Distributions and Operators for Physicists

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CHAPTER 1

Integration in Euclidean spaces

1. Euclidean spaces and functions on them

1.1. Euclidean spaces. Elements (vectors) of a real (or complex) Euclidean space $\mathbb{R}^N$ (or $\mathbb{C}^N$) are denoted by Roman letters, e.g., x, y, etc. They are ordered $N-$tuples of real (or complex) numbers, $x = (x_1, x_2, ..., x_N)$. Components of x are labeled by indices i or j. The inner (dot) product and the norm (length) are defined, respectively, by

$$(x, y) = \sum_{i=1}^{N} x_i \overline{y_i}\,, \quad |x| = \sqrt{(x, x)}$$

where $\bar{z}$ denotes the complex conjugation of z. Unless stated otherwise, Einstein's summation rule over repeated indices will be used throughout the text:

$$\sum_{j=1}^{N} \sum_{i=1}^{N} A_{ij} x_i y_j \stackrel{\text{def}}{=} A_{ij} x_i y_j$$

The norm satisfies the *triangle inequality*

$$|\,|x| - |y|\,| \leq |x - y| \leq |x| + |y|\,,$$

and the Schwartz inequality holds for the inner product

$$|(x, y)| \leq |x||y|\,, \qquad (1.1)$$

Two vectors are called *orthogonal* if their inner product vanishes.

Any N orthogonal vectors $\{\hat{n}_a\}_1^N$ form an orthogonal basis in $\mathbb{R}^N$ and, if $|\hat{n}_a| = 1$, the basis is called *orthonormal*. Any vector x has a unique expansion:

$$x = x_a \hat{n}_a\,, \quad x_a = \frac{(x, \hat{n}_a)}{|\hat{n}_a|^2}\,.$$

Orthogonal vectors $\hat{e}_a$, $a = 1, 2, ..., N$, with components $\hat{e}_{ai} = \delta_{ai}$ form the *standard basis* in $\mathbb{R}^N$. The standard basis is orthonormal, $(\hat{e}_a, \hat{e}_b) = \delta_{ab}$.

1.1.1. Sequences in $\mathbb{R}^N$. Indices n or m are used to label elements in a Euclidean space. In particular, a sequence of points is denoted by $\{x_n\}_1^\infty$ or simply $\{x_n\}$ (by default, the index enumerating elements of a point sequence ranges over all positive integers). A sequence $\{x_n\}$ is said *to converge to* x if

$$\lim_{n\to\infty} |x_n - x| = 0$$

and in this case, one also writes $x_n \to x$. It follows from the inequalities

$$|x_{nj} - x_j| \le |x_n - x| \le \sum_{i=1}^{N} |x_{ni} - x_i| \,,$$

where x_{nj} and x_j are the $j^{\rm th}$ components of the vectors x_n and x, respectively, that $x_n \to x$ in $\mathbb{R}^N$ *if and only if the sequences of components converge to the corresponding components of the limit point*, $x_{nj} \to x_j$ in $\mathbb{R}$ for every $j = 1, 2, ..., N$.

A sequence $\{x_n\}_1^\infty$ is called a *Cauchy sequence* if

$$|x_n - x_m| \to 0 \quad \text{for} \quad n, m \to \infty \,.$$

In other words, the distance $|x_n - x_m|$ can be made arbitrary small for all sufficiently large n and m. The *Cauchy criterion* states that *a sequence in a Euclidean space converges to some point if and only if it is a Cauchy sequence.* If $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}^N$, then the sequences of components $\{x_{ni}\}$ are Cauchy sequences in $\mathbb{R}$ and vice versa. The assertion follows from the Cauchy criterion for numerical sequences.

Every convergent sequence is bounded. Indeed, if $x_n \to x$, then $|x_n - x| < 1$ for all $n > m$ and some m. This implies that at most finitely many elements of a convergent sequence are outside of any ball centered at the limit point. Therefore the sequence lies in a ball centered at x and of radius $\max_{n\le m}\{|x_n - x|, 1\}$ and, hence, is bounded. This also implies that any Cauchy sequence is bounded.

The converse in not true. However, *every bounded sequence in a Euclidean space has a convergent subsequence (the Bolzano-Weierstrass theorem).*

Let us show first this is true for $\mathbb{R}$. Suppose $a \le x_n \le b$ for all elements of a numerical sequence $\{x_n\}$. If the sequence has finitely many distinct elements, then the assertion is true. So without loss of generality all elements are assumed to distinct. Put $y_1 = x_1$. Suppose there are infinitely many elements of the sequence $\{x_n\}$ that are greater than x_1. Then one can select a monotonically increasing subsequence $y_k < y_{k+1}$ by choosing y_2 to be the first element x_n, $n = 1, 2, ...,$ that is greater than x_1: $y_2 = x_{n_2} > x_1$. Then $y_3 = x_{n_3}$ is the first element that is greater than x_{n_2} and so on. Every monotonic sequence that is bounded, $y_k \leq b$, converges. If there only finitely many elements that are greater than x_1, then there should be infinitely many elements that are less than x_1. So, one can select a monotonically decreasing subsequence in the same fashion. This subsequence is bounded from below by a and, hence, converges. In $\mathbb{R}^N$, the above argument is applied to each component of a vector $x_n \in \mathbb{R}^N$. Since a sequence in $\mathbb{R}^N$ converges if and only if all sequences of the components converges, the conclusion of the theorem is also true in $\mathbb{R}^N$.

1.1.2. Basic sets in a Euclidean space. A collection of all points whose distance from x is less than $a > 0$,

$$B_a(x) = \{y \in \mathbb{R}^N \,|\, |x - y| < a\}\,,$$

is called an open ball of radius a centered at x. For brevity, $B_a(0) = B_a$. A set Ω is *bounded* if it lies in a ball of sufficiently large radius, $\Omega \subset B_a$.

A *neighborhood* of a point x is $B_a(x)$ for some $a > 0$. A point x in a set Ω is called an *interior point* if there exists a neighborhood of x that lies in Ω, $B_a(x) \subset \Omega$ for small enough a. A point x is called a *limit point* of Ω if any neighborhood of x has a point of Ω distinct from x. Clearly, a limit point of Ω may or may not be in Ω. For example, the limit points of an open interval (a, b) form the closed interval $[a, b]$. A set that contains all its limit points is called *closed*. The set obtained from Ω by adding all its limit points is called the *closure* of Ω and will be denoted by $\bar{\Omega}$. The reader is advised to show that the closure is closed. The closure $\bar{\Omega}$ is the smallest closed set that contains Ω.

An *open box* R in $\mathbb{R}^N$ is a collection of points whose coordinates span open bounded intervals, $a_j < x_j < b_j$, $j = 1, 2, ..., N$. For brevity,

$$R = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)\,.$$

A box is closed if all intervals are closed.

A collection of all interior points is called the *interior* of Ω and will be denoted by Ω^o. The interior of Ω is the largest open set that lies in Ω. By the Cauchy criterion, any Cauchy sequence in Ω converges to a limit point of Ω. So, the closure of Ω consists of limit points of all Cauchy sequences in Ω.

The set $\Omega^c = \mathbb{R}^N \backslash \Omega$ is called the *complement* of Ω. The complement of a closed set is open (a proof is left to the reader as an exercise).

1.1.3. The boundary of a set. The boundary of Ω is the difference between the closure and interior of Ω:

$$\partial\Omega = \bar{\Omega} \setminus \Omega^o$$

1.1.4. Heine-Borel theorem. Bounded and closed sets in a Euclidean space are called *compact*. They have a remarkable property. Let Ω be a set $\mathbb{R}^N$. Suppose that a collection of open sets $\{U_\alpha\}$ labeled by an index α is such that their union contains Ω:

$$\Omega \subset \bigcup_\alpha U_\alpha$$

The collection $\{U_\alpha\}$ is called an *open cover* of Ω. The nature of index α is not specified. For example, it can range over any set in a Euclidean space. If α can take infinitely many values, then the cover is called *infinite*, otherwise it is called *finite*.

The following statement, known as the *Heine-Borel theorem*, holds[1]. *Every infinite open cover of a compact set has a finite subcover.* A proof can be found in Appendix.

In particular, any set Ω is covered by the union of open balls of radius $a > 0$ centered at every point of Ω:

$$\Omega \subset \bigcup_{x\in\Omega} B_a(x)$$

If Ω is closed and bounded in $\mathbb{R}^N$, then, by the Heine-Borel theorem, one can find a finitely many points in Ω such that the union of open balls centered at these points contains Ω:

$$\bar{\Omega} = \Omega\,, \quad \Omega \subset B_R \quad \Rightarrow \quad \Omega \subset \bigcup_{j=1}^n B_a(x_j)$$

for any $a > 0$ and some $\{x_j\}_{j=1}^n$.

Intuitively, this assertion is obvious for any bounded set. For example, a bounded set Ω in $\mathbb{R}$ is contained in a bounded interval R. Let us take an open cover of R that consists of open intervals of length $2a$ labeled by the midpoints. Then it is also a cover of Ω. The length of R does not exceed $na/2$ for a large enough integer n. Then there are n points in R whose distance from its left and right neighbors does not exceed $a/2$. The intervals of length $2a$ centered at these points form a finite subcover of R and, hence, Ω. Of course, it can happen that one or more selected midpoints are not in Ω. But for any interval with such a midpoint can be replaced by a suitable interval centered at a

[1]see, e.g., W. Rudin, Principles of mathematical analysis, Chapter 2

point of Ω within the distance $a/2$ from the selected midpoint. This argument is readily extended to a bounded set in $\mathbb{R}^N$.

Why is compactness required in the Heine-Borel theorem? The reason can be illustrated by the following example. Suppose $\Omega = (0, 1] \subset \mathbb{R}$. Take a sequence $\{x_n\} \subset (0, 1]$ that converges to 0 strictly monotonically, e.g., $x_n = \frac{1}{n}$. Let $x_1 = 1$. Consider a cover of Ω by open intervals $(\frac{x_n}{2}, \frac{3x_n}{2})$ (balls of radius $\frac{1}{2}x_n$ centered at x_n). Clearly, it is not possible to select a finite subcover because for any interval in the cover there are points of Ω close enough to $x = 0$ that are not this interval. Now if $\Omega = [0, 1]$, then the constructed set of intervals is not a cover of Ω as the point $x = 0$ does not belong to the union of the intervals. To obtain a cover, one should add at least one open interval that contains $x = 0$. But this interval must also contain a small neighborhood of $x = 0$ of some radius a and, hence, this interval and finitely many intervals centered at $x_n > \frac{a}{2}$ form a finite subcover of a compact interval $[0, 1]$. A similar example is not difficult to construct in $\mathbb{R}^N$. The idea is that if Ω has a limit point that does not belong to Ω, then one can construct an open cover that has no finite subcover (e.g., using open spherical shells centered at the limit point and whose inner and outer radii are monotonically decreasing to zero).

1.1.5. A dense subset in a set. A subset Ω' of a set Ω is called *dense* if for any point $x \in \Omega$ one can find a point $y \in \Omega'$ that is arbitrary close to x. In other words, any ball centered at x contains points of Ω'. For example, rational numbers are dense in $\mathbb{R}$. Points in $\mathbb{R}^N$ with rational coordinates are dense in $\mathbb{R}^N$.

1.1.6. Parametric curves. A *vector function* is a vector-valued function on an interval, $x_i = x_i(t)$, $a \le t \le b$, $i = 1, 2, ..., N$, or, for brevity, $x = x(t)$. A vector function is continuous if every component of $x(t)$ is continuous. A continuous vector function is also called a *parametric curve* in $\mathbb{R}^N$ (think of a trajectory of a point-like particle if t is a physical time). A parametric curve is *closed* if $x(a) = x(b)$. A parametric curve is *simple* if $x(t_1) = x(t_2)$ implies $t_1 = t_2$, unless the curve is closed and $t_1 = a$ and $t_2 = b$. In other words, a simple curve has no self-intersections.

1.1.7. A region in $\mathbb{R}^N$. A set Ω is *connected* if any two points in it can be connected by a parametric curve that lies in Ω. A connected open set will be called a *region* and the closure of a region will be called a *closed region*.

1.1.8. A neighborhood of a set. The union of open balls centered at all points of a set Ω is called a *neighborhood* of Ω. By construction, a neighborhood of Ω is open. In what follows, if all balls have the same radius a, then the corresponding neighborhood is said to have radius a. For example, a neighborhood of a closed ball $|x| \le R$ is an open ball $|x| < R + a$.

1.1.9. Distance between sets. A distance between sets A and B is defined by

$$d(A, B) = \inf_{x\in A,\, y\in B} |x - y| \,.$$

which is the greatest lower bound (*infimum*) for lengths of all interval with endpoints in A and B. The function $|x - y|$ may not reach its minimum at some $x \in A$ and some $y \in B$, but the infimum always exists for any set of non-negative reals.

It is important to note that the distance can vanish even if the sets do not intersect. Let Ω be a region and let x be not in Ω. Suppose that the distance between x and Ω vanishes

$$d(\Omega, x) = \inf_{y\in\Omega} |y - x| = 0\,.$$

Then x belongs to the boundary of Ω. Indeed, take a point y_1 in Ω and put $a = |y_1 - x|$. Then one can find a point y_2 in Ω such that $|y_2 - x| \le \frac{a}{2}$. By repeating this procedure a sequence of points y_n in Ω can be obtained such that $|y_n - x| \le 2^{-n}a$. This means that x is a limit point of Ω. Since Ω is open, x must be in the boundary $\partial\Omega = \bar{\Omega} \setminus \Omega$. Furthermore, $d(\Omega, x) > 0$ if and only if x does not belong to the closure $\bar{\Omega}$, or, in other words, x lies in the complement of $\bar{\Omega}$.

The distance $d(\Omega, x)$ is a continuous function of x. Indeed, let $x_n \to x$. For any y and z in Ω,

$$\begin{aligned} |y - x_n| - |z - x| &\le |y - x| - |z - x| + |x - x_n|\,, \\ |y - x_n| - |z - x| &\ge |y - x_n| - |z - x_n| - |x - x_n|\,, \end{aligned}$$

by the triangle inequality. By taking the infimum over y and z,

$$|d(\Omega, x) - d(\Omega, x_n)| \le |x - x_n|\,.$$

Therefore $d(\Omega, x_n) \to d(\Omega, x)$ for $x_n \to x$, which means that $d(\Omega, x)$ is continuous at any x (see also Sec. **1.2**). By the *extreme value theorem*, if A is a compact set, then there exists a point x_* in A such that

$$d(\Omega, A) = d(\Omega, x_*)\,.$$

1.1.10. A proper subset. A bounded set Ω' is said to be *a proper subset* of a set Ω if its closure lies in the interior of Ω, $\overline{\Omega'} \subset \Omega^o$. A proper subset has a characteristic property that the distance between it and the boundary $\partial\Omega$ does not vanish:

$$d(\Omega', \partial\Omega) > 0\,.$$

The boundary $\partial\Omega$ is a closed set (it can be viewed as the intersection of two closed sets, $\bar{\Omega}$ and the complement of Ω^o, which is closed because Ω^o is open). If Ω is bounded, then its boundary is also bounded and, in this case, there exists $x_* \in \partial\Omega$ such that

$$d(\Omega', \partial\Omega) = d(\Omega', x_*) > 0$$

because x_* is not in Ω^o and, hence, cannot be in $\overline{\Omega'} \subset \Omega^o$. If Ω is not bounded, then its boundary can be unbounded too. In this case, consider the part of the boundary $\partial\Omega$ that lies in the closed ball of radius R, $\partial\Omega_R = \partial\Omega \cap \overline{B_R}$. Then $\partial\Omega_R$ is closed and bounded for any $R > 0$. Since Ω' is bounded, one can take R large enough so that B_R contains Ω' and

$$d(\Omega', \partial\Omega) = d(\Omega', \partial\Omega_R) = d(\Omega', x_*) > 0\,, \quad x_* \in \partial\Omega_R\,.$$

It also follows that *for any proper subset Ω' of a region Ω there exists a neighborhood of Ω' of radius $\delta > 0$ that is also a proper subset of Ω.* Indeed, by the above reasoning, one can take $\delta = \frac{1}{2}d(\Omega', \partial\Omega) > 0$.

1.2. Functions on a Euclidean space. A function $f:\ \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ is a rule that assigns a unique number $f(x)$ to every point $x \in \Omega$. The sets Ω and $f(\Omega) \subset \mathbb{R}$ are called the domain and the range of f. If $f(x)$ is a complex number (the range lies in the complex plane), then f is called complex-valued. Let y be a limit point of Ω. A function f is said to have a limit A at y if for any sequence $\{x_n\} \subset \Omega$, the sequence $\{f(x_n)\}$ converges to the number A. In this case, one writes

$$\lim_{x\to y} f(x) = A \quad \text{or} \quad f(x) \to A \text{ as } x \to y\,.$$

1.2.1. The characteristic function of a set. For any set Ω, the function defined by

$$\chi_\Omega(x) = \begin{cases} 1\,, & x \in \Omega \\ 0\,, & x \notin \Omega \end{cases}$$

is called the *characteristic function* of Ω.

1.2.2. The classes $C^p(\Omega)$ and $C^p(\bar{\Omega})$. Let Ω be open. A function f is *continuous* at a point $x \in \Omega$ if for any sequence $\{x_n\}$ in Ω that converges to x, the image sequence $\{f(x_n)\}$ converges to $f(x)$, and f is said to be *continuous on* Ω if it is continuous at every point of Ω. The class of all functions that are continuous on Ω will be denoted by $C^0(\Omega)$. The class of functions whose partial derivatives up to order p are continuous on Ω will be denoted by $C^p(\Omega)$.

Let $y \in \partial\Omega$. For an open Ω, f is not defined at any point of the boundary. Suppose that for any sequence $\{x_n\}$ in Ω that converges to a boundary point y, the sequence $\{f(x_n)\}$ has a limit. In this case, f is said to have a *continuous extension to a boundary point* y by the rule

$$f(y) = \lim_{x \to y} f(x)$$

The class of continuous functions on an open set Ω that have a continuous extension to every point of the boundary of Ω will be denoted by $C^0(\bar{\Omega})$. Similarly, the class of functions whose partial derivatives are continuous up to order p on an open set Ω and have continuous extensions to every point of the boundary of Ω will be denoted by $C^p(\bar{\Omega})$. If $\Omega = \mathbb{R}^N$ or an explicit form of Ω is irrelevant, it will be said that f is from class C^p. The class $C^\infty(\Omega)$ consists of functions whose partial derivatives of any order are continuous on Ω. If all partial derivatives of any order of a function f have continuous extensions to the boundary, then f is from class $C^\infty(\bar{\Omega})$.

In what follows, functions from classes C^1 and C^∞ will be referred to as *continuously differentiable* and *smooth* functions, respectively.

1.2.3. Uniform continuity. A function f is said to be *uniformly continuous* on a set Ω if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) >$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta$$

for all x and y in Ω. In other words, values of the function differ from each other by no more than ε whenever the arguments lie in a ball of radius δ. Clearly, every uniformly continuous function on Ω is continuous on Ω. The converse is not true. The key difference between uniform continuity and continuity is that in the latter δ depends on ε and a point at which the function is continuous, $\delta = \delta(\varepsilon, y)$ if $f(x) \to f(y)$ as $x \to y$. It is always possible to find the same δ for all points in Ω. A good candidate for such a uniform δ would be $\inf_\Omega \delta(\varepsilon, y)$ if f is continuous at every $y \in \Omega$. But the infimum can be zero. The function $f(x) = \frac{1}{x}$ is continuous on $\Omega = (0, 1)$. Fix $\varepsilon > 0$. Then $|f(x) - f(y)| = |x - y|/(xy) < \varepsilon$ always fails for any $|x - y| < \delta$ and any δ if x and y are close enough to zero. This shows that the uniform continuity depends very much on the set rather than on a point.

The following assertion provides sufficient conditions for uniform continuity[2]. *Let f be continuous on Ω. If Ω is compact, then f is uniformly continuous on Ω.*

1.2.4. Support of a function. The closure of the set on which a function f does not vanish is called the *support* of f and denoted supp f:

$$\operatorname{supp} f = \overline{A}\,, \quad A = \{\, x \mid f(x) \neq 0\}\,.$$

For example, the support of $f(x) = \sin(x)$ is $\mathbb{R}$. The support of the characteristic function of a set Ω is the closure $\bar{\Omega}$.

1.2.5. Extreme properties of a function. If the range of a function is bounded, then the function is called *bounded* (on its domain). For a function bounded on Ω

$$-\infty < \inf_{\Omega} f \leq f(x) \leq \sup_{\Omega} f < \infty\,, \quad x \in \Omega$$

However, these bounds are not generally reached by f at some points in Ω. In other words, the inequalities are strict for a generic bounded function. Sufficient conditions for a function to attain its extreme values are stated in the *extreme value theorem*. It states that *a continuous function always attains its extreme values on a compact set*. In other words, there exist $x_{\pm} \in \Omega$ such that

$$\inf_{\Omega} f = \min_{\Omega} f = f(x_-)\,, \quad \sup_{\Omega} f = \max_{\Omega} f = f(x_+)\,.$$

if f is continuous and Ω is bounded and closed in $\mathbb{R}^N$.

1.3. Notations for partial derivatives. In what follows, for brevity partial derivatives will written as

$$\frac{\partial g(x)}{\partial x_j} = D_j g = \partial_j g\,.$$

Any partial derivative of a function g of order α will be denoted by

$$D^{\alpha} g \stackrel{\text{def}}{=} D_N^{\alpha_N} \cdots D_1^{\alpha_1} g = \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}\,, \quad |\alpha| = \alpha_1 + \cdots + \alpha_N\,.$$

A monomial of order α in components of $x \in \mathbb{R}^N$ will be denoted by

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_N^{\alpha_N}\,.$$

[2]W. Rudin, Principles of mathematical analysis, Chapter 4

The Taylor polynomial of order n for g about $x = 0$ will be written as

$$p_n(x) = \sum_{|\alpha|=0}^{n} \sum_{\alpha_1+\cdots+\alpha_N=\alpha} \frac{x_1^{\alpha_1}\cdots x_N^{\alpha_N}}{\alpha_1!\cdots\alpha_N!}\Big(\frac{\partial^{|\alpha|}g}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}\Big)_{x=0}$$

$$\stackrel{\text{def}}{=} \sum_{\alpha=0}^{n} \frac{D^\alpha g(0)}{\alpha!}\, x^\alpha\,, \quad \alpha! = \alpha_1!\cdots\alpha_N!$$

Similarly, for the binomial expansion of any partial derivative of order α of the product of two functions

$$D^\alpha(fg) = D_N^{\alpha_N}\cdots D_1^{\alpha_1}(fg) = D_N^{\alpha_N}\cdots D_2^{\alpha_2}\sum_{\beta_1=0}^{\alpha_1}\binom{\alpha_1}{\beta_1}D_1^{\alpha_1-\beta_1}gD_1^{\beta_1}f$$

$$= \sum_{\beta_N=0}^{\alpha_N}\cdots\sum_{\beta_1=0}^{\alpha_1}\binom{\alpha_N}{\beta_N}\cdots\binom{\alpha_1}{\beta_1}D_N^{\alpha_N-\beta_N}gD_N^{\beta_N}f\cdots D_1^{\alpha_1-\beta_1}gD_1^{\beta_1}f$$

$$\stackrel{\text{def}}{=} \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}fD^\beta g\,, \qquad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}\,.$$

Let $g(x, y)$ be a function of two variables $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. Any partial derivative of g of order α with respect to x will be denoted by $D_x^\alpha g(x, y)$. Mixed partials are

$$D_x^\alpha D_y^\beta g(x,y) = D_x^\alpha(D_y^\beta g(x,y)) = D_y^\beta(D_x^\alpha g(x,y))$$

1.4. Smooth boundary of a region in $\mathbb{R}^N$. The boundary $\partial\Omega$ of a region Ω is called *smooth* if in a neighborhood of any point, $\partial\Omega$ is a level set of a function g from class C^1 whose gradient ∇g does not vanish:

$$\partial\Omega = \{x \in \mathbb{R}^N \mid g(x) = 0\,,\ \nabla g(x) \neq 0\}\,.$$

Recall that ∇g is a vector whose components are first partials of g. The boundary $\partial\Omega$ is said to be from class C^p if, in addition, $g \in C^p$, $p \geq 2$.

Let $x_j = x_j(t)$ be a parametric curve that lies in the level set $g(x) = 0$. Then the vector $v(t) = x'(t)$ is tangent to the curve. Since the equation $g(x(t)) = 0$ holds for all t, its differentiation shows that the gradient of g is orthogonal to v at any point of the curve:

$$0 = \frac{d}{dt}g(x(t)) = \Big(\nabla g(x(t)), v(t)\Big)\,.$$

Tangent vectors to all curves through any point x on the level set form a tangent plane to the level set through the point x, and, hence, the gradient $\nabla g(x)$ is orthogonal to all vectors in this plane. Thus, *the gradient* ∇g *is normal to the boundary* $\partial\Omega$. In particular, a *unit normal* to the boundary $\partial\Omega$ can be defined by

$$\hat{n}(x) = \frac{\nabla g(x)}{|\nabla g(x)|}, \quad x \in \partial\Omega$$

The vector $-\hat{n}(x)$ is also a unit normal to $\partial\Omega$. If f is from class $C^1(\bar{\Omega})$, then

$$\frac{\partial f}{\partial n} \stackrel{\text{def}}{=} \Big(\hat{n}(x), \nabla f(x)\Big), \quad x \in \partial\Omega,$$

is called a *normal derivative* of f on the boundary of Ω.

1.4.1. Example: Smooth boundary of a region in $\mathbb{R}^3$. Let g be from the class $C^1(\mathbb{R}^3)$ and $\nabla g \neq 0$. Without loss of generality $\partial_3 g \neq 0$ at some point $x = y$ of the level set. Then by the *implicit function theorem* the equation $g(x_1, x_2, x_3) = 0$ can be solved in a neighborhood of y with respect to x_3, that is, there exists a function $f(x_1, x_2)$ such that $g(x_1, x_2, f(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of (y_1, y_2). Moreover, the function f is from class C^1 and

$$\partial_1 f = -\frac{\partial_1 g}{\partial_3 g}\Big|_{x_3=f}, \quad \partial_2 f = -\frac{\partial_2 g}{\partial_3 g}\Big|_{x_3=f}.$$

The latter equations are known are *implicit differentiation* equations. So, *a smooth boundary of a region in* $\mathbb{R}^3$ *locally looks like a graph of a* C^1 *function of two variables*, which is a two-dimensional *surface* in space.

For example, let $\Omega = B_a$. Then its boundary is a sphere which is a level set $g(x) = |x|^2 = a^2$. The gradient $\nabla g = 2x$ is continuous and does not vanish on the sphere. It is also normal to the sphere. The derivative $\partial_3 g = 2x_3$ does not vanish if $x_3 > 0$ or $x_3 < 0$. So, in a neighborhood of any point in the upper hemisphere the sphere is a graph $x_3 = \sqrt{a^2 - x_1^2 - x_2^2}$ while it is the graph $x_3 = -\sqrt{a^2 - x_1^2 - x_2^2}$ near any point in the lower hemisphere. Near any point at which $\partial_3 g = 0$, the equation cannot be solved for x_3 and should be solved either with respect to x_1 (if $\partial_1 g \neq 0$) or x_2 (if $\partial_2 g \neq 0$).

This picture has a natural generalization to higher dimensional spaces. A smooth boundary of a region in $\mathbb{R}^N$ is locally a graph of a C^1 function of $N-1$ variables obtained by solving the equation $g(x) = 0$ with respect one of the variables. It defines an $N-1$ dimensional surface in $\mathbb{R}^N$.

1.5. Sequences and series of functions. A sequence of functions $\{f_n\}$ is said to converge *pointwise* to a function f on a set Ω if for all $x \in \Omega$

$$\lim_{n\to\infty} f_n(x) = f(x)\,.$$

In general, the limit function does not inherit properties of terms of the sequence. For example, f may not be continuous on Ω even if the terms of the sequence are continuous on Ω. It is easy to construct an example. Let $f_n(x) = 0$ if $x < 0$, $f_n(x) = nx$ if $0 \le x \le \frac{1}{n}$, and $f_n(x) = 1$ if $x > \frac{1}{n}$. The limit function is the step function $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \ge 0$. It is not continuous at $x = 0$.

Suppose a sequence of continuously differentiable functions converges pointwise to a function f on Ω. Two essential questions arise:

(i) *Is the limit function continuously differentiable?*
(ii) *If so, can the derivative of the limit function be obtained as the limit of the sequence of the derivatives of terms?*

The answer is negative to both questions. As an example, put

$$f_n(x) = \begin{cases} 0 & , \quad x < 0 \\ (nx)^2[1-(1-nx)^2] & , \quad 0 \le x \le \frac{1}{n} \\ 1 & , \quad x > \frac{1}{n} \end{cases}$$

Then $f_n \in C^1(\mathbb{R})$. The sequence f_n converges to the step function which is not differentiable at $x = 0$, whereas $f_n'(0) = 0$. A sequence $g_n(x) = \sin(nx)/n$ converges to $g(x) = 0$ for all $x \in \mathbb{R}$. The terms and the limit function are from class C^∞. However $g_n'(x)$ has no limit for all x, and $g_n'(2\pi m) = 1$ for any integer m, but $g'(x) = 0$ for all x.

A lesson here is that stronger conditions than a mere pointwise convergence are required in order for the limit function to inherit the smoothness properties of the terms as well as for rearranging the order of taking the limit and partial derivatives. The said conditions are based on the concept of *uniform convergence*.

1.5.1. Uniform convergence. A sequence of functions $\{f_n\}$ is said to *converge uniformly to a function f on a set* Ω if

$$\lim_{n\to\infty} \sup_{\Omega} |f_n(x) - f(x)| = 0\,.$$

Clearly, every uniformly convergent sequence converges pointwise. The converse is false. For example, if $f_n(x)$ is the sequence converging to the step function as defined in Sec. **1.5**. Then

$$\sup |f_n(x) - f(x)| = 1\,.$$

for all n and, hence, f_n does not converge to f uniformly on $\mathbb{R}$.

Similarly to uniform continuity, the uniform convergence depends on the set. For example, f_n converges to the step function f uniformly on $\Omega = (-\infty, -a) \cup (a, \infty)$ for any $a > 0$. If one thinks about terms of a pointwise convergent sequence as an approximation to the limit function, then $|f_n(x) - f(x)|$ is an absolute error of the approximation at a point x. The pointwise convergence means that the error can be made smaller than any positive number ε for all large enough $n > m$ where m naturally depends on ε and the point x. The uniform convergence means that the integer m is independent of x so that the error of the approximation is uniformly bounded by ε for all points in Ω. A natural candidate for such m is the largest integer for all x, that is, $\sup_\Omega m(\varepsilon, x)$. If the sequence converges pointwise but not uniformly on Ω, then the supremum is either infinite or cannot be made small by decreasing ε. So, the approximation $f(x) \approx f_n(x)$ is not good everywhere in Ω.

1.5.2. Cauchy criterion for uniform convergence. A verification of uniform convergence by the definition requires an explicit form of the limit function. The limit function can be hard to calculate or only some properties of terms of the sequence are known but not their explicit form. In this case, the *Cauchy criterion* is essential. It states that *a sequence of functions $\{f_n\}$ converges uniformly on a set Ω if and only if*

$$\sup_\Omega |f_n(x) - f_m(x)| \to 0$$

as $n, m \to \infty$.

If $\{f_n\}$ converges to a function f uniformly on Ω, then

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |f(x) - f_n(x)| + |f(x) - f_m(x)| \\ &\le \sup_\Omega |f(x) - f_n(x)| + \sup_\Omega |f(x) - f_m(x)| \end{aligned}$$

for all $x \in \Omega$. Therefore

$$\sup_\Omega |f_n(x) - f_m(x)| \le \sup_\Omega |f(x) - f_n(x)| + \sup_\Omega |f(x) - f_m(x)|$$

and the sequence satisfies the Cauchy criterion for uniform convergence.

Conversely, suppose that the sequence obeys the Cauchy criterion. Since

$$|f_n(x) - f_m(x)| \le \sup_\Omega |f_n(x) - f_m(x)|$$

the numerical sequence $\{f_n(x)\}$ is a Cauchy sequence for any x in Ω and, hence, $\{f_n\}$ converges pointwise to a function $f(x)$ in Ω. By taking the limit $m \to \infty$ in the Cauchy criterion first, it is concluded that the sequence converges uniformly to f on Ω:

$$\lim_{n\to\infty} \lim_{m\to\infty} \sup_\Omega |f_n(x) - f_m(x)| = \lim_{n\to\infty} \sup_\Omega |f_n(x) - f(x)| = 0\,.$$

This completes the proof.

1.5.3. Uniform convergence of functional series. Suppose that a function is defined by a series that converges pointwise:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)\,, \quad x \in \Omega$$

Recall a numerical series converges if the sequence of its partial sums converges. Therefore by definition

$$f(x) = \lim_{n\to\infty} s_n(x)\,, \quad s_n(x) = \sum_{k=1}^{n} f_k(x)$$

for every x in Ω. It also follows from this definition that the sum of a functional series does not generally inherit smoothness properties of terms in the series. So, the order of taking a limit $x \to x_0$ and summation in the series cannot generally be interchanged. Similarly, the order of taking partial derivatives and summation cannot also be interchanged. These operations become legitimate under additional conditions based on uniform continuity. By definition, a series converges uniformly on Ω if its sequence of partial sums converges uniformly on Ω. Here is a useful criterion for a uniform convergence of a functional series.

Let $\{f_n\}$ be a sequence of bounded functions on a set Ω such that the series of bounds converges:

$$\sum_n M_n < \infty\,, \quad M_n = \sup_{\Omega} |f_n(x)|\,.$$

Then the series $\sum_n f_n(x)$ converges uniformly on Ω.

Indeed, by the Cauchy criterion for uniform convergence in Sec.**1.5.2**, the sequence of partial sums converges uniformly on Ω to some function f because

$$\sup_{\Omega} |s_n(x) - s_m(x)| = \sup_{\Omega} \Big| \sum_{k=m+1}^{n} f_k(x) \Big| \le \sum_{k=m+1}^{n} M_k \to 0$$

as $n > m \to \infty$ because $\sum_k M_k < \infty$ and, hence, its partial sums form a Cauchy sequence.

1.5.4. Continuity and uniform convergence. It turns out that the uniform convergence is sufficient for the continuity of the limit function[3].

Suppose that a sequence of continuous functions converges to a function f uniformly on a set Ω. Then the function f is continuous on Ω.

[3]W. Rudin, Principles of Mathematical Analysis, Chapter 7

In particular, if a series of continuous functions converges uniformly, then the order of taking the limit and summation can be interchanged:

$$\lim_{x\to x_0}\sum_n f_n(x) = \sum_n \lim_{x\to x_0} f_n(x) = \sum_n f_n(x_0)\,. \tag{1.2}$$

In the theory of trigonometric Fourier series, it is proved that if f is a continuous and 2π periodic function on $\mathbb{R}$, then its trigonometric Fourier converges to f uniformly. Furthermore, if f has a jump-discontinuity at x_0, then its trigonometric Fourier series at $x = x_0$ converges to the mid-point to $(f_+ + f_-)/2$ where $f(x) \to f_\pm$ as $x \to x_0^\pm$ (the left and right limits). In this case, the limit $x \to x_0$ of the sum of the series does not exist, whereas the sum of the series of the terms at $x = x_0$ exists. The convergence of the series is not uniform on any interval containing x_0.

For example, the trigonometric Fourier series

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

converges for all x, where $f(0) = 0$, $f(-x) = -f(x)$, and $f(x) = 1 - \frac{x}{\pi}$ for $0 < x \le \pi$. The sum is not continuous at $x = 0$. Relation (**1.2**) fails for this series. The convergence is not uniform on any interval containing $x = 0$.

1.5.5. Differentiation and uniform continuity. The following theorem holds[4].

Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ such that $\{f_n(c)\}$ converges for some $c \in [a, b]$. If $\{f_n'\}$ converges uniformly to a function g on $[a, b]$, then $\{f_n\}$ converges uniformly to a function f on $[a, b]$ and $f' = g$.

By combining this assertion with Secs. **1.5.4** and **1.5.5**, the following useful criterion for continuity and differentiability of functional series can be established.

PROPOSITION 1.1. (Differentiation of a series)
Let $\{f_n\}$ be a sequence of continuous and bounded functions on a set $\Omega \subset \mathbb{R}^N$ such that the series of the bounds converge:

$$\sum_n M_n < \infty\,, \quad M_n = \sup_\Omega |f_n(x)|\,.$$

[4] W. Rudin, Principles of Mathematical Analysis, Chapter 7

Then the following series converges to a continuous function on Ω*:*

$$f(x) = \sum_n f_n(x) \, .$$

If, in addition, $f_n \in C^1(\Omega)$ *and partial derivatives are bounded on* Ω *so that the series of bounds also converges,*

$$\sum_n M_n^{(1)} < \infty \, , \quad M_n^{(1)} = \sup_\Omega \Big| Df_n(x) \Big| \, ,$$

then the function f *is from class* $C^1(\Omega)$ *and*

$$Df(x) = D \sum_n f_n(x) = \sum_n Df_n(x) \, .$$

Clearly, similar sufficient conditions for term-by-term differentiation of a convergent functional series of smooth functions can be established for higher order partials $D^p f$. The criterion is simple and easy to use. Unfortunately, it fails to detect smoothness of the sum of a series in many cases and, hence, has a limited applicability. For example, the sum of the trigonometric Fourier series discussed in Sec.**1.5.4** is from class C^∞ on any open interval that does not contain $x = 2\pi n$ with n being any integer. But the series of first derivatives of terms does not even converge pointwise everywhere, not to mention uniform convergence.

2. Sets of measure zero in a Euclidean space

2.1. Volume of a set. Let R be a rectangular box in $\mathbb{R}^N$. A component x_i of any $x \in R$ spans an interval $[a_i, b_i]$. By definition, the length of this interval is $b_i - a_i$ and the volume of R is the product of lengths of intervals spanned by each component:

$$V_N(R) = \prod_{i=1}^{N} (b_i - a_i)\,.$$

The volume of other sets can be defined by a limiting procedure in which the set is approximated by the union of rectangular boxes. This will be discussed in the next sections using the Riemann and Lebesgue theories of integration. The objective of this section is to characterize sets of zero volume which will play a significant role throughout the book.

In what follows, the volume of a ball B_a in $\mathbb{R}^N$ will be used. Recall that a volume of a region $\Omega \subset \mathbb{R}^N$ is defined by the integral

$$V(\Omega) = \int_\Omega d^N x$$

provided the integral exists (a review of Riemann integration theory is given in Section **3**). If Ω is a ball of radius a, then this integral can be evaluated in spherical coordinates. The task can however be accomplished by technically simpler means.

2.1.1. Volume of a ball in $\mathbb{R}^N$. Let $V_N(a)$ be the volume in question. First note that by the scaling transformation $x = y/a$,

$$V_N(a) = \int_{|x|<a} d^N x = a^N \int_{|y|<1} d^N y = V_N(1) a^N\,.$$

This property will be essential for what follows, while the constant $V_N(1)$ is not relevant. The scaling property also follows from a dimensional analysis because any ball is characterized by just one constant, the radius, that has dimension of length.

The constant $C_N = V_N(1)$ (the volume of a unit ball) can be calculated recursively. Let r be a coordinate along a diameter of the ball so that $-1 \le r \le 1$. A cross-section of the ball by a plane perpendicular to the diameter at a point r is an $N-1$ dimensional ball centered at r of radius $\sqrt{1-r^2}$. The volume of a portion of the ball between two such planes at a distance dr is therefore $dV_N = V_{N-1}(\sqrt{1-r^2})dr$ and, hence,

$$V_N(1) = \int_{-1}^{1} V_{N-1}(\sqrt{1-r^2})\, dr\,, \qquad V_1(1) = 2\,.$$

By the scaling property for V_{N-1}, the above equation is reduced to the recurrence relation

$$C_N = C_{N-1}\int_{-1}^{1}\left(1-s^2\right)^{\frac{N}{2}-\frac{1}{2}} ds\,, \qquad C_1 = 2$$

Evaluating the integral, one infers that

$$V_N(a) = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\, a^N$$

where Γ is Euler's gamma function:

$$\Gamma(z) = \lim_{b\to\infty}\int_0^b e^{-t}t^{z-1}\, dt$$

It has the following properties:

$$\Gamma(z+1) = z\Gamma(z)\,, \quad \Gamma(1) = 1\,, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

The first one is established by integration by parts, while the other two are proved by a direct evaluation of the integral.

If $\sigma_{_N}(a)$ is the surface area of the sphere $|x| = a$, then $dV_N(a) = \sigma_{_N}(a)da$. It follows from this relation that the surface area of the unit sphere in $\mathbb{R}^N$ reads

$$\sigma_{_N} = \sigma_{_N}(1) = \frac{N\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\,. \tag{2.1}$$

2.2. Definition of a set of measure zero in $\mathbb{R}^N$. The volume of a point in a Euclidean space is equal to zero because it is contained in a ball of arbitrary small radius. Similarly, any finite collection of points has the zero volume because it is contained in a the union of balls whose total volume can be arbitrary small. This observation is extended to all sets.

A set in $\mathbb{R}^N$ is said to be of measure zero if it can be covered by a union of open balls whose total volume can be made smaller than any preassigned positive number.

For brevity, one writes $\mu(\Omega) = 0$ if Ω is a set of measure zero where μ stands for the word "measure". A rationale for this term will be introduced later in the framework of the Lebesgue integration theory.

It is worth noting that the use of open balls in the above definition is not a necessity. One can also use *open* rectangular boxes. Any open rectangular box has non-vanishing dimensions $a_j > 0$, $j = 1, 2, ..., N$, and is contained in an open ball of radius $a/2$ where $a^2 = a_j a_j$. Conversely, any open ball of radius $a > 0$ is contained is an open rectangular box with dimensions $a_j = 2a$. So, the conditions $a_1 a_2 \cdot a_N \to 0$ and $a \to 0$ are equivalent in the definition of a set of measure zero.

2.2.1. Examples of sets of measure zero.

- A finite collection of points in space is a set of measure zero.
- A segment of a straight line of finite length L is a set of measure zero. Indeed, let us split it into n pieces of length L/n. Each such segment can be covered by a ball of radius L/n centered at the midpoint of the segment. The total volume is
$$V_n = nV_N(L/n) = C_N n(L/n)^N \to 0$$
as $n \to \infty$ for any dimension $N \geq 2$.
- Generalizing the previous example, a Euclidean space $\mathbb{R}^M$ can be viewed a hyper-plane in a higher dimensional Euclidean space $\mathbb{R}^N$, $N > M$. Any rectangular box in $\mathbb{R}^M$ is a set of measure zero in $\mathbb{R}^M$. For example, a rectangle in a plane in a three-dimensional space is a set of measure zero. A proof of this assertion is left to the reader as an exercise.

2.3. Properties of sets of measure zero. An obvious property of any set of measure is that *any subset of a set of measure zero is also a set of measure zero.*

Let us show that *a countable union of sets of measure zero is also a set of measure zero.*

Let
$$G = \bigcup_{n=1}^{\infty} G_n$$
where $\mu(G_n) = 0$ for all n. Fix $\varepsilon > 0$. Then each G_n can be contained in a union of open balls with the total volume $\varepsilon/2^n$. Therefore G is contained in the union of all such balls with the total volume being
$$V = \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \frac{\varepsilon}{2}\Big(1 + \frac{1}{2} + \Big(\frac{1}{2}\Big)^2 + \Big(\frac{1}{2}\Big)^3 + \cdots\Big) = \frac{\varepsilon}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \varepsilon$$
Since ε is arbitrary, G is a set of measure zero.

An immediate consequence of this theorem is that *a countable collection of points is a set of measure zero.* For example, all rational numbers in the interval $[0,1]$ are countable. Any positive rational number in this interval is a fraction of positive integers, n/m, $n \leq m$. The set of pairs (n,m) is countable. So, rational numbers in $[0,1]$ form a set measure zero (a set of zero length). All rational numbers is a countable union of rational numbers in intervals $[n, n+1]$ where n is an integer. This implies that all rational numbers form a set of measure zero in $\mathbb{R}$. Furthermore, points with rational coordinates in the rectangle $[0,1] \times [0,1]$ are also countable. Indeed, pairs (a_n, b_n) with a_n and b_n from countable sets $(n = 1, 2, ...)$ can be counted in the order (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), etc. So, all points with rational coordinates in $\mathbb{R}^2$ form a set of measure zero. This conclusion is readily extended to a Euclidean space of any dimension.

Remark. Are there sets of measure zero in $\mathbb{R}$ that are not countable? The answer is affirmative. There are uncountable collections of numbers which contain no interval. One of the most famous examples is the *Cantor set*.

