CHAPTER 5

Applications to PDEs

35. Fundamental solutions

35.1. Linear problem for a differential operator. Recall a basic problem in a linear algebra. If A is a matrix that defines a linear transformation of $\mathbb{R}^N$ to $\mathbb{R}^M$, find a solution to the linear system

$$Ax = b$$

where b is a given vector. If A is invertible, then the solution is unique, $x = A^{-1}b$. If A is not invertible, then by the Fredholm alternative a solution exists only if b is orthogonal to any vector annihilated by the adjoint matrix

$$(y, b) = 0\,, \quad A^*y = 0$$

and, in this case, any solution can be written in the form

$$x = Gb + x_h\,,$$

where x_h is a general solution to the associated homogeneous equation $Ax_h = 0$, and a matrix G defines a particular solution:

$$AGb = b\,.$$

The matrix G is generally not unique because its action on b can always be amended by adding a solution to the associated homogeneous equation, but it acts like the inverse of A on the subspace of vectors orthogonal to the null set of the adjoint matrix A^*. In particular, if $Ax_h = 0$ has only a trivial solution, then A is invertible and, in this case, G is unique and $G = A^{-1}$ so that $AG = I$ where I is the identity matrix ($Ix = x$ for any vector x). So, the problem of solving a linear problem is equivalent to finding a matrix G for a given matrix A.

Let $L(D)$ be a linear differential operator. Then a linear problem for the operator $L(D)$ is to find a function $u(x)$ such that

$$L(D)u(x) = f(x)$$

for a given function f. Suppose that a solution exists. Then it has the form

$$u(x) = (Gf)(x) + h(x)$$

where $h(x)$ is a general solution to the associate homogeneous equation

$$L(D)h(x) = 0$$

and the action of the operator G on f produces a particular solution

$$L(D)(Gf)(x) = f(x)\,.$$

So, G resembles the inverse of the differential operator $L(D)$. Just like in the case of linear algebra, solving a linear problem for a differential operator (or solving a non-homogeneous linear differential equation in partial derivatives) is equivalent to finding the operator G for a given differential operator. It turns out that this operator can be constructed out of the so called *fundamental solution* for a differential operator and the distributional convolution.

35.2. Fundamental solution for a differential operator. *A distributional solution $\mathcal{E}$ to a linear differential equation*

$$L(D)\mathcal{E}(x) = \delta(x)\,, \quad x \in \mathbb{R}^N$$
$$L(D) = a_\beta(x)D^\beta + a_{\beta-1}(x)D^{\beta-1} + \cdots + a_(x)D + a_0(x)$$

is called a fundamental solution *for the differential operator L, where the coefficients are smooth functions (from C^∞).* For example, the regular distribution

$$\mathcal{E}(x) = -\frac{1}{4\pi}\frac{1}{|x|}\,, \qquad x \in \mathbb{R}^3$$

is a fundamental solution for the Laplace operator $L = \Delta$ in $\mathbb{R}^3$ (see (**21.15**)).

A fundamental solution is not unique if it exists. If $\mathcal{E}$ is a fundamental solution for an operator L, then $\mathcal{E} + \mathcal{E}_0$ is also a fundamental solution where $\mathcal{E}_0$ is a solution to the associated homogeneous equation:

$$\begin{aligned} L\mathcal{E} &= \delta \\ L\mathcal{E}_0 &= 0 \end{aligned} \qquad \Rightarrow \qquad L(\mathcal{E} + \mathcal{E}_0) = \delta$$

by linearity of L. In particular, a regular distribution

$$\mathcal{E}(x) = -\frac{1}{4\pi}\frac{1}{|x|} + H(x)\,, \qquad \Delta H(x) = 0$$

is also a fundamental solution for the Laplace operator for any harmonic function $H(x)$ in $\mathbb{R}^3$.

The operator L^*

$$L^*\varphi = \sum_{\alpha=0}^{\beta}(-1)^\alpha D^\alpha(a_\alpha\varphi)$$

is called the *adjoint* of L. In particular, a test function is a regular distribution, so that for any two test functions

$$(L\varphi, \psi) = (\varphi, L^*\psi)\,, \qquad \varphi,\ \psi \in \mathcal{D}\,,$$

by the definition of multiplication of a distribution by a smooth function and by the definition of distributional derivatives (which are equal to the classical one in this case). Note that $D^* = -D$. If the coefficients in L are constant, $a_\alpha(x) = \text{const}$, then

$$L^*(D) = L(-D)\,.$$

If

$$(L\varphi, \psi) = (\varphi, L\psi)$$

then the operator L is called *Hermitian* or *symmetric* on the space of test functions. For example, the Laplace operator is symmetric on $\mathcal{D}$.

Let $\mathcal{E}$ be a fundamental solution for L, then for any test function

$$\varphi(0) = (\delta, \varphi) = (L\mathcal{E}, \varphi) = (\mathcal{E}, L^*\varphi)\,.$$

35.3. Linear differential equations with constant coefficients. Let $L(D)$ be a linear differential operator with constant coefficients, $a_\beta(x) = a_\beta = \text{const}$. Consider a linear problem in the space of distributions

$$L(D)u(x) = f(x)$$

where $f \in \mathcal{D}'(\mathbb{R}^N)$ is a given distribution. Then the distribution

$$u(x) = (G\rho)(x) = (\mathcal{E} * \rho)(x)$$

is a solution to this problem, where $\mathcal{E}(x)$ is a fundamental solution for the operator L, *provided the convolution* $\mathcal{E} * f$ *exists in* $\mathcal{D}'$. Indeed, if the convolution $\mathcal{E} * f$ exists, then using the rule for differentiation of the convolution of two distributions

$$\begin{aligned} L(D)(\mathcal{E} * f) &= \sum_\beta a_\beta D^\beta(\mathcal{E} * f) = \sum_\beta a_\beta (D^\beta \mathcal{E} * f) \\ &= \Big(\sum_\beta a_\beta D^\beta \mathcal{E}\Big) * f = \delta * f = f\,. \end{aligned}$$

Thus, *a general solution to a linear differential equation with constant coefficients and a distributional inhomogeneity is given by*

$$u(x) = (\mathcal{E} * \rho)(x) + h(x)\,, \quad L(D)h(x) = 0\,,$$

provided the convolution of a fundamental solution $\mathcal{E}$ and the inhomogeneity distribution f exists in $\mathcal{D}'$. In particular, if f has a bounded support, then the convolution always exists for any choice of a fundamental solution.

For example, for any compactly supported distribution $f(x)$, $x \in \mathbb{R}^3$, the Poisson equation is solved by the distribution

$$\Delta u(x) = f(x) \quad \Rightarrow \quad u(x) = -\frac{1}{4\pi}\frac{1}{|x|} * \rho(x) + H(x)$$

where $H(x)$ is any harmonic function in $\mathbb{R}^3$. If, in addition, $f(x)$ is a regular distribution, then the convolution is given by a potential-like integral

$$u(x) = -\frac{1}{4\pi}\frac{1}{|x|} * f(x) = -\frac{1}{4\pi}\int \frac{f(y)}{|x-y|}\, d^3y\,.$$

As shown earlier, if f is smooth enough, then $u(x)$ is a classical solution from class C^2.

35.4. Green's functions of differential operators. A fundamental solution for a differential operator is not unique. However, it is possible to impose additional conditions on a fundamental solution to make it unique. Fundamental solutions subject to additional conditions are often called *Green's functions* for a differential operator. First, note that if Ω is any open set that does not contain $x = 0$ (the support of the delta-function), then in the distributional sense

$$L(D)\mathcal{E}(x) = 0\,, \quad x \in \Omega$$

and, hence, $\mathcal{E}(x)$ is a smooth function in Ω. This implies that one can impose additional conditions on the solution in the complement of any neighborhood of $x = 0$.

For example, one can demand that a fundamental solution vanishes at spatial infinity,

$$\mathcal{E}(x) \to 0 \quad \text{as } |x| \to \infty\,.$$

This condition leads to a unique fundamental solution for the Laplace operator. The conclusion is based on the property of harmonic functions: *if $H(x)$ is a harmonic function in the whole $\mathbb{R}^3$ and $H(x)$ vanishes at infinity, then $H(x) = 0$.*

In physics, a fundamental solution for the Laplace operator is a Newton or Coulomb potential created by a point-like source (a point charge or mass). An observable quantity is the field that is the gradient of a potential. Since the field should vanish far away from the source, a fundamental solution is required to have the vanishing gradient in the asymptotic region:

$$|\nabla\mathcal{E}(x)| \to 0 \quad \text{as } |x| \to \infty\,.$$

In this case, the fundamental solution (or the corresponding Green's function) for the Laplace operator is unique up to an additive constant.

35.5. Uniqueness of distributional solutions to differential equations. Let $L(D)$ be a linear differential operator with constant coefficients and $\mathcal{E}$ be a fundamental solution for $L(D)$:

$$L(D)\mathcal{E}(x) = \delta(x)\,.$$

In the space of distributions, consider a subspace that consists of all distributions for which the convolution with $\mathcal{E}$ exists:

$$\mathcal{D}'_{\mathcal{E}} = \Big\{ u \in \mathcal{D}' \,\Big|\, \mathcal{E} * u \in \mathcal{D}' \Big\} \subset \mathcal{D}'\,.$$

For any $u \in \mathcal{D}'_{\mathcal{E}}$, the following identity holds

$$u = (L(D)u) * \mathcal{E}\,.$$

Indeed, let u be any such solution. By the hypothesis, the convolution $u*\mathcal{E}$ exists, and the assertion follows from the differentiation properties of the convolution:

$$(L(D)u) * \mathcal{E} = L(D)(u * \mathcal{E}) = u * (L(D)\mathcal{E}) = u * \delta = u\,.$$

Let $\mathcal{N}_L$ be the null space of the operator $L(D)$ which is a linear space of all solutions to the homogeneous equation:

$$\mathcal{N}_L = \{u_0 \,|\, L(D)u_0 = 0\}\,.$$

Then *the null space* $\mathcal{N}_L$ *and* $\mathcal{D}'_{\mathcal{E}}$ *have no common elements but the zero distribution, or, in other words, the homogeneous equation* $L(D)u = 0$ *has only the trivial solution in the subspace* $\mathcal{D}'_{\mathcal{E}}$:

$$\left.\begin{array}{r} L(D)u = 0 \\ u \in \mathcal{D}'_{\mathcal{E}} \end{array}\right\} \quad \Leftrightarrow \quad u = 0\,.$$

Indeed, by the hypothesis $u * \mathcal{E}$ exists and $L(D)u = 0$. Therefore

$$u = u * \delta = u * (L(D)\mathcal{E}) = L(D)(u * \mathcal{E}) = L(D)u * \mathcal{E} = 0 * \mathcal{E} = 0\,.$$

Now consider a non-homogeneous equation

$$L(D)u = f\,,$$

where $f \in \mathcal{D}'$. Let $\mathcal{E}$ be a fundamental solution for $L(D)$. Suppose that $f \in \mathcal{D}'_{\mathcal{E}}$, that is, the convolution $\mathcal{E} * f$ exists in $\mathcal{D}'$. Then $u = \mathcal{E} * f$ is a solution. However, this solution is not unique. A general solution reads

$$u = \mathcal{E} * f + u_0\,, \qquad u_0 \in \mathcal{N}_L$$

If, in addition, one demands that any two solutions u_1 and u_2 can differ only by an element from $\mathcal{D}'_{\mathcal{E}}$, then the solution is unique:

$$\left.\begin{array}{rl} L(D)u_1 = & f \\ L(D)u_2 = & f \\ u_1 - u_2 \in & \mathcal{D}'_{\mathcal{E}} \end{array}\right\} \quad \Rightarrow \quad u_1 = u_2 = \mathcal{E} * f\,.$$

In other words, the solution is unique in the class of distributions for which there exists the convolution with a (selected) fundamental solution. Note that the convolution $\mathcal{E} * f$ is required to exist but the existence does not imply that the convolution belongs to $\mathcal{D}'_{\mathcal{E}}$ (see Problem 7 in Exercises). So, the uniqueness is defined in the sense that the associated homogeneous equation has only the trivial solution in the subspace $\mathcal{D}'_{\mathcal{E}}$. The following theorem has been established.

THEOREM **35.1**. *Let $\mathcal{E}$ be a fundamental solution for a linear differential operator $L(D)$ with constant coefficients. Suppose that the convolution $\mathcal{E} * f$ exists in $\mathcal{D}'$ for a distribution $f \in \mathcal{D}'$. Then the distribution $u = \mathcal{E} * f$ is a solution to $L(D)u = f$ and this solution is unique in the subspace $\mathcal{D}'_{\mathcal{E}}$, meaning that, the associated homogeneous equation $L(D)u = 0$ has only the trivial solution in $\mathcal{D}'_{\mathcal{E}}$.*

Any two fundamental solutions differ by a solution of the associated homogeneous equation. By fixing the choice of the solution to the homogeneous equation, a particular fundamental solution (Green's function) is selected. Then the solution to the non-homogeneous equation is unique in the class of distributions for which the convolution with the selected Green's function exists. One can say that the operator $L(D)$ is invertible in the specified subspace of distributions and the convolution with the corresponding Green's function defines the inverse of $L(D)$. Clearly, the uniqueness of the convolution solution implies that the solution will satisfy some additional conditions induced by the choice of Green's function. The choice of Green's function is dictated by additional physical conditions (e.g., boundary conditions).

35.6. Regularization of distributional sources. Consider a linear differential equation with constant coefficients

$$L(D)u = f$$

where f is the source term that is a distribution with a bounded support. If G is a Green's function for the operator $L(D)$, then

$$u = G * f$$

is a solution to the equation that is unique in the class of distributions which have a convolution with G. However this solution is not a classical solution (it is not generally smooth enough) because the source term is a distribution. For example, if $L = \Delta$, then u is a potential-like integral if f is Lebesgue integrable. The solution is not smooth enough in the support of f to be a classical solution to the Poisson equation. Source terms in applications are often not smooth enough for a classical solution to exist. However, they can be regularized to make the solution smooth. The question is: what is the difference of the distributional and regularized solutions?

Let f be compactly supported distribution. Then it was shown that its regularization by the convolution with a hat function is a test function:

$$f_a = f * \omega_a\,, \quad \operatorname{supp} f \subset B_R \quad \Rightarrow \quad f_a \in \mathcal{D}$$

Therefore the convolution $G * f_a$ exists (owing to the boundedness of the support of f_a) and, by Sec. **19.5**, it is a smooth function:

$$u_a = G * f_a \in C^\infty\,, \qquad D^\beta u_a = G * D^\beta f_a$$

So it is a classical solution to the equation $L(D)u_a = f_a$. Now the distributional solution $u = G * f$ and the smooth solution u_a can be compared.

Proposition **35.1**. *Let $\mathcal{E}$ be a Green's function for a linear differential operator $L(D)$ with constant coefficients. Let f be a distribution with a bounded support and $f_a = f * \omega_a$ be its regularization. Put $u = G * f$ and $u_a = G * f_a$. Then for any $\varepsilon > 0$ and any test function φ, there exists a_0 such that*

$$|(u_a - u, \varphi)| \le \varepsilon\,, \quad a < a_0\,.$$

The support of f_a is shrinking with decreasing a because $f_a \to f$ in $\mathcal{D}'$. Since f_a have a bounded support for any $a > 0$, the support of f_a lie in the same ball for all sufficiently small a. By Theorem **31.1** the convolution $G*f_a$ is continuous with respect to f_a so that $G*f_a \to G*f$ in $\mathcal{D}'$ as $a \to 0$. This means that for any test function φ

$$\lim_{a\to 0}(u_a, \varphi) = (u, \varphi)$$

and the assertion follows the definition of the limit.

35.7. Exercises.

1. Consider the equation $L(D)u = f$ in $\mathcal{D}'(\mathbb{R})$ where $L = \frac{d^2}{dx^2}$. Investigate whether or not $u = \mathcal{E} * f$ is in $\mathcal{D}'_{\mathcal{E}}$, where $\mathcal{E}$ is a fundamental solution.

(i) Show that the regular distributions

$$\mathcal{E}_1 = x\theta(x)\,, \quad \mathcal{E}_2 = \frac{1}{2}\,|x|\,.$$

are fundamental solutions.

(ii) Let $f = \delta$. Then $\mathcal{E}_{1,2} * f = \mathcal{E}_{1,2}$. Show that the convolution $\mathcal{E}_2 * \mathcal{E}_2$

does not exist and, hence, $u = \mathcal{E}_2 * \delta$ is not in $\mathcal{D}'_{\mathcal{E}_2}$, where as

$$(\mathcal{E}_1 * \mathcal{E}_1)(x) = \frac{1}{6}\,\theta(x)x^3$$

so that $u = \mathcal{E}_1 * \delta \in \mathcal{D}'_{\mathcal{E}_1}$.
(iii) Find the null space of L. Let f have a bounded support. Show that $u_{1,2} = \mathcal{E}_{1,2} * f$ are two different solutions to $Lu = f$ and each of the solutions is unique in the corresponding class of distributions, that is, none of the convolutions $\mathcal{E}_{1,2} * u_0$ exists if u_0 is a non-zero element from the null space of L. Finally, put $f(x) = \theta(1 - |x|)$ and find a function u_0 from the null space of L such that

$$\mathcal{E}_1 * f(x) = \mathcal{E}_2 * f(x) + u_0(x)\,.$$

(iv) Show that if, in addition, the solution is required to vanish when $x \to -\infty$, then the only solution that satisfies this condition is $\mathcal{E}_2 * f$. Show that $\mathcal{E}_2$ is a unique Green's function of L that satisfies an asymptotic boundary condition $\mathcal{E}(x) \to 0$ as $x \to -\infty$.

