\leq} \frac{n}{n^2-1} \int_0^{\pi} dt = \frac{\pi n}{n^2-1} \to 0\end{aligned}$$

as $n \to \infty$. Here
(1) is obtained by using the parametric equation of S_n^+, $z = ne^{it}$,

$0 \le t \le \pi$;
(2) is obtained by moving the absolute value into the integral and by calculating $|f(ne^{it})|$;
(3) holds because $k > 0$ and $\sin(t) \ge 0$ if $0 \le t \le \pi$;
(4) follows from the triangle inequality $||z_1| - |z_2|| \le |z_1 - z_2|$ for $z_1 = 1$ and $z_2 = n^2 e^{2it}$. Thus, by taking the limit $n \to \infty$, its concluded that $F(k) = \pi e^{-k}$ if $k > 0$.

Similarly, if $k < 0$, take the closed contour C_n^- that consists of the interval $[-n, n]$ in the real axis and the circular arc $|z| = n$, $\operatorname{Im} z \le 0$, denoted S_n^-. If C_n is oriented *clockwise*, then by the residue theorem

$$\begin{aligned}\oint_{C_n^-} f(z)\,dz &= \int_{-n}^{n} f(x)\,dx + \int_{S_n^-} f(z)\,dz \\ &= -2\pi i \operatorname*{res}_{-i} f(z) = \pi e^{k}\,, \quad k < 0\end{aligned}$$

The reader is asked to show that

$$\left| \int_{S_n^-} f(z)\,dz \right| \le \frac{\pi n}{n^2 - 1}\,, \quad n > 1\,,$$

using the same line of arguments as in the case of integration over S_n^+, but with $k < 0$. Thus,

$$F(k) = \pi e^{-|k|}$$

Example 2: Fresnel's integrals. Consider the improper integral

$$\int_0^\infty e^{ix^2}\,dx \stackrel{\text{def}}{=} \lim_{n\to\infty} \int_0^n e^{ix^2} dx$$

The integral does not converge absolutely because the integral of the absolute value of the integrand, $|e^{ix^2}| = 1$, diverges. So, the value of the integral depends on regularization. This fact is emphasized by using the symbol $\stackrel{\text{def}}{=}$ (the *definition* of the left-hand side). In particular, in the regularization defined by the above limit, the integral converges and its real and imaginary parts are known as Fresnel's integrals.

In the complex plane, consider a closed contour C that is the boundary of the wedge of the disk of radius n that corresponds to the interval $[0, \frac{\pi}{4}]$ of the polar angle. It consist of three pieces. The first goes from $z = 0$ to $z = n$ along the real axis, the second from $z = n$ to $z = \sqrt{i}n$ along the circle $|z| = n$, and the third goes back to the origin along the line segment from $z = \sqrt{i}n$. Here $\sqrt{i} = e^{i\pi/4}$. Parametric equations of these three pieces can be chosen respectively as

$$\begin{aligned} C_1:&\quad z=t\,,\quad t\in[0,n]\,;\\ C_2:&\quad z=ne^{it}\,,\quad t\in[0,\pi/4]\,;\\ C_3:&\quad z=\sqrt{i}t\,,\quad t\in[n,0]\,. \end{aligned}$$

Note that C_3 must be oriented from $z=\sqrt{i}n$ to $z=0$. This is indicated by the range $[n,0]$ of the parameter: from $t=n$ to $t=0$, which corresponds to the lower and upper limits of integration. The function e^{iz^2} is analytic. Therefore its line integral over C vanishes:

$$\oint_C e^{iz^2}dz=\left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)e^{iz^2}\,dz=0$$

One has for these integrals

$$\begin{aligned} \int_{C_1}e^{iz^2}dz&=\int_0^n e^{it^2}\,dt\,,\\ \int_{C_2}e^{iz^2}dz&=\int_0^{\pi/4}e^{in^2e^{2it}}ine^{it}\,dt\,,\\ \int_{C_3}e^{iz^2}dz&=\int_n^0 e^{-t^2}\sqrt{i}\,dt=-e^{i\pi/4}\int_0^n e^{-t^2}dt \end{aligned}$$

Let us show that the integral over the circular arc vanishes in the limit $n\to\infty$. One has

$$\begin{aligned} \left|\int_{C_2}e^{iz^2}dz\right| &\overset{(1)}{\le} n\int_0^{\pi/4}e^{-n^2\sin(2t)}\,dt\overset{(2)}{\le} n\int_0^{\pi/4}e^{-4n^2t/\pi}\,dt\\ &\overset{(3)}{=}\frac{\pi}{4n}\left(1-e^{-n^2}\right)\to 0 \end{aligned}$$

as $n\to\infty$. Here
(1) is obtained by moving the absolute value into the integral and calculating $|f(ne^{it})|$;
(2) follows from the inequality $\sin(2)\ge 4t/\pi$ that holds in the interval $0\le t\le\pi/4$. Note that the graph of $\sin(2t)$ is concave downward in the interval $[0,\pi/4]$. So the secant line through the origin $(0,0)$ and the point $(\pi/4,1)$ on the graph lies below the graph, which comprises the said inequality.
(3) is obtained by evaluating the integral.

Using the Gaussian integral

$$\int_0^\infty e^{ix^2}\,dx\overset{\text{def}}{=}\lim_{n\to\infty}\int_0^n e^{ix^2}\,dx=e^{i\pi/4}\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\,e^{i\frac{\pi}{4}}$$

Example 3. Let us evaluate the conditionally convergent integral that was discussed earlier:

$$I = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx \stackrel{\text{def}}{=} \lim_{n\to\infty} \int_{-n}^{n} \frac{\sin(x)}{x}\, dx$$

Owing to the continuity of the Riemann integral

$$I = \lim_{n\to\infty} \lim_{a\to 0^+} \operatorname{Im} \left(\int_{-n}^{-a} + \int_{a}^{n} \right) \frac{e^{ix}}{x}\, dx$$

The function $f(z) = e^{iz}/z$ is analytic everywhere except at $z = 0$.

Let C be a closed contour that is oriented counterclockwise and consists of two intervals $[-n, -a]$ and $[a, n]$ on the real axis, and two circular arcs C_a, $|z| = a$, and C_n, $|z| = n$, which lie in the upper half-plane, $\operatorname{Im} z \geq 0$. The function f is analytic in the region bounded by C and, hence, its line integral over C vanishes:

$$\oint_C e^{iz} \frac{dz}{z} = \left(\int_{-n}^{-a} + \int_{a}^{n} \right) e^{ix} \frac{dx}{x} + \int_{C_a} e^{iz} \frac{dz}{z} + \int_{C_n} e^{iz} \frac{dz}{z} = 0$$

The imaginary part of the first two terms is equal to the integral in question after taking the limits $a \to 0^+$ and then $n \to \infty$. The integral over the arc C_n vanishes in the limit $n \to \infty$. Indeed,

$$\left| \int_{C_n} e^{iz} \frac{dz}{z} \right| \stackrel{(1)}{\leq} \int_0^{\pi} e^{-n\sin(t)}\, dt \stackrel{(2)}{\to} 0\,, \quad n \to \infty\,,$$

where (1) is obtained by using the parametric equation $z = ne^{it}$ and calculating $|f(ne^{it})|$; (2) follows from the Lebesgue dominated convergence theorem because $|e^{-n\sin(t)}| \leq 1 \in \mathcal{L}(0, \pi)$ and $e^{-n\sin(t)} \to 0$ a.e. as $n \to \infty$.

Let us evaluate the limit of the integral over C_a as $a \to 0^+$. Using the parametric equation $z = ae^{it}$

$$\lim_{a\to 0^+} \int_{C_a} e^{iz} \frac{dz}{z} = -i \lim_{a\to 0^+} \int_0^{\pi} e^{iae^{it}} dt = -i \int_0^{\pi} \lim_{a\to 0^+} e^{iae^{it}} dt = -i\pi$$

where the minus sign is due to the opposite orientation of C_a in the chosen parameterization, and the order of integration and taking the limit can interchanged by the Lebesgue dominated convergence theorem because $|e^{iae^{it}}| \leq 1 \in \mathcal{L}(0, \pi)$ for all $a \geq 0$. It is then concluded that

$$\lim_{n\to\infty} \int_{-n}^{n} \frac{\sin(x)}{x}\, dx = \pi\,.$$

7.4. Gaussian integrals with complex parameters. Consider the following Gaussian integral

$$I_N(A,b) = \int_{\mathbb{R}^N} e^{-(x,Ax)+2(b,x)}\, d^N x\,, \quad b_j \in \mathbb{C}$$

in which parameters b_j are complex. This integral converges absolutely because $b_j = \beta_j + i\alpha_j$ and

$$\left| e^{-(x,Ax)+2(b,x)} \right| = e^{-(x,Ax)} \left| e^{2(\beta,x)+i(\alpha,x)} \right| = e^{-(x,Ax)+2(\beta,x)}$$

so that the integral of the absolute value converges for any $\beta \in \mathbb{R}^N$.