Other uncountable sets of measure zero in $\mathbb{R}^N$ include Euclidean subspaces. A line in space is also a set measure zero because it is a union of countably many line segments of a finite length. Similarly, any subspace $\mathbb{R}^M$ of $\mathbb{R}^N$ is a set of measure zero if $M < N$ because it is a union of countably many boxes, and any box in $\mathbb{R}^M$ is a set of measure zero in $\mathbb{R}^N$, $N > M$. What about measures of curves and surfaces in space?

2.4. Smooth transformations of sets of measure zero. It is natural to ask if a curve in a Euclidean space is a set of measure zero because a curve looks like a continuous deformation of a line. If Ω is an open set in $\mathbb{R}^N$, then it cannot be a set of measure zero as it always contains an open ball of some radius. Is the boundary $\partial\Omega$ a set of measure zero? An open set can be obtained by a continuous deformation of an open rectangular box, like a combination of local stretching, compressing, and twisting, but without breaking the set into unconnected pieces. The word "local" means "in a neighborhood of a point". It will always be used in this sense. A face of a box lies in a hyperplane $\mathbb{R}^{N-1}$. So the boundary $\partial\Omega$ locally looks like a continuous deformation of a hyperplane $\mathbb{R}^{N-1}$ in $\mathbb{R}^N$. Therefore in order to answer these questions, one should investigate sets of measure under continuous transformations.

2.4.1. Curves in a Euclidean space as point sets. Intuitively, a curve as a point set in a Euclidean space can be obtained by a continuous deformation (without breaking) of a line segment that has a continuous inverse. The existence of the inverse is needed to avoid gluing parts of the segment together upon deformation. The result of such a deformation can be viewed as the range of *a continuous vector function* $x(t)$ *on an interval* $[a, b]$ *that is one-to-one except possibly at the boundary points of the interval.* If $x(a) = x(b)$, then the curve is called *closed.* So, any curve as point set is a simple parametric curve (cf. Sec.**1.1.6**). It should be noted that there exists infinitely many simple parameterizations for the same curve as a point set in a Euclidean space. If $t = t(\tau)$ is a one-to-one continuous map of $[\alpha, \beta]$ to $[a, b]$, then $y(\tau) = x(t(\tau))$ are also parametric equations of the same curve (the vector functions $y(\tau)$ and $x(t)$ have the same range). The described process is called a *reparameterization* of a curve.

A curve has a self-intersection point if two points of an interval are glued together upon deformation. This implies that any mapping whose range in the curve is not one-to-one at $t = t_1$ and $t = t_2 > t_1$ because $x(t_1) = x(t_2)$ and it is one-to-one on (a, t_1), (t_1, t_2), and (t_2, b). Clearly, in this way one can define a curve with any number of self-intersections as the range of a parametric curve (modulo a reparameterization).

2.4.2. Smooth curves. Suppose that a curve has a simple parameterization from class C^1 on $[a, b]$. Then $x'(t)$ is a tangent vector and $\hat{w}(t) = x'(t)/|x'(t)|$ is a unit tangent vector. If $x'(t)$ does not vanish anywhere, then the unit tangent vector is continuous along the curve. If $x'(c) = 0$ for some $t = c$, then $\hat{w}(c)$ is defined by the left and right limits $\hat{w}(c) = \lim_{t\to c+} \hat{w}(t) = \lim_{t\to c-} \hat{w}(t)$, provided they exists and are equal. A curve is called *smooth* if it has a continuous unit tangent vector, and a closed curve is smooth if, in addition, $\hat{w}(a) = \hat{w}(b)$. Here $\hat{w}(a)$ is defined by the right limit $t \to a^+$, while $\hat{w}(b)$ is defined by the left limit $t \to b^-$ for any parameterization. A unit tangent vector along a curve can be defined only in two ways. If $\hat{w}$ is a unit tangent vector, then $-\hat{w}$ is also a unit tangent vector. A particular choice is called the *orientation* of the curve. If $\hat{w}(a) = -\hat{w}(b)$, then the closed curve has a cusp, like the graph $y = \sqrt{|x|}$ at $x = 0$. So, to check if a given curve is smooth one should find a simple parameterization from class C^1 (if none exists, then the curve is not smooth) and check if $\hat{w}(t)$ can be continuously defined along the curve. A curve is piecewise smooth if it consists of finitely many smooth pieces.

2.4.3. M-surfaces in $\mathbb{R}^N$. By analogy with curves, an M dimensional surface (or simply M-surface) in $\mathbb{R}^N$, $M < N$, can be defined as a continuous deformation of an M dimensional rectangular box that has a continuous inverse.

Let R be an open rectangular box in $\mathbb{R}^M$ and F be a continuous mapping of a neighborhood of R into $\mathbb{R}^N$ that is one-to-one on R. Then the range $F(R) \subset \mathbb{R}^N$ is called an M-surface. The equations $x = F(\xi)$ are called *parametric equations* of the M-surface. As in the case of curves, there are infinitely many parametric equations for the same surface. Self-intersecting M-surfaces can also be described by a continuous mapping of R that is one-to-one except some points in R.

By definition, the mapping F is defined in a neighborhood of R and, hence, on the boundary ∂R. This condition ensures that F restricted to the boundary ∂R is also a continuous map and, hence, the boundary ∂S_M is a surface of dimension $M-1$ (or a finite union of such surfaces). A restriction of F to ∂R, although being continuous, is not required to be one-to-one. In this way, surfaces with non-trivial topology can be obtained through identification of certain points of the boundary of R upon deformation.

For example, a unit sphere in $\mathbb{R}^3$ is obtained by the mapping of $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$x_1 = \cos(\xi_1)\,, \quad x_2 = \sin(\xi_1)\cos(\xi_2)\,, \quad x_3 = \sin(\xi_1)\sin(\xi_2)\,.$$

Here ξ_1 and ξ_2 are the zenith and polar angles of the spherical coordinates. Therefore the mapping is one-to-one in $R = (0,\pi)\times(0,2\pi)$. The boundary lines $\xi_2 = 0$ and $\xi_2 = 2\pi$ have the same image curve (the semi-circle from the north to south pole of the sphere), whereas the boundary lines $\xi_1 = 0$ and $\xi_1 = \pi$ are mapped to single points, the north and south poles, respectively. With these identifications, the closure $\bar{R}$ is mapped into a sphere $|x| = 1$.

Similarly, the parametric equations

$$x_1 = [a + b\cos(\xi_1)]\cos(\xi_2)\,,\; x_1 = [a + b\cos(\xi_1)]\sin(\xi_2)\,,\; x_3 = b\sin(\xi_1)\,,$$

describe a torus with radii $a > b$. The cross section of the torus by the plane $x_3 = 0$ is the union of two circles of radii $a \pm b$. The cross section of the torus by a half-plane bounded by the x_3 coordinate axis is a circle of radius b. The position of the half-plane is defined by the polar angle ξ_2 in the plane $x_3 = 0$, whereas the angle ξ_1 defines a position of the point on the circle of intersection. By geometry, the map is one-to-one in $R = (0,2\pi)\times(0,2\pi)$. The map identifies the opposite boundary lines of the rectangle R. So, the range $F(\bar{R})$ is a 2-torus in $\mathbb{R}^3$.

2.4.4. Smooth M-surfaces. Let S_M be an M-surface in $\mathbb{R}^N$. In a neighborhood of any point $x_0 \in S_M$, the surface is an image of a open box R under a continuous one-to-one mapping $x = F(\xi)$, $\xi \in R$. Consider coordinate lines through $\xi_0 \in R$, where $x_0 = F(\xi_0)$. There are M such lines with parametric equations $\xi = \xi_a(t) = \xi_0 + t\hat{e}_a$, $a = 1, 2, ..., M$, where $\{\hat{e}_a\}$ is the standard basis in $\mathbb{R}^M$. The images of these lines are curves in S_M described by parametric equations $x = x_a(t) = F(\xi_a(t))$. Suppose that the components of F have continuous partial derivatives on R. The matrix of partial derivatives will be denoted by DF so that its elements are

$$(DF)_{ia} = \partial F_i/\partial\xi_a\,.$$

Suppose that the rank of DF is equal to M everywhere in R. Then $x = x_a(t)$ are smooth curves intersecting at $x_0 = F(\xi_0)$ in S_M. The tangent vectors $x'_a(t) = \partial F/\partial\xi_a$ are continuous and do not vanish anywhere because they are columns of DF and $\mathrm{rank}[DF] = M$. The M tangent vectors

$$w_a = x'_a(0) = \partial_a F(\xi_0)$$

at the point of intersection are linearly independent in $\mathbb{R}^N$ and, hence, their span is an M-plane in $\mathbb{R}^N$. It is called a *tangent space* of S_M at a point $x_0 = F(\xi_0)$.

Furthermore any smooth curve in a neighborhood of ξ_0 is mapped into a smooth curve in S_M is a neighborhood of $x_0 = F(\xi_0)$. Indeed, if $\xi = \xi(t) \in C^1$ and $\xi'(t) \neq 0$ (such parameterization always exists for a smooth curve). Then the curve $x = F(\xi(t))$ also has a non-zero continuous tangent vector $x'(t) = DF\xi'(t)$ because the matrix DF is continuous and its rank is M everywhere so that the curve has a continuous unit tangent vector. An M-surface is called smooth near a point x_0 if S_M near x_0 is an image of an open box R under a mapping from class C^1 such that $\mathrm{rank}[DF] = M$. An M-surface is smooth if it is smooth near any point. In other words, a smooth surface looks like a tangent M-plane is a sufficiently small neighborhood of any of its points, just like a smooth curve looks like its tangent line near any point.

It should be noted that if $x = F(\xi)$ are parametric equations for the whole S_M, the condition $\mathrm{rank}[DF] = M$ can be broken on the boundary ∂R because F is not required to be one-to-one on ∂R but S_M can still be smooth near any point of $F(\partial R)$. For a smooth surface one can find another parameterization $x = \tilde{F}(\xi)$ such that $\mathrm{rank}[D\tilde{F}] = M$ in a neighborhood of any "bad" point of $F(\partial R)$. For example, consider a two-sphere in $\mathbb{R}^3$. The rank of DF for the parameterization given in Sec.**2.4.3** is equal to 1 for the boundary lines $\xi_1 = 0$ or $\xi_1 = \pi$ because $\partial F/\partial\xi_2$ vanishes on them. Their images are points $x = \pm\hat{e}_1$ of the unit sphere, $|x| = 1$. However, one can take another parameterization of the sphere near $x = \pm\hat{e}_1$ in which the zenith angle ξ_1 is counted from, say, $\hat{e}_3$ not from $\hat{e}_1$, and the polar angle ξ_2 is defined in the plane orthogonal to $\hat{e}_3$. In this parameterization, the points $x = \pm\hat{e}_1$ are the images of interior points of R and, hence, the surface is smooth near $x = \pm\hat{e}_1$.

2.4.5. Smooth transformations of $\mathbb{R}^N$. A *transformation* of $\mathbb{R}^N$ is a function $F:\ \mathbb{R}^N \to \mathbb{R}^N$. If all components of F are continuously differentiable, then the transformation is said to be from class C^1. If, in addition, its Jacobian does not vanish, then the transformation is called *non-singular*:

$$F\ :\quad \mathbb{R}^N \to \mathbb{R}^N\,,\qquad J = \det[DF] \neq 0\,.$$

Since the Jacobian J does not vanish, the transformation is invertible in a neighborhood of each point by the inverse function theorem[5].

Therefore, any straight line passing through a point y_0 becomes a curve passing through the point $x_0 = F(y_0)$ in a neighborhood of x_0. Parametric equations of a line passing through y_0 and parallel to a unit vector $\hat{v}$ read $y = y_0 + t\hat{v}$ where t is a real parameter. Then the image curve is $x = x(t) = F(y_0 + t\hat{v})$ so that its tangent vector is given by the directional derivative of F, $x'(t) = (\hat{v}, \nabla)F_i = D_v F$. Since the Jacobian matrix DF is continuous and not singular, the tangent vector $x'(t)$ is continuous and does not vanish anywhere. So, the curve is smooth.

Similarly, F maps a 2-plane through y_0 into a smooth 2-surface in a neighborhood of $x_0 = F(y_0)$. Parametric equations of a 2-plane through y_0 that is parallel to two linearly independent vectors u and v are $y = y(t,s) = y_0 + su + tv$, where s and t are real parameters. Parametric equations of the image 2-surface are $x = x(t,s) = F(y(t,s))$. At every point, the surface has two non-vanishing continuous tangent vectors $\partial_t x(t,s)$ and $\partial_s x(t,s)$ that are linearly independent because the matrix DF is not singular and continuous. It is not difficult to see that the image of an M-plane through y_0 is a smooth M-surface through $x_0 = F(y_0)$.

Suppose that a set is covered by a union of open balls. A continuous transformation maps an open set into an open set. So, the transformation F from class C^1 maps an open ball B_a of radius a into an open set whose volume tends to zero as a^N, just like the volume of the ball. Indeed, recall that a volume of $F(B_a)$ is given by the integral of unit function over $F(B_a)$. This integral can be transformed to an integral over B_a by the change of variables $x = F(y)$, where $|y| < a$. Let $a < 1$ (as $a \to 0$ anyway). Then

$$V_a = \int_{F(B_a)} d^N x = \int_{B_a} J(y)\, d^N y = a^N \int_{|z|<1} J(az)\, d^N z \leq C a^N\,,$$

where $C = \max_{|y|\leq 1} J$. Since J is continuous and does not vanish anywhere, it can always be set positive by adjusting a sign of one of

[5] W. Rudin, Principles of Mathematical Analysis, Chapter 9.

components of F. If a set is covered by the union of open balls of total volume less than any preassigned positive number ε, then the image of this set is covered by the union of open sets of total volume less than $C\varepsilon$. Although $F(B)$ is not a ball, intuitively this observation suggests that a set of zero volume is mapped into a set of zero volume by a non-singular transformation from class C^1 as ε is arbitrary. If a C^1 transformation is singular at some points, then the volume element $d^N x = J(y) d^N y$ can vanish on some set. So, the volume of $F(B_a)$ can even vanish if $J = 0$ in B_a, and, hence, the conjecture seems to hold when the transformation of $\mathbb{R}^N$ is not non-singular everywhere. Although the reasoning is not rigorous, nonetheless it led us to a correct statement [6].

THEOREM **2.1**. *The image of a set of measure zero Ω in $\mathbb{R}^N$ under a transformation F of $\mathbb{R}^N$ from class C^1 is a set of measure zero:*

$$\mu(\Omega) = 0 \quad \Rightarrow \quad \mu(F(\Omega)) = 0 \,.$$

Any smooth M-surface in $\mathbb{R}^N$ can be obtained by a C^1 mapping of $\mathbb{R}^M$ into $\mathbb{R}^N$. Any M-plane in $\mathbb{R}^N$ is a set of measure zero. It turns out that any smooth deformation of such a plane also produces a set of measure zero in $\mathbb{R}^N$.

THEOREM **2.2**. *Let Ω be an open set in $\mathbb{R}^M$ and the mapping $F : \Omega \to \mathbb{R}^N$ is from class $C^1(\Omega)$ and the the rank of the Jacobian matrix DF is equal to $M < N$. Then the image of Ω is a set of measure zero in $\mathbb{R}^N$,*

$$\mu(F(\Omega)) = 0 \,.$$

Thus, any smooth M-surface is a set of measure zero in $\mathbb{R}^N$, and any transformation of $\mathbb{R}^N$ into itself from class C^1 maps this surface into a set of measure zero.

Recall that a boundary of a region is smooth if it is a level set of a function from class C^1 whose gradient does not vanish. By the implicit function theorem, the equation $g(x) = 0$ can be solved with respect to one of the components of x with respect to which the partial derivative of g does not vanish near x_0 on the boundary. For example, $x_N = f(y)$ where $x_j = y_j$, $j = 1, 2, ..., N-1$, and by the implicit function theorem f is continuously differentiable. The latter equations can be viewed as parametric equations of an $N-1$ dimensional smooth surface in $\mathbb{R}^N$. Therefore, a piecewise smooth boundary of a region in $\mathbb{R}^N$ is a set of measure zero.

[6] Proofs of Theorems **2.1** and **2.2** can be found in: J.M. Lee, Introduction to smooth manifolds

3. Riemann integral

3.1. Definition of a Riemann integral in a Euclidean space. Let a function f be bounded on a box $R = [a_1, b_1] \times \cdots \times [a_N, b_N]$ in $\mathbb{R}^N$, that is, $m \leq f(x) \leq M$ for all x in R. The volume of R is $V = (b_1 - a_1) \cdots (b_N - a_N)$ (by definition). Each coordinate interval can be partitioned into intervals of smaller lengths. By doing so, R is partitioned by boxes R_s of smaller volumes ΔV_s where s enumerates all partition boxes so that $V = \sum_s \Delta V_s$. A partition P of R is a collection of all vertices of partition boxes. For example, let R be a rectangle in a plane and each side be partitioned into 3 intervals. Then the partition P consists of 16 points being vertices of 9 partition rectangles. A partition P' is called a *refinement* of P if $P \subset P'$. A refinement can be obtained by adding just one point in the interior of a partition box. In this case, this partition box becomes the union of 2^N partition boxes in the refined partition.

Since f is bounded, one can define the *lower and upper sums* of f associated with a partition P, denoted by L and U, respectively,

$$\begin{array}{ll} L(P,f) = \sum_s m_s \Delta V_s \,, & m_s = \inf_{R_s} f(x) \\ U(P,f) = \sum_s M_s \Delta V_s \,, & m_s = \sup_{R_s} f(x) \end{array}$$

Recall the basic properties of the supremum and infimum:

$$\sup_A f(x) \leq \sup_B f(x)\,, \quad \inf_A f(x) \geq \inf_B f(x)\,, \quad A \subset B$$

These relations imply that the lower sum is increasing upon a refinement whereas the upper sum is decreasing:

$$mV \leq L(P,f) \leq L(P',f) \leq U(P',f) \leq U(P,f) \leq MV\,, \quad P \subset P'\,.$$

The values of $L(P,f)$ for all partitions form a set of reals that is bounded from above by MV and, hence, it has the least upper bound (supremum). Similarly, the values of $U(P,f)$ for all partitions form a set bounded from below by mV and therefore this set has the greatest lower bound (infimum).

A bounded function f is said to be Riemann integrable on R if the greatest lower bound of upper sums is equal to the least upper bound of lower sums and, in this case, their value is called the Riemann integral of f over R:

$$\int_R f(x)\, d^N x = \inf_P U(P,f) = \sup_P U(P,f)\,.$$

3.2. Riemann integrability. For continuous functions, the Riemann integral exists and can be evaluated via their antiderivatives.

3.2.1. The fundamental theorem of calculus. Let f be continuous on $[a, b]$ and F be an antiderivative of f, that is, $F' = f$. Then

$$\int_a^b f(x)\, dx = F(b) - F(b)\,.$$

3.2.2. Fubini's theorem. *Let $f(x)$ be continuous function on a closed rectangle R. Then it is Riemann integrable on R and*

$$\int_\Omega f(x)\, d^N x = \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} f(x_1, ..., x_N)\, dx_N \cdots dx_1$$

Here x_j, $j = 1, 2, ..., N$, are coordinates of a point x in $\mathbb{R}^N$, and the iterated integral can be computed in any order.

The Fubini allows one to calculate the integral by means of the fundamental theorem of calculus applied to each of the iterated integrals.

3.2.3. An example of a non-integrable function. Continuity is not necessary for Riemann integrability. Suppose that a function g coincides with a continuous function f on interval $[a, b]$ everywhere but a point c, $g(c) \neq f(x)$. Then g is still integrable on $[a, b]$. The lower and upper sums for f and g only differs by the term corresponding to a partition interval containing c, but this term can be made arbitrary small by refining the partition. Therefore, g is integrable and the integrals of f and g are equal. Clearly, a continuous function can be altered at finitely many points without destroying its integrability.

The Riemann integrability can be lost if a continuous function is altered at infinitely many points. A simple example is provided by the Dirichlet function defined by

$$f_D(x) = \begin{cases} 1\ , & x \in \mathbb{Q} \\ 0\ , & x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers. It is continuous nowhere and not Riemann integrable on any interval. Indeed, since any interval contains rational and irrational numbers, for any partition the lower sum is equal to zero, whereas the upper sum is equal to the length of the interval. So, the limits of the lower and upper sums cannot be equal.

3.2.4. Class of Riemann integrable functions. The class of Riemann integrable functions is described in the following theorem [7].

[7] see, e.g., S. Abbott, Understanding Analysis, Springer, 2010

THEOREM 3.1. (Lebesgue's criterion for Riemann integrability) *A bounded function is Riemann integrable on a rectangular box in a Euclidean space if and only if it is not continuous at most on a set of measure zero.*

It is worth mentioning that it is possible to construct a function on $\mathbb{R}$ that is not continuous at rational numbers but continuous otherwise (e.g., *Thomae's function*). The rational numbers form a countable set of measure zero. This function is Riemann integrable on any bounded interval. The characteristic function of the Cantor set (which is not continuous on an uncountable set of measure zero) is also Riemann integrable on any bounded interval.

3.2.5. Riemann sums. If the function is not continuous but still Riemann intgerable, then Fubini's theorem cannot be used to calculate the integral. In this case, the integral can be approximated with any desired accuracy by means of *Riemann sums*.

A *Riemann sum* for a function f and a partition P is defined by

$$R(P, f) = \sum_s f(x_s^*)\Delta V_s\,,$$

where x_s is a *sample* point in a partition box R_s. For any partition P, the inequality $m_s \leq f(x_s^*) \leq M_s$ implies that

$$L(P, f) \leq R(P, f) \leq U(P, f)$$

for any choice of sample points in the Riemann sum. Riemann sums can be used for approximations of the Riemann integral. By refining the partition, the Riemann sum converges to the integral by the squeeze principle. So, the following assertion holds.

Let f be a Riemann integrable function on a box R. For any positive number $\varepsilon > 0$, there exists a partition P_ε such that

$$\Big|\int_R f(x)\, d^N x - R(P, f)\Big| \leq \varepsilon\,, \quad P_\varepsilon \subset P$$

for any choice of sample points in the Riemann sum and any refinement P of P_ε.

3.2.6. Geometrical interpretation of a Riemann integral. Let f be continuous and non-negative on interval $[a, b]$. Let P be a partition of $[a, b]$ and R_s be the partition intervals. A planar region Ω defined by $a \leq x \leq b$ and $0 \leq y \leq f(x)$ contains the union of rectangles $R_s \times [0, m_s]$ with the total area being equal to $L(P, f)$, whereas the union of rectangles $R_s \times [0, M_s]$, with the total area being equal to $U(P, f)$, contains Ω. The Riemann integral of f *defines* the area of Ω. The lower sum is an estimate of the area from below and the upper sum is its estimate from above.

Similarly, let $f(x, y)$ be a a non-negative continuous function of two real variables on a rectangle $R = [a_1, b_1] \times [a_2, b_2]$. Let Ω be the solid above the rectangle and below the graph $z = f(x, y)$. Then the upper and lower sums are estimates of the volume of Ω from above and below, respectively, because the union of three-dimensional boxes $R_s \times [0, M_s]$ contains Ω, where R_s are partition rectangles, while Ω contains the union of boxes $R_s \times [0, m_s]$. Upon refinement the estimates tends to one another so that the integral of f over the rectangle gives, *by definition*, the volume of Ω. This geometrical interpretation of the Riemann integral can readily be extended to any Euclidean space.

3.3. Riemann integral over a set. Let Ω be a bounded set in $\mathbb{R}^N$ and f be bounded on Ω. Let us extend f to $\mathbb{R}^N$ by zero, that is, $f(x) = 0$ if $x \neq \Omega$. The extension can also be written as $\chi_\Omega(x)f(x)$ where χ_Ω is the characteristic function of Ω. The Riemann integral of f over Ω is defined by

$$\int_\Omega f(x)\, d^N x = \int_R \chi_\Omega(x) f(x)\, d^N x\,,$$

where R is any rectangle that contains Ω, provided $\chi_\Omega f$ is integrable on R.

Suppose f is not continuous on a set of measure zero in Ω. Then $\chi_\Omega f$ is not continuous also on the boundary $\partial\Omega$. If the boundary is a set of measure zero, then $\chi_\Omega f$ is integrable by Theorem **3.1**. In particular, any continuous function on a bounded closed region with piecewise smooth boundary is integrable because such a boundary is a set of measure zero.

Let Ω be a bounded region with piecewise smooth boundary and f be from class $C^0(\bar{\Omega})$. Then f is Riemann integrable on Ω. Furthermore, if f is bounded and not continuous on finitely many smooth surfaces in Ω, then f is also integrable on Ω.

3.3.1. Volume (or measure) of a set. By Fubini's theorem, the volume of a rectangular box R in a Euclidean space can be written as an integral

$$V_N(R) = a_1 a_2 \cdots a_N = \int_R d^N x$$

where a_j, $j = 1, 2, ..., N$, are lengths of the adjacent edges of the box. Similarly, the volume of any bounded set Ω is defined by

$$V(\Omega) = \int_\Omega d^N x$$

provided the unit function is Riemann integrable on Ω. In particular, a bounded region with a piecewise smooth boundary always has a volume.

By definition, the volume of a set of measure zero must be zero. However, this is not so for some sets of measure zero *if the volume is defined via the Riemann integral of the characteristic function of the set*. The problem is that the characteristic function is not Riemann integrable for some sets of measure zero. For example, the set of all points with rational coordinates in a Euclidean space has measure zero, but its characteristic function is nowhere continuous and, hence, not Riemann integrable on any box. The situation looks really paradoxical because the above observation suggests that a set that has a volume contains subsets for which the volume does not even exist! Alternatively, if a set of measure zero is removed from a set of volume V, then the volume of the resulting set cannot even be defined. Intuitively, one might expect that the volume of any subset should exists and be less than the volume of the whole set, or, if a set of zero volume is removed from a set of volume V, then the volume of the resulting set is still V.

This deficiency of the Riemann volume stems from the very definition of the Riemann integral. Riemann integrability can be destroyed by altering an integrable function on a set of measure zero. For example, the Dirichlet function, that is not Riemann integrable, can be obtained from the zero function by changing its values to unit values at rational values of the argument. This substantial drawback is eliminated in the Lebesgue integration theory which will be discussed later.

3.4. Properties of the Riemann integrals. A complex-valued function f is Riemann integrable on Ω if its real and imaginary parts are integrable and

$$\int_\Omega f(x)\,d^N x = \int_\Omega \operatorname{Re} f(x)\,d^N x + i\int_\Omega \operatorname{Im} f(x)\,d^N x$$

3.4.1. Linearity. If f and g are Riemann integrable on Ω, then their linear combination is integrable and

$$\int_\Omega \Big(\alpha f(x) + \beta g(x)\Big)\,d^N x = \alpha\int_\Omega f(x)\,d^N x + \beta\int_\Omega g(x)\,d^N x$$

for any (real or complex) numbers α and β.

3.4.2. Positivity. If $f(x) \geq 0$ and f is integrable on Ω, then

$$\int_\Omega f(x)\,d^N x \geq 0\,.$$

3.4.3. Integrability of the absolute value. If f is Riemann integrable on Ω, then its absolute value is also integrable on Ω and

$$\left|\int_\Omega f(x)\,d^N x\right| \le \int_\Omega |f(x)|\,d^N x$$

The converse is false. For example, put $f(x) = 1$ if x is rational, and $f(x) = -1$ otherwise. This function is not integrable on any interval $[a, b]$ because its lower sum is equal to $-(b-a)$ and the upper sum is equal to $b-a$ for any partition. However, the absolute value $|f(x)| = 1$ is continuous and, hence, integrable on $[a, b]$.

3.4.4. Additivity. Let subsets $\Omega_{1,2} \subset \Omega$ be closed and bounded, and $\Omega_1 \cup \Omega_2 = \Omega$ but the interiors of $\Omega_{1,2}$ do not intersect. If f is integrable on $\Omega_{1,2}$, then it is integrable on Ω and

$$\int_\Omega f(x)\,d^N x = \int_{\Omega_1} f(x)\,d^N x + \int_{\Omega_2} f(x)\,d^N x$$

In particular, if f is continuous on a bounded closed region Ω with a piecewise smooth boundary and the regions $\Omega_{1,2}$ are obtained by cutting Ω into two pieces by a smooth surface, then the above equation holds.

3.4.5. Continuity. Let Ω_n be a family of subsets of a bounded set Ω labeled by a positive integer n such that

$$\Omega_n \subset \Omega_{n+1}\,, \quad \bigcup_n \Omega_n = \Omega\,.$$

In other words, subsets Ω_n becomes larger with increasing n and in the limit $n \to \infty$, Ω_n becomes Ω. If a function f is Riemann integrable on Ω and on each Ω_n, then

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\,d^N x = \int_\Omega f(x)\,d^N x\,.$$

For example, let Ω be an open bounded set, and its boundary be smooth. Then any f from the class $C^0(\bar{\Omega})$ is integrable on Ω. The subsets Ω_n can be obtained by removing closed balls of radius $1/n$ centered at every point of the boundary of Ω. The boundaries $\partial\Omega_n$ are also smooth if $\partial\Omega$ is smooth enough. Then f is integrable on Ω_n, and the sequence of integrals of f over Ω_n converges to the integral of f over Ω.

3.5. Change of variables. Let $\Omega' \subset \mathbb{R}^N$ be a closed and bounded region with a piecewise smooth boundary. Let $x = F(y)$ be a transformation in $\mathbb{R}^N$ from class C^1 such that it is one-to-one on the interior of Ω' and its Jacobian does not vanish in Ω' except possibly on the boundary of Ω'. Let f be an integrable function on $\Omega = F(\Omega')$. Then

$$\int_\Omega f(x)\, d^N x = \int_{\Omega'} f(F(y))\, J(y)\, d^N y\,, \quad J(y) = |\det[DF]|\,.$$

3.6. Riemann integrability and uniform convergence. Suppose that a sequence $\{f_n\}$ of Riemann integrable functions on Ω converges pointwise to a function f. Then the function f is not generally Riemann integrable. Even if f happens to be Riemann integrable of Ω, then the integral of f is not generally equal to the limit of the integrals of f_n. The first assertion from the representation of the Dirichlet function by the double limit:

$$f_D(x) = \lim_{n\to\infty} \lim_{m\to\infty} \Big(\cos(\pi x n!)\Big)^{2m}\,.$$

The first limit is equal to zero if $xn!$ is not an integer and to 1 if $xn!$ is an integer. Therefore $f(x) = 0$ if x is not rational and $f(x) = 1$ if x is rational because any rational number can be written as a ratio of integers $x = p/q$ and $n!/q$ is an integer if $n \geq q$. The limit function is the Dirichlet function that is not Riemann integrable on any interval. Clearly, the terms of the sequence are continuous and, hence, integrable on any bounded interval.

To illustrate the second assertion, put $f_n(x) = 2nx(1-x^2)^n$ where $x \in [0,1]$ and $n = 1, 2, ...$ It is not difficult to verify that the sequence converges pointwise

$$f(x) = \lim_{n\to\infty} f_n(x) = 0\,, \quad 0 \leq x \leq 1$$

However,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 = \int_0^1 f(x)\, dx$$

where the second equality has been obtained by evaluating the integral. So, *the order of neither differentiation nor integration of a convergent functional sequence can be interchanged with taking the limit*, unless the functional sequence satisfies additional conditions.

THEOREM **3.2**. [8] *Let $\{f_n\}$ be a sequence of Riemann integrable functions on a bounded region Ω that converges uniformly to a function f*

[8] see, e.g., W. Rudin, Principles of Mathematical Analysis

on Ω*. Then* f *is Riemann integrable on* Ω *and*

$$\lim_{n\to\infty}\int_\Omega f_n(x)\,d^Nx = \int_\Omega \lim_{n\to\infty} f_n(x)\,d^Nx = \int_\Omega f(x)\,d^Nx\,.$$

This theorem offers a sufficient condition for interchanging the order of Riemann integration and taking the limit with respect to a parameter of the integrand.

3.7. Exercises.

1. Can a set in $\mathbb{R}^N$ be a set of measure zero in $\mathbb{R}^N$ if it has an interior point? Give an example or show that the answer is negative.

2. Let f be a function from class $C^1(\mathbb{R})$. Show that $f(\mathbb{Q})$ is a set of measure zero where $\mathbb{Q}$ denotes all rational numbers.

3. Use spherical coordinates in $\mathbb{R}^N$ to calculate the volume of an N dimensional ball.

4. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n\,, \qquad |x-x_0| < R\,.$$

(i) Show that the convergence of the series implies that $|c_n|\delta^n \to 0$ as $n\to\infty$ for any $0<\delta<R$.
(ii) Show that

$$|c_n(x-x_0)^n| \le Mq^n\,, \quad |x-x_0|\le\delta$$

for some constants $M>0$ and $0<q<1$ and any $\delta<R$. Use this inequality to show that the power series converges uniformly in the interval $|x-x_|\le\delta<R$.
(iii) Show that f is from class C^∞ by investigating uniform convergence of series of derivatives of the terms.
(iv) Prove that $c_n = f^{(n)}(x_0)/n!$.

5. (i) Use the power series representation of the exponential function to show that

$$\left|e^{i\alpha e^{ix}} - 1\right| \le e^{|\alpha|} - 1\,, \quad x\in\mathbb{R}$$

(ii) Use this inequality to show that $e^{i\alpha e^{ix}}$ converges to 1 uniformly on $\mathbb{R}$ as $\alpha \to 0$ and prove that

$$\lim_{\alpha \to 0} \int_a^b e^{i\alpha e^{ix}} dx = b - a\,.$$

4. Improper Riemann integrals

4.1. Preliminaries. The Riemann integral is defined for a bounded function f and a bounded region Ω. Intuitively, a Riemann integral over an unbounded region can be defined as the limit of integrals over bounded subregions. For example, one can take subregions that are intersections of an unbounded region with a ball of radius a, compute the integrals over these subregions, and then investigate the limit $a \to \infty$. Similarly, if a function is not bounded in a neighborhood of a point, one can reduce the region of integration by removing a ball of radius a centered at this point, compute the integral, and then investigate the limit $a \to 0^+$. If there are more then one of such points, the reduced region is obtained by removing the union of such balls centered at all singular points of the function. By combining the two ideas, one can define the integral of an unbounded function over an unbounded region.

For example, a continuous function $f(x) = e^{-x}$ can be integrated over an unbounded interval $[0, \infty)$ using the rule

$$\int_0^\infty e^{-x}\,dx \overset{\text{def}}{=} \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty}(1 - e^{-b}) = 1\,.$$

Note f is integrable on every $[0, b]$ because f is continuous. So, the rule makes sense. The function $f(x) = x^{-1/2}$ is not bounded on $[0, 1]$, but it is continuous on every $[a, 1]$ so it makes sense to define

$$\int_0^1 \frac{dx}{\sqrt{x}} \overset{\text{def}}{=} \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}(2 - 2\sqrt{a}) = 2\,.$$

Similarly, the function $f(x) = e^{-x}x^{-1/2}$ is not bounded on $[0, \infty)$, but it is continuous on any interval $[\frac{1}{a^2}, a^2]$. So, the integral can be defined by

$$\begin{aligned}\int_0^\infty e^{-x}\,\frac{dx}{\sqrt{x}} &\overset{\text{def}}{=} \lim_{a\to\infty} \int_{\frac{1}{a^2}}^{a^2} e^{-x}\,\frac{dx}{\sqrt{x}} = 2\lim_{a\to\infty}\int_{\frac{1}{a}}^{a} e^{-y^2}\,dy \\ &= 2\lim_{a\to\infty}\left(\int_{\frac{1}{a}}^{1} + \int_1^a\right) e^{-y^2}\,dy = 2\lim_{a\to\infty}\int_0^a e^{-y^2}\,dy = \sqrt{\pi}\,,\end{aligned}$$

where $x = y^2$ and the continuity of the integral was used to take the limit in the integral over $[\frac{1}{a}, 1]$.

A Riemann integral in which the integrand or region of integration or both are not bounded are referred to as an *improper Riemann integral*. A limiting procedure used to define the improper Riemann integral is called a *regularization*. A consistency of this definition requires answering the key question: *Does the value of the improper integral depend on the regularization?* The answer is not straightforward.

4.1.1. An example. Consider the following function of two real variables:

$$f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$$

Evidently it is defined everywhere except the origin. One can choose $f(0,0)$ to be any number. Regardless of this choice, f is not bounded in any neighborhood of the origin. Suppose one wants to integrate this function over a bounded closed region

$$\Omega=\{(x,y)\,|\,x^2+y^2\leq 1\,,\ x\geq 0\,,\ y\geq 0\}$$

which is the part of the unit disk that lies in the positive quadrant. So, f is not bounded on Ω. In attempt to mimic a one-dimensional improper integral, let us take a subregion $\Omega_a\subset\Omega$

$$\Omega_a=\{(x,y)\,|\,a^2\leq x^2+y^2\leq 1\,,\ x\geq 0\,,\ y\geq 0\}$$

so that Ω_a gets larger with decreasing a and becomes Ω when $a=0$. Using the polar coordinates it is not difficult to show that

$$\iint_{\Omega_a} f(x,y)\,dxdy=0\,.$$

Alternatively, the result follows from a symmetry argument. The region Ω_a is symmetric under the reflection about the line $y=x$: $(x,y)\to(y,x)$, whereas the integrand is skew-symmetric, $f(x,y)=-f(y,x)$. Can one conclude that the improper Riemann integral of f over Ω exists and is equal to zero?

It is obvious that Ω can be obtained in many ways as the limit of subregions. For example, consider a collection of subregions which are defined in polar coordinates

$$x=r\cos(\theta)\,,\quad y=r\sin(\theta)$$

by the conditions

$$\Omega_k=\{(x,y)\,|\,a_k\leq r\leq 1\,,\quad \beta_k\leq\theta\leq\pi/2\}\,,\quad k=1,2,...$$

where $\{a_k\}$ and $\{\beta_k\}$ are positive sequences that converge to 0 monotonically. In addition to removing a disk of radius $a=a_k$ as in the previous regularization, a sector with the angle β_k is removed from Ω. So, with increasing k, the region Ω_k gets larger and eventually becomes Ω in the limit $k\to\infty$:

$$\Omega_k\subset\Omega_{k+1}\subset\Omega\,,\quad \bigcup_{k=1}^{\infty}\Omega_k=\Omega$$

The latter union is a proper mathematical way of saying that Ω_k "approaches Ω and coincides with Ω in the limit $k \to \infty$". Using polar coordinates

$$\iint_{\Omega_k} f(x,y)\,dxdy = \int_{a_k}^{1} \int_{\beta_k}^{\pi/2} \frac{r^2\cos(2\theta)}{r^4}\, rdrd\theta = \frac{1}{2}\sin(2\beta_k)\ln(a_k)$$

The right-hand side is an indeterminate form "$0 \times \infty$" in the limit $k \to \infty$. The limit may or may not exist and, even if it exists, it can have any value! Indeed, take $a_k = e^{-c/\beta_k}$ where $c > 0$ so that $a_k \to 0$ monotonically if $\beta_k \to 0$ monotonically. Then

$$\lim_{k\to\infty} \iint_{\Omega_k} f(x,y)\,dxdy = -\lim_{k\to\infty} \frac{c\sin(2\beta_k)}{2\beta_k} = -c\,.$$

If $a_k = \beta_k$, then the limit is 0 and, if $a_k = e^{-c/\beta_k^2}$, $c > 0$, then the limit is $-\infty$. The reader is asked to verify that if the range of the polar angle in Ω_k is restricted to the interval $0 \le \theta \le \pi/2 - \beta_k$, then the limit can be made arbitrary positive number or $+\infty$ by a suitable choice of a_k. A similar result can be established for the integral of f over an unbounded region $x^2 + y^2 \ge 1$, $x \ge 0$, $y \ge 0$ (see Exercises).

The above example shows that the improper integral can depend on its regularization. Naturally, one wants a definitive (or unique) value of an improper integral, and, for this reason, a naive attempt to define improper Riemann integrals should be amended in some way to eliminate the noted deficiency.

4.2. Improper Riemann integrals. Let $\Omega \subset \mathbb{R}^N$ be bounded or unbounded. An *exhaustion* of Ω is a sequence of subsets $\{\Omega_k\}_1^\infty$ such that

- each Ω_k is bounded, closed, and contained in Ω;
- Ω_{k+1} contains Ω_k;
- the union of all Ω_k coincides with Ω except possibly a set of measure zero.

Examples of exhaustions were given in the previous section. If a bounded function f is Riemann integrable on a closed bounded set Ω and on each Ω_k, then by continuity of the Riemann integral

$$\lim_{k\to\infty} \int_{\Omega_k} f(x)\,d^N x = \int_{\Omega} f(x)\,d^N x\,.$$

Clearly, continuity holds for *any choice* of the sequence $\{\Omega_k\}$. If f is not bounded and/or Ω is not bounded, the improper integral is defined by demanding that the continuity property still holds.

Definition 4.1. *Let $\{\Omega_k\}_1^\infty$ be an exhaustion of Ω. Suppose that a function f on Ω is Riemann integrable on each Ω_k. Then the function f is said to be Riemann integrable on Ω if the limit*

$$\lim_{k\to\infty}\int_{\Omega_k} f(x)\,d^Nx = \int_\Omega f(x)\,d^Nx$$

exists and is independent of the choice of Ω_k. In this case, the value of the limit is called an improper Riemann integral of f over Ω.

So, by this definition, the function of two variables considered in the previous section is not integrable on the part of a disk that lies in the positive quadrant or on its complement because the value of the limit depends on the choice of exhaustions (or regularization). To show that an improper integral does not exist, it is sufficient to find two regularizations in which the limits are not equal. However, it is impossible to check the independence of the limit by *computing* the improper integral in every possible regularization. It is therefore important to establish criteria for the existence of improper integrals so that if the limit exists in one particular regularization, then it exists in any other one and has the same value.

4.3. Improper integrals of non-negative functions. Suppose that $f(x)$ is non-negative on Ω. For any exhaustion, the sequence of integrals is monotonically increasing

$$0 \le \int_{\Omega_k} f(x)\,d^Nx \le \int_{\Omega_{k+1}} f(x)\,d^Nx$$

by the positivity property of the Riemann integral and that $\Omega_k \subset \Omega_{k+1}$. Any monotonic sequence converges if only if it is bounded. So, there are only two possibilities: either the limit is a number

$$\lim_{k\to\infty}\int_{\Omega_k} f(x)\,d^Nx = \sup_k \int_{\Omega_k} f(x)\,d^Nx = I_f$$

or it is infinite, $I_f = \infty$. Suppose that $I_f < \infty$. Let $\{\Omega'_k\}$ be another exhaustion of Ω. Then the sequence of the integrals is bounded:

$$\int_{\Omega'_k} f(x)\,d^Nx \le I_f$$

because $f(x) \ge 0$ and $\Omega'_k \subset \Omega$ for any k'. Since the sequence is also increasing monotonically, it converges

$$\lim_{k\to\infty}\int_{\Omega'_k} f(x)\,d^Nx = \sup_{k'} \int_{\Omega'_k} f(x)\,d^Nx = I'_f \le I_f$$

and its limit cannot exceed I_f. On the other hand, one can swap the roles of the exhaustions and use the same argument show that

$$\int_{\Omega_k} f(x)\, d^N x \le I'_f \quad \Rightarrow \quad I_f \le I'_f$$

because $\Omega_k \subset \Omega$ and $f(x) \ge 0$. Therefore $I_f = I'_f$ and the value of the limit does not depend on the choice of the exhaustion.

Suppose now that $I_f = \infty$. Then $I'_f = \infty$. Indeed, if $I'_f < \infty$, then the sequence of integrals of f over $\{\Omega_k\}$ is bounded by $I'_f < \infty$ by the above argument (as $\Omega_k \subset \Omega$ for any k) so that, by taking the supremum over k, $I_f \le I'_f < \infty$, which is a contradiction.

THEOREM **4.1**. (Improper integral for non-negative functions) *Suppose that*

(i) $f(x) \ge 0\,, \quad \forall x \in \Omega$;
(ii) $\{\Omega_n\}$ *and* $\{\Omega'_n\}$ *are exhaustions of* Ω;
(iii) f *is Riemann integrable on each* Ω_n *and* Ω'_n

Then

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \lim_{n\to\infty} \int_{\Omega'_n} f(x)\, d^N x$$

where the limit can also be $+\infty$.