36. Harmonic functions

The following properties of harmonic functions to be discussed:
(1) Green's (distributional) identity for harmonic functions
(2) Representation of harmonic functions as the sum of Newton-type potentials
(3) The mean value property
(4) The maximum principle
(5) Asymptotic behavior of harmonics functions.
(6) Harmonic functions as distributions
(7) The analog of the Liouville theorem for harmonic functions: If $u \in \mathcal{S}'$ and $\Delta u = 0$ in $\mathbb{R}^N$, then u is a harmonic polynomial (instead of discussing this in the next section)

37. The Poisson equation in $\mathbb{R}^N$

37.1. Fundamental solutions for the Laplace operator. One of the basic techniques for finding a fundamental solution for a linear differential operator with constant coefficients is based on the Fourier transform. Here the method is illustrated with the example of the Laplace operator. Its fundamental solutions were found earlier. The Fourier transform exists for temperate distributions. Therefore a fundamental solution is sought in this class of distributions. If it exists in $\mathcal{S}'$, then it also a solution in $\mathcal{D}'$.

By taking the Fourier transform of both sides of the equation

$$\Delta G_{_N}(x) = \delta(x)\,, \qquad G \in \mathcal{S}'(\mathbb{R}^N)$$

one infers that

$$\mathcal{F}[\Delta G_{_N}](k) = -(k,k)\mathcal{F}[G_{_N}](k) = 1$$

Therefore, the problem is reduced to an *algebraic* equation. Its particular solution is given by a *distributional regularization* of $|k|^{-2}$ if $N \le 2$, and for $N > 2$, a particular solution is given by a locally integrable function.

37.1.1. Case $N = 1$. A general solution the distributional algebraic equation

$$-k^2\mathcal{F}[G_1](k) = 1\,, \quad k \in \mathbb{R}$$

was found earlier:

$$\mathcal{F}[G_1](k) = -\mathcal{P}\frac{1}{k^2} + c_0\delta(k) + c_1\delta'(k)$$

Therefore any fundamental solution to the second derivative operator has the form

$$\begin{aligned} G_1(x) &= -\frac{1}{2\pi}\mathcal{F}\Big[\mathcal{P}\frac{1}{k^2}\Big](x) + \frac{c_0}{2\pi}\mathcal{F}[\delta(-k)](x) + \frac{c_1}{2\pi}\mathcal{F}[\delta'(-k)](x) \\ &= \frac{|x|}{2} + a_0 + a_1 x \end{aligned}$$

where $a_{0,1}$ are arbitrary constants. The linear function $a_0 + a_1x$ is nothing but a general solution to the associated homogeneous equation $g''(x) = 0$.

37.1.2. Case $N = 2$. The function $|k|^{-2}$ is not locally integrable and, hence, does not define a distributional solution. One has to find a distributional extension of this function to the singular point $k = 0$.

So, a particular solution reads

$$\mathcal{F}[G_2](k) = -\mathrm{Reg}\,\frac{1}{|k|^2}\,, \quad k \in \mathbb{R}^2\,.$$

Define the principal value distribution from $\mathcal{S}'(\mathbb{R}^2)$ by the rule

$$\Big(\mathcal{P}\frac{1}{|x|^2}, \varphi\Big) = \int_{|x|<1} \frac{\varphi(x) - \varphi(0)}{|x|^2}\, d^2x + \int_{|x|>1} \frac{\varphi(x)}{|x|^2}\, d^2x$$

It is not difficult to see that for any test function whose support does not contain $x = 0$,

$$\Big(\mathcal{P}\frac{1}{|x|^2}, \varphi\Big) = \int \frac{\varphi(x)}{|x|^2}\, d^2x$$

which means that this distribution is an extension of the singular function $1/|x|^2$ to the singular point $x = 0$. The Fourier transform of this distribution reads

$$\mathcal{F}\Big[\mathcal{P}\frac{1}{|x|^2}\Big](k) = -2\pi \ln|k| + 2\pi C\,,$$

$$C = \int_1^\infty \frac{J_0(z)}{z}\, dz + \int_0^1 \frac{J_0(z) - 1}{z}\, dz\,.$$

where $J_\nu(z)$ is the Bessel function of order ν. Therefore by taking a particular solution in the form

$$\mathcal{F}[G_2](k) = -\mathcal{P}\,\frac{1}{|k|^2}$$

it is concluded that

$$G_2(x) = -\frac{1}{(2\pi)^2}\mathcal{F}\Big[\mathcal{P}\,\frac{1}{|k|^2}\Big](x) = \frac{1}{2\pi}\,\ln|x| - \frac{C}{2\pi}\,, \quad x \in \mathbb{R}^2\,.$$

The constant term is a solution to the homogeneous equation $\Delta g = 0$. So, the first term can be taken as a fundamental solution for the 2D Laplace operator. This solution satisfies the asymptotic boundary condition

$$|\nabla G_2(x)| \to 0\,, \quad |x| \to \infty\,.$$

A general solution is obtained by adding any harmonic function in a plane. Recall that all such functions are real (or imaginary) parts of holomorphic functions of a complex variable $z = x_1 + ix_2$ or of its complex conjugate variable $\bar{z}$.

37.1.3. Case $N \geq 3$. If $N \geq 3$, then a particular solution is given by a regular distribution

$$\mathcal{F}[G_{_N}](k) = -\frac{1}{|k|^2}\,, \quad N > 2\,.$$

Therefore $G_{_N}(x)$ is obtained by taking the inverse Fourier transform of this distribution. For any test function φ, one has

$$\begin{aligned}(G_{_N}, \varphi) &= (\mathcal{F}^{-1}[\mathcal{F}[G_{_N}]], \varphi) = (\mathcal{F}[G_{_N}], \mathcal{F}^{-1}[\varphi]) \\ &= -\frac{1}{(2\pi)^N}\int \frac{1}{|k|^2}\int e^{-i(k,x)}\varphi(x)\,d^N x\,d^N k \\ &= -\frac{1}{(2\pi)^N}\lim_{R\to\infty}\int_{|k|<R}\frac{1}{|k|^2}\int e^{-i(k,x)}\varphi(x)\,d^N x\,d^N k \\ &= -\frac{1}{(2\pi)^N}\lim_{R\to\infty}\int \varphi(x)\int_{|k|<R} e^{-i(k,x)}\frac{d^N k}{|k|^2}\,d^N x\end{aligned}$$

where the order of integration was changed by Fubini's theorem because the integrand is an integrable function on $\mathbb{R}^N \times \{|k| < R\}$. The integral over the ball of radius R is computed in the spherical coordinates such that $(x, k) = |x| r \cos(\phi)$ where $r = |k|$ and

$$d^N k = r^{N-1}\sin^{N-2}(\phi)\,dS_{N-2}d\phi\,dr$$

here dS_{N-2} is the surface area element on the unit $N-2$ dimensional sphere. If $\sigma_{_N}$ is the surface area of a unit sphere $|x| = 1$ in $\mathbb{R}^N$, then, after the scaling transformation $y = |x|r$

$$\int_{|k|<R} e^{-i(k,x)}\frac{d^N k}{|k|^2} = \frac{\sigma_{_{N-1}}}{|x|^{N-2}}\int_0^{|x|R}\int_0^{\pi} e^{iy\cos(\phi)}\sin^{N-2}(\phi)\,d\phi\,y^{N-3}\,dy$$

The simplest case is $N = 3$ when $\sigma_2 = 2\pi$ and using the substitution $s = \cos(\phi)$, one infers that

$$\int_{|k|<R} e^{-i(k,x)}\frac{d^3 k}{|k|^2} = \frac{2\pi}{|x|}\int_0^{|x|R}\int_{-1}^{1} e^{-iys}ds dy = \frac{4\pi}{|x|}\int_0^{R|x|}\frac{\sin(y)}{y}\,dy$$

The improper integral converges to $\pi/2$ as $R \to \infty$ for any $x \neq 0$ and, hence, is a bounded

$$\Big|\int_0^{R|x|}\frac{\sin(y)}{y}\,dy\Big| \leq M$$

for all $R > 0$. The function

$$\frac{M\varphi(x)}{|x|} \in \mathcal{L}(\mathbb{R}^3)$$

is integrable. Therefore the limit in (G_3, φ) can be computed by the Lebesgue dominated convergence theorem

$$(G_3, \varphi) = -\frac{4\pi}{(2\pi)^3} \lim_{R\to\infty} \int \frac{\varphi(x)}{|x|} \int_0^{|x|R} \frac{\sin(y)}{y}\, dy\, d^3x = -\frac{1}{4\pi} \int \frac{\varphi(x)}{|x|}\, d^3x$$

$$G_3(x) = -\frac{1}{4\pi|x|}\,, \quad x \in \mathbb{R}^3\,.$$

A similar line of arguments can be used for any $N > 3$ to show that

$$G_N(x) = -\frac{1}{(N-2)\sigma_N} \frac{1}{|x|^{N-2}}\,, \quad x \in \mathbb{R}^N\,, \quad N \geq 3\,.$$

The integral over the zenith angle can be reduced to one of integrals that can be found in tables of definite integrals. However, an easier approach is based on the observation that after the scaling transformation $y = |x|r$, G_N is shown to be proportional to $|x|^{2-N}$. The proportionality coefficient is given by the said improper iterated integral. Then the proportionality coefficient can be computed by a direct evaluation of $\Delta|x|^{2-N} \sim \delta(x)$.

37.2. Uniqueness of the fundamental solution for the Laplace operator. A general temperate distribution that is the Fourier transform of a fundamental solution for the Laplace operator is the sum of a particular solution and a general temperate distribution $g(k)$ that satisfies the equation

$$(k, k)g(k) = 0$$

This distribution has a point support $\{k = 0\}$. Indeed, if support of a test function $\varphi(k)$ does not contain $k = 0$. Then $\psi(k) = \varphi(k)/|k|^2$ is also a test function. Therefore

$$0 = (0, \psi) = \left(|k|^2 g(k), \frac{\varphi(k)}{|k|^2}\right) = (g, \varphi)$$

which means that $g(k) = 0$ in $\{|k| > 0\}$. Thus, one has to determine a structure of temperate distributions with point support. For a single variable, any distribution with point support was shown to be a linear combination of the delta function and its derivatives. It turns out the same assertion holds in higher dimensions for *temperate distributions.*

THEOREM **37.1**. *If support of a temperate distribution f is a point $\{x = 0\}$, then there exists a unique collection of coefficients c_α and an integer $p \geq 0$ such that*

$$f(x) = \sum_{\alpha=0}^{p} c_\alpha D^\alpha \delta(x)\,.$$

It follows from this theorem that any fundamental solution for the Laplace operator is unique up to an additive *harmonic polynomial* in the space $\mathcal{S}'$:

$$G_N(x) \to G_N(x) + P_h(x)\,, \quad P_h(x) = \sum_{\alpha=0}^{p} c_\alpha x^\alpha\,, \quad \Delta P_h(x) = 0$$

It is worth noting that in the space $\mathcal{D}'$, a fundamental solution for the Laplace operator is unique up to an additive harmonic function, not just a harmonic polynomial. A general harmonic function of two or more variables grows too fast in some directions to be a regular temperate distribution. For example, any harmonic polynomial of two variables is a linear combination of monomials $\operatorname{Re} z^m$ and $\operatorname{Im} z^m$, with $m = 0, 1, \ldots$, where $z = x_1 + ix_2$, whereas the real or imaginary parts of any holomorphic function of z is a harmonic function of x_1 and x_2. For example, $h(x_1, x_2) = \operatorname{Re} e^z = e^{x_1}\cos(x_2)$ or $h(x_1, x_2) = \operatorname{Re} e^{z^2} = e^{x_1^2 - x_2^2}\cos(2x_1x_2)$ which are not regular temperate distributions.

Thus, the set of all fundamental solutions for a differential operator that can be obtained by the Fourier transform is smaller than the set of all fundamental solutions for this operator. Only fundamental solutions that are also temperate distributions can be obtained.

A fundamental solution $G_N(x)$ is a harmonic function in the asymptotic region $|x| > R > 0$. Therefore it is possible to impose asymptotic boundary conditions on G to make the solution unique. For example, the condition

$$G_N(x) \to 0\,, \quad |x| \to \infty\,,$$

yields a unique solution if $N \geq 3$ because $P_h = 0$. The condition

$$|\nabla G_N(x)| \to 0\,, \quad |x| \to \infty\,,$$

makes the solution unique up to an additive constant if $N \geq 2$.

37.3. Solving the Poisson equation by the Fourier transform method. Put

$$G_1(x) = \frac{|x|}{2}\,, \quad x \in \mathbb{R}\,,$$

$$G_2(x) = \frac{1}{2\pi}\ln(|x|)\,, \quad x \in \mathbb{R}^2\,,$$

$$G_N(x) = -\frac{1}{(N-2)\sigma_N}\,\frac{1}{|x|^{N-2}}\,, \quad x \in \mathbb{R}^N\,, \quad N \geq 3$$

Suppose that ρ is a compactly supported distribution. Then $\rho \in \mathcal{S}'$. Consider the problem of finding a temperate distribution u that satisfies the Poisson equation

$$\Delta u(x) = \rho(x)\,, \quad u \in \mathcal{S}'\,.$$

By taking the Fourier transform of this equation one infers that

$$|k|^2 \mathcal{F}[u](k) = \mathcal{F}[\rho](k)$$

Therefore a particular solution reads

$$\mathcal{F}[u](k) = \mathcal{F}[G_N](k)\mathcal{F}[\rho](k) = \mathcal{F}[G_N * \rho](k)$$

It defines a temperate distribution because $\mathcal{F}[\rho]$ is a smooth function of slow growth (from class $\mathcal{O}_M$). Therefore, a general solution in $\mathcal{S}'$ is

$$u(x) = (G_N * \rho)(x) + P_h(x)$$

where P_h is a harmonic polynomial. Suppose ρ is a bounded function. Then

$$u(x) = \int_\Omega G_N(x-y)\rho(y)\, d^N y + P_h(x)$$

where Ω is a bounded set that contains the support of ρ. If $N \geq 3$, then the first term is from class C^∞ in the complement of Ω and vanishes in the asymptotic region $|x| \to \infty$. If, in addition, it is demanded that $u(x) \to 0$ as $|x| \to \infty$, then the solution is *unique* because $P_h(x) = 0$. Another asymptotic condition used in application is to demand that the solution is bounded or its gradient vanishes in the limit $|x| \to \infty$. In ether of these cases, the solution is *unique up to an additive constant*.

If $N = 2$, then the convolution

$$(G_2 * \rho)(x) = \frac{1}{2\pi}\int_\Omega \ln(|x-y|)\rho(y)\, d^2 y$$

does not vanish in the asymptotic region. However it is a smooth function in the complement of $\bar{\Omega}$ and

$$D^\alpha (G_2 * \rho)(x) \frac{1}{2\pi}\int_\Omega D_x^\alpha \ln(|x-y|)\rho(y)\, d^2 y$$

by the theorem about differentiation of a function defined by the integral. It follows from this relation that the gradient of the convolution vanishes in the asymptotic region. Indeed, let $\Omega \subset B_R$. Then

$$1 - \frac{|y|}{|x|} \geq \frac{1}{2}\,, \quad y \in \Omega\,, \quad |x| \geq 2R$$

for any such x,

$$|\nabla(G_2 * \rho)(x)| \le \frac{1}{2\pi}\Big|\int_\Omega \frac{x\rho(y)}{|x-y|^2}\,d^2y\Big| \le \frac{|x|}{2\pi}\int_\Omega \frac{|\rho(y)|}{(|x|-|y|)^2}\,d^2x$$
$$\le \frac{M}{|x|} \to 0\,, \qquad M = \frac{1}{\pi}\int_\Omega |\rho(y)|\,d^2y < \infty$$

Therefore, if one demands that the gradient of a solution vanishes in the asymptotic region, then *the solution is unique up to an additive constant.*

37.4. Examples of distributional solutions to the Poisson equation. Here some examples of distributional solutions to the Poisson equation in $\mathbb{R}^3$ are obtained that are commonly used in physics.

37.4.1. Dipole potentials. The electric charge density of a point-like electric dipole with moment p and positioned at $x = 0$ is given by the distribution

$$\rho(x) = (p,\nabla)\delta(x)\,, \quad x \in \mathbb{R}^3\,.$$

The magnitude $|p|$ is the dipole strength. So, the Coulomb potential generated by a point-like dipole is required to have the vanishing gradient in the asymptotic region, $|x| \to \infty$, and hence is given by the convolution

$$u(x) = G_3 * (p,\nabla)\delta(x) = (p,\nabla)(G_3 * \delta) = -\frac{1}{4\pi}(p,\nabla)\frac{1}{|x|} = \frac{(p,x)}{4\pi|x|^3}\,.$$

A shift of the fundamental solution by a constant does not change $u(x)$ because $1 * (p,\nabla)\delta(x) = (p,\nabla)1 = 0$.

