CHAPTER 4

Convolution and Fourier transform

29. Direct product of distributions.

29.1. Direct product of regular distributions. Let $f(x)$ and $g(y)$ be locally integrable functions of their arguments, $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. Then the function of two variables that is the product of f and g is locally integrable

$$h(x,y) = f(x)g(y) \;\in\; \mathcal{L}_{\rm loc}(\mathbb{R}^{N+M})$$

Indeed, any bounded region Ω in $\mathbb{R}^{N+M}$ is contained in the direct product $B_{R_1} \times B_{R_2}$ of two balls, $|x| < R_1$ and $|y| < B_{R_2}$. For example, if $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then any bounded region in $\mathbb{R}^2$ is contained in a rectangle

$$\Omega \subset B_{R_1} \times B_{R_2} = (-R_1, R_1) \times (-R_2, R_2)$$

In general, if $z \in \mathbb{R}^{N+M}$, then

$$|z|^2 = |x|^2 + |y|^2 \quad \Rightarrow \quad |z| \le |x| + |y|$$

so that $|z| < R_1 + R_2$ and

$$\iint_\Omega |h(x,y)| d^N x d^M y \le \int_{B_{R_1}} |f(x)|\, d^N x \int_{B_{R_2}} |g(x)|\, d^M y < \infty$$

by local integrability of f and g.

With every pair of regular distributions $f(x)$ and $g(y)$ one can associate a unique regular distribution of *two variables* $h(x,y) = f(x)g(y)$ that acts on a test function of two variables by the rule that follows from the Fubini's theorem:

$$\begin{aligned}(h,\varphi) &= \iint h(x,y)\varphi(x,y)\, d^N x d^M y \\ &= \int f(x) \left(\int g(y)\varphi(x,y)\, d^M y \right) d^N x \\ &= (f,\psi)\,, \qquad \psi(x) = \int g(y)\varphi(x,y)\, d^M y\end{aligned}$$

Or, alternatively

$$(h,\varphi) = \iint h(x,y)\varphi(x,y)\, d^N x d^M y$$
$$= \int g(y) \left(\int f(x)\varphi(x,y)\, d^N x \right) d^M y$$
$$= (g,\phi)\,, \qquad \phi(y) = \int f(x)\varphi(x,y)\, d^N x$$

This shows that the action of the distribution of two variables $h(x,y)$ on a test function is defined via the actions of distributions of single variable, provided the result of the actions of these distributions on a test function of two variables is a test function:

$$\psi(x) = \Big(g(y), \varphi(x,y)\Big) \ \in \ \mathcal{D}(\mathbb{R}^N)\,,$$
$$\phi(y) = \Big(f(x), \varphi(x,y)\Big) \ \in \ \mathcal{D}(\mathbb{R}^M)$$

This is indeed so by Proposition **19.2**. Thus, *the product of two regular distributions of distinct variables is a linear continuous functional of two variables.*

29.2. Direct (or tensor) product of distributions. Proposition **19.2** holds for all distributions. This allows us to extend the direct product to all distributions of distinct variables.

DEFINITION **29.1**. *The direct (or tensor) product of two distributions $f(x) \in \mathcal{D}'(\mathbb{R}^N)$ and $g(y) \in \mathcal{D}'(\mathbb{R}^M)$ is a distribution of two variables $h(x,y) \in \mathcal{D}'(\mathbb{R}^{N+M})$, denoted as*

$$h(x,y) = f(x) \cdot g(y)$$

that acts on any test function of two variables $\varphi(x,y) \in \mathcal{D}(\mathbb{R}^{N+M})$ by the rule

$$\Big(h(x,y), \varphi(x,y)\Big) = (f,\psi)\,, \quad \psi(x) = \Big(g(y), \varphi(x,y)\Big)$$

or, in brief,

$$(f \cdot g, \varphi) = (f, (g,\varphi))$$

This rules does define a functional on $\mathcal{D}(\mathbb{R}^{N+M})$ by Proposition **19.2**. However its linearity and continuity is yet to be established. A proof is based on interpreting the direct product distribution as the adjoint transformation of a linear continuous transformation of the space of test functions into another space of test functions. The adjoint transformation maps a space of distributions to another space of distributions and this transformation is linear and cotinuous.

For any distribution $g \in \mathcal{D}'(\mathbb{R}^N)$ the transformation (**19.3**) defines a linear map $T_g : \mathcal{D}(\mathbb{R}^{N+M}) \to \mathcal{D}(\mathbb{R}^N)$. If T_g is proved to be continuous, then its adjoint

$$T_g^* : \mathcal{D}'(\mathbb{R}^N) \to \mathcal{D}'(\mathbb{R}^{N+M})$$

is also a linear continuous transformation, and $T_g^*(f)(x,y) = f(x) \cdot g(y)$ because

$$\Big(T_g^*(f), \varphi\Big) = (f, T_g(\varphi)) = \Big(f(x), \Big(g(y), \varphi(x,y)\Big)\Big) = (f \cdot g, \varphi)$$

Thus, for consistency of the definition of the direct product it is sufficient to establish continuity of the transformation T_g.

THEOREM **29.1**. (Consistency of the direct product) *For any distribution $g \in \mathcal{D}'(\mathbb{R}^M)$, the transformation T_g of $\mathcal{D}(\mathbb{R}^{N+M})$ into $\mathcal{D}(\mathbb{R}^N)$ defined by*

$$T_g(\varphi)(x) = \Big(g(y), \varphi(x,y)\Big)$$

is linear and continuous so that its adjoint $T_g^*(f) = f \cdot g$ is a distribution from $\mathcal{D}'(\mathbb{R}^{N+M})$ for any distribution $f \in \mathcal{D}'(\mathbb{R}^N)$.

By Proposition **19.2**

$$D^\alpha T_g(\varphi)(x) = \Big(g(y), D_x^\alpha \varphi(x,y)\Big)$$

Suppose that $\varphi_n(x,y) \to 0$ in $\mathcal{D}(\mathbb{R}^{N+M})$. Let us show that

$$\psi_n = T_g(\varphi_n) \to 0 \quad \text{in } \mathcal{D}(\mathbb{R}^N) \quad \Leftrightarrow \quad \lim_{n\to\infty} \sup |D^\alpha \psi_n| = 0$$

for any $\alpha \geq 0$. Suppose that the latter is false. Then there exist β and a sequence of points $\{x_n\}$ such that

$$|D^\beta \psi_n(x_n)| \geq \delta > 0$$

for some $\delta > 0$. Supports of all ψ_n lie in an interval $|x| \leq R$ if supports of all φ_n are in a ball of radius R. Therefore the sequence $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem, every bounded sequence in $\mathbb{R}^N$ contains a convergent subsequence $x_{n_k} \to x'$ as $k \to \infty$. Since $D^\alpha \varphi_n(x,y)$ converges uniformly to zero,

$$\sup_y |D_x^\alpha \varphi_{n_k}(x_{n_k}, y)| \leq \sup_{x,y} |D_x^\alpha \varphi_{n_k}(x,y)| \to 0$$

as $k \to \infty$, from which it follows, by continuity of the functional g, that

$$\lim_{k\to\infty} D^\alpha \psi_{n_k}(x_{n_k}) = \lim_{k\to\infty} \Big(g(y), D_x^\alpha \varphi_{n_k}(x_{n_k}, y)\Big) = 0\,,$$

which is not possible because $|D^\alpha \psi_{n_k}(x_{n_k})| \geq \delta > 0$ for all k, leading to a contradiction. Thus, T_g is continuous. Then the direct product of distributions is a linear continuous functional on $\mathcal{D}(\mathbb{R}^{N+M})$ by the standard argument:

$$\lim_{n\to\infty}(f\cdot g,\varphi_n)=\lim_{n\to\infty}(T_g^*(f),\varphi_n)=\lim_{n\to\infty}(f,T_g(\varphi_n))=0$$

by continuity of f. The linearity of $f\cdot g$ is established in a similar way and follows from linearity of T_g and f.

29.3. Delta-function of several variables. Let us show that the delta-function of two variables is the direct product product of two delta-functions:

$$\delta(x,y)=\delta(x)\cdot\delta(y)\,,\qquad x\in\mathbb{R}^N\,,\quad y\in\mathbb{R}^M\,.$$

Indeed, for any test function of two variables $\varphi(x,y)$,

$$\begin{aligned}\Big(\delta(x)\cdot\delta(y),\varphi(x,y)\Big)&=\Big(\delta(x),\Big(\delta(y),\varphi(x,y)\Big)\Big)=\Big(\delta(x),\varphi(x,0)\Big)\\&=\varphi(0,0)=\Big(\delta(x,y),\varphi(x,y)\Big)\end{aligned}$$

as required. It is also clear that the direct product of delta-functions is commutative and distributive:

$$\begin{aligned}\delta(x)\cdot\delta(y)&=\delta(y)\cdot\delta(x)\\ \delta(x)\cdot\Big(\delta(y)\cdot\delta(z)\Big)&=\Big(\delta(x)\cdot\delta(y)\Big)\cdot\delta(z)\end{aligned}$$

In particular, the delta-function of $x\in\mathbb{R}^N$ is the direct product of the single-variable delta-functions:

$$\delta(x)=\delta(x_1)\cdot\delta(x_2)\cdots\delta(x_N)$$

29.4. Properties of the direct product of distributions. The dot denoting the direct product of distributions is often omitted as its properties are similar to the product of ordinary functions of different variables. Here it will be kept in what follows for consistency.

29.4.1. Approximations of test functions by polynomials. In what follows, the classical theorem about approximations of continuous functions by polynomials will be used.

THEOREM **29.2**. (Weierstrass)
Let Ω be bounded and open in $\mathbb{R}^N$ and $\psi\in C^p(\overline{\Omega})$. Then for any $\varepsilon>0$ there exists a polynomial P such that

$$\sup_{\alpha\le p,\Omega}|D^\alpha\psi(x)-D^\alpha P(x)|<\varepsilon\,.$$

The Weierstrass theorem asserts that the space of polynomials is dense in $C^p(\overline{\Omega})$. Any test function can also be approximated by polynomials in the following sense.

PROPOSITION **29.1**. *For any test function φ, there exists a sequence of test functions φ_n that converges to φ in $\mathcal{D}$ and $\varphi_n(x)$ is a polynomial for all x from the support of φ.*

Let us construct the sequence φ_n with required properties explicitly. The support of a test function φ is a closed subset in an open ball B_R. Therefore one can construct a bump function η_φ that is equal to 1 in a neighborhood Ω of $\operatorname{supp}\varphi$ and $\eta_\varphi(x) = 0$ if $|x| \geq R$. Here $\operatorname{supp}\varphi \subset \Omega \subset B_R$. By the Weierstrass theorem, for every $\varepsilon = \frac{1}{n}$, $n = 1, 2, ...$, one can find a polynomial P_n such that

$$|D^\alpha\varphi(x) - D^\alpha P_n(x)| < \frac{1}{n}\,, \quad |x| \leq R\,, \quad \alpha \leq n\,,$$

because $\varphi \in C^\infty(\overline{\Omega})$. Then the sequence

$$\varphi_n(x) = \eta_\varphi(x)P_n(x) \tag{29.1}$$

converges to φ in the topology of $\mathcal{D}$. Indeed, if $x \in \overline{\Omega}$, then $D^\alpha\eta_\varphi(x) = 0$ for any $\alpha > 0$, then

$$|D^\alpha\varphi(x) - D^\alpha\varphi_n(x)| = |D^\alpha\varphi(x) - D^\alpha P_n(x)| \leq \frac{1}{n}\,, \quad x \in \overline{\Omega} \subset B_R\,.$$

If x is not in $\overline{\Omega}$, then $D^\alpha\varphi(x) = 0$ for any α and $|D^\beta P_n(x)| < \frac{1}{n}$ for any $\beta < \alpha \leq n$. Using the binomial expansion of the derivative of the product

$$|D^\alpha\varphi(x) - D^\alpha\varphi_n(x)| \leq \frac{1}{n}\sum_{\beta<\alpha} C_\alpha^\beta \sup|D^{\alpha-\beta}\eta_\varphi| \equiv \frac{A_\alpha}{n}\,, \quad x \notin \overline{\Omega}$$

where C_α^β are the binomial coefficients. Therefore for any α and $n \geq \alpha$

$$|D^\alpha\varphi(x) - D^\alpha\varphi_n(x)| \leq \frac{C_\alpha}{n}\,, \quad |x| \leq R\,,$$

where $C_\alpha = \max\{1, A_\alpha\}$ are independent of n. Since the supports of all terms φ_n lie in the ball B_R by construction, the latter inequality implies that

$$\sup|D^\alpha\varphi(x) - D^\alpha\varphi_n(x)| \leq \frac{C_\alpha}{n} \ \to 0\,,$$

as $n \to \infty$, which means that $\varphi_n \to \varphi$ in $\mathcal{D}$.

29.4.2. Commutativity and associativity. The direct product is commutative and associative:

$$f(x)\cdot g(y) = g(y)\cdot f(x)$$
$$f(x)\cdot\Big(g(y)\cdot h(z)\Big) = \Big(f(x)\cdot g(y)\Big)\cdot h(z)$$

In the case of regular distributions, the assertion follows from Fubini's theorem. The details are left to the reader as an exercise. To prove it for all distributions, note that the commutativity and associativity of the product of distributions of distinct variables holds for test function of the form

$$\varphi_n(x,y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y)$$

where ϕ and ψ are test functions of a single variable. Indeed, by linearity of functionals f and g

$$(f\cdot g,\varphi_n) = \Big(f(x),(g(y),\varphi_n(x,y))\Big) = \sum_{j=1}^{n}\Big(f(x),\Big(g(y),\psi_j(y)\Big)\phi_j(x)\Big)$$
$$= \sum_{j=1}^{n}\Big(g(y),\Big(f(x),\phi_j(x)\Big)\psi_j(y)\Big) = (g\cdot f,\varphi_n)$$

and similarly for associativity. Next, given a test function of two variables $\varphi(x,y)$, one can construct a sequence (**29**.**1**) that converges to φ in $\mathcal{D}(\mathbb{R}^{N+M})$:

$$\varphi_n(x,y) = \eta_1(x)\eta_2(y)P_n(x,y)$$

where $\eta_{1,2}$ are test functions of one variable that take unit value in a neighborhood of a ball of radius R if the support of φ lies in a ball of radius R, $|x|^2+|y|^2 < R^2$. Since any polynomial is a linear combination of monomials of the form $x^\alpha y^\beta$, the terms of this sequence are linear combinations of products of test functions of a single variables for which the commutativity and associativity holds. The functionals $f\cdot g$ and $g\cdot f$ are continuous as any adjoint transformation of a continuous transformation of a space of test functions. Therefore by taking the limit $n\to\infty$, the commutativity of the direct product is established for all test functions of two variables:

$$(f\cdot g,\varphi) = \lim_{n\to\infty}(f\cdot g,\varphi_n) = \lim_{n\to\infty}(g\cdot f,\varphi_n) = (g\cdot f,\varphi)$$

and similarly for the associativity.

29.4.3. Differentiation of the direct product. The following rule for differentiation of the direct product of distributions holds:

$$D_x^\alpha\Big(f(x)\cdot g(y)\Big) = D^\alpha f(x)\cdot g(y)$$

just like in the case of the product of two smooth functions of different variables. Indeed, for any test function $\varphi(x,y)$, one has

$$\begin{aligned}\Big(D_x^\alpha(f(x)\cdot g(y)),\varphi(x,y)\Big) &= (-1)^\alpha\Big((f(x)\cdot g(y)),D_x^\alpha\varphi(x,y)\Big)\\ &= (-1)^\alpha\Big((f(x),\Big(g(y)),D_x^\alpha\varphi(x,y)\Big)\Big)\\ &= (-1)^\alpha\Big((f(x),D^\alpha\Big(g(y)),\varphi(x,y)\Big)\Big)\\ &= \Big(D^\alpha(f(x),\Big(g(y)),\varphi(x,y)\Big)\Big)\\ &= \Big((D^\alpha f(x)\cdot g(y)),\varphi(x,y)\Big)\end{aligned}$$

where the first equality holds by the definition of a distributional derivative, the second is by definition of the direct product, the third is by Proposition **19.2**, the forth and fifth are valid again by definitions of distributional derivatives and direct product, respectively.

29.4.4. Multiplication by a smooth function. Let $a(x)$ be from C^∞. Then

$$a(x)\Big(f(x)\cdot g(y)\Big) = \Big(a(x)f(x)\Big)\cdot g(y)$$

for any two distributions $f(x)$ and $g(y)$. A proof of this property is left to the reader as an exercise.

29.5. Change of variables in the direct product of distributions. The direct product of distributions is a distribution of two or more variables. By changing variables in the product, new distributions of several variables can be obtained. For example, let

$$f(x,y) = \theta(x)\cdot\delta(y)\,.$$

Then

$$g(x,y) = \theta(x+2y)\cdot\delta(x+y)$$

is a distribution because it is obtained from the distribution f by a non-singular linear change of variables defined by the matrix

$$A = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix} \quad\Rightarrow\quad A^{-1} = \begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}$$

and $\det A = 1$. Using the rule of changing variables in a distribution, the action of g on a test function can be computed via the action of f on the associated transform of the test function:

$$\begin{aligned}\big(g(x,y),\varphi(x,y)\big) &= \frac{1}{|\det A|}\Big(f(x,y),\varphi(2y-x,x-y)\Big)\\ &= \int_0^\infty \varphi(-x,x)\,dx\,.\end{aligned}$$

The sign of the direct product is often omitted, which may lead to the question of how to interpret products of single-variable distributions in which the arguments are replaced by a function of several variables. There should exist a change of variables such that such a product becomes the direct product of distributions of independent variables. For example, consider a singular function of two real variables $(x^2-y^2)^{-1}$. The singularity on the line $y = x$ is not locally integrable in $\mathbb{R}^2$. This function admits a distributional regularization that can be constructed as follows. Using the partial fraction decomposition,

$$\frac{1}{x^2-y^2} = \frac{1}{2y}\Big(\frac{1}{x-y} - \frac{1}{x+y}\Big)\,.$$

Consider the direct product of the principal value and Sokhotsky distributions, $\mathcal{P}\frac{1}{y}\cdot\frac{1}{x+i0}$. Then a distributional regularization of the above singular function can be obtained by a linear change of variables in each term of this product:

$$\operatorname{Reg}\frac{1}{x^2-y^2} = \frac{1}{2}\mathcal{P}\frac{1}{y}\cdot\Big(\frac{1}{x-y+i0} - \frac{1}{x+y+i0}\Big)\,.$$

29.6. Exercises.

1. Prove the commutativity and associativity of the direct product of regular distributions. *Hint*: Use Fubini's theorem.

2. Prove the rule for multiplication of the direct product of distributions by a smooth function.

3. Let x and y be real variables and $\varphi(x,y)$ be a test function of two variables. Put

$$\psi(x) = \Big(\mathcal{P}\frac{1}{y},\varphi(x,y)\Big) = P.v.\int \frac{\varphi(x,y)}{y}\,dy$$

(i) Show that $\psi(x)$ is a test function and

$$D^\alpha \psi(x) = \Big(\mathcal{P}\frac{1}{y}, D_x^\alpha \varphi(x,y)\Big)$$

by a direct use of the theorem about differentiation of an integral with respect to parameters x.
(ii) Suppose that $\varphi_n(x,y)$ is a null sequence in $\mathcal{D}(\mathbb{R}^2)$. Show that

$$\psi_n(x) = \Big(\mathcal{P}\frac{1}{y}, \varphi_n(x,y)\Big)$$

is a null sequence in $\mathcal{D}(\mathbb{R})$, by a direct verification of uniform convergence of the sequences of all partial derivatives to zero.

4 Let x and y be real variables and $a(x)$ and $b(y)$ be smooth functions. If $a(0) = a_0$, $a'(0) = a_1$, and $b(0) = b_0$, find

$$(a(x) + b(y))D_x\Big(\delta(x)\cdot\delta(y)\Big)$$

in terms of constants a_0, b_0, and a_1 and direct products of delta-functions and their derivatives.

5. If $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$, find

$$\frac{\partial}{\partial t}\Delta_x\left(\theta(t)\cdot\frac{1}{|x|}\right)$$

6. Let g and f be a distribution of a real variable. Show that $f(x)\cdot g(x-y)$ is a distribution of two real variables x and y. In particular, show that

$$\delta(x)\cdot g(y-x) = \delta(x)\cdot g(y)$$

for any distribution $g(y)$. Next show that

$$D_x[f(x)\cdot g(y-x)] = f'(x)\cdot g(y-x) - f(x)\cdot g'(y-x)\,.$$

7. Let x and t be real and c be a constant. Use the definition of distributional derivatives to show that

$$\begin{aligned} D_t[\theta(t)\theta(x+ct)] &= \delta(t)\cdot\theta(x) + c\theta(t)\cdot\delta(x+ct) \\ D_x[\theta(t)\theta(x+ct)] &= \theta(t)\cdot\delta(x+ct) \end{aligned}$$

Compare the result with the last equation in the previous problem by showing that $f(x)g(x-y) = f(x)\cdot g(x-y)$ for regular distributions f and g. Use the result to show that

$$\Big(D_t^2 - c^2 D_x^2\Big)\theta(ct-|x|) = 2c\delta(t)\cdot\delta(x)$$

Hint: Express $\theta(ct-|x|)$ in terms of products of $\theta(t)$ and $\theta(x\pm ct)$.

30. Convolution of distributions.

30.1. Convolution of regular distributions. Suppose that f and g are locally integrable functions of $x \in \mathbb{R}^N$ such that

$$\int |f(y)g(x-y)|\, d^N y \;\in\; \mathcal{L}_{\text{loc}}$$

is also a locally integrable function. *Then the convolution*

$$h(x) = (f * g)(x) = \int f(y)g(x-y)\, d^N y$$

exists and is a locally integrable function, and the convolution is commutative:

$$f * g = g * f$$

The assertion follows from Part 2 of Fubini's theorem. By the hypothesis

$$\int_{B_R} \int |f(y)g(x-y)|\, d^N y\, d^N x < \infty$$

for any ball B_R. This implies that the function

$$u(x,y) = f(y)g(x-y) \;\in\; \mathcal{L}(B_R \times \mathbb{R}^N)$$

is Lebesgue integrable on $B_R \times \mathbb{R}^N$. By Fubini's theorem the function $h(x) = \int u(x,y)\, d^N y$ exists and is Lebesgue integrable on any ball B_R. The commutativity is established by changing integration variables $y \to x - y$.