To compute the integral, consider first a one-dimensional case:

$$I(b) = \int_{-\infty}^{\infty} e^{-x^2+2ibx}$$

where b is real. Since $I(b)$ is independent of regularization,

$$I(b) = \lim_{n\to\infty} \int_{-n}^{n} e^{-x^2-2ibx}\, dx$$

Let R_b be a rectangle in the complex plane $\operatorname{Re} z \in [-n,n]$ and $\operatorname{Im} z \in [0,b]$. Since e^{-z^2} is analytic,

$$\oint_{\partial R_b} e^{-z^2}\, dz = 0$$

Rewriting this line integral as the sum of ordinary integrals over four intervals comprising the boundary of R_b, one gets

$$\int_{-n}^{n} e^{-x^2} dx - \int_{-n}^{n} e^{-(t+ib)^2} dt + \int_0^b e^{-(n+it)^2} dt - \int_0^b e^{-(n-it)^2} dt = 0$$

The second integral is the line integral of e^{-z^2} over the top horizontal boundary of R_b: $z = t + ib$, $-n \le t \le n$. The integrals over the vertical intervals vanish in the limit $n \to \infty$:

$$\left| \int_0^b e^{-(n\pm it)^2} dt \right| \le \int_0^b |e^{-(n\pm it)^2}| dt = e^{-n^2} \int_0^b e^{t^2} dt \le b e^{b^2} e^{-n^2} \to 0$$

as $n \to \infty$. Note that by monotonicity $e^{t^2} \le e^{b^2}$ if $0 \le t \le b$. Therefore

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \lim_{n\to\infty} \int_{-n}^{n} e^{-(t+ib)^2} dt$$

It follows from this relation that

$$I(b) = \lim_{n\to\infty} \int_{-n}^{n} e^{-t^2-2ibt} dt = \sqrt{\pi}\, e^{-b^2}$$

Using shift and scaling transformations of the integration variable and the above result, one can show that

$$I(a,\xi) = \int_{-\infty}^{\infty} e^{-ax^2+2\xi x}\,dx = \sqrt{\frac{\pi}{a}}\,e^{\xi^2/a}\,, \quad \xi \in \mathbb{C}\,, \quad a > 0$$

Technicalities are left to the reader as an exercise.

To compute the integral in $\mathbb{R}^N$, one can follows the same line of arguments used to evaluate the Gaussian integral $I(A,b)$ with real b. First, new integration variables are introduced in which the quadratic form is diagonal, $(x,Ax) = (y,ay)$ where a is a diagonal matrix. In doing so, the integral is proved to be the product of one-dimensional integrals so that

$$I(A,b) = \frac{\pi^{\frac{N}{2}}}{\sqrt{\det A}}\,e^{(b,A^{-1}b)}\,, \quad b \in \mathbb{C}^N$$

for any positive definite matrix A. Technicalities are left to the reader as an exercise.

7.5. Exercises.

1. Let

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n\,, \quad |z-z_0| < R$$

(i) Show that the convergence of the series implies that $|c_n|\delta^n \to 0$ as $n \to \infty$ for any $\delta < R$.
(ii) Show that

$$|c_n(z-z_0)^n| \le Mq^n\,, \quad q < 1\,, \quad |z-z_0| \le \delta < R$$

for some constants M and q and any δ, then use Proposition **1.1** to show that the series converges uniformly.
(iii) Show that the series obtained by term-by-term differentiation of the power series any number of times also converge uniformly in the disk $|z-z_0| < R$ so that f is from class C^∞ and $c_n = f^{(n)}(z_0)/n!$.

2. Prove the equation for the Gaussian integral $I(a,\xi)$.

3. Prove the equation for the Gaussian integral $I(A,b)$.

4. Evaluate

$$I(a,b) = \int_{-\infty}^{\infty} e^{iax^2+bx}dx \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} e^{-\varepsilon x^2+iax^2+bx}dx\,,$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

5. Evaluate

$$\int_{R^N} e^{i(x,Ax)} d^N x \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0^+} \int_{R^N} e^{i(x,Ax)-\varepsilon(x,x)} d^N x \,, \quad \det(A) \neq 0 \,.$$

Express the answer in terms of the matrix A.

6. Let $\Omega \subset \mathbb{C}$ be closed, bounded, and simply connected, and its boundary $\partial\Omega$ is piecewise smooth and oriented counterclockwise. Use the residue theorem to prove the identity

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(w)}{w-z}\, dw$$

for any function f that is analytic on Ω and any point z that is in the interior of Ω.

7. Suppose that f is analytic everywhere. Let z and z' be two points in the complex, and C_n be a circle of radius $n > 2|z - z'|$ centered at z and oriented counterclockwise.
(i) Show that for any $w \in C_n$

$$|w - z'| > \frac{n}{2} \,.$$

(ii) Put $w = z + ne^{it}$, where $0 \leq t \leq 2\pi$. Use the identity from Problem 6 to show that

$$|f(z) - f(z')| \leq \frac{|z-z'|}{\pi n} \int_0^{2\pi} |f(ne^{it})|\, dt$$

(iii) Prove Liouville's theorem which states that *a function that is analytic in $\mathbb{C}$ and bounded*

$$|f(z)| \leq M \,, \quad \forall z \in \mathbb{C}$$

is constant. In particular, *if f is analytic everywhere and $f(z) \to 0$ as $|z| \to \infty$, then $f(z) = 0$.*
Hint: Show that $|f(z) - f(z')|$ is smaller than any preassigned positive number.

8. Potential-like integrals

8.1. Preliminaries. Let $E(x)$ be a conservative vector field in $\mathbb{R}^3$, that is, $E(x) = -\nabla u(x)$ where u is a potential of E. The divergence of a vector field is proportional to the density of sources of E. If the density $\rho(x)$ is known, then the potential satisfies the *Poisson equation*:

$$\operatorname{div} E(x) = -\Delta u(x) = 4\pi G\rho(x)$$

where Δ is the Laplace operator, G is a constant, and 4π is a convention factor whose significance will be clarified later.

For example, a static electric or gravitational field is conservative. A point-like particle located at $y \in \mathbb{R}^3$ creates an electric (or gravitational) potential at a point x

$$du(x) = G\,\frac{dm(y)}{|x-y|}$$

where $dm(y)$ is an electric charge (or mass) of the particle, and the constant G is a universal constant for the electromagnetic theory (or the gravity theory). The law is known as the Coulomb law in electricity and as the Newton gravity law in the gravity theory. If electric charges or masses are distributed over a region Ω, then $du(x)$ with $dm(y) = \rho(y)d^3y$ is the potential at x created by an element of volume d^3y at a point y. By the superposition principle (the total field at a point is the vector sum of the fields created by all sources), the potential at x is

$$u(x) = G\int_\Omega \frac{\rho(y)}{|x-y|}\,d^3y$$

This suggests that the potential defined by this integral should be a solution to the Poisson equation. To verify that that is the case, one has to show that u has necessary partial derivatives and to figure out a way for computing them.

8.2. Potential-like integrals. Let Ω be a bounded region in $\mathbb{R}^N$. Put

$$u(x) = \int_\Omega \frac{\rho(y)}{|x-y|^\alpha}\,d^Ny\,, \qquad x \in \mathbb{R}^N$$

If $x \in \bar{\Omega}$, then the integrand is singular at $y = x$. A sufficient condition for the integral to exist is to require that ρ is bounded, $|\rho(y)| \leq M$, and $\alpha < N$ because by the comparison test

$$\frac{|\rho(y)|}{|x-y|^\alpha} \leq \frac{M}{|x-y|^\alpha} \in \mathcal{L}(\Omega) \quad \text{if} \quad \alpha < N$$

for a bounded region Ω. If x does not belong to the closure $\overline{\Omega}$, then $|x-y|^{-\alpha}$ is continuous on $\overline{\Omega}$ and, hence, bounded. In this case, the integral exists if $\rho \in \mathcal{L}(\Omega)$. Without loss of generality, $\overline{\Omega}$ can be viewed as support of the density ρ and the integral is taken over $\mathbb{R}^N$. The objective is to investigate smoothness of $u(x)$ in $\mathbb{R}^N$.

In what follows, the following result will be used.

PROPOSITION **8.1**. *Let B_R be a ball of radius R, $|y| < R$. Then there exists a constant C_α such that*

$$\int_{B_R} \frac{d^N y}{|x-y|^\alpha} \leq C_\alpha R^{N-\alpha} . \tag{8.1}$$

To prove this relation, consider two cases. First, suppose $|x| > 2R$. In this case, using the triangle inequality and that $|y| < R$, it is concluded that

$$|x-y| \geq |x| - |y| > R \quad \Rightarrow \quad \frac{1}{|x-y|^\alpha} < \frac{1}{R^\alpha}$$

Therefore

$$\int_{B_R} \frac{d^N y}{|x-y|^\alpha} \leq \frac{1}{R^\alpha} \int_{B_R} d^N y = \frac{\sigma_N}{R^\alpha} \int_0^R r^{N-1}\, dr = \frac{\sigma_N}{N} R^{N-\alpha}$$

where σ_N is the surface area of the unit sphere $|y| = 1$. Suppose that $|x| \leq 2R$. Using the new variables $z = y - x$,

$$\begin{aligned}\int_{B_R} \frac{d^N y}{|x-y|^\alpha} &= \int_{B_R(x)} \frac{d^N z}{|z|^\alpha} \leq \int_{B_{3R}} \frac{d^N z}{|z|^\alpha} \leq \sigma_N \int_0^{3R} r^{N-\alpha-1}\, dr \\ &= \frac{3^{N-\alpha}\sigma_N}{N-\alpha} R^{N-\alpha}\end{aligned}$$

here it was used that the ball of radius R centered at x is contained in the ball of radius $3R$ centered at the origin if $|x| \leq 2R$. The assertion follows if

$$C_\alpha = \max\left\{ \frac{\sigma_N}{N} ,\ \frac{3^{N-\alpha}\sigma_N}{N-\alpha} \right\}$$

8.3. Smoothness on the complement of support of the density. Here it is proved that *if x in the complement of $\overline{\Omega}$, that is, x is neither in Ω or in its boundary $\partial\Omega$, the potential integral has partial derivatives of any order,*

$$u(x) \in C^\infty\Big(\mathbb{R}^N \setminus \overline{\Omega}\Big) , \tag{8.2}$$

and

$$D_x^\beta u(x) = \int_\Omega \rho(y)\, D_x^\beta \frac{1}{|x-y|^\alpha}\, d^N y \tag{8.3}$$

The distance between two non-intersecting regions is not zero only if their boundaries do not have common points. Let $\Omega_\delta \subset \mathbb{R}^N \setminus \overline{\Omega}$ be such that

$$d(\Omega_\delta, \Omega) = \delta > 0$$

For example, if $\Omega = B_a$ is a ball of radius a, then Ω_δ is the complement of the ball of radius $a+\delta$, that is, $x \in \Omega_\delta$ if $|x| > a + \delta$.