37.4.2. Potentials of single and double layers. Suppose that electric charges are distributed over a smooth bounded surface S in $\mathbb{R}^3$ with a surface density μ that is continuous on S. Then the electric charge density is given by a simple layer distribution

$$\rho(x) = (\mu\delta_S)(x)\,.$$

This distribution has a bounded support (which is the surface S) and, hence, the convolution $G_3 * \rho$ exists. It is an electric potential created by a simple layer of charges. Let η_S be a test (bump) function that is equal to 1 in a neighborhood of S. Then using the theorem about the convolution of a distribution with bounded support

$$\begin{aligned}(G_3 * \rho, \varphi) &= \Big(\rho(y), \eta_S(y)\Big(G_3(z), \varphi(z+y)\Big)\Big)\\ &= \int_S \mu(y)\eta_S(y) \int \frac{\varphi(z+y)}{4\pi|z|}\, d^3z\, dS_y\\ &= \int_S \mu(y) \int \frac{\varphi(x)}{4\pi|x-y|}\, d^3x\, dS_y\\ &= \int\int_S \frac{\mu(y)}{4\pi|x-y|}\, dS_y\, \varphi(x)\, d^3x\,,\end{aligned}$$

where the latter equality follows from the Fubini theorem (the integrand is integrable on $S \times B_R$ where $\operatorname{supp}\varphi \subset B_R$). Therefore

$$u(x) = \frac{1}{4\pi}\int_S \frac{\mu(y)}{|x-y|}\, dS_y$$

is a potential of a single layer with density μ that satisfies the distributional Poisson equation

$$\Delta u = \mu\delta_S\,.$$

Note that the above analysis also shows that $u(x)$ is locally integrable. Recall the analysis of functions defined by potential-like integrals in the previous chapter. By this analysis the potential of a single layer is a smooth function in the complement of any neighborhood of the surface S, and

$$D^\alpha u(x) = \frac{1}{4\pi}\int_S \mu(y) D_x^\alpha \frac{1}{|x-y|}\, dS_y\,, \quad x \in \mathbb{R}^3 \setminus S\,.$$

As noted a fundamental solution with the vanishing gradient in the asymptotic region is unique up an additive constant. If G_3 is shifted by a constant, then u is also changed by an additive constant because

$$1 * \mu\delta_S = \int_S \mu(y)\, dS$$

This integral is equal to the total electric charge of the surface S.

Similarly, a distributional solution to the Poisson equation

$$\Delta u = -\frac{\partial}{\partial n}\Big(\nu\delta_S\Big)$$

whose gradient vanishes as $|x| \to \infty$, is an electric potential created by electric dipoles with moments parallel to the unit normal $\hat{n}$ to the surface S and distributed with a surface density ν which is assumed to be continuous on S,

$$u(x) = \frac{1}{4\pi}\int_S \nu(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y = \frac{1}{4\pi}\int_S \nu(y)\,\frac{(\hat{n}_y,(x-y))}{|x-y|^3}\,dS_y\,.$$

Indeed, for any test function φ and a bump function η_S for a bounded smooth surface S

$$\begin{aligned}(u,\varphi) &= \Big(G_3*\rho,\varphi\Big) = \Big(\rho*G_3,\varphi\Big) = \Big(\rho(y)\cdot G_3(z),\eta_s(y)\varphi(y+z)\Big)\\ &= -\Big((\hat{n}_y,\nabla_y)\nu(y),\eta_s(y)\int\frac{\varphi(y+z)}{|z|}\,d^3z\Big)\\ &= \int_S \nu(y)(\hat{n}_y,\nabla_y)\int\frac{\varphi(x)}{|x-y|}\,d^3x\,dS_y\\ &= \int_S \nu(y)\int\varphi(x)(\hat{n}_y,\nabla_y)\frac{1}{|x-y|}\,d^3x\,dS_y\\ &= \int\varphi(x)\int_S \nu(y)(\hat{n}_y,\nabla_y)\frac{1}{|x-y|}\,dS_y\,d^3x\end{aligned}$$

The second equality from the bottom is justified by the theorem about differentiation of potential-like integrals studied earlier (in this case the density $\varphi(x)$ has a bounded support and smooth and, hence, is bounded, which is sufficient for the integral to be from class C^1 in $\mathbb{R}^3$). The last equality follows from the Fubini theorem. Note that the integrand is an integrable function on $S\times B_R$ where $\operatorname{supp}\varphi\subset B_R$ because

$$\Big|\varphi(x)\nu(y)(\hat{n}_y,\nabla_y)\frac{1}{|x-y|}\Big| \le \frac{M}{|x-y|^2}\in\mathcal{L}(S\times B_R)$$

where

$$M = \sup|\varphi|\sup_S|\nu| < \infty$$

since S is bounded and ν is continuous on S.

The equations for the potentials of single and double layers can be obtained from the superposition principle in physics. For example, the dipole moment of an infinitesimal part of the surface of area dS_y at a point y is given by $dp_y = \nu(y)d\Sigma_y$, where $d\Sigma_y = \hat{n}_y dS_y$. The potential is nothing but a superposition of potentials created by point-like dipoles distributed over S at a point x. Indeed a single dipole at a point y creates a potential at a point x that is given by

$$du(x) = \frac{(x-y,dp_y)}{4\pi|x-y|^3}\,.$$

A summation of partition of the surface means taking the superposition of all potential

$$u(x) = \int_S du(x) = \int_S \frac{(x-y, dp_y)}{4\pi|x-y|^3}\,.$$

37.5. The multipole expansion of the Coulomb potential of an extended source. The examples given above shows that in many cases an explicit form of distributional solutions is easier to find, especially when the source term is a combination of the delta function and its derivatives. The corresponding distributional solution is close to a classical solution when dimensions of the support of sources can be neglected (see Proposition **35.1**). In practical applications, they, in fact, cannot even be distinguished. For example, a fundamental solution is a solution with a source being a delta function. But any measurement of coordinates (or position in general) has an uncertainty. The length can be measured with the smallest uncertainty of 10^{-18} cm (the limit reached in the hadron collider in CERN). So, it is pointless to discuss how an electric charge is distributed in a ball of radius 10^{-18} cm because it cannot be measured anyway. A fundamental solution to the Poisson equation gives an excellent approximation.

Let $\omega_a(x)$ be a smooth regularization of $\delta(x)$ with support in a ball of radius a (the radius is about the uncertainty of distance measurements). The convolution

$$u(x) = -\int \frac{\omega_a(y)}{4\pi|x-y|}\,d^3y = -\int_{B_a} \frac{\omega_a(y)}{4\pi|x-y|}\,d^3y$$

is a classical solution to the Poisson equation. Consider the solution in the region where $|x|$ is much larger than a so that $a/|x|$ is a small number. For example, a size of a hydrogen atom is about 10^{-8} cm, which is roughly a radius of the orbit of an electron moving in the Coulomb potential created by a proton whose dimension is about $a \sim 10^{-12}$ cm. In this case, $a/|x| \sim 10^{-4}$. Let us expand the fundamental solution into a power series

$$\begin{aligned}\frac{1}{|x-y|} &= \frac{1}{|x|} - y_j\frac{\partial}{\partial x_j}\frac{1}{|x|} + \frac{1}{2}y_iy_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{|x|} + \cdots\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(y_i\frac{\partial}{\partial x_i}\Big)^n\frac{1}{|x|}\end{aligned}$$

where the summation over repeated indices is assumed (the Einstein summation rule). The Taylor series converges if $|y| \le a$ and $a/|x|$ is sufficiently small. A convergent power series can be integrated term-by-term over a ball $|y| < a$ and the result is a convergent series such that

$$u(x) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{4\pi n!} \int_{B_a} \omega_a(y) \Big(y_i \frac{\partial}{\partial x_i} \Big)^n \frac{1}{|x|}\, d^3y$$

Put

$$Q^{(n)}_{j_1 \cdots j_n} = \frac{1}{n!} \int_{B_a} \omega_a(y) y_{j_1} \cdots y_{j_n}\, d^3y$$

These quantities are called *the moments of the density* ω_a. In particular, for $n = 0$, $Q^{(0)} = 1$ is the total charge (recall that ω_a is a regularization of $\delta(x)$), then $Q^{(1)}_j$ is known as *the dipole moment*, $Q^{(2)}_{ij}$ as *the quadrupole moment*, etc. Then

$$\begin{aligned} u(x) &= -\frac{1}{4\pi|x|} - \sum_{n=1}^{\infty} (-1)^n Q^{(n)}_{j_1 \cdots j_n} \frac{\partial^n}{\partial x_{j_1} \cdots \partial x_{j_n}} \frac{1}{4\pi|x|} \\ &= -\frac{1}{4\pi|x|} - \frac{Q^{(1)}_j x_j}{4\pi|x|^3} - Q^{(2)}_{ij} \frac{3x_i x_j - \delta_{ij}|x|^2}{|x|^5} - \cdots \end{aligned}$$

The second term is a dipole correction to the Coulomb potential of a point-like source and the third one is a quadrupole correction. This expansion is known as a *multipole expansion of the Coulomb potential of an extended source*.

So, with every classical (smooth) compactly supported source ρ, one can associate the distributional source

$$\rho_m(x) = Q^{(0)}\delta(x) + \sum_{n=1}^{m} (-1)^n Q^{(n)}_{j_1 \cdots j_n} \frac{\partial^n}{\partial x_{j_1} \cdots \partial x_{j_n}}\, \delta(x)$$

where $Q^{(0)} = \int \rho(x)\, d^3x$ (a total charge) and $Q^{(n)}$ are moments of ρ. Then the distributional solution

$$u_m(x) = -\frac{1}{4\pi|x|} * \rho_m(x)$$

matches the first $m+1$ terms of the multipole expansion of the classical solution by the differentiation properties of the convolution.

If the classical source has support in a ball of radius a, then

$$|Q^{(n)}_{j_1 \cdots j_n}| \le \frac{1}{n!} \int_{B_a} \int_{B_a} |\rho(y)||y_{j_1} \cdots y_{j_n}|\, d^3y \le M_0 \frac{a^n}{n!}\,, \quad M_0 = \int |\rho(y)|\, d^3y\,.$$

Define a constant C_n as the smallest constant for which the following inequality holds:

$$\Big| \frac{\partial^n}{\partial x_{j_1} \cdots \partial x_{j_n}} \frac{1}{|x|} \Big| \le \frac{C_n}{|x|^{n+1}}\,.$$

Then

$$|u(x) - u_{m-1}(x)| \leq \frac{1}{4\pi|x|}\left(\frac{C_m}{m!}\Big(\frac{a}{|x|}\Big)^m + O\Big(\Big(\frac{a}{|x|}\Big)^{m+1}\Big)\right)$$

If a is roughly an absolute uncertainty of length measurement, then even u_0 is an accurate solution because a relative uncertainty of the distributional solution is about the same as the length relative uncertainty:

$$\frac{|u(x) - u_0(x)|}{u_0(x)} \sim \frac{a}{|x|}\,.$$

Even in the case when a is greater than an absolute uncertainty of length measurements but a solution is studied for $|x|$ much larger than a (the size of sources), the distributional solution u_m is accurate for large enough m (because the amplitude $u(x)$ is also measured with some uncertainty).

37.6. Exercises.

1. Let ρ be a bounded function with support in an interval $[-R, R]$. Show that that $u(x) = (G_1 * \rho)(x)$ is a unique solution to $u''(x) = \rho(x)$ up to an additive constant if $|u'(x)| \leq M$ for all x.

2. Use the Fourier transform method to find a solution to the problem

$$(\nabla, A\nabla)G(x) = \delta(x)\,, \quad G \in \mathcal{S}'(\mathbb{R}^N)$$

where A is a strictly positive symmetric $N \times N$ matrix.
Hint: Find a linear change of variables to reduce the problem to finding a fundamental solution for the Laplace operator.

3. Let $\rho(x)$ be a bounded function with bounded support. Use the Fourier transform method to find an integral representation of the most general distributional solution from to the problem

$$(\nabla, A\nabla)u(x) = \rho(x)\,, \quad |u(x)| \leq M\,, \quad x \in \mathbb{R}^3$$

where A is a strictly positive symmetric 3×3 matrix. Find a direction in which the solution is decreasing most rapidly with increasing $|x|$ in the asymptotic region $|x| \to \infty$.

4. Show that

$$G_2(x) = -\frac{1}{2\pi}\,\ln(|x|) - \frac{C}{2\pi}$$

by the direct evaluation of the Fourier transform as suggested above in this section.

5. Consider the single and double layer distributions $\mu\delta_S$ and $-\frac{\partial}{\partial n}(\nu\delta_S)$ in $\mathbb{R}^N$, $N > 3$. Assume that the $N-1$ dimensional surface S is smooth and bounded, and the densities μ and ν are continuous on S. Find an integral representation to the solution to the Poisson equation

$$\Delta u(x) = \rho(x)\,, \quad |\nabla u(x)| \to 0 \quad \text{as } |x| \to \infty$$

where ρ is either the single or double layer distribution in $\mathbb{R}^N$.

38. The Helmholtz equation

Consider a wave equation with a special inhomogeneity

$$(D_t^2 - c^2\Delta_x)u(x,t) = e^{-i\omega t}\cdot \rho(x)\,, \quad t\in\mathbb{R}\,, \quad x\in\mathbb{R}^N$$

where $\rho(x)$ is a distribution (the density of sources). A related physical problem is formulated as follows. If $u(x,t)$ represents a deviation of air pressure from its equilibrium (e.g., an atmospheric pressure), then the equation describes a generation of sound waves by monochromatic sources of frequency ω distributed with a density $\rho(x)$. A solution is sought in the form of the direct product

$$u(x,t) = e^{-i\omega t}\cdot v(x)$$

where the distribution $v(x)$ satisfies the Helmholtz equation

$$(\Delta + k^2)v(x) = c^{-2}\rho(x)\,, \quad k^2 = \frac{\omega^2}{c^2}$$

The differential operator in the left-hand side of the equation is called the Helmholtz operator in $\mathbb{R}^N$. A solution to this equation can be found as a convolution of its fundamental solution with a distributional density ρ. In particular, the convolution always exists if the distribution ρ has a bounded support.

38.1. Fundamental solutions in $\mathbb{R}^3$. Let us show that the regular distributions

$$\mathcal{E}^{\pm}(x) = -\frac{1}{4\pi}\,\frac{e^{\pm ik|x|}}{|x|}\,, \quad x\in\mathbb{R}^3\,,$$

are fundamental solutions for the 3D Helmholtz operator

$$(\Delta + k^2)\mathcal{E}^{\pm}(x) = \delta(x)\,.$$

One way of doing this is to show that for any test function

$$\begin{aligned}\Big((\Delta + k^2)\mathcal{E}^{\pm},\varphi\Big) &= \Big(\mathcal{E}^{\pm},(\Delta + k^2)\varphi\Big)\\ &= -\frac{1}{4\pi}\int \frac{e^{\pm ik|x|}}{|x|}(\Delta + k^2)\varphi(x)\,d^3x\\ &= \varphi(0) = (\delta,\varphi)\end{aligned}$$

by means of the Green's identity and that

$$\Big(\Delta + k^2\Big)\,\mathcal{E}^{\pm}(x) = 0\,, \quad |x|\geq a>0$$

similarly to the proof of (**21.15**).