By the integrability of the absolute value, the functions

$$f_\pm(x) = \frac{1}{2}\Big(|f(x)| \pm f(x)\Big) \ge 0$$

are Riemann integrable if f is Riemann integrable. The function $f_+(x)$ coincides with $f(x)$ whenever $f(x) \ge 0$ and vanishes otherwise, whereas $f_-(x)$ coincides with $-f(x)$ whenever $f(x) \le 0$ and vanishes otherwise. Thus, any Riemann integrable function can be written as the difference of two non-negative integrable functions:

$$f(x) = f_+(x) - f_-(x)\,,$$
$$\int_\Omega f(x)\, d^N x = \int_\Omega f_+(x)\, d^N x - \int_\Omega f_-(x)\, d^N x$$

and vice versa (integrability of $f_\pm$ implies integrability of f and $|f| = f_+ + f_-$ by the linearity of the integral).

Using the limit laws, the following theorem can be established from the above representation.

THEOREM **4.2**. *Suppose that the improper integrals of* $f_\pm$ *over* Ω *exist. Then the improper integral of* f *over* Ω *exists and can be computed in any exhaustion* $\{\Omega_n\}$ *of* Ω.

Indeed, by the limit laws and the existence of the improper integral of $f_\pm$,

$$\begin{aligned}\int_\Omega f(x)\,d^Nx &= \lim_{n\to\infty}\int_{\Omega_n}\Big(f_+(x)-f_-(x)\Big)\,d^Nx\\ &= \lim_{n\to\infty}\int_{\Omega_n} f_+(x)\,d^Nx - \lim_{n\to\infty}\int_{\Omega_n} f_-(x)\,d^Nx\\ &= \int_\Omega f_+(x)\,d^Nx - \int_\Omega f_-(x)\,d^Nx\end{aligned}$$

and, by Theorem **4.1** the values of the limits in the right side of the equation do not depend on the choice of the exhaustion (or regularization) of the integrals.

COROLLARY **4.1**. *Let $\{\Omega_n\}$ be an exhaustion of Ω. Suppose that f and its absolute $|f|$ are integrable on each Ω_n and*

$$\lim_{n\to\infty}\int_{\Omega_n}|f(x)|\,d^Nx = \int_\Omega |f(x)|\,d^Nx < \infty$$

Then the improper integral of f over Ω exists and

$$\int_\Omega f(x)\,d^Nx = \lim_{n\to\infty}\int_{\Omega_n} f(x)\,d^Nx$$

In other words, *if the improper integral of the absolute value of f converges in any particular regularization, then the improper integral of f exists and can be computed in any suitable regularization.* Indeed, since

$$0 \le f_\pm(x) \le |f(x)|$$

It is concluded that monotonic sequences of integrals of $f_\pm$ over Ω_n are bounded:

$$0 \le \int_{\Omega_n} f_\pm(x)\,d^Nx \le \int_{\Omega_n}|f(x)|\,d^Nx \le \int_\Omega |f(x)|\,d^Nx < \infty$$

and, hence, converge. By Theorem **4.1** the limits are independent of the choice of Ω_n. By Theorem **4.2**, the improper integral of f over Ω exists (it is independent of regularization).

4.4. Absolutely and conditionally convergent integrals. Am improper Riemann integral of a function f over a region Ω is called *absolutely convergent* if

$$\lim_{n\to\infty}\int_{\Omega_n}|f(x)|\,d^Nx = I_f < \infty$$

The absolute convergence of the Riemann integral implies the existence of the improper Riemann integral.

If the limit

$$\lim_{n\to\infty}\int_{\Omega_n} f(x)\, d^N x\,.$$

exists for some exhaustion (regularization) $\{\Omega_n\}$ but the integral does not converge absolutely, then the integral of f is said to be *conditionally convergent* in the exhaustion $\{\Omega_n\}$. Absolutely and conditionally convergent integrals are analogous to absolutely and conditionally convergent series as illustrated below.

4.4.1. Conditionally convergent integrals. Let the integral of f over Ω be conditionally convergent. In this case, the integrals of $f_\pm$ must diverge, and the value of a conditionally convergent integral is an indeterminate form "$\infty - \infty$" which can happen to be a number in a particular regularization:

$$\lim_{n\to\infty}\int_{\Omega_n} f(x)\, d^N x = \lim_{n\to\infty}\left(\int_{\Omega_n} f_+(x)\, d^N x - \int_{\Omega_n} f_-(x)\, d^N x\right)$$

Indeed, the divergence of the integral of $|f| = f_+ + f_-$ implies that either the integral of f_+, or f_-, or both diverge because $f_\pm \geq 0$. The conditional convergence of the integral of f (the existence of the limit in the left side) is only possible when the integrals of $f_\pm$ diverge.

The integrals of $f_\pm$ resemble the (divergent) series of positive and negative terms of a conditionally convergent series. The sum of such a series depends on the *arrangement* of terms (the order in which the terms are added). In the case of conditionally convergent integrals, the value depends on the choice of the exhaustion (or regularization). In other words, by choosing a suitable exhaustion one can always make the difference of the integrals of f_+ and f_- over Ω_n to be convergent to any desired number even though both the sequences diverges to $+\infty$, similarly to that the sum of a conditionally convergent numerical series can be made any number or infinity by a suitable rearrangement of terms [9]. This is illustrated with the following example.

Consider the improper integral

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \lim_{n\to\infty}\int_0^{b_n} \frac{\sin(x)}{x}\, dx$$

where $\{b_n\}$ is positive, monotonically increasing, unbounded sequence. Here the integrand extended to $x = 0$ by continuity (the integrand approaches 1 as $x \to 0^+$). In particular, let us take

$$b_n = \pi n\,, \quad n = 1, 2, \ldots$$

[9] This is known as the Riemann theorem about rearrangements (see, e.g., W. Rudin, Principles of mathematical analysis, Chapter 3).

This regularization corresponds to the exhaustion:

$$\Omega_n = [0, \pi n]$$

so that

$$\Omega_n = \Omega_{n-1} \cup S_n\,, \quad S_n = [\pi(n-1), \pi n]\,.$$

If the limit exists, then is equal to the sum of the series

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \sum_{n=1}^\infty \int_{S_n} \frac{\sin(x)}{x}\,dx = \sum_{n=1}^\infty \int_{\pi(n-1)}^{\pi n} \frac{\sin(x)}{x}\,dx$$

This is an alternating series because the integrand is positive on S_{2k-1} and negative on S_{2k}. It follows from the inequality

$$\frac{|\sin(x)|}{\pi n} \le \frac{|\sin(x)|}{x} \le \frac{|\sin(x)|}{\pi(n-1)}\,, \quad n > 1$$

that

$$\frac{2}{\pi n} \le a_n \le \frac{2}{\pi(n-1)}\,, \quad a_n = \int_{\pi(n-1)}^{\pi n} \frac{|\sin(x)|}{x}\,dx > 0$$

and

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \sum_{n=1}^\infty (-1)^{n+1} a_n$$

The sequence $\{a_n\}$ is positive and converges to 0 monotonically because

$$a_{n+1} \le \frac{2}{\pi n} \le a_n$$

By the alternating series test, the series converges.

However, by the comparison test:

$$\frac{2}{\pi}\sum_{k=1}^n \frac{1}{k} \le \int_0^{\pi n} \frac{|\sin(x)|}{x} = \sum_{k=1}^n a_k \quad \Rightarrow \quad \int_0^\infty \frac{|\sin(x)|}{x}\,dx = \infty$$

the series and the integral do not converge absolutely because $\sum \frac{1}{n} = \infty$. So, the integral is only conditionally convergent and does not exist in the sense of Definition **4.1**. Its value depends on the choice of regularization. In particular, the sum can be made equal to any desired number by a suitable rearrangement in the series. A rearrangement corresponds to a different exhaustion made of unions of the intervals S_n.

Consider a rearrangement $\{S'_n\}$ of the sequence of intervals $\{S_n\}$ and put

$$\Omega'_1 = S'_1\,, \quad \Omega'_{n+1} = \Omega'_n \cup S'_{n+1}$$

So, Ω'_n is a collection of any n intervals from $\{S_n\}$, and Ω'_{n+1} is obtained by adding *any* of remaining intervals in the collection $\{S_n\}$. In other words, the order in which the intervals from $\{S_n\}$ are added to obtain an exhaustion is changed, but

$$\bigcup_{n=1}^{\infty} \Omega_n = \bigcup_{n=1}^{\infty} \Omega'_n = [0, \infty)\,.$$

The function is still integrable on any finite collection of intervals Ω'_n. Therefore in this exhaustion (regularization)

$$\int_0^{\infty} \frac{\sin(x)}{x}\, dx = \sum_{n=1}^{\infty} \int_{S'_n} \frac{\sin(x)}{x}\, dx$$

The series in the right-hand side is a rearrangement of the alternating series $\sum_n (-1)^{n+1} a_n$.

Let us show that a rearrangement can converge to any number or $\pm\infty$. Fix a number $I_f > 0$. Make Ω'_1 to be the union of odd intervals, S_1, S_3 and so on until S_{2n-1} where n is the smallest integer for which the integral over the union becomes greater than I_f. Then start adding even intervals, S_2, S_4, and so on until the integral becomes less than I_f. The union of Ω'_1 and the added even intervals is Ω'_2. Then begin to add remaining odd intervals until the integral becomes greater than I_f again. The union of Ω'_2 and the added shells makes Ω'_3. In each step, the overshot or undershot necessarily occurs because the integrals over all odd and all even intervals diverge to $+\infty$ and $-\infty$, respectively. In this way, the sequence of integrals

$$\int_{\Omega'_n} f(x)\, d^N x\,, \quad \Omega'_n \subset \Omega'_{n+1}\,,$$

oscillates about I_f and converges to I_f because the overshot or undershot of the integral is decreasing with increasing the number of iterations:

$$\left| I_f - \int_{\Omega'_n} f(x)\, d^N x \right| \leq \int_{S_{k_n}} |f(x)|\, d^N x \to 0$$

for some $k_n \geq n$ so that $k_n \to \infty$ as $n \to \infty$. Since $\{\Omega'_n\}$ is an exhaustion of Ω by construction, the integral conditionally converges to a preassigned positive number I_f. A similar exhaustion can be constructed to make the integral converging to any negative number. The reader is asked to construct exhaustions in which the integral converges to either $+\infty$ or $-\infty$, or does not converge at all (e.g., oscillates between any two numbers).

4.5. Absolutely convergent integrals. There are tests for absolute convergence of improper integrals that are analogous to the corresponding tests for absolute convergence of series.

4.5.1. The comparison test. Let f and g be Riemann integrable on Ω, and

$$|f(x)| \leq g(x)\,, \quad \forall x \in \Omega$$

Then

$$\int_\Omega |f(x)|\, d^N x \leq \int_\Omega g(x)\, d^N x$$

If now f and g are not integrable in the proper sense, then for any exhaustion $\{\Omega_n\}$

$$\int_{\Omega_n} |f(x)|\, d^N x \leq \int_{\Omega_n} g(x)\, d^N x$$

If the improper integral of g converges, then the integral f converges absolutely because

$$\int_{\Omega_n} g(x)\, d^N x \leq \int_\Omega g(x)\, d^N x < \infty$$

$$\Rightarrow \lim_{n\to\infty} \int_{\Omega_n} |f(x)|\, d^N x \leq \int_\Omega g(x)\, d^N x$$

By Theorem **4.2** the limit does not depend on the choice of the exhaustion and the improper integral of f exists.

THEOREM **4.3**. (Comparison test for absolute convergence) *Let $\{\Omega_n\}$ be an exhaustion of a region Ω and a function f be integrable on any Ω_n. If the absolute value $|f(x)|$ is bounded on Ω by a function whose improper Riemann integral over Ω exists,*

$$|f(x)| \leq g(x)\,, \quad x \in \Omega\,, \quad \lim_{n\to\infty} \int_{\Omega_n} g(x)\, d^N x < \infty\,,$$

then the improper integral of f over Ω also exists and converges absolutely.

4.5.2. Integrals over unbounded regions. Suppose $\Omega = \mathbb{R}^N$ and f is a continuous function. Clearly it is integrable on any ball $\Omega_n = B_n$ (that is, $|x| \leq n$, $n = 1, 2, ...$). So the existence of the improper integral would depend on how fast f falls off as $|x| \to \infty$.

PROPOSITION **4.1**. *Let f be integrable on any ball and*

$$|f(x)| \leq \frac{M}{|x|^p}\,, \quad |x| \geq R$$

for some positive constants M and R, and $p > N$, then the improper integral of f over the whole space exists and

$$\int_{\mathbb{R}^N} f(x)\, d^N x = \lim_{n\to\infty} \int_{|x|\le n} f(x)\, d^N x < \infty\,.$$

Consider the case $N = 2$. The integral over the whole plane is split into the integral over the disk B_R and the rest of the plane $\mathbb{R}^2 \setminus B_R$. Since the integral over B_R is a regular integral, one has to investigate the convergence of the improper integral over the rest of the plane. Since $|f(x)| \ge 0$, if it converges in a particular regularization, then it converges in any other regularization to the same value. Let Ω_n be an annulus $R \le |x| \le n$. Then

$$\int_{\Omega_n} |f(x)|\, d^2x \le \int_{\Omega_n} \frac{M}{|x|^p}\, d^2x = \int_0^{2\pi} \int_R^n \frac{M}{r^p}\, r dr d\theta$$
$$= \frac{2\pi M}{p-2}\left(\frac{1}{R^{p-2}} - \frac{1}{n^{p-2}}\right)$$

The right side converges if $p > N = 2$ when $n \to \infty$. Therefore the integral of f converges absolutely and, hence, the improper integral of f exists by the comparison test (it can be computed in any suitable regularization).

For $N > 2$ note that the volume of a spherical shell of thickness dr and radius r is the differential of the volume of the ball of radius r:

$$dV_N(r) = \sigma_N r^{N-1} dr$$

where σ_N is the area of the unit sphere in $\mathbb{R}^N$. Then using spherical coordinates

$$\int_{\Omega_n} |f(x)|\, d^2x \le \int_{\Omega_n} \frac{M}{|x|^p}\, d^N x = \sigma_N \int_R^n \frac{M}{r^p}\, r^{N-1} dr$$

The integral converges in the limit $n \to \infty$ if $p > N$.

4.5.3. Integrals of unbounded functions. Suppose f is not bounded in any neighborhood of a particular point, and it is continuous otherwise. Without loss of generality, the singular point can be chosen to be the origin $x = 0$ (values of $|f(x)|$ becomes infinitely large as x approaches 0). Then the absolute integrability depends on how fast $|f(x)|$ diverges as $x \to 0$.

Proposition **4.2**. *Suppose that f is not bounded in any ball B_a and integrable on $\Omega \setminus B_a$ where Ω contains $x = 0$. If*

$$|f(x)| \le \frac{M}{|x|^p}\,, \quad |x| \le a$$

for some constants M and R, and $p < N$, then the improper integral of f exists and

$$\int_{\Omega} f(x)\, d^N x = \lim_{a\to 0^+} \int_{\Omega\setminus B_a} f(x)\, d^N x\,.$$

A proof of this assertion can also be done by using spherical coordinates in $\mathbb{R}^N$. Let $\Omega_{a,R}$ be the intersection of Ω with the spherical shell $a^2 \le |x|^2 \le R^2$. The integral of f over $\Omega \setminus B_R$ exists by continuity of f. Then

$$\begin{aligned}\int\limits_{\Omega_{a,R}} |f(x)|\, d^N x &\le \int\limits_{a\le|x|\le R} \frac{M}{|x|^p}\, d^N x = \sigma_N \int_a^R \frac{M}{r^p}\, r^{N-1} dr \\ &= \frac{\sigma_N M}{N-p}\Big(R^{N-p} - a^{N-p}\Big)\end{aligned}$$

So the integral converges in the limit $a \to 0^+$ if $p < N$. Therefore the integral of f converges absolutely by the comparison test, and, hence, the improper integral of f exists.

4.6. Improper integrals of complex-valued functions. Let f be a complex-valued function of N real variables. If $\{\Omega_n\}$ is an exhaustion of Ω, then the integral of f over Ω is said to converge in this exhaustion if the integrals of the real and imaginary parts of f converge, and in this case

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \lim_{n\to\infty} \int_{\Omega_n} \operatorname{Re} f(x)\, d^N x + i \lim_{n\to\infty} \int_{\Omega_n} \operatorname{Im} f(x)\, d^N x$$

It follows from the inequalities

$$|\operatorname{Re} f| \le |f|\,, \quad |\operatorname{Im} f| \le |f|$$

that the integrals of the real and imaginary parts of f converge absolutely if the integral of the absolute value converges. The converse follows from the inequality

$$|f| \le |\operatorname{Re} f| + |\operatorname{Im} f|$$

that is,

- *the integral of a complex-valued function converges absolutely if and only if the integral of the absolute value converges.*

4.7. Gaussian integrals. The objective is to prove that

$$I_N(A,b) = \int_{\mathbb{R}^N} e^{-(x,Ax)+(b,x)}\, d^N x = \frac{\pi^{N/2}}{\det(A)}\, e^{\frac{1}{4}(b,A^{-1}b)} \tag{4.1}$$

where the quadratic form

$$(x, Ax) = \sum_{k,n=1}^{N} A_{kn} x_k x_n > 0\,, \quad \forall x \neq 0$$

is strictly positive if $x \neq 0$. The integrals of this type are known as *Gaussian integrals*. They are routinely used in various applications. Note that the integrand is positive and, hence, if the integral converges, then it converges absolutely. Therefore it can be computed in any convenient regularization.

4.7.1. A special case. Consider a two-dimensional Gaussian integral

$$I_2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dxdy$$

Let Ω_n be a disk of radius n, $x^2+y^2 \leq n^2$. Then using polar coordinates

$$\begin{aligned} I_2 &= \lim_{n\to\infty} \iint_{\Omega_n} e^{-x^2-y^2}\, dxdy = \lim_{n\to\infty} \int_0^{2\pi} \int_0^n e^{-r^2} r dr d\theta \\ &= \pi \lim_{n\to\infty} \int_0^{n^2} e^{-s}\, ds = \pi \end{aligned}$$

Since the value of the absolutely convergent integral does not depend on the regularization, put $\Omega_n' = [-n, n] \times [-n, n]$ so that by Fubini's theorem

$$\pi = \lim_{n\to\infty} \iint_{\Omega_n'} e^{-x^2-y^2}\, dxdy = \lim_{n\to\infty} \int_{-n}^{n} e^{-x^2} dx \int_{-n}^{n} e^{-y^2} dy$$

Therefore, by the limit laws,

$$I_1 = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

because $I_1^2 = \pi$. Using a scaling transformation, $y = \sqrt{a}x$

$$I_1(a) = \int_{-\infty}^{\infty} e^{-ax^2}\, dx = \frac{I_1(1)}{\sqrt{a}} = \frac{\sqrt{\pi}}{\sqrt{a}}$$

Furthermore using the scaling and shift transformation

$$y = \sqrt{a}\, x\,, \quad s = y - \frac{b}{2\sqrt{a}}$$

one infers that

$$\begin{aligned} I_1(a,b) &= \int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx = \lim_{n\to\infty}\int_{-n}^{n} e^{-ax^2+bx}\,dx \\ &= \frac{1}{\sqrt{a}}\lim_{n\to\infty}\int_{-n\sqrt{a}}^{n\sqrt{a}} e^{-y^2+\frac{b}{\sqrt{a}}\,y}\,dy \\ &= \frac{e^{\frac{b^2}{4a}}}{\sqrt{a}}\lim_{n\to\infty}\int_{-n\sqrt{a}-\frac{b}{2\sqrt{a}}}^{n\sqrt{a}+\frac{b}{2\sqrt{a}}} e^{-s^2}\,ds \\ &\overset{(1)}{=} \frac{I_1(1,0)}{\sqrt{a}}\,e^{\frac{b^2}{4a}} = \frac{\sqrt{\pi}}{\sqrt{a}}\,e^{\frac{b^2}{4a}}\,. \end{aligned}$$

Since the value of $I_1(1,0) = I_1$ does not depend on the choice of the exhaustion, the final equality (1) holds. The result (**4.1**) is established for $N = 1$.

4.7.2. General case. Let an exhaustion $\{\Omega_n\}$ of $\mathbb{R}^N$ be rectangular boxes, $|x_j| \le n$, $j = 1,2,...,N$. Let A be a diagonal matrix with diagonal elements a_j. The condition $(x,Ax) > 0$ implies that the diagonal elements are strictly positive, $a_j > 0$. By Fubini's theorem one infers that

$$\begin{aligned} I_N(A,b) &= \int_{\mathbb{R}^N} \exp\Big(-\sum_{j=1}^{N}(a_jx_j^2 - b_jx_j)\Big)\,d^Nx \\ &= \lim_{n\to\infty}\prod_{j=1}^{N}\int_{-n}^{n} e^{-a_jx_j^2+b_jx_j}\,dx_j \\ &= \prod_{j=1}^{N} I_1(a_j,b_j) = \frac{\pi^{N/2}}{\sqrt{a_1a_2\cdots a_N}}\exp\left(\frac{1}{4}\sum_{j=1}^{N}\frac{b_j^2}{a_j}\right) \end{aligned}$$

Any matrix A can be written as a sum of symmetric and skew-symmetric matrix:

$$A = \frac{1}{2}\Big(A + A^T\Big) + \frac{1}{2}\Big(A - A^T\Big) \equiv B + C$$

where B is symmetric, $B^T = B$ (here B^T denotes the transposed matrix B), and C is skew-symmetric, $C^T = -C$. A quadratic form vanishes identically for a skew-symmetric matrix because

$$(x,Cx) = (C^Tx,x) = -(Cx,x) = -(x,Cx) \quad \Rightarrow \quad (x,Cx) = 0$$

Therefore without loss of generality $A = A^T$ (a symmetric matrix). Any symmetric matrix A is diagonalizable, and there exists an orthogonal matrix U,

$$U^T = U^{-1}$$

such that

$$A = U^T aU\,, \quad a_{ij} = a_j\delta_{ij}$$

where a is a real diagonal matrix. The diagonal elements a_j are eigenvalues of A and the columns of U are the corresponding unit eigenvectors. The positivity of a quadratic form requires that all eigenvalues of A are strictly positive because

$$(x, Ax) = (x, U^T aUx) = (Ux, aUx) = (y, ay) = \sum_{j=1}^{N} a_j y_j^2\,.$$

The transformation $y = Ux$ preserves the distance in $\mathbb{R}^N$ because

$$|y|^2 = (y, y) = (Ux, Ux) = (x, U^T Ux) = (x, x) = |x|^2$$

because $U^T U = UU^T = I$ is the unit matrix $I_{ij} = \delta_{ij}$. An orthogonal transformations is a composition of rotations and reflections ($x_j \to p_j x_j$ where $p_j = \pm 1$). Owing to the absolute convergence of the Gaussian integral with a diagonal matrix A and that the Jacobian of an orthogonal transformation is equal to one,

$$d^N x = \left|\det\left(\frac{\partial x_j}{\partial y_i}\right)\right| d^N y = |\det U^T|\, d^N y = d^N y$$

one infers that

$$\begin{aligned} I_N(A, b) &= \lim_{n\to\infty}\int_{\Omega_n} e^{-(x,Ax)+(b,x)} d^N x = \lim_{n\to\infty}\int_{U(\Omega_n)} e^{-(y,ay)+(Ub,y)} d^N y \\ &= \int_{\mathbb{R}^N} e^{-(y,ay)+(Ub,y)} d^N y \\ &= \frac{\pi^{N/2}}{\sqrt{a_1 a_2 \cdots a_N}} \exp\left(\frac{1}{4}\sum_{j=1}^{N}\frac{c_j^2}{a_j}\right) \end{aligned}$$

where $c = Ub$. Since the transformation U preserves the distances between points, for any exhaustion $\{\Omega_n\}$, the image $\{U(\Omega_n\}$ is also an exhaustion of $\mathbb{R}^N$.

Next, note that

$$a_1 a_2 \cdots a_N = \det a = \det(UAU^T) = (\det U)^2 \det A = \det A$$

Therefore $\det A \neq 0$ and the inverse A^{-1} exists and

$$A^{-1} = (U^T aU)^{-1} = U^{-1}a^{-1}(U^T)^{-1} = U^T a^{-1} U\,.$$

It is then concluded that

$$\sum_{j=1}^{N} \frac{c_j^2}{a_j} = (c, a^{-1}c) = (Ub, a^{-1}Ub) = (b, U^T a^{-1} Ub) = (b, A^{-1}b)$$

and the relation (**4.1**) follows.

4.8. Exercises.

1. Consider the improper integral

$$\iint_\Omega \frac{x^2 - y^2}{(x^2+y^2)^2}\, dxdy$$

where

$$\Omega = \{(x,y) \,|\, x^2 + y^2 \geq 1\,,\ x \geq 0\,,\ y \geq 0\}$$

Take an exhaustion $\{\Omega_n\}$ which is a rectangle in the polar coordinates $(r,\theta) \in [1, a_n] \times [\alpha_n, \pi/2 - \beta_n]$ where α_n and β_n are positive and tend to 0 monotonically, while $1 < a_n$ increases monotonically to infinity, as $n \to \infty$. Show that there is a choice of α_n, β_n, and a_n such that the sequence of integrals over Ω_n can converge to any real number or $\pm\infty$.

2. For the integrand in Problem 1, find $f_\pm(x,y)$ and show that for any exhaustion the improper integrals of $f_\pm$ diverge

$$\iint_\Omega f_\pm(x,y)\, dxdy = \infty$$

3. Prove Proposition **4.1** for $N = 3$ using spherical coordinates.

4. Prove Proposition **4.2** for $N = 3$ using spherical coordinates.

5. Let p and q be positive integers. Do the following improper integrals exist in the sense of Definition **4.1**?

$$\text{(i)} \quad \int_0^\infty \frac{\sin^2(x)}{x^p}\, dx$$

$$\text{(ii)} \quad \int_1^\infty \frac{\cos^q(x)}{x^p}\, dx\,,$$

$$\text{(iii)} \quad \int_0^1 \sin^p\left(\frac{1}{x}\right) dx$$

6. Put

$$I_n(a) = \int_0^\infty x^n e^{-ax^2}\, dx\,, \quad n = 0, 1, ..$$

(i) Show that the integral converges absolutely.
(ii) Use integration by parts to prove the recurrence relation

$$I_{n+2} = \frac{n+1}{2a} I_n$$

(iii) Find $I_0(a)$ and $I_1(a)$. Use the above recurrence relation to find $I_n(a)$.

7. Let σ_N be the surface area of a unit sphere $|x| = 1$ in $\mathbb{R}^N$.
(i) Let Ω_n be an exhaustion of $\mathbb{R}^N$ made of balls $|x| \leq n$. Show that

$$\int_{\Omega_n} e^{-(x,x)} \, d^N x = \sigma_N \int_0^n e^{-r^2} r^{N-1} \, dr$$

(ii) Use this result and the result of Problem 6 to find σ_N and the volume $V_N(a)$ of a ball of radius a in terms of Euler's gamma function.

5. Lebesgue integral

5.1. Piecewise continuous functions on $\mathbb{R}$. Suppose a function f is not continuous at a point $x = c$ and has a *jump discontinuity* at $x = c$. The latter means that the right and left limits of $f(x)$ at $x = c$ exist but are not equal:

$$\lim_{x\to c^+} f(x) = f_+(c)\,, \quad \lim_{x\to c^-} f(x) = f_-(c)\,, \quad f_+(c) \neq f_-(c)$$

A piecewise continuous function is a function that is not continuous at finitely many points in any bounded interval and has jump discontinuities at these points.

First note that points at which a piecewise continuous function has jump discontinuities form a countable set. Indeed a real line can be viewed as the union of countable many intervals and in each such interval the function has finitely many jump discontinuities. So, a collection $\{c_n\}$ of all such points is either finite or form a sequence. The sequence cannot have any limit point because otherwise the function would have infinitely many jump discontinuities in any open interval containing the limit point. In each interval (c_n, c_{n+1}), the function is continuous and has a *continuous extension* to $[c_n, c_{n+1}]$.

Put

$$m = \inf\{c_n\}\,, \qquad M = \sup\{c_n\}$$

If the sequence $\{c_n\}$ is not bounded from below, then $m = -\infty$ and otherwise m is the smallest number in $\{c_n\}$. If the sequence $\{c_n\}$ has no upper bound, then $M = \infty$ and otherwise M is the largest number in the collection $\{c_n\}$. Clearly, if $-\infty < m \leq M < \infty$, that is, the collection $\{c_n\}$ has the smallest and largest number, then the collection must be finite. Let $\{\Omega_n\}$ denote a collection of open intervals (c_n, c_{n+1}) together with $(-\infty, m)$ and (M, ∞) (if these intervals are not empty). This collection of intervals has the following characteristic properties:

(i) the intervals do not overlap:

$$\Omega_n \cap \Omega_{n'} = \emptyset\,, \quad n \neq n'\,,$$

(ii) any bounded interval (a, b) is covered by finitely many closed intervals $\bar{\Omega}_n$:

$$(a,b) \subset \bigcup_{n=j}^{k} \bar{\Omega}_n\,,$$

(iii) the union of closures of the intervals coincides with whole real line:

$$\bigcup_n \bar{\Omega}_n = \mathbb{R}\,.$$

This observation allows us to give an alternative definition of a piecewise continuous function which can be extended to the multivariable case.

5.1.1. Definition of a piecewise continuous function. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is said to be* piecewise continuous *on* $\mathbb{R}$ *if there exists an at most countable collection of open intervals* Ω_n *with no common points such that any bounded interval is covered by finitely many closed intervals* $\bar{\Omega}_n$*, and* $f \in C^0(\bar{\Omega}_n)$.

A piecewise continuous function is not continuous on $\{c_n\}$ which is a set measure zero. One can also say that *a piecewise continuous function is continuous almost everywhere*. Therefore any piecewise continuous function is Riemann integrable on any $[a, b]$. The value of the Riemann integral does not depend on the values of a piecewise continuous function at the points where it is not continuous.

5.2. Measurable functions on $\mathbb{R}$. Let $\mathcal{A}$ be a set of functions that is defined by some characteristic property (e.g., continuity, or integrability, etc.). Then the limit function of a pointwise convergent sequence $\{f_n\} \subset \mathcal{A}$ does not in general belong to $\mathcal{A}$. One can ask how large the set $\mathcal{A}$ should be in order to be *complete* in the sense that the limit function of every pointwise convergent sequence in $\mathcal{A}$ belongs to $\mathcal{A}$. It turns out that such a set of functions exists and is known as a set of *measurable functions*.

Suppose that a sequence $\{f_n\}$ of functions on $\mathbb{R}$ converges pointwise almost everywhere. In other words, a numerical sequence $\{f_n(x)\}$ can have no limit for some points x that form a set of measure zero. In this case, one writes

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e.$$

For example,

$$\lim_{n\to\infty} [\cos(\pi x)]^n = 0 \quad a.e.$$

Note that the limit does not exist if x is an integer. If x is not an integer, then $|\cos(\pi x)| < 1$ and the limit is equal to zero. But the integers form a set of measure zero.

A function f is called measurable *if it coincides almost everywhere with the limit of an almost everywhere convergent sequence of piecewise continuous functions.*

5.2.1. Measurable sets. A set of real numbers is called *measurable* if its characteristic function is measurable. Clearly, any interval (bounded or unbounded, closed or open or semi-open) is measurable. Any set of measure zero is measurable. The following properties of measurable sets can also be established:

- *The complement of a measurable set is measurable.*
- *The union or intersection of countably many measurable sets is measurable.*
- *Every open or closed set is measurable.*

5.3. Properties of measurable functions. [10] Evidently, every piecewise continuous function f is measurable because one can take a sequence of piecewise continuous functions $f_n(x) = f(x)$ of identical terms which obviously converges to $f(x)$. Suppose that f is a measurable function and g coincides with f almost everywhere. Then g is also measurable. Indeed, Let f_n be a sequence of piecewise continuous functions that converges to f almost everywhere. Since f and g differ only on a set of measure zero, f_n converges to g almost everywhere, too:

$$\left.\begin{array}{l} f(x) \text{ is measurable} \\ f(x) = g(x) \ a.e. \end{array}\right\} \quad \Rightarrow \quad g(x) \text{ is measurable}$$

5.3.1. Algebraic operations with measurable functions. Using the basic limit laws, it is not difficult to see that the set of measurable functions is *closed* relative to algebraic operations of addition, multiplication, and division:

$$\left.\begin{array}{l} f(x) \text{ is measurable} \\ g(x) \text{ is measurable} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} f(x)+g(x) \text{ is measurable} \\ f(x)g(x) \text{ is measurable} \\ f(x)/g(x),\ g(x)\neq 0,\ \text{ is measurable} \end{array}\right.$$

Indeed, if $f_n(x)$ and $g_n(x)$ are sequences of piecewise continuous functions, then the functions $f_n(x) + g_n(x)$, $f_n(x)g_n(x)$, and $f_n(x)/g_n(x)$, $g_n(x) \neq 0$, also form sequences of piecewise continuous functions, and the above assertion follows from the basic laws of limits. This also implies that linear combinations of measurable functions are measurable. Sets that are complete relative to additions and multiplications by a number are called a linear space. Thus, *the set of measurable functions is a linear space.*

[10] Proofs of the listed properties of measurable functions can be found in: A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis

5.3.2. Absolute value of a measurable function. Given two functions f and g, define the following functions

$$\max(f,g)(x) = \begin{cases} f(x)\ , & f(x) > g(x) \\ g(x)\ , & f(x) \le g(x) \end{cases}$$

$$\min(f,g)(x) = \begin{cases} g(x)\ , & f(x) > g(x) \\ f(x)\ , & f(x) \le g(x) \end{cases}$$

One can prove that *the functions* $\max(f,g)$ *and* $\min(f,g)$ *are measurable, if* f *and* g *are measurable.* It follows that *the absolute value*

$$|f(x)| = \max(f,0)(x) - \min(f,0)(x)$$

of a measurable function f *is measurable.*

5.3.3. Measurable and Riemann integrable functions. One can prove the following property

- *A function that is not continuous on a set of measure zero is measurable.*

Therefore every Riemann integrable function is measurable by Theorem **3.1**. Furthermore, every function for which the improper Riemann integral exists is also measurable. So, *the set of measurable functions contains all Riemann integrable functions (either in the proper or improper sense).*

There are measurable functions that are not Riemann integrable. For example, the Dirichlet function introduced in Section **3.2.3** is measurable but not Riemann integrable on any interval. The set $\mathbb{Q}$ of rational numbers has measure zero in $\mathbb{R}$. Therefore $f_D(x) = 0$ a.e., but any constant function and, in particular, $g(x) = 0$ is measurable and, hence, so is the Dirichlet function.

5.3.4. Composition of measurable functions. A composition of measurable functions is measurable

5.3.5. Completeness of the set of measurable functions.

THEOREM **5.1**. *A function that coincides almost everywhere with the limit of an almost everywhere convergent sequence of measurable functions is measurable.*

5.3.6. Non-measurable sets and functions. Thus, the set of measurable functions is quite large. Are there non-measurable functions and sets? It appears that one can prove that they exist[11] using the so called *axiom of choice*:

[11] see, e.g., A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5, Sec. 1.3

- Let $\{E_a\}$ be a collection of subsets of a set E (the indexing set a is of arbitrary nature). Then there exists a *choice function*, $a \to x(a)$ where $x(a) \in E_a$ for all a.

No example of an explicit non-measurable functions has been constructed so far. This suggests that all functions and sets that can possibly be used in applications or otherwise are measurable. For this reason, in what follows all sets are assumed to be measurable and all functions are assumed to be measurable and bounded almost everywhere. A function f is not bounded at x_0 if $|f(x)| \to \infty$ as $x \to x_0$. The set of all such points is a set of measure zero for a function that is bounded *a.e.*

5.4. Definition of the Lebesgue integral. To avoid any confusion between Riemann and Lebesgue integrals, the Riemann integral (proper or improper) will be denoted as

$$\mathcal{R}\int_\Omega f(x)\,dx\,, \quad \Omega \subseteq \mathbb{R}\,,$$

in what follows.

The main deficiency of the Riemann integral is its sensitivity to alterations of an integrable function on sets of measure zero (cf. Sec.**3.3.1**). To find a remedy, one should get rid of the definition of the integral via the lower and upper sums because there are sets of measure zero densely defined in any interval (cf. Sec.**1.1.5**) so that by altering values of an integrable function on a set of measure zero, one can drive the upper (or lower) sum to any limit, thus destroying Riemann integrability. Since practically any function can be obtained as the pointwise limit of an almost everywhere convergent sequence of piecewise continuous functions, that are integrable on any interval, one could define a new integral of a function f as the limit of Riemann integrals of piecewise continuous functions that converge to f. Then by properties of measurable functions (see Sec. **5.3**), alterations of the function f on a set of measure zero would not change the value of the integral because the latter is defined by the limit of the *same* sequence of Riemann integrals of piecewise continuous functions.

To make this work, one should make sure that the said sequence of Riemann integrals converges. If the function is bounded from below (e.g., $f(x) \geq 0$ a.e.), then it can be obtained as the limit of non-decreasing sequence of piecewise continuous functions. Therefore the sequence of Riemann integrals is monotonically increasing and has a limit, provided it is bounded. Similarly, the integral of any measurable function that is bounded from above can be defined as the limit of a non-increasing sequence of Riemann integrals if the latter is bounded. Furthermore, any measurable function can be written as a difference of two measurable functions bounded from above and below (see Sec.**5.3.2**). Therefore, the integral of any measurable function can be defined as the difference of two integrals, just like the improper Riemann integral in Sec. **4.3**. By construction, the new and Riemann integrals coincide for piecewise continuous functions on any bounded set. Furthermore, one would anticipate that the new integral also coincides with *an absolutely convergent Riemann integral whenever the latter exists*. Finally, the new integral does not change if the integrand is altered on any set of measure zero. Let us formalize the idea.

5.4.1. The space $\mathcal{L}_+$. Let a real function $f(x)$ be the limit of a non-decreasing sequence of piecewise continuous functions $f_n(x)$ almost everywhere such that the sequence of Riemann integrals is bounded:

$$\begin{aligned} &f_n(x) \le f_{n+1}(x)\,, \quad n = 1, 2, ..., \,, \quad \forall x \in \mathbb{R}\,, \\ &\mathcal{R}\!\int f_n(x)\,dx \le M\,, \quad n = 1, 2, \dots\,, \end{aligned}$$

for some number M. The limit of the non-decreasing sequence of Riemann integrals is called *the Lebesgue integral* of f and is denoted by the symbol $\int f(x)dx$ so that

$$\int f(x)\,dx = \lim_{n\to\infty} \mathcal{R}\!\int f_n(x)\,dx\,.$$

The set of all such functions is denoted by $\mathcal{L}_+$.

5.4.2. The space of Lebesgue integrable function $\mathcal{L}$. A function f is called *Lebesgue integrable* if it can be represented as the difference of two functions from the set $\mathcal{L}_+$:

$$f(x) = f_1(x) - f_2(x)\,, \quad f_1 \in \mathcal{L}_+\,,\ f_2 \in \mathcal{L}_+$$

The number

$$\int f_1(x)\,dx - \int f_2(x)\,dx = \int f(x)\,dx$$

is called *the Lebesgue integral* of the function f. The set of all Lebesgue integrable functions is denoted by $\mathcal{L}$.

5.4.3. The Lebesgue integral over a set. A function f is said to be Lebesgue integrable on a measurable set $\Omega \subset \mathbb{R}$, if $f\chi_\Omega \in \mathcal{L}$, where χ_Ω is the characteristic function of Ω, and the number

$$\int f(x)\chi_\Omega(x)\,dx = \int_\Omega f(x)\,dx$$

is called the Lebesgue integral of f over Ω. The class of all Lebesgue integrable functions is denoted by $\mathcal{L}(\Omega)$.

5.4.4. Consistency of the definition. Definition **5.4.1** makes sense only if the Lebesgue integral does not depend on the choice of the sequence $\{f_n\}$. Similarly, Definition **5.4.2** is consistent if the Lebesgue integral is independent of the choice of f_1 and f_2. This is indeed so as shown in Appendix.

5.5. Riemann and Lebesgue integrals in $\mathbb{R}$. If the Lebesgue integral of a piecewise continuous function f over any bounded interval coincides with the Riemann integral because any such function is from class $\mathcal{L}_+$:

$$f \in C^0[a,b] \quad \Rightarrow \quad \int_a^b f(x)\,dx = \mathcal{R}\int_a^b f(x)\,dx$$

Note that one can take $f_n(x) = f(x)\chi_{[a,b]}(x)$ in Definition **5.4.1**.

5.5.1. Lebesgue integrability and sets of measure zero. One of the key differences between the Lebesgue and Riemann integrals is that alterations of an integrable function on a set measure zero does not affect integrability and the value of the integral does not change. Let $f(x) = 0$ a.e. Put $f_n(x) = 0$ in Definition **5.4.1**. Clearly f_n converges to f almost everywhere. Therefore

$$f(x) = 0 \text{ a.e.} \quad \Rightarrow \quad \int_\Omega f(x)\,dx = 0$$

In particular, the Lebesgue integral of the Dirichlet function vanishes over any (measurable) set

$$\int_\Omega f_D(x)\,dx = 0$$

because $f_D(x) = 0$ a.e.

One can show that the converse is also true if f is a non-negative function[12]

Proposition **5.1**. *Let $f(x) \geq 0$. Then its Lebesgue integral vanishes if and only if $f(x) = 0$ almost everywhere.*

It follows from linearity of Lebesgue integral that if $f \in \mathcal{L}$ and g differs from f only on a set of measure zero, then g is also integrable

[12] see, e.g., A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5.

and its integral is equal to the integral of f:

$$f \in \mathcal{L}\,, \quad g(x) = f(x) \text{ a.e.} \quad \Rightarrow \quad g \in \mathcal{L}\,, \ \int g(x)\,dx = \int f(x)\,dx$$

Thus, in full contrast to the Riemann integral, the Lebesgue integral is insensitive to alterations of an integrable function on sets of measure zero. Note that if f is continuous and $g(x) = f(x)$ a.e., then g can be continuous nowhere, just like the Dirichlet function, and hence g may not even be Riemann integrable.

5.5.2. Lebesgue integrability of Riemann integrable functions. Let us show that *any function f that is Riemann integrable on $[a, b]$ is Lebesgue integrable and*

$$\mathcal{R}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

As noted earlier, for any Riemann integrable function f there exists a sequence of partitions P_n such that P_{n+1} is a refinement of P_n and

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = \mathcal{R}\int_a^b f(x)\,dx$$

Define two sequences of piecewise constant functions

$$L_n(x) = m_s\,, \quad x \in R_s\,, \qquad U_n(x) = M_s\,, \quad x \in R_s$$

where R_s are partition intervals for P_n. Then

$$L_n(x) \le L_{n+1}(x) \le f(x) \le U_{n+1}(x) \le U_n(x)$$

The sequence $\{L_n(x)\}$ is monotonically increasing and bounded from above, and the $\{U_n(x)\}$ is monotonically decreasing and bounded from below. Therefore they converge for all x:

$$\lim_{n\to\infty} L_n(x) = L(x)\,, \quad \lim_{n\to\infty} U_n(x) = U(x)\,.$$

and

$$L(x) \le f(x) \le U(x)\,, \quad a \le x \le b\,.$$

The limit function L is Lebesgue integrable because the sequence of its Riemann integrals is nothing but the sequence of lower sums for f:

$$\int_a^b L(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int_a^b L_n(x) = \lim_{n\to\infty} L(P_n, f)$$

Similarly, the function $-U(x)$ is also Lebesgue integrable because the sequence $\{-U_n\}$ satisfies the conditions in Definition **5.4.1**. Therefore U is Lebesgue integrable and

$$\int_a^b U(x)\,dx = \lim_{n\to\infty} \mathcal{R}\int_a^b U_n(x)\,dx = \lim_{n\to\infty} U(P_n, f)$$

Thus, the Lebesgue integrals of U and L are equal, and therefore the integral of a non-negative function $U(x) - L(x) \geq 0$ vanishes. By Proposition **5.1**, this implies that $U(x) = L(x)$ a.e. and, hence,

$$f(x) = L(x) \quad \text{a.e.}$$

from which it follows that f is Lebesgue integrable and

$$\int_a^b f(x)\,dx = \mathcal{R}\int_a^b f(x)\,dx\,.$$

5.5.3. Lebesgue and improper Riemann integrals. Suppose that f is not bounded on (a,b) but $f \in C^0(a,b)$ (f is singular at one or both endpoints of the interval). Suppose that the improper Riemann integral of f over (a,b). This implies that

$$\mathcal{R}\lim_{n\to\infty}\int_{a_n}^{b_n} f_\pm(x)\,dx = \mathcal{R}\int_a^b f_\pm(x)\,dx < \infty$$

for an exhaustion $[a_n, b_n] \subset (a,b)$ where a_n and b_n converge monotonically to a and b, respectively. It follows that $f_\pm$ are from class $\mathcal{L}_+$ because they are limits of monotonically increasing sequences of piecewise continuous functions $\chi_n(x)f_\pm(x)$ where χ_n is the characteristic function of $[a_n, b_n]$ whose Riemann integrals are bounded. Since $f(x) = f_+(x) - f_-(x)$, it is concluded that f is Lebesgue integrable on (a,b) and its Lebesgue integral is equal to the improper Riemann integral. Conversely, if a continuous function is Lebesgue integrable, then its Riemann integral converges absolutely is equal to the Lebesgue integral. Clearly, the argument can readily be extended to a continuous (or piecewise continuous) function on an unbounded interval. So, for *continuous (or piecewise continuous) functions, the Lebesgue and absolutely convergent Riemann integrals are equivalent.*

In fact, a more general assertion is true (see Sec. **6.10**).