Define a function of two variables

$$f(x,y) = \frac{\rho(y)}{|x-y|^\alpha}$$

Since $|x-y| \geq \delta > 0$ for any $y \in \overline{\Omega}$ and $x \in \Omega_\delta$, its partial derivatives of any order are continuous at any $x \in \Omega_\delta$ for any y:

$$\begin{aligned}
\frac{\partial}{\partial x_i} f(x,y) &= \alpha\rho(y)\, \frac{y_i - x_i}{|x-y|^{\alpha+2}} \\
\frac{\partial^2}{\partial x_j \partial x_i} f(x,y) &= \alpha\rho(y) \left(\frac{(\alpha+2)(x_i-y_i)(x_j-y_j)}{|x-y|^{\alpha+4}} - \frac{\delta_{ij}}{|x-y|^{\alpha+2}} \right)
\end{aligned}$$

and similarly for $D_x^\beta f$. Furthermore, they are bounded by Lebesgue integrable functions independent of x:

$$\begin{aligned}
\left| \frac{\partial}{\partial x_i} f(x,y) \right| &\leq \alpha|\rho(y)|\, \frac{1}{|x-y|^{\alpha+1}} \leq \frac{\alpha}{\delta^{\alpha+1}}\, |\rho(y)| \in \mathcal{L}(\Omega) \\
\left| \frac{\partial^2}{\partial x_j \partial x_i} f(x,y) \right| &\leq \alpha|\rho(y)| \frac{(\alpha+2)+\delta_{ij}}{|x-y|^{\alpha+2}} \leq \frac{\alpha(\alpha+3)}{\delta^{\alpha+2}}\, |\rho(y)| \in \mathcal{L}(\Omega)
\end{aligned}$$

where $x \in \Omega_\delta$ and $y \in \Omega$. Here the inequality $|x_j| \leq |x|$ was used. In general,

$$|D_x^\beta f(x,y)| \leq \frac{M_\beta}{\delta^{\alpha+\beta}}\, |\rho(y)| \in \mathcal{L}(\Omega)$$

for some constant M_β (that depends on α). By Theorem **5.2** u has continuous partial derivatives of any order in Ω_δ for any $\delta > 0$, and the conclusions (**8.2**) and (**8.3**) follow.

8.4. Continuity on support of the density. *If the density is bounded,*

$$|\rho(y)| \leq M\,, \quad \forall y \in \Omega$$

then the potential integral is a continuous function everywhere, $u \in C^0$.

Let x_0 and x be two points in $\overline{\Omega}$. One has to show that $u(x)$ can get arbitrary close to $u(x_0)$ and stay arbitrary close for all x that are close enough to x_0. Put

$$g(x,y) = \left| \frac{1}{|x_0 - y|^\alpha} - \frac{1}{|x - y|^\alpha} \right|$$

Then

$$\begin{aligned} |u(x_0) - u(x)| &\le \int_\Omega |\rho(y)|\, g(x,y)\, d^N y \le M \int_\Omega g(x,y)\, d^N y \\ &= M \left(\int_{\Omega \setminus B_R(x_0)} + \int_{B_R(x_0)} \right) g(x,y)\, d^N y \end{aligned}$$

Let us show that the integrals can be made arbitrary small for sufficiently small radius R such that $|x_0 - x| < R$. This would prove the assertion.

Using (**8.1**),

$$\begin{aligned} \int_{B_R(x_0)} g(x,y)\, d^N y &\le \int_{B_R(x_0)} \frac{d^N y}{|x_0 - y|^\alpha} + \int_{B_R(x_0)} \frac{d^N y}{|x_0 - y|^\alpha} \\ &\le 2C_\alpha R^{N-\alpha} \to 0 \end{aligned}$$

as $R \to 0$ because $N > \alpha$.

To show that the other integral is also small, note that the function $g(x,y)$ is a continuous function in the set

$$|x - x_0| \le \frac{R}{2}\,, \quad |y - x_0| \ge R\,,\ y \in \overline{\Omega}$$

This set is bounded and closed. By the extreme value theorem, g attains its extreme values in the set. In particular, since $g(x,y) \ge 0$, its absolute minimum is reached at $x = x_0$, $g(x_0, y) = 0$. Its maximum is reached at some point that depends on R, $x = x_R$ and $y = y_R$. The maximal value depends on R:

$$\max g = g(x_R, y_R) = C(R)$$

If $R \to 0$, then $x \to x_0$. Therefore by continuity of g, the maximal value $C(R)$ tends to 0 as $R \to 0$. Hence,

$$\begin{aligned} \int_{\Omega \setminus B_R(x_0)} g(x,y)\, d^N y &\le C(R)\mu\Big(\Omega \setminus B_R(x_0)\Big) \\ &\le C(R)\mu(\Omega) \to 0 \end{aligned}$$

as $R \to 0$. Note that the measure (volume) $\mu(\Omega) < \infty$ is finite because Ω is bounded. Therefore the integral can be made arbitrary small if $R \to 0$ for all x close enough to x_0: $|x - x_0| \le \frac{R}{2}$.

8.5. Differentiability on support of the density. *If the density is bounded, $|\rho(y)| \leq M$, then the potential integral has continuous partial derivatives up to order p everywhere with p being the largest integer such that $\alpha + p < N$, and, in this case,*

$$u \in C^p(\mathbb{R}^N)\,, \quad D_x^\beta u(x) = \int_\Omega \rho(y)\, D_x^\beta \frac{1}{|x-y|^\alpha}\, d^N y\,, \quad \beta \leq p\,.$$

Put

$$u_j(x) = \int_\Omega \rho(y)\, \frac{\partial}{\partial x_j} \frac{1}{|x-y|^\alpha}\, d^N y = \alpha \int_\Omega \rho(y) \frac{x_j - y_j}{|x-y|^{\alpha+2}}\, d^N y\,.$$

If u_j are proved to be continuous and $D_j u = u_j$, then the assertion is true if $p = 1$. The continuity of u_j is proved in the same way as the continuity of u. One has

$$\begin{aligned} |u_j(x_0) - u_j(x)| &\leq \alpha M \int_\Omega g_j(x,y)\, d^N y \\ &= \alpha M \left(\int_{\Omega \setminus B_a(x_0)} + \int_{B_a(x_0)} \right) g_j(x,y)\, d^N y\,, \end{aligned}$$

where

$$g_j(x,y) = \left| \frac{x_{0j} - y_j}{|x_0 - y|^{\alpha+2}} - \frac{x_j - y_j}{|x-y|^{\alpha+2}} \right|$$

The integral over the ball $B_a(x_0)$ can be made arbitrary small for all $|x_0 - x| < a$ with small enough a. This conclusion follows from (**8.1**) and the inequality

$$\left| \frac{x_j - y_j}{|x-y|^{\alpha+2}} \right| \leq \frac{1}{|x-y|^{\alpha+1}}$$

Note that (**8.1**) holds if $\alpha + 1 < N$ in this case. Using the continuity of $g_j(x,y)$ in the same way as the continuity of $g(x,y)$ when proving the continuity of u, one can show that the integral over the complement of the ball, $\Omega \setminus B_a(x_0)$, can also be made arbitrary small for all $|x_0 - x| < \frac{a}{2}$ and small enough $a > 0$.

A proof of the equality $D_j u = u_j$ is analogous to the proof of Theorem **5.2**. Since u_j is continuous, by the fundamental theorem of calculus

$$\frac{\partial}{\partial \xi_j} \int_{x_{0j}}^{\xi_j} u_j(x_1, ..., x_j, ..., x_N)\, dx_j = u_j(x_1, ..., \xi_j, ..., x_N)$$

On the other hand, using Fubini's theorem

$$\begin{aligned}\int_{x_{0j}}^{\xi_j} u_j(x)\,dx_j &= \int_{x_{0j}}^{\xi_j}\int_\Omega \rho(y)\,\frac{\partial}{\partial x_j}\,\frac{1}{|x-y|^\alpha}\,d^N y dx_j\\ &\overset{(1)}{=} \int_\Omega \rho(y)\int_{x_{0j}}^{\xi_j}\frac{\partial}{\partial x_j}\,\frac{1}{|x-y|^\alpha}\,dx_j\,d^N y\\ &= u(x_1,...,\xi_j,...,x_N)-u(x_1,...,x_{0j},...,x_N)\end{aligned}$$

Taking the partial derivative $\frac{\partial}{\partial \xi_j}$ of both sides of this relation, it is concluded that the partial derivatives of u coincide with u_j. Here (1) holds because the integrand in the iterated integral is Lebesgue integrable on $\Omega\times(x_{0j},\xi_j)$ and, by Fubini's theorem the order of integration can be changed. Indeed the iterated integral of the absolute value is finite:

$$\begin{aligned}\int_{x_{0j}}^{\xi_j}\int_\Omega\left|\rho(y)\,\frac{\partial}{\partial x_j}\,\frac{1}{|x-y|^\alpha}\right|\,d^N y dx_j &\le \alpha M\int_{x_{0j}}^{\xi_j}\int_\Omega \frac{d^N y}{|x-y|^{\alpha+1}}\,dx_j\\ &\le \alpha M\int_{x_{0j}}^{\xi_j}\int_{B_R}\frac{d^N y}{|x-y|^{\alpha+1}}\,dx_j\\ &\le \alpha M|\xi_j - x_{0j}|C_{\alpha+1}R^{N-\alpha-1}\end{aligned}$$

where the boundedness of Ω was used, $\Omega\subset B_R$ for large enough radius R, and the latter inequality follows from Proposition **8.1** if $\alpha+1<N$.