It should be noted that the (classical) convolution does not exist for any two locally integrable functions. For example, take $f(x) = g(x) = 1$, then the convolution integral diverges. If $f(x)$ has a bounded support, then the classical convolution always exist. In particular, the convolution of two test functions always exists.

Since f and g can also be viewed as regular distributions, one can define the convolution of regular distributions and attempt to extend the definition to all distributions just like it was done for the direct product. Suppose that $f * g$ exists (e.g. under the above sufficient conditions on f and g). Then for any test function $\varphi(x)$, one infers that

$$
\begin{aligned}
(f * g, \varphi) &= \int (f * g)(z)\varphi(z)\, d^N z = \int \int g(y) f(z-y)\, d^N y \varphi(z)\, d^N z \\
&= \int g(y) \int f(z-y)\varphi(z)\, d^N z\, d^N y \\
&= \int g(y) \int f(x)\varphi(x+y)\, d^N x\, d^N y \\
&= \Big(f(x) \cdot g(y), \varphi(x+y) \Big)
\end{aligned}
$$

where Fubini's theorem was used to change the order of integration, and $x = z - y$ so that $d^N z = d^N x$ for any y. This shows that the convolution can be interpreted as the direct product of the distributions of different variables acting on a smooth function of two variables

$$
\psi(x, y) = \varphi(x+y)\,.
$$

The problem with this interpretation is such that $\psi(x, y)$ is not a test function of two variables! Since the support of φ is bounded and lies in a ball of radius R, the support of $\psi(x, y)$ lies in an unbounded cylinder $|x + y| < R$ because the function $\psi(x, y)$ takes a constant value on any hyper-plane $x + y = k$ in $\mathbb{R}^{2N}$. Therefore it is not possible to use the right side of the last equality as a definition of the convolution even for any two regular distributions.

30.2. Unit sequences in the space of test functions. Let $\eta_n(x)$ be a sequence of test functions with the following properties:

(i) *For any closed and bounded region $\Omega \subset \mathbb{R}^N$, there exist an integer m such that*

$$
\eta_n(x) = 1\,, \quad x \in \Omega\,, \quad n \geq m
$$

(ii) *All partial derivatives of η_n are bounded uniformly for all n:*

$$
\sup |D^\alpha \eta_n(x)| \leq C_\alpha
$$

where the constant C_α is independent of n.

Any sequence of test functions with the stated properties is called *a unit sequence* in $\mathcal{D}(\mathbb{R}^N)$, and one writes

$$
\eta_n(x) \to 1 \quad \text{in} \quad \mathbb{R}^N \quad \text{as} \quad n \to \infty\,.
$$

Let us show that such a sequence exists. Recall that one can always construct a test function that has unit value in any ball. For example, let

$$
\eta \in \mathcal{D}\,, \qquad \eta(x) = 1\,, \quad |x| < 1\,.
$$

Put

$$\eta_n(x) = \eta\Big(\frac{x}{n}\Big)\,.$$

Then η_n is a test function for any n. If the support of η lies in a ball of radius R, then the support of η_n lies in the ball of radius nR. Evidently, for any bounded region one can an integer m such that the region lies in any ball of radius $n \geq m$ so that $\eta_n = 1$ in this region for all $n \geq m$. Furthermore,

$$|D^\alpha \eta_n(x)| = \frac{1}{n^\alpha}|D^\alpha \eta(x)| \leq |D^\alpha \eta(x)| \leq \sup|D^\alpha \eta| = C_\alpha$$

Thus, all derivatives of η_n are uniformly bounded, and η_n is a unit sequence.

Let $f(x)$ and $g(x)$ be regular distributions. Suppose that the integral

$$\Big(f(x)\cdot g(y), \varphi(x+y)\Big) = \int f(x) \int g(y)\varphi(x+y)\, d^N y d^N x$$

exists for any test function $\varphi(x)$, that is, the classical convolution $(f * g)(x)$ exists. *Then for any unit sequence $\eta_n(x,y)$ in $\mathbb{R}^{2N}$*

$$\lim_{n\to\infty} \Big(f(x)\cdot g(y), \eta_n(x,y)\varphi(x+y)\Big) = \Big(f(x)\cdot g(y), \varphi(x+y)\Big)$$

The assertion follows from the Lebesgue dominated convergence theorem. Indeed the integrand has a Lebesgue integrable bound independent of n:

$$|f(x)g(y)\eta_n(x,y)\varphi(x+y)| \leq C_0|f(x)g(y)\varphi(x+y)| \ \in \mathcal{L}(\mathbb{R}^{2N})$$

by the hypothesis and that $|\eta_n(x,y)| \leq C_0$ for all n. Therefore the order of integration and taking the limit can be interchanged and the conclusion follows because $\eta_n(x,y) \to 1$ in $\mathbb{R}^{2N}$.

Note that $\eta_n(x,y)\varphi(x+y)$ is a test function of two variables. So, the value of the direct product of any two distributions at it exists for any n. All these values form a numerical sequence. If the sequence converges, then it can be used to define the convolution of the corresponding distributions, just like in the case of regular distributions. If it does not converge, then the convolution of the corresponding distributions does not exist. In this approach, whenever the classical convolution of two locally integrable functions is locally integrable, the distributional convolution also exists and is equal to the classical one.

30.3. Definition of the convolution of distributions. *Let f and g be distributions from $\mathcal{D}'(\mathbb{R}^N)$. If the limit*

$$\lim_{n\to\infty}\Big(f(x)\cdot g(y), \eta_n(x,y)\varphi(x+y)\Big)$$

exists for any test function φ and any unit sequence $\eta_n(x,y)$ of two variables and is independent of the choice of η_n, then it defines a distribution $f*g\in\mathcal{D}'(\mathbb{R}^N)$, called the convolution of distributions f and g, that acts on a test function by the rule

$$(f*g,\varphi)=\lim_{n\to\infty}\Big(f(x)\cdot g(y), \eta_n(x,y)\varphi(x+y)\Big)\,.$$

A consistency of this definition requires proving that $f*g$ is a linear continuous functional on $\mathcal{D}(\mathbb{R}^N)$. Consider a sequence of linear functionals on $\mathcal{D}(\mathbb{R}^2)$ defined by the rule

$$(h_n,\varphi)=\Big(f(x)\cdot g(y), \eta_n(x,y)\varphi(x+y)\Big)\,,\quad n=1,2,...$$

By the completeness theorem for distributions, if h_n is a sequence of distributions (linear continuous functionals) that converges in $\mathcal{D}'$, then its limit is a distribution. Thus, it is sufficient to show continuity of functionals h_n.

Let $\varphi_m\to 0$ in $\mathcal{D}(\mathbb{R}^N)$ as $m\to\infty$. Then

$$\psi_m(x,y)=\eta_n(x,y)\varphi_m(x+y)\to 0 \quad\text{in}\quad \mathcal{D}(\mathbb{R}^{2N})$$

as $m\to\infty$ for every n. This follows from the binomial expansion of derivatives of the product and from the estimate:

$$\sup|D^\alpha\eta_n(x,y)D^\beta\varphi_m(x+y)|\le C_\alpha\sup|D^\beta\varphi_m|\to 0\quad\text{as}\quad m\to\infty\,.$$

By continuity of the direct product $(h_n,\varphi_m)\to 0$ as $m\to\infty$ for every n. Thus, h_n is a continuous functional on $\mathcal{D}(\mathbb{R}^N)$ and, hence, $h_n\to h=f*g\in\mathcal{D}'$ by the completeness theorem.

30.3.1. Convolution with a test function. Any test function ω defines regular regular distribution. In Sec.**19.3** the convolution $\omega*f$ was defined for any distribution f. Let us show that this convolution is consistent with the above general definition. One has to show that

$$(\omega*f,\varphi)=\lim_{n\to\infty}\Big(f(x),\Big(\omega(y),\eta_n(x,y)\varphi(x+y)\Big)\Big)=(f,\omega_-*\varphi)$$

The function $\omega(y)\varphi(x+y)$ vanishes if $|y|>R$ and $|x+y|>R$ for some large enough R and, hence, it is a test function of two variables because. For any test function of two variables $\psi(x,y)$ and any unit sequence $\eta_n(x,y)$, the sequence of test functions $\eta_n\psi$ converges to ψ

in $\mathcal{D}(\mathbb{R}^{2N})$ because $D^\alpha(\eta_n\psi) = D^\alpha\psi$ for all large enough n (such that $\eta_n = 1$ in a neighborhood of $\operatorname{supp}\psi$). Then

$$\Big(\omega(y), \eta_n(x,y)\varphi(x+y)\Big) = \int \eta_n(x,y)\omega(y)\varphi(x+y)\,d^N y$$
$$\to \int \omega(y)\varphi(x+y)\,d^N y = (\omega_- * \varphi)(x)$$

in $\mathcal{D}(\mathbb{R}^N)$ as $n \to \infty$ because the transformation of reduction of the number of variables in test functions defined in Sec.**??** is continuous. The conclusion follows from continuity of the functional f.

30.4. Convolution with a delta-function. The convolution of any distribution with the delta-function is equal to that distribution:

$$f * \delta = \delta * f = f\,, \qquad f \in \mathcal{D}'$$

Let $\eta_n(x,y)$ be a unit sequence in $\mathbb{R}^{2N}$ and $\varphi(x)$ be a test function from $\mathcal{D}(\mathbb{R}^N)$. By the definition of convolution

$$\begin{aligned}(f*\delta,\varphi) &= \lim_{n\to\infty}\Big(f(x)\cdot\delta(y), \eta_n(x,y)\varphi(x+y)\Big)\\ &= \lim_{n\to\infty}\Big(f(x), \Big(\delta(y), \eta_n(x,y)\varphi(x+y)\Big)\Big)\\ &= \lim_{n\to\infty}\Big(f(x), \eta_n(x,0)\varphi(x)\Big) = (f,\varphi)\end{aligned}$$

by continuity of f. Note that the sequence of test functions $\varphi_n(x) = \eta_n(x,0)\varphi(x)$ converges to $\varphi(x)$ in the topology of $\mathcal{D}$. Indeed, since the support of φ is bounded, for all sufficiently large n, $\varphi_n = \varphi$, by the properties of a unit sequence so that $D^\alpha\varphi_n = D^\alpha\varphi$. Similarly, one can show that $(\delta * f, \varphi) = (f,\varphi)$.

30.5. Properties of the convolution. The convolution defines a product on the space of distributions. As noted, this product does exist for distributions from $\mathcal{D}'$. Let us investigate basic properties of this product.

30.5.1. Commutativity and distributivity. The convolution is a commutative product of two distributions

$$f * g = g * f$$

if it exists. It follows from the commutativity of the direct product of distributions. Put $\psi(x,y) = \varphi(x+y)$ for a test function φ for brevity. Then

$$(f*g,\varphi) = \lim_{n\to\infty}(f\cdot g, \eta_n\psi) = \lim_{n\to\infty}(g\cdot f, \eta_n\psi) = (g*f,\varphi)\,.$$

The space of distributions is linear and the convolution defines a distributive product

$$f * (g + h) = f * g + f * h$$

provided the convolutions $f * g$ and $f * h$ exists. This follows from the limit laws and distributivity of the direct product:

$$\begin{aligned}(f * (g + h), \varphi) &= \lim_{n\to\infty} (f \cdot (g + h), \eta_n \psi) \\ &= \lim_{n\to\infty} \Big[(f \cdot g, \eta_n \psi) + (f \cdot h, \eta_n \psi)\Big] \\ &= \lim_{n\to\infty} (f \cdot g, \eta_n \psi) + \lim_{n\to\infty} (f \cdot h, \eta_n \psi) \\ &= (f * g, \varphi) + (f * h, \varphi)\,.\end{aligned}$$

Note well that the existence of $f * (g + h)$ alone does not imply the distributive law because $f * g$ and $f * h$ may not exist.

30.5.2. Differentiation of the convolution. Suppose that $f * g$ exists. Then

$$D^\alpha (f * g) = D^\alpha f * g = f * D^\alpha g$$

It is sufficient to show that the rule holds for a first-order partial derivative. For any unit sequence, the sequence

$$\tilde{\eta}_n(x, y) = \eta_n(x, y) + D_x\, \eta_n(x, y)$$

is also a unit sequence because η_n is constant in any bounded region for all sufficiently large n, and all derivatives are uniformly bounded:

$$\sup |D^\alpha \tilde{\eta}_n| \le C_\alpha + C_{\alpha+1}$$

Since $f * g$ exists,

$$\begin{aligned}(f * g, \varphi) &= \lim_{n\to\infty} \Big(f(x) \cdot g(y), \eta_n(x, y)\varphi(x + y)\Big) \\ &= \lim_{n\to\infty} \Big(f(x) \cdot g(y), \tilde{\eta}_n(x, y)\varphi(x + y)\Big)\end{aligned}$$

It follows from this equality that

$$\lim_{n\to\infty} \Big(f(x) \cdot g(y), D_x \eta_n(x, y)\varphi(x + y)\Big) = 0$$

for any test function φ.

$$\begin{aligned}
\Big(D(f*g),\varphi\Big) &= -\Big(f*g, D\varphi\Big) \\
&= -\lim_{n\to\infty}\Big(f(x)\cdot g(y), \eta_n(x,y)D_x\varphi(x+y)\Big) \\
&= -\lim_{n\to\infty}\Big[\Big(f(x)\cdot g(y), D_x[\eta_n(x,y)\varphi(x+y)]\Big) \\
&\qquad -\Big(f(x)\cdot g(y), D_x\eta_n(x,y)\varphi(x+y)\Big)\Big] \\
&= \lim_{n\to\infty}\Big(D_x(f(x)\cdot g(y)), \eta_n(x,y)\varphi(x+y)\Big) \\
&= \lim_{n\to\infty}\Big(Df(x)\cdot g(y), \eta_n(x,y)\varphi(x+y)\Big) \\
&= (Df*g,\varphi)\,.
\end{aligned}$$

Since $D_x\varphi(x+y) = D_y\varphi(x+y)$, the derivative D_x in the above lines of equations can be changed to D_y so that

$$\Big(D(f*g),\varphi\Big) = \Big(f*Dg,\varphi\Big)\,.$$

It is important to stress that the rule of differentiation of the convolution holds under the assumption that the convolution exist. *If the convolution $f*g$ does not exist, but convolutions with derivatives, $Df*g$ and $f*Dg$, exist, then in general*

$$Df*g \neq f*Dg$$

For example,

$$1 = \delta * 1 = \theta' * 1$$

but

$$0 = \theta * 0 = \theta * 1'$$

Note that in this case the convolution $\theta * 1$ does not exist. Indeed, the step and unit functions are regular distributions. So, their convolution should be the classical convolution if it exists, but the convolution integral diverges

$$\int \theta(y)1(x-y)\,dy = \int_0^\infty dy = \infty$$

Thus, *the existence of $f*g$ implies the existence of $Df*g$ and $f*Dg$. But the converse is false.*

30.5.3. Associativity. *The convolution is not generally an associative multiplication of distributions.* Here is an example. Take a unit distribution:

$$1 = \delta * 1 = \theta' * 1 = (\theta' * \delta) * 1 = (\theta * \delta') * 1$$

Take the zero distribution:

$$0 = \theta * 0 = \theta * (\delta * 0) = \theta * (\delta * 1') = \theta * (\delta' * 1)$$

Since $0 \neq 1$,

$$(\theta * \delta') * 1 \neq \theta * (\delta' * 1)$$

However, there are subspaces in the space of distributions for which the convolution exists and is commutative and associative. Let f, g, and h be distributions from $\mathcal{D}'$. Define the *double convolution* by the rule

$$\Big(f * g * h, \varphi\Big) = \lim_{n\to\infty} \Big(f(x) \cdot g(y) \cdot h(z), \eta_n(x, y, z)\varphi(x + y + z)\Big)$$

where η_n is a unit sequence in $\mathbb{R}^{3N}$, provided the limit exists and is independent of the choice of a unit sequence. In fact, one can define a *multiple convolution* of any number of distributions in this way. By commutativity and associativity of the direct product of distributions, the multiple convolution does not depend on the order $f*g*h = h*g*f$ etc.

Proposition **30.1**. *Suppose that the convolutions $f * g$ and $f * g * h$ exists in $\mathcal{D}'$. Then the convolution $(f * g) * h$ exists and*

$$(f * g) * h = f * g * h$$

so that for a class of distributions for which the convolution and double convolution exist, the convolution is commutative and associative.

Let η_n and ξ_n be unit sequences in $\mathbb{R}^{2N}$, then the double sequence

$$\psi_{kn}(x, y, z) = \eta_k(x, y)\xi_n(x + y, z)$$

is a unit sequence in $\mathbb{R}^{3N}$. Since the double convolution exists, the following double limit exists

$$\lim_{n,k\to\infty} \Big(f(x) \cdot g(y) \cdot h(z), \psi_{kn}(x, y, z)\varphi(x + y + z)\Big) = (f * g * h, \varphi)$$

for any test function φ. This means that the repeated limits also exist. Put $s = x + y$

$$
\begin{aligned}
(f * g * h, \varphi) &= \lim_{n\to\infty} \lim_{k\to\infty} \Big(f(x) \cdot g(y) \cdot h(z), \eta_k(x,y)\xi_n(s,z)\varphi(s+z) \Big) \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \Big(f(x) \cdot g(y), \eta_k(x,y) \Big(h(z), \xi_n(s,z)\varphi(s+z) \Big) \Big) \\
&= \lim_{n\to\infty} \Big((f * g)(s), \Big(h(z), \xi_n(s,z)\varphi(s+z) \Big) \Big) \\
&= \lim_{n\to\infty} \Big((f * g)(s) \cdot h(z), \xi_n(s,z)\varphi(s+z) \Big) \\
&= \Big((f * g) * h, \varphi \Big)
\end{aligned}
$$

as required.

30.5.4. Shift of the argument. Let $f_h(x) = f(x+h)$ be the shifted distribution f. If $f * g$ exists, then the convolution $f_h * g$ exists and

$$(f_h * g)(x) = (f * g_h)(x) = (f * g)(x+h)$$

A proof of this assertion is left to the reader as an exercise. For example,

$$f(x) * \delta(x+h) = f(x+h)$$

for any distribution f.

30.5.5. Non-continuity. For every $g \in \mathcal{D}'$, the convolution $g * f$ can be viewed a transformation of $\mathcal{D}'$ into itself. Note that this transformation is not defined on the whole $\mathcal{D}'$ but on its subset that consists of distributions for which their convolution with g exists. *This transformation is not continuous in general.* This means the following.

Suppose f_n is a sequence of distributions that converges to a distribution f in the sense of distributions,

$$\lim_{n\to\infty} (f_n, \varphi) = (f, \varphi), \qquad \varphi \in \mathcal{D}$$

Suppose that the convolutions $g * f$ and $g * f_n$ exist for all n and some distribution g. Then *the sequence $g * f_n$ does not generally converge to $g * f$ in $\mathcal{D}'$. So, in practical terms, the order of taking the limit and the convolution is not generally interchangeable.*

Here is an example. Put $f_n(x) = \delta(x-n)$ so that

$$\lim_{n\to\infty} (f_n, \varphi) = \lim_{n\to\infty} \varphi(n) = 0$$

because the support of φ is bounded. Therefore $f_n \to 0$ in the sense of distributions. On the other hand, take $g(x) = 1$ so that

$$(g * f_n)(x) = g(x) * \delta(x-n) = g(x-n) = 1$$

and hence $g * f_n$ converges to the unit distribution.

30.6. Exercises.

1. Show that the convolution

$$f * D^\alpha \delta$$

exists for any integer $\alpha \geq 0$ and any distribution f, and find the convolution in terms of distributional derivatives of f.

2. Prove the shift property of the convolution.

3. Consider a space of locally integrable functions with support in a half-line, $x \geq 0$.
(i) Show that the convolution exists in this space of regular distributions and

$$(f * g)(x) = \int_0^x g(y)f(x-y)\,dy$$

Hint: Sketch the support of $f(x)g(y)\varphi(x+y)$ in the plane where φ is a test function.
(ii) Show that the convolution is associative. *Hint*: Investigate the double convolution.

4. Let $g(x) = \sum_k c_k \delta(x-k)$.
(i) Show that the series converges in the sense of distributions for any choice of the sequence c_k. If $g_n(x)$ is a partial sum of the series (the summation is taken over $|k| < n$), then $g_n \to g$ in the distributional sense. Let f be a distribution with unbounded support.
(ii) Show that $f * g_n$ exists for any n. Does the limit of the distributional sequence $f * g_n$ always exist?
(iii) Does the convolution $f * g$ always exist?
Hint: Use the definition of the convolution. Take $f(x) = \theta(x)$ as an example. Investigate the limit in the definition of the convolution for a particular test function, e.g., the hat function $\varphi = \omega_a(x)$ with $a = \frac{1}{2}$.

5. Find the double convolution $\theta * \delta' * 1$ or show that it does not exist.

31. Existence of convolution

The convolution does not exist for all distributions. Here some classes of distributions for which the convolution exists and is continuous are described. They are important in applications.

31.1. Convolution of a distribution with bounded support.

THEOREM **31.1**. *Let f and g be distributions from $\mathcal{D}'(\mathbb{R}^N)$ and the support of g is bounded. Then the convolution $f * g$ exists, and*

$$(f * g, \varphi) = \Big(f(x) \cdot g(y), \eta_g(y)\varphi(x+y)\Big) \tag{31.1}$$

where η_g is a test function that has unit value in a neighborhood of the support of g. The convolution is also continuous in both variables, that is, for any sequence f_n that converges to f in $\mathcal{D}'$,

$$f_n * g \to f * g \quad \text{in } \mathcal{D}'$$

and for any sequence of distributions whose supports lies in a ball with radius independent of n, $\operatorname{supp} g_n \subset B_R$, that converges to a distribution g

$$f * g_n \to f * g \quad \text{in } \mathcal{D}'$$

As the support of g is bounded, there exists a test function η_g that has unit value in a neighborhood of the support of g. Then for any test function φ

$$(g, \varphi) = (g, \eta_g \varphi)$$

Therefore for any unit sequence $\eta_n(x, y)$ in $\mathbb{R}^{2N}$,

$$\begin{aligned}(f * g, \varphi) &= \lim_{n\to\infty} \Big(f(x), \Big(g(y), \eta_n(x,y)\varphi(x+y)\Big)\Big) \\ &= \lim_{n\to\infty} \Big(f(x), \Big(g(y), \eta_g(y)\eta_n(x,y)\varphi(x+y)\Big)\Big)\end{aligned}$$

Note that $\psi(x, y) = \eta_g(y)\varphi(x + y)$ is a test function of two variables because its support lies in a region defined by $|y| < R_1$ and $|x+y| < R_2$ by boundedness of supports of φ and η_g. Since $\eta_n\psi \to \psi$ in $\mathcal{D}(\mathbb{R}^{2N})$, by continuity of g,

$$\Big(g(y), \eta_n(x,y)\psi(x,y)\Big) \to \Big(g(x), \psi(x,y)\Big) \in \mathcal{D}(\mathbb{R}^N)$$

by the consistency theorem for the direct product. This shows that the convolution exists and can be computed by the rule (**31.1**).