The result can also be obtained by the Leibniz rule. Note that the imaginary part of the fundamental solution is from $C^\infty(\mathbb{R}^3)$

$$\frac{\sin(k|x|)}{|x|} = k - \frac{k^3}{6}\,|x|^2 + O(|x|^4)$$

The function is represented by a *power series* in three rectangular coordinates that has infinite radius of convergence. In spherical coordinates, the function depends only on the radial variable $r = |x|$ so that

$$\Delta\frac{\sin(k|x|}{|x|} = \frac{1}{r^2}\frac{\partial}{\partial r}\,r^2\frac{\partial}{\partial r}\frac{\sin(kr)}{r} = -k^2\,\frac{\sin(kr)}{r}$$

where the Laplace operator was written in spherical coordinates. So, the imaginary part is a smooth solution to the Helmholtz equation:

$$\Big(\Delta + k^2\Big)\,\frac{\sin(k|x|)}{|x|} = 0\,, \qquad x\in\mathbb{R}^3$$

The real part contains a C^∞ function

$$\cos(k|x|) = 1 - \frac{k}{2}\,|x|^2 + O(|x|^4)$$

because it is given by a power series in three variables with infinite radius of convergence. So, the real part is the product of a C^∞ function and a distribution, and, hence, the Leibniz rule applies to calculate the action of the Laplace operator on it:

$$\Delta\,\frac{\cos(k|x|)}{|x|} = \frac{\Delta\cos(k|x|)}{|x|} + 2\Big(\nabla\frac{1}{|x|},\nabla\cos(k|x|)\Big) + \cos(k|x|)\,\Delta\frac{1}{|x|}$$

The distributional and classical gradients of $|x|^{-1}$ are equal

$$\nabla\frac{1}{|x|} = -\frac{x}{|x|^3} = \left\{\nabla\frac{1}{|x|}\right\}$$

because the gradient is locally integrable in $\mathbb{R}^3$. Indeed, let ∂_j denotes the partial derivative with respect to x_j. For any test function φ with support in a ball B_R,

$$\begin{aligned}
\Big(\partial_j\frac{1}{|x|},\varphi\Big) &= -\Big(\frac{1}{|x|},\partial_j\varphi\Big) = -\int_{B_R}\frac{\partial_j\varphi(x)}{|x|}\,d^3x \\
&= -\lim_{a\to 0^+}\int_{B_R\setminus B_a}\frac{\partial_j\varphi(x)}{|x|}\,d^3x \\
&= -\lim_{a\to 0}\Big(\oint_{|x|=a}+\oint_{|x|=R}\Big)\frac{n_j\varphi(x)}{|x|}\,dS \\
&\quad -\lim_{a\to 0}\int_{B_R\setminus B_a}\frac{x_j}{|x|^3}\,\varphi(x)\,d^3x \\
&= -\int_{B_R}\frac{x_j}{|x|^3}\,\varphi(x)\,d^3x = \Big(\Big\{\partial_j\frac{1}{|x|}\Big\},\varphi\Big)
\end{aligned}$$

where the integration by parts was carried out, n_j is the jth components of the outward unit normal, and the integral over the boundary sphere $|x| = R$ vanished because $\varphi(x) = 0$ on the sphere. The integral over the boundary sphere $|x| = a$ vanishes in the limit $a \to 0$. Indeed, let $M = \sup|\varphi|$. Using $|n_j| \le 1$,

$$\Big|\oint_{|x|=a}\frac{n_j\varphi}{|x|}\,dS\Big| \le \frac{1}{a}\oint_{|x|=a}|\varphi|\,dS \le \frac{M}{a}\oint_{|x|=a}dS = 4\pi aM$$

which tends to 0 as $a \to 0$. Calculating

$$\begin{aligned}
\nabla\cos(k|x|) &= -\frac{kx\sin(k|x|}{|x|^2}\,, \\
\Delta\cos(k|x|) &= (\nabla,\nabla\cos(k|x|)) = -k^2\cos(k|x|) + 2k\frac{\sin(k|x|}{|x|}
\end{aligned}$$

it is deduced that

$$(\Delta + k^2)\frac{\cos(k|x|)}{|x|} = \cos(k|x|)\Delta\frac{1}{|x|} = -4\pi\cos(k|x|)\,\delta(x) = -4\pi\delta(x)$$

as required. The analysis also shows that the fundamental solutions differs by a solution to the homogeneous Helmholtz equation

$$\mathcal{E}^+(x) - \mathcal{E}^-(x) = \frac{i\sin(k|x|)}{2\pi|x|}\,.$$

38.1.1. Physical significance of $\mathcal{E}^{\pm}$. Let us analyze a physical significance of these fundamental solutions. For $|x| > 0$, $\mathcal{E}^{\pm}(x)$ are smooth and represent classical solutions to the wave equation

$$u_+(x,t) = -\frac{e^{-i\omega t+ik|x|}}{4\pi|x|}\,, \qquad u_-(x,t) = -\frac{e^{-i\omega t-ik|x|}}{4\pi|x|}\,, \quad |x| > 0\,.$$

The phase of u_+ is constant on the sphere $|x| = ct$ for any moment of time t. So, with increasing $t > 0$, the sphere of the constant phase is expanding, hence, u_+ describes a spherical wave outgoing with the speed c to spatial infinity (recall that $k = \omega/c$). The solution $u_-(x,t)$ can be interpreted as a spherical wave that is expanding with the speed $-c$, or *collapsing* with the speed c. One can think of a spherical wave that comes from infinity and collapses to the origin. The fundamental solution u_+ describes a sound wave coming from a point-like source producing a single sound note corresponding to the wavelength $\lambda = 2\pi/k$, like a tuning fork.

Suppose that $\rho(x)$ is a bounded function with a bounded support. Then the convolutions $\mathcal{E}^\pm * \rho$ exist so that the original wave equation has two solutions

$$u_\pm(x,t) = -\frac{e^{-i\omega t}}{4\pi c^2}\int \frac{e^{\pm ik|x-y|}}{|x-y|}\,\rho(y)\,d^3y$$

The solution u_+ can be viewed as a "superposition" of outgoing waves

$$du_+(x,t) = -\frac{e^{-i\omega t+ik|x-y|}}{4\pi c^2|x-y|}\,\rho(y)d^3y$$

each of which is the outgoing spherical wave from a point-source at a point y of strength $\rho(y)d^3y$. The support of ρ is partitioned into small volumes, each volumes acts as a point source of a spherical wave. The sum over partition is a superposition of these waves that becomes an integral over the support of ρ as the partition is refined. In contrast, the solution u_- can be viewed as a superposition of incoming waves.

38.2. Sommerfeld radiation condition. If a solution is required to describe a physical process of emitting waves by a given source, then a solution cannot contain any incoming waves. It is not difficult to see that the fundamental $\mathcal{E}^+$ satisfies the *Sommerfeld radiation condition* in $\mathbb{R}^3$:

$$r\Big(\frac{\partial}{\partial r} - ik\Big)\mathcal{E}^+(x) \;\to\; 0 \quad \text{as} \quad r = |x| \to \infty$$

while $\mathcal{E}^-$ does not. The reader is asked to show that the solution $u_+(x,t)$ satisfies the Sommerfeld radiation condition

$$r\Big(\frac{\partial}{\partial r} - ik\Big)u^+(x) \;\to\; 0 \quad \text{as} \quad r = |x| \to \infty$$

and, hence, describes the process of emitting waves by the source. This shows that the choice of a fundamental solution and its use to construct solutions to the associated non-homogeneous equation with a distributional source depends on additional (physical) conditions imposed on the solution. Recall that a particular fundamental solution satisfying additional conditions is called a Green's function for a differential operator. In this sense, $\mathcal{E}^+$ is the Green's function of the 3D Helmholtz operator for outgoing waves (or satisfying the Sommerfeld radiation condition).

38.3. Homogeneous Helmholtz equation. Make an analogy with harmonic functions:
(1) any distributional solution is from class C^∞;
(2) representation of a solution in a bounded region Ω via its values and values of its normal derivatives on $\partial\Omega$, like Green's identities for harmonic functions;
(3) Proof that if u solves the homogeneous equation and satisfies Sommerfeld radiation condition, then $u(x) = 0$;

38.4. The non-homogeneous equation. The following topics to be discussed:
(1) Analysis of smoothness of potentials (convolutions);
(2) The attenuation principle and the Fourier method for solving

38.5. Radiation of a dipole-like source. Suppose that the source is given by

$$\rho(x) = (p, \nabla)\delta(x)$$

where p is a constant vector in $\mathbb{R}^3$. In this case,

$$\begin{aligned} u_\pm(x,t) &= \frac{e^{-i\omega t}}{c^2}\,(\mathcal{E}^\pm * (p,\nabla)\delta)(x) = \frac{e^{-i\omega t}}{c^2}\,(p,\nabla)\mathcal{E}^\pm(x) \\ &= -\frac{e^{-i\omega t \pm ik|x|}}{4\pi c^2|x|}\,\frac{(p,x)}{|x|}\Big(\pm ik - \frac{1}{|x|}\Big) \end{aligned}$$

are solutions to the wave equation. One can see that in the limit $|x| \to \infty$, the leading term of the solution is

$$v_\pm(x) \sim \frac{e^{-i\omega t \pm ik|x|}}{|x|}$$

because $|(p,x)| \le |p||x|$. So, the outgoing waves produced by the dipole source are described by u_+ that satisfies the Sommerfeld radiation condition, while u_- does not satisfy it.

38.6. Exercises.

1. An alternative proof that the distributions $\mathcal{E}^\pm$ are fundamental solutions for the Helmholtz operator. Let $x \in \mathbb{R}^3$. Show first that

$$(\Delta + k^2)\frac{e^{ik|x|}}{|x|} = 0\,, \quad x \neq 0\,.$$

If φ is a test function with support in a ball B_R of radius R, justify the following chain of equalities

$$\Big(\Delta\frac{e^{ik|x|}}{|x|}, \varphi\Big) = \int_{B_R} \frac{e^{ik|x|}}{|x|}\Delta\varphi(x)\,d^3x = \lim_{a\to 0}\int_{B_R\setminus B_a} \frac{e^{ik|x|}}{|x|}\Delta\varphi(x)\,d^3x$$

Use Green's formula to show that

$$\Big((\Delta + k^2)\frac{e^{ik|x|}}{|x|}, \varphi\Big) = -\lim_{a\to 0}\oint\limits_{|x|=a} \Big(\varphi(x)\frac{\partial}{\partial n}\frac{e^{ik|x|}}{|x|} - \frac{e^{ik|x|}}{|x|}\frac{\partial\varphi}{\partial n}\Big)\,dS$$

where $n = -x/a$ is the unit normal on the sphere $|x| = a$. Show that the second surface integral vanishes in the limit $a \to 0$ and use the integral mean value theorem to show that

$$\lim_{a\to 0}\oint\limits_{|x|=a} \varphi(x)\frac{\partial}{\partial n}\frac{e^{ik|x|}}{|x|}\,dS = 4\pi\varphi(0) = 4\pi(\delta, \varphi)$$

Conclude that in the sense of distributions

$$(\Delta + k^2)\frac{e^{ik|x|}}{|x|} = -4\pi\delta(x)$$

Show that the above line of arguments remains valid if k changed to $-k$.

2. (i) Find a distributional solution to the Helmholtz equation for $|x| \geq a > 0$ with a "quadrupole" point-like source:

$$(\Delta + k^2)v(x) = \sum_{i,j=1}^{3} p_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\,\delta(x) = (\nabla, p\nabla)\delta(x)\,, \quad x \in \mathbb{R}^3\,,$$

that satisfies the Sommerfeld radiation condition.
(ii) Show that the solution found for $x \neq 0$ is not locally integrable in the whole $\mathbb{R}^3$. So, a distributional solution in $\mathbb{R}^3$ is an extension of the classical solution to the singular point $x = 0$. Find this extension.

3. Let $\rho(x)$ be bounded and have a bounded support in $\mathbb{R}^3$. Use the theory of functions defined by potential-like integrals to prove that the solution $u_+(x,t) = c^{-2}e^{-i\omega t}(\mathcal{E}^+ * \rho)(x)$ to the 4D wave equation satisfies the Sommerfeld radiation condition.

4. Let ω_a be a regularization of $\delta(x)$ with support in a ball $|x| \leq a$.

Find the multipole expansion of the solution $u_+(x) = \mathcal{E}^+ * \omega_a$ to the Helmholtz equation in $\mathbb{R}^3$ in a region where $a/|x|$ is small. Estimate a relative uncertainty of the fundamental solution $\mathcal{E}^+(x)$ relative to the classical solution $u_+(x)$ in terms of $a/|x|$.

39. General linear PDEs in $\mathcal{S}'$

Let $L(D) = \sum_{\alpha<p} a_\alpha D^\alpha$ be a general linear differential operator with constant coefficients a_α. Let us find a fundamental solution for it in the space of temperate distributions:

$$L(D)G(x) = \delta(x)\,, \qquad G \in \mathcal{S}'(\mathbb{R}^N)\,.$$

Suppose that G is a temperate distribution that satisfies this equation. Then by taking the Fourier transform of both sides,

$$\begin{aligned}\mathcal{F}[L(D)G](k) &= \sum_{\alpha<p} a_\alpha \mathcal{F}[D^\alpha G](k) = \sum_{\alpha<p} a_\alpha (-ik)^\alpha \mathcal{F}[G](k)\\ &= L(-ik)\mathcal{F}[G](k)\end{aligned}$$

one infers that its Fourier transform satisfies the algebraic equation:

$$L(-ik)\mathcal{F}[G](k) = 1$$

Conversely, suppose the Fourier transform of G is a temperate distribution that satisfies this algebraic equation. Then by taking the inverse Fourier transform of both sides of the algebraic equation, it is concluded that G is a fundamental solution for L in $\mathcal{S}'$. Thus, the following proposition holds.

Proposition **39.1**. *In order for a temperate distribution G to be a fundamental solution for a linear differential operator $L(D)$ with constant coefficients, it is necessary and sufficient that its Fourier transform satisfies the algebraic equation $L(-ik)\mathcal{F}[G](k) = 1$.*

If the polynomial $L(-ik)$ has no real roots, then a particular solution to this distributional equation is a regular temperate distribution

$$\mathcal{F}[G](k) = \frac{1}{L(-ik)}\,, \quad L(-ik) \neq 0\,, \quad k \in \mathbb{R}^N$$

If the polynomial $L(-ik)$ vanishes at some $k \in \mathbb{R}^N$, then the reciprocal of $L(-ik)$ is generally a singular function (not locally integrable) and, hence, a solution, if it exists in $\mathcal{S}'$, must be given by a distributional extension of the reciprocal to the set of real zeros of $L(-ik)$. Any such extension will be denoted as

$$\mathcal{F}[G](k) = \mathcal{R}\,\frac{1}{L(-ik)} \in \mathcal{S}'\,.$$

Its characteristic property is that in the distributional sense

$$L(-ik)\mathcal{R}\frac{1}{L(-ik)} = 1$$

This implies that

$$\mathcal{R}\,\frac{1}{L(-ik)} = \frac{1}{L(-ik)}\,, \qquad k \in \mathbb{R}^N \setminus \mathcal{N}_L\,,$$

where $\mathcal{N}_L$ is the set of all real zeros of $L(-ik)$:

$$\mathcal{N}_L = \{k \in \mathbb{R}^N \,|\, L(-ik) = 0\}$$

The following theorem give an answer to a natural question about the existence of such an extension.

THEOREM **39.1**. (L. Hörmander)
Let $P(k)$ be a complex polynomial of $k \in \mathbb{R}^N$. Then the equation

$$P(k)f(k) = 1$$

always has a solution f in the space of temperate distributions.

By this theorem, a fundamental solution always exists in the space of temperate distributions and is given by

$$G(x) = \mathcal{F}^{-1}\Big[\mathcal{R}\frac{1}{L(-ik)}\Big](x)$$

39.1. The inhomogeneous problem. Recall that a general non-homogeneous problem

$$L(D)u(x) = f(x)\,, \quad f \in \mathcal{D}'$$

has a solution in $\mathcal{D}'$, given by $u = G * f$, provided the convolution $G * f$ exists in $\mathcal{D}'$. A solution is unique in the class of distributions for which the convolution with G exists in $\mathcal{D}'$.

Suppose that the inhomogeneity f is a temperate distribution. Then a particular solution to the problem is also a temperate distribution u whose Fourier transform is obtained by dividing a temperate distribution $\mathcal{F}[f]$ by a polynomial:

$$\mathcal{F}[u](k) = \mathcal{R}\frac{1}{L(-ik)}\mathcal{F}[f](k)\,.$$

If the polynomial $L(-ik)$ has no real zeros, then its reciprocal is a smooth temperate function and its product with a temperate distribution $\mathcal{F}[f]$ is a temperate distribution. If $\mathcal{N}_L$ is not empty, then its reciprocal must be extended to $\mathcal{N}_L$ so that the extension is a temperate distribution. Then the inverse Fourier transform of the product gives a particular solution:

$$u(x) = \mathcal{F}^{-1}\Big[\mathcal{R}\frac{1}{L(-ik)}\mathcal{F}[f]\Big](x)\,.$$

39.2. Basic methods to construct a distributional extension. There is no universal method for constructing a distributional extension of the reciprocal of a polynomial. However, there are two basic approaches that can be used to solve this problem.

39.2.1. Partial fraction decomposition. Suppose $k \in \mathbb{R}$. Then the roots of $L(-ik)$ are isolated points, and the reciprocal of $L(-ik)$ can be decomposed into a sum of partial fractions. Each term of the sum is singular at one root and, hence, can be extended by using a suitable principal value distribution. Suppose $k = a$ is a real root of multiplicity $n+1$, $n \geq 0$. Then the partial fraction expansion has the following terms

$$\frac{1}{L(-ik)} = \frac{A_n(k)}{(k-a)^{n+1}} + \frac{A_{n-1}(k)}{(k-a)^n} + \cdots + \frac{A_0}{k-a} + \cdots$$

where $A_m(k)$ is a polynomial of degree $m = n, n-1, ..., 1, 0$. For $k \neq a$

$$\frac{d^n}{dk^n}\frac{1}{k-a} = (-1)^n \frac{n!}{(x-a)^{n+1}}$$

A multiplication of a temperate distribution by a polynomial produces a temperate distribution. So, put

$$\mathcal{R}\frac{1}{(k-a)^{n+1}} = \frac{(-1)^n}{n!}\frac{d^n}{dk^n}\mathcal{P}\frac{1}{k-a}$$

Since the principal value distribution $\mathcal{P}\frac{1}{k}$ is a temperate distribution, a shift of its argument also defines a temperate distribution, and any derivative of it is a temperate distribution. Since distributional derivatives coincide with the corresponding classical ones wherever the latter exist, the above rule defines a distributional extension of the singular function $(k-a)^{-n-1}$ to the singular point $k = a$ so that for any test function

$$\begin{aligned}\Big(\mathcal{R}\frac{1}{(k-a)^{n+1}}, \varphi\Big) &= \frac{1}{n!}\Big(\mathcal{P}\frac{1}{k-a}, \varphi^{(n)}\Big) = \frac{1}{n!}\, v.p. \int \frac{\varphi^{(n)}(k+a)}{k}\, dk \\ &= \frac{1}{n!}\lim_{a\to 0^+}\int_{|k|>a} \frac{\varphi^{(n)}(k+a)}{k}\, dk\,, \qquad \varphi \in \mathcal{S}\end{aligned}$$

Sokhotsky's distributions can also be used to obtain a similar extension:

$$\mathcal{R}\frac{1}{(k-a)^{n+1}} = \frac{(-1)^n}{n!}\frac{d^n}{dk^n}\frac{1}{k-a \pm i0}$$

It differs from the principal value extension only by a linear combination of $\delta(k-a)$ and its derivatives.