If $\alpha+2<N$, then the above arguments can be applied to the functions $u_j(x)$ (instead of u) to show that partial derivatives of u_j (or second partials of u) are continuous and the conclusion of the stated theorem holds. This iterative process holds as long as $\alpha+\beta<N$, $\beta=0,1,...,p$.

8.5.1. Smooth density with a bounded support. If the density ρ is from class $C^q(\mathbb{R}^N)$ and has a bounded support Ω with a piecewise smooth boundary $\partial\Omega$, then the potential integral defines a smoother function than in the case discussed above. A proof is based on the integration by parts in the integral

$$\begin{aligned}Du(x) &= D\int\frac{\rho(y)}{|x-y|^\alpha}\,d^N y = \int\rho(y)D_x\frac{1}{|x-y|^\alpha}\,d^N y\\ &= -\int_\Omega \rho(y)D_y\frac{1}{|x-y|^\alpha}\,d^N y\,.\end{aligned}$$

After integration by parts, the surface integral over $\partial\Omega$ vanishes because $\rho=0$ on $\partial\Omega$, and Du has the same integral representation that is studied Sec.**8.5** with a new bounded density $D\rho$ because $D\rho$ is continuous and has a bounded support. This implies that $Du \in C^p$ by Sec.**8.5**. The argument can be repeated q times, showing that $u \in C^{p+q}$.

However the derivatives of $|x-y|^{-\alpha}$ are not continuous if $x \in \Omega$ and, hence, the hypotheses for integration by parts are not fulfilled. To justify the integration by parts in this case, note that the integral exists (that is, it converges absolutely) and, hence, its value does not depend on a regularization near the singular point $y = x$. Let us regularize the integral by removing a ball $B_a(x)$ from Ω where $a \to 0^+$. Then ρ and $|x-y|^{-\alpha}$ are continuously differentiable in $\Omega \setminus B_a(x)$ for any $a > 0$ and the integration by parts is justified:

$$\begin{aligned} Du(x) &= -\lim_{a\to 0^+} \int_{\Omega\setminus B_a(x)} \rho(y) D_y \frac{1}{|x-y|^\alpha}\, d^N y \\ &= \lim_{a\to 0^+} \left(\oint_{\partial B_a(x)} \frac{\rho(y)(y-x)}{|x-y|^{\alpha+1}}\, dS_y + \int_{\Omega\setminus B_a(x)} \frac{D_y\rho(y)}{|x-y|^\alpha}\, d^N y \right), \end{aligned}$$

where the boundary of $\Omega \setminus B_a(x)$ consists of $\partial\Omega$ and the sphere $|y-x| = a$. The surface integral over $\partial\Omega$ vanishes as noted. The outward unit normal on the sphere is $n = (x-y)/a$. Let us show that the surface integral vanishes in the limit $a \to 0^+$. Since $|\rho(y)| \le M$, one has

$$\Big| \oint_{\partial B_a(x)} \frac{\rho(y)(y-x)}{|x-y|^{\alpha+1}}\, dS_y \Big| \le \frac{M}{a^\alpha} \int_{|z|=a} dS = \frac{M\sigma_N a^{N-1}}{a^\alpha} \to 0$$

because by assumption $\alpha + p < N$ for some integer $p \ge 1$. Therefore by continuity of the Lebesgue integral

$$Du(x) = \lim_{a\to 0^+} \int_{\Omega\setminus B_a(x)} \frac{D_y\rho(y)}{|x-y|^\alpha}\, d^N y = \int_\Omega \frac{D_y\rho(y)}{|x-y|^\alpha}\, d^N y\,,$$

as desired. The above equation is valid for all derivatives up order q, that is, one can replace D by D^β, $\beta \le q$. For the derivatives of order higher than q the integration by parts cannot be justified because $\rho \in C^q$, and they must be computed by the equation from Sec.**8.5**:

$$D^\beta D^q u(x) = \int_\Omega D_y^q \rho(y) D_x^\beta \frac{1}{|x-y|^\alpha}\, d^N y\,,$$

where $\beta \le p$. In particular, if the density ρ and all its partial derivatives of any order are continuous in the whole space and have bounded support, then u is from class C^∞.

The condition that ρ is from $C^q(\mathbb{R}^N)$ and has a bounded support is crucial for the conclusion. Suppose that ρ is defined on a bounded open Ω and $\rho \in C^q(\bar\Omega)$. Then it does not generally have a C^q extension to the whole $\mathbb{R}^N$ with support $\overline{\Omega}$ unless ρ and its partial derivatives up to order q vanish at the boundary $\partial\Omega$. In the latter case, the extension is defined by $\rho(x) = 0$ for all x in the complement of $\overline{\Omega}$ and the extended density is from $\rho \in C^q(\mathbb{R}^N)$. If $D^\beta \rho$, $\beta \le q$, do not vanish on the boundary $\partial\Omega$, the surface integral over $\partial\Omega$ in the integration by parts is not zero and the class of Du depends on the smoothness of this integral. This kind of surface integrals are called *surface potentials*:

$$v(x) = \int_{\partial\Omega} \frac{\rho(y)}{|x-y|^\alpha}\, dS_y\,.$$

If the density ρ is bounded on $\partial\Omega$, then $v \in C^p$ where p is the largest integer for which $\alpha + p < N-1$ (see Exercises), which is more restrictive than the condition in Sec.**8.5**. So, derivatives of Du may not exist at the boundary $\partial\Omega$ due to the lack of differentiability of the corresponding surface potentials.

8.6. Exercises.

1. Show that if ρ is a bounded function, then the function defined by the line integral in $\mathbb{R}^2$

$$v(x) = \int_S \frac{\rho(y)}{|x-y|^\alpha}\, ds_y$$

where S is a circle $|y| = a$, is continuous in $\mathbb{R}^2$ if $0 < \alpha < 1$, and u is from class C^∞ in $\mathbb{R}^2 \setminus S$ and, in this case,

$$D_x^\beta v(x) = \int_S D_x^\beta \frac{\rho(y)}{|x-y|^\alpha}\, ds_y$$

Hint: If $y_1 = a\cos(\theta)$ and $y_2 = a\sin(\theta)$, then $ds_y = ad\theta$ and $0 \le \theta \le 2\pi$ for S.

2. Extend the conclusion of Problem 1 to the case when S a curve from class C^1.

3. Surface potentials. Let S be an M surface in $\mathbb{R}^N$ from class C^1, $M \le N-1$. Define a function v by the surface integral (called a surface potential)

$$v(x) = \int_S \frac{\rho(y)}{|x-y|^\alpha}\, dS_y\,, \quad 0 < \alpha < M$$

(i) Show that $v \in C^\infty(\mathbb{R}^N \setminus S)$;
(ii) Show that $v \in C^p(\mathbb{R}^N)$ where p is the largest integer such that

$\alpha + p < M$, and

$$D_x^\beta v(x) = \int_S D_x^\beta \frac{\rho(y)}{|x-y|^\alpha}\, dS_y$$

where $\beta \geq 0$ in (i) and $0 \leq \beta \leq p$ in (ii).

4. Solution to the Poisson equation. Suppose that $\rho \in C^1(\mathbb{R}^3)$ and has a bounded support Ω. Suppose that the boundary $\partial\Omega$ is smooth (or piecewise smooth). Prove that

$$\Delta u(x) = -4\pi\rho(x)\,, \quad x \in \mathbb{R}^3\,, \quad u(x) = \int \frac{\rho(y)}{|x-y|}\, d^3y$$

by justifying each of the following assertions:

(i) $u \in C^1(\mathbb{R}^3)\,, \quad u \in C^\infty\Big(\mathbb{R}^3 \setminus \Omega\Big)\,,$

(ii) $\Delta_x \dfrac{1}{|x-y|} = 0\,, \quad \forall x \neq y$

(iii) $x \notin \Omega \quad \Rightarrow \quad \Delta u(x) = 0\,,$

(iv)
$$x \in \Omega \quad \Rightarrow \quad \Delta u(x) = -\left(\nabla, \int_\Omega \rho(y) \nabla_y \frac{1}{|x-y|}\, d^3y\right)$$
$$= -\int_\Omega \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y$$
$$= -\left(\int_{\Omega\setminus B_\varepsilon(x)} + \int_{B_\varepsilon(x)}\right) \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y$$

(v)
$$\lim_{\varepsilon\to 0} \int_{\Omega\setminus B_\varepsilon(x)} \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y = 4\pi\rho(x)\,, \quad x \in \Omega$$

(vi)
$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon(x)} \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y = 0$$

where ∇ denotes the gradient.

9. Functions defined by improper integrals

9.1. Conditionally convergent integrals. Let $\{\Omega_n\}$ be an exhaustion of Ω. Then the limit of integrals of a locally integrable function $f \in \mathcal{L}_{\rm loc}(\Omega)$ over Ω_n,

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x,$$

does not exist if f is not integrable on Ω, and even if it exists then its value depends on the choice of the exhaustion because the integral does not converge absolutely:

$$f \notin \mathcal{L}(\Omega) \quad \Rightarrow \quad \int_\Omega |f(x)|\, d^N x = \lim_{n\to\infty} \int_{\Omega_n} |f(x)|\, d^N x = \infty$$

where the first equality is by the continuity of the Lebesgue integral. Suppose there exists an exhaustion such that the limit of integrals of f over Ω_n exists. Then f is said to be *conditionally integrable* on Ω and the value of the limit is called a *conditional integral* of f over Ω. Note that the word "conditional" refers to that the limit is computed in a particular exhaustion (or regularization) of the integral, and its value depends on the choice of the exhaustion (or regularization).