PROPOSITION **5.2**. *If the function $f(x)$ and $|f(x)|$ are Riemann integrable on a set Ω (possibly in the improper sense), then they are Lebesgue integrable on Ω, and their Lebesgue and Riemann integrals are equal:*

$$\mathcal{R}\int_\Omega f_\pm(x) < \infty\,dx \quad \Rightarrow \quad \int_\Omega f(x)\,dx = \mathcal{R}\int_\Omega f(x)\,dx$$

where $f_\pm(x) = \frac{1}{2}(|f(x)| \pm f(x))$.

Thus, any function g that coincides almost everywhere with an absolutely Riemann integrable function f is Lebesgue integrable and, in this case, the Lebesgue integral of g is equal to the Riemann integral of f.

5.5.4. The Lebesgue integral of a complex-valued function. A complex-valued function $f(x)$, $x \in \mathbb{R}$, is said to be integrable if its real and imaginary parts are integrable, and in this case

$$\int f(x)dx = \int \mathrm{Re}\, f(x)\, dx + i \int \mathrm{Im}\, f(x)\, dx\,.$$

It follows from Proposition **5.2** that if the Riemann integral of a complex-valued function converges absolutely, then the function is Lebesgue integrable and its Lebesgue and Riemann integrals are equal.

5.6. Lebesgue integral in $\mathbb{R}^N$. The Lebesgue integral in any Euclidean space is defined in a similar way, that is, as a limit of Riemann integrals of piecewise continuous functions. However, the notion of a piecewise continuous function of several variables requires a refinement related to the boundary of sets of continuity of the function.

5.6.1. Piecewise continuous functions on $\mathbb{R}^N$. Recall that a region is an open connected set $\mathbb{R}^N$. A function f is called *piecewise continuous* in $\mathbb{R}^N$ if

(i) there is at most countably many non-intersecting regions Ω_n, $n = 1, 2, ...,$
(ii) with piecewise smooth boundaries $\partial\Omega_n$,
(iii) any ball is contained in the union of finitely many closed regions $\overline{\Omega}_n$,
(iv) the union of $\overline{\Omega}_n$ coincides with $\mathbb{R}^N$, and
(v) $f \in C^0(\overline{\Omega}_n)$

This definition is to be compared with the definition of a piecewise continuous function on $\mathbb{R}$. Regions Ω_n are analogs of open intervals.

A piecewise continuous function is continuous almost everywhere and at any point where it is not continuous the function can only have a jump discontinuity. A piecewise continuous function is bounded on any ball. Therefore a piecewise continuous function with a bounded support is Riemann integrable on $\mathbb{R}^N$.

5.6.2. Definition of the Lebesgue integral in $\mathbb{R}^N$. Let a real-valued function f coincide almost everywhere with the limit of a non-decreasing sequence of piecewise continuous functions $f_n(x)$,

$$f_n(x) \le f_{n+1}(x)\,, \quad \forall x \in \mathbb{R}^N\,, \quad n = 1, 2, ..$$

such that the sequence of the Riemann integrals is bounded:

$$\mathcal{R}\int f_n(x)\, d^N x \le M\,,$$

for all n, where the Riemann integral is understood in the improper sense if supports of f_n are not finite. The limit

$$\lim_{n\to\infty} \mathcal{R}\int f_n(x) d^N x = \int f(x) d^N x < \infty$$

of this non-decreasing bounded sequence is called the Lebesgue integral of f. The set of such functions is denoted by $\mathcal{L}_+$. A real function f is called Lebesgue integrable if it can be represented as the difference of two functions from $\mathcal{L}_+$, $f = f_1 - f_2$, $f_{1,2} \in \mathcal{L}_+$ and

$$\int f(x) d^N x = \int f_1(x) d^N x - \int f_2(x) d^N x\,.$$

The set of Lebesgue integrable functions is denoted by $\mathcal{L}$. The proof of consistency of the Lebesgue integral over $\mathbb{R}$ given in Appendix **1.2** is easily extended to $\mathbb{R}^N$ by replacing all intervals in $\mathbb{R}$ by the corresponding balls in $\mathbb{R}^N$.

Similarly to the one dimensional case, a function f is said to be from $\mathcal{L}(\Omega)$ if the function $\chi_\Omega f \in \mathcal{L}$, where χ_Ω is the characteristic function of the set Ω and, in this case,

$$\int_\Omega f\, d^N x = \int \chi_\Omega f\, d^N x\,.$$

5.6.3. Lebesgue and Riemann integrability in $\mathbb{R}^N$. Let Ω be a region in $\mathbb{R}^N$ and $f \in C^0(\Omega)$. *Then $f \in \mathcal{L}(\Omega)$ if and only if its Riemann integral over Ω converges absolutely, that is, if and only if*

$$\lim_{n\to\infty} \mathcal{R}\int_{\Omega_n} |f(x)|\, d^N x < \infty$$

for some exhaustion (or regularization) $\{\Omega_n\}$ of Ω, and, in this case,

$$\int_\Omega f(x)\, d^N x = \lim_{n\to\infty} \mathcal{R}\int_{\Omega_n} f(x)\, d^N x\,.$$

A proof of this assertion is left to the reader (cf. Sec. **5.5.3**).

Proposition **5.2** can be extended to integrals in $\mathbb{R}^N$. Let f be an absolutely Riemann integrable function and $g(x) = f(x)$ a.e.. Then g is Lebesgue integrable and its integral is equal to the Riemann (improper) integral of f.

5.7. Exercises.

1. Let $f(x) = 0$ if x is rational and $f(x) = e^{-x}$ otherwise. Find the Lebesgue integral

$$\int_0^\infty f(x)\,dx$$

or show that it does not exist.

2. Let L_Q be a collection of lines through the origin in $\mathbb{R}^2$ such that the angle between any two lines is equal πq where q is a rational number. Let $f(x) = 0$ if $x \in L_Q$ and $f(x) = e^{-|x|^2}$ otherwise. Investigate the existence of the integrals

$$\int f(x)\,d^2x\,, \quad \mathcal{R}\int f(x)\,d^2x$$

and, if an integral exists, find its value.

3. Which of the following functions are Lebesgue integrable on $\mathbb{R}$:

$$\frac{\sin(x)}{x}\,, \quad \frac{e^{ikx}}{x}\,, \quad \frac{\cos(x)}{\sqrt{|x|}}\,, \quad e^{-x}\,, \quad x^{100}e^{-x^2}$$

4. A function is said to be Lebesgue square integrable on Ω, or from the space $\mathcal{L}_2(\Omega)$, if $|f|^2 \in \mathcal{L}(\Omega)$. Which of the functions from Problem 3 are square integrable?

5. Let f be continuous and Lebesgue integrable on $\mathbb{R}^N$. Show that its Fourier transform

$$\mathcal{F}[f](k) = \int e^{i(k,x)}\,f(x)\,d^Nx$$

exists for any $k \in \mathbb{R}^N$.

6. Properties of the Lebesgue integral in $\mathbb{R}^N$

Properties of the Lebesgue integral are analogous to the properties of the Riemann integral (cf. Sec. **3.4**) because the Lebesgue integral coincides with the absolutely convergent Riemann integral whenever the latter exists (Proposition **5.2**).

The key difference between the Lebesgue and Riemann integrals is that the Lebesgue integral is insensitive to alterations of the integrand on sets of measure zero, whereas the Riemann integrability can be lost after such alterations. Apart from eliminating the aforementioned deficiency in the definition of volume of a set (cf. Sec.**3.3.1**), this also leads to simplifications of theorems about integrability of the limit function of a functional sequence. In particular, the hypotheses of the uniform convergence can be weakened and simplified, which is a major advantage of the Lebesgue integral in applications.

In what follows, a Lebesgue integrable function will be called just integrable and integrals are always understood in the Lebesgue sense (unless stated otherwise) and the term integrability means integrability in the Lebesgue sense. In mathematical literature, Lebesgue integrable functions are often called *summable* to distinguish them from integrable functions in the Riemann sense.

6.1. The set $\mathcal{L}$ is a linear space. If f and g are integrable, then their linear combination is also integrable and

$$\int \Big(c_1 f(x) + c_2 g(x)\Big)\, d^N x = c_1 \int f(x)\, d^N x + c_2 \int g(x)\, d^N x\,.$$

So, the set $\mathcal{L}(\Omega)$ of Lebesgue integrable functions on $\Omega \subset \mathbb{R}^N$ is a linear space. This property follows from the limit laws. If $\{f_n\}$ and $\{g_n\}$ are sequences of piecewise continuous functions that define the integrals of f and g, then by linearity of the Riemann integral the sequence $c_1 f_n + c_2 g_n$ defines the integral of the linear combination $c_1 f + c_2 g$.

6.2. Monotonicity. Suppose that f and g are integrable. Then[13]

$$f(x) \geq 0 \quad \Rightarrow \quad \int f(x)\, d^N x \geq 0$$

and, as a consequence,

$$f(x) \geq g(x) \quad \Rightarrow \quad \int f(x)\, d^N x \geq \int g(x)\, d^N x$$

[13]see Proposition **1.1** in Appendix

6.3. Integrals on sets of measure zero. The Lebesgue integral is insensitive to alterations of a function on sets of measure zero. If $f \in \mathcal{L}$, then every function that coincides with f almost everywhere is also integrable and its Lebesgue integral has the same value. Similarly, if f is not Lebesgue integrable, then any other function that differs from f on a set of measure zero is also non-integrable. In other words,

$$f(x) = g(x) \ \ a.e. \quad \Rightarrow \quad \int f(x)\, d^N x = \int g(x)\, d^N x$$

and both the integrals either exist or do not exist simultaneously. As noted earlier, *this property is not true for the Riemann integral.*

In particular, if the integral of any (measurable) function over a set of measure zero vanishes:

$$\chi_\Omega(x) f(x) = 0 \ a.e. \quad \Rightarrow \quad \int_\Omega f(x)\, d^N x = \int \chi_\Omega(x) f(x)\, d^N x = 0\,.$$

6.4. Additivity of the Lebesgue integral. Suppose that f is integrable on Ω and Ω' and the intersection $\Omega \cap \Omega'$ is a set of measure zero. Then f is integrable on the union $\Omega \cup \Omega'$ and

$$\int_{\Omega\cup\Omega'} f(x)\, d^N x = \int_\Omega f(x)\, d^N x + \int_{\Omega'} f(x)\, d^N x\,.$$

This follows from that

$$\chi_{\Omega\cup\Omega'}(x) = \chi_\Omega(x) + \chi_{\Omega'}(x) \quad \text{a.e.}$$

and the linearity of the Lebesgue integral.

6.5. Lebesgue measure of a set. If the characteristic function of a set $\Omega \subset \mathbb{R}^N$ is integrable, then the number

$$\mu(\Omega) = \int \chi_\Omega(x) d^N x$$

is called the *Lebesgue measure* of Ω. For example, if Ω is a bounded region with a smooth boundary, then its characteristic function is piecewise continuous and the Lebesgue measure is equal to the volume of Ω defined by the Riemann integral. If Ω is not bounded, then the volume is defined by the improper Riemann integral. The volume can be infinite if the improper Riemann integral diverges.

In general, a characteristic function of a measurable set is measurable. So, every bounded measurable set Ω has the Lebesgue measure. If a measurable set is not bounded and its characteristic function is not integrable, then the set is said to have infinite measure $\mu(\Omega) = \infty$ (similarly to sets of infinite volume). Thus, in contrast to the volume, the Lebesgue measure is defined on all measurable sets if it is allowed to have the infinite value. The Lebesgue measure has the following properties similarly to the volume[14].

6.5.1. Positivity. The Lebesgue measure is non-negative function of a set:

$$\mu(\Omega) \geq 0$$

and it vanishes if and only if Ω is a set of measure zero.

6.5.2. Monotonicity. The Lebesgue measure is increasing with enlarging the set:

$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \mu(\Omega_1) \leq \mu(\Omega_2)\,.$$

In particular,

$$\mu(\Omega) \leq \mu(\bar{\Omega})$$

and the strict inequality is also possible. Let $\Omega \subset \mathbb{R}$ consists of all rational numbers in an interval $[a, b]$. Evidently $\chi_\Omega(x) = 0$ a.e. and $\mu(\Omega) = 0$. However, $\bar{\Omega} = [a, b]$ and $\mu(\bar{\Omega}) = b - a > 0$.

6.5.3. Countable additivity. If a set is the union of countably many non-intersecting sets, then its measure is the sum of measures of sets in the union:

$$\Omega = \bigcup_n \Omega_n\,, \quad \Omega_n \cap \Omega_m = \emptyset\,,\ n \neq m \quad \Rightarrow \quad \mu(\Omega) = \sum_n \mu(\Omega_n)$$

In particular, the Lebesgue measure of Ω does not change when a set of measure zero is removed from Ω:

$$\mu(\Omega \setminus \Omega') = \mu(\Omega) \quad \text{if} \quad \mu(\Omega') = 0$$

For example, if Ω is a bounded region with a piecewise smooth boundary, then $\bar{\Omega} = \Omega \cup \partial\Omega$ has the same Lebesgue measure.

6.5.4. Measure of an unbounded set. Let Ω be an unbounded set. Let Ω_n a sequence of subsets of a finite measure such that $\Omega_n \subset \Omega_{n+1}$ for any n and the union of all Ω_n coincides with Ω up to a set of measure zero. Then it follows from the countable additivity that

$$\Omega = \bigcup_n \Omega_n\,,\ \Omega_n \subset \Omega_{n+1} \quad \Rightarrow \quad \mu(\Omega) = \lim_{n\to\infty} \mu(\Omega_n)\,.$$

This procedure can be used to evaluate the measure of unbounded sets. The limit either exists or is infinite and does not depend on the choice

[14] Proofs of the properties of the Lebesgue measure can be found in: A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5, Sec. 1

of exhaustion of Ω (similarly to the absolutely convergent Riemann integrals). For example, $\Omega_n = \Omega \cap B_n$ where B_n is a ball of radius n.

6.5.5. Continuity. Let $\{\Omega_n\}$ be a sequence of set embedded into one another $\Omega_{n+1} \subset \Omega_n$ and Ω is the intersection of all Ω_n. Then

$$\Omega = \bigcap_n \Omega_n\,,\ \Omega_{n+1} \subset \Omega_n \quad \Rightarrow \quad \mu(\Omega) = \lim_{n\to\infty} \mu(\Omega_n)$$

For example, if Ω is a bounded set, then one can take Ω_a to be the union of open ball of radius a that are centered at every point of Ω. Then $\mu(\Omega_a) \to \mu(\Omega)$ as $a \to 0^+$.

6.5.6. Geometrical properties of measurable sets. The *symmetric difference* of two sets A and B is defined by

$$A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$$

So, it consists of elements that are either in A or B but not in their intersection. Therefore, if $\mu(A \bigtriangleup B) = 0$, then A and B differs at most by a set of measure zero.

Two rectangular boxes are called *almost disjoint* if their interiors do not intersect. In other words, two almost disjoint boxes can have a non-empty intersection of their boundaries. A set is called *elementary* if it can be represented as a finite union of almost disjoint rectangular boxes. The measure of an elementary set is just its volume. One can show that

- the union, intersection, difference, and symmetric difference of two elementary sets is elementary;
- the union, intersection, set difference, and symmetric difference of two elementary sets is elementary.

Any measurable set has the following characteristic property. For any measurable set A and any $\varepsilon > 0$, there exists an elementary set B such that

$$\mu(A \bigtriangleup B) < \varepsilon$$

In other words, any measurable set in $\mathbb{R}^N$ can be "approximated with any desired accuracy" by a finite collection of almost disjoint boxes.

6.6. Upper and lower bounds. Suppose that $f \in \mathcal{L}(\Omega)$ and f is bounded almost everywhere in Ω, then

$$m \le f(x) \le M \quad a.e. \quad \Rightarrow \quad m\mu(\Omega) \le \int_\Omega f(x)\, d^N x \le M\mu(\Omega)\,.$$

A similar property also holds for the Riemann integral over an interval (without a.e.).

6.7. Integrability of the absolute value. If $f \in \mathcal{L}$, then $|f| \in \mathcal{L}$. If f is measurable and $|f| \in \mathcal{L}$, then $f \in \mathcal{L}$ and

$$\left|\int f(x) d^N x\right| \leq \int |f(x)| d^N x \,.$$

In view of the early remark about non-measurable functions, the integrability of f and $|f|$ is practically equivalent in the Lebesgue theory. So, *if f is measurable, then the integrals*

$$\int f(x)\, dx \quad \text{and} \quad \int |f(x)|\, dx$$

exist or do not exist simultaneously. This property does not hold for the Riemann integral (see Sec. **3.4.3**).

6.8. Vanishing integral of the absolute value. Recall that, if f is continuous and the Riemann integral of the absolute value $|f|$ vanishes, then $f(x) = 0$. The converse is obviously true. The Lebesgue integral has a similar property that follows from Proposition **5.1**: *if $f \in \mathcal{L}$ and the integral of $|f|$ vanishes, then $f(x) = 0$ almost everywhere (and the converse is obviously true)*:

$$f \in \mathcal{L}\,, \quad \int |f(x)| dx = 0 \qquad \Leftrightarrow \qquad f(x) = 0\,, \quad a.e.$$

6.9. Comparison test for integrability. If a function g is integrable on Ω and $|f(x)| \leq g(x)$ a.e., then f is also integrable on Ω:

$$|f(x)| \leq g(x)\ a.e.\,, \quad g \in \mathcal{L}(\Omega) \quad \Rightarrow \quad f \in \mathcal{L}(\Omega)$$

This implies that *any bounded (and measurable) function is Lebesgue integrable on any bounded (and measurable) set.* Indeed,

$$|f(x)| \leq M\ a.e. \quad \Rightarrow \quad \int_\Omega |f(x)|\, d^N x \leq M\mu(\Omega) < \infty$$

because Ω is bounded.

For example,

$$\left|\sin\left(\frac{1}{|x|^p}\right)\right| \leq 1 \quad \text{a.e.}$$

for any real p. Therefore $\sin(|x|^{-p})$ in integrable on any bounded interval. If $p > 0$, the function is not defined at $x = 0$. One can assign any value to the function at $x = 0$. The Lebesgue integral does not change.

Let $|f(x)| \to \infty$ for $x \to x_0$ and f be integrable on $\Omega \setminus B_R(x_0)$, then f is integrable on $\Omega \subset \mathbb{R}^N$ if

$$|f(x)| \le \frac{M}{|x - x_0|^p} \quad a.e.\,, \quad p < N\,,\ x \in B_R(x_0)$$

because the Riemann integral of the right side of this inequality was shown to converge absolutely. Similarly, let Ω be not bounded and $|f(x)| \to \infty$ for $|x| \to \infty$. Let f be integrable on $\Omega \cap B_R$ for some ball B_R. Then f is integrable on Ω if

$$|f(x)| \le \frac{M}{|x|^p} \quad a.e., \quad |x| > R\,,\ p > N$$

for some M.

6.10. Absolute continuity of the Lebesgue integral. Consider the Lebesgue integral as a function of the integration set:

$$F(\Omega) = \int_\Omega f(x)\, d^N x\,. \tag{6.1}$$

The function F has the following properties[15].

THEOREM **6.1**. *Suppose that*

$$\Omega = \bigcup_n \Omega_n\,, \quad \Omega_k \cap \Omega_n = \emptyset\,,\ k \ne n$$

and f is integrable on Ω. Then f is integrable on any Ω_n and

$$\int_\Omega f(x)\, d^N x = \sum_n \int_{\Omega_n} f(x)\, d^N x \tag{6.2}$$

where there the series converges absolutely. Conversely, if f is integrable on every Ω_n and the series

$$\sum_n \int_{\Omega_n} |f(x)|\, d^N < \infty$$

*converges, then f is integrable on Ω and relation (**6.2**) holds.*

There are a few consequences that can be deduced from this theorem. A measurable set Ω in Theorem **6.1** is represented as the union of arbitrary non-intersecting measurable sets. Therefore *the Lebesgue integrability on Ω implies the Lebesgue integrability on any measurable subset of* Ω:

$$f \in \mathcal{L}(\Omega)\,, \quad \Omega' \subset \Omega \quad \Rightarrow \quad f \in \mathcal{L}(\Omega')$$

[15] A proof can be found in: A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter 5, Sec. 5

Note that if f is bounded almost everywhere and Ω is bounded, then this conclusion follows from the comparison test $|f(x)| \leq M\chi_\Omega(x)$ a.e. and that $\mu(\Omega') \leq \mu(\Omega) < \infty$.

The convergence of the series in (**6.2**) implies that the terms of the series must tend to zero. Therefore for any function $f \in \mathcal{L}(\Omega)$ one can find a measurable subset $\Omega' \subset \Omega$ such that the integral of f over Ω' is arbitrary small. This property is known as the *absolute continuity of the Lebesgue integral*.

THEOREM **6.2**. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left|\int_{\Omega'} f(x)\, d^N x\right| < \varepsilon \quad \text{whenever} \quad \mu(\Omega') < \delta\,,\ \Omega' \subset \Omega\,.$$

The assertion is obvious if f is *bounded almost everywhere* on Ω because

$$\left|\int_{\Omega'} f(x)\, d^N x\right| \leq \int_{\Omega'} |f(x)|\, d^N x \leq M\mu(\Omega')$$

if $|f(x)| \leq M$ *a.e.*. In this case, $\delta = \varepsilon/M$.

By the absolute continuity of the Lebesgue integral, if $f \in \mathcal{L}(\Omega)$, then for any $\varepsilon > 0$ one can find a proper subregion Ω' or a region Ω (cf. Sec. **1.1.10**) such that

$$\int_{\Omega\setminus\Omega'} |f(x)|\, d^N x < \varepsilon\,.$$

For example, if Ω is bounded, then Ω' can be constructed by removing closed balls of sufficiently small radius from Ω whose centers are on the boundary $\partial\Omega$.

6.10.1. Lebesgue integral over unbounded regions. Let $\{\Omega_n\}$ be an exhaustion of a region Ω. If f is integrable on Ω, then

$$\lim_{n\to\infty} \int_{\Omega_n} f(x)\, d^N x = \int_\Omega f(x)\, d^N x\,. \tag{6.3}$$

In contrast to the continuity of the Riemann integral, the integrability of f on every Ω_n is redundant because f is integrable on any measurable subset of Ω. Conversely, suppose one wants to investigate integrability of f on Ω. If $f \in \mathcal{L}(\Omega_n)$ for all n, then its absolute value is also integrable on every Ω_n. By the second part of Theorem **6.1**, if the limit

$$\lim_{n\to\infty} \int_{\Omega_n} |f(x)|\, d^N x < \infty$$

exists (not infinite), then f is integrable on Ω and (**6.3**) holds. In what follows, this continuity property will often be used to investigate integrability in combination with comparison tests.

6.10.2. Asymptotic behavior of integrable functions. If a function f of a real variable x is integrable on $\mathbb{R}$, then

$$\int |f(x)|\,dx = \lim_{n\to\infty} \int_{\Omega_n} |f(x)|\,dx < \infty$$

for any exhaustion $\{\Omega_n\}$ of $\mathbb{R}$. Does this imply that $f(x) \to 0$ as $|x| \to \infty$? The answer is *negative*. It is not difficult to find an example of an integrable function that has no limit as $|x| \to \infty$.

Let $f(x) = 0$ if $x < 0$ and for $x > 0$ the function is piecewise constant. In each interval $(n-1, n)$, $n = 1, 2, ...$, $f(x)$ is not zero in a subinterval of length $a_n \le 1$ where $f(x) = h_n > 0$. Let the sequence $h_n \to \infty$ for $n \to \infty$. Then, evidently, $f(x)$ has no limit as $x \to \infty$. It is not even bounded on (R, ∞) for any $R > 0$. But with a suitable choice of a_n, f is integrable. For example, let $h_n = n$. Put $a_n = \frac{1}{n^3}$. Then

$$\int f(x)\,dx = \sum_{n=1}^{\infty} a_n h_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty\,.$$

It is clear that similar examples can be constructed for smooth functions.

So, an integrable function does not tend to zero in the asymptotic region, but its average over any interval $(R, R+a)$ and the average of its absolute value tend to zero when $R \to \infty$ for any length a. Indeed, given $a > 0$, one has

$$\int f(x)\,dx = \sum_{n=-\infty}^{\infty} \int_{(n-1)a}^{an} f(x)\,dx\,.$$

Since $|f(x)|$ is integrable for any integrable f, the same series representation holds for the integral of $|f(x)|$. By the convergence of the series, their terms must tend to zero:

$$\lim_{|n|\to\infty} \int_{(n-1)a}^{an} |f(x)|\,dx = 0\,, \qquad \lim_{|n|\to\infty} \int_{(n-1)a}^{an} f(x)\,dx = 0\,.$$

This observation can be extended to Lebesgue integrable functions in $\mathbb{R}^N$. For example, let $R(x_c)$ be a box with fixed dimensions centered at x_c (the point of intersection of main diagonals of the box). Then the average of an integrable function over $R(x_c)$ tends to zero when $|x_c| \to \infty$. The integral of an integrable function in $\mathbb{R}^N$ over a spherical shell $B_{R+a}\backslash B_R$ tends to zero for $R \to \infty$ for any $a > 0$ but the values of $f(x)$ may not even be bounded for $|x| > R$.

6.11. The space $\mathcal{L}_{\rm loc}(\Omega)$ of locally integrable functions. Let $\Omega \subset \mathbb{R}^N$ be an open set and f be continuous on Ω. Then f is integrable on any proper subset Ω' of Ω because $\overline{\Omega'} \subset \Omega$ and Ω' is bounded (see Sec. **1.1.10**). However, f is not necessarily integrable on Ω. Recall that in this case, the integrability means that the Riemann integral of f converges absolutely on Ω if f has singular points on the boundary of Ω, or Ω is not bounded, or both. A function f is called *locally integrable on an open set* $\Omega \subseteq \mathbb{R}^N$ if it is integrable on any proper bounded subset of Ω. The class of such function is denoted by $\mathcal{L}_{\rm loc}(\Omega)$ or simply by $\mathcal{L}_{\rm loc}$ if $\Omega = \mathbb{R}^N$:

$$f \in \mathcal{L}_{\rm loc}(\Omega) \;\; : \quad \int_{\Omega'} |f(x)|\, d^N x < \infty$$

for any proper subset Ω' of Ω.

6.12. The space $\mathcal{L}(\Omega;\sigma)$. Let σ be a non-negative integrable function on Ω. Then the function

$$\mu_\sigma(\Omega) = \int_\Omega \sigma(x)\, d^N x$$

has the same properties at the Lebesgue measure $\mu(\Omega)$. It is defined on all measurable sets, it is non-negative, monotonic, and countably additive, and the condition $\mu(\Omega) = 0$ implies $\mu_\sigma(\Omega) = 0$

Let $\sigma(x) \geq 0$. A function f is called *Lebesgue integrable on* Ω *with weight (or measure)* σ if the product $f\sigma$ is integrable on Ω. The space of all integrable functions on Ω with weight σ is denoted by $\mathcal{L}(\Omega;\sigma)$.

6.13. Interchanging the order of taking the limit and integration. It was shown that the limit of a pointwise convergent sequence of Riemann integrable functions is not generally Riemann integrable. A uniform convergence of the sequence is sufficient for the Riemann integrability of the limit function (cf. Theorem **3.2**). In the Lebesgue theory, taking limits under the integral sign is simpler (requires weaker conditions). This stems from insensitivity of the Lebesgue integral to alterations of the integrand on sets of measure zero. Here a few theorems stating sufficient conditions for interchanging the order of taking the limit and integral are discussed (their proofs can be found in[16])

[16] A.N. Kolmogorov nad S.V. Fomin, Elements of the theory functions and functional analysis

6.13.1. The Lebesgue dominated convergence theorem. *Let a sequence of (measurable) functions* $\{f_n\}_1^\infty$ *converge to* f *a.e.,*

$$\lim_{n\to\infty} f_n(x) = f(x) \ a.e.$$

If there exists an integrable function g *independent of* n *such that*

$$|f_n(x)| \le g(x) \ a.e., \quad g \in \mathcal{L},$$

then $f \in \mathcal{L}$ *and*

$$\lim_{n\to\infty} \int f_n(x)dx = \int \lim_{n\to\infty} f_n(x)dx = \int f(x)dx\,. \tag{6.4}$$

This theorem is perhaps one of the most useful theorems in mathematical analysis. To illustrate it, recall the first example in Sec. **3.6**. The sequence is bounded by $g(x) = 1$ that is integrable on any bounded interval and, hence, (**6.4**) holds for any such interval. The limit function is the Dirichlet function that is nowhere continuous and, hence, not Riemann integrable, but it is Lebesgue integrable because it is zero almost everywhere.

6.13.2. Example. Let

$$f_n(x) = \frac{n\sin(x^2/n)}{x^2(x^2+a_n^2)}\,, \quad x \neq 0\,,$$

where $a_n > 0$ and $a_n \to a > 0$ as $n \to \infty$. The functions f_n are not defined at $x = 0$. However they can either be extended by continuity $f_n(x) \to 1/a_n^2 = f_n(0)$ as $x \to 0$ or one can set $f_n(0) = b_n$ for some sequence $\{b_n\}$. Then

$$\lim_{n\to\infty} f_n(x) = \frac{1}{x^2+a^2} \quad \text{a.e.}$$

Indeed, the limit may or may not exist at $x = 0$, and for $x \neq 0$, the limit follows from that

$$\lim_{y\to 0} \frac{\sin(y)}{y} = 1$$

where $y = x^2/n \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$$

provided there exists a Lebesgue integrable bound g, $|f_n(x)| \le g(x)$ a.e., that is *independent* of the parameter n. To find $g(x)$, note first that $|\sin(y)| \le |y|$, and it follows that

$$|f_n(x)| \le \frac{1}{x^2+a_n^2} \quad \text{a.e.}$$

A positive sequence a_n converges to $a > 0$ and, hence, its greatest lower bound cannot be equal to zero $a_0 = \inf_n\{a_n\} > 0$. Indeed, any interval $|x - a| < \delta < a$ contains all but finitely many terms of the sequence $\{a_n\}$. Since $a_n > 0$, a_0 is the smallest among finitely many terms outside the interval. If $a_n \neq a$, then for small enough δ there will always be terms outside the interval. Therefore

$$|f_n(x)| \leq \frac{1}{x^2 + a_n^2} \leq \frac{1}{x^2 + a_0^2} = g(x) \quad \text{a.e.}, \quad a_0 = \inf_n\{a_n\} > 0\,.$$

6.13.3. An example of a convergent sequence with no integrable bound. If there exists no integrable bound, then (**6**.**4**) can be false. Consider the sequence

$$f_n(x) = \frac{n}{1 + n^2x^2}\,, \quad x \in \mathbb{R}$$

Then for any n,

$$\int f_n(x)\,dx = \lim_{b\to\infty}\int_{-b}^{b}\frac{ndx}{1+n^2x^2} = \lim_{b\to\infty}\int_{-bn}^{bn}\frac{dy}{1+y^2} = \pi$$

However the integral of the limit function is zero. Indeed, the sequence converges to zero if $x \neq 0$ and to infinity if $x = 0$. Therefore

$$\lim_{n\to\infty} f_n(x) = 0 \quad a.e.$$

and

$$\lim_{n\to\infty}\int f_n(x)\,dx = \pi \neq 0 = \int \lim_{n\to\infty} f_n(x)\,dx\,.$$

Note that $\frac{2}{3}n \leq f_n(x) \leq n$ if $|x| \leq \frac{1}{n}$. This implies that if $f_n(x) \leq g(x)$ for all x and all n, then $g(x) \sim \frac{1}{|x|}$ near $x = 0$ which is not integrable.

6.13.4. Levi's theorem. If the sequence has no integrable bound, then the integrability of the limit function can be established by means of Levi's theorem: *Let $\{f_n\}$ be an almost everywhere non-decreasing sequence of integrable functions, $f_n \in \mathcal{L}(\Omega)$, and the sequence of the integrals of f_n is bounded,*

$$f_n(x) \leq f_{n+1}(x)\ a.e.$$
$$\left|\int f_n(x)\,d^Nx\right| \leq M\,,$$

for all n. Then there exists $f \in \mathcal{L}(\Omega)$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e.}$$

*and the relation (**6**.**4**) holds.*

As an example, put

$$f_n(x) = \left(1 + \frac{x f_D(x)}{n}\right)^n$$

where f_D is the Dirichlet function. Recall the sequence $(1 + p/n)^n$ converges to e^p and it is monotonically increasing if $p > 0$. Therefore $f_n(x) \leq f_{n+1}(x)$ if $x \geq 0$ and

$$0 \leq \int_a^b f_n(x)\,dx \leq \int_a^b e^x\,dx < \infty\,, \qquad 0 \leq a < b < \infty\,.$$

The limit function is

$$f(x) = \lim_{n\to\infty} f_n(x) = 1 \quad \text{a.e.}$$

because $f(x) = e^x$ if x is rational and $f(x) = 1$ otherwise so that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx = b - a\,.$$

In Levi's theorem the hypothesis of the boundedness of a sequence by an integrable function is replaced by the hypothesis of monotonicity of the sequence and boundedness of the sequence of integrals. The monotonicity hypothesis is essential. For example, the sequence of functions in Sec. **6.13.3** has a bounded sequence of integrals. But, by graphing $f_n(x)$, it is not difficult to see that the sequence is not monotonic: if $n > m$, then $f_n(x) > f_m(x)$ near $x = 0$ and $f_n(x) < f_m(x)$ for all large enough $|x|$.

There is a simple consequence of Levi's theorem for functions defined by functional series of non-negative terms that allows one to interchange the summation and integration signs.

Corollary **6.1**. *If* $f_n(x) \geq 0$ *and*

$$\sum_{n=1}^{\infty} \int_\Omega f_n(x)\,d^N x < \infty$$

then there exists $f \in \mathcal{L}(\Omega)$ *such that*

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{a.e.}$$

and

$$\int_\Omega \Big(\sum_{n=1}^{\infty} f_n(x)\Big)\,d^N x = \sum_{n=1}^{\infty} \int_\Omega f_n(x)\,d^N x$$

Note that partial sums of the series $\sum_n f_n(x)$ form a sequence satisfying the hypotheses of Levi's theorem.

For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n(1+n^2x^2)}$$

converges almost everywhere to $f(x)$ that is integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^3}{12}$$

The series converges for any $x \neq 0$ and diverges for $x = 0$. So, f exists almost everywhere. Its integrability follows from that $f_n(x) > 0$ and

$$\int_{-\infty}^{\infty} f_n(x)\,dx = \frac{\pi}{2}\cdot\frac{1}{n^2}\,, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty\,.$$

6.14. Change of variables in the Lebesgue integral. Let Ω be an open set in $\mathbb{R}^N$ and $x = x(y)$ be a transformation of $\mathbb{R}^N$ from class $C^1(\bar{\Omega})$ that is one-to-one on Ω. Let $\Omega' = x(\Omega)$ and $J(y) = \det[Dx(y)]$ be the Jacobian of the transformation. *Then in order for $f(x)$ to be integrable on Ω, it is necessary and sufficient that $f(x(y))J(y)$ is integrable on Ω' and, in this case,*

$$\int_\Omega f(x)\,d^N x = \int_{\Omega'} f(x(y))|J(y)|\,d^N y\,.$$

6.15. Exercises.

1. Can the Lebesgue measure of an unbounded region be finite? If so, construct an example. *Hint*: Think of the area under the graph of a non-negative continuous function on $\mathbb{R}$.

2. Construct an example of set in $\mathbb{R}^N$ that contains an open ball $|x| < R$ whose measure is equal to the volume of the ball but the closure of the set has measure that twice as much as the volume of the ball.

3. Are there any values of p for which the function

$$f(x) = \frac{\sin^2(|x|)}{|x|^p}\,, \quad x \in \mathbb{R}^N$$

is integrable on

(i) a bounded set that contains $x = 0$;

(ii) $\mathbb{R}^N$;
(iii) on the complement of a region containing $x = 0$

4. Suppose that

$$|f(x)| \le \frac{M}{1 + |x|^p}$$

For what values of p does f have a Fourier transform

$$\mathcal{F}[f](k) = \int e^{i(k,x)} f(x)\, d^N x\,, \quad k \in \mathbb{R}^N$$

5. Suppose $|f(x)| \le M|x|^p$ a.e., where $p > 0$. For what values of p is the function $e^{-|x|} f(x)$ is integrable on $\mathbb{R}^N$? Give an upper bound of the value of the integral.

6. Let $f_n(x) = \Big(1 - x/n\Big)^n$, $n = 1, 2, \ldots.$
(i) Show that $f_n(x)$ converges to e^{-x} uniformly on $[0, 1]$, that is,

$$\lim_{n\to\infty} \sup_{[0,1]} \big| f_n(x) - e^{-x} \big| = 0$$

Note that $f_n(x) - e^{-x}$ is continuous on $[0, 1]$ and, hence, attains its extreme values on $[0, 1]$. Find them and compute the limit. Conclude that

$$\lim_{n\to\infty} \int_0^1 f_n(x) = \int_0^1 e^{-x}\, dx$$

(ii) Show that $|f_n(x)| \le M$ for all $x \in [0, 1]$, where M is some constant independent of n. Use the Lebesgue dominated convergence theorem to established the same result.

7. Let $\varphi \in C^1(\mathbb{R})$ and the support of φ is bounded. Show that

$$\lim_{n\to\infty} \int e^{inx} \varphi(x)\, dx = 0$$

Hint: Use integration by parts in combination with the Lebesgue dominated convergence theorem (or with the theorem about the uniform convergence and integrability).

8. Let $f \in \mathcal{L}(\mathbb{R})$ such that $\int f(x)\, dx = 1$ and φ be a continuous function with bounded support. Put $f_n(x) = nf(nx)$, $n = 1, 2, \ldots.$ Show that

$$\lim_{n\to\infty} \int f_n(x)\varphi(x)\, dx = \varphi(0)$$

Hint: Use the Lebesgue dominated convergence theorem and that any continuous function with bounded support is bounded.

9. Use the Lebesgue dominated convergence theorem to find the following limit

$$\lim_{n\to\infty} n \int_0^{\frac{\pi}{4}} e^{-n^2 \sin(2t)}\, dt$$

7. Functions defined by Lebesgue integrals

Let $f(x, y)$ be a function of two variables $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. Suppose that f is Lebesgue integrable with respect to y for any x. Then the integral defines a function

$$u(x) = \int f(x, y)\, d^M y\,.$$

Under what conditions on the function f is the function u integrable, or continuous, or differentiable? These questions will be answered in this section.

7.1. Fubini's theorem. *Suppose that the iterated integral of $|f(x, y)|$ exists, then f is Lebesgue integrable on $\mathbb{R}^{N+M}$:*

$$\int \left(\int |f(x, y)|\, d^N x \right) d^M y < \infty \quad \Rightarrow \quad f(x, y) \in \mathcal{L}(\mathbb{R}^{N+M})$$

Conversely, if f is Lebesgue integrable, then the function defined by the integrals of f either with respect to x or y

$$h(x) = \int f(x, y)\, d^M y\,, \quad g(y) = \int f(x, y)\, d^N x$$

exist almost everywhere and are Lebesgue integrable:

$$f \in \mathcal{L}(\mathbb{R}^{N+M}) \quad \Rightarrow \quad h \in \mathcal{L}(\mathbb{R}^N)\,, \quad g \in \mathcal{L}(\mathbb{R}^M)$$

and, in this case, the integral of f is equal to the iterated integrals:

$$\begin{aligned} \iint f(x, y)\, d^N x d^M y &= \int \left(\int f(x, y)\, d^N x \right) d^M y \\ &= \int \left(\int f(x, y)\, d^M y \right) d^N x \end{aligned}$$

Funini's theorem also holds if f is defined on $\Omega \times \Omega'$, that is, $x \in \Omega \subset \mathbb{R}^N$ and $y \in \Omega' \subset R^M$. Indeed, one can replace $f(x, y)$ by $\chi_\Omega(x)\chi_{\Omega'}(y) f(x, y)$ in the above formulation and use the definition of the Lebesgue integral over a region.

It should be noted that if f is not integrable, then its iterated integrals either do not exist or, if they exist, they are not equal. The latter can happens if f has a conditionally convergent Riemann integral. For example, consider

$$\begin{aligned} h(x) &= \lim_{b \to 0^+} \int_b^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy = \lim_{b \to 0^+} \int_b^1 \frac{\partial}{\partial y} \frac{y}{x^2 + y^2}\, dy \\ &= \lim_{b \to 0^+} \frac{y}{x^2 + y^2}\Big|_b^1 = \frac{1}{1 + x^2}\,, \quad x \neq 0 \end{aligned}$$

Similarly,

$$g(y) = \lim_{a\to 0^+} \int_a^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx = -\lim_{a\to 0^+} \int_b^1 \frac{\partial}{\partial x} \frac{x}{x^2+y^2}\, dx$$
$$= -\lim_{a\to 0^+} \frac{x}{x^2+y^2}\Big|_a^1 = -\frac{1}{y^2+1}\,, \quad y \neq 0$$

Therefore, the functions

$$h(x) = \int_0^1 f(x,y)\, dy = \frac{1}{1+x^2} \quad a.e.$$
$$g(y) = \int_0^1 f(x,y)\, dx = -\frac{1}{1+y^2} \quad a.e.$$

are integrable on $(0,1)$ and

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx = \int_0^1 h(x)\, dx = \frac{\pi}{4}\,,$$
$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = \int_0^1 g(y)\, dy = -\frac{\pi}{4}$$

It was shown earlier that the improper Riemann integral of f over any bounded closed region that contains the origin does not converge absolutely so that f is not Lebesgue integrable.

The first part of Fubini's theorem is a criterion for Lebesgue integrability of a function of two variables, while the second part gives a criterion for changing the order of integration. If

$$\int_\Omega \int_{\Omega'} |f(x,y)|\, d^N x d^M y < \infty$$

in any particular order, then

$$\int_\Omega \int_{\Omega'} f(x,y)\, d^M y d^N x = \int_{\Omega'} \int_\Omega f(x,y)\, d^N x d^M y$$

In the above example

$$\int_0^1 \left(\int_0^1 |f(x,y)| dx \right) dy = \infty$$

This is left to the reader as an exercise.

7.2. Continuity of a function defined by an integral. Recall that g is continuous at a point y if and only if for any sequence $\{y_n\}$ converging to y, the sequence $\{g(y_n)\}$ converges to $g(y)$. If $g(y)$ is defined by an integral of $f(x,y)$ with respect to x, then under what conditions on $f(x,y)$ the function g is continuous.

THEOREM 7.1. *Let $f(x, y)$ be defined on $\mathbb{R}^N \times \Omega$, $\Omega \subset \mathbb{R}^M$. Suppose f is continuous in $y \in \Omega$ for almost all $x \in \mathbb{R}^N$, and there exists an integrable function $F(x)$ such that $|f(x, y)| \le F(x)$ a.e. for every $y \in \Omega$. Then the function*

$$g(y) = \int f(x, y) d^N x$$

is continuous on Ω, that is,

$$\lim_{z \to y} \int f(x, z) d^N x = \int_\Omega \lim_{z \to y} f(x, z) d^N x = \int f(x, y) d^N x$$

for any $y \in \Omega$.

The assertion follows from the Lebsgue dominated convergence theorem. Consider the sequence of functions $f_n(x) = f(x, y_n)$. Then

$$\lim_{n \to \infty} f_n(x) = f(x, y) \quad a.e.$$

because $f(x, y)$ is continuous in y for almost every $x \in \mathbb{R}^N$. The sequence $\{f_n\}$ is bounded for all n by a Lebesgue integrable function

$$|f_n(x)| \le F(x)$$

for any choice of $\{y_n\}$. By the Lebesgue dominated convergence theorem

$$\lim_{n \to \infty} g(y_n) = \lim_{n \to \infty} \int f_n(x)\, d^N x = \int \lim_{n \to \infty} f_n(x) d^N x = g(y)\,,$$

which holds for any y in Ω and, hence, $g \in C^0(\Omega)$.