A disadvantage of this method is that it does not have a universal extension to higher dimensions. Roots of $L(-ik)$ may no longer be isolated. They can form hyper-surfaces of various dimensions not exceeding N in $\mathbb{R}^N$.

39.2.2. The $i\epsilon$ prescription method. The idea of this method is to find the distributional limit

$$\mathcal{R}\,\frac{1}{L(-ik)} = \lim_{\epsilon\to 0^+} \frac{1}{L(-ik) \pm i\epsilon}$$

if it exists. It can be applied in any dimensions. A necessary condition for this method to work is that $L(-ik) \pm i\epsilon$ has no real roots if $0 < \epsilon < a$ for some $a > 0$. In this case, its reciprocal is a regular temperate distribution and its limit can be investigated. This condition is always fulfilled in one particularly important case when $L(-ik)$ is real

$$\overline{L(-ik)} = L(-ik)$$

This is true for the Laplace, Helmholtz, or wave operators in any dimensions. For any temperate test function φ, the integral

$$\Big(\frac{1}{L(-ik) \pm i\epsilon}, \varphi\Big) = \int \frac{\varphi(k)}{L(-ik) \pm i\epsilon}\, d^N k = \int \frac{L(-ik) \mp i\epsilon}{|L(-ik)|^2 + \epsilon^2}\, \varphi(k)\, d^N k$$

exists for $\epsilon > 0$, and its limit can be investigated. If the limit exists, then owing to the continuity of the Fourier transform, the following limit exists and vice versa:

$$G_{\pm}(x) = \lim_{\epsilon\to 0} \mathcal{F}^{-1}\Big[\frac{1}{L(-ik) \pm i\epsilon}\Big](x)$$

So, the above equation defines a fundamental solution if the limit exists. The procedure will be illustrated with examples below.

39.3. Simple examples. Consider the problem

$$G''(x) - \omega^2 G(x) = \delta(x)\,, \qquad x \in \mathbb{R}$$

where $\omega > 0$. Then

$$\mathcal{F}[G](k) = -\mathcal{R}\frac{1}{k^2 + \omega^2} = -\frac{1}{k^2 + \omega^2}$$

because the denominator has no real zeros. The Fourier transform of a fundamental solution is integrable on $\mathbb{R}$. Therefore

$$\begin{aligned} G(x) &= -\mathcal{F}^{-1}\Big[\frac{1}{k^2+\omega^2}\Big](x) = -\frac{1}{2\pi}\mathcal{F}\Big[\frac{1}{(-k)^2+\omega^2}\Big](x) \\ &= -\frac{1}{2\pi}\int \frac{e^{ikx}}{k^2+\omega^2}\,dk = -\frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ikx}}{k^2+\omega^2}\,dk \\ &= -\frac{2\pi i}{2\pi}\left(\theta(x)\operatorname*{res}_{z=i\omega}\frac{e^{izx}}{z^2+\omega^2} - \theta(-x)\operatorname*{res}_{z=-i\omega}\frac{e^{izx}}{z^2+\omega^2}\right) = -\frac{e^{-\omega|x|}}{2\omega} \end{aligned}$$

where the integral was evaluated by the residue theorem. So, $G(x)$ is integrable on $\mathbb{R}$. Note that G is unique solution in $\mathcal{S}'$. A general solution to the homogeneous equation reads

$$G_0(x) = Ae^{-\omega x} + Be^{\omega x}$$

This function is not a regular temperate distribution, unless $A = B = 0$, and hence cannot be added to G to get another fundamental solution in $\mathcal{S}'$. The Fourier transform of G_0 does not exist and the homogeneous equation $(k^2 + \omega^2)\mathcal{F}[G_0] = 0$ has only the trivial solution. However, $G(x)+G_0(x)$ is a fundamental solution in $\mathcal{D}'$ for any choice of constants A and B. In particular, a fundamental solution found earlier by a different method reads

$$\mathcal{E}(x) = \theta(x)\,\frac{\sinh(\omega x)}{\omega} = G(x) + \frac{1}{2\omega}\,e^{\omega x} \;\in\; \mathcal{D}'$$

39.3.1. Harmonic oscillator. Consider the problem

$$G''(x) + \omega^2 G(x) = \delta(x)\,, \qquad x \in \mathbb{R}$$

where $\omega > 0$. Then

$$\mathcal{F}[G](k) = \mathcal{R}\frac{1}{\omega^2 - k^2}$$

which is a distributional extension of the singular function $(\omega^2 - k^2)^{-1}$ to singular points $k = \pm\omega$. The roots $k = \pm\omega$ are simple, and the partial fraction decomposition method gives the following extension

$$\mathcal{F}[G](k) = -\frac{1}{2\omega}\left(\mathcal{P}\frac{1}{k-\omega} - \mathcal{P}\frac{1}{k+\omega}\right)$$

By taking the inverse Fourier transform and using the property of the Fourier transform under a shift of the argument, a fundamental solution is obtained:

$$\begin{aligned} G(x) &= \mathcal{F}^{-1}[\mathcal{F}[G](k)](x) = \frac{1}{2\pi}\mathcal{F}[\mathcal{F}[G](-k)](x) \\ &= \frac{1}{4\pi\omega}\left(\mathcal{F}\Big[\mathcal{P}\frac{1}{k+\omega}\Big](x) - \mathcal{F}\Big[\mathcal{P}\frac{1}{k-\omega}\Big](x)\right) \\ &= \frac{1}{4\pi\omega}\Big(e^{-i\omega x} - e^{i\omega x}\Big)\mathcal{F}\Big[\mathcal{P}\frac{1}{k}\Big](x) \\ &= \frac{\sin(\omega x)}{2\omega}\,\mathrm{sign}\,(x) = \frac{\sin(\omega|x|)}{2\omega} = \frac{\sin(\omega x)}{2\omega x}\,|x| \end{aligned}$$

A general solution of the homogeneous equation reads

$$G_0(x) = A\sin(\omega x) + B\cos(\omega x)$$

which is a regular temperate distribution for any constants A and B. So, any fundamental solution in $\mathcal{S}'$ has the form

$$G(x) = \frac{\sin(\omega x)}{2\omega}\,\mathrm{sign}\,(x) + A\sin(\omega x) + B\cos(\omega x)$$

In particular, setting $A = \frac{1}{2\omega}$ and $B = 0$, the fundamental solution from the algebra $\mathcal{D}'_+$ is obtained:

$$G(x) = \frac{\sin(\omega x)}{\omega}\,\theta(x)$$

Let us illustrate the $i\epsilon$ prescription method. For example, put

$$\mathcal{F}[G_+](k) = \mathcal{R}\,\frac{1}{\omega^2 - k^2} = -\lim_{\epsilon\to 0^+}\frac{1}{k^2 - \omega^2 + i\epsilon}$$

For $0 < \epsilon < a$, the new roots are complex:

$$k_\pm = \pm\sqrt{\omega^2 - i\epsilon} = \pm\omega\sqrt{1 - i\epsilon/\omega^2} = \pm(\omega - i\xi) + O(\xi^2)\,, \qquad \xi = \frac{\epsilon}{2\omega}$$

Therefore using the partial fraction decomposition, the limit can be expressed via shifted Sokhotsky's distributions:

$$\begin{aligned} \mathcal{F}[G_+](k) &= -\lim_{\xi\to 0^+}\frac{1}{k_+ - k_-}\left(\frac{1}{k - k_+} - \frac{1}{k - k_-}\right) \\ &= \frac{1}{2\omega}\left(\frac{1}{k + \omega - i0} - \frac{1}{k - \omega + i0}\right) \end{aligned}$$

Using the Fourier transform of the step function, it is not difficult to infer that

$$\mathcal{F}^{-1}\Big[\frac{1}{k \pm i0}\Big](x) = \mp i\theta(\pm x)$$

It follows from the shift-of-argument property of the Fourier transform that

$$G_+(x) = \frac{i}{2\omega}\Big(e^{i\omega x}\theta(-x) + e^{-i\omega x}\theta(x)\Big) = \frac{i}{2\omega}\, e^{-i\omega|x|}$$

It is not difficult to see that this fundamental solution differ from the others obtained above only by an additive solution to the homogeneous equation. This can also be concluded from the Sokhotsky's equation. The Fourier transforms differ by a linear combination of shifted delta-functions, $\delta(k \pm \omega)$, whose inverse Fourier transforms are solutions to the homogeneous equation, $e^{\pm i\omega x}$.

39.4. Helmholtz operator. Consider the problem

$$(\Delta + \omega^2)G(x) = \delta(x)\,, \qquad G \in \mathcal{S}'(\mathbb{R}^N)$$

By taking the Fourier transform

$$(\omega^2 - |k|^2)\mathcal{F}[G](k) = 1$$

Real zeros of the polynomial $\omega^2 - |k|^2$ form a sphere $|k| = \omega$ in $\mathbb{R}^N$. Let us use the $i\epsilon$ prescription to solve this equation

$$\mathcal{F}[G](k) = \mathcal{R}\,\frac{1}{\omega^2 - |k|^2} = -\lim_{\epsilon\to 0^+} \frac{1}{|k|^2 - \omega^2 + i\epsilon}$$

and investigate if the limit exists in $\mathcal{S}'$.

The technical details are given for the case $N = 3$. Other dimensions can be studied similarly. For any temperate test function $\varphi(x)$, one has

$$\begin{aligned}
\Big(\mathcal{F}^{-1}\Big[\frac{1}{|k|^2 - \omega^2 + i\epsilon}\Big], \varphi\Big) &= \frac{1}{(2\pi)^3}\Big(\frac{1}{|k|^2 - \omega^2 + i\epsilon}, \mathcal{F}[\varphi](k)\Big) \\
&= \frac{1}{(2\pi)^3}\int \frac{1}{|k|^2 - \omega^2 + i\epsilon}\int e^{i(k,x)}\varphi(x)\, d^3x d^3k \\
&= \frac{1}{(2\pi)^3}\lim_{R\to\infty}\int_{|k|<R} \frac{1}{|k|^2 - \omega^2 + i\epsilon}\int e^{i(k,x)}\varphi(x)\, d^3x d^3k \\
&= \frac{1}{(2\pi)^3}\lim_{R\to\infty}\int \varphi(x)\int_{|k|<R} \frac{e^{i(k,x)}}{|k|^2 - \omega^2 + i\epsilon}\, d^3k\, d^3x
\end{aligned}$$

where the order of integration has been changed by Fubini's theorem as the integrand is integrable on $B_R \times \mathbb{R}^3$ for any $\epsilon > 0$. The integral over the Fourier variable k is evaluated in spherical coordinates such that the zenith angle ϕ is counted from the vector $x \in \mathbb{R}^3$ so that $(k, x) = |x| r \cos(\phi)$ where $r = |k|$ and $d^3k = r^2 \sin(\phi) dr d\phi d\theta$ with θ

being the polar angle in the plane perpendicular to x:

$$\begin{aligned}\int_{|k|<R} \frac{e^{i(k,x)}}{|k|^2-\omega^2+i\epsilon}\,d^3k &= 2\pi\int_0^R\int_0^\pi \frac{e^{i|x|r\cos(\phi)}}{r^2-\omega^2+i\epsilon}\,r^2\sin(\phi)\,d\phi\,dr\\ &= \frac{4\pi}{|x|}\int_0^R \frac{r\sin(|x|r)}{r^2-\omega^2+i\epsilon}\,dr\\ &= \frac{2\pi}{|x|}\int_{-R}^R \frac{r\sin(|x|r)}{r^2-\omega^2+i\epsilon}\,dr\\ &= \frac{2\pi}{|x|}\mathrm{Im}\int_{-R}^R \frac{re^{i|x|r}}{r^2-\omega^2+i\epsilon}\,dr\\ &= \frac{2\pi}{|x|}\,\mathrm{Im}\,I(\epsilon,R,x)\end{aligned}$$

The function

$$f(z)=\frac{ze^{i|x|z}}{z^2-\omega^2+i\epsilon}\,,\quad z\in\mathbb{C}\,,$$

has two simple poles

$$z=z_\pm=\pm\sqrt{\omega^2-i\epsilon}=\pm(\omega-i\xi)+O(\xi^2)\,,\qquad \xi=\frac{\epsilon}{2\omega}$$

and is analytic otherwise. Since $\omega>0$ and $\epsilon>0$ can be taken arbitrary small, the pole $z_+=\omega-i\xi+O(\xi^2)$ lies below the real axis, while the pole $z_-=-\omega+i\xi+O(\xi^2)$ lies above it for all $0<\epsilon<a$ and some $a>0$. Take a closed contour C that consists of the line segment $|\mathrm{Re}\,z|\le R$ and the circular arc $|z|=R$, $\mathrm{Im}\,z\ge 0$, denoted by C_R. The contour is positively oriented. Then by the residue theorem

$$\oint_C f(z)\,dz=2\pi i\,\operatorname*{res}_{z=z_-} f(z)=\frac{2\pi i z_-}{z_- - z_+}\,e^{i|x|z_-}=\pi i\,e^{i|x|z_-}$$

Therefore

$$I(\epsilon,R,x)=-\int_{C_R} f(z)dz+\pi i\,e^{i|x|z_-}$$

Let us show that the integral over the arc C_R vanishes in the limit $R\to\infty$. The integrand has the following estimate. If $z=Re^{it}$ on C_R, then $dz=Rie^{it}dt$ and

$$|f(Re^{it})Rie^{it}|=\frac{R^2e^{-R|x|\sin(t)}}{|R^2e^{2it}-\omega^2+i\epsilon|}\le\frac{R^2}{R^2-|\omega^2-i\epsilon|}\le 2$$

for all $R^2>2|\omega^2-i\epsilon|$. A constant function is integrable on $(0,\pi)$ and therefore by the Lebesgue dominated convergence theorem

$$\lim_{R\to\infty}\int_0^\pi f(Re^{it})Rie^{it}dt=\int_0^\pi \lim_{R\to\infty} f(Re^{it})Rie^{it}dt=0\,,\quad x\ne 0$$

because $e^{-R|x|\sin(t)} \to 0$ as $R \to \infty$ for $0 < t < \pi$ and $x \neq 0$. Thus,

$$\lim_{\epsilon\to 0^+} \lim_{R\to\infty} I(\epsilon, R, x) = \pi i\, e^{-i|x|\omega}\,, \quad x \neq 0\,,$$

that is, almost everywhere in $\mathbb{R}^3$. Let us find an upper bound for $|I(\epsilon, R, x)|$ that is independent of R and ϵ. If the product of the bound and $|\varphi(x)|/|x|$ is integrable on $\mathbb{R}^3$, then the limits and the integration over x can be swapped by the Lebesgue dominated convergence theorem. Using the above estimate of the integrand in the integral over C_R

$$|I(\epsilon, R, x)| \leq 2\pi + \pi|e^{i|x|z_-}| \leq 3\pi$$

because the pole z_- lies above the real axis and, hence, $|e^{i|x|z_-}| \leq 1$. By the Lebesgue dominated convergence theorem

$$\begin{aligned}\Big(G(x), \varphi(x)\Big) &= -\frac{1}{(2\pi)^3} \lim_{\epsilon\to 0^+} \lim_{R\to\infty} \int \varphi(x) \frac{2\pi}{|x|} \operatorname{Im} I(\epsilon, R, x)\, d^3x \\ &= -\frac{1}{4\pi} \int \varphi(x)\, \frac{\cos(\omega|x|)}{|x|}\, d^3x\end{aligned}$$

Thus, a fundamental solution is given by

$$G(x) = -\frac{\cos(\omega|x|)}{4\pi|x|} = \frac{1}{2}\Big(\mathcal{E}_+(x) + \mathcal{E}_-(x)\Big)\,, \quad \mathcal{E}_\pm(x) = -\frac{e^{\pm i\omega|x|}}{4\pi|x|}$$

where $\mathcal{E}_\pm$ are fundamental solutions satisfying the outgoing or incoming radiation conditions. It is not difficult to verify that the $-i\epsilon$ prescription leads to the same answer, that is, $G_+ = G_- = G$ for the Helmholtz operator in $\mathbb{R}^3$.