Since $f_n \to f$ in $\mathcal{D}'$, for any test function ϕ

$$\lim_{n\to\infty} (f_n, \phi) = (f, \phi)$$

By the consistency theorem for the direct product the function

$$\phi(x) = \Big(g(y), \eta_g(y)\varphi(x+y)\Big)$$

is a test function for any test function φ. Therefore

$$\lim_{n\to\infty}\Big(f_n(x), \Big(g(y), \eta_g(y)\varphi(x+y)\Big)\Big) = \Big(f(x), \Big(g(y), \eta_g(y)\varphi(x+y)\Big)\Big)$$

or, by the first part of theorem,

$$\lim_{n\to\infty}(f_n * g, \varphi) = (f * g, \varphi)$$

for any test function φ, which means that $f_n * g \to f * g$ in $\mathcal{D}'$.

Let η_g be a test function such that $\eta_g(x) = 1$ if $|x| < R$ and the ball B_R contains support of any g_n. Then

$$(g_n, \varphi) = (g_n, \eta_g \varphi)$$

for any test function. Let $\phi_n \to \phi$ in $\mathcal{D}$. By continuity of f,

$$\lim_{n\to\infty}(f, \phi_n) = (f, \phi)$$

Consider the test function of two variables

$$\psi(x,y) = \eta_g(y)\varphi(x+y)$$

Put

$$\phi_n(x) = \Big(g_n(y), \psi(x,y)\Big)$$

If the support of φ lies in a ball of radius R_0, then $\psi(x,y) = 0$ for all $|x| > R + R_0$ and, hence, the support of all terms in the sequence of test functions ϕ_n lies in a ball B_{R+R_0}. Then it also follows that

$$\lim_{n\to\infty}\phi_n(x) = \phi(x) = \Big(g(y), \psi(x,y)\Big)$$

for any x. By the consistency theorem for the direct product

$$\begin{aligned}\lim_{n\to\infty} D^\alpha \phi_n(x) &= \lim_{n\to\infty}\Big(g_n(y), D_x^\alpha \psi(x,y)\Big) = \Big(g(y), D_x^\alpha\psi(x,y)\Big)\\ &= D^\alpha \phi(x)\end{aligned}$$

The stated properties of the sequence ϕ_n imply that $\phi_n \to \phi$ in $\mathcal{D}$. Therefore

$$\lim_{n\to\infty}\Big(f(x), \Big(g_n(y), \psi(x,y)\Big)\Big) = \Big(f(x), \Big(g(y), \psi(x,y)\Big)\Big)$$

or, for any test function φ,

$$\lim_{n\to\infty}(f * g_n, \varphi) = (f * g, \varphi)$$

as required.

31.2. Convolution algebra $\mathcal{D}'_+$. Consider a subspace of the space of distributions of one real variable x that consists of all distributions with support in the positive half-line $x \geq 0$. It is denoted by $\mathcal{D}'_+$:

$$f \in \mathcal{D}'_+ \quad \Rightarrow \quad \operatorname{supp} f \subset [0, \infty)$$

or

$$(f, \varphi) = 0\,, \quad \operatorname{supp} \varphi \subset (-\infty, -\delta]$$

for some $\delta > 0$. For example,

$$\delta \in \mathcal{D}'_+\,, \qquad \theta \in \mathcal{D}'_+$$

It appears that the convolution defines a commutative and associative product on $\mathcal{D}'_+$. For this reason $\mathcal{D}'_+$ is also called a *convolution algebra*.

THEOREM **31.2**. *Let f and g be distributions from $\mathcal{D}'_+$. Then their convolution exists and belongs to $\mathcal{D}'_+$. It can be computed by the rule*

$$(f * g, \varphi) = \Big(f(x) \cdot g(y), \eta_1(x)\eta_2(y)\varphi(x+y)\Big) \tag{31.2}$$

where $\eta_{1,2} \in C^\infty$ and $\eta_{1,2}(x) = 1$ if $x > -\delta$ for some $\delta > 0$ and $\eta_{1,2}(x) = 0$ if $x < -a$ for some $a > \delta$. The convolution is continuous:

$$f_n \to f \text{ in } \mathcal{D}'_+ \quad \Rightarrow \quad f_n * g \to f * g \text{ in } \mathcal{D}'_+\,.$$

The convolution is associative on $\mathcal{D}'_+$:

$$(f * g) * h = f * (g * h)\,, \qquad f, g, h \in \mathcal{D}'_+$$

A proof of this theorem is analogous to the previous case of convolutions with a distribution with bounded support. So, it will be sketched leaving some of technical details to the reader as an exercise. Let us first show that smooth functions $\eta_{1,2}(x)$ exist. Take a shifted step function $\theta(x + \delta)$ for some $\delta > 0$, then its convolution with the hat function

$$\eta(x) = \int \omega_a(y)\theta(x - y + \delta)\,dy = \int_{x+\delta}^{\infty} \omega_a(y)\,dy$$

is a C^∞ function such that $\eta(x) = 1$ if $x > -\delta$ and $\eta(x) = 0$ if $x < a+\delta$. For any distribution f from $\mathcal{D}'_+$

$$(f, \varphi) = (f, \eta\varphi)$$

Therefore

$$\begin{aligned}(f*g,\varphi) &= \lim_{n\to\infty}\Big(f(x),\Big(g(y),\eta_n(x,y)\varphi(x+y)\Big)\Big)\\ &= \lim_{n\to\infty}\Big(f(x),\eta_1(x)\Big(g(y),\eta_2(y)\eta_n(x,y)\varphi(x+y)\Big)\Big)\\ &= \lim_{n\to\infty}\Big(f(x),\Big(g(y),\eta_n(x,y)\eta_1(x)\eta_2(y)\varphi(x+y)\Big)\Big)\\ &= \Big(f(x),\Big(g(y),\eta_1(x)\eta_2(y)\varphi(x+y)\Big)\Big)\\ &= \Big(f(x)\cdot g(y),\eta_1(x)\eta_2(y)\varphi(x+y)\Big)\Big)\,.\end{aligned}$$

for any choice of smooth functions $\eta_{1,2}$ with properties stated above. Note that the product $\psi(x,y)=\eta_1(x)\eta_2(y)\varphi(x+y)$ is a test function of two variables. If the support of φ lies in the interval $[-R,R]$, then the support of $\psi(x,y)$ lies in the triangle $|x+y|\le R$, $x\ge 0$, and $y\ge 0$. Associativity of the convolution follows from the associativity of the direct product and the rule (**31**.**2**).

31.3. Convolution equations. A convolution equation has the form

$$g*u=f$$

where g and f are given distributions from $\mathcal{D}'(\mathbb{R}^N)$, and the problem is to find a distribution u. In particular, *all linear partial differential equations with constant coefficients can be formulated as the convolution equation*. Put

$$g(x)=\sum_{\beta=0}^{m}a_\beta D^\beta\delta(x)=L(D)\delta(x)\,.$$

By the properties of convolution

$$g*u=\sum_{\beta=0}^{m}a_\beta D^\beta u=L(D)u$$

Therefore

$$L(D)u=f\quad\Leftrightarrow\quad g*u=f\,.$$

If g is locally integrable, then the convolution equation is called an *integral equation of the first kind*:

$$g*u(x)=\int u(y)g(x-y)\,d^Ny=f(x)$$

and if $g(x) = \delta(x) + h(x)$ where h is locally integrable, the convolution equation is called *an integral equation of the second kind*:

$$g * u(x) = u(x) + \int u(y)h(x-y)\, d^N y = f(x)\,.$$

Linear finite-difference equations with constant coefficients

$$\sum_\beta a_\beta u(x - x_\beta) = f(x)$$

can also be viewed as convolution equations

$$g(x) = \sum_\beta a_\beta \delta(x - x_\beta) \quad \Rightarrow \quad g * u(x) = \sum_\beta a_\beta u(x - x_\beta)\,.$$

So, techniques for solving convolution equations are important in application.

31.3.1. Solving a convolution equation. The inverse (or reciprocal) g^{-1} of a distribution g relative the convolution multiplication is called a *fundamental solution for the convolution operator* $g*$. So, by definition,

$$g^{-1} * g = g * g^{-1} = \delta$$

because the delta function plays the role of a unit element in the convolution multiplication ($\delta * f = f * \delta = f$ for any distribution f). The significance of a fundamental solution is that the convolution $u = g^{-1} * f$ is a solution to the convolution equation with any right-hand side:

$$g * u = g * (g^{-1} * f) = (g * g^{-1}) * f = \delta * f = f\,.$$

This conclusion is based on several assumptions that need be verified. First, note that g^{-1} is not unique even if it exists. For example, if $g = \delta'$, then a general solution to $g * u = \delta$ or $u' = \delta$ is $g^{-1} = \theta + C$ where C is a constant distribution. Second, suppose that g^{-1} and $u = g^{-1} * f$ exist in $\mathcal{D}'$. However, the distribution u cannot always be a solution to the convolution equation because the convolution is not associative in general:

$$g * u = g * (g^{-1} * f) \neq (g * g^{-1}) * f = \delta * f = f$$

For example, let $g = \theta$. Then $g^{-1} = \delta'$. Indeed,

$$\theta * \delta' = (\theta * \delta)' = \theta' = \delta$$

Let $f = 1$. The convolution $u = g^{-1} * f = \delta' * 1 = (\delta * 1)' = 1' = 0$ exists, but it is not a solution to $g * u = f$ or $\theta * u = 1$ because $\theta * 0 = 0 \neq 1$. The origin of this problem is in the non-associativity of the convolution of the distributions θ, δ', and 1 noted earlier.

Suppose that g^{-1} exists. Define a subset $\mathcal{D}'_g \subset \mathcal{D}'$ that consists of distributions for which the convolution $g^{-1} * f$ and the double convolution $g * g^{-1} * f$ exist

$$f \in \mathcal{D}'_g \subset \mathcal{D}' \ : \quad g^{-1} * f \in \mathcal{D}' \,, \quad g * g^{-1} * f \in \mathcal{D}' \,. \qquad (\mathbf{31}.\mathbf{3})$$

The existence of the double convolution guarantees (see Proposition **30**.**1**) that the distribution $u = g^{-1} * f$ satisfies the equation $g * u = f$ because the convolution is associative in this case. Furthermore, the associated homogeneous equation has only trivial solution in the subspace $\mathcal{D}'_g$:

$$\left.\begin{array}{r} g * u = 0 \\ u \in \mathcal{D}'_g \end{array}\right\} \quad \Leftrightarrow \quad u = 0$$

Indeed, let u be from $\mathcal{D}'_g$ and also a solution to the homogeneous equation, then

$$\begin{aligned} u &= u * \delta = u * (g * g^{-1}) = u * g * g^{-1} = (u * g) * g^{-1} = (g * u) * g^{-1} \\ &= 0 * g^{-1} = 0 \end{aligned}$$

The associativity of convolution is crucial for the conclusion. The following theorem has been proved.

Theorem **31**.**3**. *Let $f \in \mathcal{D}'_g$ where $\mathcal{D}'_g$ is defined in (**31**.**3**). Then the equation $g * u = f$ has a solution that is given by*

$$u = g^{-1} * f$$

and the solution is unique in $\mathcal{D}'_g$.

31.3.2. Equations in the convolution algebra $\mathcal{D}'_+$. The space of distributions $\mathcal{D}'_+$ is closed relative to the multiplication defined by the convolution and the convolution is associative. By Theorem **31**.**3** the following assertion holds.

Corollary **31**.**5**. *If the reciprocal g^{-1} exists in $\mathcal{D}'_+$, then the equation $g * u = f$ has a unique solution given by $u = g^{-1} * f$ for any $f \in \mathcal{D}'_+$.*

The inverse of any distribution in $\mathcal{D}'_+$ has the same properties as the inverse in reals. In particular,

$$(g_1 * g_2)^{-1} = g_1^{-1} * g_2^{-1}$$

A proof is based on a direct verification of the equality

$$(g_1 * g_2)^{-1} * (g_1^{-1} * g_2^{-1}) = \delta$$

using the associativity and commutativity of the convolution in $\mathcal{D}'_+$ and is left to the reader as an exercise.

31.3.3. Example. Forced vibrations of a harmonic oscillator. Consider a differential equation for a harmonic oscillator

$$u''(t) + \omega^2 u(t) = f(t)\,, \quad f(t) = 0\,, \quad t < 0\,.$$

where $\omega > 0$ is a numerical parameter (the frequency of the oscillator), and $f(t)$ is an external force that starts acting on the oscillator at $t = 0$. This equation can be cast as a convolution equation in $\mathcal{D}'_+$:

$$(\delta'' + \omega^2\delta) * u = f\,, \qquad u \in \mathcal{D}'_+\,.$$

This implies that the oscillator was at rest for $t < 0$ (because $u(t) = 0$ if $u \in \mathcal{D}'_+$) and then makes forced vibrations under the action of the external force. This problem is known to have a unique solution in mechanics.

Let us show that the reciprocal of the distribution $g(t) = \delta'' + \omega^2\delta$ is given by

$$g^{-1}(t) = \theta(t)\,\frac{\sin(\omega t)}{\omega} = \theta(t)Z(t)$$

where $Z''(t) + \omega^2 Z(t) = 0$. Indeed, recall that $\theta'(t) = \delta(t)$ and, since $\sin(\omega t)$ is from class C^∞, one infers that

$$(\delta'' + \omega^2\delta) * (\theta Z) = (\theta Z)'' + \omega^2\theta Z = \delta + \theta\Big(Z'' + \omega^2 Z\Big) = \delta$$

because by the Leibniz rule

$$\begin{aligned}(\theta Z)' &= Z(0)\delta(t) + \theta(t)Z'(t) = \theta(t)Z'(t)\,,\\ (\theta Z)'' &= (\theta Z')' = Z'(0)\delta(t) + \theta Z''(t) = \delta(t) + \theta(t)Z''(t)\,.\end{aligned}$$

If $f(t)$ is a regular distribution, then the problem has a unique solution that is given by

$$\begin{aligned}u(t) &= (g^{-1} * f)(t) = \int_0^\infty g^{-1}(t-\tau)f(\tau)\,d\tau\\ &= \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\,f(\tau)\,d\tau\end{aligned}$$

If f is continuous on $[0,\infty)$, then u is from class $C^2(t > 0)$ and satisfies the equation for all $t > 0$. The function u and its derivative u' have continuous extensions to $t = 0$ such that $u(0) = 0$ and $u'(0) = 0$. These facts are readily established by the theorem about differentiation of the integral with respect to parameters. In mechanics, the said initial conditions mean that the oscillator was at rest at $t = 0$ (and it would remain so if $f(t) = 0$).

31.4. Exercises.

1. Show each of the convolutions exists and find its value

(i) $|x| * \delta''(x)$

(ii) $\theta * \theta$

(iii) $(|\sin(x)|\theta(x)) * \sum_{k\geq 0} \delta'(x - \pi k)$

2. Let f and g be distributions from $\mathcal{D}'_+$. Show that

$$(e^{ax} f(x)) * (e^{ax} g(x)) = e^{ax}(f * g)(x)$$

3. Let $x \in \mathbb{R}^3$. Find

$$\frac{1}{|x|} * \Delta\delta(x)$$

4. Let $f_a(x) \in \mathcal{D}'_+$ be defined by

$$f_a(x) = \frac{\theta(x)}{\Gamma(a)}\, x^{a-1} \quad \text{if} \quad a > 0, \quad f_a(x) = f'_{a+1}(x) \quad \text{if} \quad a \leq 0$$

where the latter relation is understood recursively (e.g., if $a = -3/2$, then $f_{-3/2} = f'_{-1/2} = f''_{1/2}$).

(i) Prove that

$$f_a * f_b = f_{a+b}$$

(ii) and, in particular, for an integer n,

$$f_{-n} * u(x) = D^n u(x)$$

so that for a positive n, the convolution operator $f_{-n}*$ is equal to the nth order derivative, and for negative n, $f_{-n}*$ is the nth order anti-derivative. For non-integer a, the operator f_a* is called an operator of *fractional differentiation, if $a < 0$, and fractional integration if $a > 0$.*

(iii) Show that

$$D^{1/2}u = D(f_{1/2} * u) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x \frac{u(y)dy}{\sqrt{x-y}}$$

(iv) Show that the function

$$u(x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}\, dy$$

is a solution to Abel's integral equation:

$$\int_0^x \frac{u(y)}{(x-y)^\alpha}\, dy = f(x)\,, \quad f(0) = 0\,, \quad f \in C^1(x \geq 0)\,, \quad 0 < \alpha < 1$$

Hint: To find $f_a * f_b$, recall

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Show next that $f_0 = \delta$ and therefore $f_{-a} * f_a = \delta$. Take $a = -n$, $n > 0$, then $f_{-n} = f_0^{(n)} = \delta^{(n)}$. Next show that $(f_n * u)^{(n)} = u$, $n > 0$. In Part (iv) restate Abel's integral equation as a convolution equation and use properties of the convolution operator $f_a *$.

5. Solve the system of equations in $\mathcal{D}'_+$:

$$\begin{aligned} \delta'' * f_1 + \delta' * f_2 &= \delta \\ \delta' * f_1 + \delta'' * f_2 &= 0 \end{aligned}$$

6. Let $f(x,t)$ be locally integrable function, $(x,t) \in \mathbb{R}^2$. Show that the following convolution exists and find its integral representation:

$$\theta(ct - |x|) * \Big(\theta(t) f(x,t)\Big)$$

where $c > 0$.

7. Let $f(x,t)$ be locally integrable function, $x \in \mathbb{R}^3$ and t is real. Show that the following convolution exists and find its integral representation:

$$\delta(ct - |x|) * \Big(\theta(t) f(x,t)\Big)$$

where $c > 0$ and

$$\Big(\delta(ct - |x|), \varphi(x,t)\Big) = \int_0^\infty \int_{|x|=ct} \varphi(x,t)\, dS_x\, dt$$

8. Show that

$$(g_1 * g_2)^{-1} = g_1^{-1} * g_2^{-1}$$

for any distributions $g_{1,2}$ from $\mathcal{D}'_+$ that have the inverse in $\mathcal{D}'_+$.

32. Temperate distributions

32.1. Extension of the space of test functions. The following notation for a power function of several variables will be used in what follows:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}\,, \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_N = \alpha\,, \qquad x \in \mathbb{R}^N\,.$$

Consider a space of functions that are smooth and all their derivatives decrease to zero at infinity ($|x| \to \infty$) faster than any reciprocal power function. This space is called the *Schwartz space* and will be denoted as $\mathcal{S}(\mathbb{R}^N)$ or simply $\mathcal{S}$:

$$\varphi \in \mathcal{S}\ : \quad \varphi \in C^\infty\,, \quad \sup|x^\alpha D^\beta \varphi| < \infty\,, \quad \alpha, \beta \geq 0\,.$$

It is clear that any test function from $\mathcal{D}$ belongs to $\mathcal{S}$ because of boundedness of the support:

$$\mathcal{D} \subset \mathcal{S}\,.$$

Any smooth function that is exponentially decreasing with increasing $|x|$ belongs to $\mathcal{S}$. For example, the Gaussian function $e^{-|x|^2}$ belongs to $\mathcal{S}$ but it does not belong to $\mathcal{D}$. For any function from $\mathcal{S}$ and any $p > 0$ there is a positive constant M_β such that

$$|D^\beta \varphi(x)| \leq \frac{M_\beta}{1 + |x|^p}\,, \quad \varphi \in \mathcal{S}\,.$$

32.1.1. Topology on the Schwartz space. *A sequence $\{\varphi_n\} \subset \mathcal{S}$ is said to converge to $\varphi \in \mathcal{S}$ if for any α and β, the sequences $x^\alpha D^\beta \varphi_n$ converge uniformly to $x^\alpha D^\beta \varphi$*:

$$\varphi_n \to \varphi \text{ in } \mathcal{S}\ : \qquad \lim_{n\to\infty} \sup \Big| x^\alpha D^\beta \varphi_n - x^\alpha D^\beta \varphi \Big| = 0\,.$$

If $\varphi_n \to \varphi$ in $\mathcal{D}$, then $\varphi_n \to \varphi$ in $\mathcal{S}$ by boundedness of the support of functions from $\mathcal{D}$.

PROPOSITION **32.1**. *The space $\mathcal{D}$ of test functions with bounded supports is a dense subset in $\mathcal{S}$.*

To prove this assertion, let us show that for any $\varphi \in \mathcal{S}$ there exists a sequence in $\mathcal{D}$ that converges to φ in the topology of $\mathcal{S}$. Let $\eta \in \mathcal{D}$ be a bump function for the ball $|x| < 1$. Then for any $\varphi \in \mathcal{S}$ the terms of the sequence $\varphi_n(x) = \varphi(x)\eta(x/n)$ belong to $\mathcal{D}$, and $\varphi_n(x) = \varphi(x)$ if $|x| < n$. Since $D\eta(x/n) = 0$ if $|x| < n$ so that $D(\varphi_n - \varphi) = 0$ if $|x| < n$, $|D^\gamma \eta(x/n)| \leq M_\gamma n^{-\gamma}$ (cf. Theorem **14.1**, Property (v)), and $D^\gamma \varphi$ tends to zero faster than any power function as $|x| \to \infty$, it is concluded that

$$\lim_{n\to\infty} \sup |x^\alpha D^\beta(\varphi_n - \varphi)| = 0$$

for any α and β. Therefore $\varphi_n \to \varphi$ in $\mathcal{S}$.