9.2. Abel's theorem. Abel's theorem for conditionally convergent integrals is similar to Abel's theorem for conditionally convergent series. Hypotheses of the theorem are:

(i) $f(x) = \alpha(x)\beta(x)\,, \quad \forall x > a$

(ii) $\alpha(x) > 0\,, \quad \alpha(x) \to 0$ monotonically as $x \to \infty$

(iii) $\beta(x) \in C^0[a,\infty)$

(iv) $\left| \int_c^d \beta(x)\, dx \right| \le \sigma\,, \quad \forall c, d \ge a$

The latter condition means that integrals of β over any finite interval are bounded and the bound σ is independent of the interval. The conclusion of the theorem is that the limit

$$I = \int_a^\infty f(x)\, dx \stackrel{\rm def}{=} \lim_{R\to\infty} \int_a^R f(x)\, dx$$

exists, and for any $b > a$

$$\left| \int_b^\infty f(x)\, dx \right| \le \sigma\alpha(b)$$

The latter relation provides an estimate of the rate of convergence in the following sense:

$$\left| I - \int_a^R f(x)\, dx \right| \le \sigma\alpha(R) \to 0 \text{ as } R \to \infty$$

The hypothesis for the function β requires that the mean value of β over an interval is decreasing with increasing the length of the interval. This happens when β is bounded and oscillates about zero, like trigonometric functions. For example, the integrals

$$\left| \int_c^d e^{ikx}\, dx \right| = \left| \frac{e^{ikd} - e^{ikc}}{ik} \right| \le \frac{2}{k} = \sigma$$

are bounded and the bound is independent of the interval of integration. The monotonic decrease of α and boundedness of β does not guarantee integrability of $\beta\alpha$ on (a, ∞). But owing to monotonicity of α and oscillations of β, there are cancellations in the integral of the product $\alpha\beta$ over an ever increasing interval so that the integral conditionally converges. Abel's theorem offers sufficient conditions for conditional convergence of the integral.

A proof is given under a simplified assumption that α is continuously differentiable. In this case, an integration by parts can be used. Let

$$\alpha \in C^1$$

Since α is monotonically decreasing, $\alpha'(x) \le 0$. Put

$$\sigma_a(x) = \int_a^x \beta(y)\, dy$$

By continuity of β, the function σ_a is continuously differentiable, and $\sigma_a'(x) = \beta(x)$ by the fundamental theorem of calculus. Furthermore, σ_a is bounded by the hypothesis

$$|\sigma_a(x)| \le \sigma\,, \quad \forall x > a$$

Using the integration by parts

$$\int_a^R f(x)\, dx = \int_a^R \alpha(x)\, d\sigma_a(x) = \alpha(R)\sigma_a(R) - \int_a^R \sigma_a(x)\alpha'(x)\, dx$$

because $\sigma_a(a) = 0$. Since $\alpha(R) \to 0$ as $R \to \infty$, it is concluded that $|\alpha(R)\sigma_{a,R}| \le \sigma\alpha(R) \to 0$ and, therefore the integral of f converges if and only if the integral of $\sigma_a\alpha'$ converges. But $\alpha'(x) \le 0$, and, hence, the integral converges absolutely because

$$\begin{aligned}\int_a^R |\sigma_a(x)\alpha'(x)|\,dx \le \sigma \int_a^R |\alpha'(x)|\,dx &= -\sigma \int_a^R \alpha'(x)\,dx \\ &= \sigma\alpha(a) - \sigma\alpha(R) \\ &\le \sigma\alpha(a) < \infty\end{aligned}$$

Furthermore it follows that for any $b > a$

$$\left|\int_b^R |\sigma_a(x)\alpha'(x)\,dx\right| \le \int_b^R |\sigma_a(x)\alpha'(x)|\,dx \le \sigma\alpha(b)$$

for all R. By taking the limit $R \to \infty$, it is concluded that

$$\left|\int_b^\infty |\sigma_a(x)\alpha'(x)\,dx\right| \le \sigma\alpha(b)$$

as required.

9.3. Differentiability of Fourier transforms revisited. It was shown in Sec. **5.3.2** that the Fourier transform

$$F(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

is from class C^p if $x^p f(x)$ is integrable on $\mathbb{R}$. It turns out that even if $x^p f(x)$ is not integrable it is possible to show that F can be from class C^p at least in some interval without evaluating the integral explicitly, provided the Fourier integral of $x^p f(x)$ converges conditionally. The technique is based on combining Abel's theorem and Theorem **1.3**. The basic idea is illustrated with Example (**7.3**).

The function (**7.3**) is from class C^∞ on any interval that does not contain $k = 0$. However it is shown in Sec. **5.3.1** that the hypotheses of Theorem **5.2** are too restrictive to detect differentiability. Can the differentiability for all $k \neq 0$ be detected *without* evaluation of the integral?

Define a sequence

$$F_n(k) = \int_{-n}^{n} g(k,x)\,dt = \int_{-n}^{n} \frac{e^{ikx}}{1+x^2}\,dx$$

Since $g(k,x)$ is integrable on $\mathbb{R}$ for any x, the sequence $F_n(k)$ converges to $F(k)$ for any k. Let us show that $F_n(k)$ is continuously differentiable for any n. Indeed, although $|D_x g(k,x)|$ is not integrable on $\mathbb{R}$, it is integrable on any bounded interval $(-n,n)$. Therefore By Theorem

5.2, F_n is continuously differentiable and

$$F_n'(k) = \int_{-n}^{n} D_x g(k,x)\,dx = \int_{-n}^{n} \frac{ixe^{ikx}}{1+x^2}\,dx = -2\int_0^n \frac{x\sin(kx)}{1+x^2}\,dx$$

Next, one should show that the sequence of derivatives converges to some function $G(x)$ and then try to find an interval on which this convergence is uniform. Then by Theorem **1.3**, $F'(k)$ exists and $F'(k) = G(k)$ in this interval.

A pointwise convergence of $F_n'(k)$ can be investigated by means of Abel's theorem. The integral that defines $F_n'(k)$ contains the product of a function $\alpha(x) = x/(1+x^2)$, that is positive and monotonically decreasing to zero in the interval $(1,\infty)$, and the function $\beta(x) = \sin(kx)$ whose integrals over any bounded interval are bounded by a number independent of the interval:

$$\Big|\int_c^d \beta(x)\,dx\Big| = \Big|\int_c^d \sin(kx)\,dx\Big| = \Big|\frac{\cos(ck)-\cos(dk)}{k}\Big| \le \frac{2}{|k|} = \sigma\,.$$

provided $k \neq 0$. By Abel's theorem, the sequence of derivatives has a limit for any $k \neq 0$:

$$\lim_{n\to\infty} F_n'(k) = G(k)\,, \quad k \neq 0\,.$$

Let us estimate of the rate of convergence by means of the second part of Abel's theorem to show that F_n' converges to G uniformly on any set $|x| \ge \delta > 0$ and, hence, by Theorem **1.3** $F'(x) = G(x)$ in this set. Indeed, by Abel's theorem

$$|G(k) - F_n'(k)| \le 2\sigma\alpha(n) \le \frac{4}{\delta}\cdot\frac{n}{1+n^2}\,, \quad \forall |k| \ge \delta$$

Since the above inequality holds for any $|k| \ge \delta > 0$ any n, one can take the supremum in the left side and then the limit $n\to\infty$ in both sides. The limit in the right side vanishes so that

$$\lim_{n\to\infty} \sup_{|k|\ge\delta>0} |G(k) - F_n'(k)| = 0\,.$$

This means that F_n' converges to G uniformly on the set $|k| \ge \delta > 0$ and therefore $F'(k) = G(k)$. Since $\delta > 0$ is arbitrary,

$$F'(k) = \frac{d}{dk}\int_{\infty}^{\infty} \frac{e^{ikx}}{1+k^2}\,dk = \lim_{n\to\infty}\int_{-n}^{n} \frac{\partial}{\partial k}\frac{e^{ikx}}{1+x^2}\,dx\,, \qquad k \neq 0$$

This example shows that the lack of an integrable bound of partial derivatives with respect to a parameter that is independent of the parameter does not mean that the integral is not differentiable with respect to that parameter; it can still be differentiable on a smaller set and its derivatives can be given by improper integrals of the corresponding partial derivatives with respect to parameters:

PROPOSITION **9.2**. *Suppose that $f \in \mathcal{L}$ but $xf(x)$ is not integrable on $\mathbb{R}$. If, in addition, $xf(x)$ is monotonic for all $|x| \geq a > 0$ and $|xf(x)| \to 0$ as $|x| \to \infty$, then the Fourier transform of f is continuously differentiable for all non-zero values of the argument and*

$$F'(k) = \frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \lim_{n\to\infty}\int_{-n}^{n} ixf(x)e^{ikx}\,dx\,, \quad k \neq 0$$

A proof of this proposition is left to the reader as an exercise.

9.4. Exercises.

1. Prove Proposition **9.2**. Put

$$F_n(k) = \int_{-n}^{n} e^{ikx} f(x)\,dx\,, \qquad n = 1, 2, ...$$

(i) Show that F_n converges to the Fourier transform F of f;
(ii) Prove that $F_n \in C^1$ for all n, and

$$F_n'(k) = \int_{-n}^{n} ixf(x)\,e^{ikx}\,dx\,.$$

(iii) Use Abel's theorem to prove that the sequence $F_n'(k)$ converges to some $G(k)$ for any $k \neq 0$.
(iv) Show that there exists a constant C such that

$$|F_n'(k) - G(k)| \leq \frac{C}{|k|}\,n\Big(|f(n)| + |f(-n)|\Big)\,,$$

for all $k \neq 0$ and all $n > a$.
(v) Prove that $F(k)$ is continuously differentiable for all $k \neq 0$ and $F'(k) = G(k)$.