39.5. Wave operators. Let us find a fundamental solution for the wave operator:

$$L(D) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_x\,, \qquad t \in \mathbb{R}\,, \quad x \in \mathbb{R}^N$$

Put $x_0 = ct$ and let $k_0 \in \mathbb{R}$ be the Fourier variable for x_0. Then a fundamental solution is given by

$$G(x) = \mathcal{F}^{-1}[\mathcal{F}[G]](x) = -\mathcal{F}^{-1}\Big[\mathcal{R}\, \frac{1}{k_0^2 - |k|^2}\Big](x)$$

There are four $i\epsilon$ prescriptions used to construct a distributional extension of $(k_0^2 - |k|^2)^{-1}$ to singular points which form an $N-$dimensional double cone in $\mathbb{R}^{N+1}$. For a given $k \neq 0$, the function

$$\frac{1}{k_0^2 - |k|^2} = \frac{1}{(k_0 - |k|)(k_0 + |k|)}$$

has two poles $k_0 = \pm|k|$ and is analytic otherwise in the complex k_0 plane of k_0. Each of the poles can be shifted up or down using the $i\epsilon$ prescription, thus producing four possibly different fundamental solutions whose Fourier transform are distributional limits

$$\mathcal{F}[G](k) = -\lim_{\epsilon\to 0^+} \frac{1}{(k_0 - |k| \pm i\epsilon)(k_0 + |k| \pm i\epsilon)}$$

These four possibilities corresponds to the following Green's functions for the wave operator:

$$\begin{array}{lll} k_0 = \pm|k| - i\epsilon\,, & G = G_R^{(N)} & (\text{causal retarded}) \\ k_0 = \pm|k| + i\epsilon\,, & G = G_A^{(N)} & (\text{causal advanced}) \\ k_0 = \pm(|k| - i\epsilon)\,, & G = G_F^{(N)} & (\text{Feynman propagator}) \\ k_0 = \pm(|k| + i\epsilon)\,, & G = G_T^{(N)} & (\text{anti} - \text{time ordered}) \end{array}$$

The latter two are used in quantum field theory for calculating Feynman diagrams. They can also be obtained by the standard $i\epsilon$ prescription

$$\mathcal{F}[G](k) = -\lim_{\epsilon\to 0^+} \frac{1}{k_0^2 - |k|^2 \pm i\epsilon}$$

where $-i\epsilon$ corresponds to the Feynman propagator. The Causal retarded Green's function is used for solving the Cauchy problem for the wave equation.

Let us find $G_R^{(3)}(x_0, x)$. Put $z_1 = |k| - i\epsilon$ and $z_2 = -|k| - i\epsilon$

39.6. Ordinary linear differential operator with constant coefficients. For ordinary differential operators, there exists a simpler method of finding a fundamental solution. This method will also be used to find causal Green's functions for partial differential operators that are used to solve a Cauchy problem.

Let $x \in \mathbb{R}$ and

$$L = \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}\frac{d}{dx} + a_n$$

where a_k are constants, $k = 0, 1, ..., n$. A general solution to the homogeneous equation

$$LZ(x) = 0\,, \qquad Z \in C^\infty$$

is a smooth function. It can be found by the method of undermined coefficients. Let $Z(x) = e^{\lambda x}$, then λ is a root of the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_0 = 0$$

For each root λ (real or complex) of multiplicity m, the equation has m linearly independent (smooth) solutions $x^k e^{\lambda x}$, $k = 0, 1, ..., m$. Since a general solution is a linear combination of all linearly independent solutions, it is concluded that any solution to the homogeneous equation is from C^∞. There are exactly n linearly independent solutions and, hence, *the initial value problem*

$$LZ(x) = 0\,, \quad Z^{(k)}(0) = c_k\,, \quad k = 0, 1, ..., n-1$$

has a unique solution. The coefficients in the linear combination of n linearly independent solutions are uniquely determined by the initial conditions.

Proposition **39.2**. *Let $Z(x)$ be the solution to the initial value problem*

$$LZ(x) = 0\,, \quad Z(0) = Z'(0) = \cdots = Z^{(n-2)}(0) = 0\,, \quad Z^{(n-1)}(0) = 1$$

Then

$$\mathcal{E}(x) = \theta(x)Z(x)\,,$$

where θ is the step function, is a fundamental solution for the operator L, that is,

$$L\mathcal{E}(x) = \delta(x)\,.$$

Proof. The assertion is proved by the Leibniz rule for distributions

$$\begin{aligned}(\theta(x)Z(x))' &= Z(x)\delta(x) + \theta(x)Z'(x) = Z(0)\delta(x) + \theta(x)Z'(x)\\ &= \theta(x)Z'(x)\end{aligned}$$

because $Z(0) = 0$. Similarly,

$$(\theta(x)Z(x))'' = (\theta(x)Z'(x))' = \theta(x)Z''(x)$$

because $Z'(0) = 0$. So that

$$(\theta(x)Z(x))^{(k)} = \theta(x)Z^{(k)}(x)\,, \quad k = 0, 1, ..., n-1$$

and

$$(\theta(x)Z(x))^{(n)} = \delta(x)Z^{(n-1)}(x) + \theta(x)Z^{(n)}(x) = \delta(x) + \theta(x)Z^{(n)}(x)$$

because $Z^{(n-1)}(0) = 1$. Therefore

$$L(\theta Z) = \theta LZ + \delta = \delta$$

as required. □

39.6.1. Pendulum and electric circuits. Vibrations of a small amplitude of a mass on a spring, or small oscillations of a pendulum under an external force $g(t)$ are described by the equation

$$f''(t) + \omega^2 f(t) = g(t)$$

where the constant ω is called a frequency. If $g(t) = 0$, the amplitude of oscillations a function of time t is

$$f(t) = A\cos(\omega t) + B\sin(\omega t)$$

so that the period is $T = 2\pi/\omega$ and the frequency is equal to $2\pi/T = \omega$. If vibrations are also suppressed by a friction force, then the vibrations are described by the equation

$$f''(t) + 2\gamma f'(t) + \omega^2 f(t) = g(t)$$

where γ is a damping coefficients. In the so-called under-damped regime when $\nu^2 = \omega^2 - \gamma^2 > 0$, free oscillations ($g(t) = 0$) decay exponentially

$$f(t) = e^{-\gamma t}\Big(A\cos(\nu t) + B\sin(\nu t)\Big)$$

Recall from mechanics that the integral of a force acting on a system is the net change of the momentum

$$p(t_2) - p(t_1) = \int_{t_1}^{t_2} g(t)\,dt$$

Here units are such that the system has a unit mass. Let $t_1 = 0$ and $t_2 \to 0^+$, while $g(t)$ is such that the integral is equal to 1, that is, the system gets a finite push (momentum) in an infinitesimally small time. This situation can be modeled by $g(t) = \delta(t)$. So, a fundamental solution for a mechanical oscillator is a special solution that describes vibrations of the oscillator that gets a finite push in an arbitrary small time.

Similarly, consider a circuit that consists of a capacitor, inductor, and resistor connected consecutively to an external electric power source with voltage $U(t)$. If $I(t)$ is the electric current in the circuit, then by Ohm's law

$$L\frac{dI}{dt} + RI + \frac{Q(t)}{C} = U(t)$$

where L, R, and C are the inductance, resistance, and capacitance, respectively, and $Q(t)$ is the electric charge of the capacitor at a time t. Since $I(t) = Q'(t)$, it follows from differentiation of the above equation that

$$I''(t) + 2\gamma I'(t) + \omega^2 I(t) = g(t)$$

where $g(t) = U'(t)/L$, $\omega = 1/\sqrt{LC}$, and $\gamma = \frac{1}{2}(R/L)$. If the voltage $U(t)$ was constant for $t < 0$, then at $t = 0$ it is suddenly changed to a different constant value, then its derivative is proportional to a delta-function (so is $g(t)$). So, a fundamental solution in this case describes an electric current in a basic electric circuit caused by a sudden jump in the external voltage.

39.6.2. The initial value problem. Let $D = d/dt$ and t is a physical time. Suppose that the external force $g(t)$ is a distribution $g(t)$ with support in $[0, \infty)$. Then the convolution of the fundamental solution $\mathcal{E}(t) = \theta(t)Z(t)$ and $g(t)$ always exists in the algebra $\mathcal{D}'_+$:

$$L(D)u(t) = g(t) \quad \Rightarrow \quad u(t) = (\mathcal{E} * g)(t) \in \mathcal{D}'_+ \,.$$

According to the previous section, this distributional solution is unique in the subspace of all distributions that have the convolution with $\mathcal{E}$. Indeed, any solution to the homogeneous equation is a linear combination of $e^{\lambda t}$, for some complex λ, but the convolution of $\mathcal{E}$ with $e^{\lambda t}$ does not exist because the integral

$$\int \mathcal{E}(t-\tau)e^{\lambda\tau}d\tau = \int_{-\infty}^{t} Z(t-\tau)e^{\lambda\tau}d\tau$$

does not exist for any complex λ. Therefore, any non-trivial solution to the homogeneous equation does not belong to the class $\mathcal{D}'_{\mathcal{E}}$, and, hence, the convolution $\mathcal{E} * g$ is the unique distributional solution in the class $\mathcal{D}'_{\mathcal{E}}$.

Its physical significance can be understood if the external force is a regular function. In particular, if $g(t)$ is a locally integrable, then

$$(\mathcal{E} * g)(t) = \theta(t) \int_0^t Z(t-\tau)g(\tau)\,d\tau$$

If in addition $g(t)$ is bounded, then the convolution is a continuous function for $t > 0$, and

$$\lim_{t\to 0^+} (\mathcal{E} * g)(t) = 0 \,.$$

In other words, the solution satisfies the zero initial condition at $t = 0$. If $g(t)$ is continuous in $[0, \infty)$, then the convolution is continuously differentiable function for $t > 0$ so that

$$\frac{d}{dt}(\mathcal{E} * g)(t) = \int_0^t Z'(t-\tau)g(\tau)\,d\tau \,, \quad t > 0 \,,$$

from which it follows, by induction, that

$$\frac{d^k}{dt^k}(\mathcal{E} * g)(t) = \int_0^t Z^{(k)}(t-\tau)g(\tau)\,d\tau\,, \quad t>0\,, \quad k=1,2,...,n-1$$

Therefore,

$$\lim_{t\to 0^+}\frac{d^k}{dt^k}(\mathcal{E} * g)(t) = 0\,, \quad k=0,1,...,n-1$$

Thus, the convolution $u = \mathcal{E} \star g$ for a continuous g is the classical solution to the initial value problem

$$L(D)u(t) = g(t)\,,\ t>0\,, \quad u^{(k)}(0) = 0\,, \quad k=0,1,...,n-1\,.$$

which is unique. A general solution is obtained by adding a general solution to the homogeneous equation to the constructed convolution. The freedom in choosing a solution to the homogeneous equation can be used to satisfy general (non-zero) initial conditions.

39.7. Exercises.

1. Find a fundamental solution for each of the following differential operators in one real variable x:

(i) $L = \dfrac{d^2}{dx^2}$

(ii) $L = \dfrac{d^2}{dx^2} + \omega^2$

(iii) $L = \dfrac{d^2}{dx^2} - \omega^2$

(iv) $L = \dfrac{d^2}{dx^2} + 2\gamma\,\dfrac{d}{dx}$

(v) $L = \dfrac{d^2}{dx^2} + 2\gamma\,\dfrac{d}{dx} + \omega^2$

2. Use the fundamental solution from $\mathcal{D}'_+$ to find an integral representation of the solution to the initial value problem

$$u''(t) + 2\gamma u'(t) + \nu^2 u(t) = \omega_a'(t-a)\,, \quad u(0) = u'(0) = 0$$

where ω_a is the hat function and $\nu^2 > \gamma^2$. Use the properties of the hat function and the distributional convolution to find the distributional limit of the solution as $a \to 0^+$.

3. Let ω_a be a regularization of $\delta(x)$ with support in a ball $|x| \le a$. Find the multipole expansion of the solution $u_+(x) = \mathcal{E}^+ * \omega_a$ to the Helmholtz equation in $\mathbb{R}^3$ in a region where $a/|x|$ is small by analogy with the multipole expansion of a solution to the Poisson equation. Estimate a relative uncertainty of the fundamental solution $\mathcal{E}^+(x)$ relative to the classical solution $u_+(x)$ in terms of $a/|x|$.

4. Calculate the retarded and advanced Green's function for the 4D wave operator.

5. Calculate the Feynman propagator for the 4D wave operator.

40. The Cauchy problem

40.1. The initial value problem for ODE. Consider the initial value problem for a pendulum

$$
\begin{aligned}
&Lu(t) = u''(t) + \omega^2 u(t) = f(t)\,, \quad t > 0\,, \\
&u\Big|_{t=0} = \lim_{t\to 0^+} u(t) = u_0\,, \quad u'\Big|_{t=0} = \lim_{t\to 0^+} u'(t) = u_1\,,
\end{aligned}
$$

Clearly a classical solution to this problem must be from class $u \in C^2(t > 0) \cap C^1(t \geq 0)$, and the function f should be at least continuous for $t \geq 0$. Suppose $u(t)$ is a classical solution. Consider regular distributions from $\mathcal{D}'_+$

$$
v(t) = \theta(t)u(t)\,, \quad g(t) = \theta(t)f(t)
$$

One can say continuous functions $u(t)$ and $f(t)$ has been extended to $t < 0$ by zero. Let us calculate the second derivative of v. For any test function

$$
\begin{aligned}
(v'', \varphi) = (v, \varphi'') &= \int_0^\infty u(t)\varphi''(t)\,dt = \lim_{a\to 0^+} \int_a^\infty u(t)\varphi(t)\,dt \\
&= \lim_{a\to 0^+} \left(u(t)\varphi'(t)\Big|_a^\infty - u'(t)\varphi(t)\Big|_a^\infty + \int_a^\infty u''(t)\varphi(t)\,dt \right) \\
&= -u_0\varphi'(0) + u_1\varphi(0) + \lim_{a\to 0^+} \int_a^\infty [f(t) - \omega^2 u(t)]\varphi(t)\,dt \\
&= \Big(u_1\delta(t) + u_0\delta'(t), \varphi\Big) + \Big(g(t) - \omega^2 v(t), \varphi\Big)
\end{aligned}
$$

This shows that the distribution $v(t) \in \mathcal{D}'_+$ satisfies the equation

$$
Lv(t) = v''(t) + \omega^2 v(t) = g(t) + u_1\delta(t) + u_0\delta'(t)
$$

On the other hand, the equation

$$
v''(t) + \omega^2 v(t) = h(t)
$$

has a unique solution in the algebra $\mathcal{D}'_+$ for any distribution $h \in \mathcal{D}'_+$. The solution is given by the convolution

$$
v(t) = (\mathcal{E} * h)(t)\,, \quad \mathcal{E}(t) = \theta(t)\,\frac{\sin(\omega t)}{\omega} \in \mathcal{D}'_+
$$

Since any classical solution to the initial value problem is also a distributional solution, it is concluded that the classical initial value problem has a unique solution. It can be obtained from the distributional solution if $f \in C^0(t \geq 0)$:

$$\begin{aligned} v(t) &= (\mathcal{E} * g)(t) + u_1 \mathcal{E}(t) + u_0 \mathcal{E}'(t) \\ &= \frac{\theta(t)}{\omega} \left(\int_0^t \sin(\omega(t-\tau)) f(\tau)\, d\tau + u_1 \sin(\omega t) + \omega u_0 \cos(\omega t) \right) \end{aligned}$$

Therefore

$$u(t) = \int_0^t \sin(\omega(t-\tau)) f(\tau)\, d\tau + u_1 \frac{\sin(\omega t)}{\omega} + u_0 \cos(\omega t)\,, \quad t > 0\,.$$

This method for solving the initial value problem can be extended to any linear ordinary differential equation with constant coefficients:

$$\begin{aligned} &Lu(t) = u^{(n)}(t) + a_1 u^{(n-1)}(t) + \cdots + a_n u(t) = f(t)\,, \quad t > 0 \\ &u^{(k)}\Big|_{t=0} = \lim_{t \to 0^+} u^{(k)}(t) = u_k\,, \quad k = 0, 1, ..., n-1\,. \end{aligned}$$

Consider the associated distributional differential equation in the algebra $\mathcal{D}'_+$

$$Lv(t) = \theta(t) f(t) + \sum_{k=0}^{n-1} c_k \delta^{(k)}(t)$$

where the coefficients c_k are given by

$$\begin{aligned} c_{n-1} &= u_0 \\ c_{n-2} &= a_1 u_0 + u_1 \\ c_{n-3} &= a_2 u_0 + a_1 u_1 + u_2 \\ &\cdots \\ c_0 &= a_{n-1} u_0 + \cdots + a_1 u_{n-2} + u_{n-1} \end{aligned}$$

Let $Z(t)$ be the solution to the initial value problem

$$LZ(t) = 0\,, \quad Z^{(k)}(0) = 0\,,\ k = 0, 1, ..., n-2\,,\ Z^{(n-1)}(0) = 1\,.$$

Then $\mathcal{E}(t) = \theta(t) Z(t)$ is a fundamental solution for L in $\mathcal{D}'_+$, and the distributional solution is unique in $\mathcal{D}'_+$ and given by

$$\begin{aligned} v(t) &= \mathcal{E}(t) * \Big(\theta(t) f(t) + \sum_{k=0}^{n-1} c_k \delta^{(k)}(t) \Big) = \theta(t) u(t) \\ u(t) &= \int_0^t Z(t-\tau) f(\tau)\, d\tau + \sum_{k=0}^{n-1} c_k Z^{(k)}(t) \end{aligned}$$

40.2. The Cauchy problem for PDEs. Let $x \in \mathbb{R}^N$ and t be a real variable. Suppose that a function $u(x,t)$ describes an evolution of some quantity distributed in space $\mathbb{R}^N$. The variable t is an evolution parameter, like a physical time. The evolution is governed by a partial differential equation, and a classical initial value problem is to find a solution $u(x,t)$ for $t > 0$ under some conditions at the initial moment of time $t = 0$. So, the initial data are formulated on a hyperplane $t = 0$ in space-time $\mathbb{R}^{N+1}$. A problem like this is known as a Cauchy problem in the theory of PDEs.