Proposition **32.1** implies, in particular, that any function from $\mathcal{S}$ can be uniformly approximated by a test function with bounded support with any desired accuracy: for any $\varepsilon > 0$ and any $\varphi \in \mathcal{S}$ one can find $\phi \in \mathcal{D}$ such that

$$\sup |\varphi - \phi| < \varepsilon$$

32.1.2. Space of slowly increasing smooth functions. Recall that the product of a test function with bounded support with a smooth function is a test function with bounded support. This property does not hold for the space $\mathcal{S}$. For example, $\varphi(x) = e^{-x^2} \in \mathcal{S}(\mathbb{R})$ and $a(x) = e^{2x^2} \in C^\infty$, but the product $a(x)\varphi(x) = e^{x^2}$ is not in $\mathcal{S}$. Let us describe a subset of smooth functions whose product with a test function from $\mathcal{S}$ is a function from $\mathcal{S}$. Since any function from $\mathcal{S}$ cannot grow faster than any reciprocal power function with increasing $|x|$, the growth of a smooth function in the product must be restricted by the growth of a power function. Define a subset $\mathcal{O}_M \subset C^\infty$ as a collection of functions whose derivatives cannot grow faster than a power function with $|x| \to \infty$:

$$a \in \mathcal{O}_M \;\; : \quad a \in C^\infty\,, \quad |D^\beta a(x)| \leq C_\beta \Big(1 + |x|\Big)^{m_\beta}$$

for some constant C_β and $m_\beta \geq 0$. Here $x \in \mathbb{R}^N$. Then

$$a(x)\varphi(x) \in \mathcal{S}\,, \qquad a \in \mathcal{O}_M\,, \quad \varphi \in \mathcal{S}$$

The linear space $\mathcal{O}_M$ will be called a *space of slowly increasing smooth functions.*

32.2. Linear and continuous transformations of $\mathcal{S}$. A linear transformation T of $\mathcal{S}(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^M)$ is continuous if it maps a null sequence in $\mathcal{S}(\mathbb{R}^N)$ to a null sequence in $\mathcal{S}(\mathbb{R}^M)$. The results of Sec.**??** are readily extended to the space of temperate test functions.

32.2.1. Differentiation in $\mathcal{S}$. Differentiation is a linear and continuous transformation of $\mathcal{S}$ into itself:

$$T \;:\; \varphi \in \mathcal{S} \;\to\; D^\alpha \varphi \in \mathcal{S}\,.$$

If φ is a test function from $\mathcal{S}$, then any its partial derivative is also a test function, $D^\alpha \varphi \in \mathcal{S}$. Linearity of this transformation is obvious. If φ_n is a null sequence in $\mathcal{S}$, then $D^\alpha \varphi_n$ is also a null sequence because

$$\lim_{n\to\infty} \sup |x^p D^{\alpha+q} \varphi_n(x)| = 0$$

for any non-negative p and q.

32.2.2. Multiplication by a slowly increasing smooth function. Let

$$T:\ \varphi \in \mathcal{S}\ \to T(\varphi) = a\varphi \in \mathcal{S}$$

where $a \in \mathcal{O}_M$. Then T is linear and continuous. The assertion follows from the binomial expansion of the derivatives of a product and the characteristic properties of functions from $\mathcal{O}_M$, similarly to the case of multiplication of test functions from $\mathcal{D}$ by a smooth function discussed earlier. Linearity is obvious. Let φ_n be a null sequence in $\mathcal{S}$. Then $|x^p D^\gamma(a\varphi_n)|$ does not exceed a linear combinations of terms

$$|x^p D^\beta a D^\alpha \varphi_n| \le C_\beta (1+|x|)^{m_\beta} |x^p D^\alpha \varphi_n(x)|$$

where $\alpha + \beta = \gamma$. Each such term converges uniformly to zero because $\sup |x^q D^\alpha \varphi_n| \to 0$ for any α and q.

32.2.3. Affine transformations of the argument. Let

$$T:\ \varphi(x) \in \mathcal{S}\ \to T(\varphi)(x) = \varphi(Ax+b) \in \mathcal{S}$$

where $\det A \neq 0$. Then T is linear and continuous. The assertion follows from the chain rule and its proof is left to the reader as an exercise.

32.2.4. Convolution transformations. Let

$$T:\ \varphi \in \mathcal{S}\ \to T(\varphi) = \omega * \varphi \in \mathcal{S}$$

where $\omega \in \mathcal{S}$ is a temperate test function. Then T is linear and continuous. The function $\omega(y)\varphi(x-y)$ is Lebesgue integrable with respect to y because it falls off faster than any reciprocal power as $|y| \to \infty$ and it is from class C^∞ in the variable x. So the convolution

$$(\omega * \varphi)(x) = \int \omega(y)\varphi(x-y)\, d^N y$$

exists and is from class C^∞ because the derivatives of the integrand have an integrable bound independent of x:

$$|\omega(y) D_x^\alpha \varphi(x-y)| \le M_\alpha |\omega(y)| \in \mathcal{L}\,, \quad M_\alpha = \sup |D^\alpha \varphi|$$

Next, one has to show that $|x|^\alpha |(\omega * \varphi)(x)| \to 0$ as $|x| \to \infty$. This would mean that $\omega * \varphi \in \mathcal{S}$. To this end, note that for any y, $\varphi(x-y)$ is a temperate test function so that $|x|^\alpha D^\beta \varphi(x-y)| \to 0$ as $|x| \to \infty$. To interchange the order of the limit $|x| \to \infty$ and the integration with respect to y, it is sufficient show that the integrand has a Lebesgue integrable bound independent of x and the conclusion would follow from the Lebesgue dominated convergence theorem. Using the binomial expansion, the required bound is obtained

$$|x|^\alpha |\omega(y)\varphi(x-y)| \leq \Big(|y| + |x-y|\Big)^\alpha |\omega(y)\varphi(x-y)|$$
$$\leq \sum_{\beta=0}^{\alpha} C_\beta^\alpha M_\beta |y|^{\alpha-\beta} |\omega(y)| \in \mathcal{L}\,,$$

where $M_\beta = \sup |z|^\beta |\varphi(z)|$, and C_β^α are the binomial coefficients.

The linearity of T follows from the linearity of the integral. Let φ_n is a null sequence in $\mathcal{S}$, then its image is also a null sequence:

$$|x^\alpha D^\gamma T(\varphi_n)| \leq \sum_{\beta=0}^{\alpha} C_\beta^\alpha \int |y|^{\alpha-\beta} |\omega(y)|\, d^N y \sup |z^\beta D^\gamma \varphi_n|$$

where the identity $x^\alpha = (y + (x-y))^\alpha = (y+z)^\alpha$ and its binomial expansion were used again. This inequality holds for all x and by taking the supremum in the left-hand side, it is concluded that the convergence $\varphi_n \to 0$ in $\mathcal{S}$ implies the convergence $T(\varphi_n) \to 0$ in $\mathcal{S}$.

32.2.5. Transformation of $\mathcal{S}$ into $\mathcal{D}$. Consider a transformation of $\mathcal{S}$ defined by the rule

$$T:\ \varphi \in \mathcal{S}\ \to\ T(\varphi)(x) = \omega(x)\varphi(x) \in \mathcal{D}$$

where $\omega \in \mathcal{D}$. This transformation is linear and continuous. Let φ_n be a null sequence in $\mathcal{S}$, then supports of all $T(\varphi_n)$ lies in one ball that contains the support of ω. By the binomial expansion, the derivatives $D^\gamma T(\varphi_n)$ are bounded by a linear combination of terms

$$|D^\alpha \omega(x) D^\beta \varphi_n(x)| \leq \sup |D^\alpha \omega|\ \sup |D^\beta \varphi_n|$$

where $\alpha + \beta = \gamma$. Therefore if $\varphi_n \to 0$ in $\mathcal{S}$, then $T(\varphi_n) \to 0$ in $\mathcal{D}$.

32.2.6. Injection of $\mathcal{D}$ into $\mathcal{S}$. Consider a linear transformation of $\mathcal{D}$ into $\mathcal{S}$ defined by the injection

$$T:\ \varphi \in \mathcal{D}\ \to\ T(\varphi) = \varphi \in \mathcal{S}$$

Then T is continuous because the convergence $\varphi_n \to 0$ in $\mathcal{D}$ implies the convergence $T(\varphi_n) = \varphi_n \to 0$ in the topology of $\mathcal{S}$ (see Sec.**32.1.1**).

32.3. Temperate distributions. A linear continuous functional on the Schwartz space is called a *temperate distribution*. A space of all temperate distributions is denoted by $\mathcal{S}'$:

$$\begin{aligned} & f \ : \quad \mathcal{S} \to \mathbb{R} \\ & (f, c_1\varphi_1 + c_2 f_2) = c_1(f,\varphi_1) + c_2(f,\varphi_2) \\ & \varphi_n \to \varphi \text{ in } \mathcal{S} \quad \Rightarrow \quad \lim_{n\to\infty}(f,\varphi_n) = (f,\varphi)\,. \end{aligned}$$

Note that the space $\mathcal{S}$ is larger than the space $\mathcal{D}$ and, hence, not every distribution from $\mathcal{D}'$ can be extended from $\mathcal{D}$ to $\mathcal{S}$, just like for ordinary functions, a rule to find the value of the function in an interval cannot always be extended to the whole real axis. For example, $f(x) = e^x$ is a regular distribution from $\mathcal{D}'$ and its action on a test function φ is defined by the integral of the product $e^x\varphi(x)$ which exists thanks to the boundedness of the support of φ. If $\varphi \in \mathcal{S}$, then the product $e^x\varphi(x)$ is not integrable in general because $\varphi(x)$ can, for example, decrease as $e^{-x/2}$ as $x \to \infty$. Therefore $f(x) = e^x$ is not a temperate distribution, $f \notin \mathcal{S}'$. The term "temperate" refers to regular distributions with somewhat moderate growth.

On the other hand, every temperate distribution can be *reduced* to the domain $\mathcal{D} \subset \mathcal{S}$. Thus,

$$\mathcal{S}' \subset \mathcal{D}'\,.$$

The reduction of $f \in \mathcal{S}'$ on $\mathcal{D}$ can be viewed as the adjoint transformation $T^* :\ \mathcal{S}' \to \mathcal{D}'$ of the injection $T :\ \mathcal{D} \to \mathcal{S}$ defined by $T(\varphi) = \varphi$ (see Sec.**32.2.6**):

$$(T^*(f),\varphi) = (f, T(\varphi)) = (f,\varphi)\,, \quad \varphi \in \mathcal{D}$$

By linearity and continuity of T, $T^*(f)$ is a linear and continuous functional on $\mathcal{D}$, i.e., $T^*(f) \in \mathcal{D}'$. Furthermore, If $f_n \to f$ in $\mathcal{S}'$, then this distributional sequence also converges in the topology of the larger space $\mathcal{D}'$ because $T^*(f_n) \to T^*(f)$ in $\mathcal{D}'$ by continuity of the adjoint transformation.

32.3.1. Support of a temperate distribution. Since any temperate distribution $f \in \mathcal{S}'$ can always be reduced on $\mathcal{D}(\Omega)$ for any open Ω, the support of f is defined in the same way as the support of distributions from $\mathcal{D}'$. A temperate distribution f is said to vanish in an open set Ω if $(f,\varphi) = 0$ for any $\varphi \in \mathcal{D}(\Omega)$. The support of f is the complement of the largest open set on which $f(x) = 0$.

32.3.2. Delta function as a temperate distribution. The delta-function is a linear functional on $\mathcal{S}$:

$$(\delta, \varphi) = \varphi(0)\,, \quad \varphi \in \mathcal{S}$$

It is also continuous. Indeed, the convergence $\varphi_n \to 0$ in $\mathcal{S}$ implies that φ_n converges to 0 uniformly and, hence, $\varphi_n(0) \to 0$ as $n \to \infty$. Therefore $\delta \in \mathcal{S}'$.

32.3.3. The principal value distribution. Let us show that the principal value distribution is a temperate distribution. For any test function $\varphi \in \mathcal{S}$, put

$$\psi(x) = \frac{\varphi(x) - \varphi(0)}{x}$$

Then $\psi(x) \in C^\infty$. It follows that

$$\begin{aligned}
\Big(\mathcal{P}\frac{1}{x}, \varphi\Big) &= \lim_{a\to 0^+} \int_{|x|>a} \frac{\varphi(x)}{x}\, dx = \lim_{a\to 0^+} \Big(\int_{a<|x|<R} + \int_{|x|>R}\Big) \frac{\varphi(x)}{x}\, dx \\
&= \lim_{a\to 0^+} \Big(\int_{a<|x|<R} \frac{\varphi(x)-\varphi(0)}{x}\, dx + \int_{|x|>R} \frac{\varphi(x)}{x}\, dx\Big) \\
&= \int_{|x|<R} \psi(x)\, dx + \int_{|x|>R} \frac{\varphi(x)}{x}\, dx
\end{aligned}$$

This shows that $\mathcal{P}\frac{1}{x}$ is a linear functional on $\mathcal{S}$. Let φ_n is a null sequence in $\mathcal{S}$. This implies, in particular, that

$$M_n = \sup |x||\varphi_n(x)| \to 0\,, \qquad M_n' = \sup |\varphi_n'| \to 0$$

as $n \to \infty$ The former, in turn, implies that

$$|\varphi_n(x)| \le \frac{M_n}{|x|}\,, \quad |x| > R > 1$$

Therefore

$$\Big|\int_{|x|>R} \frac{\varphi_n(x)}{x}\, dx\Big| \le 2M_n \int_R^\infty \frac{dx}{x^2} = \frac{2M_n}{R} \to 0$$

as $n \to \infty$. Next, $\psi_n(x) = (\varphi_n(x) - \varphi_n(0))/x = \varphi_n'(x_n)$ for some x_n between 0 and x. Hence, $|\psi_n(x)| \le \sup |\varphi_n'|$ for all $|x| < R$. Therefore

$$\begin{aligned}
\Big|\Big(\mathcal{P}\frac{1}{x}, \varphi_n\Big)\Big| &\le \Big|\int_{|x|<R} \psi(x)\, dx\Big| + \Big|\int_{|x|>R} \frac{\varphi(x)}{x}\, dx\Big| \\
&\le 2RM_n' + \frac{2M_n}{R} \to 0
\end{aligned}$$

as $n \to \infty$. Thus, the principal value functional is continuous on $\mathcal{S}$.

32.3.4. Extension of distributions with bounded support to $\mathcal{S}$.

PROPOSITION **32.2**. *Any distribution from $\mathcal{D}'$ that have a bounded support can be uniquely extended to $\mathcal{S}$ and, hence, defines a temperate distribution, by the rule*

$$(f, \varphi) = (f, \eta_f \varphi)\,, \quad \varphi \in \mathcal{S}$$

where η_f is any test function from $\mathcal{D}$ such that $\eta_f(x) = 1$ in a neighborhood of the support of f.

This assertion follows from that the extension rule is nothing but the adjoint transformation $T^* : \mathcal{D}' \to \mathcal{S}'$ of the transformation $T : \mathcal{S} \to \mathcal{D}$ defined in Sec.**32.2.5** where $\omega = \eta_f$. Indeed, by the definition of the adjoint transformation

$$(T^*(f), \varphi) = (f, T(\varphi)) = (f, \eta_f \varphi)\,.$$

By linearity and continuity of T, the adjoint is also linear and continuous and, hence, defines a temperate distribution. The distribution $T^*(f)$ does not depend on the choice of η_f because $(f, (\eta_f - \eta'_f)\varphi) = 0$ for any two test functions η_f and η'_f that take unit value in a neighborhood of the support of f.

32.3.5. On an extension of a distribution to $\mathcal{S}$. Not every distribution from $\mathcal{D}'$ has a continuous extension to the larger set of test functions $\mathcal{S}$. Since the space $\mathcal{D}$ is dense in $\mathcal{S}$ (see Proposition **32.1**), a continuous extension of a functional f can be obtained by

$$(f, \varphi) = \lim_{n \to \infty} (f, \phi_n)\,, \quad \{\phi_n\} \subset \mathcal{D}\,, \quad \phi_n \to \varphi \text{ in } \mathcal{S}\,,$$

provided the limit exists for any $\varphi \in \mathcal{S}$ and is independent of the choice of the sequence $\{\phi_n\}$. If f has a bounded support, then this limit process reproduces the extension rule stated in Proposition **32.2**. If the limit does not exist for a particular choice of the sequence or it exists for any two sequences but has different values, then f cannot be extended to $\mathcal{S}$ because it does not define a continuous functional on $\mathcal{S}$.

To illustrate this assertion, let $f(x) = e^x$. As noted earlier, f is a regular distribution in $\mathcal{D}'$ but not in $\mathcal{S}'$. Let $\eta(x) \geq 0$ be a test function of one real variable x with support in $|x| < 1$. Consider a sequence $\phi_n(x) = M_n \eta(x - n)$. Then

$$|D^\beta \eta(x - n)| \leq \sup |D^\beta \eta(x)| = C_\beta$$

for all n and x, and therefore

$$|x|^\alpha |D^\beta \phi_n(x)| \leq M_n (n+1)^\alpha |D^\beta \eta(x - n)| \leq C_\beta M_n (n+1)^\alpha$$

for all x, where the first inequality follows from that the support of $\eta(x-n)$ lies in $|x-n|<1$. Therefore if $M_n n^\alpha \to 0$ for any $\alpha \geq 0$ as $n \to \infty$, then $\phi_n \to 0$ in $\mathcal{S}$, and for any such ϕ_n

$$\begin{aligned}(f, \phi_n) &= \int e^x \phi_n(x)\,dx = M_n \int e^x \eta(x-n)\,dx \\ &= M_n e^n \int e^x \eta(x)\,dx = C M_n e^n\,.\end{aligned}$$

If f had a continuous extension to $\mathcal{S}$, then the sequence (f, ϕ_n) should converge to zero for *any* M_n that decreases faster than any reciprocal power of n with increasing n. This is not so. The limit depends on the choice of the sequence. For example, if $M_n = e^{-2n}$, then $M_n e^n \to 0$, but if $M_n = e^{-n}$, then $M_n e^n \to 1$ or, if $M_n = e^{-\sqrt{n}}$, then $M_n e^n \to \infty$. Thus, f cannot be extended to $\mathcal{S}$.

32.4. Differentiation of temperate distributions. Differentiation is a linear continuous transformation of $\mathcal{S}$ onto itself as shown in Sec.**32.2.1**. Therefore the adjoint transformation of any $f \in \mathcal{S}'$, denoted by $D^\alpha f$, is a temperate distribution defined by the rule

$$(D^\alpha f, \varphi) = (-1)^\alpha (f, D^\alpha \varphi)$$

It is called the derivative of f. Clearly, all properties of differentiation on $\mathcal{D}'$ are readily extended to $\mathcal{S}'$. In particular, if $f \in \mathcal{D}'$ has an extension to $\mathcal{S}$, then $D^\alpha f$ also have extensions to $\mathcal{S}$ and the derivatives of the extension are equal to extensions of the corresponding derivatives, that is, if $(D^\alpha f)_s \in \mathcal{S}'$, $\alpha \geq 0$, is an extension of $D^\alpha f \in \mathcal{D}'$, then

$$D^\alpha (f)_s = (D^\alpha f)_s\,.$$

For example, $\frac{1}{|x|}$ defines a regular temperate distribution in $\mathbb{R}^3$, that is, $(\frac{1}{|x|})_s = \frac{1}{|x|}$, so that Eq. (**21.15**) holds in $\mathcal{S}'$, too.

32.5. Multiplication by a slowly increasing smooth function. Multiplication of tests functions by a slowly increasing smooth function is a linear continuous transformation (see Sec. **32.2.2**). Therefore its adjoint is a linear continuous transformation on the space of tempered distribution. If $f \in \mathcal{S}'$ and $a \in \mathcal{O}_M$, then the product af is a tempered distribution defined by the adjoint rule

$$(af, \varphi) = (f, a\varphi)\,, \quad \varphi \in \mathcal{S}$$

Since $D^\alpha a \in \mathcal{O}_M$, the Leibniz rule for differentiation also holds for tempered distributions:

$$D^\alpha(af) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} a D^\beta f\,, \quad f\in\mathcal{S}'\,, \quad a\in\mathcal{O}_M\,.$$

Any bump function η for a set Ω is from class $\mathcal{O}_M$ (a C^∞ function whose values are between 0 and 1 and $\eta(x) = 1$ in a neighborhood of Ω). If η_f is a bump function for the support of a temperate distribution f, then

$$f(x) = \eta_f(x) f(x)\,.$$

32.6. Regular temperate distributions and their derivatives. Any bounded function $|f(x)| \le M$ defines a regular temperate distribution

$$(f,\varphi) = \int f(x)\varphi(x)\, d^N x$$

because $|f(x)\varphi(x)| \le M|\varphi(x)|$ is Lebesgue integrable. Continuity of this functional follows from the inequality

$$|\varphi_n(x)| \le \frac{M_n}{1+|x|^\alpha}\,, \qquad M_n = \sup|(1+|x|^\alpha)\varphi_n(x)|$$

Indeed, if $\varphi_n \to 0$ in $\mathcal{S}$, then $M_n \to 0$ as $n\to\infty$ for any $\alpha > 0$. By taking $\alpha > N+1$, it is concluded that

$$|(f,\varphi_n)| \le M\int |\varphi_n(x)|\, d^N x \le MM_n \int \frac{d^N x}{1+|x|^\alpha} = CMM_n \to 0$$

as $n\to\infty$, where C is the value of the integral for $\alpha > N+1$.

Furthermore, any locally integrable function f defines a regular temperate distribution if it grows no faster than a power function

$$|f(x)| \le M(1+|x|^p)\,, \quad |x| > R\,.$$

In this case, for any temperate test function φ, the integral

$$\int_{|x|>R} |f(x)||\varphi(x)| \le M\int_{|x|>R} (1+|x|^p)|\varphi(x)|\, d^N x < \infty$$

exists because $\varphi(x)$ decreases faster than any power function as $|x| \to \infty$. Therefore the integral

$$(f,\varphi) = \int f(x)\varphi(x)\, d^N x$$

converges absolutely and defines a linear continuous functional on $\mathcal{S}$.

However one should not get an impression that locally integrable functions that grow faster than any power function cannot define a tempered distribution. Note that any such function defines a regular distribution from $\mathcal{D}'$ and some of such distributions can have an extension to $\mathcal{S}$. As an example, put $f(x) = e^x \cos(e^x)$. Then $|f(x)|$ grows exponentially as $x \to \infty$. Nonetheless this regular distribution in $\mathcal{D}'$ can be extended to $\mathcal{S}$. Let ϕ_n be a sequence of test functions from $\mathcal{D}$ that converges to $\varphi \in \mathcal{S}$ in the topology of $\mathcal{S}$. Differentiation is continuous on $\mathcal{S}$ so that $\phi_n' \to \varphi'$ in $\mathcal{S}$. Since $f(x) = g'(x)$ where $g(x) = \sin(e^x)$, using integration by parts

$$(f, \phi_n) = \int_{-R}^{R} f(x)\phi_n(x)\, dx = -\int_{-R}^{R} g(x)\phi_n'(x)\, dx = -(g, \phi_n')$$

because the support of any ϕ_n is bounded (R can depend on n). Since $g(x)$ is bounded, it defines a regular tempered distribution. Therefore by continuity of the functional g on $\mathcal{S}$

$$\begin{aligned}(f, \varphi) &= \lim_{n\to\infty} (f, \phi_n) = -\lim_{n\to\infty} (g, \phi_n') = -(g, \varphi') \\ &= -\int_{-\infty}^{\infty} \sin(e^x)\varphi'(x)\, dx\end{aligned}$$

and the limit exists for any $\varphi \in \mathcal{S}$ and does not depend on the choice of $\{\phi_n\} \subset \mathcal{D}$. It defines a linear continuous functional on $\mathcal{S}$ and its reduction on $\mathcal{D}$ coincides with $f(x)$. Note that the integration by parts is not permitted in the last integral if $\varphi \in \mathcal{S}$ but permitted if $\varphi \in \mathcal{D}$.