2. Consider the function defined by the Fourier integral

$$F(k) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{1+x^4}\,dx$$

(i) Show that $F \in C^2(\mathbb{R})$
(ii) Show that $F \in C^3(|k| \geq \delta)$ for any $\delta > 0$.
(iii) Use the residue theorem to find an explicit form of $F(k)$. Compute $F'''(k)$. Does it exist for all k?
(iv) Can $F'''(k)$ be obtained by interchanging the order of D_k^3 and integration with respect to x? If so, evaluate the integral after differentiation of the integrand with respect to k.

10. Space of square integrable functions $\mathcal{L}_2(\Omega)$

10.1. Metric spaces. The distance between two points x and y in $\mathbb{R}^N$ is defined by $d(x,y) = |x-y|$. The distance defines a numerical measure of that how two points are close to one another. Consider a collection of elements of any nature, denoted $\mathcal{X}$. Let us define a *distance* on $\mathcal{X}$ as a function of a pair elements that satisfies the *distance axioms*: The distance is a symmetric and non-negative function and vanishes if and only if the pair contains identical elements, and it obeys the triangle inequality:

$$\begin{aligned} &d(f,g) = d(g,f) \geq 0\,, \\ &d(f,g) = 0 \quad \Leftrightarrow \quad f = g\,, \\ &d(f,g) \leq d(f,h) + d(h,g) \end{aligned}$$

for any f, g, and h from $\mathcal{X}$. A set $\mathcal{X}$ with the distance function is called a *metric space* and the distance function is called a *metric* on $\mathcal{X}$.

A sequence $\{f_n\}$ is said to converge to f in $\mathcal{X}$ if $d(f_n, f) \to 0$ as $n \to \infty$

$$f_n \to f \quad \text{in } \mathcal{X}\ : \quad \lim_{n\to\infty} d(f_n, f) = 0$$

Similarly, one can define Cauchy sequences in $\mathcal{X}$. A sequence $\{f_n\}$ in a metric space is called a *Cauchy sequence* if for any $\varepsilon > 0$ one can find an integer m such that

$$d(f_n, f_k) < \varepsilon\,, \quad n, k > m$$

In other words, the distance $d(f_n, f_k)$ can be made arbitrary small for all sufficiently large n and k. It follows from the triangle inequality

$$d(f_n, f_k) \leq d(f_n, f) + d(f_k, f)$$

that every sequence that converges in $\mathcal{X}$ is a Cauchy sequence. But in contrast to $\mathbb{R}^N$, a Cauchy sequence in a general metric space may or may not have a limit element in $\mathcal{X}$. As an example, consider the set of all rational numbers. It is a metric space with the usual distance function. Take a sequence of rational numbers $\{q_n\}$ where q_n is an approximation of $\sqrt{2}$ with n decimal places, $q_1 = 1.4$, $q_2 = 1.41$, $q_3 = 1.414$, $q_4 = 1.4142$, etc. This sequence is a Cauchy sequence but it has no limit in the set of rational numbers.

A metric space is called *complete* if all Cauchy sequences have limits in it.

There are many ways to define a distance on the same set. Properties of a metric space depend on the metric even if the metric spaces contain the same elements. In particular, the completeness of a metric space depends on the metric.

10.1.1. Space of bounded functions as a metric space. Let $\mathcal{B}(\Omega)$ be a set of all bounded functions:

$$f \in \mathcal{B}(\Omega) : \quad \sup_{\Omega} |f(x)| < \infty$$

The number

$$\|f\|_\infty = \sup_{\Omega} |f(x)|$$

is called the *supremum norm* of a bounded function f. This set is a *linear space* because a linear combination of bounded functions is a bounded function. In other words, the set $\mathcal{B}(\Omega)$ is closed relative to addition of functions and multiplication them by a number. Define the distance on $\mathcal{B}(\Omega)$ by

$$d(f,g) = \sup_{\Omega} |f(x) - g(x)| = \|f - g\|_\infty$$

It satisfies the distance axioms. So, the norm of f can be interpreted as the distance of f from the zero function, just like the length of a vector in a Euclidean space.

Every Cauchy sequence of functions in $\mathcal{B}(\Omega)$ has a pointwise limit. Indeed, for any $x \in \Omega$, a numerical sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ because

$$|f_n(x) - f_j(x)| \leq \sup_{\Omega} |f_n(x) - f_j(x)| = d(f_n, f_j)\,.$$

Therefore by the Cauchy criterion for numerical sequences there exists a function f defined by the pointwise limit:

$$f(x) = \lim_{n\to\infty} f_n(x)\,, \quad x \in \Omega$$

Let us show that $f \in \mathcal{B}(\Omega)$, that is, f is bounded. Fix $\varepsilon > 0$ and find m such that $d(f_n, f_k) < \varepsilon$ for all $n, k > m$. Put

$$M = \max_{k=1,2,\ldots,m} \{\|f_k\|_\infty, \varepsilon\}$$

Then for any n and all x in Ω

$$\begin{aligned} |f_n(x)| &\leq |f_m(x)| + |f_n(x) - f_m(x)| \\ &\leq \sup_{\Omega} |f_m(x)| + \sup_{\Omega} |f_n(x) - f_m(x)| \leq 2M \end{aligned}$$

Therefore by taking the limit in the left side of the inequality, it is concluded that the limit function is bounded

$$|f(x)| \leq 2M\,, \quad x \in \Omega \quad \Rightarrow \quad f \in \mathcal{B}(\Omega)\,.$$

Thus, the space of bounded functions is complete relative to the supremum metric.

10.2. Metric spaces $C^0(\Omega)$ and $C_2^0(\Omega)$. Let Ω be a bounded closed region in $\mathbb{R}^N$. The space of continuous functions on Ω is a subset of $\mathcal{B}(\Omega)$. Therefore the distance defined by the supremum is a distance function on $C^0(\Omega)$. The space $C^0(\Omega)$ is complete because the limit of a uniformly convergent sequence of continuous functions on Ω is a continuous function on Ω by Theorem **1.2**.

There is another way to define a distance in the space of continuous functions. Any continuous function on a bounded closed region is square integrable. Put

$$\|f\|_2 = \left(\int_\Omega |f(x)|^2\, d^N x\right)^{1/2} < \infty$$

Let $C_2^0(\Omega)$ be the space of continuous functions in which the distance is defined by

$$d(f,g) = \|f-g\|_2$$

It also satisfies the distance axioms for all f and g from $C^0(\Omega)$. Indeed, it is non-negative and symmetric and vanishes if and only if two continuous functions f and g are equal. To see the latter, assume that $f(x_0) \neq g(x_0)$ at some x_0 in Ω while $d(f,g) = 0$. By continuity of $f-g$, there exists a ball $B_a(x_0)$ of some radius a where $|f(x) - g(x)| > 0$ so that integral cannot vanish as the ball has a non-zero measure, which contradicts to the condition $d(f,g) = 0$. The triangle inequality follows from the *Cauchy-Schwartz inequality*

$$\int_\Omega |f(x)g(x)|\, d^N x \leq \|f\|_2\|g\|_2 \tag{10.1}$$

which will be proved in the next section. Indeed, for any complex-valued functions

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

by squaring this inequality and integrating both sides, one infers from the Cauchy-Schwartz inequality that

$$\begin{aligned}\|f-g\|_2^2 &\leq \|f-h\|^2 + \|h-g\|^2 + 2\int_\Omega |f(x)-h(x)||h(x)-g(x)|\, d^N x \\ &\leq \Big(\|f-h\|_2 + \|h-g\|_2\Big)^2\end{aligned}$$

and the triangle inequality follows.

The convergence of a sequence in $C_2^0(\Omega)$ is called a *convergence in the mean. Uniform convergence implies convergence in the mean if* Ω *has a finite measure* because

$$\|f_n - f\|_2^2 \le \|f_n - f\|_\infty^2 \int_\Omega d^N x = \|f_n - f\|_\infty^2 \mu(\Omega)\,. \tag{10.2}$$

Similarly, if $\{f_n\}$ is a Cauchy sequence in $C^0(\Omega)$, it is a Cauchy sequence in $C_2^0(\Omega)$ if $\mu(\Omega) < \infty$. However, the converse is false, and there are Cauchy sequences in $C_2^0(\Omega)$ that do not have a limit in it, that is, the space of continuous square integrable functions is not complete.

It is not difficult to construct a Cauchy sequence in $C_2^0(\Omega)$ whose pointwise limit is a function that is not from $C_2^0(\Omega)$. Let $\Omega = [-1,1]$ and $f_n(x) = nx$ if $|x| < \frac{1}{n}$ and $f_n(x) = 1$ otherwise. Then for any $x \neq 0$, $f_n(x) \to 1$ as $n \to \infty$, and $f_n(0) = 0$. So the limit function is not continuous at $x = 0$ and, hence, does not belong to $C_2^0[-1,1]$. On the other hand,

$$\|f_n - f_k\|_2^2 = 2(n-k)^2 \int_0^{\frac{1}{n}} x^2 dx + 2\int_{\frac{1}{n}}^{\frac{1}{k}} (1-kx)^2 dx \to 0$$

for $n > k \to \infty$.