7.3. Differentiation of an integral with respect to parameters.

THEOREM 7.2. *Let $f(x, y)$ be defined on $\mathbb{R}^N \times (a, b)$. Suppose that the partial derivative $D_y f(x, y)$ is continuous in $y \in (a, b)$ for almost all $x \in \mathbb{R}^N$. Furthermore, there exists an integrable function $F(x)$ such that for every $y \in (a, b)$, $|D_y f(x, y)| \le F(x)$ almost everywhere, and the integral of $f(x, y)$ with respect to x exists for some particular $y_0 \in (a, b)$. Then the function*

$$g(y) = \int f(x, y) d^N x \in C^1(a, b)$$

has the derivative continuous in (a, b) and the following equality holds

$$h'(y) = \frac{d}{dy} \int f(x, y) d^N x = \int D_y f(x, y)\, d^N x\,. \tag{7.1}$$

Put

$$\phi(y) = \int D_y f(x,y)\, d^N x$$

Since $D_y f(x,y)$ is continuous in y for almost every x and is bounded by an integrable function:

$$D_y f(x,y) \in C^0(a,b) \;\; \forall x\,; \qquad |D_y f(x,y)| \le F(x) \in \mathcal{L}$$

by Theorem **7.1**, the function $\phi(y)$ is continuous on (a,b). Therefore for any y and y_0 in (a,b), its integral

$$\Phi(y) = \int_{y_0}^{y} \phi(t)\, dt \in C^1(a,b)$$

is continuously differentiable in (a,b) and, by the Fundamental theorem of calculus,

$$\Phi'(y) = \phi(y)$$

Since $F \in \mathcal{L}$, one infers that

$$\begin{aligned} \int_a^b \int |D_y f(x,y)|\, d^N x\, dy &\le \int_a^b \int F(x)\, d^N x\, dy \\ &= (b-a) \int F(x)\, d^N x < \infty \end{aligned}$$

Therefore the function $D_y f(x,y)$ is Lebesgue integrable on $\mathbb{R}^N \times (a,b)$ by the first part of Fubini's theorem. By the second part of Fubini's theorem, the order of integration can be changed:

$$\begin{aligned} \Phi(y) &= \int_{y_0}^{y} \int D_t f(x,t)\, d^N x dt = \int \int_{y_0}^{y} D_t f(x,t)\, dt\, d^N x \\ &= \int [f(x,y) - f(x,y_0)]\, d^N x = g(y) - g(y_0) \end{aligned}$$

This shows that $g(y)$ is continuously differentiable and $g'(y) = \Phi'(y) = \phi(y)$ as required. This completes the proof of the theorem.

It is clear from the proof that the same result holds if $y \in \Omega \subset \mathbb{R}^M$. If Ω is open, then each coordinate y_i ranges over some open interval for given values of the other coordinates. Similarly, $g(y)$ is from class C^p if partial derivatives $D_y^\beta f(x,y)$, $\beta = 1, 2, ..., p$, are continuous with respect to y for almost every x and are bounded by Lebesgue integrable functions, $|D_y^\beta f(x,y)| \le F_\beta(x) \in \mathcal{L}$:

$$D_y^\beta f(x,y) \in C^0(\Omega) \quad \forall x\,; \quad |D_y^\beta f(x,y)| \le F_\beta(x) \in \mathcal{L}\,,\ \beta \le p\,,$$

$$\Rightarrow \quad g(y) = \int f(x,y)\, d^N x \in C^p(\Omega)$$

$$\Rightarrow \quad D_y^\beta g(y) = \int D_y^\beta f(x,y)\, d^N x\,, \quad \beta \le p$$

So, *the order of differentiation with respect to parameters and the integration with respect to other variables can be interchanged if partial derivatives of the integrand with respect to parameters are bounded by a Lebesgue integrable function that is independent of the parameters.*

7.3.1. On interchanging the order of integration and differentiation. Theorem **7.2** states sufficient but not necessary conditions for differentiation of the integral with respect to a parameter. In fact, the integral can be differentiable infinitely many times while partial derivatives with respect to parameters do not have integrable bounds independent of parameters. This implies that *in general the order of differentiation and integration cannot be interchanged.* For example, one can show that[17],

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\, dx = \pi\, e^{-|k|}$$

This function is infinitely many times differentiable on any interval that does not contain $k = 0$. However, the derivatives of the integrand with respect to k are not even integrable:

$$\Big|\frac{d^n}{dk^n}\frac{e^{ikx}}{1+x^2}\Big| = \frac{|x|^n}{1+x^2} \notin \mathcal{L}\,, \quad n > 0\,.$$

7.4. The Fourier transform. The Fourier transform of a Lebesgue integrable function is defined by

$$\mathcal{F}[f](k) = \int e^{i(k,x)} f(x)\, d^N x\,, \quad k \in \mathbb{R}^N\,.$$

First note that the Fourier transform exists for any $k \in \mathbb{R}^N$ because $|e^{i(k,x)} f(x)| = |f(x)|$ and the absolute value is integrable if $f \in \mathcal{L}$. Let $g(x,k) = e^{i(k,x)} f(x)$. The exponential $e^{i(k,x)}$ is continuous with respect to k for any x and so is $g(x,k)$. So,

$$g(x,k) \in C^0\,, \quad \forall x\,; \qquad |g(x,k)| = |f(x)| \in \mathcal{L}$$

By Theorem **7.1**, *the Fourier transform is continuous.*

[17]Example 1 in Sec. **9.2.2**

7.4.1. Differentiability of the Fourier transform. Using the notations introduced in Sec.**1.3**,

$$\left|D_k^\beta e^{i(k,x)f(x)}\right| = \left|i^\beta x^\beta e^{i(k,x)}f(x)\right| \leq |x|^{|\beta|}|f(x)|$$

where the latter inequality follows from $|x_j| \leq |x|$. By Theorem **7.2** the Fourier transform is from class C^p if $|x|^p|f(x)| \in \mathcal{L}$ and

$$D_k^\beta \mathcal{F}[f](k) = D_k^\beta \int e^{i(k,x)}f(x)\,d^Nx = i^\beta \int e^{i(k,x)}x^\beta f(x)\,d^Nx$$
$$= \mathcal{F}[(ix)^\beta f(x)](k)\,, \quad \beta = 1, 2, ..., p$$

So, differentiability of the Fourier transform depends on the asymptotic behavior of the function, that is, when $|x| \to \infty$.

Recall that if f is integrable on any ball $|x| < R$ and $|f(x)| = O(|x|^{-m})$, $m > N$, in the asymptotic region $|x| \to \infty$, then $f \in \mathcal{L}$. Therefore the Fourier transform $\mathcal{F}[f]$ is p times continuously differentiable if

$$|f(x)| = O(|x|^{-n})\,, \quad n > N + p$$

in the asymptotic region $|x| \to \infty$. In particular, if f decreases faster than any power function, its Fourier transform is from class C^∞:

$$\lim_{|x|\to\infty} |x|^p|f(x)| = 0 \quad \text{for all } p \quad \Rightarrow \quad \mathcal{F}[f](k) \in C^\infty\,.$$

7.5. Differentiability of Gaussian integrals. Let us prove that for $n = 1, 2, ...,$ and $t > 0$

$$\int_{-\infty}^{\infty} x^{2n}e^{-tx^2}dx = (-1)^n \frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{-tx^2}dx = \sqrt{\pi}\,(-1)^n \frac{d^n}{dt^n}\,t^{-1/2}\,.$$

Let $f(x,t) = e^{-tx^2}$. Fix $t_0 > 0$ (an arbitrary positive number). Using the power series for e^x about $x = 0$ is not difficult to establish the inequality

$$|x|^m \leq m!e^{|x|}\,, \qquad x \in \mathbb{R}\,.$$

Therefore for every positive integer n

$$|D_t^n f(t,x)| = x^{2n}e^{-tx^2} \leq (2n)!e^{-tx^2+|x|} \leq (2n)!e^{-t_0x^2+|x|} = g_n(x)$$

for all $0 < t_0 \leq t < \infty$ and $x \in \mathbb{R}$. The function $g_n(x)$ is integrable. Therefore the order of differentiation and integration can be interchanged leading to the desired result:

$$(-1)^n \frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{-tx^2}dx = (-1)^n \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n}e^{-tx^2}dx = \int_{-\infty}^{\infty} x^{2n}e^{-tx^2}dx\,.$$

for all $t > 0$.

7.6. Fundamental theorem of Lebesgue integral calculus. In the Riemann integration theory, the fundamental theorem of calculus states that if a function f is from class C^1, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

The statement is also true for the Lebesgue integral. The Lebesgue integral exists even if $f'(x)$ does not exist on a set of measure zero, but the assertion may be false as the function f may lack continuity. Let $f(x) = \arctan(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$. This function is not continuous at $x = 0$. Its derivative exists for $x \neq 0$, $f'(x) = -(1+x^2)^{-1}$, and $f'(0)$ does not exist (because u is not continuous at $x = 0$). However, the function $f'(x)$ has a continuous extension to $x = 0$ because $f'(x) \to -1$ as $x \to 0^{\pm}$ (the left and right limits are equal). So, the Riemann integral of f' exists, as an improper integral, over any interval containing $x = 0$, and it is equal to the Lebesgue integral. But the fundamental theorem of calculus fails. For example, the integral of $f'(x)$ over $(-1, 1)$ is equal to $-\pi/2$, while $f(1) - f(-1) = \pi/2$.

What is the largest set of functions for which the fundamental theorem of calculus holds if the integral is understood in the Lebesgue sense? Consider the equation

$$f'(x) = g(x) \quad a.e.$$

where g is a "nice" function, e.g., g is continuous. This is a linear equation. Therefore its general solution is the sum of a particular solution and a general solution of the associated homogeneous equation ($g = 0$). A particular solution is easy to find using the fundamental theorem of calculus so that

$$f(x) = h(x) + \int_a^x g(y)\,dy\,, \quad h'(x) = 0 \quad a.e.$$

If $h'(x) = 0$ *everywhere*, then $h(x) = c$ is a constant and $f(b) - f(a)$ does not depend on c for any choice of a and b. Otherwise,

$$f(b) - f(a) = h(b) - h(a) + \int_a^b g(y)\,dy\,.$$

There are non-zero functions with the vanishing almost everywhere derivative. So, if the fundamental theorem of calculus holds for f, then it does not hold for $f + h$ where $h' = 0$ a.e. because the integral of h' vanishes over any interval.

For example, take a piecewise constant function h. Then its derivative is zero almost everywhere and its integral vanishes on any interval, whereas $h(b) - h(a) \neq 0$ if a and b lie in different intervals of continuity. Evidently, the function $f + h$ is not continuous. So, continuity is *necessary* just like in the Riemann theory. However, it is not sufficient because *there are continuous functions whose derivative is zero almost everywhere*. The most famous one is the so called *Cantor ladder*, a function that is continuous, monotonically increasing, and whose derivative vanishes almost everywhere[18]:

$$h(x) \in C^0\,, \quad h'(x) = 0 \ a.e.\,, \quad h(x) < h(y)\,,\ x < y$$

One can always set $h(0) = 0$ and, in this case, by monotonicity

$$0 = \int_0^x h'(y)dy < h(x)\,,$$

and the inequality is strict. So, the class of functions in question is smaller than the space of continuous functions but larger than the space of continuously differentiable functions.

7.6.1. Absolutely continuous functions. The answer to the question posed in the previous section is provided by *absolutely continuous functions*.

DEFINITION 7.1. (Absolutely continuous functions)
A function f is called absolutely continuous on an interval I if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of non-overlapping intervals $I_j = (a_j, b_j) \subset I$, $j = 1, 2, ..., n$, $I_j \cap I_k = \emptyset$, $j \neq k$, the total absolute variation of u on these intervals does exceed ε whenever the total length of the intervals does not exceed δ:

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon \quad \text{whenever} \quad \sum_{j=1}^{n} |b_j - a_j| < \delta\,.$$

Clearly every absolutely continuous function is continuous. If a function is absolutely continuous on a closed interval $[a, b]$, then it is uniformly continuous on it. Roughly speaking, an absolutely continuous function cannot oscillate "too fast" so that it would have a finite variation on a finite set of intervals of an arbitrary small total length. For example, $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ is uniformly continuous on any bounded interval containing $x = 0$, but not absolutely continuous.

Absolutely continuous function are proved to have the following characteristic properties [19].

[18]A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter VI, Sec. 4

[19]A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter VI

THEOREM **7.3**. *If f is absolutely continuous on $[a,b]$, then it is differentiable almost everywhere, the derivative is locally integrable and*

$$f(x) = f(a) + \int_a^x f'(y)\, dy\,.$$

Conversely, for every absolutely continuous function f, there exists a a Lebesgue integrable function $g \in \mathcal{L}(a,b)$ such that

$$f(x) = f(a) + \int_a^x g(y)\, dy\,,$$

and in this case

$$f'(x) = g(x) \quad a.e.$$

The first part of this theorem is known as the *fundamental theorem of Lebesgue integral calculus* which is due to Lebesgue. The second part can be used as an alternative definition of absolutely continuous functions (in which case, Definition **7.1** becomes a theorem). In applications, the latter characteristic property is far more convenient and easier to use than Definition **7.1**.

By Theorem **2.1**, any set of measure zero in $\mathbb{R}$ is mapped to a set of measure zero by a function from class C^1. Absolutely continuous functions have the same property. *If $\Omega \subset [a,b]$ and $\mu(\Omega) = 0$, then $\mu(f(\Omega)) = 0$ for any absolutely continuous f on $[a,b]$*. This assertion is known as the *Luzin theorem.*

The set of absolutely continuous functions on an interval I will be denoted as $AC^0(I)$. It is a linear space with respect the usual addition of functions and multiplication of a function by a number. A function f is absolutely continuous on $\mathbb{R}$ if it is absolutely continuous on any bounded interval. The space of such functions is denoted by $AC^0(\mathbb{R})$ or AC^0 for brevity. Functions from class AC^0 play a fundamental role in the analysis of self-adjoint extensions of differential operators (like momentum and kinetic energy operators in quantum mechanics).

7.6.2. Integration by parts. Using Theorem **7.3**, it is not difficult to show that the product of two absolutely continuous functions is also absolutely continuous, and if an absolutely continuous function does not vanish anywhere, then its reciprocal is absolutely continuous. Therefore, by integrating the identity $(uv)' = u'v + uv'$ that holds almost everywhere, the integration by part is extended to absolutely continuous functions,

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\, dx\,, \quad u, v \in AC^0\,,$$

where the integral is understood in the Lebesgue sense.

7.7. Exercises.

1. Let

$$f(x,y) = \frac{xy}{(x^2+y^2)^2}\,, \quad (x,y) \in (-1,1)\times(-1,1) = \Omega \subset \mathbb{R}^2$$

(i) Show that

$$h(x) = \int_{-1}^{1} f(x,y)\,dy = 0\,, \quad g(y) = \int_{-1}^{1} f(x,y)\,dx = 0$$

so that these functions are integrable on $(-1,1)$ and their integrals vanish.
(ii) Show that the function f is not integrable on the rectangle Ω. Explain why Fubini's theorem does not apply in this case.

2. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $[0,1]$ that converge to 0 monotonically, $x_1 = 1 > x_2 > \cdots$ and $y_1 = 1 > y_2 > \cdots$. Put $\Delta x_n = x_n - x_{n+1}$ and $\Delta y_n = n_n - y_{n+1}$, and suppose that

$$p = \frac{\Delta x_n}{\Delta x_{n+1}}\,, \quad q = \frac{\Delta y_n}{\Delta y_{n+1}}$$

for any n. Consider the function of two real variables

$$f(x,y) = \begin{cases} p^n q^n\,, & (x,y) \in [x_{n+1}, x_n] \times (y_{n+1}, y_n]\,, \\ -p^n q^{n+1}\,, & (x,y) \in [x_{n+2}, x_{n+1}) \times (y_{n+1}, y_n]\,, \\ 0\,, & \text{otherwise} \end{cases}$$

(i) Is the function f piecewise continuous? Explain.
(ii) Evaluate the iterated integrals of f:

$$\int\!\!\int f(x,y)\,dxdy\,, \quad \int\!\!\int f(x,y)\,dydx\,.$$

(iii) Is the function f integrable on $\mathbb{R}^2$?

3. (i) Show that the function defined by the integral

$$h(x) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{k^2+m^2}\,dk\,,$$

where m is a positive constant, exists and is continuous for all $x \in \mathbb{R}$.
(ii) Find an explicit form of $h(x)$. Is $h(x)$ differentiable for all x? Is it true that

$$h'(x) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\frac{\cos(kx)}{k^2+m^2}\,dk\,,$$

if $h'(x)$ exists for some x?

4. Let $f(u) = 1 - |u|$ if $|u| \leq 1$ and f is extended periodically to all $u \in \mathbb{R}$, $f(u+2) = f(u)$. Define a function

$$F(t) = \int_0^\infty \frac{f(tx)}{1+x^p}\,dx\,, \quad p > 2\,, \quad t \in \mathbb{R}$$

(i) Show that $F(t)$ exists and $F(-t) = F(t)$;
(ii) Show that $F \in C^1(a,b)$ for any $0 < a < b$ and

$$F'(t) = \int_0^\infty \frac{xf'(tx)}{1+x^p}\,dx\,, \quad 0 < a \leq t \leq b$$

Hint: Consider a change of the integration variable $u = tx$. Use the theorem about differentiation of a function defined by a Lebesgue integral.
(iii) Show that

$$\left|\frac{f(xt) - f(0)}{t}\right| \leq |x|$$

(iv) Use the above inequality and the Lebesgue dominated convergence theorem to show that the left and right limits

$$\lim_{t\to 0^\pm} \frac{F(t) - F(0)}{t}$$

exist but are not equal. Is F differentiable at $t = 0$?

5. Let A be a positive matrix (all eigenvalues are strictly positive). Define the function

$$J(y) = \int e^{-(x,Ax)+(x,y)}\,d^N x\,, \qquad y \in \mathbb{R}^N$$

(i) Show that $J \in C^\infty$ and

$$D_y^\beta J(y) = \int D_y^\beta e^{-(x,Ax)+(x,y)}\,d^N x\,.$$

(ii) Calculate $J(y)$ and show that for any polynomial $P(x)$

$$\int P(x)e^{-(x,Ax)}\,d^N x = P(D_y)J(y)\Big|_{y=0}$$

6. Let

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}\,, \quad (x,y) \in \Omega = (1,\infty)\times(1,\infty)$$

(i) Calculate the iterated integral

$$\int_1^\infty \left(\int_1^\infty |f(x,y)|\, dx \right) dy$$

Is it true that $f \in \mathcal{L}(\Omega)$?
(ii) Calculate and compare the iterated integrals

$$\int_1^\infty \left(\int_1^\infty f(x,y)\, dx \right) dy\,, \quad \int_1^\infty \left(\int_1^\infty f(x,y)\, dy \right) dx$$

8. Line and surface integrals

8.1. Line integrals in a Euclidean space. Let C be a curve in $\mathbb{R}^N$ from a point x_a to a point x_b ($x_a = x_b$ is admissible). A finite collection of points $P = \{x_n\}$ of C that contain x_a and x_b is called a partition of C. The points in P are ordered along the curve. P can be viewed as the image of a partition of a line segment from which C is obtained by a continuous deformation. A refinement of P is a partition that contains P. Put

$$L(P, C) = \sum_n |x_n - x_{n-1}| \,.$$

The number $L(P, C)$ is the length of a polygonal path with vertices at x_n on the curve. Upon a refinement of P, the polygon path gets closer to the curve but its length is increasing by the triangle inequality:

$$L(P, C) \le L(P', C)\,, \quad P \subset P'$$

for any refinement P' of P. The quantity

$$L_C = \sup_P L(P, C)$$

is called the *arclength* of C. Note that L_C can be infinite.

If the curve is smooth, then there exists a simple parameterization of it from class C^1. Let $x = x(t)$, $a \le t \le b$, be such a parameterization. Then[20]

$$L_C = \int_a^b |x'(t)|\, dt < \infty \,.$$

In physics, this equation has a simple meaning. If $x = x(t)$ is the trajectory of a point-like particle, then $x'(t)$ is the velocity vector, and its magnitude $|x'(t)|$ is the speed. The distance traveled along path C is the integral of the speed with respect to time.

8.1.1. Natural parameterization of a smooth curve. Let $x = x(t)$ be a parameterization of a smooth curve C such that $x'(t) \ne 0$. Define an *arclength* parameter by

$$s = s(t) = \int_a^t |x'(\tau)|\, d\tau \,.$$

Then $s'(t) = |x'(t)| > 0$ and, hence, $s(t)$ is monotonic and maps $[a, b]$ onto $[0, L_C]$. The map is invertible, $t = t(s)$. A parameterization of C in terms of the arclength, $x = X(s) = x(t(s))$, is called a *natural parameterization* of C. Note that $X'(s)$ is a unit tangent vector, $|X'(s)| = 1$, as one infers from the chain rule, $dX(s)/ds = (dx(t)/dt)(ds/dt)^{-1}$.

[20] W. Rudin, Principles of mathematical analysis, Theorem 6.27

8.1.2. Line integral of a scalar function. Let $x(s)$ be a natural parameterization of a smooth curve C and $f(x)$ be continuous in a neighborhood of C. Then the integral

$$\int_C f\,ds = \int_0^{L_C} f(x(s))\,ds$$

exists and is called a *line integral of a function f over a curve C*. A line integral can be defined for any f as long as the function $f(x(s))$ is integrable on $[0, L_C]$. If $x = x(t)$ is any parameterization of a smooth curve, then using the change of variables, $ds = |x'(t)|dt$,

$$\int_C f\,ds = \int_a^b f(x(t))\,|x'(t)|\,dt\,.$$

Recall that the center of mass a collection of point-like particles with positions x_p and masses m_p (the index p labels the particles) is

$$x_c = \frac{1}{m}\sum_p m_p x_p\,, \quad m = \sum_p m_p\,.$$

Suppose these particles are assembled into a smooth curve C with a linear mass density $\sigma(x)$ so that $dm(x_p) = \sigma(x_p)ds$ is the mass of a segment of the curve of length ds at a sample point x_p. Then it follows that the coordinates of the center of mass of this wire are given by the following line integrals:

$$x_{0j} = \frac{1}{m}\int_C \sigma(x)x_j\,ds\,, \quad m = \int_C \sigma(x)\,ds\,.$$

8.1.3. Line integral of a vector field. Let T be a unit tangent vector at some point of a smooth curve. Then the vector $-T$ is also a unit tangent vector. Since a unit tangent vector is continuous for a smooth curve, there are only two ways of defining a continuous T for a smooth curve. If a curve connects points x_a and x_b, then a natural parameter can be counted either from x_a or from x_b, the derivatives of the corresponding natural parameterizations are opposite unit tangent vectors at any point of the curve. This choice defines an *orientation* of the curve C.

Let C be a smooth curve oriented by a unit tangent vector T. Let $F(x)$ be a continuous vector field (its components $F_j(x)$ are continuous). The dot product

$$F_T(x) = \Big(F(x), T(x)\Big)\,, \quad x \in C$$

is the tangent component of the vector field F at a point x of C. If $x = x(s)$ is a natural parameterization of C such that $x'(s) = T(x(s))$, then $F_T = (F, x')$. The line integral

$$\int_C F_T(x)\,ds = \int_C F_j(x)dx_j$$

where Einstein's summation rule over repeated indices j is assumed, is called the *integral of a vector field F along a curve C*. The integral is extended to all vector fields whose tangent component is integrable on C.

Let $x = x(t)$ be a simple parameterization of a curve C such that $x'(t)$ defines the correct orientation of C. Then by changing variable in the line integral, $dx_j = x'(t)\,dt$, one infers that

$$\int_C F_j(x)dx_j = \int_a^b F_j(x(t))x_j'(t)\,dt$$

Let $-C$ be the curve C with the opposite orientation, then

$$\int_{-C} F_j(x)dx_j = -\int_C F_j(x)dx_j\,.$$

Consequently, any parameterization can be used to evaluate the line integral over a curve from class C^1 (if a parameterization defines an opposite parameterization, the sign of the integral should be changed after evaluating it). It should be noted that the line integral is defined for any piecewise smooth parametric that is not simple. In other words, the vector function $x = x(t)$ may trace out some part of the curve multiple times, like, e.g., a particle moving back and forth along the same path. In this case, the line integral is nothing but the sum of line integrals over smooth simple pieces oriented accordingly.

8.1.4. Line integrals as a work done by a force. If $F(x)$ is a force acting on a point-like particle at a point x, then the work done by F in moving the particle along an infinitesimal straight-line segment from x to $x+dx$ is given by

$$dW(x) = F_j(x)dx_j = |F(x)|\cos(\theta)\,ds\,,$$

where θ is the angle between $F(x)$ and $T(x)$ and $|dx| = ds$. The line integral of F along C is nothing but the total work done by F in moving the particle along the curve C.

If $x = x(t)$ is a physical trajectory of a point-particle of mass m, then the trajectory satisfies Newton's Law $mx''(t) = F(x(t))$ (recall that the second derivative $x''(t)$ is the acceleration of the particle). Then the work done by F in moving the particle is a net change of the kinetic energy $\frac{1}{2}m|v|^2$, where $v(t) = x'(t)$ is the velocity of the particle,

$$W = \int_C F_j(x)dx_j = \int_a^b mv_j'(t)v_j(t)dt = \frac{1}{2}\,m|v(b)|^2 - \frac{1}{2}\,m|v(a)|^2\,.$$

8.1.5. Fundamental theorem for line integrals. A vector field is said to be *conservative* in an open set Ω if it is the gradient of a function $U \in C^1(\Omega)$, that is, $F = \nabla U$ in Ω. The function U is called a *potential* of F. Note that U is not unique as it can be changed by an additive constant.

Let C be a smooth curve in Ω outgoing from a point x_a and ending at a point x_b, and a vector field F be conservative in Ω. Then using a parameterization of C such that $x = x(t)$, $a \le t \le b$, $x(a) = x_a$, and $x(t_b) = x_b$, one infers that

$$\int_C F_j dx_j = \int_a^b \frac{\partial U}{\partial x_j}\frac{dx_j}{dt}\,dt = \int_a^b dU(x(t)) = U(x_b) - U(x_a)\,.$$

Thus, *for a conservative vector field, its line integral does not depend on the curve and is determined by the difference of its potential at the endpoints of the curve.* This comprises the fundamental theorem for line integrals. In particular, the line integral of a conservative vector field vanishes for a closed curve.

8.1.6. Conservative forces in physics. In physics, the fundamental theorem for line integrals is interpreted as the energy conservation for a particle moving under a conservative force $F = -\nabla U$. The minus sign in the latter relation is a convention used in physics to make the total energy to be the *sum* of the kinetic and potential energy U as is shown shortly. The work done by a conservative force is determined by the net change change of a potential energy U:

$$W = \int_C F_j dx_j = U(x_a) - U(x_b)\,.$$

Combining this relation with the work being the net change of kinetic energy established in Sec.**8.1.4**, it is concluded that the energy of a particle

$$E(t) = \frac{m}{2}\,|v(t)|^2 + U(x(t))$$

remains constant along any trajectory of the particle moving under a conservative force, $E(a) = E(b)$ for any time interval $a \le t \le b$, or $E'(t) = 0$. The latter can also be established by a direct evaluation of the derivative $E'(t)$ and invoking the Newton's Law, $mv' = -\nabla U$.

8.2. Surface area. Let S be a smooth M-surface and $x = F(y)$ be its parameterization, $y \in R$ (see Sec.**2.4.3**). Define a matrix W whose columns are the tangent vectors $w_a \in \mathbb{R}^N$:

$$W_M = [w_1\, w_2\, \cdots\, w_M]\,, \qquad w_a = \frac{\partial F}{\partial y_a}\,.$$

Then the area of S is defined by the integral

$$A_S = \int_R J(y)\, d^M y\,, \qquad J = \sqrt{\det(W_M^T W_M)}\,, \tag{8.1}$$

where W_M^T is the transposed matrix W_M. Note that $J(y)$ is continuous on $\bar{R}$ and, hence, the integral exists for any smooth M-surface in $\mathbb{R}^N$.

The Jacobian $J(y)$ is the volume of an M dimensional parallelepiped with adjacent sides being vectors w_a at a point $F(y)$. It is easy to verify the assertion for $M = 2$. If $0 \le \theta \le \pi$ is the angle between w_1 and w_2, then the area of parallelogram with adjacent sides w_1 and w_2 is

$$V_2 = |w_1||w_2|\sin(\theta) = \sqrt{|w_1|^2|w_2|^2 - (w_1, w_2)^2} = \sqrt{\det(W_2^T W_2)}$$

because

$$W_2^T W_2 = \begin{pmatrix} |w_1|^2 & (w_1, w_2) \\ (w_1, w_2) & |w_2|^2 \end{pmatrix}$$

Consider an n dimensional parallelepiped with adjacent sides w_1, w_2,..., w_n. Its volume is denoted by V_n. Its base is the $n-1$ dimensional parallelepiped with adjacent sides w_1, w_2,..., w_{n-1} with volume V_{n-1}. Then

$$V_n = V_{n-1} h\,,$$

where h is the height. If $w_n^{\|}$ is the orthogonal projection of w_n onto $\mathrm{span}\{w_1, ..., w_{n-1}\}$, then by the Pythagorean theorem

$$h^2 = |w_n|^2 - |w_n^{\|}|^2\,.$$

One has

$$w_n^{\|} = c_1 w_1 + c_2 w_2 + \cdots + c_{n-1} w_{n-1}\,,$$

where the constants c_a are such that $w_n - w_n^{\|}$ is orthogonal to all w_a, $a = 1, 2, ..., n-1$, so that

$$\sum_{b=1}^{n-1} (w_a, w_b) c_b = (w_a, w_n)\,.$$

If $c \in \mathbb{R}^{n-1}$ with components c_a satisfying the above equation, then

$$c = (W_{n-1}^T W_{n-1})^{-1} W_{n-1}^T w_n$$

because (w_a, w_b) are matrix elements of $W_{n-1}^T W_{n-1}$ and this matrix is invertible because w_a are linearly independent. Therefore

$$|w_n^{\|}|^2 = (w_n^{\|}, w_n^{\|}) = c^T W_{n-1}^T W_{n-1} c = w_n^T W_{n-1} (W_{n-1}^T W_{n-1})^{-1} W_{n-1}^T w_n \,.$$

On the other hand, W_n is obtained from W_{n-1} by adding an extra column w_n so that in the block-matrix notation, $W_n = [W_{n-1}\, w_n]$. Using the block-matrix multiplication

$$W_n^T W_n = \begin{bmatrix} W_{n-1}^T \\ w_n^T \end{bmatrix} [W_{n-1}\, w_n] = \begin{bmatrix} W_{n-1}^T W_{n-1} & W_{n-1}^T w_n \\ w_n^T W_{n-1} & |w_n|^2 \end{bmatrix}$$

Suppose that the equation for the volume is correct for $n-1$. It follows from the determinant of a block-matrix

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B)$$

that the equation is also correct for n

$$\begin{aligned} \det(W_n^T W_n) &= \det(W_{n-1}^T W_{n-1})(|w_n|^2 - |w_n^{\|}|^2) \\ &= V_{n-1}^2(|w_n|^2 - |w_n^{\|}|^2) = V_n^2 \end{aligned}$$

By mathematical induction, the equation is true for any $n \leq N$.

A *linearization* of F an a point y^* is the linear function $L:\ D \to \mathbb{R}^N$ defined by

$$L(y) = F(y^*) + \sum_{a=1}^{M} w_a(y^*)(y_a - y_a^*)\,, \quad w_a(y^*) = \frac{\partial F}{\partial y_a}\Big|_{y=y^*}\,.$$

It maps R into the tangent space of S at a point $x^* = F(y^*)$. Equation (**8**.**1**) has a simple geometrical meaning. For any partition box R_p in R, the surface area of $F(R_p)$ is approximated by the volume of a parallelepiped that is the image $L(R_p)$ of a partition box R_p in the tangent space taken at a sample point $x_p = F(y_p)$, $y_p \in R_p$. The total volume depends on the choice of sample points. But since it depends continuously on them for a C^1 surface, variations of the volume related to different choice of sample points do not contribute in the limit when dimensions of all partition boxes tend to zero uniformly, just like a Riemann sum converges to the integral of a continuous function for any choice of sample points.

If F is a continuous map such that the Jacobian J exists almost everywhere, then the integral (**8**.**1**) understood in the Lebesgue sense is also called the surface Lebesgue measure of S. This is not to be confused with the Lebesgue measure of S in $\mathbb{R}^N$. If S is smooth, then its Lebesgue measure is zero (as an $\mathbb{R}^N$ volume of S).

8.3. Surface integrals in $\mathbb{R}^N$ of a scalar function. Let S be a smooth M-surface. Let $x = F(y)$, $y \in R$, be a parameterization of S. The surface integral of a function f is defined by

$$\int_S f(x)\, dS \stackrel{\text{def}}{=} \int_R f(F(y))\, J(y)\, d^M y$$

if $f(F(y))$ is integrable on R.

8.3.1. Integration over a sphere in $\mathbb{R}^N$. A sphere of radius a in $\mathbb{R}^N$ is defined by

$$|x|^2 = x_1^2 + x_2^2 + \cdots + x_N^2 = a^2$$

Its parameterization

$$\begin{aligned} x_1 &= a\cos(\xi_1)\,, \\ x_2 &= a\sin(\xi_1)\cos(\xi_2)\,, \\ x_{N-1} &= a\sin(\xi_1)\sin(\xi_2)\cdots\sin(\xi_{N-2})\cos(\xi_{N-1})\,, \\ x_N &= a\sin(\xi_1)\sin(\xi_2)\cdots\sin(\xi_{N-2})\sin(\xi_{N-1})\,, \end{aligned}$$

where $\xi_p \in [0, \pi]$ for $p < N-1$ and $\xi_{N-1} \in [0, 2\pi]$, can be obtained using spherical coordinates. Here $0 \le \xi_1 \le \pi$ is the angle between the standard basis vector $\hat{e}_1$ and the position vector x. The vector $\hat{e}_1$ sets an *axis of a spherical coordinate system*. Let $x_\perp$ be the orthogonal projection of x onto the $N-1$ dimensional plane orthogonal to $\hat{e}_1$. Then the length of the projection is $|x_\perp| = a\sin(\xi_1)$. With this choice of the axis, x_2 is the scalar projection of $x_\perp$ onto $\hat{e}_2$, where $0 \le \xi_2 \le \pi$ is the angle between $x_\perp$ and $\hat{e}_2$. Then the length of the orthogonal projection of $x_\perp$ onto the plane orthogonal to $\hat{e}_2$ is $a\sin(\xi_1)\sin(\xi_2)$, and x_3 is the scalar projection of this vector projection onto $\hat{e}_3$. This procedure is repeated N times to get all x_j as functions of the angles ξ_a. The angle between any two vectors changes from 0 (parallel vectors) to π (anti-parallel vectors), which explains the range of ξ_b for $b < N-1$, and ξ_{N-1} is nothing but a polar angle a 2-plane spanned by $\hat{e}_{N-1}$ and $\hat{e}_N$.

The tangent vectors to curves that are images of the coordinate lines of parameters ξ are orthogonal:

$$\begin{aligned} w_b &= \frac{\partial x}{\partial \xi_b} \quad \Rightarrow \quad (w_b, w_{b'}) = |w_b|^2 \delta_{bb'}\,, \\ |w_1| &= a\,, \quad |w_b| = a\sin(\xi_1)\sin(x_2)\cdots\sin(\xi_{b-1})\,, \quad b = 2, 3, ..., N-1 \\ J(\xi) &= a^{N-1}\sin^{N-2}(\xi_1)\sin^{N-3}(\xi_2)\cdots\sin(\xi_{N-2})\,. \end{aligned}$$

Then an integral of a function over the sphere is reduced to the following iterated integral

$$\int_{|x|=a} f(x)\,dS = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} f(x(\xi))\,J(\xi)\,d\xi_1\cdots d\xi_{N-2}\,d\xi_{N-1}\,.$$

By construction of the spherical coordinate system, the integral over a unit sphere in $\mathbb{R}^N$ can also be written as an iterated integral over a sphere in $\mathbb{R}^{N-1}$ of radius $|x_\perp| = \sin(\xi_1)$ and an integral over the angle with the axis of the spherical coordinates ξ_1

$$\int\limits_{|x|=1} f(x)\,dS_N = \int_0^{\pi}\int\limits_{|y|=1} f\Big(\hat{e}_1\cos(\xi_1) + y\sin(\xi_1)\Big)\,\sin^{N-2}(\xi_1)\,dS_{N-1}\,d\xi_1$$

where dS_M is the surface area element of a unit sphere in $\mathbb{R}^M$, and $\hat{e}_1$ is the unit vector parallel to the axis of the spherical coordinates. If the function $f(x)$ is invariant under rotations about the axis of the spherical coordinate system, then it depends only on $x_1 = \cos(\xi_1)$ and $|x_\perp| = \sin(\xi_1)$, that is, $f(x) = g(\xi_1)$. In this case,

$$\int_{|x|=1} f(x)\,dS = \sigma_{N-1}\int_0^{\pi} g(\xi_1)\sin^{N-2}(\xi_1)\,d\xi_1\,.$$

where σ_N is the surface are of a unit sphere in $\mathbb{R}^N$. In particular, if $f(x) = 1$ so that $g(\xi_1) = 1$, then this equation gives the recurrence relation to compute σ_N (cf. (**2.1**)):

$$\sigma_N = \sigma_{N-1}B\Big(\frac{N-1}{2},\frac{1}{2}\Big) = \sigma_{N-1}\frac{\Gamma(\frac{N}{2}-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{N}{2})}\,,$$

where

$$B(a,b) = 2\int_0^{\pi/2}(\sin(\xi))^{2a-1}(\cos(\xi))^{2b-1}\,d\xi$$

is the *beta function*. It is proved that

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,.$$

8.3.2. Levi-Civita symbol. Let the symbol $\varepsilon_{j_1j_2\cdots j_N}$ be defined so that it is skew-symmetric under permutation of any two indices, and $\varepsilon_{12\cdots N} = 1$. This symbol is called the *Levi-Civita symbol* in $\mathbb{R}^N$. Any symbol with N indices has N^N indexed values. But the Levi-Civita symbol has only one independent value because its indexed values vanish if any two indices are equal and

$$\varepsilon_{j_1j_2\cdots j_N} = (-1)^P\varepsilon_{12\cdots N} = (-1)^P\,,$$

where P is the number of permutations needed to convert the set $j_1j_2\cdots j_N$ to $12\cdots N$ by permutations.

The product of two symbols can be expressed in terms the Kronecker delta symbol:

$$(\mathbf{8.2})\qquad \varepsilon_{i_1i_2\cdots i_N}\varepsilon_{j_1j_2\cdots j_N} = \det\begin{pmatrix} \delta_{i_1j_1} & \delta_{i_1j_2} & \cdots\delta_{i_1j_N} \\ \delta_{i_2j_1} & \delta_{i_2j_2} & \cdots\delta_{i_2j_N} \\ \vdots & \vdots & \vdots \\ \delta_{i_Nj_1} & \delta_{i_Nj_2} & \cdots\delta_{i_Nj_N} \end{pmatrix} \stackrel{\text{def}}{=} \delta^{j_1j_2\cdots j_N}_{i_1i_2\cdots i_N}\,.$$

The determinant of Kronecker deltas, denoted by $\delta^{j\cdots}_{i\cdots}$, is called the *generalized Kronecker delta symbol*. This symbol is convenient to write any *contraction* of indices in the product of Levi-Civita symbols:

$$\sum_{i_1,i_2,\ldots,i_n=1}^{N} \varepsilon_{i_1i_2\cdots i_n i_{n+1}\cdots i_N}\varepsilon_{i_1i_2\cdots i_n j_{n+1}\cdots j_N} = n!\,\delta^{j_{n+1}j_{n+2}\cdots j_N}_{i_{n+1}i_{n+2}\cdots i_N}\,.$$

Note that free indices in the contraction take integer values from 1 to N whereas the generalized Kronecker delta symbol in this equation is defined by the determinant of an $(N-n)\times(N-n)$ matrix.

For example, the 3-dimensional Levi-Civita symbol defines the components of the cross product of two vectors

$$(a\times b)_i = \varepsilon_{ijk}a_jb_k\,.$$

The relation

$$\varepsilon_{ijk}\varepsilon_{iln} = \det\begin{pmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{pmatrix} = \delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\,,$$

is convenient for expanding the double cross product:

$$\begin{aligned}(a\times(b\times c))_i &= \varepsilon_{ijk}a_j\varepsilon_{kln}b_lc_n = \varepsilon_{kij}\varepsilon_{kln}a_jb_lc_n \\ &= (\delta_{il}\delta_{jn} - \delta_{in}\delta_{jk})a_jb_lc_n = b_i(a,c) - c_i(a,b)\,,\end{aligned}$$

which is the so called "*bac-minus-cab*" rule for the double cross product. Similarly, for the dot product of vectors being cross products:

$$(a\times b, c\times d) = \varepsilon_{ijk}\varepsilon_{iln}a_jb_kc_ld_n = (a,c)(b,d) - (a,d)(b,c)\,.$$

Let A_{ij} be an $N\times N$ matrix. By setting $\{i_1i_2\cdots i_N\} = \{12\cdots N\}$ in (**8.2**) and multiplying it by $A_{1j_1}A_{2j_2}\cdots A_{Nj_N}$ with the subsequent summation over repeated indices, one infers that

$$\det A = \varepsilon_{j_1j_2\cdots j_N}A_{1j_1}A_{2j_2}\cdots A_{Nj_N}\,.$$

The absolute value $|\det A|$ is the volume on an $N-$dimensional parallelepiped with adjacent sides being the vectors defined by the columns of the matrix A.

For any $N-1$ vectors w_a, $a = 1, 2, ..., N-1$, define a vector n with components

$$n_i = \varepsilon_{ij_1\cdots j_{N-1}} w_{1j_1} w_{2j_2} \cdots w_{N-1 j_{N-1}} . \tag{8.3}$$

This vector is not zero if and only if the non-zero vectors w_a are linearly independent, and n is orthogonal to the span of vectors w_a because $(n, w_a) = 0$. This follows from the skew-symmetry of the Levi-Civita symbol under a permutation of two indices.

8.3.3. Oriented surface area element of the boundary of a region. Let Ω be an open set in $\mathbb{R}^N$ with a smooth boundary. If $x = F(\xi)$ is a parameterization of the boundary $\partial\Omega$, then the span of vectors $w_a = \partial_a F$ is the tangent space to $\partial\Omega$ which is an $N-1$ dimensional plane that is orthogonal to the vector n defined in (**8.3**). So, the vector n will be called a *normal* to the boundary $\partial\Omega$.

Using the contraction formula for one index in the product of Levi-Civita symbols, one infers that

$$\begin{aligned} |n|^2 = n_i n_i &= \delta^{j_1 j_2 \cdots j_{N-1}}_{i_1 i_2 \cdots i_{N-1}} w_{1j_1} w_{2j_2} \cdots w_{N-1 j_{N-1}} w_{1i_1} w_{2i_2} \cdots w_{N-1 i_{N-1}} \\ &= \det(W^T W) = J^2(\xi) . \end{aligned}$$

So, the length of n is equal to the volume of the parallelepiped with adjacent sides being vectors w_a. The vector

$$d\Sigma_i = n_i d^{N-1}\xi$$

is called an *oriented surface area element* on the boundary $\partial\Omega$ and

$$|d\Sigma| = |n| d^{N-1}\xi = J(\xi)\, d^{N-1}\xi .$$

8.3.4. Orientable surfaces. Let S be a smooth $N-1$ surface in $\mathbb{R}^N$ and $x = F(\xi)$ be its parameterization, $\xi \in R \subset \mathbb{R}^{N-1}$. If the Jacobian J does not vanish at a point ξ, then a continuous unit normal vector $\hat{n} = n/|n|$ can be defined in a neighborhood of the point $x = F(\xi)$ of S. If J vanishes at a particular point, then $\hat{n}$ is defined by a continuous extension which exists for a smooth surface (by definition, cf. Secs. **2.4.2** and **2.4.4**). The choice of a unit normal vector is not unique because $-\hat{n}$ is also a unit normal vector that is continuous near $x = F(\xi)$. Therefore a unit normal vector can be continuously defined in two ways on every patch of a smooth surface which is, by definition, the image of a proper subset of an open box R. One can say that each such patch has two sides, one side is seen from the tip of a chosen unit normal vector, and the other can be seen only if the direction of the unit normal is reversed. Indeed, a proper subset of R is a portion of an $N-1$ plane that is deformed into a portion of S by the map F without any boundary identifications because any identification can only occur on the boundary of the box R. So the image has two sides just like two sides of a portion of the plane.