32.7. General structure of temperate distributions. The analysis given at the very end of the previous section can be extended to define a large class of distributions from $\mathcal{D}'$ that can be extended to $\mathcal{S}$. A *temperate continuous function* is a continuous function which grows no faster than a power function:

$$g \in C^0, \quad |g(x)| \le M(1 + |x|^p), \quad x \in \mathbb{R}^N$$

for some constants M and $p > 0$. Then any temperate continuous function defines a regular temperate distribution:

$$(g, \varphi) = \int g(x)\varphi(x)\, d^N x, \quad \varphi \in \mathcal{S}.$$

Any such function is also a regular distribution from $\mathcal{D}'$ and, hence, the distributional derivatives $D^\alpha g$ are also from $\mathcal{D}'$. The derivatives can be extended to $\mathcal{S}$. Let $\varphi \in \mathcal{S}$. Put

$$(D^\alpha g, \varphi) \overset{\text{def}}{=} (-1)^\alpha (g, D^\alpha \varphi) = (-1)^\alpha \int g(x) D^\alpha \varphi(x)\, d^N x.$$

By linearity and continuity of differentiation on $\mathcal{S}$, this rule defines a linear continuous functional on $\mathcal{S}$. This is an extension of $D^\alpha g \in \mathcal{D}'$ to $\mathcal{S}$. If g is from C^∞, the integration by parts is not permitted if $\varphi \in \mathcal{S}$ because the derivatives $D^\alpha g$ are not temperate functions, in general, as shown in the previous section. However, if $\varphi \in \mathcal{D}$, then the integration by parts is permitted and the distributional and classical derivatives of g coincide in $\mathcal{D}'$.

The analysis shows any distribution from $\mathcal{D}'$ that is a sum of derivatives of temperate continuous functions can be extended to $\mathcal{S}$ and defines a tempered distribution. It turns out that the converse is also true. In other words, any tempered distribution can be written in this form.

THEOREM **32.1**. (L. Schwartz)
For any temperate distribution f, there exist temperate continuous functions g_α that vanish outside a neighborhood of the support of f of arbitrary small radius, and

$$f(x) = \sum_{\alpha \le p} D^\alpha g_\alpha(x)$$

for some integer $p \ge 0$.

Thus, for any $f \in \mathcal{S}'$ one can find a finite collection of temperate continuous functions such that

$$(f, \varphi) = \sum_{\alpha \le p} (-1)^\alpha \int g_\alpha(x) D^\alpha \varphi(x)\, d^N x$$

The Schwartz theorem **32.1** is also known as the structure for temperate distributions. Note that singular distributions like delta functions, principal value distributions, and Sokhotsky type singular distributions are tempered distributions and, hence, can be written as a linear combinations of distributional derivatives of continuous functions. Practically, all distributions used quantum field theory and Green's functions for linear differential operators are of this type! A proof of Theorem **32.1** is based on another theorem due to L. Schwartz

THEOREM **32.2**. *In order for a linear functional f on $\mathcal{S}$ to be continuous (to be a temperate distribution) it is necessary and sufficient that for any test function $\varphi \in \mathcal{S}$ there exists an number $C > 0$ and an integer p such that*

$$|(f, \varphi)| \le C \sup_{\alpha \le p,\, x} \Big[(1 + |x|)^p |D^\alpha \varphi(x)| \Big] .$$

32.7.1. Completeness of the space of tempered distributions. Using the Schwartz theorem one can show that the space $\mathcal{S}'$ is complete. *Let* $\{f_n\}$ *be a sequence of temperate distributions such that* (f_n, φ) *converges for any test function* $\varphi \in \mathcal{S}$. *Then the functional* f *defined by the rule*

$$(f, \varphi) = \lim_{n\to\infty} (f_n, \varphi)$$

is linear and continuous on $\mathcal{S}$, *that is,* f *is a temperate distribution.*

32.8. Topology in $\mathcal{S}'$. *A sequence of temperate distributions is said to converge to a temperate distribution if for any test function from* $\mathcal{S}$ *the numerical sequence of values of terms of the sequence converges to the value of the limit distribution on the test function:*

$$f_n \to f \text{ in } \mathcal{S}' \ : \qquad \lim_{n\to\infty} (f_n, \varphi) = (f, \varphi)\,, \quad \forall \varphi \in \mathcal{S}$$

It follows from this definition that the convergence in $\mathcal{S}'$ implies convergence in $\mathcal{D}'$:

$$f_n \to f \text{ in } \mathcal{S}' \quad \Rightarrow \quad f_n \to f \text{ in } \mathcal{D}'\,.$$

Indeed, recall that the injection of $\mathcal{D}$ into $\mathcal{S}$ is linear and continuous (see Sec.**32.2.6**). Therefore its adjoint $T^* : \mathcal{S}' \to \mathcal{D}'$ is reduction of any temperate distribution on $\mathcal{D}$, and T^* must also be linear and continuous. For any test function $\varphi \in \mathcal{D}$ (so that $T(\varphi) \in \mathcal{S}$)

$$\lim_{n\to\infty} (T^*(f_n), \varphi) = \lim_{n\to\infty} (f_n, T(\varphi)) = (f, T(\varphi)) = (T^*(f), \varphi)$$

which means that the convergence $f_n \to f$ in $\mathcal{S}'$ implies that convergence of the reduction of f_n on $\mathcal{D}$ to the reduction of f on $\mathcal{D}$, that is, $T^*(f_n) \to T^*(f)$.

32.8.1. Example. Let us show that

$$\frac{\sin(nx)}{x} \ \to \ \pi\delta(x) \quad \text{in } \mathcal{S}'(\mathbb{R}) \text{ as } n \to \infty\,.$$

Put $f_n(x) = \sin(nx)/x$. Then $|f_n(x)| \leq n$ is bounded for any n and, hence, defines a regular temperate distribution. Consider the sequence

$$F_n(x) = \int_0^x f_n(y)\,dy = \int_0^{nx} \frac{\sin(z)}{z}\,dz\,.$$

For every n, $F_n \in C^1$ is continuously differentiable function and $F_n'(x) = f_n(x)$. Its pointwise limit reads

$$\lim_{n\to\infty} F_n(x) = \frac{\pi}{2}\,\text{sign}\,(x)\,, \qquad F_n(0) = 0$$

Therefore $F_n(x)$ is a bounded sequence $|F_n(x)| \le M$ and M is independent of n (recall also Abel's theorem about conditionally convergent integrals in this regard). By integrating by parts one infers that

$$(f_n, \varphi) = \int f_n(x)\varphi(x)\,dx = \int \varphi(x)\,dF_n(x) = -\int F_n(x)\varphi'(x)\,dx$$

the boundary terms vanish because φ_n vanishes at infinity and F_n is bounded. Since the integrand is bounded by an integrable function

$$|F_n(x)\varphi'(x)| \le M|\varphi'(x)| \in \mathcal{L}$$

that is independent of n, by the Lebesgue dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty}(f_n,\varphi) &= -\frac{\pi}{2}\int \operatorname{sign}(x)\varphi'(x)\,dx\\ &= \frac{\pi}{2}\int_{-\infty}^{0}\varphi'(x)\,dx - \frac{\pi}{2}\int_0^\infty \varphi'(x)\,dx\\ &= \pi\varphi(0) = \pi(\delta,\varphi)\end{aligned}$$

Thus, $f_n \to \delta$ in $\mathcal{S}'$.

32.9. Direct product of temperate distributions. The direct product of temperate distributions is defined in the same way as the direct product of distributions from $\mathcal{D}'$. First, one has to construct a linear continuous transformation of temperate test functions of two variables to the space of temperate test functions of one variable induced by a temperate distribution. Then its adjoint defines the direct product of temperate distributions.

32.9.1. Reduction of variables in temperate test functions. Let $g(y) \in \mathcal{S}'(\mathbb{R}^M)$. Consider the transformation

$$T_g:\ \varphi(x,y) \in \mathcal{S}(\mathbb{R}^{N+M}) \ \to\ T_g(\varphi)(x) = \Big(g(y), \varphi(x,y)\Big)\,. \tag{32.1}$$

Proposition **32.3**. *The transformation (***32.1***) is a linear and continuous transformation of $\mathcal{S}(\mathbb{R}^{N+M})$ into $\mathcal{S}(\mathbb{R}^N)$ and*

$$D^\alpha T_g(\varphi)(x) = \Big(g(y), D_x^\alpha \varphi(x,y)\Big) \tag{32.2}$$

Using the same line of arguments as in the proof of Proposition **19.2**, one shows that $T_g(\varphi) \in C^\infty$ and (**32.2**) holds. Next, one has to show that $T_g(\varphi)$ is a temperate test function, that is, $D^\alpha T_g(\varphi)$ decreases to zero faster than any reciprocal power function as $|x| \to \infty$ for all $\alpha \ge 0$.

By Theorem **32.2** applied to g, there exist numbers $p \geq 0$ and $C > 0$ such that

$$\begin{aligned}|x^\gamma D^\alpha T_g(\varphi)| &= \Big|x^\gamma \Big(g(y), D_x^\alpha \varphi(x,y)\Big)\Big| \\ &\leq C \sup_{y,\beta\leq p} (1+|y|)^p |x^\gamma D_y^\beta D_x^\alpha \varphi(x,y)|\end{aligned}$$

Since $D_y^\beta D_x^\alpha \varphi(x,y)$ decreases to zero faster than any reciprocal power function of two variables x and y, the limit of the left-hand side is zero when $|x| \to \infty$. Thus, $T_g(\varphi) \in \mathcal{S}$.

Let us prove continuity of T_g. Let φ_n be a null sequence in $\mathcal{S}(\mathbb{R}^{N+M})$. One has to show that its image $T_g(\varphi_n)$ is a null sequence in $\mathcal{S}(\mathbb{R}^N)$. By replacing in the above inequality $T_g(\varphi)$ by $T_g(\varphi_n)$ and taking the supremum over x in the right-hand side, it is concluded that

$$|x^\gamma D^\alpha T_g(\varphi_n)| \leq M_n = C \sup_{x,y,\beta\leq p} (1+|y|)^p |x^\gamma D_y^\beta D_x^\alpha \varphi_n(x,y)| \ \to 0$$

as $n \to \infty$ for any non-negative α and γ by the definition of convergence $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^{N+M})$. This implies that $x^\gamma D^\alpha T_g(\varphi_n)$ converges uniformly to zero as $n \to \infty$ for any γ and α, or $T_g(\varphi_n) \to 0$ in $\mathcal{S}(\mathbb{R}^N)$.

Consider the adjoint transformation T_g^*. It is a linear continuous transformation of $\mathcal{S}'(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^{N+M})$ defined by the rule

$$(T_g^*(f), \varphi) = (f, T_g(\varphi)) = \Big(f(x), \Big(g(y), \varphi(x,y)\Big)\Big)\,.$$

The tempered distribution of two variables defined by this rule is called the *direct product* of tempered distributions $f(x)$ and $g(x)$:

$$T_g^*(f)(x,y) = f(x) \cdot g(y)\,.$$

By construction, it is a linear and continuous functional on $\mathcal{S}(\mathbb{R}^{N+M})$.

PROPOSITION **32.4**. (Properties of the direct product in $\mathcal{S}'$) *The direct product is commutative and associative in $\mathcal{S}'$:*

$$\begin{aligned}f(x) \cdot g(y) &= g(y) \cdot f(x) \\ (f(x) \cdot g(y)) \cdot h(z) &= f(x) \cdot (g(y) \cdot h(z))\end{aligned}$$

for any tempered distributions f, g, and h of independent variables x, y, and z spanning Euclidean spaces.

A proof of this assertion follows from the commutativity and associativity of the direct product in $\mathcal{D}'$. Since $\mathcal{D}$ is dense in $\mathcal{S}$ for any $\varphi(x,y) \in \mathcal{S}$ there exists a sequence $\varphi_n(x,y) \in \mathcal{D}$ that converges to $\varphi(x,y)$ in $\mathcal{S}$. Therefore for any $\varphi \in \mathcal{S}$,

$$(f \cdot g, \varphi) = \lim_{n\to\infty} (f \cdot g, \varphi_n) = \lim_{n\to\infty} (g \cdot f, \varphi_n) = (g \cdot f, \varphi)$$

and similarly for the associativity.

In particular, $f(x)\cdot 1(y) = 1(y)\cdot f(x)$ implies that

$$\Big(f(x), \int \varphi(x,y)\, d^M y\Big) = \int \Big(f(x), \varphi(x,y)\Big)\, d^M y \tag{32.3}$$

for any test function of two variables φ.

Owing to the commutativity and associativity, in what follows, the dot indicating the direct product is often omitted. For example

$$\delta(x) = \delta(x_1)\cdot\delta(x_2)\cdots\delta(x_n) = \delta(x_1)\delta(x_2)\cdots\delta(x_n)\,, \quad x \in \mathbb{R}^N$$
$$\theta(x)\cdot\delta(y) = \theta(x)\delta(y)\,, \quad x, y \in \mathbb{R}\,.$$

32.10. Convolution of temperate distributions. Suppose that f and g are temperate distributions so they can always be reduced to the subspace $\mathcal{D} \subset \mathcal{S}$. Suppose that their convolution $f * g$ exists in $\mathcal{D}'$. Then their convolution in $\mathcal{S}'$ can be defined as an extension of $f * g$ to $\mathcal{S}$. As was shown, not every distribution from $\mathcal{D}'$ can be extended. So, the question is: Under what conditions $f * g \in \mathcal{S}'$ and the convolution transformation $f \in \mathcal{S}' \to f * g \in \mathcal{S}'$ is continuous? This is a difficult question in general. It turns out that for all three important cases in which the convolution exists in $\mathcal{D}'$ covered by Theorems **31.1**, **31.2**, and **43.1** the answer to this question is affirmative.

32.10.1. Convolution with distributions with bounded support. By Proposition **32.2** any distribution $g \in \mathcal{D}'$ with bounded support has an extension to $\mathcal{S}$, and the convolution any two distributions one of which has a bounded support always exists in $\mathcal{D}'$. It turns out that this convolution is also a tempered distribution.

THEOREM **32.3**. *Let f and g be temperate distribution and the support of g be bounded. Then the convolution $f * g$ is a temperate distribution defined by the rule*

$$(f * g, \varphi) = \Big(f(x)\cdot g(y), \eta(y)\varphi(x+y)\Big)\,, \quad \varphi \in \mathcal{S}\,,$$

where η is any test function from $\mathcal{D}$ that has unit value in a neighborhood of $\operatorname{supp} g$. *Furthermore, the convolution is continuous with respect to both arguments, that is,*

$$f_n \to f \quad \text{in } \mathcal{S}' \quad \Rightarrow \quad f_n * g \to f * g \quad \text{in } \mathcal{S}'$$
$$g_n \to g \quad \text{in } \mathcal{D}' \quad \Rightarrow \quad f * g_n \to f * g \quad \text{in } \mathcal{S}'$$

By Theorem **31.1** the convolution $f * g$ exists in $\mathcal{D}'$ and is defined by the same rule where $\varphi \in \mathcal{D}$. One has to show that the rule can be extended to $\mathcal{S}$ and defines a linear continuous functional on $\mathcal{S}$. Let

$\varphi \in \mathcal{S}$. Then $\eta(y)\varphi(x+y) \in \mathcal{S}(\mathbb{R}^{2N})$ is a temperate test function of two variables. Indeed, for any given x, this function vanishes for all large enough y (by the properties of $\eta(y)$), and for any given y, it and all its derivatives are decreasing faster than any reciprocal power function (by the properties of $\varphi(x+y)$). So, by linearity of the direct product, the convolution is a linear functional on $\mathcal{S}(\mathbb{R}^N)$. To show continuity, let $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^N)$. Then $\eta(y)\varphi_n(x+y) \to 0$ in $\mathcal{S}(\mathbb{R}^{2N})$. Indeed, one has

$$\sup |z^\alpha y^\beta D^\gamma \eta(y) D^\delta \varphi_n(z)| \leq M_{\beta\gamma} \sup |z^\alpha D^\delta \varphi_n(z)| \to 0$$

where $M_{\beta\gamma} = \sup |y^\beta D^\gamma \eta(y)|$, because $\varphi_n \to 0$ in $\mathcal{S}$. Using the binomial expansion of $x^\alpha = (z-y)^\alpha$ and of the derivative D^γ of the product, one infers that that above uniform convergence implies that

$$\lim_{n\to\infty} \sup |x^\alpha y^\beta D^\gamma(\eta(y)\varphi_n(x+y))| = 0$$

for any α, β, and γ, as required. Thus, $f * g \in \mathcal{S}'$. The direct product of two temperate distributions is the adjoint transformation for (**32.1**). So, the convolution is continuous by continuity of the adjoint transformation.

32.10.2. Convolution algebra $\mathcal{S}'_+$. Put $\mathcal{S}'_+ = \mathcal{D}'_+ \cap \mathcal{S}'$, that is, $\mathcal{S}'_+$ consists of all temperate distributions of one variable whose reduction on $\mathcal{D}$ is a distribution from $\mathcal{D}'_+$. Theorem **31.2** describes the convolution in $\mathcal{D}'_+$. Then the convolution of any two distributions f and g from $\mathcal{S}'_+$ can be extended to $\mathcal{S}$ by

$$(f * g, \varphi) = \Big(f(t) \cdot g(\tau), \eta_1(t)\eta_2(\tau)\varphi(t+\tau)\Big), \quad \varphi \in \mathcal{S}, \tag{32.4}$$

where $\eta_{1,2}$ are any bump functions for the half-line $[0, \infty)$, and the convolution is continuous, that is,

$$f_n \to f \quad \text{in } \mathcal{S}' \quad \Rightarrow \quad f_n * g \to f * g \quad \text{in } \mathcal{S}'$$

where the sequence f_n, its limit f, and g are from $\mathcal{S}'_+$. In other words, Theorem **31.2** holds for all tempered distributions whose support lies in a half-line. In this sense, $\mathcal{S}'_+$ is the convolution subalgebra of $\mathcal{D}'_+$.

A proof of this assertion goes along the same lines as the proof of Theorem **32.3**. For any temperate test function $\varphi(t)$ of one variable, the function $\eta_1(t)\eta_2(\tau)\varphi(t+\tau)$ is a temperate test function of two variables so that the convolution is a linear functional on $\mathcal{S}$ because so is the direct product. The convergence $\varphi_n \to 0$ in $\mathcal{S}$, that is, $\sup |t^\alpha D^\beta \varphi(t)| \to 0$ as $n \to \infty$, implies that

$$\lim_{n\to\infty} \sup |t^\alpha \tau^\beta D^\gamma(\eta_1(t)\eta_2(\tau)\varphi_n(t+\tau))| = 0\,.$$

So, $f * g \in \mathcal{S}'$. Finally, $(f * g, \varphi) = 0$ for any $\varphi \in \mathcal{D}$ whose support lies in $(-\infty, 0)$. This means that $(f * g)(x) = 0$ in $(-\infty, 0)$ or $f * g \in \mathcal{S}'_+$. The continuity of the convolution follows from the continuity of the direct product.

32.10.3. Convolution of tempered distributions with support in a light cone. Let $f(x,t)$ and $g(x,t)$, $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$ be temperate distributions such that

$$f(x,t) = 0\,, \quad t < 0\,, \qquad \operatorname{supp} g \subseteq \overline{\Gamma^+}$$

where Γ^+ is the positive light cone, $ct > |x|$ and $t > 0$ for some $c > 0$. Then the convolution of their reductions on $\mathcal{D}$ exists in $\mathcal{D}'$ by Theorem **43.1**. Then the convolution can be extended to $\mathcal{S}$ by the rule (**43.1**) where $\varphi \in \mathcal{S}(\mathbb{R}^{N+1})$, and $f * g = g * f$ is continuous with respect to both arguments:

$$\begin{aligned} f_n \to f \quad \text{in } \mathcal{S}' \quad &\Rightarrow \quad f_n * g \to f * g \quad \text{in } \mathcal{S}' \\ g_n \to f \quad \text{in } \mathcal{S}' \quad &\Rightarrow \quad f * g_n \to f * g \quad \text{in } \mathcal{S}' \end{aligned}$$

where $\operatorname{supp} g_n \subseteq \overline{\Gamma^+}$ and $f_n(x,t) = 0$, $t < 0$, for all n. A proof of this assertion is similar to the previous case in Sec.**32.10.2**. Put

$$T(\varphi)(x,t) = \eta(t)\eta(\tau)\eta(c^2\tau^2 - |y|^2)\varphi(x+y, t+\tau)$$

where η is defined in Theorem **43.1**. One has to show that T is a linear continuous transformation of $\mathcal{S}$ into $\mathcal{S}$, that is

$$\begin{aligned} \varphi \in \mathcal{S} \quad &\Rightarrow \quad T(\varphi) \in \mathcal{S} \\ \varphi_n \to 0 \quad \text{in } \mathcal{S} \quad &\Rightarrow \quad T(\varphi_n) \to 0 \quad \text{in } \mathcal{S} \end{aligned}$$

while the linearity of T is evident. The technical details are left to the reader as an exercise. Once these properties of T are established, that is, $f * g \in \mathcal{S}'$, the continuity of the convolution follows from the continuity of the direct product in $\mathcal{S}'$.

32.10.4. Convolution with a test function. Let $\omega \in \mathcal{S}$ and $f \in \mathcal{S}'$. Then ω and f can be viewed as distributions from $\mathcal{D}'$. Their convolution in $\mathcal{D}'$ can be found by the limit process defined in Sec.**30.3**, provided the limit exists. Then one can try to extend the convolution $\omega * f$ to $\mathcal{S}$.