The first-order Cauchy problem. Let $L(D)$ be a linear differential operator in the variable x. Consider the following initial value (Cauchy) problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + L(D)u = f(x,t)\,, \quad t > 0\,, \quad x \in \mathbb{R}^N \\ &u\Big|_{t=0} = \lim_{t\to 0^+} u(x,t) = u_0(x)\,, \quad x \in \mathbb{R}^N\,. \end{aligned}$$

One has to find $u(x,t)$ that solves the equation in the open half-space $t > 0$ and satisfies the said initial condition at the hyperplane $t = 0$. The function u In both problems u must have sufficiently many continuous partial derivatives in x in order for $L(D)u$ to be continuous. It also must be continuously differentiable for $t > 0$ and have a continuous extension to $t = 0$:

$$u \in C^1(t > 0) \cap C^0 t \geq 0$$

This is a classical solution to the Cauchy problem. Of course, its existence depends on the smoothness of the inhomogeneity f. It is necessary that $f \in C^0(t > 0)$ in order for the problem to make sense. Finally, one has to show that the solution is unique if it exists. Clearly, the uniqueness is essential in mathematical modeling of physical processes.

Suppose that u_0 and f are such that a classical solution exists. Let $u(x,t)$ be a solution. Consider distributions from $\mathcal{D}'(\mathbb{R}^{N+1})$ defined by

$$v(x,t) = \theta(t)u(x)\,, \quad g(x,t) = \begin{cases} f(x,t) & , t > 0 \\ 0 & , t < 0 \end{cases}$$

The function $g(x,t)$ is an extension of a continuous function $f(x,t)$ to the half-space $t < 0$ by zero. The extension is assumed to be locally integrable in $\mathbb{R}^{N+1}$ so that g is a regular distribution. Note that in general, f is not required to have a continuous extension to the hyperplane $t = 0$ or be bounded near it. The distribution $v(x)$ is a smooth function for $t > 0$ that has a jump discontinuity at $t = 0$. Using the relation between the distributional and classical derivatives, one infers that

$$\frac{\partial v}{\partial t} = u_0(x)\cdot\delta(t) + \left\{\frac{\partial v(x,t)}{\partial t}\right\} = u_0(x)\cdot\delta(t) + \theta(t)\,\frac{\partial u}{\partial t}$$
$$L(D)v = \theta(t)L(D)u(x,t)$$

It follows from these relations that v a solution to the distributional problem

$$\frac{\partial v}{\partial t} + L(D)v = g(x,t) + u_0(x)\cdot\delta(t)\,, \qquad v(x,t) = 0\,,\ t<0\,.$$

It should be pointed out that, if $L(D)$ is a differential operator of order p, then the above equation is an *identity* holding for *any* function $v(x,t)\in C^p(t>0)\cap C^0(t\geq 0)$ that vanishes for $t<0$ and for which the combination $f=\frac{\partial v}{\partial t}+L(D)v$, $t>0$, is locally integrable (for example, $f\in C^0(t\geq 0)$). This follows from that for any test function φ

$$\begin{aligned}
&\Big(\frac{\partial v}{\partial t} + L(D)v, \varphi\Big) = \Big(v, -\frac{\partial \varphi}{\partial t} + L^*(D)\varphi\Big)\\
&= \int_0^\infty v(x,t)\Big(-\frac{\partial \varphi}{\partial t} + L^*(D)\varphi\Big)\,d^Nx\,dt\\
&= \lim_{a\to 0^+}\int_a^\infty\int v(x,t)\Big(-\frac{\partial \varphi}{\partial t} + L^*(D)\varphi\Big)\,d^Nx\,dt\\
&= \lim_{a\to 0^+}\left(\int_a^\infty\!\!\int\Big(\frac{\partial v}{\partial t} + L(D)v\Big)\varphi(x,t)\,d^Nx\,dt - \int v(x,a)\varphi(x,a)\,d^Nx\right)
\end{aligned}$$

Here L^* is the Hermitian conjugation of L, and the last equality follows from integration by parts. Note that $v(x,t)$ is smooth enough for this in the half-space $t\geq a>0$. This shows that the limit exists (and the stated identity holds if

$$\frac{\partial v}{\partial t} + L(D)v \ \in \mathcal{L}_{\rm loc}(t>0)$$

This is the reason for that the inhomogeneity $f(x,t)$ was required to be such that its extension $g(x,t)$ is locally integrable (e.g., $f\in C^0(t\geq 0)$).

Let $\mathcal{E}(x,t)$ be a fundamental solution for the operator $\partial/\partial t + L(D)$ that vanishes in the open half-space $t<0$ in the distributional sense:

$$\frac{\partial \mathcal{E}}{\partial t} + L(D)\mathcal{E} = \delta(x)\cdot\delta(t)\,, \quad \mathcal{E}(x,t) = 0\,, \quad t<0\,.$$

Then

$$v(x,t) = (\mathcal{E} * h)(x,t)\,, \quad h(x,t) = g(x,t) + u_0(x)\cdot\delta(t)$$

provided the convolution exist. Indeed, let us show that the distributional solution vanishes if $t < 0$. Let $\eta_n(x, y, t, \tau)$ be a unit sequence in $\mathbb{R}^{2N+2}$ and $\eta(s) = 1$ if $s > -a$, $\eta(s) = 0$ if $s < -b$ for some $0 < a < b$, and $\eta \in C^\infty$. Then by the definition of the convolution

$$(v, \varphi) = \lim_{n\to\infty} \Big(\mathcal{E}(x,t)\cdot h(y,\tau), \eta(\tau)\eta(t)\eta_n(x,y,t,\tau)\varphi(x+y,t+\tau)\Big)$$

because supports of $\mathcal{E}$ and h are in the half-space $t \geq 0$. If support of $\varphi(x,t)$ lies in the half-space $t < -\epsilon$ for some $\epsilon > 0$, then the product $\eta(\tau)\eta(t)\varphi(x+y,t+\tau)$ vanishes for small enough ϵ. This means that the distributional solution vanishes in the open half-space

$$v(x,t) = (\mathcal{E} * h)(x,t) = 0\,, \quad t < 0\,,$$

because ϵ is arbitrary.

Thus, every solution to the classical Cauchy problem is a solution to the *generalized Cauchy problem* which is to find a distribution $v(x,t)$ that satisfies the given equation and vanishes for $t < 0$:

$$\frac{\partial v}{\partial t} + L(D)v = h(x,t)\,, \qquad v(x,t) = 0\,,\ t < 0$$

for a given distribution $h(x,t)$ supported in the closed half-space $t \geq 0$. As shown earlier, the distributional solution $\mathcal{E} * h$ is unique in the class of distributions for which the convolution with $\mathcal{E}$ exists. In what follows, a fundamental solution that is supported in the half-space $t \geq 0$ will also be called a *causal Green's function*. The analysis shows that the following approach can be adopted for solving the classical Cauchy problem:

(i) Find the associated generalized Cauchy problem;
(ii) Find the causal Green's function for the differential operator in the problem;
(iii) Investigate conditions under which the generalized Cauchy problem has a solution, meaning that, find conditions on a distributional inhomogeneity under which its convolution with the causal Green's function exists;
(iv) Calculate the convolution if the distributional inhomogeneity has a special form for given inhomogeneity and initial data in the classical Cauchy problem, that is, find an integral representation of the convolution that is a regular distributional solution;
(v) Find conditions on the inhomogeneity and initial data so that the regular distributional solution is smooth enough to be a classical solution. Uniqueness of the classical solution follows from the uniqueness of the distributional solution.

40.3. Causal Green's function. If $L(D)$ is a linear operator with constant coefficients, one can try to find the causal Green's function by the Fourier method. In what follows, the causal Green's function will be denoted by G to distinguish it from a general fundamental solution. Consider the Fourier transform $\mathcal{F}_x[G](k,t)$ of the Green's function $G(x,t)$ in the variable x. It is a distribution defined by the rule

$$\Big(\mathcal{F}_x[G](k,t), \varphi(k,t)\Big) = \Big(G(x,t), \mathcal{F}_k[\varphi](x,t)\Big)$$

where

$$\mathcal{F}_k[\varphi](x,t) = \int e^{i(x,k)} \varphi(k,t)\, d^N k\,.$$

Of course, the Fourier transform exists only for temperate distributions, and this means that the Green's function is sought in a subspace $\mathcal{S}'$ of the space of all distributions $\mathcal{D}'$. So, it should be kept in mind that such a Green's function may not always exist in the space of temperate distributions. It follows from the rules of differentiation of the Fourier transforms of distributions that $\mathcal{F}_x[G](k,t)$ satisfies that the following generalized initial value problem for the first-order ordinary differential equation:

$$\Big(\frac{d}{dt} + L(ik)\Big)\mathcal{F}_x[G](k,t) = \delta(t)\,, \quad \mathcal{F}_x[G](k,t) = 0\,, \quad t < 0\,.$$

Its solution is found by the standard method introduced earlier

$$\mathcal{F}_x[G](k,t) = \theta(t)e^{-tL(ik)}$$

Therefore,

$$G(x,t) = \theta(t)\mathcal{F}_k^{-1}[e^{-tL(ik)}](x,t)\,.$$

The Fourier transform $\mathcal{F}_x[G](k,t)$ is a C^∞ function for $t > 0$ because $L(ik)$ is a polynomial. So, it must be a regular temperate distribution, which is true only if the real part of $L(ik)$ is non-negative because otherwise the solution would have exponential grows as $|k| \to \infty$ for $t > 0$ and, hence, cannot be a temperate distribution. In the latter case, the causal Green's function either does not exist or cannot be found by the Fourier transform in the variable x. Other methods should be invoked. For example, $L(D) = -\Delta$ so that $e^{-tL(ik)} = e^{t|k|^2}$, and the inverse Fourier transform of $e^{t|k|^2}$ does not exist.

40.4. The Cauchy problem for a transfer equation. Let us apply the developed approach to solve the Cauchy problem for the transfer or flow equation

$$\frac{1}{c}\frac{\partial u(x,t)}{\partial t} + (s,\nabla_x)u(x,t) + \alpha u(x,t) = f(x,t)\,, \quad t>0\,, \quad x\in\mathbb{R}^N$$
$$u\Big|_{t=0} = u_0(x)$$

where $s\in\mathbb{R}^N$, $|s|=1$, $c>0$, and $\alpha>0$. A classical solution must be from class $u\in C^1(t>0)\cap C^0(t\geq 0)$ if it exists (under some smoothness conditions on the inhomogeneity f and the initial data u_0).

Generalized Cauchy problem. Suppose that the problem has a solution. Let u be a classical solution. Consider the distributions from $\mathcal{D}'(\mathbb{R}^{N+1})$:

$$v(x,t) = \theta(t)u(x,t)\,, \quad g(x,t) = \theta(t)f(x,t)$$

The distribution $v(x,t)$ satisfies the equation

$$Lv \stackrel{\text{def}}{=} \Big[\frac{1}{c}\partial_t + (s,\nabla_x) + \alpha\Big]v(x,t) = g(x,t) + \frac{1}{c}u_0(x)\cdot\delta(t)\,.$$

Thus, every classical solution is a distributional solution to the generalized Cauchy problem which is to find a distribution from $\mathcal{D}'(\mathbb{R}^{N+1})$ that vanishes in the open half-space $t<0$ and satisfies the equation

$$Lv(x,t) = h(x,t)\,, \quad v(x,t)=0\,,\ t<0\,, \quad v(x,t)\in\mathcal{D}'(\mathbb{R}^{N+1})$$

for a given distribution $h(x,t)$ that with support in the half-space $t\geq 0$.

Fourier transform with respect to a selected variable. The Fourier transform of a temperate distribution of two variables $f(x,y)$ with respect to the variable x is defined by the rule

$$\Big(\mathcal{F}_x[f](k,y),\varphi(k,y)\Big) = \Big(f(x,y),\mathcal{F}_k[\varphi](x,y)\Big)$$

where $\varphi(k,y)$ is a temperate test function and its Fourier transform with respect to k reads

$$\mathcal{F}_k[\varphi](x,y) = \int e^{i(k,x)}\varphi(k,y)\,d^Nk\,, \quad y\in\mathbb{R}^M\,.$$

It is not difficult to verify that the Fourier transform of a test function of two variables with respect to any of the variables is a temperate test function from $\mathcal{S}(\mathbb{R}^{N+M})$.

Causal Green's function for the transfer operator. Let G be a fundamental solution for the transfer operator L that vanishes for $t < 0$:

$$LG(x,t) = \delta^N(x) \cdot \delta(t)\,, \quad G(x,t) = 0\,, \quad t < 0\,, \quad G \in \mathcal{D}'(\mathbb{R}^{N+1})\,,$$

Let us try to find the Green's function of L in the space of temperate distributions. If it exists, then it can be found by the Fourier method. By taking the Fourier transform of both sides of the equation with respect to the variable x, it is concluded that the Fourier transform $\mathcal{F}[G](k,t)$ satisfies the ordinary differential equation

$$\Big[\frac{1}{c}\frac{d}{dt} - i(s,k) + \alpha\Big]\mathcal{F}_x[G](k,t) = \delta(t)\,, \quad \mathcal{F}_x[G](k,t) = 0\,,\ t < 0$$

Its solution is found by solving the associated initial value problem for the homogeneous equation:

$$\mathcal{F}_x[G](k,t) = c\theta(t)e^{-\alpha ct + ic(s,k)t} \ \in \mathcal{S}'(\mathbb{R}^{N+1})$$

Since $\alpha > 0$, the solution is a smooth function in $t > 0$ that is bounded

$$|\mathcal{F}_x[G](k,t)| \le c$$

and, hence, it is a regular temperate distribution. To find the inverse Fourier transform, recall that

$$\mathcal{F}[\delta(x - x_0)](k) = e^{i(k,x_0)}\,, \quad \mathcal{F}^{-1}[e^{i(k,x_0)}] = \delta(x - x_0)\,.$$

Therefore for any test function $\varphi(x,t) \in \mathcal{S}$,

$$\begin{aligned}
(G,\varphi) &= \Big(\mathcal{F}_x[G_c], \mathcal{F}_x^{-1}[\varphi]\Big) = c\int_0^\infty\!\!\int e^{-\alpha t}e^{ic(s,k)t}\mathcal{F}_x^{-1}[\varphi](k,t)\,d^Nk dt \\
&= c\int_0^\infty e^{-\alpha t}\Big(e^{ic(s,k)t}, \mathcal{F}_x^{-1}[\varphi](k,t)\Big)\,dt \\
&= c\int_0^\infty e^{-\alpha t}\Big(\mathcal{F}_k^{-1}[e^{ic(s,k)t}](x), \varphi(x,t)\Big)\,dt \\
&= c\int_0^\infty e^{-\alpha t}\Big(\delta(x - cst), \varphi(x,t)\Big)\,dt \\
&= c\int_0^\infty e^{-\alpha ct}\varphi(cst,t)\,dt
\end{aligned}$$

So, G is a line-delta function supported on the half-line in $\mathbb{R}^{N+1}$ with parametric equations $x = cst$, $t = t$, $t > 0$. Since $\mathcal{S}' \subset \mathcal{D}'$, the causal Green's function exists in $\mathcal{D}'(\mathbb{R}^{N+1})$. For brevity, it can be written via the delta function on the line:

$$G(x,t) = c\theta(t)e^{-\alpha ct}\delta(x - cts) \ \in \ \mathcal{D}'(\mathbb{R}^{N+1})\,.$$

The convention is that the delta function acts first on the variable x of the test function, the result is a test function in the variable t on which the regular distribution $\theta(t)e^{-\alpha t}$ acts in the standard way:

$$(G,\varphi) = \Big(c\theta(t)e^{-\alpha t}, \Big(\delta(x-cst), \varphi(x,t)\Big)\Big) = c\int_0^\infty e^{-\alpha ct}\varphi(cst,t)\,dt\,.$$

Solution to the generalized Cauchy problem. A solution to the generalized Cauchy problem is given by the convolution

$$v(x,t) = (G*h)(x,t)$$

provided the convolution exists. The solution is unique in the class of distributions for which the convolution with G exists. Recall that $v(x,t)=0$ for $t<0$. Let us investigate the existence of the convolution $v = G*h$. Both distributions G and h are supported in the upper half-space, $t \geq 0$. It follows from the definition of the convolution and the explicit form of G that

$$\begin{aligned}(v,\varphi) &= \lim_{n\to\infty}\Big(h(y,\tau)\cdot G(y,\tau), \eta(\tau)\eta(t)\eta_n(x,y,t,\tau)\varphi(x+y,t+\tau)\,dt\Big)\\ &= \lim_{n\to\infty}\Big(h(y,\tau), \eta(\tau)\int_0^\infty e^{-\alpha ct}\eta_n(cst,y,t,\tau)\varphi(y+cst,t+\tau)\,dt\Big)\end{aligned}$$

The function $\psi(t,\tau,y) = \eta(\tau)\varphi(y+cst,t+\tau)$ is smooth and not zero only in a bounded range of t and τ because $t>0$ and $\tau>0$, and the support of φ is bounded in the time variable $|t+\tau|<R$. If the range of t is bounded, then the range of y is also bounded because $\varphi(y+cst,t+\tau)=0$ if $|y+cst|>R$ for some R. This means that $\psi(t,\tau,y)$ is a test function of three variables. The action of a regular distribution $\theta(t)e^{-\alpha ct}$ on $\psi(t,\tau,y)$ defines a test function in two variables τ and y by the consistency theorem for the direct product of distributions. Since $\eta_n\psi = \psi$ for all large enough n, the limit always exists for any distribution h supported in $t\geq 0$.