Suppose $\hat{n}$ can be continuously extended to the whole S. This implies that a net variation of $\hat{n}$ along any closed curve in S must be zero. In this case, S must have two sides, one is defined by $\hat{n}$ and the other by $-\hat{n}$, like a sphere or a portion of a plane. Imagine that each side of a patch of S is colored, say, in red (for $\hat{n}$) and blue (for $-\hat{n}$). It would be impossible to flip the colors of the patch by moving the latter along any closed curve in S because it would contradict the continuity of $\hat{n}$.

A surface is said to be *orientable* if a continuous unit normal vector can be defined on it. In this case, the surface is *oriented* by the unit normal vector. An orientable surface can have two orientations.

If a smooth surface is one-sided or non-orientable, then there should exist a closed curve such that the net change of a unit normal vector along it is not zero. Imagine an ant carrying a flagpole as a unit normal always pointing up from the surface. Since a surface is one-sided, it is possible to find a closed path in the surface such that, when the ant is back to the initial point, the pole points in the direction opposite the initial one. This implies that it is not possible to define a *continuous* unit normal vector on a one-sided surface. It turns out that non-orientable surface do exist. Here is the simplest example.

8.3.5. Möbius strip. Consider a circle in a plane. Take a pole perpendicular to the plane. If the midpoint of the pole is moved along the circle while keeping the pole orthogonal to the plane, the pole sweeps a portion of a cylinder. Let z be a vector perpendicular to the plane, and x_m be the position vector of the pole midpoint relative to the center of the circle. The vectors z and x_m are orthogonal and their span is a plane normal to the circle at any point. At every point of the circle, the pole occupies that same position in this plane.

Now image that the pole is rotated in this plane about the midpoint while the midpoint moves around the circle. Suppose that when the midpoint returns to the initial point, the pole net rotation angle is π so that it will occupy the same (staring) position but the endpoints are swapped. If the pole is oriented by a vector T, then the final position is oriented by $-T$. The surface swept by the pole is smooth and one-sided by construction. If in the beginning of the motion, the swept surface is colored so that one side is red, and the other is blue, then at the end of the motion, the red side is glued to the blue one and vice versa. So, it is impossible to continuously define the either "red" or "blue" normal on this surface because, after making around the circle, the normal becomes the opposite to that at the starting point. This surface is known as a *Möbius strip*.

It is not difficult to find its parametric equations. Let ξ_1 be a parameter that labels points on the pole, so that the straight line segment $x_1 = a$, $x_2 = 0$, and $x_3 = \frac{1}{2}\xi_1$, $-b \le \xi \le b$, is the initial position of the pole of length b. So, the pole is parallel to the x_3 axis and its midpoint at a distance a from the origin on the x_1 axis. The midpoint moves along a circle $x_1 = a\cos(\xi_2)$, $x_2 = a\sin(\xi_2)$, $x_3 = 0$, making one full turn when $0 \le \xi_2 \le 2\pi$. Suppose that the pole is rotated through the angle $\frac{1}{2}\theta$ and the midpoint rotates through the angle ξ_2 so that the pole rotates through the angle π as the midpoint returns to the initial position. Then the projection of the position vector of a point of the pole relative to the midpoint onto the x_3 axis is $\frac{1}{2}\xi_1\cos(\xi_2/2)$, and its projection on the axis from the origin to the midpoint is $\frac{1}{2}\xi_1\sin(\xi_2/2)$. Therefore the position vector of a point of the pole relative to the origin reads

$$\begin{aligned} x_1 &= \Big(a + \frac{\xi_1}{2}\sin(\xi_2/2)\Big)\cos(\xi_2)\,, \\ x_2 &= \Big(a + \frac{\xi_1}{2}\sin(\xi_2/2)\Big)\sin(\xi_2)\,, \\ x_3 &= \frac{\xi_1}{2}\cos(\xi_2/2)\,, \end{aligned}$$

where $(\xi_1, \xi_2) \in R = [-b, b] \times [0, 2\pi]$. These are the parametric equations of a Möbius strip.

Let us investigate continuity of a unit normal vector along the circle traversed by the midpoint of the pole, that is, when $\xi_1 = 0$. The vector

$$n(\xi) = \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2}$$

is normal to the surface, where $\times$ denotes the cross (or vector) product in $\mathbb{R}^3$. Recall that $(x \times y)_i = \varepsilon_{ijk}x_j y_k$ where ε_{ijk} is the Levi-Civita symbol in $\mathbb{R}^3$. By evaluating the derivatives, the cross product, and setting $\xi_1 = 0$, the normal is found to be

$$\begin{aligned} n_1(0, \xi_2) &= \frac{1}{2}\sin(\xi_2/2)\cos(\xi_2)\,, \\ n_2(0, \xi_2) &= \frac{1}{2}\sin(\xi_2/2)\sin(\xi_2)\,, \\ n_3(0, \xi_2) &= \frac{1}{2}\cos(\xi_2/2)\,. \end{aligned}$$

so that $|n(0,\xi_2)| = a/2$. The values $\xi_2 = 0$ and $\xi_2 = 2\pi$ correspond to the same point of the circle. It follows from this equations that

$$n(0,2\pi) = -n(0,0)$$

Therefore a unit normal cannot be continuously defined on the Möbius strip.

The parametric equations define a smooth map of the rectangle R to $\mathbb{R}^3$ that is one-to-one in the interior of R, but maps the boundaries $\xi = 0$ and $\xi = 2\pi$ onto the same set. This identification is done with a twist (obtained by rotating the boundary through the angle π before the identification), which leads to a one-sided smooth surface. One can easily construct a similar map that sends the boundaries $\xi = 0$ and $\xi = 2\pi$ to the same line segment with any numbers of twists. A surface with an odd number of twists is one-sided.

It is worth noting that there are surfaces without boundaries (like a sphere) that are one-sided. An example is provided by the famous *Klein bottle*.

8.4. Flux of a vector field. Consider a vector field $F : \mathbb{R}^N \to \mathbb{R}^N$. Let S be a smooth orientable $N-1$ surface described by parametric equations $x_j = x_j(\xi)$, $\xi \in R$. The surface integral

$$\Phi = \int_S (F, d\Sigma) = \int_R F_j(x(\xi)) n_j(\xi)\, d^{N-1}\xi$$

is called a *flux* of the vector field F across the surface S. *The flux of a vector field can only be defined across an orientable surface, and its changes its sign when the orientation is changed.*

Suppose that F describes a flow of some quantity. For example, consider a moving air with the velocity vector field $v(x,t)$ and mass density $\rho(x,t)$ where t is time and x is a position in space. Then $F = \rho v$ is a mass flow. If S is a smooth orientable surface, then there exists a continuous unit tangent vector $\hat{n}$ on S. By construction

$$d\Phi(x,t) = \Big(F(x,t), d\Sigma(x)\Big) = \rho(x,t)\Big(v(x,t), \hat{n}(x)\Big) dS(x)$$

is the mass carried by the flow per unit time across the surface of area dS at a sample point x in the direction of $\hat{n}(x)$. Let $m(\Omega,t)$ be the total mass in a bounded region Ω. Then the flux of F cross the boundary $\partial\Omega$ defines the rate of change of mass with respect to time:

$$\frac{dm(\Omega,t)}{dt} = \frac{d}{dt}\int_\Omega \rho(x,t)\, d^3x = -\int_{\partial\Omega} \rho(x,t)\Big(v(x,t), \hat{n}(x)\Big) dS(x)\,.$$

where $\hat{n}$ points outward from Ω. Observe the minus sing in the right-hand side. If the flux is positive, then the mass stored in Ω is decreasing because there is a net mass carried away from Ω by the flow across the boundary $\partial\Omega$. This is *the integral form of the mass conservation law*.

Note that $(v, \hat{n})$ is the normal component of the velocity vector field (the scalar projection of v on $\hat{n}$ at a point x). If the vector field is orthogonal to the normal, the flux vanishes. Therefore, if the flow is tangential to the surface, the flux vanishes. This implies that the mass cannot be carried *across* S by such a flow. Therefore the surface must have two sides (or be orientable) in order for the concept of the flux to make sense. For one-sided surface, a mass flow tangential to the surface can carry a non-zero mass across the surface. For example, a particle can get to the other side of a surface patch $d\Sigma(x)$ of a Möbius band at a point x by moving tangentially to the band along a closed curve so that $d\Phi = 0$ on the curve but a mass is transferred across the patch $d\Sigma(x)$.

8.4.1. Flux integral in $\mathbb{R}^3$. A parameterization of a 2-surface S in $\mathbb{R}^3$ is defined by a C^1 map of a rectangle in $\mathbb{R}^2$ to $\mathbb{R}^3$

$$x = x(\xi_1, \xi_2)\,, \quad (\xi_1, \xi_2) \in [a, b] \times [c, d] = R$$

such that the normal vector

$$n(\xi) = \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2}$$

is not zero in R except possibly on the boundary of R. The surface area is given by

$$A(S) = \int_S dS = \int_a^b \int_c^d |n(\xi)|\, d\xi_2 d\xi_1$$

A surface integral of a function $f(x)$ over S is given by

$$\int_S f(x)\, dS = \int_a^b \int_c^d f(x(\xi))\, |n(\xi)|\, d\xi_2 d\xi_1$$

and a flux of a vector field $F(x)$ across S reads

$$\int_S (F, d\Sigma) = \int_a^b \int_c^d \Big(F(x(\xi)), n(\xi) \Big)\, d\xi_2\, d\xi_1$$

8.5. The divergence (Gauss-Ostrogradsky) theorem. Let Ω be a bounded region with a smooth boundary which is a level set of a C^1 function g with the non-vanishing gradient. The boundary divides $\mathbb{R}^N$ into two non-intersecting regions. Then unit normal $\hat{n} = \nabla g / |\nabla g|$ is continuous on the boundary (cf. Sec. **1.4**). The boundary $\partial\Omega$ is said to be oriented *positively* if the unit normal points outward from Ω. The other orientation is called *negative*. Unless stated otherwise, $\partial\Omega$ will always denote the positively oriented boundary of Ω. The *divergence* of a vector field F is defined by

$$\operatorname{div} F(x) = \sum_{j=1}^{N} \frac{\partial F_j(x)}{\partial x_j} = (\nabla, F)\,,$$

where ∇ is a formal vector with components being $\partial/\partial x_j$.

THEOREM 8.1. *Let Ω be an open bounded set in $\mathbb{R}^N$ such that its boundary piecewise smooth. Suppose that a vector field F and a function u are from class $C^1(\overline{\Omega})$. Then*

$$\int_\Omega u\,(\nabla, F)\,d^N x = -\int_\Omega (F, \nabla u)\,d^N x + \int_{\partial\Omega} u\,(F, d\Sigma)\,,$$

where $d\Sigma = \hat{n} dS$ is the surface area element on $\partial\Omega$ oriented positively.

In particular, if $u(x) = 1$, then

$$\int_\Omega \operatorname{div} F\,d^N x = \int_{\partial\Omega} (F, \hat{n})\,dS\,.$$

This statement is known as *the divergence or Gauss-Ostrogradsky theorem*. Recall that if F describes a flow of some quantity, then the divergence of F is the density of sources of the flow. The divergence theorem states that the net flux of a vector field across the boundary of a bounded region is equal to the sum of all sources of the field in the region. It should be noted that the boundary $\partial\Omega$ can have several disjoint pieces. For example, Ω can have several "cavities" obtained by removing proper open subsets from Ω. All separate parts of $\partial\Omega$ are oriented outward and the surface integral is the sum over all separate parts.

8.5.1. Green's theorem. In a two-dimensional Euclidean space, consider a bounded region Ω whose boundary is a C^1 closed curve without self-intersections. Suppose that the boundary curve is oriented counterclockwise (the x_2 axis is directed upward, while the x_1 is directed to the right). If $x_j = x_j(t)$ are parametric equations of the boundary, then the unit normal to the boundary directed outward is

$$n_j(t) = \varepsilon_{ji} T_i(t)\,, \quad T_j(t) = \frac{x_j'(t)}{|x'(t)|}$$

Indeed, suppose the origin is in the interior of Ω and $T_1 < 0$ and $T_2 > 0$ (for a counterclockwise orientation). Since $n_1 = T_2$ and $n_2 = -T_1$,

$n_{1,2} > 0$. This implies that T is obtained by rotating n counterclockwise through the angle $\frac{\pi}{2}$. Any continuous deformation of the boundary preserves this property of T and n. So the equation holds for any shape of Ω that can be continuously deformed to a disk.

The dot product of any two vectors A_j and B_j is equal to the dot product of the (dual) vectors $\varepsilon_{jk}A_k$ and $\varepsilon_{jk}B_k$:

$$\varepsilon_{jk}A_k\varepsilon_{jn}B_n = \delta_{kn}A_kB_n = A_kB_k\,,$$

by the properties of the Levi-Civita symbol in $\mathbb{R}^2$ so that

$$F_jdx_j = F_jT_jds = \varepsilon_{jk}F_kn_jds = \varepsilon_{jk}F_kd\Sigma_j$$

Therefore by the divergence theorem for a vector field $\varepsilon_{jk}F_k$

$$\text{(8.4)}\qquad \oint_{\partial\Omega} F_jdx_j = \int_\Omega \varepsilon_{jk}\partial_jF_k\,d^2x = \int_\Omega\left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right)d^2x$$

This statement is known as *Green's theorem.*

It is also valid if Ω has holes, that is, if its boundary contains several closed curves without self-intersections. In this case, the outer boundary must be oriented counterclockwise, while all the inner boundaries must be oriented clockwise. Indeed, let us cut a region Ω without any holes by a smooth curve C into two regions Ω_1 and Ω_2. Then Green's theorem can be applied to both of them. Note that the curve C is a part of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, but it has opposite orientation in them so that for any line integral of a vector field

$$\oint_{\partial\Omega_1} + \oint_{\partial\Omega_2} = \oint_{\partial\Omega}$$

because the line integral over the cut curve C is cancelled. The line integrals in the left side also contain integration over the inner boundary of Ω (over the boundary of the hole) that must be oriented clockwise if $\partial\Omega_1$ and $\partial\Omega_2$ are oriented counterclockwise. Evidently, this argument can be extended to any number of holes.

8.6. Integration by parts in $\mathbb{R}^N$. The fundamental theorem of calculus has a multi-variable generalization. Let b be a non-zero constant vector and v is a function from class $C^1(\overline{\Omega})$. Put $F = bv$ in Theorem **8.1**. Then

$$\int_\Omega u(b,\nabla)v\,d^Nx = \oint_{\partial\Omega} vu(b,d\Sigma) - \int_\Omega v(b,\nabla)u\,d^Nx$$

Since the vector b is arbitrary, the integration by parts can be stated in the form

$$\int_\Omega u(x)D_jv(x)\,d^Nx = \oint_{\partial\Omega} v(x)u(x)d\Sigma_j - \int_\Omega v(x)D_ju(x)\,d^Nx\,.$$

where $D_j = \partial/\partial x_j$, $j = 1, 2, ..., N$. If Ω is not bounded, the integration by parts can still be used with a suitable regularization. For example, it can be applied to a part of Ω that lies in a ball of radius R and the limit $R \to \infty$ should be taken after evaluation of the integrals. The answer may depend on the regularization if the improper integral does not converge absolutely.

Suppose that u and v are from class C^p and their supports are in Ω. This implies that any partial derivative of u and v up order p vanishes on the boundary $\partial\Omega$. Then by applying the integration by parts several times

$$\int_\Omega u D^\beta v \, d^N x = (-1)^\beta \int_\Omega v D^\beta u \, d^N x \,, \quad 0 \le \beta \le p \,,$$

where $D^\beta u$ stands for any partial derivative of order β. The surface integrals arising upon integration by parts vanish because of the said properties of the functions u and v.

8.6.1. Green's identity. Let the boundary of a bounded region Ω be oriented outward by a unit normal $\hat{n}$, and let u and v be functions from class $C^2(\overline{\Omega})$. Integrating the identity

$$u\Delta v - \Delta u v = \partial_j \Big(u \partial_j v - \partial_j u v \Big)$$

where $\Delta = \partial_j \partial_j$ is the Laplace operator, over Ω and using the divergence theorem to transform the integral in the right-hand side to a surface integral, one infers that

$$\int\limits_\Omega \Big(u\Delta v - \Delta u v \Big) \, d^N x = \int\limits_{\partial\Omega} \Big(u \partial_j v - \partial_j u v \Big) \, d\Sigma_j$$

$$(8.5) \qquad = \int\limits_{\partial\Omega} \Big(u \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \Big) \, dS$$

This is known as Green's first identity. Here $\frac{\partial v}{\partial n} = (\hat{n}, \nabla v)$ is the normal derivative of v.

8.7. Exercises.

1. Suppose S is a surface in $\mathbb{R}^3$ obtained by a revolution of the graph $x_3 = f(s)$, $a \le s \le b$, about the x_3 axis.
(i) Show that its parametric equations can be written in the form

$$x_1 = s\cos(\phi) \,, \; x_2 = s\sin(\phi) \,, \quad x_3 = f(s) \,, \quad (s, \phi) \in [a, b] \times [0, 2\pi]$$

or

$$x_1(s,t) = \frac{s(1-t^2)}{1+t^2}\,,\ x_2 = \frac{2st}{1+t^2}\,,\ x_3 = f(s)\,,\quad (s,t) = [a,b]\times\mathbb{R}$$

(ii) Find the normal vectors $n(s,\phi)$ and $n(s,t)$ for both parameterizations. Express the surface area in terms of the function f.

2. Let σ_N be the surface are of a unit sphere, $|x| = 1$, in $\mathbb{R}^N$. Suppose f is continuous on $\mathbb{R}^N$. Show that

$$\lim_{a\to 0^+} \frac{1}{\sigma_N a^{N-1}} \int\limits_{|x|=a} f(x)\,dS = f(0)$$

3. Show that the volume of a bounded region in $\mathbb{R}^N$ with a piecewise smooth boundary is given by the surface integral

$$V(\Omega) = \frac{1}{N}\oint_{\partial\Omega} x_j d\Sigma_j$$

Use this relation to show that the volume V_N and the surface area σ_N of an $N-$ball of radius a are related by

$$V_N(a) = \frac{a}{N}\,\sigma_N(a)\,.$$

4. Suppose that u and its partial derivative $\partial_j u$ are continuous and

$$|u(x)| \le \frac{M_0}{|x|^\alpha}\,,\quad \Big|\frac{\partial u}{\partial x_j}\Big| \le \frac{M_1}{|x|^\beta}\,,\quad |x| > R > 0$$

for some constants $M_{0,1}$. Show that if $\alpha > N-1$ and $\beta > N$, then

$$\int \frac{\partial u}{\partial x_j}\, d^N x = 0$$

Hint: Reduce the integration domain to $[-a,a]\times\mathbb{R}^{N-1}$ and use continuity of the Lebesgue integral as $a\to\infty$. Use Fubini's theorem to evaluate the integral with respect to x_j and then investigate the limit.

5. Put

$$u(x,y) = \frac{\arctan(x)}{1+y^2}\,,\quad x,y\in\mathbb{R}$$

Show that the partial derivative $D_x u = \frac{\partial u}{\partial x}$ is integrable in the plane $\mathbb{R}^2$ spanned by real variables x and y, and find the value of the integral of $D_x u(x,y)$ over the plane. Does the answer contradict to the result of Problem 4?

6. Suppose that u and v are from class C^1 and

$$|u(x)| \leq \frac{A_0}{|x|^{\alpha_0}}, \quad |Du(x)| \leq \frac{A_1}{|x|^{\alpha_1}},$$
$$|v(x)| \leq \frac{b_0}{|x|^{\beta_0}}, \quad |Dv(x)| \leq \frac{B_1}{|x|^{\beta_1}},$$

for all $|x| > R > 0$ and constants $A_{0,1}$ and $B_{0,1}$. Find a condition on parameters $\alpha_{0,1}$ and $\beta_{0,1}$ under which

$$\int u(x)Dv(x)\, d^N x = -\int Du(x)v(x)\, d^N x$$

9. Cauchy line integrals of analytic functions

9.1. Functions of a complex variable. A function $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ is called a function of a complex variable. A function $f(z)$ has a limit $w \in \mathbb{C}$ at z_0 if $|f(z) - w| \to 0$ as $|z - z_0| \to 0$, and in this case one writes

$$\lim_{z \to z_0} f(z) = w \quad \text{or} \quad f(z) \to w \quad \text{as} \quad z \to z_0 \,.$$

A function f is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$, and f is continuous on a set Ω is it is continuous at all point of Ω. The derivative defined by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided the limit exists. In particular, $(z^n)' = nz^{n-1}$ for any integer n. A function f is (complex) differentiable on a set Ω if the derivative exists at every point of Ω.

9.1.1. Analytic functions. A function of a complex variable is said to be *analytic* at a point z_0 if in a neighborhood of z_0 it is given by a power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \,, \quad |z - z_0| < R \,, \quad R > 0 \,.$$

A function is *analytic on a set* Ω if it is analytic at every point of Ω. Using Proposition **1.1** one can prove the Taylor theorem that f is from class C^∞ and its derivatives can be obtained by term-by-term differentiation of the series and $c_n = f^{(n)}(z_0)/n!$ (see Exercises). By the Taylor theorem,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n \,, \quad |z - z_1| < R_1$$

for some $R_1 > 0$ and any z_1 in the disk, $|z_1 - z_0| < R$. Therefore analyticity at a point implies analyticity in a neighborhood of the point.

For example,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is analytic in the whole complex plane because the above series has infinite radius of convergence. The function

$$f(z) = \frac{1}{1 - z}$$

is analytic everywhere except the point $z = 1$. Recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\,, \quad |z| < 1$$

This shows that the function is analytic in the open disk $|z| < 1$. Let $z_0 \neq 1$. Then the following identity holds

$$\frac{1}{1-z} = \frac{1}{1-z_0}\frac{1}{1-\frac{z-z_0}{1-z_0}}$$

Therefore near z_0, the function is represented by a power series

$$\frac{1}{1-z} = \frac{1}{1-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{1-z_0}\right)^n\,, \qquad |z-z_0| < |1-z_0|$$

whose radius of convergence coincides with the distance from 1 to z_0.

The set of analytic functions is closed under basic algebraic operations with functions. The sum and product of two analytic functions on Ω is analytic on Ω. The reciprocal of an analytic function is analytic except points where the functions vanishes. A composition of two analytic functions is analytic. These properties readily follow from basic algebraic rules for addition, multiplication, and division of power series.

9.1.2. Holomorphic functions. A function $f(z)$ is said to be *holomorphic* on an open set Ω of the complex plane if it is differentiable in a neighborhood of every point of Ω. In particular, the functions e^z and $\frac{1}{1-z}$ are holomorphic on their domains.

A major theorem in complex analysis states that *every holomorphic function is analytic and vice versa*[21]. Note that every analytic function is differentiable so it is holomorphic. It turns out that a complex differentiability (the existence of the derivative $f'(z)$) implies that all derivatives exist so that the function can be given by a Taylor series (which is a power series) so that the function is analytic.

9.1.3. Cauchy-Riemann equations. Let $f(z)$ be analytic. Put $z = x+iy$ in the power series representation of f so that

$$f(z) = u(x,y) + iv(x,y)\,, \quad x = \frac{1}{2}(z+\bar{z})\,, \quad y = \frac{1}{2i}(z+\bar{z})\,.$$

[21]see, e.g.,

Since $f(z)$ is independent of $\bar{z}$, it follows from the chain rules $\partial_{\bar{z}} = \frac{1}{2}\partial_x - \frac{1}{2i}\partial_y$ that

$$\frac{\partial f(z)}{\partial \bar{z}} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{9.1}$$

These relations are known as *the Cauchy-Riemann equations.*

The functions u and v are from class C^∞ by analyticity of f. In particular, using the Cauchy-Riemann equations it is not difficult to show that that real and imaginary parts of an analytic function in Ω are *harmonic functions*, that is, they are solutions to the Laplace equation

$$\Delta u(x, y) = 0, \quad \Delta v(x, y) = 0.$$

The assertion follows from Clairaut's theorem $\partial_x \partial_y u = \partial_y \partial_x u$ (and similarly for v) that holds for smooth functions.

9.1.4. Poles. A function $f(z)$ is said to have a *pole* at $z = z_0$ of order n if near z_0

$$f(z) = \sum_{k=1}^{n} \frac{a_k}{(z - z_0)^k} + g(z), \tag{9.2}$$

where g is analytic at z_0. If $n = 1$, the pole is called *simple*. The coefficient a_1 is called the *residue* of f at the pole z_0 and is denoted by

$$a_1 = \operatorname*{res}_{z_0} f.$$

If the pole is simple, then

$$\operatorname*{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

For example,

$$\frac{1}{1 + z^2} = \frac{1}{2i} \left(\frac{1}{z - i} - \frac{1}{z + i} \right).$$

So, the function has two simple poles at $z = \pm i$ and

$$\operatorname*{res}_{\pm i} \frac{1}{1 + z^2} = \pm \frac{1}{2i}.$$

The reciprocal of $\sin(z)$ has simple poles at $z = z_n = \pi n$, where n is an integer, and

$$\operatorname*{res}_{z_n} \frac{1}{\sin(z)} = \lim_{z \to z_n} \frac{z - z_n}{\sin(z)} = 1.$$

by the power series representation of $\sin(z)$ about $z = z_n$.

9.2. The line integral in a complex plane. A curve in the complex plane is defined in the same as a curve in $\mathbb{R}^2$. It is the range of a continuous mapping of an interval to $\mathbb{C}$ which is one-to-one except possibly some points of the interval. Any such mapping is described by parametric equations $z = w(t)$, $a \leq t \leq b$. The curve is closed if $w(a) = w(b)$. If $w(t_1) = w(t_2)$ implies that $t_1 = t_2$ (except possibly for $t_1 = a$ and $t_2 = b$ for a closed curve), then the curve is called *simple* (otherwise the curve has a self-intersection at $z = w(t_1) = w(t_2)$). If there exists a parameterization from class C^1 such that $w'(t)$ does not vanish and, for a closed curve, $w'(a) = w'(b)$, then the curve is smooth. The direction in which the curve is traversed by $z = w(t)$ with increasing t is called an orientation of the curve.

For example

$$z = w(t) = ae^{it}\,, \quad 0 \leq t \leq 2\pi$$

is a circle of radius a centered at the origin because $|w(t)| = a$. The circle is a smooth curve because $w'(t)$ is continuous and $|w'(t)| = a \neq 0$. The circle is oriented counterclockwise. Parametric equations $z = ae^{-it}$ describe the same circle oriented clockwise.

Let $z = w(t)$, $a \leq t \leq b$, be parametric equations of a smooth curve C in the complex plane. Let $f(z)$ be a continuous function of a complex variable z. Then the integral

$$\int_C f(z)\,dz = \int_a^b f(w(t))w'(t)\,dt$$

is called the *Cauchy line integral of f over the curve C*. Note that the integral depends on the orientation of C in full contrast to the line integral of a scalar function. The Cauchy line integral changes its sign if the orientation of the curve is changed. Parametric equations $z = w(\tau(t))$, $a \leq t \leq b$, $\tau(t) = b + a - t$, describe the same curve but with opposite orientation, denoted by $-C$. Then by changing the integration variable

$$\begin{aligned}\int_{-C} f(z)\,dz &= \int_a^b f(w(\tau(t)))w'(\tau(t))\tau'(t)\,dt \\ &= \int_b^a f(w(\tau))w'(\tau)\,d\tau = -\int_C f(z)\,dz\,.\end{aligned}$$

For example, if C is a circle $|z| = a$ oriented counterclockwise, then for any integer n

$$\oint_C z^n \, dz = \int_0^{2\pi} a^n e^{int} \, iae^{it} \, dt = ia^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt$$

$$= \begin{cases} 0 \,, & n \neq -1 \\ 2\pi i \,, & n = -1 \end{cases} \qquad (\mathbf{9.3})$$

By the 2π periodicity of the integrand.

9.2.1. Cauchy's integral theorem. A region in a Euclidean space is called *simply connected* if any closed curve can be continuously contracted to a point in the region without crossing its boundary. In particular, a simply connected region in the complex plane has no holes.

THEOREM **9.1**. *Let f be analytic in a simply connected region and C be a simple, closed, and smooth curve in this region. Then the line integral of f over C vanishes*

$$\oint_C f(z) \, dz = 0 \, .$$

This theorem follows from Green's theorem (**8.4**) and the Cauchy-Riemann equations (**9.1**). The hypotheses of Green's theorem are met for the Cauchy integral if $C = \partial\Omega$ (the boundary of some simply connected Ω). Therefore by Green's theorem and the Cauchy-Riemann equations

$$\begin{aligned} \oint_C f(z)dz &= \oint_C (udx - vdy) + i \oint_C (vdx + udy) \\ &= -\iint_\Omega \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + i \iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy \\ &= 0 \, . \end{aligned}$$

Green's theorem holds for piecewise simple curves and, hence, Cauchy's integral theorem is valid for simply connected regions with piecewise smooth boundaries.

9.2.2. The residue theorem. If the function is analytic except some points where it has poles, the Cauchy integral is determined by the residues of the poles.

THEOREM **9.2**. *Let f have finitely many poles at $z = z_k$, $k = 1, 2, ..., n$, in a simply connected region Ω, the boundary $\partial\Omega$ is smooth* *and oriented counterclockwise. Then*

$$\oint_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z_k} f(z)\,.$$

Take z_k and connect it to some point on the boundary by a smooth curve C_k. Let Ω_a be the region obtained from Ω by removing the curves C_k and the disks $|z - z_k| \le a$. Then the boundary of Ω_a consists of $\partial\Omega$, the circles $|z - z_k| = a$, denoted by S_k, and the curves C_k. If $\partial\Omega$ is oriented counterclockwise, then $\partial\Omega_a$ must also be oriented counterclockwise. This implies that the circles S_k are oriented clockwise, and the curves C_k must be traversed twice (from the boundary toward the pole and back after traversing S_k clockwise). Let C_k^+ denote C_k oriented from the boundary to the pole, and C_k^- from the pole to the boundary. The function f is analytic in Ω_a. By the Cauchy integral theorem

$$\int_{\partial\Omega_a} f(z)\,dz = 0\,.$$

On the other hand,

$$\int_{\partial\Omega_a} f(z)\,dz = \int_{\partial\Omega} f(z)\,dz + \sum_{k=1}^{n} \left(\int_{C_k^+} + \int_{C_k^-} + \oint_{S_k} \right) f(z)\,dz$$

The integrals over $C_k^\pm$ are taken along the same curve but with opposite orientations. So, they cancel each other in the sum. Near z_k, f has the form (**9.2**). So, only the term proportional to $(z - z_k)^{-1}$ contributes to the integral over S_k according to (**9.3**). Since the circles S_k are oriented clockwise, there is an extra minus sign as compared to (**9.3**) so that

$$\int_{\partial\Omega_a} f(z)\,dz = \int_{\partial\Omega} f(z)\,dz - 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z_k} f(z)$$

and the conclusion of the residue theorem follows.

Example 1. Let k be real parameter. Put

$$F(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + x^2}\,dx\,. \tag{9.4}$$

The integral converges absolutely for any k because

$$\lim_{n\to\infty} \int_{-n}^{n} \left| \frac{e^{ikx}}{1 + x^2} \right| dx = \lim_{n\to\infty} \int_{-n}^{n} \frac{dx}{1 + x^2} = \pi < \infty\,.$$

Therefore $F(k)$ can be computed in any suitable regularization. In particular,

$$F(k) = \lim_{n\to\infty} \int_{-n}^{n} \frac{e^{ikx}}{1+x^2}\,dx$$

One has $F(0) = \pi$. Put $f(z) = e^{ikz}(1+z^2)^{-1}$. The function f has two simple poles at $z = \pm i$ and analytic otherwise. If $k > 0$, it decays exponentially with increasing $|z|$ if the upper half-plane, $\operatorname{Im} z > 0$. If $k < 0$, it decays exponentially with increasing $|z|$ in the lower half-plane, $\operatorname{Im} z < 0$.

Let C_n^+ be the closed contour that consists of the interval $[-n, n]$ in the real axis and the circular arc $|z| = n$, $\operatorname{Im} z \geq 0$, denoted S_n^+. If C_n^+ is oriented counterclockwise, then by the residue theorem

$$\begin{aligned}\oint_{C_n^+} f(z)\,dz &= \int_{-n}^{n} f(x)\,dx + \int_{S_n^+} f(z)\,dz \\ &= 2\pi i \operatorname*{res}_{i} f(z) = \pi e^{-k}\,, \quad k > 0\,.\end{aligned}$$

The integral over S_n^+ vanishes in the limit $n \to \infty$:

$$\begin{aligned}\left|\int_{S_n^+} f(z)\,dz\right| &\overset{(1)}{=} \left|\int_0^{\pi} \frac{e^{ikne^{it}}}{1+n^2e^{2it}} ine^{it}\,dt\right| \overset{(2)}{\leq} n\int_0^{\pi} \frac{e^{-kn\sin(t)}}{|1+n^2e^{2it}|}\,dt \\ &\overset{(3)}{\leq} n\int_0^{\pi} \frac{dt}{|1+n^2e^{2it}|} \overset{(4)}{\leq} \frac{n}{n^2-1}\int_0^{\pi} dt = \frac{\pi n}{n^2-1} \to 0\end{aligned}$$

as $n \to \infty$. A justification for this chain of inequalities is as follows: (1) is obtained by using the parametric equation of S_n^+, $z = ne^{it}$, $0 \leq t \leq \pi$; (2) is obtained by moving the absolute value into the integral and by calculating $|f(ne^{it})|$; (3) holds because $k > 0$ and $\sin(t) \geq 0$ if $0 \leq t \leq \pi$; (4) follows from the triangle inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$ for $z_1 = 1$ and $z_2 = n^2e^{2it}$. Thus, by taking the limit $n \to \infty$, its concluded that $F(k) = \pi e^{-k}$ if $k > 0$.

Similarly, if $k < 0$, take the closed contour C_n^- that consists of the interval $[-n, n]$ in the real axis and the circular arc $|z| = n$, $\operatorname{Im} z \leq 0$, denoted S_n^-. If C_n is oriented *clockwise*, then by the residue theorem

$$\begin{aligned}\oint_{C_n^-} f(z)\,dz &= \int_{-n}^{n} f(x)\,dx + \int_{S_n^-} f(z)\,dz \\ &= -2\pi i \operatorname*{res}_{-i} f(z) = \pi e^{k}\,, \quad k < 0\end{aligned}$$

The reader is asked to show that

$$\left|\int_{S_n^-} f(z)\,dz\right| \leq \frac{\pi n}{n^2-1}\,, \quad n > 1\,,$$

using the same line of arguments as in the case of integration over S_n^+, but with $k < 0$. Thus,

$$F(k) = \pi e^{-|k|}\,.$$

Example 2: Fresnel's integrals. Consider the improper integral

$$\int_0^\infty e^{ix^2}\,dx \stackrel{\text{def}}{=} \lim_{n\to\infty}\int_0^n e^{ix^2}dx$$

The integral does not converge absolutely because the integral of the absolute value of the integrand, $|e^{ix^2}| = 1$, diverges. So, the value of the integral depends on regularization. This fact is emphasized by using the symbol $\stackrel{\text{def}}{=}$ (the *definition* of the left-hand side). In particular, in the regularization defined by the above limit, the integral converges and its real and imaginary parts are known as *Fresnel's integrals*.

In the complex plane, consider a closed contour C that is the boundary of the wedge of the disk of radius n: $z = \rho e^{i\theta}$ where $0 \leq \rho \leq n$ and $0 \leq \theta \leq \frac{\pi}{4}$. It consist of three smooth pieces. The first goes from $z = 0$ to $z = n$ along the real axis, the second from $z = n$ to $z = \sqrt{i}n$ along the circle $|z| = n$, and the third goes back to the origin along the line segment from $z = \sqrt{i}n$. Here $\sqrt{i} = e^{i\pi/4}$. Parametric equations of these three pieces can be chosen respectively as

$$\begin{aligned}
C_1 :&\quad z = t\,,\quad t \in [0, n]\,;\\
C_2 :&\quad z = ne^{it}\,,\quad t \in [0, \pi/4]\,;\\
C_3 :&\quad z = \sqrt{i}t\,,\quad t \in [n, 0]\,.
\end{aligned}$$

Note that C_3 must be oriented from $z = \sqrt{i}n$ to $z = 0$. This is indicated by the range $[n, 0]$ of the parameter: from $t = n$ to $t = 0$, which corresponds to the lower and upper limits of integration. The function e^{iz^2} is analytic. Therefore its line integral over C vanishes:

$$\oint_C e^{iz^2}dz = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) e^{iz^2}\,dz = 0\,.$$

One has for these integrals

$$\begin{aligned}
\int_{C_1} e^{iz^2}dz &= \int_0^n e^{it^2}\,dt\,,\\
\int_{C_2} e^{iz^2}dz &= \int_0^{\pi/4} e^{in^2e^{2it}}ine^{it}\,dt\,,\\
\int_{C_3} e^{iz^2}dz &= \int_n^0 e^{-t^2}\sqrt{i}\,dt = -e^{i\pi/4}\int_0^n e^{-t^2}dt\,.
\end{aligned}$$

Let us show that the integral over the circular arc vanishes in the limit $n \to \infty$. One has

$$\left|\int_{C_2} e^{iz^2}dz\right| \overset{(1)}{\leq} n\int_0^{\pi/4} e^{-n^2\sin(2t)}\,dt \overset{(2)}{\leq} n\int_0^{\pi/4} e^{-4n^2t/\pi}\,dt$$
$$\overset{(3)}{=} \frac{\pi}{4n}\left(1-e^{-n^2}\right) \to 0$$

as $n \to \infty$. The inequalities are justified as follows: (1) is obtained by moving the absolute value into the integral and calculating $|f(ne^{it})|$; (2) follows from the inequality $\sin(2t) \geq 4t/\pi$ that holds in the interval $0 \leq t \leq \pi/4$. Note that the graph of $\sin(2t)$ is concave downward in the interval $[0, \pi/4]$. So the secant line through the origin $(0,0)$ and the point $(\pi/4, 1)$ on the graph lies below the graph, which comprises the said inequality; (3) is obtained by evaluating the integral.

Thus the integrals over C_3 and C_1 are equal in the limit:

$$\int_0^\infty e^{ix^2}\,dx \overset{\text{def}}{=} \lim_{n\to\infty}\int_0^n e^{ix^2}\,dx = e^{i\pi/4}\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\,e^{i\frac{\pi}{4}}\,.$$

Example 3. Let us evaluate the conditionally convergent integral that was discussed in Sec.**4.4.1**:

$$I = \int_{-\infty}^{\infty}\frac{\sin(x)}{x}\,dx \overset{\text{def}}{=} \lim_{n\to\infty}\int_{-n}^{n}\frac{\sin(x)}{x}\,dx\,.$$

Owing to the continuity of the Riemann integral (cf. Sec.**3.4.5**)

$$I = \lim_{n\to\infty}\lim_{a\to 0^+}\operatorname{Im}\left(\int_{-n}^{-a} + \int_a^n\right)\frac{e^{ix}}{x}\,dx\,.$$

The function $f(z) = e^{iz}/z$ is analytic everywhere except at $z = 0$.

Let C be a closed contour that is oriented counterclockwise and consists of two intervals $[-n, -a]$ and $[a, n]$ on the real axis, and two circular arcs C_a, $|z| = a$, and C_n, $|z| = n$, which lie in the upper half-plane, $\operatorname{Im} z \geq 0$. The function f is analytic in the region bounded by C and, hence, its line integral over C vanishes:

$$\oint_C e^{iz}\frac{dz}{z} = \left(\int_{-n}^{-a} + \int_a^n\right)e^{ix}\frac{dx}{x} + \int_{C_a} e^{iz}\frac{dz}{z} + \int_{C_n} e^{iz}\frac{dz}{z} = 0$$

The imaginary part of the first two terms is equal to the integral in question after taking the limits $a \to 0^+$ and then $n \to \infty$. The integral over the arc C_n vanishes in the limit $n \to \infty$. Indeed,

$$\left|\int_{C_n} e^{iz}\frac{dz}{z}\right| \overset{(1)}{\leq} \int_0^\pi e^{-n\sin(t)}\,dt \overset{(2)}{\to} 0\,, \quad n \to \infty\,,$$

where (1) is obtained by using the parametric equation $z = ne^{it}$ and calculating $|f(ne^{it})|$; (2) follows from the Lebesgue dominated convergence theorem because $|e^{-n\sin(t)}| \leq 1 \in \mathcal{L}(0,\pi)$ and $e^{-n\sin(t)} \to 0$ a.e. as $n \to \infty$.

Let us evaluate the limit of the integral over C_a as $a \to 0^+$. Using the parametric equation $z = ae^{it}$

$$\lim_{a\to 0^+} \int_{C_a} e^{iz}\frac{dz}{z} = -i \lim_{a\to 0^+} \int_0^{\pi} e^{iae^{it}} dt = -i \int_0^{\pi} \lim_{a\to 0^+} e^{iae^{it}} dt = -i\pi$$

where the minus sign in the first equality is due to the opposite orientation of C_a in the chosen parameterization, while the order of integration and taking the limit can be interchanged by the Lebesgue dominated convergence theorem because $|e^{iae^{it}}| \leq 1 \in \mathcal{L}(0,\pi)$ for all $a \geq 0$. It is then concluded that

$$\lim_{n\to\infty} \int_{-n}^{n} \frac{\sin(x)}{x}\, dx = \pi\,.$$

9.3. Gaussian integrals with complex parameters. Consider the following Gaussian integral

$$I_N(A,b) = \int_{\mathbb{R}^N} e^{-(x,Ax)+2(b,x)}\, d^N x\,, \quad b_j \in \mathbb{C}\,,$$

in which parameters b_j are complex and the matrix A is positive, that is, $(x,Ax) > 0$ for any $x \neq 0$. This integral converges absolutely because $b_j = \beta_j + i\alpha_j$ and

$$\left|e^{-(x,Ax)+2(b,x)}\right| = e^{-(x,Ax)}\left|e^{2(\beta,x)+i(\alpha,x)}\right| = e^{-(x,Ax)+2(\beta,x)}$$

so that the integral of the absolute value converges for any $\beta \in \mathbb{R}^N$.

To compute the integral, consider first a one-dimensional case:

$$I(b) = \int_{-\infty}^{\infty} e^{-x^2+2ibx}$$

where b is real. Since $I(b)$ is independent of regularization, put

$$I(b) = \lim_{n\to\infty} \int_{-n}^{n} e^{-x^2-2ibx}\, dx\,.$$

Let R_b be a rectangle in the complex plane $\operatorname{Re} z \in [-n,n]$ and $\operatorname{Im} z \in [0,b]$. Since e^{-z^2} is analytic,

$$\oint_{\partial R_b} e^{-z^2}\, dz = 0\,.$$

Rewriting this line integral as the sum of ordinary integrals over four intervals comprising the boundary of R_b, one gets

$$\int_{-n}^{n} e^{-x^2}dx - \int_{-n}^{n} e^{-(t+ib)^2}dt + \int_{0}^{b} e^{-(n+it)^2}dt - \int_{0}^{b} e^{-(n-it)^2}dt = 0\,.$$

The second integral is the line integral of e^{-z^2} over the top horizontal boundary of R_b: $z = t+ib$, $-n \le t \le n$. The integrals over the vertical intervals vanish in the limit $n \to \infty$:

$$\left|\int_{0}^{b} e^{-(n\pm it)^2}dt\right| \le \int_{0}^{b} |e^{-(n\pm it)^2}|dt = e^{-n^2}\int_{0}^{b} e^{t^2}dt \le be^{b^2}e^{-n^2} \to 0$$

as $n \to \infty$. Note that by monotonicity $e^{t^2} \le e^{b^2}$ if $0 \le t \le b$. Therefore

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \lim_{n\to\infty}\int_{-n}^{n} e^{-(t+ib)^2}dt$$

It follows from this relation that

$$I(b) = \lim_{n\to\infty}\int_{-n}^{n} e^{-t^2-2ibt}dt = \sqrt{\pi}\,e^{-b^2}\,.$$

Using shift and scaling transformations of the integration variable and the above result, one can show that

$$I(a,\xi) = \int_{-\infty}^{\infty} e^{-ax^2+2\xi x}\,dx = \sqrt{\frac{\pi}{a}}\,e^{\xi^2/a}\,,\quad \xi\in\mathbb{C}\,,\quad a>0\,.$$

Technicalities are left to the reader as an exercise.

To compute the integral in $\mathbb{R}^N$, one can follows the same line of arguments used to evaluate the Gaussian integral $I(A,b)$ with real b. First, new integration variables are introduced in which the quadratic form is diagonal, $(x, Ax) = (y, ay)$ where a is a diagonal matrix. In doing so, the integral is proved to be the product of one-dimensional integrals so that

$$I(A,b) = \frac{\pi^{\frac{N}{2}}}{\sqrt{\det A}}\,e^{(b,A^{-1}b)}\,,\quad b\in\mathbb{C}^N$$

for any positive definite matrix A. Technicalities are left to the reader as an exercise.