10.2.1. Proof of the Cauchy-Schwartz inequality. Consider a quadratic non-negative function of a real variable t defined by

$$h(t) = \||f| - t|g|\|_2^2 = A - 2Bt + Ct^2 \ge 0\,,$$
$$A = \|f\|_2^2\,, \quad C = \|g\|_2^2\,, \quad B = \int_\Omega |f(x)g(x)|\, d^N x$$

If $C = 0$, then the inequality holds. If $C \neq 0$, then $t(t)$ attains its absolute minimum at $t = t^* = B/C$. The inequality follows from $h(t^*) \ge 0$:

$$h(t^*) = A - \frac{B^2}{C} \ge 0 \quad \Rightarrow \quad B \le \sqrt{AC} = \|f\|_2\, \|g\|_2$$

10.3. Lebesgue square integrable functions. The space $\mathcal{L}_2(\Omega)$. Recall that all real numbers are constructed by *completion* of rational numbers by limits of all Cauchy sequences of rational numbers. The same question can be asked about completion of the space $C_2^0(\Omega)$. The extension to the set of all Riemann square integrable functions also does not produce a complete space because there are sequences of continuous functions converging to Riemann non-integrable functions. It turns out that an extension of the set of square integrable continuous functions by the set of all Lebesgue square integrable functions gives a complete space.

The space of all Lebesgue square integrable functions on a set Ω will be denoted by $\mathcal{L}_2(\Omega)$ or simply by $\mathcal{L}_2$ if $\Omega = \mathbb{R}^N$:

$$f \in \mathcal{L}_2(\Omega) \ : \quad \int_\Omega |f(x)|^2 \, d^N x < \infty \, .$$

and the number $\|f\|_2$ will be called the $\mathcal{L}_2$*−norm of* f.

10.3.1. $\mathcal{L}_2(\Omega)$ is a linear space. If f is square integrable, then its multiplication by a constant produces a square integrable function. Let f and g be square integrable. For any two complex square integrable functions f and g

$$\left| \int_\Omega \overline{f(x)} g(x) \, d^N x \right| \le \|f\|_2 \, \|g\|_2$$

which follows from (**10.1**), and, as a consequence,

$$\|f + g\|_2 \le \|f\|_2 + \|g\|_2 \, .$$

The latter inequality is known as the *Minkowski inequality.* It implies that the sum $f(x) + g(x)$ is square integrable because $\|f + g\|_2 < \infty$ and, hence, $\mathcal{L}_2(\Omega)$ is a linear space. To prove the Minkowski inequality, note that

$$\begin{aligned} \|f + g\|_2^2 &= \|f\|_2^2 + \|g\|_2^2 + 2\mathrm{Re} \int \overline{f(x)} g(x) \, d^N x \\ &\le \Big(\|f\|_2 + \|g\|_2 \Big)^2 \end{aligned}$$

by $\mathrm{Re}\, z \le |z|$ and the Cauchy-Schwartz inequality.

10.3.2. Relation between $\mathcal{L}(\Omega)$ and $\mathcal{L}_2(\Omega)$. Let us show that any square integrable function on Ω is integrable on Ω if the measure of Ω is finite:

$$f \in \mathcal{L}_2(\Omega) \, , \quad \mu(\Omega) < \infty \quad \Rightarrow \quad f \in \mathcal{L}(\Omega)$$

In the Cauchy-Schwartz inequality, let g be the characteristic function of Ω. This implies that

$$\int_\Omega |f(x)| \, d^N x \le \|1\|_2 \|f\|_2 = \sqrt{\mu(\Omega)} \|f\|_2 < \infty$$

Since $|f|$ is integrable so is f by Sec. **4.7**. The converse is not true. As an example, consider $f(x) = x^{-1/2}$ on $\Omega = (0, 1)$. Then $f \in \mathcal{L}(0, 1)$ but $f^2(x) = \frac{1}{x}$ is not integrable on $(0, 1)$. Thus,

$$\mathcal{L}_2(\Omega) \subset \mathcal{L}(\Omega) \, , \qquad \mu(\Omega) < \infty \, ,$$

If $\mu(\Omega) = \infty$, then $\mathcal{L}_2(\Omega)$ contains functions that are not integrable. For, example $f(x) = (1 + x^2)^{-1/2}$ is not integrable on $\mathbb{R}$, whereas it is square integrable on $\mathbb{R}$.

10.3.3. $\mathcal{L}_2(\Omega)$ as a metric space. Let us define the distance in $\mathcal{L}_2(\Omega)$ in the same way as in $C_2^0(\Omega)$. Then the distance satisfies all the distance axioms but the second one because the distance vanishes for any two functions that are equal almost everywhere:

$$d(f, g) = 0 \quad \Leftrightarrow \quad f(x) = g(x) \text{ a.e.}$$

To resolve this problem, let us split all Lebesgue integrable functions into *equivalence classes* where each class contains all functions that differ from one another on sets of measure zero. Then the space $\mathcal{L}_2(\Omega)$ is defined as a collection of all such equivalence classes. In other words, by saying that f is an element of $\mathcal{L}_2(\Omega)$, it is meant that f is a collection of all functions that differ from one another on a set of measure zero so that

$$f = g \text{ in } \mathcal{L}_2(\Omega) \quad \Leftrightarrow \quad f(x) = g(x) \text{ a.e.}$$

The distance between any two classes is defined as the $\mathcal{L}_2$−distance between any two representatives of these classes. The distance does not depend on the choice of the representatives as the Lebesgue integral cannot be change by alterations of the integrand on a set of measure zero. With this agreement, the second distance axiom is satisfied. In particular, the zero element in $\mathcal{L}_2(\Omega)$ is the collection of all functions that are zero almost everywhere:

$$f = 0 \quad \text{in } \mathcal{L}_2(\Omega) \quad \Leftrightarrow \quad f(x) = 0 \text{ a.e.}$$

10.3.4. Completeness of $\mathcal{L}_2(\Omega)$. It turns out that every Cauchy sequence in $\mathcal{L}_2(\Omega)$ has a limit in it so that $\mathcal{L}_2(\Omega)$ is a complete metric space.

THEOREM **10.1**. (Riesz-Fisher)[20]
Let $\{f_n\}$ be a sequence in the space of square integrable functions. Then in order that there exists an element f toward which the sequence converges in the mean, it is necessary and sufficient that $\|f_n - f_k\|_2 \to 0$ for $n, k \to \infty$.

The space of square integrable functions plays a fundamental role in quantum theory. According to one of the postulates of quantum mechanics, all possible states of any physical system are elements of some space $\mathcal{L}_2(\Omega)$, which is a *Hilbert space* of the system, and a time evolution of any physical system is governed by the Schroedinger equation in the

[20] F. Riesz and B. Sz.-Nagy, Functional analysis, Sec. 28

Hilbert space of the system. Hilbert spaces and the Schroedinger equation will be discussed later in detail. It will be shown that the space $\mathcal{L}_2(\Omega)$ admits an inner product, analogous to the dot (or scalar) product in Euclidean spaces, and there exist orthogonal functional bases in $\mathcal{L}_2(\Omega)$ such that any element of $\mathcal{L}_2(\Omega)$ can be expanded over them into a unique series (called a Fourier series). In this sense and owing to the completeness of $\mathcal{L}_2(\Omega)$, the space $\mathcal{L}_2(\Omega)$ is an infinite dimensional generalization of Euclidean (real or complex) spaces.

10.4. Dense subsets in a metric space. In practical calculations, it is sufficient to use only rational numbers because any irrational number can be approximated by a rational one with any desired accuracy. Naturally, it is interesting to investigate subsets in a metric space whose elements can approximate any element in a metric space with any desired accuracy.

A subset $\mathcal{A} \subset \mathcal{X}$ in a metric space is called *dense* is for any element f in $\mathcal{X}$ one can find an element g from $\mathcal{A}$ that is arbitrary close to f (the distance $d(f,g)$ can be made arbitrary small with a suitable choice of g). Putting this in a more formal way, *a subset $\mathcal{A}$ is dense in a metric space $\mathcal{X}$ if for any $f \in \mathcal{X}$, there exists a sequence $\{f_n\} \in \mathcal{A}$ such that*

$$\lim_{n\to\infty} d(f, f_n) = 0\,.$$

In other words, for any $f \in \mathcal{X}$ one can find an element from $\mathcal{A}$ that is arbitrary close to f (the distance $d(f,g)$ can be made arbitrary small with a suitable choice of g). For example, the set of rational numbers is dense in the space of reals. The set of all vectors in $\mathbb{R}^N$ with rational components is dense in $\mathbb{R}^N$.

If $\mathcal{A}$ is dense in $\mathcal{X}$, then any larger subset of $\mathcal{X}$ is dense in $\mathcal{X}$. If $\mathcal{A}$ is dense in $\mathcal{B}$ and $\mathcal{B}$ is dense in $\mathcal{X}$, then $\mathcal{A}$ is dense in $\mathcal{X}$. This follows from the triangle inequality. Fix $f \in \mathcal{X}$ and $\varepsilon > 0$. Since $\mathcal{B}$ is dense in $\mathcal{X}$, there exists $g \in \mathcal{B}$ that is arbitrary close to f, that is, $d(f,g) < \varepsilon$. Having found g, one can find $h \in \mathcal{A}$ that is arbitrary close to g, $d(h,g) < \varepsilon$. By the triangle inequality, h is arbitrary close to f:

$$d(h,f) \le d(h,g) + d(g,f) < 2\varepsilon\,.$$

as ε is arbitrary.

10.4.1. Polynomials in $C^0[a,b]$. Let $\mathcal{P}$ be a set of all polynomials. Then $\mathcal{P} \subset C^0[a,b]$. The following theorem shows that *$\mathcal{P}$ is dense in the space of continuous functions on any bounded closed interval*, and any continuous function can be approximated by a polynomial with any desired accuracy relative to the supremum metric.

THEOREM **10.2**. (Weierstrass)[21] *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n that converges to f uniformly on $[a, b]$:*

$$\lim_{n\to\infty} \|P_n - f\|_\infty = 0$$

If f is real, then P_n may be taken real.