PROPOSITION **32.5**. *If $\omega \in \mathcal{S}$ and $f \in \mathcal{S}'$, then the convolution $\omega * f$ exists in $\mathcal{S}'$ and is defined by the rule*

$$(\omega * f, \varphi) = (f, \psi)\,, \quad \psi(x) = \int \omega(y)\varphi(x+y)\,d^N y\,.$$

The convolution is continuous

$$f_n \to f \quad \text{in } \mathcal{S}' \quad \Rightarrow \quad \omega * f_n \to \omega * f \quad \text{in } \mathcal{S}'$$
$$\omega_n \to \omega \quad \text{in } \mathcal{S} \quad \Rightarrow \quad \omega_n * f \to \omega * f \quad \text{in } \mathcal{S}'$$

Furthermore the convolution is a regular tempered distribution defined by a function from $\mathcal{O}_M$

$$(\omega * f)(y) = (f(x), \omega(y-x)), \quad D^\beta(\omega * f)(y) = (f(x), D_y^\beta \omega(y-x))$$

for all $\beta > 0$.

First, note that $\psi(y) = T(\varphi)(y)$ where T is the convolution transformation of $\mathcal{S}$ into $\mathcal{S}$ induced by a test function $\omega(-x)$. In Sec.**32.2.4**, this transformation is proved to be linear and continuous. Therefore the adjoint $T^*(f)$ is a temperate distribution for any $f \in \mathcal{S}'$ that is defined by the rule

$$(T^*(f), \varphi) = (f, T(\varphi)) = (f, \psi) \quad \Rightarrow \quad T^*(f) = \omega * f\,.$$

Furthermore, T^* is a linear continuous transformation so that $T^*(f_n) \to T^*(f)$ in $\mathcal{S}'$ if $f_n \to f$ in $\mathcal{S}'$. If $\omega_n(x) \to \omega(x)$ in $\mathcal{S}$, then $\omega_n(-x)$ converges to $\omega(-x)$ because non-singular linear transformations of the argument are linear and continuous transformations on $\mathcal{S}$. So, $\psi_n(x) = \omega_n(-x) * \varphi(x) \to \omega(-x) * \varphi(x) = \psi(x)$ in $\mathcal{S}$ by continuity of the convolution transformation induced by a test function φ. By continuity of the functional f, $(f, \psi_n) \to (f, \psi)$ and, hence, $\omega_n * f \to \omega * f$ in $\mathcal{S}'$.

Second, note that $\omega(y)\varphi(x+y) \in \mathcal{S}(\mathbb{R}^{2N})$ is a test function of two variables. It is obtained from $\omega(y)\varphi(x) \in \mathcal{S}(\mathbb{R}^{2N})$ by a non-singular linear transformation of the argument, $y \to y$ and $x \to x+y$. Therefore by (**32.3**)

$$\begin{aligned}(\omega * f, \varphi) &= \Big(f(x), \int \omega(z-x)\varphi(z)\, d^N z\Big) \\ &= \int \Big(f(x), \omega(z-x)\Big)\varphi(z)\, d^N z\,,\end{aligned}$$

where the change of variables $y = z - x$ has been done. This shows that $(\omega * f)(z) = (f(x), \omega(z-x))$ is a regular distribution. By Proposition **32.3** the integrand in the last integral is a test function. Therefore the convolution is a smooth function. Let us calculate its derivatives. One has

$$\begin{aligned}(D(\omega * f), \varphi) &= -(\omega * f, D\varphi) = -\Big(f(x), \int \omega(z-x) D_z\varphi(z)\, d^N z\Big) \\ &= \Big(f(x), \int D_z\omega(z-x)\varphi(z)\, d^N z\Big) \\ &= \int \Big(f(x), D_z\omega(z-x)\Big)\varphi(z)\, d^N z\,.\end{aligned}$$

Therefore

$$D^\alpha(\omega * f)(x) = \Big(f(x), D_z^\alpha \omega(z-x)\Big)\,.$$

Finally, let us show that $\omega * f \in \mathcal{O}_M$. By Theorem **32.2**

$$\begin{aligned}|D^\alpha(\omega * f)(x)| &= |(f(y), D_x^\alpha \omega(y-x))| \\ &\le C \sup_{\beta\le p, y} (1+|y|)^p |D_x^\alpha D_y^\beta \omega(x-y)| \\ &= C \sup_{\beta \le p, z} (1+|x-z|)^p |D^{\alpha+\beta}\omega(z)| \\ &\le 2^p C (1+|x|)^p \sup_{\beta\le p, z} (1+|z|)^p |D^{\alpha+\beta}\omega(z)|\end{aligned}$$

where the inequality $(a+b)^p \le 2^p a^p b^p$, $a \ge 1$ and $b \ge 1$, has been used to obtain the last inequality. Therefore all derivatives of the convolution are bounded by a power function, which means that the convolution is a smooth slowly increasing function.

32.10.5. Example. Let $\omega(x) = e^{-|x|^2}$ where $x \in \mathbb{R}^2$ and $f(x) = \delta_C(x)$ is a surface delta function with support on a circle $|x| = a$. Let us calculate the convolution $\omega * \delta_C$. Since ω is a test function, by Proposition **32.5**

$$\begin{aligned}(\omega * \delta_C)(x) &= \Big(\delta_C(y), \omega(x-y)\Big) = \int_{|y|=a} \omega(x-y)\, ds_y \\ &= ae^{-|x|^2-a^2} \int_{-\pi}^{\pi} e^{-2a|x|\cos(\theta)} d\theta = 2\pi a e^{-|x|^2-a^2} I_0(2a|x|)\end{aligned}$$

where the line integral is evaluated using a parameterization in which $(x, y) = |x||y|\cos(\theta) = a|x|\cos(\theta)$, $ds_y = a d\theta$, and $I_0(z)$ is the modified Bessel function. Note that I_0 is analytic and its power series representation contains only even powers. Therefore the convolution is an analytic function of $|x|^2$ and, hence, from $C^\infty(\mathbb{R}^2)$. For large values of the argument $I_0(z) = (2\pi z)^{-1/2} e^z (1+O(\frac{1}{z}))$. Therefore the convolution is from $\mathcal{S} \subset \mathcal{O}_M$.

32.11. Regularization of temperate distributions. Suppose that ω_n is a sequence of test functions from $\mathcal{D}$ such that their supports are in a ball B_R for all n and $\omega_n \to \delta$ in $\mathcal{D}'$. Then ω_n are regular distributions that can be extended to $\mathcal{S}$ because they have a bounded support. Then by Theorem **32.3**

$$\omega_n * f \to \delta * f = f \quad \text{in } \mathcal{S}'$$

for any temperate distribution f. By Proposition **32.5**

$$f_n(x) = (\omega_n * f)(x) = (f(y), \omega_n(x-y)) \in \mathcal{O}_M\,.$$

So, any temperate distribution can be viewed as a distributional limit of a sequence of smooth slowly increasing functions. This also means that $\mathcal{O}_M$ is dense in the space of temperate distributions. Furthermore, for any $\eta \in \mathcal{O}_M$,

$$e^{-\varepsilon|x|^2}\eta(x) \to \eta(x) \quad \text{in } \mathcal{S}'$$

as $\varepsilon \to 0^+$. This implies that the subspace of regular temperate distributions defined by test functions is dense in $\mathcal{O}_M \subset \mathcal{S}'$, and, hence, $\mathcal{S}$ is dense in $\mathcal{S}'$. So, for any temperate distribution f one can find a sequence of test functions that converges to f in $\mathcal{S}'$. For example, one can take $f_\varepsilon(x) = e^{-\varepsilon|x|^2}(\omega_\varepsilon * f)(x)$ where ω_ε is the hat function. Then $f_\varepsilon \in \mathcal{S}$ and $f_\varepsilon \to f$ in $\mathcal{S}'$ as $\varepsilon \to 0^+$.

32.12. Exercises.

1. Let $\{a_n\}$ be a numerical sequence. Consider the series

$$\sum_{n=1}^{\infty} a_n \delta(x-n)$$

(i) Show that the series converges in $\mathcal{D}'$ but not necessarily in $\mathcal{S}'$. Give an example of a_n such that the series does not converge in $\mathcal{S}'$.

(ii) Show that the series converges in $\mathcal{S}'$ if

$$|a_n| \le M n^p$$

for some $p > 0$.

2. Show that the distribution $\mathcal{P}\frac{1}{|x|} \in \mathcal{D}'(\mathbb{R})$,

$$\Big(\mathcal{P}\frac{1}{|x|}, \varphi\Big) = \int_{|x|<1} \frac{\varphi(x)-\varphi(0)}{|x|}\,dx + \int_{|x|>1} \frac{\varphi(x)}{|x|}\,dx$$

has an extension to $\mathcal{S}$ defined by the same rule with $\varphi \in \mathcal{S}$.

3. Show that the distribution $\mathcal{P}\frac{1}{|x|^2} \in \mathcal{D}'(\mathbb{R}^2)$,

$$\Big(\mathcal{P}\frac{1}{|x|^2}, \varphi\Big) = \int_{|x|<1} \frac{\varphi(x) - \varphi(0)}{|x|^2}\, d^2x + \int_{|x|>1} \frac{\varphi(x)}{|x|^2}\, d^2x$$

has an extension to $\mathcal{S}(\mathbb{R}^2)$ defined by the same rule with $\varphi \in \mathcal{S}$.

4. Let C be a smooth curve in $\mathbb{R}^N$ and $\mu(x)$ be a continuous function. Suppose that C is not bounded but any ball contains a part of C that has a finite length.
(i) Show that the linear functional $\mu\delta_C$ defined by the rule

$$(\mu\delta_C, \varphi) = \int_C \mu(x)\varphi(x)\, ds$$

is a distribution from $\mathcal{D}'(\mathbb{R}^N)$.
(ii) Show that the functional $\mu\delta_C$ does not generally have an extension to $\mathcal{S}$ for arbitrary μ and C. Give an example of the density μ and a smooth curve C for which $\mu\delta_C$ is not a temperate distribution. Show that if there exists some $k > 0$ and $p > 0$ such that

$$\int_C \frac{ds}{1 + |x|^k} < \infty\,, \quad |\mu(x)| \le M(1 + |x|^p)\,,$$

then $\mu\delta_C \in \mathcal{S}'$.

5. Write the following distributions as a linear combination of distributional derivatives of continuous functions:
(i) $\delta(x)$, $\mathcal{P}\frac{1}{x}$, $\mathcal{P}\frac{1}{|x|}$, where $x \in \mathbb{R}$
(ii) $\nu(x)\delta_{S_a}(x)$ (the spherical delta function in $\mathbb{R}^N$ with density ν)
(iii) $(|x|^2 - m^2 \pm i0)^{-1}$ where $x \in \mathbb{R}^3$.

6. Find the following convolutions in $\mathcal{S}'$ or show that they do not exist:

(i) $\mathcal{P}\dfrac{1}{x} * \delta''(x)\,, \quad x \in \mathbb{R}$

(ii) $\mathcal{P}\dfrac{1}{|x|} * \delta'(x)\,, \quad x \in \mathbb{R}$

(iii) $\mathcal{P}\dfrac{1}{|x|} * [\theta(x-a) - \theta(x-b)]\,, \quad x \in \mathbb{R}\,, \quad a < b$

(iv) $\mathcal{P}\dfrac{1}{x^n} * [\theta(x-a) - \theta(x-b)]\,, \quad x \in \mathbb{R}\,, \quad a < b$

33. Fourier transform of distributions

33.1. Preliminaries. Let f be an integrable function on $\mathbb{R}^N$:

$$\int |f(x)|\, d^N x < \infty \,.$$

Then its Fourier transform

$$\mathcal{F}[f](k) = \int e^{i(k,x)} f(x)\, d^N x$$

is a continuous function for all $k \in \mathbb{R}^N$. Therefore f and $\mathcal{F}[f]$ are regular distributions in $\mathcal{D}'$, and for any test function $\varphi \in \mathcal{D}$

$$\begin{aligned}(\mathcal{F}[f], \varphi) &= \int \mathcal{F}[f](k)\varphi(k)\, d^N k = \int\int e^{i(k,x)} f(x)\varphi(k)\, d^N x\, d^N k \\ &= \int\int e^{i(k,x)} f(x)\varphi(k)\, d^N k\ d^N x\ = \int f(x)\mathcal{F}[\varphi](x)\, d^N x \\ &= (f, \mathcal{F}[\varphi])\end{aligned}$$

where the order of integration can be changed by Fubini's theorem because

$$\int\int \left| e^{i(k,x)} f(x)\varphi(k)\right| d^N x\, d^N k = \int |f(x)|\, d^N x \int |\varphi(k)|\, d^N k < \infty \,.$$

The obtained relation defines the value of the regular distribution $\mathcal{F}[f]$ on a test function via the value of the original distribution f on the Fourier transform of the test function.

If one wants to extend the Fourier transform of ordinary functions (or regular distributions) to any distribution, then this rule looks appropriate. However, the Fourier transform of a test function from $\mathcal{D}$ does not belong to $\mathcal{D}$

$$\varphi \in \mathcal{D} \quad \Rightarrow \quad \mathcal{F}[\varphi] \notin \mathcal{D}\,,$$

because the function $\mathcal{F}[\varphi](k)$ does not vanish for all $|k| > R$ and some R. Therefore, the value $(f, \mathcal{F}[\varphi])$ is not defined for all $f \in \mathcal{D}'$. To make the definition consistent, one has to expand the domain of distributions so that the Fourier transform of any test function would also be a test function. It turns out that the space of temperate test functions has the required properties.

33.2. Fourier transform on $\mathcal{S}$.

PROPOSITION **33.1**. *The Fourier transform*

$$\mathcal{F}:\ \varphi \in \mathcal{S}\ \rightarrow\ \mathcal{F}[\varphi] \in \mathcal{S}$$

is a linear and continuous transformation of $\mathcal{S}$ *onto itself, and*

$$D^\alpha \mathcal{F}[\varphi](k) = \mathcal{F}[(iD)^\beta \varphi](k) \tag{33.1}$$

$$k^\beta \mathcal{F}[\varphi](k) = \mathcal{F}[(iD)^\beta \varphi](k)\,. \tag{33.2}$$

Let us first prove that $\mathcal{F}[\varphi] \in \mathcal{S}$. The function $e^{i(k,x)}\varphi(x)$ is from C^∞ in the variable k for any x and its derivatives have integrable bounds independent of k:

$$|D_k^\alpha e^{i(k,x)}\varphi(x)| \le |x|^\alpha |\varphi(x)| \in \mathcal{L}$$

Therefore $\mathcal{F}[\varphi] \in C^\infty$ and the order of differentiation and integration can be interchanged:

$$D^\alpha \mathcal{F}[\varphi](k) = \int D_k^\alpha e^{i(k,x)}\varphi(x)\, d^N x = \mathcal{F}[(ix)^\alpha \varphi](k)\,.$$

Using the integration by parts

$$\begin{aligned} k^\beta \mathcal{F}[\varphi](k) &= \int (-iD_x)^\beta e^{i(k,x)}\varphi(x)\, d^N x = \int e^{i(k,x)} (iD)^\beta \varphi(x)\, d^N x \\ &= \mathcal{F}[(iD)^\beta \varphi](k)\,. \end{aligned}$$

It follows from the established properties of the Fourier transform that for any non-negative α and β

$$\begin{aligned} \Big| k^\beta D^\alpha \mathcal{F}[\varphi](k) \Big| &= \Big| \mathcal{F}\Big[(iD)^\beta \Big((ix)^\alpha \varphi \Big) \Big](k) \Big| \\ &\le \int |D^\beta (x^\alpha \varphi)|\, d^N x = M < \infty \end{aligned}$$

for all $k \in \mathbb{R}^N$. This inequality implies that $\mathcal{F}[\varphi]$ and all its partial derivatives are decreasing to zero faster than any reciprocal power $|k|^{-\beta}$ as $|k| \to \infty$. Thus, $\mathcal{F}[\varphi] \in \mathcal{S}$.

Next let us prove that the Fourier transform is linear and continuous transformation. The linearity follows from the linearity of the integral

$$\mathcal{F}[c_1\varphi_1 + c_2\varphi_2] = c_1\mathcal{F}[\varphi_1] + c_2\mathcal{F}[\varphi_2]\,, \qquad \varphi_{1,2} \in \mathcal{S}\,, \quad c_{1,2} \in \mathbb{R}$$

Therefore it is sufficient to show that any null sequence in $\mathcal{S}$ is mapped to a null sequence by $\mathcal{F}$.

Let $\varphi_n \to 0$ in $\mathcal{S}$. Using (**33**.**1**) and (**33**.**2**) one infers that

$$\begin{aligned}|k^\alpha D^\beta \mathcal{F}[\varphi_n](k)| &= |\mathcal{F}[(iD)^\alpha (ix)^\beta \varphi_n](k)| \\ &\le \int |D^\alpha(x^\beta \varphi_n(x))|\, d^N x \\ &= \int |D^\alpha(x^\beta \varphi_n(x))|\, \frac{1+|x|^p}{1+|x|^p}\, d^N x \\ &\le M_p \sup\Big\{|D^\alpha(x^\beta\varphi_n(x))|(1+|x|^p)\Big\} \\ M_p &= \int \frac{d^N x}{1+|x|^p} < \infty\,, \quad p > N\,.\end{aligned}$$

Since $x^p D^q \varphi_n$ converges uniformly to zero for any non-negative p and q, it is concluded that

$$\lim_{n\to\infty} \sup |k^\alpha D^\beta \mathcal{F}[\varphi_n](k)| = 0$$

for any non-negative α and β. So, $\mathcal{F}$ is continuous.

33.2.1. The inverse Fourier transform on $\mathcal{S}$. Let us show that the inverse Fourier transform exists and is defined by the rule

$$\begin{aligned}\mathcal{F}^{-1}[\varphi(k)](x) &= \frac{1}{(2\pi)^N}\int e^{-i(k,x)}\varphi(k)\, d^N k = \frac{1}{(2\pi)^N}\int e^{i(k,x)}\varphi(-k)\, d^N k \\ &= (2\pi)^{-N}\mathcal{F}[\varphi(-k)](x)\end{aligned}$$

One has to show that

$$\mathcal{F}^{-1}[\mathcal{F}[\varphi]] = \varphi$$

for any test function $\varphi \in \mathcal{S}$. Consider the one-dimensional case first, $N = 1$. One has

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \frac{1}{2\pi}\int e^{-ikx}\int e^{iky}\varphi(y)\, dy\, dk \\ &= \frac{1}{2\pi}\lim_{n\to\infty}\int_{-n}^{n}\int e^{ik(y-x)}\varphi(y)\, dy\, dk \\ &= \frac{1}{2\pi}\lim_{n\to\infty}\int \varphi(y)\int_{-n}^{n} e^{ik(y-x)}\varphi(y)\, dk\, dy \\ &= \frac{1}{\pi}\lim_{n\to\infty}\int \varphi(y)\frac{\sin(n(y-x))}{y-x}\, dy \\ &= \frac{1}{\pi}\lim_{n\to\infty}\Big(\frac{\sin(ny)}{y}, \varphi(x+y)\Big) = \varphi(x)\end{aligned}$$

where the order of integration has been changed by Fubini's theorem because

$$|e^{ik(y-x)}\varphi(y)| = |\varphi(y)| \in \mathcal{L}\Big((-n,n)\times\mathbb{R}\Big)$$

and the last equality follows from that the sequence $\sin(ny)/y$ converges to $\pi\delta(y)$ in $\mathcal{S}'$ (see Sec.**32.8.1**). A generalization to the N dimensional case is simple. The above line of arguments is applied to integrations over each variable, $dydk = dy_j dk_j$, $j = 1, 2, ..., N$.

33.3. Fourier transform of temperate distributions. The adjoint T^* of the Fourier transform $T = \mathcal{F}$ of $\mathcal{S}$ onto $\mathcal{S}$ is a linear continuous transformation of the space temperate distributions $\mathcal{S}'$ onto itself. The Fourier transform of a temperate distribution $f \in \mathcal{S}'$ is defined as the adjoint transformation:

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi])\,, \quad \varphi \in \mathcal{S}\,.$$

By linearity and continuity of $T = \mathcal{F}$, $\mathcal{F}[f]$ is a linear continuous functional on $\mathcal{S}$ for any temperate distribution f. Furthermore, if $f_n \to f$ in $\mathcal{S}'$, then $\mathcal{F}[f_n] \to \mathcal{F}[f]$ in $\mathcal{S}'$ by continuity of the adjoint transformation:

$$\lim_{n\to\infty}(\mathcal{F}[f_n], \varphi) = \lim_{n\to\infty}(f_n, \mathcal{F}[\varphi]) = (f, \mathcal{F}[\varphi)] = (\mathcal{F}[f], \varphi)\,.$$

33.3.1. The inverse Fourier transform of a temperate distribution. The inverse Fourier transform $T = \mathcal{F}^{-1}$ is also a linear and continuous transformation of $\mathcal{S}$ onto itself. Therefore its adjoint is a linear continuous transformation of $\mathcal{S}'$ onto itself:

$$\mathcal{F}^{-1}[f(k)] = (2\pi)^{-N}\mathcal{F}[f(-k)]\,, \quad f \in \mathcal{S}'\,.$$

It follows from this definition that

$$\mathcal{F}^{-1}[\mathcal{F}[f]] = \mathcal{F}[\mathcal{F}^{-1}[f]] = f$$

for any temperate distribution f. Indeed,

$$\begin{aligned}(\mathcal{F}^{-1}[\mathcal{F}[f](k)], \varphi) &= (2\pi)^{-N}(\mathcal{F}[\mathcal{F}[f](-k)], \varphi)\\ &= (2\pi)^{-N}(\mathcal{F}[f](-k), \mathcal{F}[\varphi](k))\\ &= (2\pi)^{-N}(\mathcal{F}[f](k), \mathcal{F}[\varphi](-k)) = (\mathcal{F}[f], \mathcal{F}^{-1}[\varphi])\\ &= (f, \mathcal{F}[\mathcal{F}^{-1}[\varphi]]) = (f, \varphi)\,.\end{aligned}$$

The other relation is proved by repeating the above equalities backward after using $\mathcal{F}[\mathcal{F}^{-1}[\varphi]] = \mathcal{F}^{-1}[\mathcal{F}[\varphi]]$.

The inverse Fourier transform is linear and continuous on $\mathcal{S}'$:

$$f_n \to f \quad \text{in } \mathcal{S}' \quad \Rightarrow \quad \mathcal{F}^{-1}[f_n] \to \mathcal{F}^{-1}[f] \quad \text{in } \mathcal{S}'\,.$$

33.4. Examples of distributional Fourier transforms. Here a few examples of calculating the Fourier transforms of basic distributions are considered.