9.4. Exercises.

1. Let

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n\,,\quad |z-z_0|<R$$

(i) Show that the convergence of the series implies that $|c_n|\delta^n \to 0$ as $n \to \infty$ for any $\delta < R$.
(ii) Show that

$$|c_n(z-z_0)^n| \leq Mq^n\,, \quad q<1\,, \quad |z-z_0| \leq \delta < R$$

for some constants M and q and any δ, then use Proposition **1.1** to show that the series converges uniformly.
(iii) Show that the series obtained by term-by-term differentiation of the power series any number of times also converge uniformly in the disk $|z-z_0|<R$ so that f is from class C^∞ and $c_n = f^{(n)}(z_0)/n!$.

2. Prove the equation for the Gaussian integral $I(a,\xi)$.

3. Prove the equation for the Gaussian integral $I(A,b)$.

4. Evaluate

$$I(a,b) = \int_{-\infty}^{\infty} e^{iax^2+bx}dx \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} e^{-\varepsilon x^2+iax^2+bx}dx\,,$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

5. Evaluate

$$\int e^{i(x,Ax)}d^Nx \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int e^{i(x,Ax)-\varepsilon(x,x)}d^Nx\,, \quad \det(A) \neq 0\,.$$

Express the answer in terms of the matrix A.

6. Let $\Omega \subset \mathbb{C}$ be closed, bounded, and simply connected, and its boundary $\partial\Omega$ is piecewise smooth and oriented counterclockwise. Use the residue theorem to prove the identity

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{f(w)}{w-z}\,dw$$

for any function f that is analytic on Ω and any point z that is in the interior of Ω.

7. Suppose that f is analytic everywhere. Let z and z' be two points in the complex, and C_n be a circle of radius $n > 2|z-z'|$ centered at z and oriented counterclockwise.
(i) Show that for any $w \in C_n$

$$|w-z'| > \frac{n}{2}\,.$$

(ii) Put $w = z + ne^{it}$, where $0 \le t \le 2\pi$. Use the identity from Problem 6 to show that

$$|f(z) - f(z')| \le \frac{|z - z'|}{\pi n} \int_0^{2\pi} |f(ne^{it})|\, dt$$

(iii) Prove Liouville's theorem which states that *a function that is analytic in* $\mathbb{C}$ *and bounded*

$$|f(z)| \le M\,, \quad \forall z \in \mathbb{C}$$

is constant. In particular, *if* f *is analytic everywhere and* $f(z) \to 0$ *as* $|z| \to \infty$, *then* $f(z) = 0$.
Hint: Show that $|f(z) - f(z')|$ is smaller than any preassigned positive number.

8. Let $u(x, y)$ be a real harmonic function in $\mathbb{R}^2$.
(i) Show that $u(x, y)$ can always be written as a real or imaginary part of $f(z) + g(\bar{z})$ where f and g are analytic functions in the complex plane and $z = x + iy$. *Hint*: rewrite the Laplace operator in terms of complex derivatives with respect to z and $\bar{z}$.
(ii) Use Liouville's theorem to show that a harmonic function that vanishes in the limit $x^2 + y^2 \to \infty$ is the zero function.

10. Potential-like integrals

10.1. Preliminaries. Let $E(x)$ be a conservative vector field in $\mathbb{R}^3$, that is, $E(x) = -\nabla u(x)$ where u is a potential of E. The divergence of a vector field is proportional to the density of sources of E. If the density $\rho(x)$ is known, then the potential satisfies the *Poisson equation*:

$$\text{div}\, E(x) = -\Delta u(x) = 4\pi G\rho(x)\,, \tag{10.1}$$

where Δ is the Laplace operator, G is a constant, and 4π is a convention factor whose significance will be clarified later.

For example, a static electric or gravitational field is conservative. A point-like particle located at $y \in \mathbb{R}^3$ creates an electric (or gravitational) potential at a point x

$$du(x) = G\,\frac{dm(y)}{|x-y|}$$

where $dm(y)$ is an electric charge (or mass) of the particle, and the constant G is a universal constant for the electromagnetic theory (or the gravity theory). The law is known as the Coulomb law in electricity and as the Newton gravity law in the gravity theory. If electric charges or masses are distributed over a region Ω, then $du(x)$ with $dm(y) = \rho(y)d^3y$ is the potential at x created by an element of volume d^3y at a point y. By the superposition principle (the total field at a point is the vector sum of the fields created by all sources), the potential at x is

$$u(x) = G\int_\Omega \frac{\rho(y)}{|x-y|}\, d^3y\,. \tag{10.2}$$

This suggests that the potential defined by this integral should be a solution to the Poisson equation. To verify the assertion, one has to show that u has second partials and to figure out a way for computing them. Note that if one formally interchange the order of taking the integral and the Laplacian, then the result is obviously wrong because

$$\Delta_x \frac{1}{|x-y|} = 0\,, \quad x \neq y\,, \tag{10.3}$$

that is, for any given x the integrand vanishes a.e., but $\Delta u \neq 0$ (see Exercise 4). Thus one has to investigate if the potential integral (**10.2**) has second partials and find a method to calculate them in order to verify the Poisson equation.

10.2. Potential-like integrals. Let Ω be a bounded region in $\mathbb{R}^N$. Put

$$u(x) = \int_\Omega \frac{\rho(y)}{|x-y|^\alpha}\, d^Ny\,, \qquad x \in \mathbb{R}^N\,.$$

If $x \in \bar{\Omega}$, then the integrand is singular at $y = x$. A sufficient condition for the integral to exist is to require that ρ is bounded, $|\rho(y)| \leq M$, and $\alpha < N$ because by the comparison test

$$\frac{|\rho(y)|}{|x-y|^\alpha} \leq \frac{M}{|x-y|^\alpha} \in \mathcal{L}(\Omega) \quad \text{if} \quad \alpha < N$$

for a bounded region Ω. If x does not belong to the closure $\overline{\Omega}$, then $|x-y|^{-\alpha}$ is continuous on $\overline{\Omega}$ and, hence, bounded. In this case, the integral exists if $\rho \in \mathcal{L}(\Omega)$. Without loss of generality, $\overline{\Omega}$ can be viewed as support of the density ρ and the integral is taken over $\mathbb{R}^N$. The objective is to investigate smoothness of $u(x)$ in $\mathbb{R}^N$.

In what follows, the following result will be used.

PROPOSITION **10.1**. *Let B_R be a ball of radius R, $|y| < R$. Then there exists a constant C_α such that*

$$\int_{B_R} \frac{d^N y}{|x-y|^\alpha} \leq C_\alpha R^{N-\alpha} \,. \tag{10.4}$$

To prove this relation, consider two cases. First, suppose $|x| > 2R$. In this case, using the triangle inequality and that $|y| < R$, it is concluded that

$$|x-y| \geq |x| - |y| > R \quad \Rightarrow \quad \frac{1}{|x-y|^\alpha} < \frac{1}{R^\alpha} \,.$$

Therefore

$$\int_{B_R} \frac{d^N y}{|x-y|^\alpha} \leq \frac{1}{R^\alpha} \int_{B_R} d^N y = \frac{\sigma_N}{R^\alpha} \int_0^R r^{N-1}\, dr = \frac{\sigma_N}{N} R^{N-\alpha} \,,$$

where σ_N is the surface area of the unit sphere $|y| = 1$. Suppose that $|x| \leq 2R$. Using the new variables $z = y - x$,

$$\begin{aligned} \int_{B_R} \frac{d^N y}{|x-y|^\alpha} &= \int_{B_R(x)} \frac{d^N z}{|z|^\alpha} \leq \int_{B_{3R}} \frac{d^N z}{|z|^\alpha} \leq \sigma_N \int_0^{3R} r^{N-\alpha-1}\, dr \\ &= \frac{3^{N-\alpha}\sigma_N}{N-\alpha} R^{N-\alpha} \,, \end{aligned}$$

here it was used that the ball of radius R centered at x is contained in the ball of radius $3R$ centered at the origin if $|x| \leq 2R$. The assertion follows if

$$C_\alpha = \max\left\{ \frac{\sigma_N}{N} \,,\ \frac{3^{N-\alpha}\sigma_N}{N-\alpha} \right\} \,.$$

10.3. Smoothness on the complement of support of the density. Here it is proved that *if x in the complement of $\overline{\Omega}$, that is, x is neither in Ω or in its boundary $\partial\Omega$, the potential integral has partial derivatives of any order,*

$$u(x) \in C^{\infty}\left(\mathbb{R}^N \setminus \overline{\Omega}\right), \tag{10.5}$$

and

$$D_x^{\beta} u(x) = \int_{\Omega} \rho(y)\, D_x^{\beta} \frac{1}{|x-y|^{\alpha}}\, d^N y \tag{10.6}$$

The distance between two non-intersecting regions is not zero only if their boundaries do not have common points. Let $\Omega_\delta \subset \mathbb{R}^N \setminus \overline{\Omega}$ be such that

$$d(\Omega_\delta, \Omega) = \delta > 0$$

For example, if $\Omega = B_a$ is a ball of radius a, then Ω_δ is the complement of the ball of radius $a + \delta$, that is, $x \in \Omega_\delta$ if $|x| > a + \delta$.

Define a function of two variables

$$f(x, y) = \frac{\rho(y)}{|x-y|^{\alpha}}$$

Since $|x - y| \geq \delta > 0$ for any $y \in \overline{\Omega}$ and $x \in \Omega_\delta$, its partial derivatives of any order are continuous at any $x \in \Omega_\delta$ for any y:

$$\begin{aligned} \frac{\partial}{\partial x_i} f(x, y) &= \alpha\rho(y)\, \frac{y_i - x_i}{|x-y|^{\alpha+2}} \\ \frac{\partial^2}{\partial x_j \partial x_i} f(x, y) &= \alpha\rho(y) \left(\frac{(\alpha+2)(x_i - y_i)(x_j - y_j)}{|x-y|^{\alpha+4}} - \frac{\delta_{ij}}{|x-y|^{\alpha+2}} \right) \end{aligned}$$

and similarly for $D_x^{\beta} f$. Furthermore, they are bounded by Lebesgue integrable functions independent of x:

$$\begin{aligned} \left| \frac{\partial}{\partial x_i} f(x, y) \right| &\leq \alpha|\rho(y)|\, \frac{1}{|x-y|^{\alpha+1}} \leq \frac{\alpha}{\delta^{\alpha+1}}\, |\rho(y)| \in \mathcal{L}(\Omega) \\ \left| \frac{\partial^2}{\partial x_j \partial x_i} f(x, y) \right| &\leq \alpha|\rho(y)| \frac{(\alpha+2) + \delta_{ij}}{|x-y|^{\alpha+2}} \leq \frac{\alpha(\alpha+3)}{\delta^{\alpha+2}}\, |\rho(y)| \in \mathcal{L}(\Omega) \end{aligned}$$

where $x \in \Omega_\delta$ and $y \in \Omega$. Here the inequality $|x_j| \leq |x|$ was used. In general,

$$|D_x^{\beta} f(x, y)| \leq \frac{M_\beta}{\delta^{\alpha+\beta}}\, |\rho(y)| \in \mathcal{L}(\Omega)$$

for some constant M_β (that depends on α). By Theorem **7.2** u has continuous partial derivatives of any order in Ω_δ for any $\delta > 0$, and the conclusions (**10.5**) and (**10.6**) follow.

10.4. Asymptotic behavior. Let us investigate the asymptotic behavior of the potential-like integral when $|x| \to \infty$. Since Ω is bounded there is a ball B_R that contains it. It follows from the triangle inequality

$$|x-y| \geq |x| - |y|\,, \quad |x| > R\,, \quad y \in \Omega$$

and $|\rho(y)| \leq M$ that

$$|u(x)| \leq \int_\Omega \frac{M}{(|x|-|y|)^\alpha}\, d^N y \leq \frac{M}{|x|^\alpha} \int_\Omega \frac{d^N y}{(1-|y|/R)^\alpha} = \frac{C_{\alpha,0}}{|x|^\alpha}\,,$$

where $|x| > R$. So, the potential falls off to zero as $|x| \to 0$. A similar analysis applied to Eq. (**10.6**) shows that

$$|D^\beta u(x)| \leq \frac{C_{\alpha,\beta}}{|x|^{\alpha+\beta}}\,, \quad |x| > R\,.$$

Technical details are left to the reader as an exercise.

10.5. Continuity on support of the density. *If the density is bounded,*

$$|\rho(y)| \leq M\,, \quad \forall y \in \Omega$$

then the potential integral is a continuous function everywhere, $u \in C^0$.

Let x_0 and x be two points in $\overline{\Omega}$. One has to show that $u(x)$ can get arbitrary close to $u(x_0)$ and stay arbitrary close for all x that are close enough to x_0. Put

$$g(x,y) = \left| \frac{1}{|x_0-y|^\alpha} - \frac{1}{|x-y|^\alpha} \right|$$

Then

$$\begin{aligned}|u(x_0) - u(x)| &\leq \int_\Omega |\rho(y)|\, g(x,y)\, d^N y \leq M \int_\Omega g(x,y)\, d^N y \\ &= M \left(\int_{\Omega \setminus B_R(x_0)} + \int_{B_R(x_0)} \right) g(x,y)\, d^N y\end{aligned}$$

Let us show that the integrals can be made arbitrary small for sufficiently small radius R such that $|x_0 - x| < R$. This would prove the assertion.

Using (**10.4**),

$$\begin{aligned}\int_{B_R(x_0)} g(x,y)\, d^N y &\leq \int_{B_R(x_0)} \frac{d^N y}{|x_0-y|^\alpha} + \int_{B_R(x_0)} \frac{d^N y}{|x_0-y|^\alpha} \\ &\leq 2C_\alpha R^{N-\alpha} \to 0\end{aligned}$$

as $R \to 0$ because $N > \alpha$.

To show that the other integral is also small, note that the function $g(x,y)$ is a continuous function in the set

$$|x-x_0|\le \frac{R}{2}\,,\quad |y-x_0|\ge R\,,\ y\in\overline{\Omega}$$

This set is bounded and closed. By the extreme value theorem, g attains its extreme values in the set. In particular, since $g(x,y)\ge 0$, its absolute minimum is reached at $x=x_0$, $g(x_0,y)=0$. Its maximum is reached at some point that depends on R, $x=x_R$ and $y=y_R$. The maximal value depends on R:

$$\max g = g(x_R,y_R)=C(R)$$

If $R\to 0$, then $x\to x_0$. Therefore by continuity of g, the maximal value $C(R)$ tends to 0 as $R\to 0$. Hence,

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} g(x,y)\,d^Ny &\le C(R)\mu\Big(\Omega\setminus B_R(x_0)\Big)\\ &\le C(R)\mu(\Omega)\to 0\end{aligned}$$

as $R\to 0$. Note that the measure (volume) $\mu(\Omega)<\infty$ is finite because Ω is bounded. Therefore the integral can be made arbitrary small if $R\to 0$ for all x close enough to x_0: $|x-x_0|\le \frac{R}{2}$.

10.6. Differentiability on support of the density. *If the density is bounded, $|\rho(y)|\le M$, then the potential integral has continuous partial derivatives up to order p everywhere with p being the largest integer such that $\alpha+p<N$, and, in this case, $u\in C^p(\mathbb{R}^N)$ and*

$$D_x^\beta u(x)=\int_\Omega \rho(y)\,D_x^\beta \frac{1}{|x-y|^\alpha}\,d^Ny\,,\quad \beta\le p\,. \tag{10.7}$$

Put

$$u_j(x)=\int_\Omega \rho(y)\,\frac{\partial}{\partial x_j}\,\frac{1}{|x-y|^\alpha}\,d^Ny=\alpha\int_\Omega \rho(y)\frac{x_j-y_j}{|x-y|^{\alpha+2}}\,d^Ny\,.$$

If u_j are proved to be continuous and $D_ju=u_j$, then the assertion is true if $p=1$. The continuity of u_j is proved in the same way as the continuity of u. One has

$$\begin{aligned}|u_j(x_0)-u_j(x)| &\le \alpha M\int_\Omega g_j(x,y)\,d^Ny\\ &=\alpha M\left(\int_{\Omega\setminus B_a(x_0)}+\int_{B_a(x_0)}\right) g_j(x,y)\,d^Ny\,,\end{aligned}$$

where

$$g_j(x,y) = \left| \frac{x_{0j} - y_j}{|x_0 - y|^{\alpha+2}} - \frac{x_j - y_j}{|x - y|^{\alpha+2}} \right| .$$

The integral over the ball $B_a(x_0)$ can be made arbitrary small for all $|x_0 - x| < a$ with small enough a. This conclusion follows from (**10.4**) and the inequality

$$\left| \frac{x_j - y_j}{|x - y|^{\alpha+2}} \right| \le \frac{1}{|x - y|^{\alpha+1}} .$$

Note that (**10.4**) holds if $\alpha+1 < N$ in this case. Using the continuity of $g_j(x,y)$ in the same way as the continuity of $g(x,y)$ when proving the continuity of u, one can show that the integral over the complement of the ball, $\Omega \backslash B_a(x_0)$, can also be made arbitrary small for all $|x_0 - x| < \frac{a}{2}$ and small enough $a > 0$.

A proof of the equality $D_j u = u_j$ is analogous to the proof of Theorem **7.2**. Since u_j is continuous, by the fundamental theorem of calculus

$$\frac{\partial}{\partial \xi_j} \int_{x_{0j}}^{\xi_j} u_j(x_1, ..., x_j, ..., x_N)\, dx_j = u_j(x_1, ..., \xi_j, ..., x_N)$$

On the other hand, using Fubini's theorem

$$\begin{aligned} \int_{x_{0j}}^{\xi_j} u_j(x)\, dx_j &= \int_{x_{0j}}^{\xi_j} \int_\Omega \rho(y) \frac{\partial}{\partial x_j} \frac{1}{|x - y|^\alpha}\, d^N y dx_j \\ &\overset{(1)}{=} \int_\Omega \rho(y) \int_{x_{0j}}^{\xi_j} \frac{\partial}{\partial x_j} \frac{1}{|x - y|^\alpha}\, dx_j\, d^N y \\ &= u(x_1, ..., \xi_j, ..., x_N) - u(x_1, ..., x_{0j}, ..., x_N) \end{aligned}$$

Taking the partial derivative $\frac{\partial}{\partial \xi_j}$ of both sides of this relation, it is concluded that the partial derivatives of u coincide with u_j. Here (1) holds because the integrand in the iterated integral is Lebesgue integrable on $\Omega \times (x_{0j}, \xi_j)$ and, by Fubini's theorem the order of integration can be changed. Indeed the iterated integral of the absolute value is finite:

$$\begin{aligned} \int_{x_{0j}}^{\xi_j} \int_\Omega \left| \rho(y) \frac{\partial}{\partial x_j} \frac{1}{|x - y|^\alpha} \right| d^N y dx_j &\le \alpha M \int_{x_{0j}}^{\xi_j} \int_\Omega \frac{d^N y}{|x - y|^{\alpha+1}}\, dx_j \\ &\le \alpha M \int_{x_{0j}}^{\xi_j} \int_{B_R} \frac{d^N y}{|x - y|^{\alpha+1}}\, dx_j \\ &\le \alpha M |\xi_j - x_{0j}| C_{\alpha+1} R^{N-\alpha-1} \end{aligned}$$

where the boundedness of Ω was used, $\Omega \subset B_R$ for large enough radius R, and the latter inequality follows from Proposition **10.1** if $\alpha+1 < N$.

If $\alpha + 2 < N$, then the above arguments can be applied to the functions $u_j(x)$ (instead of u) to show that partial derivatives of u_j (or second partials of u) are continuous and the conclusion of the stated theorem holds. This iterative process holds as long as $\alpha + \beta < N$, $\beta = 0, 1, ..., p$.

10.6.1. Smooth density with a bounded support. In applications, one often deal with densities that are compactly supported and smooth in the interior of the support. In this case, the potential integral can be a smoother function than that found in the previous section.

Let a compact set Ω with a smooth boundary be a support of ρ and partial derivatives of ρ be continuous on Ω. In this case, the integration by parts in the integral representation (**10.7**) of the partials Du is permitted,

$$\begin{aligned} Du(x) &= \int \rho(y) D_x \frac{1}{|x-y|^\alpha}\, d^N y = -\int_\Omega \rho(y) D_y \frac{1}{|x-y|^\alpha}\, d^N y \\ &= -\int_{\partial\Omega} \frac{\rho(y)\, d\Sigma_y}{|x-y|^\alpha} + \int_\Omega \frac{D\rho(y)}{|x-y|^\alpha}\, d^N x\,, \end{aligned} \tag{10.8}$$

despite that the integrand is singular for any $x \in \Omega$. This is established by the standard trick based on using the continuity of the Lebesgue integral. Suppose that x is in the interior of Ω. Then a ball $B_a(x)$ lies in the interior of Ω for any small enough radius a. The density ρ and $|x-y|^{-\alpha}$ are from class C^1 in $\Omega \setminus B_a(x)$ and the integration by parts is justified:

$$\begin{aligned} Du(x) = &-\lim_{a\to 0^+} \int_{\Omega\setminus B_a(x)} \rho(y) D_y \frac{1}{|x-y|^\alpha}\, d^N y = -\int_{\partial\Omega} \frac{\rho(y)\, d\Sigma_y}{|x-y|^\alpha} \\ &+ \lim_{a\to 0^+} \left(\oint_{\partial B_a(x)} \frac{\rho(y)(y-x)}{|x-y|^{\alpha+1}}\, dS_y + \int_{\Omega\setminus B_a(x)} \frac{D_y \rho(y)}{|x-y|^\alpha}\, d^N y \right), \end{aligned}$$

where the boundary of $\Omega \setminus B_a(x)$ is oriented outward. It consists of $\partial\Omega$ and the sphere $|y-x| = a$. The outward unit normal on the sphere is $n_y = (x-y)/a$ (it is directed toward the center of the ball) and $d\Sigma_y = n_y dS_y$. Since $D\rho$ are continuous on Ω, they are bounded and, hence, by continuity of the Lebesgue integral

$$\lim_{a\to 0^+} \int_{\Omega\setminus B_a(x)} \frac{D_y \rho(y)}{|x-y|^\alpha}\, d^N y = \int_\Omega \frac{D_y \rho(y)}{|x-y|^\alpha}\, d^N y\,.$$

It remains to show that the integral over the sphere vanishes in the limit $a \to 0^+$. Since $|\rho(y)| \leq M$, one has

$$\Big| \oint_{\partial B_a(x)} \frac{\rho(y)(y-x)}{|x-y|^{\alpha+1}}\, dS_y \Big| \leq \frac{M}{a^\alpha} \int_{|z|=a} dS = \frac{M\sigma_N a^{N-1}}{a^\alpha} \to 0$$

because by assumption $\alpha + p < N$ for some integer $p \geq 1$. This competes a proof of Eq. (**10.8**).

There are two consequences of (**10.8**). First, if the density vanishes on the boundary $\partial\Omega$, that is, ρ is from class C^0 in the whole space (it does not have a jump-discontinuity at $\partial\Omega$), then the surface integral in (**10.8**) vanishes. Therefore the second partials D^2u are also continuous in the whole space and can be computed by rule (**10.7**) applied to the representation (**10.8**) of Du. Furthermore if ρ is from class $C^q(\mathbb{R}^N)$, then all its partials up to order q vanish on $\partial\Omega$. Therefore the integration by parts can be carried out q times, and all surface integrals arising upon this procedure vanish. It follows that the potential is from class C^{p+q} and

$$D^\beta D^q u(x) = \int_\Omega D^q_y \rho(y) D^\beta_x \frac{1}{|x-y|^\alpha}\, d^N y\,, \tag{10.9}$$

where $\beta \leq p$. In particular, if the density ρ and all its partial derivatives of any order are continuous in the whole space and have bounded support, then u is from class C^∞.

Second, if ρ is not continuously extendable to whole space, then the smoothness of Du also depends on the smoothness of the surface integral in (**10.8**). This kind of surface integrals are called *surface potentials*:

$$v(x) = \int_S \frac{\sigma(y)}{|x-y|^\alpha}\, dS_y\,,$$

where S is a smooth M surface in $\mathbb{R}^N$, $M \leq N-1$. Evidently, if x is not in the surface S, then partials of any order of the integrand with respect to x are bounded on $\partial\Omega$ (there is no singularity in the integrand) and, hence, by Theorem **7.2** v is from class C^∞ in $\mathbb{R}^N \setminus S$, provided the density σ is bounded on S and S has a finite surface area. A loss of smoothness happens on S. One can show that *if the surface density σ is bounded on S, then $v \in C^p$ where p is the largest integer for which $\alpha + p < M$* (see Exercises), which is more restrictive than the condition in (**10.7**). For example, if $\alpha + 1 < N$ but $\alpha + 2 \geq N$, then the surface potential in (**10.8**) is not differentiable at $\partial\Omega$ (here $M = N-1$), whereas the second integral is differentiable everywhere if $|D\rho|$ is bounded.

10.6.2. Constant density. To further illustrate the point, let $\rho(x) = \rho_0$ is a non-zero constant in a ball $\Omega = B_R \subset \mathbb{R}^3$ and vanishes otherwise. The density is not continuous at the boundary sphere $|x| = R$ where it has a jump discontinuity, while all its partials vanish for $|x| \neq R$ and, hence, have continuous extensions to $|x| = R$. Let $\alpha = 1$ so that $u(x)$ is an electric potential of a uniformly charged ball or a gravitational potential of a ball with uniformly distribution mass. The potential is easy to calculate in spherical coordinates with the axis parallel to x:

$$\begin{aligned} u(x) &= 2\pi\rho_0 \int_0^R \int_{-\pi}^{\pi} \frac{\sin(\phi)\, d\phi\, r^2\, dr}{\sqrt{r^2 + |x|^2 - 2\cos(\phi) r|x|}} \\ &= \frac{\pi\rho_0}{|x|} \int_0^R \Big(r + |x| - |r - |x||\Big)\, r\, dr \\ &= \frac{\pi\rho_0}{3} \begin{cases} 3R^2 - |x|^2 & , |x| < R \\ \frac{2R^3}{|x|} & , |x| \geq R \end{cases} \end{aligned}$$

Here $\alpha + 1 < N$ but $\alpha + 2 = N$. So, u and its gradient are continuous everywhere, which also readily follows from the explicit form. However, the second partials have jump discontinuities on the sphere $|x| = R$. Since $\nabla\rho = 0$, the gradient of u (or the field) is defined only by the surface integral

$$\nabla u(x) = \rho_0 \oint_{|y|=R} \frac{d\Sigma_y}{|x - y|}\,.$$

By the aforementioned assertion, the field is continuous because $\alpha < N - 1$, but the partials of the field components are generally not continuous because $\alpha + 1 = N - 1$, which is indeed so by the explicit form of u.

10.7. Solutions to the Poisson equation in $\mathbb{R}^3$. The Newton gravity law or the Coulomb law are obtained from experimental observations and, hence, are mathematical models for the physical reality. The solution (**10.2**) is heuristically derived from these models and the superposition principle. A standard objective of mathematical modeling of reality is to investigate a mathematical consistency of any such model. If the Poisson equation indeed describes a potential of a static gravitational or electric field created by distributed sources, it is then expected that

- (i) *a solution exists;*
- (ii) *it is unique (up to an additive constant);*
- (iii) *it depends continuously on the density, that is, small variations of the density produce small variations of the solution.*

A problem whose solution always exists, is unique, and depends continuously on parameters is called a *well-posed problem*. So, mathematical models of the physical reality should lead to well-posed problems.

A classical problem for the Poisson equation (**10**.**1**) is to find a function u from class C^2 that satisfies the equation for a given density ρ that is smooth enough in order for a solution to exist. Let ρ be from class C^1 and supported in a bounded region Ω with a smooth boundary. Let us show that (**10**.**2**) is a solution to (**10**.**1**). If x is in the complement of Ω, then it follows from (**10**.**3**) and (**10**.**7**) that

$$\Delta u(x) = G\int_\Omega \rho(y)\Delta_x \frac{1}{|x-y|}\, d^3y = 0\,, \quad x \in \Omega^c$$

as required. By (**10**.**9**),

$$\Delta u(x) = -G\int_\Omega \Big(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\Big)\, d^3y\,, \quad x \in \Omega\,.$$

because $\rho(y) = 0$ if $y \in \partial\Omega$, ρ is continuously differentiable and $\nabla_x|x - y| = -\nabla_y|x-y|$. Let $B_a(x)$ be a ball of radius $a > 0$ centered at x and x be in the interior of Ω. By continuity of the Lebesgue integral and integration by parts

$$\begin{aligned}\Delta u(x) &= -G\lim_{a\to 0^+}\int_{\Omega\setminus B_a(x)} \left(\nabla_y\rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y \\ &= -G\lim_{a\to 0^+}\left(\int_{|y-x|=a} \rho(y)\Big(\nabla_y \frac{1}{|x-y|}, d\Sigma_y\Big)\right. \\ &\qquad \left. - \int_{\Omega\setminus B_a(x)} \rho(y)\Delta_y \frac{1}{|x-y|}\, d^3y\right)\end{aligned}$$

By (**10**.**3**) the volume integral vanishes. Changing variables in the surface integral $y = az + x$ so that the integration sphere becomes the unit sphere $|z| = 1$ and $d\Sigma = -za^2dS_z$ (the sphere is oriented by the unit normal directed toward the center), one infers that

$$\Delta u(x) = -G\lim_{a\to 0^+}\int_{|z|=1} \rho(x+az)dS_z = -4\pi G\rho(x)$$

because by the Lebesgue dominated convergence theorem the order of taking the limit and integration can be interchanged. Note that ρ is bounded and continuous. The same conclusion can be reached if $x \in \partial\Omega$. In this case, $B_a(x)$ is not contained in the interior of Ω for all small enough a so that the surface integral is taken over the part of the sphere that lies in Ω. However, this does not change the conclusion because the Lebesgue dominated convergence theorem still applies to this part of the sphere. Note that $\rho(x) = 0$ if $x \in \partial\Omega$. Thus, (**10.2**) is a solution to (**10.1**).

The solution (**10.2**) is not unique because any solution can be changed by adding to it a general solution to the Laplace equation (any harmonic function in space). The uniqueness is achieved by imposing a boundary condition in the asymptotic region $|x| \to \infty$. The asymptotic condition must be such that the only harmonic function that satisfies it is a constant function because there is no field in space without sources. It will be shown later that if ρ is compactly supported, then it is sufficient to demand that any solution is bounded in the asymptotic region. This is the case for (**10.2**) because it is falling off as $|x|^{-1}$ when $|x| \to \infty$. Thus, the solution is unique.

Let us investigate its continuity with respect to ρ. Let $u(x)$ and $\tilde{u}(x)$ be solutions (**10.2**) for the densities $\rho(x)$ and $\tilde{\rho}(x)$, respectively. Their supports are bounded and lie in a ball of radius R. Suppose that

$$\sup |\rho(x) - \tilde{\rho}(x)| \leq \varepsilon_0 \,.$$

Then it follows from (**10.4**) that

$$|u(x) - \tilde{u}(x)| \leq \int_{|y|<R} \frac{|\rho(y) - \tilde{\rho}(y)|}{|x-y|}\, d^3y \leq C_1 R^2 \varepsilon_0 \,,$$

which holds for any $x \in \mathbb{R}^3$. So, small variations of the density produce small changes in the potential. The Poisson equation supplemented by appropriate boundary conditions provides a good mathematical model for static gravity and electrostatics.

If the density is compactly supported and smooth, but not continuous on the boundary of its support Ω, the Poisson equation (**10.1**) does not make sense on $\partial\Omega$. However, the function (**10.2**) satisfies (**10.1**) for any x that is not in $\partial\Omega$. The reasonings go along the same line as for a continuous density but in this case Eq. (**10.8**) must be used to calculate the gradient of u instead of (**10.9**). The surface integral is a smooth function for all x in the interior of Ω (Exercise 3) so that Δu can be computed in the same way as for a continuous density with an extra term given by the surface integral. The surface integral is canceled by the boundary term arising from the integration by parts in the integral over $\Omega \setminus B_a(x)$ (Exercise 4). A general solution differs from (**10.2**) in Ω^c by a harmonic function and in the interior of Ω also by a harmonic function. So, a general solution may not decay to zero when $|x| \to \infty$ and it is not necessarily continuous at $\partial\Omega$ in contrast to (**10.2**). It will be shown later that if the solution is required to be bounded in space and continuous at $\partial\Omega$, then (**10.2**) is the unique solution to the Poisson equation in this case. It depends continuously on the density just as in the previously considered case. Thus, the problem is well posed in this case as well.

10.8. Exercises.

1. Show that if ρ is a bounded function, then the function defined by the line integral in $\mathbb{R}^2$

$$v(x) = \int_S \frac{\rho(y)}{|x-y|^\alpha}\, ds_y$$

where S is a circle $|y| = a$, is continuous in $\mathbb{R}^2$ if $0 < \alpha < 1$, and u is from class C^∞ in $\mathbb{R}^2 \setminus S$ and, in this case,

$$D_x^\beta v(x) = \int_S D_x^\beta \frac{\rho(y)}{|x-y|^\alpha}\, ds_y$$

Hint: If $y_1 = a\cos(\theta)$ and $y_2 = a\sin(\theta)$, then $ds_y = ad\theta$ and $0 \le \theta \le 2\pi$ for S.

2. Extend the conclusion of Problem 1 to the case when S is a smooth simple curve of a finite length.

3. Surface potentials. Let S be an smooth M surface in $\mathbb{R}^N$, $M \le N-1$. Define a function v by the surface integral (called a surface potential)

$$v(x) = \int_S \frac{\rho(y)}{|x-y|^\alpha}\, dS_y\,, \quad 0 < \alpha < M$$

(i) Show that $v \in C^\infty(\mathbb{R}^N \setminus S)$;
(ii) Show that $v \in C^p(\mathbb{R}^N)$ where p is the largest integer such that $\alpha + p < M$, and

$$D_x^\beta v(x) = \int_S D_x^\beta \frac{\rho(y)}{|x-y|^\alpha}\, dS_y$$

where $\beta \ge 0$ in (i) and $0 \le \beta \le p$ in (ii).

4. Solution to the Poisson equation.
Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a smooth boundary. Suppose that $\rho \in C^1(\bar{\Omega})$ and $\rho(x) = 0$ if x is not in $\bar{\Omega}$. Show that

$$\Delta u(x) = -4\pi\rho(x)\,, \quad x \in \mathbb{R}^3 \backslash \partial\Omega\,, \quad u(x) = \int \frac{\rho(y)}{|x-y|}\, d^3y$$

and u is continuous at $\partial\Omega$ by justifying each of the following assertions:

(i) $u \in C^1(\mathbb{R}^3)$,

(ii) $\Delta_x \dfrac{1}{|x-y|} = 0, \quad \forall x \neq y$

(iii) $x \notin \bar{\Omega} \quad \Rightarrow \quad \Delta u(x) = 0$,

(iv) $x \in \Omega \quad \Rightarrow \quad \Delta u(x) = -\left(\nabla, \int_{\partial\Omega} \dfrac{\rho(y) d\Sigma_y}{|x-y|}\right)$

$$- \lim_{a \to 0^+} \int_{\Omega \setminus B_a(x)} \left(\nabla_y \rho(y), \nabla_y \frac{1}{|x-y|}\right) d^3y$$

(v) $\Delta u(x) = -\lim_{a \to 0^+} \int_{|y-x|=a} \rho(y) \left(\nabla_y \dfrac{1}{|x-y|}, d\Sigma_y\right) = -4\pi\rho(x)$.

5. The Poisson equation in $\mathbb{R}^N$. Let ρ be a smooth compactly supported function in $\mathbb{R}^N$, $N \geq 3$.
(i) Show that

$$\Delta_x \frac{1}{|x-y|^{N-2}} = 0, \quad x \neq y.$$

(ii) Show that

$$\Delta u(x) = -G_N \rho(x), \quad u(x) = \int \frac{\rho(y)\, d^N y}{|x-y|^{N-2}}.$$

where $G_N = (N-2)\sigma_N$ and σ_N is the area of the unit sphere $|x| = 1$.

(iii) Let Ω be a bounded region in $\mathbb{R}^N$ with a smooth boundary. Let the density ρ vanish outside of Ω and be smooth in Ω. Suppose, in addition, that ρ is piecewise continuous (ρ is from class $C^0(\bar{\Omega})$). Investigate continuity of partials $D^\beta u$ and find the largest order β for which the partials are continuous and can be calculated by interchanging the order of differentiation and integration. In particular, show that u satisfies the Poisson equation for any $x \notin \partial\Omega$.

11. Functions defined by improper integrals

11.1. Conditionally convergent integrals. Let $\{\Omega_n\}$ be an exhaustion of Ω. Let f be locally integrable on Ω. Consider a sequence of integrals

$$\int_{\Omega_n} f(x)\, d^N x\,.$$

If f is integrable on Ω, then the sequence always converges to the integral of f over Ω by continuity of the Lebesgue integral. If f is not integrable on Ω, then the sequence may still converge but the limit depends on the choice of the exhaustion because it does not converges absolutely

$$f \notin \mathcal{L}(\Omega) \quad \Rightarrow \quad \int_\Omega |f(x)|\, d^N x = \lim_{n\to\infty} \int_{\Omega_n} |f(x)|\, d^N x = \infty\,.$$

In this case, f is said to be *conditionally integrable* on Ω and the value of the limit is called a *conditional integral* of f over Ω. The word "conditional" refers to that the limit must be computed in a specified exhaustion (or regularization) of the integral.

11.2. Abel's theorem. Abel's theorem for conditionally convergent integrals is similar to Abel's theorem for conditionally convergent series. *Hypotheses of the theorem are*:

(i) $f(x) = \alpha(x)\beta(x)\,, \quad \forall x > a$

(ii) $\alpha(x) > 0\,, \quad \alpha(x) \to 0$ monotonically for $x \to \infty$

(iii) $\beta \in C^0[a,\infty)\,, \quad \alpha \in AC^0[0,\infty)\,.$

(iv) $\left|\int_c^d \beta(x)\, dx\right| \le \sigma\,, \quad \forall c, d \ge a$

The latter condition means that integrals of β over any finite interval are bounded and the bound σ is independent of the interval. *The conclusion of the theorem is that the limit*

$$I(a) = \lim_{R\to\infty} \int_a^R f(x)\, dx$$

exists, and for any $b > a$

$$|I(b)| \le \sigma\alpha(b)\,.$$

The latter relation provides an estimate of the rate of convergence in the following sense:

$$\left| I(a) - \int_a^R f(x)\, dx \right| \le \sigma\alpha(R) \to 0 \text{ as } R \to \infty$$

Note that the integrability of f on (a, ∞) is not required. So, the integral of $|f(x)|$ over (a, ∞) can diverge. The hypothesis for the function β implies that the mean value of β over an interval is decreasing with increasing the length of the interval. This happens when β is bounded and oscillates about zero, like trigonometric functions. For example, the integrals

$$\left|\int_c^d e^{ikx}\,dx\right| = \left|\frac{e^{ikd} - e^{ikc}}{ik}\right| \le \frac{2}{k} = \sigma$$

are bounded and the bound is independent of the interval of integration. The monotonic decrease of α and boundedness of β does not guarantee integrability of $\beta\alpha$ on (a, ∞). But owing to monotonicity of α and oscillations of β, there are cancellations in the integral of the product $\alpha\beta$ over an ever increasing interval so that the integral conditionally converges. Abel's theorem offers sufficient conditions for conditional convergence of the integral.

Let us prove Abel's theorem. Put

$$\sigma_a(x) = \int_a^x \beta(y)\,dy\,.$$

By continuity of β, the function σ_a is continuously differentiable, and $\sigma_a'(x) = \beta(x)$ by the fundamental theorem of calculus. By the hypothesis (iv), the function σ_a is bounded:

$$|\sigma_a(x)| \le \sigma\,, \quad x > a\,.$$

Since α is absolutely continuous, the integration by parts is permitted in the integral of f over (a, R) (see Sec.**7.6.2**):

$$\int_a^R f(x)\,dx = \int_a^R \alpha(x)\,d\sigma_a(x) = \alpha(R)\sigma_a(R) - \int_a^R \sigma_a(x)\alpha'(x)\,dx$$

because $\sigma_a(a) = 0$. Since $\alpha(R) \to 0$ as $R \to \infty$, it is concluded that $|\alpha(R)\sigma_{a,R}| \le \sigma\alpha(R) \to 0$ and, therefore the integral of f converges if and only if the integral of $\sigma_a\alpha'$ converges. But $\alpha'(x) \le 0$ a.e. because α is monotonically decreasing, which implies that the said integral converges absolutely:

$$\begin{aligned}\int_a^R |\sigma_a(x)\alpha'(x)|\,dx &\le \sigma\int_a^R |\alpha'(x)|\,dx = -\sigma\int_a^R \alpha'(x)\,dx \\ &= \sigma\alpha(a) - \sigma\alpha(R) \le \sigma\alpha(a) < \infty\,,\end{aligned}$$

by Theorem **7.3**. The above inequalities also hold if the integration interval (a, R) is changed to (b, R) where $b > a$. By taking the limit

$R \to \infty$, it is concluded that

$$|I(b)| = \left| \lim_{R\to\infty} \int_b^R \sigma_a(x)\alpha'(x)\,dx \right| \le \lim_{R\to\infty} \int_b^R |\sigma_a(x)\alpha'(x)|\,dx \le \sigma\alpha(b)\,,$$

as required.

11.3. Differentiability of Fourier transforms revisited. It was shown in Sec. **7.4.1** that the Fourier transform

$$F(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx$$

is from class C^p if $x^p f(x)$ is integrable on $\mathbb{R}$. It turns out that even if $x^p f(x)$ is not integrable it is possible to show that F can be from class C^p at least in some interval without evaluating the integral explicitly, *provided the Fourier integral of $x^p f(x)$ converges conditionally.* The technique is based on combining Abel's theorem and Theorem **1.5.5**. The basic idea is illustrated with Example (**9.4**).

The function (**9.4**) is from class C^∞ on any interval that does not contain $k = 0$. However it is shown in Sec. **7.3.1** that the hypotheses of Theorem **7.2** are too restrictive to detect differentiability. Can the differentiability for all $k \ne 0$ be detected *without* evaluation of the integral?

Define a sequence

$$F_n(k) = \int_{-n}^{n} g(k,x)\,dt = \int_{-n}^{n} \frac{e^{ikx}}{1+x^2}\,dx\,.$$

Since $g(k,x)$ is integrable on $\mathbb{R}$ for any x, the sequence $F_n(k)$ converges to $F(k)$ for any k. Let us show that $F_n(k)$ is continuously differentiable for any n. Indeed, although $|D_x g(k,x)|$ is not integrable on $\mathbb{R}$, it is integrable on any bounded interval $(-n,n)$. Therefore By Theorem **7.2**, F_n is continuously differentiable and

$$F_n'(k) = \int_{-n}^{n} D_x g(k,x)\,dx = \int_{-n}^{n} \frac{ixe^{ikx}}{1+x^2}\,dx = -2\int_0^n \frac{x\sin(kx)}{1+x^2}\,dx\,.$$

Next, one should show that the sequence of derivatives converges to some function $G(x)$ and then try to find an interval on which this convergence is uniform. Then by Theorem **1.5.5**, $F'(k)$ exists and $F'(k) = G(k)$ in this interval.