10.4.2. Dense subsets in $\mathcal{L}_2(a, b)$. Let us show that *the space of polynomials $\mathcal{P}$ is dense in $\mathcal{L}_2(a, b)$ for any bounded interval.*

Let C^0_{pw} denote a set of piecewise continuous functions. Then

$$\mathcal{P} \subset C^0[a, b] \subset C^0_{pw} \subset \mathcal{L}_2(a, b)$$

If $\mathcal{P}$ is proved to be dense in $C^0[a, b]$, $C^0[a, b]$ in C^0_{pw}, and C^0_{pw} in $\mathcal{L}_2(a, b)$ relative to the $\mathcal{L}_2$ distance, then $\mathcal{P}$ is dense in $\mathcal{L}_2(a, b)$.

C^0_{pw} is dense in $\mathcal{L}_2(a, b)$. Let $f \in \mathcal{L}_2(a, b)$. Then $f_\pm(x) = \frac{1}{2}(|f(x) \pm f(x)) \geq 0$ are also square integrable on (a, b) and, hence, they are integrable on (a, b) by Sec.**10.3.2**. Therefore by Definition **3.3** there exist monotonically increasing sequences $h^\pm_n$ of piecewise continuous functions such that

$$\lim_{n\to\infty} h^\pm_n(x) = f_\pm(x) \quad \text{a.e.}$$

Let $m_\pm \leq h^\pm_1(x)$ for all x in $[a, b]$. Since $h^\pm_n(x)$ is increasing with increasing n and $f_\pm(x) \geq 0$,

$$\Big(f_\pm(x) - h^\pm_n(x)\Big)^2 \leq \Big(f_\pm(x) - m_\pm\Big)^2 \in \mathcal{L}(a, b)$$

where the inequality holds almost everywhere. By the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \|f_\pm - h^\pm_n\|_2^2 = \lim_{n\to\infty} \int_a^b \Big(f_\pm(x) - h^\pm_n(x)\Big)^2 dx = 0$$

By the triangle inequality, the sequence of piecewise continuous functions $h_n = h^+_n - h^-_n$ converges to f in $\mathcal{L}_2(a, b)$:

$$\|f - h_n\|_2 \leq \|f_+ - h^+_n\|_2 + \|f_- - h^-_n\|_2 \to 0$$

when $n \to \infty$.

$C^0[a, b]$ is dense in C^0_{pw} relative to the $\mathcal{L}_2$ metric. Suppose f has a jump discontinuity at $x = c \in (a, b)$ and is continuous otherwise in $[a, b]$. If $f(c+)$ and $f(c-)$ are the right and left limits of f at c, construct a continuous function $h_n(x)$ such that $h_n(x) = f(x)$ if $|x - c| < \frac{d_0}{n}$ where d_0 is the smallest number of $b - c$ and $c - a$. In the interval $[x^-_n, x^+_n]$,

[21] W. Rudin, Principles of mathematical analysis, Theorem 7.26

where $x_n^\pm = c \pm d_0/n$, $n = 1, 2, ...$, $h_n(x)$ is the linear function such that $h_n(x_n^\pm) = f(x_n^\pm)$. Then

$$\|f - h_n\|_2^2 = \int_{x_n^-}^{x_n^+} |f(x) - h_n(x)|^2 \, dx \le \frac{8M^2 d_0}{n} \to 0$$

when $n \to 0$ because $|h_n(x)| \le M$ where $M = \sup |f(x)| < \infty$. A general piecewise continuous function f has finitely many jump discontinuities in (a, b). A sequence of continuous functions converging to f in the mean is constructed in the same way by interpolating f by linear functions in small intervals containing the points where f is not continuous and letting the total length of these intervals tend to zero. $\mathcal{P}$ is dense in $C^0[a, b]$ relative to the $\mathcal{L}_2$ metric. The assertion follows from Eq. (**10.2**) and Weierstrass theorem.

10.4.3. Dense subsets in $\mathcal{L}_2(\mathbb{R})$. Let C_0^0 denote the space of all continuous functions on $\mathbb{R}$ (or $\mathbb{R}^N$) with a bounded support. For any function from C_0^0, there exists $R > 0$ such that $f(x) = 0$ if $|x| > R$.

PROPOSITION **10.1**. *The space C_0^0 is dense in $\mathcal{L}_2$*

Let f be square integrable. By continuity of the Lebesgue integral

$$\int |f(x)|^2 \, dx = \lim_{R \to \infty} \int_{|x|<R} |f(x)|^2 \, dx$$

This implies that for any ε there exists $R > 0$ such that

$$\|f - f_R\|_2 < \varepsilon \,, \quad f_R(x) = \chi_R(x) f(x) \,,$$

with χ_R being the characteristic function of $[-R, R]$. Since $C^0[-R, R]$ is dense in $\mathcal{L}_2(-R, R)$, there exists a continuous function h on $[-R, R]$ such that

$$\|f_R - \chi_R h\|_2 < \varepsilon$$

Let $g(x) = 0$ if $|x| > R + \delta$ for some $\delta > 0$ and $g(x) = h(x)$ if $|x| \le R$. On the intervals $[-R - \delta, -R]$ and $[R, R + \delta]$, $g(x)$ coincides with any continuous monotonic interpolation from $g(-R - \delta) = 0$ to $g(-R) = h(-R)$ and from $g(R) = h(R)$ to $g(R + \delta) = 0$, respectively. By construction, g is a continuous function with a bounded support. Then it follows from $|g(x)| \le M = \max\{|h(-R)|, |h(R)|\}$ if $R \le |x| \le R + \delta$ that

$$\|g - \chi_R h\|_2 = \left(\int_{R<|x|<R+\delta} |g(x)|^2 \, dx \right)^{1/2} \le M\sqrt{2\delta} < \varepsilon$$

if $\delta < \varepsilon^2/(2M^2)$. By the triangle inequality,

$$\|f - g\|_2 \le \|f - f_R\|_2 + \|f_R - \chi_R h\|_2 + \|\chi_R h - g\|_2 < 3\varepsilon$$

and since ε is arbitrary, it is concluded that for any $f \in \mathcal{L}_2$ one can find a continuous function with bounded support that is arbitrary close to f in the mean, that is, C_0^0 is dense in $\mathcal{L}_2$.

Other dense subsets of $\mathcal{L}_2$ will be discussed in detail in the next chapter.

10.5. Exercises.

1. Space $C^p(\Omega)$ as a metric space.

(i) For any function from class $C^p(a,b)$ whose derivatives are bounded, put

$$\|f\|_{C^p} = \sup_{\beta \le p, x \in (a,b)} |f^{(\beta)}(x)|$$

where a can be $-\infty$ and b can be ∞. Define the distance by

$$d(f,g) = \|f - g\|_{C^p} .$$

Show that all distance axioms are satisfied.

(ii) Show that the space of $p-$times continuously differentiable functions with bounded derivatives is complete.

Hint: Take a Cauchy sequence $\{f_n\}$. Show that the sequence converges uniformly to a continuous function f by noting that the convergence in C^p-metric implies convergence in the C^0-metric. Next show that the sequence of derivatives $\{f'_n\}$ converges uniformly to some g and $f' = g$. Repeat the argument to show that f is from the class C^p.

(iii) Show that the set of $p-$times continuously differentiable functions with bounded derivatives is not complete with respect to C^0-distance by giving an example of a Cauchy sequence of continuously differentiable functions that converges to a function that is not from C^1.

(iv) Let Ω be a region in $\mathbb{R}^N$. Put

$$d(f,g) = \|f - g\|_{C^p} = \sup_{\beta \le p,\ \Omega} |D^\beta f(x) - D^\beta f_a(x)| .$$

for any f and g from class $C^p(\Omega)$ whose partial derivatives are bounded. Show that the distance axioms are satisfied and the constructed metric space is complete.

2. Space $\mathcal{L}(\Omega)$ as a metric space.

(i) Consider the space $\mathcal{L}(\Omega)$ as a collection of equivalence classes where two functions belongs to the same class if the functions are equal almost everywhere. Use the properties of the Lebesgue integral to show that $\mathcal{L}(\Omega)$ is a linear metric space where the distance is defined by

$$d(f,g) = \int_\Omega |f(x) - g(x)|\, d^N x$$

by verifying the distance axioms.
(ii) Prove that the space of polynomials is dense in $\mathcal{L}(a, b)$ for any bounded interval (a, b).
(iii) Prove that the space of continuous functions with bounded support is dense in $\mathcal{L}(\mathbb{R})$.

CHAPTER 2

Distributions

13. Basic idea of distributions

A distribution is a generalization of the concept of a classical function. This generalization allows us to introduce a density of some quantity distributed over sets of zero measure (volume), like the mass or charge density of a point-like particle, or the electric charge density of dipoles distributed over a surface, or an intensity of instant and point-like source of waves in a mathematically correct way. On the other hand, the very notion of "instant" and "point-like" is a mathematical idealization because any physical process has a duration in time and is extended in space, and, hence, only mean values can be measured. A distribution describing a force applied to a particle that creates a finite momentum change of the particle during an arbitrary small interval of time can be viewed as the limit of the mean values of the force measured over successively smaller periods of time. Similarly, a density density of some physical quantity possessed by a point particle (like mass or electric charge) can be viewed as the limit of mean values of the density over successively smaller regions of space.

13.1. Dirac delta-function. Define a sequence of piecewise constant functions of a real variable:

$$f_n(x) = \begin{cases} \frac{n}{2}\,, & |x| \le \frac{1}{n} \\ 0\,, & |x| > \frac{1}{n} \end{cases}$$

Clearly,

$$\lim_{n\to\infty} f_n(x) = \delta(x) = \begin{cases} \infty\ , & x = 0 \\ 0\ \ , & x \neq 0 \end{cases}$$

Therefore, the limit function vanishes almost everywhere, $\delta(x) = 0$ a.e. However, the limit integral value is not zero:

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \lim_{n\to\infty} 1 = 1$$

for any interval $a < 0 < b$. There is no contradiction here. The observation merely means that the order of taking the limit and integration cannot be interchanged for this functional sequence. Evidently, the