33.4.1. The shifted delta function. Let us show that

$$\mathcal{F}[\delta(x-x_0)](k) = e^{i(k,x_0)}\,,$$
$$\mathcal{F}[e^{-i(k,x_0)}](x) = (2\pi)^N\delta(x-x_0)\,.$$

The first equality follows from

$$\Big(\mathcal{F}[\delta(x-x_0)],\varphi\Big) = \Big(\delta(x-x_0),\mathcal{F}[\varphi](x)\Big) = \mathcal{F}[\varphi](x_0) = \int e^{i(k,x_0)}\varphi(k)\,d^Nk = \Big(e^{i(k,x_0)},\varphi(k)\Big)$$

The second is proved by applying the inverse Fourier transform to the first equation:

$$\delta(x-x_0) = \mathcal{F}^{-1}[e^{i(k,x_0)}](x) = (2\pi)^{-N}\mathcal{F}[e^{-i(k,x_0)}](x)$$

In particular, setting $x_0 = 0$, one infers that

$$\mathcal{F}[1](x) = (2\pi)^N\delta(x)$$

33.4.2. Poisson summation formula. The series

$$\sum_n \delta(x-2\pi n) \quad \text{and} \quad \sum_n e^{inx}$$

converge in $\mathcal{S}'$. Indeed, Terms of the sequences of partial sums for these series are are temperate distributions. So, for any test function $\varphi\in\mathcal{S}$,

$$\Big(\sum_{|n|<m}\delta(x-2\pi n),\varphi(x)\Big) = \sum_{|n|<m}\varphi(2\pi n) \to \sum_n \varphi(2\pi n)$$

because a test function is decreasing to zero faster than any reciprocal power function with increasing the argument and, in particular,

$$|\varphi(2\pi n)| \le \frac{M}{1+n^2}\,, \qquad \sum_n \frac{M}{1+n^2} < \infty$$

For the other series, one has

$$\Big(\sum_{|n|<m} e^{inx},\varphi(x)\Big) = \sum_{|n|<m}\int e^{inx}\varphi(x)\,dx = \sum_{|n|<m}\mathcal{F}[\varphi](n) \to \sum_n \mathcal{F}[\varphi](n)$$

where the series converges because $\mathcal{F}[\varphi]$ is a test function. The convergence in $\mathcal{S}'$ implies the convergence in $\mathcal{D}'$. Thus, the Poisson summation formula holds in $\mathcal{S}'$:

$$\sum_n \delta(x - 2\pi n) = \frac{1}{2\pi}\sum_n e^{inx} = \frac{1}{2\pi}\sum_n \mathcal{F}[\delta(k-n)](x)$$

so that for any test function φ,

$$2\pi \sum_n \varphi(2\pi n) = \sum_n \mathcal{F}[\varphi](n)$$

This relation is known as the classical Poisson summation formula.

33.4.3. Gaussian distributions. Let $a > 0$. Then $f(x) = e^{-ax^2}$ is a regular temperate distribution. Its Fourier transform is obtained from the Gaussian integrals discussed earlier

$$\mathcal{F}[e^{-ax^2}](k) = \int e^{-ax^2+ikx}\,dx = \sqrt{\frac{\pi}{a}}\exp\Big(-\frac{k^2}{4a}\Big)$$

Let A be an $N \times N$ symmetric positive matrix. Then $e^{-(x,Ax)}$ is a temperate distribution of N real variables, and its Fourier transform is given by the Gaussian integral:

$$\mathcal{F}[e^{-(x,Ax)}](k) = \int e^{-(x,Ax)+i(k,x)}\,d^N x = \frac{\pi^{N/2}}{\sqrt{\det A}}\exp\Big(-\frac{1}{4}(k, A^{-1}k)\Big).$$

33.4.4. Complex Gaussian distributions. The function $f(x) = e^{ix^2}$ is bounded and, hence, defines a regular temperate distribution. Consider a family of regular temperate distribution $f_a(x) = e^{ix^2-ax^2}$ where $a > 0$. Let us show that $f_a \to f$ in $\mathcal{S}'$ as $a \to 0^+$. For any test function $\varphi \in \mathcal{S}$, one has by the Lebesgue dominated convergence theorem

$$\begin{aligned}\lim_{a\to 0^+}(f_a, \varphi) &= \lim_{a\to 0^+}\int e^{ix^2-ax^2}\varphi(x)\,dx = \int \lim_{a\to 0^+} e^{ix^2-ax^2}\varphi(x)\,dx \\ &= \int e^{ix^2}\varphi(x)\,dx = (f, \varphi)\end{aligned}$$

because the integrand has an integrable bound independent of parameter a:

$$|e^{ix^2-ax^2}\varphi(x)| \le |\varphi(x)| \in \mathcal{L}$$

By continuity of the Fourier transform on $\mathcal{S}'$

$$\begin{aligned}\mathcal{F}[e^{ix^2}](k) &= \lim_{a\to 0^+} \mathcal{F}[e^{ix^2-ax^2}](k) = \int e^{-(a-i)x^2+ikx}\,dx \\ &= \lim_{a\to 0^+}\Big(\frac{\pi}{a-i}\Big)^{1/2}\exp\Big(-\frac{k^2}{4(a-i)}\Big) \\ &= \sqrt{i\pi}\exp\Big(-\frac{ik^2}{4}\Big)\end{aligned}$$

where the Gaussian integral was used. Similarly, for any positive symmetric matrix A

$$\mathcal{F}[e^{i(x,Ax)}](k) = \lim_{a\to 0^+}\mathcal{F}[e^{i(x,Ax)-a(x,x)}] = \frac{(\pi i)^{N/2}}{\sqrt{\det A}}\exp\Big(-\frac{i}{4}(k,A^{-1}k)\Big)\,.$$

33.4.5. Step function. Consider a family of regular temperate distribution $f_a(x) = \theta(x)e^{-ax}$ where $a > 0$. Then $f_a \to \theta$ in $\mathcal{S}'$ as $a \to 0^+$ because

$$\lim_{a\to 0^+}(f_a,\varphi) = \lim_{a\to 0^+}\int_0^\infty e^{-ax}\varphi(x)\,dx = \int_0^\infty \varphi(x)\,dx = (\theta,\varphi)$$

where the Lebesgue dominated convergence theorem was used:

$$|e^{-ax}\varphi(x)| \le |\varphi(x)| \ \in \mathcal{L}(0,\infty)$$

By continuity of the Fourier transform in $\mathcal{S}'$

$$\begin{aligned}\mathcal{F}[\theta](k) &= \lim_{a\to 0^+}\mathcal{F}[f_a](k) = \lim_{a\to 0^+}\int_0^\infty e^{-ax+ikx}dx \\ &= \lim_{a\to 0^+}\frac{i}{k+ia} = \frac{i}{k+i0^+} \\ &= \pi\delta(k) + i\mathcal{P}\frac{1}{k}\,.\end{aligned}$$

So, the Fourier transform of the step function is proportional to the Sokhotsky's distribution. It follows also from the above relation that

$$\mathcal{F}[\theta(-x)](k) = \pi\delta(k) - i\mathcal{P}\frac{1}{k}\,.$$

33.4.6. The principal value distribution. The sign function $\varepsilon(x) = \theta(x) - \theta(-x)$ is a regular temperate distribution. Then using the Fourier transform of the step function one infers that

$$\mathcal{F}[\varepsilon(x)](k) = 2i\mathcal{P}\frac{1}{k}$$

Applying the inverse Fourier transform to this relation

$$\varepsilon(x) = 2i\mathcal{F}^{-1}\Big[\mathcal{P}\frac{1}{k}\Big](x) = \frac{i}{\pi}\mathcal{F}\Big[\mathcal{P}\frac{1}{(-k)}\Big](x) = -\frac{i}{\pi}\mathcal{F}\Big[\mathcal{P}\frac{1}{k}\Big](x)$$

Therefore

$$\mathcal{F}\Big[\mathcal{P}\frac{1}{x}\Big](k) = i\pi\varepsilon(k)$$

33.4.7. Spherical delta function. Let us find the Fourier transform of the spherical delta function δ_{S_a} in $\mathbb{R}^3$. For any test function, one has

$$\begin{aligned}
(\mathcal{F}[\delta_{S_a}], \varphi) &= (\delta_{S_a}, \mathcal{F}[\varphi]) = \oint_{|x|=a} \mathcal{F}[\varphi](x)\, dS \\
&= \oint_{|x|=a} \int e^{i(k,x)}\varphi(k)\, d^3k\, dS_x \\
&= \int \varphi(k) \oint_{|x|=a} e^{i(k,x)}\, dS_x\, d^3k \\
&= \int \varphi(k) \int_0^{2\pi} \int_0^{\pi} e^{i|k|a\cos(\phi)} a^2 \sin(\phi) d\phi\, d\theta\, d^3k \\
&= 4\pi a \int \varphi(k) \frac{\sin(a|k|)}{|k|}\, d^3k
\end{aligned}$$

The order of integration can be changed by Fubini's theorem because

$$\int \oint_{|x|=a} \Big| e^{i(k,x)}\varphi(k)\Big| d^3k\, dS_x \leq 4\pi a^2 \int |\varphi(k)|\, d^3k < \infty\,.$$

This shows that

$$\mathcal{F}[\delta_{S_a}](k) = 4\pi a\, \frac{\sin(a|k|)}{|k|}$$

Note that the Fourier transform is a smooth function. It will be shown below that this is true for the Fourier transform of any distribution with compact support.

33.4.8. Retarded propagator. Let us find the Fourier transform of the fundamental solution to the 2D wave operator (see Sec.**??**):

$$G_R(x_0, x) = \frac{1}{2}\theta(x_0)\theta(x_0^2 - x^2)\,, \quad x_0 \in \mathbb{R}\,, \quad x \in \mathbb{R}\,.$$

Up to the factor of $\frac{1}{2}$, this regular distribution coincides with the charactericic function of the positive cone $\Gamma^+ :\ x_0 > |x| > 0$. To do so, let us show first that the distribution $f_a(x_0, x) = e^{-x_0 a} G_R(x_0, x)$ converges to G_R in $\mathcal{S}'$ as $a \to 0^+$. For any test function $\varphi \in \mathcal{S}$

$$\begin{aligned}\lim_{a\to 0^+}(f_a, \varphi) &= \frac{1}{2}\lim_{a\to 0^+}\iint_{\Gamma^+} e^{-ax_0}\varphi(x_0, x)\,dx\,dx_0 \\ &= \frac{1}{2}\iint_{\Gamma^+}\varphi(x_0, x)\,dx\,dx_0 = (G_R, \varphi)\,.\end{aligned}$$

The order of taking the limit and the integral can be interchanged by the Lebesgue dominated convergence theorem because the integrand has an integrable bound independent of the parameter $a > 0$, $|e^{-ax_0}\varphi(x_0, x)| \leq |\varphi(x_0, x)| \in \mathcal{L}(\Gamma^+)$.

The continuity of the Fourier transform on $\mathcal{S}'$ is used to find $\mathcal{F}[G_R]$:

$$\mathcal{F}[f_a] \to \mathcal{F}[G_R] \quad \text{in } \mathcal{S}'$$

as $a \to 0^+$. Since $f_a \in \mathcal{L}(\mathbb{R}^2)$, its Fourier transform is given by

$$\begin{aligned}\mathcal{F}[f_a](k_0, k) &= \iint f_a(x_0, x)e^{ik_0x_0+ikx}\,dx\,dx_0 \\ &= \frac{1}{2}\iint_{\Gamma^+} e^{-ax_0}e^{ik_0x_0+ikx}dx\,dx_0 \\ &= \frac{1}{2}\int_0^\infty e^{-ax_0+ik_0x_0}\int_{-x_0}^{x_0} e^{ikx}dx\,dx_0 \\ &= \frac{1}{2ik}\Big(\frac{1}{i(k_0+ia+k)} - \frac{1}{i(k_0+ia-k)}\Big) \\ &= \frac{1}{k^2-(k_0+ia)^2}\,.\end{aligned}$$

In the limit $a \to 0^+$, this distribution becomes a distributional regularization of the singular function $(k_0^2 - k^2)^{-1}$ that is obtained by shifting the poles $k_0 = \pm|k|$ in the complex k_0 plane down into the half-plane $\operatorname{Im} k_0 < 0$ similarly to Sokhotsky distributions. For brevity,

$$\mathcal{F}[G_R](k_0, k) = \frac{1}{k^2-(k_0+i0)^2}\,.$$

The Green function G_R of the wave operator is called a *retarded propagator*. As shown earlier, it describes a causal wave propagation. If the poles are shifted up, $k_0^2 \to (k_0 - i0)^2$, then the corresponding Green function is called an *advanced propagator*. It is proportional to the characteristic function of the cone $\Gamma^- : -k_0 > |k|$ where $k_0 < 0$. These and other Green functions of wave operators will be discussed in detail in the next chapter devoted to applications.

33.5. Properties of the Fourier transform of distributions. Let f be a temperate distribution. Then

$$D^\alpha \mathcal{F}[f] = \mathcal{F}[(ix)^\alpha f]\,, \tag{33.3}$$

$$\mathcal{F}[D^\alpha f] = (-ik)^\alpha \mathcal{F}[f] \tag{33.4}$$

These properties follow from the corresponding properties of the Fourier transform of test functions (**33.1**) and (**33.2**). For example,

$$\begin{aligned}(D^\alpha \mathcal{F}[f], \varphi) &= (-1)^\alpha (\mathcal{F}[f], D^\alpha \varphi) = (-1)^\alpha \Big(f(x), \mathcal{F}[D^\alpha \varphi](x)\Big) \\ &= (-1)^\alpha \Big(f(x), (-ix)^\alpha \mathcal{F}[\varphi](x)\Big) = \Big((ix)^\alpha f(x), \mathcal{F}[\varphi](x)\Big) \\ &= \Big(\mathcal{F}[(ix)^\alpha f], \varphi\Big)\,.\end{aligned}$$

33.5.1. Linear transformations of the argument. The Fourier transform of a distribution with a shifted argument is given by

$$\mathcal{F}[f(x-x_0)](k) = e^{i(k,x_0)} \mathcal{F}[f](k)\,. \tag{33.5}$$

Indeed, for any test function φ,

$$\begin{aligned}\Big(\mathcal{F}[f(x-x_0)], \varphi\Big) &= \Big(f(x-x_0), \mathcal{F}[\varphi](x)\Big) = \Big(f(x), \mathcal{F}[\varphi](x+x_0)\Big) \\ &= \Big(f(x), \mathcal{F}[e^{i(k,x_0)}\varphi](x)\Big) = \Big(\mathcal{F}[f](k), e^{i(k,x_0)}\varphi\Big) \\ &= \Big(e^{i(k,x_0)}\mathcal{F}[f](k), \varphi\Big)\,.\end{aligned}$$

A shift of the argument of the Fourier transform is given by

$$\mathcal{F}[f](k+k_0) = \mathcal{F}[e^{i(k_0,x)} f(x)](k)\,. \tag{33.6}$$

which can be proved in a similar fashion.

There are generalizations of these relations to a general linear transformation of the argument, $x \to Ax - b$ where A is non-singular square matrix:

$$\mathcal{F}[f(Ax-b)](k) = \frac{e^{i(Ak,b)}}{|\det A|}\mathcal{F}[f(x)](A^{-1T}k) \tag{33.7}$$

where A^T denotes the transposed matrix A. Let φ be a test function. Then

$$\begin{aligned}\mathcal{F}[\varphi](Ax+b) &= \int e^{i(\xi,Ax)+i(\xi,b)}\varphi(\xi)\, d^N\xi \\ &= \frac{1}{|\det A|}\int e^{i(k,x)+i(k,A^{-1T}b)}\varphi(A^{-1T}k)\, d^N k \\ &= \frac{1}{|\det A|}\mathcal{F}\Big[e^{i(k,A^{-1T}b)}\varphi(A^{-1T}k)\Big](x)\end{aligned}$$

where $k = A^T\xi$. It follows from this relation that

$$\begin{aligned}(\mathcal{F}[f(Ax-b)],\varphi) &= \Big(f(Ax-b),\mathcal{F}[\varphi](x)\Big)\\ &= |\det A|^{-1}\Big(f(x),\mathcal{F}[\varphi(k)](A^{-1}(x+b))\Big)\\ &= \Big(f(x),\mathcal{F}[e^{i(k,A^Tb)}\varphi(A^Tk)](x)\Big)\\ &= \Big(\mathcal{F}[f](k),e^{i(k,A^Tb)}\varphi(A^Tk)\Big)\\ &= \frac{e^{i(Ak,b)}}{|\det A|}\Big(\mathcal{F}[f](A^{-1T}k),\varphi(k)\Big)\end{aligned}$$

In particular, for a scaling transformation of the argument, $x \to sx$ where $s \neq 0$

$$\mathcal{F}[f(sx)](k) = \frac{1}{|s|^N}\mathcal{F}[f(x)](\tfrac{k}{s})\,.$$

33.5.2. Distributions invariant under linear transformations. Suppose that a temperate distribution $f(x)$ is invariant under transformations $x \to Ax$, that is,

$$f(Ax) = f(x)\,.$$

Then it follows from (**33.7**) that

$$\mathcal{F}[f](A^{-1T}k) = |\det(A)|\mathcal{F}[f](k)$$

In particular, if A is an orthogonal matrix, $A^T = A^{-1}$ and $\det A = \pm 1$, then the Fourier transform is also invariant under this transformation. For example, the Fourier transform of any distribution invariant under rotations or Lorenz transformations is invariant under these transformations too.

33.6. The Fourier transform of distributions with compact support. *Let f be a distribution with compact support. Then its Fourier transform is from $\mathcal{O}_M$ and*

$$\mathcal{F}[f](k) = \Big(f(x),\eta_f(x)e^{i(k,x)}\Big)\,, \qquad (\mathbf{33.8})$$

where η is any test function from $\mathcal{D}$ such that $\eta(x) = 1$ in a neighborhood of the support of f.

Let us show first $\mathcal{F}[f]$ is a smooth function. For any test function φ, one has

$$\begin{aligned}(\mathcal{F}[f],\varphi) &\overset{(1)}{=} (f,\mathcal{F}[\varphi]) \overset{(2)}{=} (f,\eta_f\mathcal{F}[\varphi])\\ &\overset{(3)}{=} \Big(f(x),\int e^{i(k,x)}\eta_f(x)\varphi(k)\,d^Nk\Big)\\ &\overset{(4)}{=} \int \Big(f(x),e^{i(k,x)}\eta_f(x)\varphi(k)\Big)\,d^Nk\\ &\overset{(5)}{=} \int \Big(f(x),e^{i(k,x)}\eta_f(x)\Big)\varphi(k)\,d^Nk\,.\end{aligned}$$

Here (1) is by definition of the Fourier transform of a distribution, (2) by Proposition **32.2**, (3) by definition of the Fourier transform of a test function, (4) follows from (**32.3**) applied to the test function of two variables $\phi(x,k)=e^{i(k,x)}\eta_f(x)\varphi(k)\in\mathcal{S}(\mathbb{R}^{2N})$, (5) follows from linearity of f. This proves (**33.8**) and that $\mathcal{F}[f]$ is a smooth function because the integrand in the last integral is a test function. By property (**33.3**)

$$D^\alpha\mathcal{F}[f](k)=\Big((ix)^\alpha f(x),e^{i(k,x)}\eta_f(x)\Big)=\Big(f(x),e^{i(k,x)}(ix)^\alpha\eta_f(x)\Big)\,.$$

To show that $\mathcal{F}[f]$ is a smooth slowly increasing function, Theorem **32.2** is applied to the above relation:

$$\begin{aligned}|D^\alpha\mathcal{F}[f](k)| &\le C\sup_{\beta\le p,x}(1+|x|)^\beta|D_x^\beta(e^{i(k,x)}x^\alpha\eta_f(x))|\\ &\le C_\alpha(1+|k|)^p\end{aligned}$$

for some constants C_α because the derivatives D_x produce powers of k up order p and $\sup(1+|x|)^\beta|D^\gamma\psi|<\infty$ for any test function and, in particular, for $\psi=x^\alpha\eta_f$. Thus, the Fourier transform and all its derivatives cannot increase faster than a power function and, hence, $\mathcal{F}[f]\in\mathcal{O}_M$.

33.6.1. Fourier transforms of simple and double layer distributions. Let us use the above equation for the Fourier transform of a distribution with compact support to find integral representation for the Fourier transforms of the single and double layer distributions. Let S be a bounded smooth surface oriented by the unit vector $\hat{n}$, μ and ν be continuous functions on S, and η_S be a test function from $\mathcal{D}$ that takes unit value in a neighborhood of S. Then

$$\mathcal{F}[\mu\delta_S](k) = \Big(\mu\delta_S, \eta_S e^{i(k,x)}\Big) = \int_S \mu(x) e^{i(k,x)}\, dS_x\,,$$
$$\mathcal{F}\Big[-\frac{\partial}{\partial n}(\nu\delta_S)\Big](k) = \Big(-\frac{\partial}{\partial n}(\nu\delta_S), \eta_S e^{i(k,x)}\Big) = \int_S \nu(x)\frac{\partial}{\partial n}\, e^{i(k,x)}\, dS_x$$
$$= i\int_S \nu(x)\,(\hat{n},x)\, e^{i(k,x)}\, dS_x$$

33.7. Fourier transform of the convolution. Suppose that ϕ and φ are test functions. Then their convolution is also a test function. Let us find the Fourier transform of the convolution

$$\mathcal{F}[\varphi * \phi](k) = \int e^{i(k,x)} \int \varphi(y)\phi(x-y)\, d^N y d^N x$$
$$= \int e^{i(k,y)}\varphi(y) \int e^{i(k,x-y)}\phi(x-y)\, d^N x\, d^N y$$
$$= \mathcal{F}[\varphi](k)\mathcal{F}[\phi](k)$$

where the order of integration has been changed by Fubini's theorem (because $\varphi(y)\phi(x-y)$ is integrable on $\mathbb{R}^{2N}$). Can the relation

$$\mathcal{F}[f*g] = \mathcal{F}[f]\mathcal{F}[g]\,, \qquad f,g\in\mathcal{S}'\,, \tag{33.9}$$

be extended to distributions? First note that the right-hand side is not defined as a distribution, unless *the Fourier transform of one of the distributions is from $\mathcal{O}_M$ so that the product of Fourier transforms is well defined in $\mathcal{S}'$*. The latter is not generally true even if the convolution exists. Therefore, the relation (**33**.**9**) cannot be extended to any temperate distributions. However, if the convolution exists in $\mathcal{S}'$, its Fourier transform can be used to defined the product of distributions. Let us analyze the Fourier transform of the convolution for four cases in which the convolution exists.