A pointwise convergence of $F_n'(k)$ can be investigated by means of Abel's theorem. The integral that defines $F_n'(k)$ contains the product of a function $\alpha(x) = x/(1+x^2)$, that is positive and monotonically decreasing to zero in the interval $(1,\infty)$, and the function $\beta(x) =$

$\sin(kx)$ whose integrals over any bounded interval are bounded by a number independent of the interval:

$$\Big|\int_c^d \beta(x)\,dx\Big| = \Big|\int_c^d \sin(kx)\,dx\Big| = \Big|\frac{\cos(ck)-\cos(dk)}{k}\Big| \le \frac{2}{|k|} = \sigma\,.$$

provided $k \neq 0$. By Abel's theorem, the sequence of derivatives has a limit for any $k \neq 0$:

$$\lim_{n\to\infty} F_n'(k) = G(k)\,, \quad k \neq 0\,.$$

Let us estimate of the rate of convergence by means of the second part of Abel's theorem to show that F_n' converges to G uniformly on any set where $|x| \ge \delta > 0$ and, hence, by Theorem **1.5.5** $F'(x) = G(x)$ in this set. Indeed, by Abel's theorem

$$|G(k) - F_n'(k)| \le 2\sigma\alpha(n) \le \frac{4}{\delta}\cdot\frac{n}{1+n^2}\,, \quad |k| \ge \delta\,.$$

Since the above inequality holds for any $|k| \ge \delta > 0$ any n, the inequality is preserved if one first takes the supremum in the left side and then the limit $n \to \infty$ in both sides. The limit in the right side vanishes so that

$$\lim_{n\to\infty} \sup_{|k|\ge\delta>0} |G(k) - F_n'(k)| = 0\,.$$

This means that F_n' converges to G uniformly on the set $|k| \ge \delta > 0$ and therefore $F'(k) = G(k)$. Since $\delta > 0$ is arbitrary,

$$F'(k) = \frac{d}{dk}\int_{\infty}^{\infty} \frac{e^{ikx}}{1+k^2}\,dk = \lim_{n\to\infty}\int_{-n}^{n} \frac{\partial}{\partial k}\,\frac{e^{ikx}}{1+x^2}\,dx\,, \qquad k \neq 0\,.$$

This example shows that the lack of an integrable bound of partial derivatives with respect to a parameter that is independent of the parameter does not imply that the integral is not differentiable with respect to that parameter. It can be differentiable on a smaller set and its derivatives can be given by improper integrals of the corresponding partial derivatives with respect to parameters:

PROPOSITION **11.2**. *Suppose that $f \in \mathcal{L}$ but $xf(x)$ is not integrable on $\mathbb{R}$. If, in addition, $xf(x)$ is monotonic for all $|x| \ge a > 0$ and $|xf(x)| \to 0$ as $|x| \to \infty$, then the Fourier transform of f is continuously differentiable for all non-zero values of the argument and*

$$F'(k) = \frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \lim_{n\to\infty}\int_{-n}^{n} ixf(x)e^{ikx}\,dx\,, \quad k \neq 0\,.$$

A proof of this proposition is left to the reader as an exercise.

11.4. Exercises.

1. Prove Proposition **11.2**. Put

$$F_n(k) = \int_{-n}^{n} e^{ikx} f(x)\, dx\,, \qquad n = 1, 2, \ldots$$

(i) Show that F_n converges to the Fourier transform F of f;
(ii) Prove that $F_n \in C^1$ for all n, and

$$F_n'(k) = \int_{-n}^{n} ixf(x)\, e^{ikx}\, dx\,.$$

(iii) Use Abel's theorem to prove that the sequence $F_n'(k)$ converges to some $G(k)$ for any $k \neq 0$.
(iv) Show that there exists a constant C such that

$$|F_n'(k) - G(k)| \leq \frac{C}{|k|}\, n\Big(|f(n)| + |f(-n)|\Big)\,,$$

for all $k \neq 0$ and all $n > a$.
(v) Prove that $F(k)$ is continuously differentiable for all $k \neq 0$ and $F'(k) = G(k)$.

2. Consider the function defined by the Fourier integral

$$F(k) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{1 + x^4}\, dx$$

(i) Show that $F \in C^2(\mathbb{R})$
(ii) Show that $F \in C^3(|k| \geq \delta)$ for any $\delta > 0$.
(iii) Use the residue theorem to find an explicit form of $F(k)$. Compute $F'''(k)$. Does it exist for all k?
(iv) Can $F'''(k)$ be obtained by interchanging the order of D_k^3 and integration with respect to x? If so, evaluate the integral after differentiation of the integrand with respect to k.

12. Metric and norm functional spaces

12.1. Metric spaces. The distance between two points x and y in $\mathbb{R}^N$ is defined by $d(x, y) = |x - y|$. The distance defines a numerical measure of that how two points are close to one another. Consider a collection of functions, denoted $\mathcal{X}$. Let us define a *distance* on $\mathcal{X}$ as a function of a pair elements that satisfies the *distance axioms*: The distance is a symmetric and non-negative function and vanishes if and only if the pair contains identical elements, and it obeys the triangle inequality:

$$d(f, g) = d(g, f) \geq 0\,,$$
$$d(f, g) = 0 \quad \Leftrightarrow \quad f = g\,,$$
$$d(f, g) \leq d(f, h) + d(h, g)$$

for any f, g, and h from $\mathcal{X}$. A set $\mathcal{X}$ with the distance function is called a *metric space* and the distance function is called a *metric* on $\mathcal{X}$.

A sequence $\{f_n\}$ is said to converge to f in $\mathcal{X}$ if $d(f_n, f) \to 0$ as $n \to \infty$ and, in this case, one writes

$$\lim_{n\to\infty} f_n = f \quad \text{or} \quad f_n \to f \quad \text{in } \mathcal{X}\,.$$

Similarly, one can define Cauchy sequences in $\mathcal{X}$. A sequence $\{f_n\}$ in a metric space is called a *Cauchy sequence* if for any $\varepsilon > 0$ one can find an integer m such that

$$d(f_n, f_k) < \varepsilon\,, \quad n, k > m$$

In other words, the distance $d(f_n, f_k)$ can be made arbitrary small for all sufficiently large n and k. It follows from the triangle inequality

$$d(f_n, f_k) \leq d(f_n, f) + d(f_k, f)$$

that every sequence that converges in $\mathcal{X}$ is a Cauchy sequence. But in contrast to $\mathbb{R}^N$, a Cauchy sequence in a general metric space may or may not have a limit element in $\mathcal{X}$. As an example, consider the set of all rational numbers. It is a metric space with the usual distance function. Take a sequence of rational numbers $\{q_n\}$ where q_n is an approximation of $\sqrt{2}$ with n decimal places, $q_1 = 1.4$, $q_2 = 1.41$, $q_3 = 1.414$, $q_4 = 1.4142$, etc. This sequence is a Cauchy sequence but it has no limit in the set of rational numbers.

12.1.1. Functional norm spaces. Let $\mathcal{X}$ be a linear functional space, that is, linear combinations of elements from $\mathcal{X}$ belong to $\mathcal{X}$. For example, functions from class $C^p(\Omega)$ form a linear space. A linear functional space always contains the zero function that is obtained by multiplying any function from $\mathcal{X}$ by zero. Let us define a *norm* on $\mathcal{X}$ as a function: $\mathcal{X} \to \mathbb{R}$, denoted by $\|f\|$ for any $f \in \mathcal{X}$, that satisfies the *norm axioms*:

$$\begin{aligned}
&\|f\| \geq 0\,, \quad \|f\| = 0 \quad \Leftrightarrow \quad f = 0\,,\\
&\|\alpha f\| = |\alpha|\,\|f\|\,,\\
&\|f + g\| \leq \|f\| + \|g\|\,,
\end{aligned}$$

for any f and g from $\mathcal{X}$ and any number α (it can be complex if $\mathcal{X}$ consists of complex-valued functions). A linear functional space $\mathcal{X}$ with the norm function is called a *norm space*.

Any norm space is also a metric space with respect to the distance function defined by the norm

$$d(f,g) = \|f - g\|\,.$$

The distance axioms readily follow from the norm axioms. In what follows, functional spaces are usually linear so that the convergence in them will always be understood with respect to the distance defined by the norm.

12.1.2. Dense subsets in a metric space. In practical calculations, it is sufficient to use only rational numbers because any irrational number can be approximated by a rational one with any desired accuracy. Naturally, it is interesting to investigate subsets in a metric space whose elements can approximate any element in a metric space with any desired accuracy.

A subset $\mathcal{A} \subset \mathcal{X}$ in a metric space is called *dense* if for any element f in $\mathcal{X}$ one can find an element g from $\mathcal{A}$ that is arbitrary close to f. In other words, for any $f \in \mathcal{X}$ and any $\varepsilon > 0$, there exists $g \in \mathcal{A}$ such that

$$d(f,g) < \varepsilon\,.$$

This also means that for any $f \in \mathcal{X}$, there exists a sequence $\{f_n\} \subset \mathcal{A}$ such that

$$\lim_{n\to\infty} d(f, f_n) = 0\,.$$

Indeed, if $\mathcal{A}$ is dense in $\mathcal{X}$, then for any $f \in \mathcal{X}$ there always exists $f_n \in \mathcal{A}$ such that $d(f, f_n) < 2^{-n}$, $n = 1, 2, \ldots$. By construction, $f_n \to f$ in $\mathcal{X}$.

If $\mathcal{A}$ is dense in $\mathcal{X}$, then any larger subset of $\mathcal{X}$ is dense in $\mathcal{X}$. If $\mathcal{A}$ is dense in $\mathcal{B}$ and $\mathcal{B}$ is dense in $\mathcal{X}$, then $\mathcal{A}$ is dense in $\mathcal{X}$. This follows from the triangle inequality. Fix $f \in \mathcal{X}$ and $\varepsilon > 0$. Since $\mathcal{B}$ is dense in $\mathcal{X}$, there exists $g \in \mathcal{B}$ that is arbitrary close to f, that is,

$d(f, g) < \varepsilon$. Having found g, one can find $h \in \mathcal{A}$ that is arbitrary close to g, $d(h, g) < \varepsilon$. By the triangle inequality, h is arbitrary close to f:

$$d(h, f) \leq d(h, g) + d(g, f) < 2\varepsilon\,.$$

as ε is arbitrary.

12.1.3. Complete functional spaces. A Cauchy sequence in a Euclidean space always has a limit in it. As noted earlier, in a functional metric or norm space, every convergent sequence is a Cauchy sequence, but the converse is false. A metric space is called *complete* if all Cauchy sequences have limits in it.

Suppose that a functional space $\mathcal{X}$ is not complete. Can it be enlarged so that the enlarged space is complete? The answer is affirmative. The resulting space space is called a *completion* of $\mathcal{X}$. However finding a completion is not straightforward. Suppose that a Cauchy sequence $\{f_n\} \subset \mathcal{X}$ has a pointwise limit $f_n(x) \to f(x)$. If f is not in $\mathcal{X}$, the distance between f and elements of $\mathcal{X}$ is not defined. So, a completion requires an extension of the distance function to a larger set of functions. It is proved that such an extension always exists and $\mathcal{X}$ is a dense subset in the completion of $\mathcal{X}$. In the easiest case, when $\mathcal{X}$ is a subset in a larger metric space $\mathcal{Y}$ and all Cauchy sequences in $\mathcal{X}$ have limits in $\mathcal{Y}$. The completion of $\mathcal{X}$ is obtained by adding all such limit functions to $\mathcal{X}$, and the distance function does not require any extension.

It will be clear from what follows that complete functional spaces play a fundamental role in mathematical modeling of the real world. However a metric can be defined in many ways on the same space of functions, producing different metric spaces. A verification of completeness can be a tedious task on its own, not to mention, the task of finding a completion. Here the discussion will be limited to specific examples of functional spaces relevant for applications in physics.

12.2. Space of bounded functions as a metric space. Let $\mathcal{B}(\Omega)$ be a set of all bounded functions:

$$f \in \mathcal{B}(\Omega) : \quad \sup_{\Omega} |f(x)| < \infty\,.$$

$\mathcal{B}(\Omega)$ is a *linear space* because a linear combination of bounded functions is a bounded function. The number

$$\|f\|_\infty = \sup_{\Omega} |f(x)|$$

is called the *supremum norm* of a bounded function f. The norm axioms are easy to verify. This space is a metric space with the distance defined by the supremum norm:

$$d(f,g) = \sup_{\Omega} |f(x) - g(x)| = \|f - g\|_\infty \,.$$

PROPOSITION **12.1**. *The space of bounded functions is complete with respect to the supremum norm.*

To prove the assertion, one has to show, first, that any Cauchy sequence $\{f_n\} \subset \mathcal{B}(\Omega)$ converges to some function f on Ω, second, that f is bounded, that is, belongs to $\mathcal{B}(\Omega)$, and, finally, that $d(f_n, f) \to 0$ as $n \to \infty$.

Let us show that f exists. By the hypothesis, the distance $d(f_n, f_k)$ can be made arbitrary small for all large enough n and k. Therefore for any $x \in \Omega$, a numerical sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ because

$$|f_n(x) - f_k(x)| \le \sup_{\Omega} |f_n(x) - f_k(x)| = d(f_n, f_k)\,.$$

By the Cauchy criterion for numerical sequences there exists a function f defined by the pointwise limit:

$$f(x) = \lim_{n\to\infty} f_n(x)\,, \quad x \in \Omega\,.$$

Let us show that f is bounded. Fix $\varepsilon > 0$ and find m such that $d(f_n, f_k) < \varepsilon$ for all $n, k > m$. Put

$$M = \max_{k=1,2,\ldots,m} \{\|f_k\|_\infty, \varepsilon\}$$

Then for all n and all $x \in \Omega$

$$\begin{aligned} |f_n(x)| &\le |f_m(x)| + |f_n(x) - f_m(x)| \\ &\le \sup_{\Omega} |f_m(x)| + \sup_{\Omega} |f_n(x) - f_m(x)| \le 2M\,. \end{aligned}$$

By taking the limit $n \to \infty$ in the left side of the inequality, it is concluded that the limit function is bounded

$$|f(x)| \le 2M\,, \quad x \in \Omega \quad \Rightarrow \quad f \in \mathcal{B}(\Omega)\,.$$

Since the inequality

$$|f_n(x) - f_k(x)| < \varepsilon$$

holds for all $k > m$ and all $x \in \Omega$, one can take the limit $k \to \infty$ in it, so that $|f_n(x) - f(x)| \le \varepsilon$ for any $x \in \Omega$. By taking the supremum in the left side one infers that

$$\|f_n - f\|_\infty \le \varepsilon\,,$$

which means that $d(f_n, f) \to 0$ as $n \to \infty$, as required.

12.2.1. The subspace $C^0(\bar{\Omega})$ in $\mathcal{B}(\Omega)$. Functions from class $C^0(\bar{\Omega})$ form a linear space and belong to $\mathcal{B}(\Omega)$ and, hence, $C^0(\bar{\Omega})$ can also be viewed as a norm space with respect to the supremum norm. If $\{f_n\} \subset C^0(\bar{\Omega})$ is a Cauchy sequence, then by the same argument as for bounded functions, it converges pointwise to some bounded function f on $\bar{\Omega}$. Moreover, f_n converges to f uniformly because

$$|f_n(x) - f(x)| \leq \sup_{\bar{\Omega}} |f_n(x) - f(x)| .$$

By Theorem **1.5.4**, the limit function f is continuous on $\overline{\Omega}$. This shows that $C^0(\bar{\Omega})$ is complete with respect to the supremum distance.

12.2.2. Polynomials in C^0. Let $\mathcal{P}$ be a set of all polynomials on N real variables. $\mathcal{P}$ is a linear space and $\mathcal{P} \subset C^0(\bar{\Omega})$ for some bounded open $\Omega \subset \mathbb{R}^N$. So, $\mathcal{P}$ is a norm space. However, $\mathcal{P}$ is not complete with respect to the supremum norm. For example, for $N = 1$ consider

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

The series converges uniformly on any interval $[a, b]$. Partial sums of the series are polynomials from $\mathcal{P}$, whereas e^x is not from $\mathcal{P}$. A completion of $\mathcal{P}$ with respect to the supremum norm on $[a, b]$ is the space $C^0(\bar{\Omega})$.

THEOREM **12.1**. (Weierstrass)[22] *Let Ω be a bounded region in $\mathbb{R}^N$ and $f \in C^0(\bar{\Omega})$. Then there exists a sequence of polynomials P_n that converges to f uniformly on $\bar{\Omega}$*

$$\lim_{n\to\infty} \|P_n - f\|_\infty = 0 .$$

If f is real, then P_n may be taken real. If, in addition, $f \in C^p(\bar{\Omega})$, then the sequences of partial derivatives $D^\beta P_n$ converge uniformly to the corresponding partials of f uniformly for any $\beta \leq p$:

$$\lim_{n\to\infty} \|D^\beta P_n - D^\beta f\|_\infty = 0 , \quad \beta \leq p .$$

Weierstrass theorem states that the space of polynomials is dense is the space of continuous functions on any compact region in a Euclidean space relative to the supremum norm. Since the space $C^0(\bar{\Omega})$ is complete, the completion of $\mathcal{P}$ gives the whole space $C^0(\bar{\Omega})$.

[22]W. Rudin, Principles of mathematical analysis

12.3. Space $C^p(\bar{\Omega})$ as a norm space. Let Ω be open and bounded set in $\mathbb{R}^N$. Consider the space of all functions that have continuous partial derivatives up to order p on Ω and all these derivatives also have continuous extensions to the boundary $\partial\Omega$. The space is linear and $C^p(\bar{\Omega}) \subset C^0(\bar{\Omega})$. Therefore $C^p(\bar{\Omega})$ is a norm space with respect to the supremum norm.

Let $\{f_n\}$ be a sequence of functions from $C^p(\bar{\Omega})$. If it is a Cauchy sequence with respect to the supremum norm, then it converges pointwise to a continuous function $f \in C^0(\bar{\Omega})$, but f does not necessarily have continuous derivatives. So, the space $C^p(\bar{\Omega})$, $p > 0$, *is not complete with respect to the supremum norm*. By Theorem **1.5.5**, the uniform convergence of the sequence of all derivatives $\{D^\alpha f_n\}$ up to order p is sufficient to ensure that the limit function belongs to $C^p(\bar{\Omega})$. The metric should be modified if one wants $C^p(\bar{\Omega})$ to be a complete space.

For any $f, g \in C^p(\bar{\Omega})$, put

$$\begin{aligned} d(f,g) &= \sup_{\alpha \le p, \bar{\Omega}} |D^\alpha f(x) - D^\alpha g(x)| \\ &= \max_{\alpha \le p} \|D^\alpha f - D^\alpha g\|_\infty \stackrel{\text{def}}{=} \|f - g\|_{C^p} . \end{aligned}$$

All the distance axioms are satisfied. The number $\|f\|_{C^p}$ will be called the C^p norm of f, and the distance defined by it will be called the C^p distance (or metric). If now $\{f_n\}$ is a Cauchy sequence with respect to this metric, then the sequences of any partials $\{D^\alpha f_n\}$ converge uniformly to some continuous functions g_α for any $\alpha \le p$ because

$$|D^\alpha f_n(x) - D^\alpha f_k(x)| \le \|D^\alpha f_n - D^\alpha f_k\|_\infty \le \|f_n - f_k\|_{C^p} ,$$

for all $\alpha \le p$ and all $x \in \bar{\Omega}$. If f is the limit function of $\{f_n\}$, then by Theorem **1.5.5**, f is from class $C^p(\bar{\Omega})$ and $g_\alpha = D^\alpha f$. Thus, *the space $C^p(\bar{\Omega})$ is complete with respect to the C^p metric.*

12.3.1. Polynomials in the space C^p. Since $C^p \subset C^0$, $p > 0$, the space of polynomials $\mathcal{P}$ is dense in $C^p(\bar{\Omega})$ relative to the supremum norm for any bounded $\Omega \subset \mathbb{R}^N$. It also follows from the Weierstrass theorem that $\mathcal{P}$ is dense in $C^p(\bar{\Omega})$ relative to the C^p norm, and the completion of $\mathcal{P}$ relative to C^p norm is $C^p(\bar{\Omega})$ because $C^p(\bar{\Omega})$ is complete.

So, polynomials play the same role in a space of functions from class C^p on compact regions as rational numbers in the space of reals. For this reason, polynomial approximations are of fundamental significance in applications.

12.4. Lebesgue functional spaces $\mathcal{L}_p(\Omega)$. The space of all functions f whose powers $|f|^p$, $p \ge 1$, are Lebesgue integrable on a set Ω will be denoted by $\mathcal{L}_p(\Omega)$ or simply by $\mathcal{L}_p$ if $\Omega = \mathbb{R}^N$:

$$f \in \mathcal{L}_p(\Omega) \ : \quad \int_\Omega |f(x)|^p \, d^N x < \infty \, , \quad p \geq 1 \, ,$$

and the number

$$\|f\|_p = \Big(\int_\Omega |f(x)|^p \, d^N x \Big)^{\frac{1}{p}}$$

will be called the $\mathcal{L}_p$ *norm of* f.

12.4.1. The space $\mathcal{L}(\Omega)$ as a norm space. For brevity $\mathcal{L}_1 = \mathcal{L}$. $\mathcal{L}(\Omega)$ is a linear space by linearity of the Lebesgue integral. The $\mathcal{L}$ norm satisfies the second and third norm axioms by the properties of the Lebesgue integral but fails the first one because

$$\|f\|_1 = \int_\Omega |f(x)| \, d^N x = 0 \quad \Leftrightarrow \quad f(x) = 0 \quad \text{a.e.}$$

that is, $f(x) \neq 0$ for x from some set of measure zero. This is not the zero function obtained by multiplying any function by zero.

To resolve this problem, let us split all Lebesgue integrable functions into *equivalence classes* where each class contains all functions that differ from one another on sets of measure zero. Then *the space* $\mathcal{L}(\Omega)$ *is defined as a collection of all such equivalence classes.* In other words, by saying that f is an element of $\mathcal{L}(\Omega)$, it is meant that f is a collection of all functions that differ from one another on a set of measure zero so that

$$f = g \quad \text{in} \quad \mathcal{L}(\Omega) \quad \Leftrightarrow \quad f(x) = g(x) \quad \text{a.e.}$$

In particular, the zero element $f = 0$ from $\mathcal{L}(\Omega)$ is the set of functions that vanish almost everywhere in Ω. With this redefinition of the space $\mathcal{L}(\Omega)$, the first axiom is fulfilled. Furthermore, representatives from the same equivalence class have the same $\mathcal{L}$ norm so that the second and third norm axiom are not affected by this redefinition of $\mathcal{L}(\Omega)$.

The distance between any two elements of $\mathcal{L}(\Omega)$ is defined as the distance between any two representatives of the corresponding equivalence classes induced by the $\mathcal{L}$ norm:

$$d(f, g) = \|f - g\|_1 = \int_\Omega |f(x) - g(x)| \, d^N x \, .$$

It does not depend on the choice of the representatives as the Lebesgue integral cannot be changed by alterations of the integrand on any set of measure zero. The vanishing $\mathcal{L}$ distance implies that the functions are equal pointwise almost everywhere, not everywhere. The difference between the functions belongs to the equivalence class of the zero function and, in this sense, the functions represent the same element of $\mathcal{L}(\Omega)$.

12.4.2. The space $\mathcal{L}_2(\Omega)$ as a norm space. The space of square integrable functions plays a fundamental role in quantum physics (as a Hilbert space of all states of quantum systems with finitely many degrees of freedom). In contrast to the space of integrable functions, the linearity of the space of square integrable function is not so obvious, and the definition of a metric induced by the $\mathcal{L}_2$ norm by analogy with $\mathcal{L}$ is not possible unless the linearity is established.

The set $\mathcal{L}_2$ is a linear space. If $f \in \mathcal{L}_2(\Omega)$, then for any complex number c, $cf(x)$ is square integrable. Let f and g be square integrable, then

$$|f(x)+g(x)|^2 \leq \Big(|f(x)|+|g(x)|\Big)^2 \leq 2|f(x)|^2 + 2|g(x)|^2$$

for any x. By the comparison test, $f+g$ is square integrable and, hence, $\mathcal{L}_2(\Omega)$ is a linear space.

The product of square integrable functions is integrable. This property follows from the *Cauchy-Schwartz inequality* that asserts that

$$\left|\int_\Omega \overline{f(x)}g(x)\,d^Nx\right| \leq \|f\|_2\,\|g\|_2\,. \tag{12.1}$$

It is a functional (infinite dimensional) analog of (**1.1**). To prove (**12.1**), consider a quadratic non-negative function of a real variable t defined by

$$h(t) = \||f|-t|g|\|_2^2 = A - 2Bt + Ct^2 \geq 0\,,$$
$$A = \|f\|_2^2\,,\quad C = \|g\|_2^2\,,\quad B = \int_\Omega |f(x)g(x)|\,d^Nx$$

If $C=0$, then the inequality holds. If $C \neq 0$, then $h(t)$ attains its absolute minimum at $t = t^* = B/C$. The inequality follows from $h(t^*) \geq 0$:

$$h(t^*) = A - \frac{B^2}{C} \geq 0 \quad \Rightarrow \quad B \leq \sqrt{AC} = \|f\|_2\,\|g\|_2$$

Note that the absolute value of the integral in the left-hand side of (**12.1**) cannot exceed B.

The $\mathcal{L}_2$ norm satisfies the triangle inequality, which is known as the *Minkowski inequality*

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2\,.$$

Indeed, one has

$$\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2 + 2\mathrm{Re}\int \overline{f(x)}g(x)\,d^N x \le \Big(\|f\|_2 + \|g\|_2\Big)^2$$

by $\mathrm{Re}\, z \le |z|$ and the Cauchy-Schwartz inequality. By taking the square root, the Minkowski inequality is established.

The distance in $\mathcal{L}_2(\Omega)$ is defined by

$$d(f,g) = \|f-g\|_2 \,.$$

The second distance axiom is fulfilled if, as in the case of $\mathcal{L}$, elements of $\mathcal{L}_2$ are equivalence classes. Each class consists of functions that are equal almost everywhere. The $\mathcal{L}_2$ distance between any two elements can be computed as the distance between any two functions representing the corresponding equivalence classes. It does not depend on the choice of the functions in each class. The triangle inequality follows from the Minkowski inequality. So, the distance axioms are verified.

12.4.3. Relation between $\mathcal{L}(\Omega)$ and $\mathcal{L}_2(\Omega)$. Let us show that any square integrable function on Ω is integrable on Ω if the measure of Ω is finite:

$$f \in \mathcal{L}_2(\Omega)\,, \quad \mu(\Omega) < \infty \quad \Rightarrow \quad f \in \mathcal{L}(\Omega)$$

In the Cauchy-Schwartz inequality, let g be the characteristic function of Ω. This implies that

$$\int_\Omega |f(x)|\,d^N x \le \|1\|_2 \|f\|_2 = \sqrt{\mu(\Omega)}\|f\|_2 < \infty$$

Since $|f|$ is integrable so is f by Sec. **6.7**. The converse is not true. As an example, consider $f(x) = x^{-1/2}$ on $\Omega = (0,1)$. Then $f \in \mathcal{L}(0,1)$ but $f^2(x) = \frac{1}{x}$ is not integrable on $(0,1)$. Thus,

$$\mathcal{L}_2(\Omega) \subset \mathcal{L}(\Omega)\,, \qquad \mu(\Omega) < \infty\,,$$

If $\mu(\Omega) = \infty$, then $\mathcal{L}_2(\Omega)$ contains functions that are not integrable. For, example $f(x) = (1+x^2)^{-1/2}$ is not integrable on $\mathbb{R}$, whereas it is square integrable on $\mathbb{R}$.

12.4.4. Hölder's inequality. Let f and g be functions on $\Omega \subseteq \mathbb{R}^N$ such that the powers $|f|^p$ and $|g|^q$ are integrable on Ω where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \ge 1$. Then

$$\int_\Omega |f(x)g(x)|\,d^N x \le \|f\|_p \|g\|_q \tag{12.2}$$

This inequality is called *Hölder's inequality*. The Cauchy-Schwartz inequality is a particular case of it when $p = q = \frac{1}{2}$.

Let a and b be non-negative numbers. Then

$$ab = \min_{t>0}\left(\frac{t^p a^p}{p} + \frac{t^{-q} b^q}{g}\right), \quad \frac{1}{p} + \frac{1}{q} = 1\,.$$

This can be verified by a direct calculation of the minimum of a smooth function. Therefore for $t = 1$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

This inequality is known as *Young's inequality for products.* Hölder's inequality By setting $a = |f(x)|/\|f\|_p$, $b = |g(x)|/\|g\|_q$ in Young's inequality and integrating both sides, one infers that

$$\frac{\|fg\|_1}{\|f\|_p\|g\|_q} \le \frac{\|f\|_p^p}{p\|f\|_p^p} + \frac{\|g\|_q^q}{q\|g\|_q^q} = \frac{1}{p} + \frac{1}{g} = 1\,,$$

from which Hölder's inequality follows.

12.4.5. The space $\mathcal{L}_p$ as a norm space. The set $\mathcal{L}_p(\Omega)$ is a linear space. One has to show that $|f+g|^p$ in integrable if $|f|^p$ and $|g|^p$ are integrable. First, let show that for any $b > a > 0$ and $p > 1$

$$(a+b)^p \le 2^{p-1}(a^p + b^p)\,.$$

This inequality follows from the convexity of the function x^p for $x > 0$. The secant line for the graph $y = x^p$ through points $x = a$ and $x = b$ lies above the graph. In particular for the midpoint $x = (a+b)/2$, the convexity implies that

$$\left(\frac{a+b}{2}\right)^p \le \frac{1}{2}a^p + \frac{1}{2}b^p\,,$$

from which the desired inequality follows. By setting $a = |f(x)|$ and $b = |g(x)|$, the integrability of $|f(x) + g(x)|^p$ follows from the comparison test.

The Minkowski inequality holds for functions from class $\mathcal{L}_p$

$$\|f+g\|_p \le \|f\|_p + \|g\|_p\,, \quad p \ge 1\,.$$

Therefore the $\mathcal{L}_p$ norm satisfies the triangle inequality. The Minkowski inequality follows from Hölder's inequality. One has

$$\begin{aligned}\|f+g\|_p^p &= \int |f(x)+g(x)|^p d^N x \\ &\leq \int \Big(|f(x)|+|g(x)|\Big)|f(x)+g(x)|^{p-1} d^N x \\ &= \int |f(x)||f(x)+g(x)|^{p-1} d^N x + \int |g(x)||f(x)+g(x)|^{p-1} d^N x\end{aligned}$$

By Hölder's inequality $\||f||f+g|^{p-1}\|_1 \leq \|f\|_p \||f+g|^{p-1}\|_q$ with $q = p/(p-1)$ applied to the two above integrals, one gets

$$\|f+g\|_p^p \leq \Big(\|f\|_p + \|g\|_g\Big)\|f+g\|_p^{p-1}$$

from which the Minkowski inequality follows.

The distance can be defined by the $\mathcal{L}_p$ norm:

$$d(f,g) = \|f-g\|_p \,.$$

It satisfies the distance axioms if $\mathcal{L}_p$ is viewed as a space of equivalence classes where each class consists of all functions that are equal almost everywhere.

12.5. Completeness of $\mathcal{L}_p(\Omega)$. It turns out that every Cauchy sequence in $\mathcal{L}_p(\Omega)$ has a limit in it so that $\mathcal{L}_p(\Omega)$ is a complete metric space.

THEOREM **12.2.** (Riesz-Fisher)[23]
Let $\{f_n\}$ be a sequence in the space $\mathcal{L}_p(\Omega)$. Then in order that there exists an element f toward which the sequence converges in the $\mathcal{L}_p$ norm, it is necessary and sufficient that $\|f_n - f_k\|_p \to 0$ for $n, k \to \infty$.

12.5.1. Spaces C_2^0 and $\mathcal{R}_2$. Let Ω be open and bounded. Then the space of all continuous functions on $\bar{\Omega}$ is a subspace of $\mathcal{L}_2(\Omega)$. This suggests that there is another way to define a distance in the space of continuous functions. Any continuous function on a bounded closed region is square integrable. Put

$$d(f,g) = \|f-g\|_2 \,, \quad f,g \in C^0(\bar{\Omega}) \,.$$

It also satisfies the distance axioms for all f and g that are continuous on $\bar{\Omega}$. Indeed, it is non-negative and symmetric and vanishes if and only if two continuous functions f and g are equal. To see the latter, assume that $f(x_0) \neq g(x_0)$ at some x_0 in Ω while $d(f,g) = 0$. By continuity of $f-g$, there exists a ball $B_a(x_0)$ of some radius a where

[23] F. Riesz and B. Sz.-Nagy, Functional analysis,

$|f(x) - g(x)| > 0$ so that integral cannot vanish as the ball has a non-zero measure, which contradicts to the condition $d(f, g) = 0$. The triangle inequality follows from the Minkowski inequality. *The space of continuous functions equipped with the $\mathcal{L}_2$ metric will be denoted $C_2^0(\bar{\Omega})$* to distinguish it from the space $C^0(\bar{\Omega})$ with the distance defined by the supremum norm.

For any two continuous function f and g, one infers that

$$\|f - g\|_2^2 \leq \|f - g\|_\infty^2 \int_\Omega d^N x = \|f - g\|_\infty^2 \mu(\Omega)\,. \tag{12.3}$$

This inequality implies that every Cauchy sequence $\{f_n\}$ in $C^0(\bar{\Omega})$ is a Cauchy sequence in $C_2^0(\bar{\Omega})$ (if $\mu(\Omega) < \infty$ which is always the case if Ω is bounded). However, the converse is false, and there are Cauchy sequences in $C_2^0(\bar{\Omega})$ that do not have a limit in it, that is, *the space $C_2^0(\bar{\Omega})$ is not complete*.

It is not difficult to construct a Cauchy sequence in $C_2^0(\bar{\Omega})$ whose pointwise limit is a function that is not continuous. Let $\Omega = [-1, 1]$ and $f_n(x) = nx$ if $|x| < \frac{1}{n}$ and $f_n(x) = 1$ otherwise. Then for any $x \neq 0$, $f_n(x) \to 1$ as $n \to \infty$, and $f_n(0) = 0$. So the limit function is not continuous at $x = 0$ and, hence, does not belong to $C_2^0[-1, 1]$. On the other hand,

$$\|f_n - f_k\|_2^2 = 2(n-k)^2 \int_0^{\frac{1}{n}} x^2 dx + 2 \int_{\frac{1}{n}}^{\frac{1}{k}} (1 - kx)^2 dx \to 0$$

for $n > k \to \infty$.

Let $\mathcal{R}_2(\Omega)$ be the space of Riemann square integrable functions on Ω in which the distance is defined in the same way as in $\mathcal{L}_2(\Omega)$ but the integral is understood in the Riemann sense. Then $C_2^0(\bar{\Omega}) \subset \mathcal{R}_2(\Omega)$. In the above example, the limit function belongs to $\mathcal{R}_2(\Omega)$. Therefore one might conjecture that a completion of $C_2^0(\bar{\Omega})$ should give $\mathcal{R}_2(\Omega)$. However, this is not so. There are Cauchy sequences in $C_2^0(\bar{\Omega})$ that converge to functions that are not from $\mathcal{R}_2(\Omega)$, and $\mathcal{R}_2(\Omega)$ is not complete either. For example, a double sequence of continuous functions, $f_{mn}(x) = [\cos(\pi m! x)]^{2n}$, converges to the Dirichlet function pointwise

$$\lim_{m\to\infty} \big(\lim_{n\to\infty} f_{mn}(x)\big) = f_D(x)\,.$$

Indeed, if x is rational, then $x = p/q$ for some integers p and q. Therefore $m!x$ is an integer for $m \geq q$ and the limit is equal to one. If x is not an integer, then $\cos^2(\pi m! x) < 1$ for any m and the limit is equal to 0. Since $f_D(x) = 0$ a.e., the $\mathcal{L}_2$ norm of f_{nm} vanishes in the limit. This means that $f_{nm} \to 0$ in $\mathcal{L}_2(a,b)$ for any interval (a,b). As any convergent sequence, $\{f_{nm}\}$ is a Cauchy sequence in $\mathcal{L}_2(a,b)$. On the other hand, the Riemann and Lebesgue integrals are equal for continuous functions. Hence the sequence $\{f_{nm}\}$ is a Cauchy sequence in $\mathcal{R}_2[a,b]$ for any interval $[a,b]$. But its pointwise limit $f_D(x)$ is not Riemann integrable.

Thus, *in order to construct a completion of the space $\mathcal{R}_2$, the distance function should be extended to some functions that are not Riemann square integrable.* Evidently, this requires a generalization of the very concept of the Riemann integral. The extension of the distance function is achieved by replacing the Riemann integral by the Lebesgue integral. Then the completion of $C_2^0(\bar{\Omega})$ or $\mathcal{R}_2(\Omega)$ is the space $\mathcal{L}_2(\Omega)$. The spaces $C_2^0(\bar{\Omega})$ and $\mathcal{R}_2(\Omega)$ become dense subsets in $\mathcal{L}_2(\Omega)$ (see the next section).

The space $\mathcal{L}_2(\Omega)$ is one of the pillars of mathematical foundations of quantum physics. It is interesting to note that it is impossible to construct a consistent mathematical model of quantum mechanics using the Riemann integration theory.

12.5.2. Polynomials in $\mathcal{L}_2(a,b)$. Let us show that *the space of polynomials $\mathcal{P}$ is dense in $\mathcal{L}_2(a,b)$ for any bounded interval.*

Let C_{pw}^0 denote a set of piecewise continuous functions. Then

$$\mathcal{P} \subset C_2^0[a,b] \subset C_{pw}^0 \subset \mathcal{L}_2(a,b)\,.$$

If one shows that C_{pw}^0 is dense in $\mathcal{L}_2(a,b)$, $C_2^0[a,b]$ is dense in C_{pw}^0, and $\mathcal{P}$ is dense in $C_2^0[a,b]$, then $\mathcal{P}$ is dense in $\mathcal{L}_2(a,b)$.

C_{pw}^0 is dense in $\mathcal{L}_2(a,b)$. Let $f \in \mathcal{L}_2(a,b)$. Then $f_\pm(x) = \frac{1}{2}(|f(x)| \pm f(x)) \geq 0$ are also square integrable on (a,b) and, hence, they are integrable on (a,b) by Sec.**12.4.3**. Therefore by Definition **5.4.1** there exist monotonically increasing sequences $h_n^\pm$ of piecewise continuous functions such that

$$\lim_{n\to\infty} h_n^\pm(x) = f_\pm(x) \quad \text{a.e.}$$

Let $m_\pm \leq h_1^\pm(x)$ for all x in $[a,b]$. Since $h_n^\pm(x)$ is increasing with increasing n and $f_\pm(x) \geq 0$,

$$\Big(f_\pm(x) - h_n^\pm(x)\Big)^2 \leq \Big(f_\pm(x) - m_\pm\Big)^2 \in \mathcal{L}(a,b)\,,$$

where the inequality holds for all n almost everywhere. By the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \|f_\pm - h_n^\pm\|_2^2 = \lim_{n\to\infty} \int_a^b \Big(f_\pm(x) - h_n^\pm(x)\Big)^2 dx = 0\,.$$

By the triangle inequality, the sequence of piecewise continuous functions $h_n = h_n^+ - h_n^-$ converges to f in $\mathcal{L}_2(a,b)$:

$$\|f - h_n\|_2 \le \|f_+ - h_n^+\|_2 + \|f_- - h_n^-\|_2 \to 0$$

when $n \to \infty$.

$C^0[a,b]$ is dense in C^0_{pw}. Suppose f has a jump discontinuity at $x = c \in (a,b)$ and is continuous otherwise in $[a,b]$. If $f(c+)$ and $f(c-)$ are the right and left limits of f at c, construct a continuous function $h_n(x)$ such that $h_n(x) = f(x)$ if $|x-c| < \frac{d_0}{n}$ where d_0 is the smallest number of $b-c$ and $c-a$. In the interval $[x_n^-, x_n^+]$, where $x_n^\pm = c \pm d_0/n$, $n = 1, 2, ...$, $h_n(x)$ is the linear interpolation function such that $h_n(x_n^\pm) = f(x_n^\pm)$. Then

$$\|f - h_n\|_2^2 = \int_{x_n^-}^{x_n^+} |f(x) - h_n(x)|^2\, dx \le \frac{8M^2 d_0}{n} \to 0$$

when $n \to 0$ because $|h_n(x)| \le M$ where $M = \sup|f(x)| < \infty$. A general piecewise continuous function f has finitely many jump discontinuities in (a,b). A sequence of continuous functions converging to f is constructed in the same way by interpolating f by linear functions in small intervals containing the points where f is not continuous and letting the total length of these intervals tend to zero.

$\mathcal{P}$ is dense in $C^0_2[a,b]$. The assertion follows from Eq. (**12.3**) and the Weierstrass theorem.

Thus, $\mathcal{P}$ is dense in $\mathcal{L}_2(a,b)$. The assertion can be extended to $\mathcal{L}_2(\Omega)$ for any bounded $\Omega \subset \mathbb{R}^N$. *Any square integrable function on a bounded region in a Euclidean space can be approximated by a polynomial with any desired accuracy with respect to the $\mathcal{L}_2$ norm.*

12.5.3. Dense subsets in $\mathcal{L}_2$. Let C^0_0 denote the space of all continuous functions on $\mathbb{R}^N$ with a bounded support. Any function from C^0_0 vanishes outside a ball of a large enough radius. *The space C^0_0 is dense in $\mathcal{L}_2$, that is, any square integrable function in a Euclidean space can be approximated by a compactly supported continuous function with any desired accuracy with respect to the $\mathcal{L}_2$ norm.*

Let f be square integrable in $\mathbb{R}^N$. By continuity of the Lebesgue integral

$$\int |f(x)|^2\,dx = \lim_{R\to\infty}\int_{|x|<R} |f(x)|^2\,d^Nx\,.$$

This implies that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\|f - f_R\|_2 < \varepsilon\,, \quad f_R(x) = \chi_R(x)f(x)\,,$$

with χ_R being the characteristic function of the ball $|x| \le R$. Since $C^0(|x| \le R)$ is dense in $\mathcal{L}_2(|x| < R)$, there exists a continuous function h on $|x| \le R$ such that

$$\|f_R - \chi_R h\|_2 < \varepsilon\,.$$

The function $\chi_R h$ is compactly supported but not continuous in $\mathbb{R}^N$ because it can have a jump discontinuity on the sphere $|x| = R$. But there exists a continuous compactly supported function that is arbitrary close to $\chi_R h$ with respect to the $\mathcal{L}_2$ norm. Indeed, let $g(x) = 0$ if $|x| > R + \delta$ for some $\delta > 0$ and $g(x) = h(x)$ if $|x| \le R$. In the spherical shell $R < |x| < R + \delta$, $g(x)$ is a continuous interpolation such that $g(x) = 0$ when $|x| = R+\delta$ and $g(x) = h(x)$ when $|x| = R$. For example, $g(x) = h(R\hat{x})(R + \delta - |x|)/\delta$ where $\hat{x} = x/|x|$ for $R < |x| < R + \delta$. By construction, g is from class C_0^0. Therefore it is bounded, $|g(x)| \le M$, and

$$\|g - \chi_R h\|_2 = \left(\int_{R<|x|<R+\delta} |g(x)|^2\,dx\right)^{1/2} \le M\sqrt{\mu(\delta)}$$

where $\mu(\delta)$ is the volume of the spherical shell $R < |x| < R + \delta$. Since $\mu(\delta) = O(\delta) \to 0$ as $\delta \to 0$, the $\mathcal{L}_2$ distance between g and $\chi_R h$ can be made smaller than any $\varepsilon > 0$ for all small enough δ.

Thus, given $\varepsilon > 0$, one can find R and a function h with the properties stated above. Having found R and h, one can find a function $g \in C_0^0$ for which $M\sqrt{\mu(\delta)} < \varepsilon$. By the triangle inequality,

$$\|f - g\|_2 \le \|f - f_R\|_2 + \|f_R - \chi_R h\|_2 + \|\chi_R h - g\|_2 < 3\varepsilon\,.$$

Since ε is arbitrary, this inequality means that any $f \in \mathcal{L}_2$ can be approximated by a continuous compactly supported function with any desired accuracy with respect to the $\mathcal{L}_2$ norm.

The assertion can be extended to all Lebesgue spaces $\mathcal{L}_p$, that is, the space C_0^0 is dense in $\mathcal{L}_p$.

12.5.4. Polynomials in $\mathcal{L}_2$. There exists no good polynomial approximation for functions from class $\mathcal{L}_2(\Omega)$ if Ω is not bounded. The reason is that a polynomial is not integrable on a complement of a ball because it grows with increasing $|x|$. However polynomials can still be used to approximate functions from class $\mathcal{L}_2$ (or $\mathcal{L}_p$).

For any $\varepsilon > 0$ and any $f \in \mathcal{L}_2$, there exists $R > 0$ and a polynomial $p \in \mathcal{P}$ such that

$$\|f - \chi_R p\|_2 < \varepsilon ,$$

where χ_R is the characteristic function of the ball $|x| < R$.

In other words, any square integrable function on an unbounded region Ω can be approximated by a function that coincides with a polynomial in the part of Ω that lies in a ball of some radius R and vanishes otherwise. A proof follows the same line of reasoning as in the previous section and the Weierstrass theorem applied to the function $h \in C^0(|x| \leq R)$. Technical details are left to the reader as an exercise. The statement is also true for $\mathcal{L}_p$.

12.6. Exercises.

1. (i) Show that the space of polynomials is dense in $\mathcal{L}(a, b)$ for any bounded interval (a, b).
(ii) Show that the space of continuous compactly supported functions is dense in $\mathcal{L}(\mathbb{R})$.
(iii) Prove the assertion in Sec.**12.5.4** for functions from class $\mathcal{L}(\mathbb{R})$.

2. Show that the space C_0^0 is dense in $\mathcal{L}_p(\mathbb{R}^N)$.

3. Prove the assertion in Sec.**12.5.4** for functions from class $\mathcal{L}_p(\mathbb{R}^N)$.