PROPOSITION **33**.**2**. *Let $\omega\in\mathcal{S}$ and $f\in\mathcal{S}'$. Then the relation (**33**.**9**) holds.*

Note first that the convolution exists and is a function from class $\mathcal{O}_M$. So it is a regular temperate distribution, and, hence its Fourier transform is a temperate distribution. For any $\varphi \in \mathcal{S}$ one has

$$\begin{aligned}(\mathcal{F}[\omega * f], \varphi) &\stackrel{(1)}{=} (\omega * f, \mathcal{F}[\varphi]) \stackrel{(2)}{=} \Big(f(x), \int \omega(y) \int e^{i(k,x+y)} \varphi(k)\, d^N k\, d^N y\Big) \\ &\stackrel{(3)}{=} \Big(f(x), \int e^{i(k,x)} \varphi(k) \int e^{i(k,y)} \omega(y)\, d^N y\, d^N k\Big) \\ &\stackrel{(4)}{=} (f, \mathcal{F}[\varphi \mathcal{F}[\omega]]) \stackrel{(5)}{=} (\mathcal{F}[f], \mathcal{F}[\omega]\varphi) \\ &\stackrel{(6)}{=} (\mathcal{F}[\omega]\mathcal{F}[f], \varphi)\,,\end{aligned}$$

as required. Here (1) is by the definition of the Fourier transform on $\mathcal{S}'$, (2) follows from Proposition **32.5**, (3) holds by Fubini's theorem because $|\omega(y)\varphi(k)| \in \mathcal{L}(\mathbb{R}^{2N})$, (4) and (5) are by the definition of the Fourier transform on $\mathcal{S}$ and on $\mathcal{S}'$, respectively, and (6) follows from the rule of multiplication of a distribution by a smooth function $\mathcal{F}[\omega] \in \mathcal{S} \subset \mathcal{O}_M$.

PROPOSITION **33.3**. *Let $f \in \mathcal{S}'$ and g be a distribution with bounded support. Then the relation (**33.9**) holds.*

Note that $\mathcal{F}[g] \in \mathcal{O}_M$ by by Theorem **32.3**. Therefore the product of Fourier transforms in the right-hand side is a temperate distribution. For any test function φ, the following chain of equalities holds

$$\begin{aligned}(\mathcal{F}[f * g], \varphi) &\stackrel{(1)}{=} (f * g, \mathcal{F}[\varphi]) \\ &\stackrel{(2)}{=} \Big(f(x), \Big(g(y), \eta_g(y) \int e^{i(k,x+y)} \varphi(k)\, d^N k\Big)\Big) \\ &\stackrel{(3)}{=} \Big(f(x), \int \Big(g(y), \eta_g(y) e^{i(k,x+y)}\Big)\, \varphi(k)\, d^N k\Big) \\ &\stackrel{(4)}{=} \Big(f(x), \int \mathcal{F}[g](k)\, \varphi(k)\, e^{i(k,x)}\, d^N k\Big) \\ &\stackrel{(5)}{=} (f, \mathcal{F}[\mathcal{F}[g]\varphi]) = (\mathcal{F}[f], \mathcal{F}[g]\varphi) \\ &\stackrel{(6)}{=} (\mathcal{F}[g]\mathcal{F}[f], \varphi)\,,\end{aligned}$$

as required. Here (1) is by the definition of the Fourier transform on $\mathcal{S}'$, (2) is by Theorem **32.3**, (3) follows from (**32.3**) because $\eta_g(y)\varphi(k)e^{i(k,x+y)}$ is a test function of two variables for any x, (4) is by (**33.8**), and (5) and (6) follow from the definition of the Fourier transform and that $\mathcal{F}[g] \in \mathcal{O}_M$ and, hence, its product with any temperate distribution exists in $\mathcal{S}'$.

33.7.1. Example. Let us find the Fourier transform of the convolution $\theta(R - |x|)$ and $\mathcal{P}\frac{1}{x}$, where $x \in \mathbb{R}$. Since $\theta(R - |x|)$ has a bounded support

$$\mathcal{F}[\theta(R-|x|)](k) = \int_{-R}^{R} e^{ikx}dx = \frac{2\sin(kR)}{k}$$

Therefore by Proposition **33**.**3**

$$\mathcal{F}\Big[\theta(R-|x|) * \mathcal{P}\frac{1}{x}\Big](k) = \frac{2\sin(kR)}{k}\,\pi i\varepsilon(k) = 2\pi i\,\frac{\sin(kR)}{|k|}\,.$$

33.7.2. Fourier transform of the convolution in the algebra $\mathcal{S}'_+$. If $f \in \mathcal{S}_+$, then the relation (**33**.**9**) holds for any $g \in \mathcal{S}'_+$ by Proposition **33**.**2**. However, it can fails for convolutions of any two distributions from $\mathcal{S}'_+$.

Put

$$f(t) = \theta(t)\,, \quad g(t) = tf(t) = t\theta(t)\,.$$

They are regular distributions from $\mathcal{S}'_+$ and their convolution reads

$$(f*g)(t) = \int f(\tau)g(t-\tau)\,d\tau = \theta(t)\int_0^t (t-\tau)\,d\tau = \frac{t^2}{2}\,\theta(t)$$

Let find the Fourier transforms. Note that the Fourier transform maps $\mathcal{S}'_+$ to $\mathcal{S}'$. In other words, the Fourier transform of any distribution from $\mathcal{S}'_+$ does not generally belong to $\mathcal{S}'_+$. Using the Fourier transform of the step function (see Sec.**33.4.5**) and the properties of the Fourier transform, one infers that

$$\begin{aligned}
\mathcal{F}[f](k) &= \pi\delta(k) + i\mathcal{P}\frac{1}{k} = \frac{i}{k+i0}\,,\\
\mathcal{F}[g](k) &= -i\frac{d}{dk}\mathcal{F}[f](k) = -i\pi\delta'(k) - \mathcal{P}\frac{1}{k^2} = -\frac{1}{(k+i0)^2}\,,\\
\mathcal{F}[f*g](k) &= -\frac{1}{2}\frac{d^2}{dk^2}\mathcal{F}[f](k) = -\frac{\pi}{2}\delta''(k) + i\mathcal{P}\frac{1}{k^3} = -\frac{i}{(k+i0)^3}\,.
\end{aligned}$$

The singular supports of $\mathcal{F}[f]$ and $\mathcal{F}[g]$ consist of the single point $k=0$. Therefore their product cannot be defined in $\mathcal{S}'$ by the localization method. Thus, the relation (**33**.**9**) fails in this case.

33.7.3. Distributions supported in a light cone. A higher dimensional analog of the convolution in the algebra $\mathcal{S}'_+$ is the convolution of distributions supported in a light cone (see Sec.**32.10.3**). In this case, the relation (**33**.**9**) can fail too. As an example, consider the convolution $G_R * G_R$, where G_R is the retarded propagator for the 2D wave operator whose support lies in the future light cone Γ^+. Therefore the convolution exists in $\mathcal{S}'$ and so does its Fourier transform. However the Fourier transform $\mathcal{F}[G_R]$, found in Sec.**33.4.8**, is a singular distribution whose singular support is the double cone $k_0^2 = k^2$. Therefore the product $\mathcal{F}[G_R]\mathcal{F}[G_R]$ cannot be defined by the localization method and the relation (**33**.**9**) makes no sense.

33.8. Product of distributions via the Fourier transform. Let us try to define a product of distributions in $\mathcal{S}$ if their singular supports have common points. Clearly, the localization method would fail. However, the examples studied above when the relation (**33**.**9**) fails have one common feature. The convolution exists in $\mathcal{S}'$ and, hence, so does its Fourier transform, but the product of the Fourier transforms does not exist in $\mathcal{S}'$ only because the product of distributions with overlapping singular supports is not defined. This suggests that the relation (**33**.**9**) can be used as a definition of this product in this case.

Let f and g be tempered distributions. Suppose that the convolution $\mathcal{F}^{-1}[f] * \mathcal{F}^{-1}[g]$ exists in $\mathcal{S}'$. Then put

$$f(x)g(x) \stackrel{\text{def}}{=} \mathcal{F}\Big[\mathcal{F}^{-1}[f] * \mathcal{F}^{-1}[g]\Big](x)\,.$$

By construction, the product is a temperate distribution even if the singular supports of f and g have common points. This definition can also be written in terms of the Fourier transforms using the relation between $\mathcal{F}$ and $\mathcal{F}^{-1}$:

$$\mathcal{F}[fg](k) \stackrel{\text{def}}{=} (2\pi)^{-N}\Big(\mathcal{F}[f] * \mathcal{F}[g]\Big)(k)\,. \tag{33.10}$$

Here the product fg is defined via its Fourier transform. So, the product fg exists in $\mathcal{S}'$ whenever the convolution of the Fourier transforms of f and g exists in $\mathcal{S}'$.

33.8.1. Example. Let us try to find the product fg using (**33**.**10**) if

$$f(x) = \frac{1}{x+i0}\,, \quad g(x) = -f'(x) = \frac{1}{(x+i0)^2}\,.$$

The singular supports of these distributions coincide and contain the single point $x = 0$. One has

$$\begin{aligned}
\mathcal{F}[f](k) &= \mathcal{F}\Big[-i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big](k) = -2\pi i\theta(-k)\,,\\
\mathcal{F}[g](k) &= \mathcal{F}[-f'](k) = ik\mathcal{F}[f](k) = 2\pi k\theta(-k)\,,\\
(\mathcal{F}[f] * \mathcal{F}[g])(k) &= -4\pi^2 i \int \theta(-p)(k-p)\theta(p-k)\,dp\\
&= -4\pi^2 i\theta(-k)\int_k^0 (k-p)\,dp = 2\pi^2 ik^2\theta(-k)\,.
\end{aligned}$$

Since the convolution of the Fourier transforms exists in $\mathcal{S}'$, the product fg also exists in $\mathcal{S}'$ and is given by

$$\begin{aligned} f(x)g(x) &= \pi i\mathcal{F}^{-1}[k^2\theta(-k)](x) = \frac{i}{2}\mathcal{F}[k^2\theta(k)] \\ &= -\frac{i}{2}\frac{d^2}{dx^2}\mathcal{F}[\theta(k)](x) = \frac{1}{2}\frac{d^2}{dx^2}\frac{1}{x+i0} \\ &= \frac{1}{2}f''(x) \in \mathcal{S}' \end{aligned}$$

This example is to be compared with the discussion in Sec.**33.7.2**. The definition (**33**.**10**) has been motivated by the example in Sec.**33.7.2**.

33.8.2. Extension of the product to $\mathcal{D}'$. Not every distribution from $\mathcal{D}'$ has a Fourier transform. A direct extension of (**33**.**10**) to $\mathcal{D}'$ is not possible. However the Fourier transform of $\eta(x)f(x)$, $\eta \in \mathcal{D}$, exists because ηf has a bounded support. So, the product $f(x)g(x)$ can be defined in a neighborhood of a point x_0, provided the convolution of the Fourier transforms of ηf and ηg exists near x_0 for some η that is a bump function for a neighborhood $U(x_0)$. If this product can be defined for any $x_0 \in \Omega$, then by the localization theorem the product exists in $\mathcal{D}'(\Omega)$. The Fourier transform of ηf is a smooth function from $\mathcal{O}_M$ (see Sec.**33.6**). Its growth is bounded by a polynomial. Therefore $\mathcal{F}[\eta f] * \mathcal{F}[\eta g]$ is the classical convolution of functions from $\mathcal{O}_M$ which may or may not exist. By Fubini's theorem, it exists if the convolution integral converges absolutely (see Sec.**30.1**). If, in addition, the inverse Fourier transform of the convolution exists, then the relation (**33**.**10**) can be used to define the product near x_0.

Definition **33**.**1**. *A distribution $h \in \mathcal{D}'(\Omega)$ is the product of distributions f and g if for any point in Ω there exists a test function $\eta \in \mathcal{D}$ such that it is equal to 1 near the point and*

$$\begin{aligned} \mathcal{F}[\eta^2 h](k) &= (2\pi)^{-N}\Big(\mathcal{F}[\eta f] * \mathcal{F}[\eta g]\Big)(k) \\ &= (2\pi)^{-N}\int \mathcal{F}[\eta f](p)\mathcal{F}[\eta g](k-p)\,d^N p \end{aligned}$$

where the convolution integral converges absolutely for all $k \in \mathbb{R}^N$.

It should be noted that the product does not exist for any distributions. For example, if $f(x) = g(x) = \delta(x)$, then $\mathcal{F}[\delta](k) = 1$, but the convolution of two unit functions does not exist. So, $\delta(x)\delta(x)$ is not defined as a distribution.

A consistency of Definition **33**.**1** requires proving uniqueness of the product if it exists.

PROPOSITION **33**.4. *For any two distributions f and g, there exists at most one distribution h that satisfy Definition* **33**.1

By the localization theorem, it is sufficient to show the uniqueness in a neighborhood of any point. Let h_1 and h_2 be two distributions satisfying Definition **33.1** with some test functions η_1 and η_2, respectively, that have unit value near x_0. One has to show that $h_1 = h_2$ near x_0. Since x_0 is arbitrary, by the localization theorem $h_1 = h_2$.

Let ω be a test function. The assertion follows from the identity

$$(\mathbf{33.11}) \quad \mathcal{F}[\omega\eta^2 h] = (2\pi)^{-N}\mathcal{F}[\omega\eta f] * \mathcal{F}[\eta g] = (2\pi)^{-N}\mathcal{F}[\eta f] * \mathcal{F}[\omega\eta g]$$

Indeed, setting $\omega = \eta_2^2$ in $\mathcal{F}[\omega\eta_1^2 h_1]$ and $\omega = \eta_1^2$ in $\mathcal{F}[\omega\eta_2^2 h_2]$, it follows from (**33.11**) that $\mathcal{F}[\eta_1^2\eta_2^2 h_1] = \mathcal{F}[\eta_1^2\eta_2^2 h_2]$. Therefore $h_1 = h_2$ near x_0 as required.

Let us prove (**33.11**). Put $G = (2\pi)^N\mathcal{F}[\eta^2 h]$ for brevity. For any test function $\varphi \in \mathcal{S}$ the distribution in the left-hand side of (**33.11**) has the following value

$$\begin{aligned}\Big(\mathcal{F}\Big[\omega\mathcal{F}^{-1}[G]\Big], \varphi\Big) &= \Big(G, \mathcal{F}^{-1}\Big[\omega\mathcal{F}[\varphi]\Big]\Big) \overset{(1)}{=} \Big(G, \mathcal{F}[\omega^-] * \varphi\Big) \\ &\overset{(2)}{=} \Big(\mathcal{F}[\omega] * G, \varphi\Big)\end{aligned}$$

where (1) is obtained using the explicit form of the Fourier transform of a test function:

$$\begin{aligned}(2\pi)^N\mathcal{F}^{-1}\Big[\omega\mathcal{F}[\varphi]\Big](k) &= \int\int e^{-i(x,k-p)}\omega(x)\varphi(p)\, d^N p\, d^N x \\ &= \int \varphi(p)\mathcal{F}[\omega^-](k-p)\, d^N p = \Big(\mathcal{F}[\omega^-] * \varphi\Big)(k)\end{aligned}$$

where $\omega^-(x) = \omega(-x)$, and the order of integration has been changed by Fubini's theorem (because the integrand is an integrable function of two variables x and p for any k). The equality (2) follows from the definition of the convolution of a distribution with a test function (see Sec.**19.3**). Using the explicit form of G,

$$\begin{aligned}\Big(\mathcal{F}[\omega] * G\Big)(k) &= \int\mathcal{F}[\omega](q)\int\mathcal{F}[\eta f](p)\mathcal{F}[\eta g](k-q-p)\, d^N p\, d^N q \\ &= \int\mathcal{F}[\eta f](p)\int\mathcal{F}[\omega](q)\mathcal{F}[\eta g](k-q-p)\, d^N q\, d^N p \\ &\overset{(1)}{=} \Big(\mathcal{F}[\eta f] * (\mathcal{F}[\omega] * \mathcal{F}[\eta g])\Big)(k) \\ &= \Big(\mathcal{F}[\eta f] * \mathcal{F}\Big[\mathcal{F}^{-1}\Big[\mathcal{F}[\omega] * \mathcal{F}[\eta g]\Big]\Big]\Big)(k) \\ &\overset{(2)}{=} \frac{1}{(2\pi)^N}\Big(\mathcal{F}[\eta f] * \mathcal{F}[\omega\eta g]\Big)(k)\end{aligned}$$

as required. Here the order of integration in (1) has been changed by Fubini's theorem because the integral for the convolution G converges absolutely by the hypothesis and, hence the integrand is an integrable function of two variables, p and q, for any k. The equality (2) follows from the definition of $\mathcal{F}^{-1}$ and the Fourier transform of the convolution of a test function with a distribution. Finally, if the roles of f and g are swapped in the above calculations, the second part of (**33**.**11**) is established.

33.8.3. Properties of the product. The product of distributions given in Definition **33**.**1** has the following properties

(i) The product is compatible with multiplication of a distribution by a smooth function. If $g = a \in C^\infty$ in Definition **33**.**1**, then the product af exists for any $f \in \mathcal{D}'$ and

$$(af, \varphi) = (f, a\varphi)\,, \quad \varphi \in \mathcal{D}\,.$$

(ii) The product is commutative and associative

$$fg = gf\,, \quad f(gh) = (fg)h$$

provided fg, gf, $f(gh)$, and $(fg)h$ exist.

(iii) If the singular supports of distributions f and g are disjoint, then fg exists and coincides with the product fg defined by the localization method.

33.8.4. Sufficient conditions for the existence of the product. Suppose that f and g in Definition **33**.**1** have bounded supports. Then their Fourier transforms are from class $\mathcal{O}_M$. In this case, Definition **33**.**1** is reduced to (**33**.**10**). *If the integral*

$$\Big(\mathcal{F}[f] * \mathcal{F}[g]\Big)(k) = \int \mathcal{F}[f](p)\mathcal{F}[g](k-p)\, d^N p$$

converges absolutely and it is a slowly increasing function

$$\Big|\Big(\mathcal{F}[f] * \mathcal{F}[g]\Big)(k)\Big| \le M(1+|k|)^p$$

for some constants $M > 0$ and $p \ge 0$, then the product fg exists in $\mathcal{D}$. By the absolute convergence of the integral $G = \mathcal{F}[f] * \mathcal{F}[g]$ is locally integrable and for any $\varphi \in \mathcal{S}$

$$|(G, \varphi)| = \int |G(k)||\varphi(k)|\, d^N k \le M \int (1+|x|)^p |\varphi(k)|\, d^N k$$

$$\le M \sup \Big|(1+|k|)^{p+N+1}\varphi(k)\Big| \int \frac{d^N k}{(1+|k|)^{N+1}} < \infty$$

By the Schwartz theorem **32.2**, $G \in \mathcal{S}'$ and, hence, its Fourier transform is also a temperate distribution, that is, $fg = (2\pi)^{-N}\mathcal{F}^{-1}[G]$ exists in $\mathcal{S}'$.

The same reasonings in combination with the localization theorem lead to the following assertion. *If for every point $x_0 \in \Omega$ there exists a test function η such that $\eta(x_0) \neq 0$ and the convolution integral in Definition* **33.1** *converges absolutely and defines a slowly increasing function of k, then the product fg exists in $\mathcal{D}'(\Omega)$.*

33.9. Exercises.

1. Show that

$$\mathcal{F}\Big[\mathcal{P}\frac{1}{x^2}\Big](k) = -\pi|k|\,, \quad \mathcal{F}[|x|] = -2\mathcal{P}\frac{1}{k^2}$$

2. Find the Fourier transform of $f(x) = \theta(x)x^n$ where n is a non-negative integer.

3. Let $|a_n| \leq Mn^p$. Show that the series

$$\sum_{n=1}^{\infty} a_n \delta(x-n)$$

converges in $\mathcal{S}'$ and find its Fourier transform.

4. One can show that if the support of a distribution f is a point $x = 0$, then this distribution is uniquely represented by linear combination of the delta function and its derivatives, that is, there exist a unique collection of coefficients c_α such that

$$f(x) = \sum_{\alpha \leq p} c_\alpha D^\alpha \delta(x)$$

Use this fact, to prove that any distribution f with support $\{x = 0\}$ that is invariant under orthogonal transformations is uniquely represented by

$$f(x) = L(\Delta)\delta(x)$$

where Δ is the Laplace operator, and L is a polynomial.

5. In the fundamental solution $\mathcal{E}(x,t) = \frac{1}{2c}\theta(ct - |x|)$ for the 2D wave operator, put $x_0 = ct$. Find the Fourier transform $\mathcal{F}[\mathcal{E}](k, k_0)$ where k_0 is the Fourier variable for x_0.

6. Find the Fourier transform of $\delta_{S_a}(x)$ if $x \in \mathbb{R}^2$.

7. Find the Fourier transform of the double layer distribution if $\nu(x) = (b, x)^2$ and S is a sphere $|x| = R$ in $\mathbb{R}^3$.

8. Let G_R be the retarded propagator for the 2D wave operator (see Sec.**33.4.8**).
(i) Show that

$$(G_R * G_R)(x_0, x) = \frac{1}{8}\theta(x_0)\theta(2x_0 - |x|)g(x_0, x)\,,$$

$$g(x_0, x) = \begin{cases} (2x_0 - x)^2 & , \quad x_0 < x < 2x_0\,, \\ 2x_0(x_0 + x) - x^2 & , \quad 0 < x < x_0\,, \\ 2x_0(x_0 + x) + x^2 & , \quad -x_0 < x < 0\,, \\ -x(2x_0 + x) & , \quad -2x_0 < x < -x_0 \end{cases}$$

(ii) Find the Fourier transform $\mathcal{F}[G_R * G_R]$. Hint: Use the continuity of the Fourier transform for the distribution $e^{-ax_0}G_R * G_R \to G_R * G_R$ in $\mathcal{S}'$ as $a \to 0^+$, and the properties of derivatives of the Fourier transform.
(iii) Use the product of distribution defined via the Fourier transform to find the product in $\mathcal{S}'$

$$\frac{1}{(x_0 + i0)^2 - x^2} \cdot \frac{1}{(x_0 + i0)^2 - x^2}\,.$$

9. Does the Leibniz rule hold for differentiation of the product of distributions defined by (**33.10**)?

34. Laplace transform of distributions