Thus, the solution to the generalized Cauchy problem always exists in $\mathcal{D}'(\mathbb{R}^{N+1})$ and is given by the rule

$$\begin{aligned}(v,\varphi) &= \Big(h(y,\tau), \eta(\tau)\int_0^\infty e^{-\alpha ct}\varphi(y+cst,t+\tau)\,dt\Big)\\ &= \int_0^\infty e^{-\alpha ct}\Big(h(y,\tau), \eta(\tau)\varphi(y+cst,t+\tau)\Big)\,dt\end{aligned}$$

where the latter equality holds by commutativity of the direct product $h(y,\tau)\cdot\theta(t)e^{-\alpha ct} = \theta(t)e^{-\alpha ct}\cdot h(y,\tau)$. The solution is unique in the class of distributions that vanish in the open half-space $t<0$.

Limit properties of the Green's function. For every $t>0$, $G(x,t)$ can be viewed as a distribution in the variable x. Its action on a test function $\varphi(x)$ is given by the rule

$$\Big(G(x,t),\varphi(x)\Big) = ce^{-\alpha ct}\varphi(cst)\,,\ \ t>0\,.$$

Therefore one can investigate the distributional limit of $G(x,t)$ as $t\to 0^+$ in $\mathcal{D}'(\mathbb{R}^N)$. Owing to the continuity of test functions

$$\lim_{t\to 0^+}\Big(G(x,t),\varphi(x)\Big) = c\varphi(0) = c(\delta(x),\varphi(x))$$

This means that

$$\lim_{t\to 0^+} G_c(x,t) = c\delta(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

Homogeneous Cauchy problem with distributional initial data. Consider a solution to the generalized Cauchy problem with $h(x,t)=c^{-1}u_0(x)\cdot\delta(t)$ where $u_0\in\mathcal{D}'(\mathbb{R}^N)$. Then the convolution $G_c * h$ is given by the rule

$$\begin{aligned}(G_c * h,\varphi) &= \int_0^\infty e^{-\alpha ct}\Big(u_0(y),\varphi(y+cst,t)\Big)\,dt\\ &= \int_0^\infty e^{-\alpha ct}\Big(u_0(y-cst),\varphi(y,t)\Big)\,dt\\ (G*h)(x,t) &= \theta(t)e^{-\alpha ct}\cdot u_0(x-cst)\in\mathcal{D}'(\mathbb{R}^{N+1})\end{aligned}$$

where the definition of a shifted distribution was used for $u_0(x)$. Note that for any $t>0$ the convolution defines a distribution in the variable x just like $G(x,t)$.

For any distribution $g(x)$, define the convolution $G(x,t)*g(x)$ in the variable x by the rule

$$\Big(G(x,t)*g(x),\phi(x)\Big) = \lim_{n\to\infty}\Big(G(x,t)\cdot g(y),\eta_n(x,y)\phi(x+y)\Big)\,,\quad t>0\,,$$

where η_n is a unit sequence in $\mathbb{R}^{2N}$ and $\phi\in\mathcal{D}(\mathbb{R}^N)$. Then the convolution $G*h$ in two variables x and t can also be interpreted as the convolution of $c^{-1}G(x,t)*u_0(x)$ in the variable x because

$$\begin{aligned}c^{-1}\Big(G(x,t)*u_0(x),\phi(x)\Big) &= \frac{1}{c}\lim_{n\to\infty}\Big(G(x,t)\cdot u_0(y),\eta_n(x,y)\phi(x+y)\Big)\\ &= \theta(t)e^{-\alpha ct}\Big(u_0(y),\phi(y+cst)\Big)\end{aligned}$$

for any unit sequence η_n in $\mathbb{R}^{2N}$ and any test function ϕ. So,

$$G*\Big(u_0(x)\cdot\delta(t)\Big) = G(x,t)*u_0(x) = \theta(t)e^{-\alpha ct}\cdot u_0(x-cst)\,.$$

By continuity of the functional u_0,

$$\lim_{t\to 0^+}\Big(G(x,t)*u_0(x),\phi(x)\Big) = c\lim_{t\to 0^+} e^{-\alpha ct}\Big(u_0(y),\phi(y+cst)\Big) = c\Big(u_0(y),\phi(y)\Big)$$

for any test function $\phi\in\mathcal{D}(\mathbb{R}^N)$. This means that

$$\lim_{t\to 0^+} G(x,t)*u_0(x) = cu_0(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

The analysis shows that the distribution $v(x,t)=\frac{1}{c}G(x,t)*u_0(x)$ solves a homogeneous Cauchy problem with *distributional initial data*:

$$Lv(x,t)=0\,,\quad v(x,t)=0\,,\quad t<0\,,$$
$$\lim_{t\to 0^+} v(x,t) = u_0(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

Regular distributional and classical solutions. Let $u_0(x)$ be a regular distribution. Put $h(x,t)=u_0(x)\cdot\delta(t)$. Then the solution to the generalized Cauchy problem is a regular distribution $v(x,t)$:

$$v(x,t)=\theta(t)e^{-\alpha ct}u_0(x-cst)\,.$$

The initial condition holds pointwise

$$\lim_{t\to 0^+} v(x,t)=u_0(x)\,,\quad x\in\mathbb{R}^N$$

if u_0 is a continuous function. Furthermore this solution is a classical one (continuously differentiable for $t>0$) if $u_0\in C^1$, that is, the initial data must be continuously differentiable function.

Let $h(x,t)=f(x,t)$ be a regular distribution that vanishes for $t<0$. Then the solution to the generalized Cauchy problem has the following integral representation

$$(v,\varphi) = \int_0^\infty\int f(y,\tau)\int_0^\infty e^{-\alpha ct}\varphi(y+cst,t+\tau)\,dt\,d^Ny\,d\tau$$
$$= \int_0^\infty\int_\tau^\infty e^{-\alpha c(t-\tau)}\int f(x-cs(t-\tau),\tau)\varphi(x,t)\,d^Nx\,dt\,d\tau$$

where $x=y+cst$ and the integration variable t was shifted by τ. Let us reverse the order of integration with respect to t and τ:

$$(v,\varphi) = \int_0^\infty\int\int_0^t e^{-\alpha c(t-\tau)}f(x-cs(t-\tau),\tau)\,\varphi(x,t)\,d\tau\,d^Nx\,dt$$
$$v(x,t) = \int_0^t e^{-\alpha c(t-\tau)}f\Big(x-cs(t-\tau),\tau\Big)\,d\tau\,,\quad t>0\,.$$

It is clear that the initial condition holds pointwise

$$\lim_{t\to 0^+} v(x,t) = 0\,, \quad x \in \mathbb{R}^N\,,$$

if f is continuous for $t \geq 0$. Furthermore $v(x,t)$ is a classical solution with the zero initial condition if $f(x,t)$ is from class $C^1(t \geq 0)$ (in order for the partial derivatives to be continuous). Thus, the classical solution is unique and given by the integral representation

$$u(x,t) = e^{-\alpha ct}u_0(x-cst) + \int_0^t e^{-\alpha c(t-\tau)} f\Big(x - cs(t-\tau), \tau\Big)\, d\tau\,.$$

if $u_0 \in C^1$ and $f \in C^1(t \geq 0)$.

Well-posedness of the Cauchy problem. A PDE problem is said to be *well-posed* if the following three conditions are fulfilled:

(i) The problem has a solution
(ii) The solution is unique
(iii) The solution changes continuously with initial data

Let us investigate if the Cauchy problem for the transfer equation is well-posed. It has been shown that a solution exists and is unique. Let $u(x,t)$ and $\tilde{u}(x,t)$ be two classical solutions for initial data u_0 and $\tilde{u}_0$ and inhomogeneities f and $\tilde{f}$, respectively. Suppose that

$$|u_0(x) - \tilde{u}_0(x)| \leq \varepsilon_0\,,$$
$$|f(x,t) - \tilde{f}(x,t)| \leq \varepsilon_1$$

for all x and t. Then for $0 \leq t \leq T$ and any x

$$\begin{aligned} |u(x,t) - \tilde{u}(x,t)| &\leq e^{-\alpha ct}\varepsilon_0 + \varepsilon_1 \int_0^t e^{-\alpha c(t-\tau)}\, d\tau \\ &\leq \varepsilon_0 + T\varepsilon_1 \end{aligned}$$

This shows that small variations of the initial data and inhomogeneity produce small variations of the solution. Therefore the Cauchy problem for the transfer equation is well posed.

40.5. Smoothness of a distribution in a particular variable. When analyzing the homogeneous Cauchy problem with general distributional initial data, it was noted that a distribution of several variables can also be viewed as a distribution of fewer variables, while the other variables are regarded as parameters. Let $f(x,y)$ be a distribution in two variables $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. By definition, *a distribution* $f(x,y)$

is said to be from class C^p in the variable y in a set $\Omega \subset \mathbb{R}^M$ if the function

$$g(y) = \Big(f(x,y), \varphi(x)\Big) \in C^p(\Omega)$$

is from class $C^p(\Omega)$ for any test function $\varphi \in \mathcal{D}(\mathbb{R}^N)$. In this case,

$$D^\alpha g(y) = \Big(D^\alpha_y f(x,y), \varphi(x)\Big), \quad \alpha \le p .$$

Indeed, $g(y)$ is a regular distribution. Since $g \in C^p$, its classical and distributional derivatives are equal, and for any test function $\phi(y)$,

$$\begin{aligned}(D^\alpha g, \phi) &= (-1)^\alpha (g, D^\alpha \phi) = (-1)^\alpha \Big(f(x,y), \varphi(x) D^\alpha \phi(y)\Big)\\ &= \Big(D^\alpha_y f(x,y), \varphi(x)\phi(y)\Big) = \Big(\Big(D^\alpha_y f(x,y), \varphi(x)\Big), \phi(y)\Big)\end{aligned}$$

because the product $\varphi(x)\phi(y)$ is a test function of two variables.

For example, the Green's function of the transfer operator is a distribution of two variables $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. For any test function $\varphi(x)$,

$$g(t) = \Big(G(x,t), \varphi(x)\Big) = \theta(t) e^{-\alpha c t} \varphi(cst) \in C^\infty(0,\infty)$$

So, $G(x,t)$ is from class $C^\infty(0,\infty)$ in the variable t. Furthermore, the function $g(t)$ has a continuous extension to $t = 0$ because

$$\lim_{t\to 0^+} g(t) = \varphi(0)$$

This means that $g \in C^0([0,\infty))$ and accordingly $G \in C^0([0,\infty))$ in the variable t. For any $t > 0$,

$$g'(t) = -c\alpha g(t) + c e^{-\alpha c t} (s, \nabla_x)\varphi(x)\Big|_{x=cst}$$

so that

$$\lim_{t\to 0^+} g'(t) = -c\alpha\varphi(0) + (s, \nabla_x)\varphi(x)\Big|_{x=0}$$

It is clear that $g \in C^\infty([0,\infty))$ because $\varphi \in C^\infty$ and, hence, the Green's function $G(x,t)$ is from class $C^\infty([0,\infty))$ in the variable t.

Homogeneous Cauchy problem with distributional initial data. Using the concept of smoothness of a distribution in a particular variable, the homogeneous Cauchy problem with distributional initial data is to find a distribution $v(x,t)$ from class $C^1(0,\infty) \cap C^0([0,\infty))$ in the variable t such that

$$\frac{\partial v(x,t)}{\partial t} + L(D_x) v(x,t) = 0\,, \quad t > 0\,, \quad v\Big|_{t=0} = u_0(x) \in \mathcal{D}'(\mathbb{R}^N)$$

In particular, this problem has a unique solution for the transfer equation given by the distribution

$$v(x,t) = G(x,t) * u_0(x) = \theta(t)e^{-\alpha ct}u_0(x - cst)\,.$$

The solution is from class $C^\infty([0,\infty))$ in the variable t because because for any test function $\varphi(x)$ and $t > 0$

$$\begin{aligned} g(t) &= \Big(v(x,t), \varphi(x)\Big) = e^{-\alpha ct}\Big(u_0(x), \varphi(x + cst)\Big)\,, \\ g'(t) &= -\alpha c g(t) + ce^{-\alpha ct}\Big(u_0(x), (s, \nabla_x)\varphi(x + cst)\Big)\,, \end{aligned}$$

so that

$$\begin{aligned} g(0) &= \lim_{t\to 0^+} g(t) = (u_0, \varphi)\,, \\ g'(0) &= \lim_{t\to 0^+} g(t) = -\alpha c g(0) + c(u_0, (s, \nabla)\varphi) \end{aligned}$$

The existence of $g^{(p)}(0)$ for $p > 1$ is established similarly.

41. The heat equation

Consider the Cauchy problem for a heat equation:

$$\frac{\partial u(x,t)}{\partial t} = a^2\Delta_x u(x,t) + f(x,t)\,, \quad t > 0\,, \quad x \in \mathbb{R}^N$$

$$u\Big|_{t=0} = u_0(x)$$

If $u(x,t)$ is a temperature at a point $x \in \mathbb{R}^N$ and time $t > 0$, then a solution to this problem describes a time evolution of the temperature field in space with an external heat energy source $t(x,t)$ if the initial temperature field was given by $u_0(x)$. The coefficient $a^2 > 0$ is proportional to a heat conductance. A solution is required to be from class $C^2(t>0) \cap C^0(t \geq 0)$ (under some smoothness condition on the inhomogeneity f and the initial data u_0).

41.1. Generalized Cauchy problem. Let $u(x,t)$ be a solution. Then the regular distribution $v(x,t) = \theta(t)u(x,t)$ satisfies the equation

$$Lv(x,t) \stackrel{\text{def}}{=} \frac{\partial v}{\partial t} - a^2\Delta_x v(x,t) = \delta(t)\cdot u_0(x) + g(x,t)$$

where $g(x,t)$ is an extension of $f(x,t)$ to the open half-space $t < 0$ by zero. It is assumed to be a regular distribution. So, the associated generalized Cauchy problem is to find a distribution $v(x,t)$ that vanishes for $t < 0$ and satisfies the equation

$$Lv(x,t) = h(x,t)\,, \quad v(x,t) = 0\,, \quad t < 0\,,$$

where a distribution $h \in \mathcal{D}'(\mathbb{R}^{N+1})$ is supported in the half-space $t \geq 0$.

41.2. Causal Green's function. A fundamental solution for L that vanishes in the half-space $t < 0$,

$$LG(x,t) = \delta(t)\cdot\delta(x)\,, \quad G(x,t) = 0\,,\ t < 0\,,$$

is sought as a temperate distribution. If such a temperate distribution exists, then its Fourier transform in the variable x satisfy the ordinary differential equation

$$\Big(\frac{d}{dt} + a^2|k|^2\Big)\mathcal{F}_x[G](k,t) = \delta(t)\,, \quad \mathcal{F}_x[G](k,t) = 0\,,\ t < 0\,.$$

Its solution reads

$$\mathcal{F}_x[G](k,t) = \theta(t)e^{-a^2|k|^2 t}$$

It is bounded $|\mathcal{F}_x[G](k,t)| \leq 1$ and, hence, define a regular temperate distribution.

The inverse Fourier transform is given by the Fourier transform of a Gaussian distribution:

$$\begin{aligned}
(G,\varphi) &= \Big(\mathcal{F}_x[G](k,t), \mathcal{F}_x^{-1}[\varphi](k,t)\Big) \\
&= \int_0^\infty \int e^{-a^2|k|^2t}\mathcal{F}_x^{-1}[\varphi](k,t)\, d^Nk\, dt \\
&= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \int e^{-a^2|k|^2t}\mathcal{F}_x^{-1}[\varphi](k,t)\, d^Nk\, dt \\
&= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \Big(e^{-a^2|k|^2t}, \mathcal{F}_x^{-1}[\varphi](k,t)\Big)\, dt \\
&= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \Big(\mathcal{F}_k^{-1}[e^{-a^2|k|^2t}](x,t), \varphi(x,t)\Big)\, dt \\
&= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \Big(\mathcal{F}_k^{-1}[e^{-a^2|k|^2t}](x,t), \varphi(x,t)\Big)\, dt \\
&= (2\pi)^{-N}\lim_{\epsilon\to 0^+}\int_\epsilon^\infty \Big(\mathcal{F}_k[e^{-a^2|k|^2t}](x,t), \varphi(x,t)\Big)\, dt \\
&= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \int \frac{e^{-\frac{|x|^2}{4a^2t}}}{(2a\sqrt{\pi t})^N}\,\varphi(x,t)\, d^Nx\, dt
\end{aligned}$$

The limit exists by the properties of the Fourier transform even though the singularity $t^{-N/2}$ is not locally integrable if $N \geq 2$. It is also possible to investigate the limit directly. Put $x = 2a\sqrt{t}y$ in the integral with respect to x, then

$$\begin{aligned}
\Big|\int \frac{e^{-\frac{|x|^2}{4a^2t}}}{(2a\sqrt{\pi t})^N}\,\varphi(x,t)\, d^Nx\Big| &\leq \pi^{-N/2}\int e^{-|y|^2}|\varphi(2a\sqrt{t}y,t)|\, d^Ny \\
&\leq M\pi^{-N/2}\int e^{-|y|^2} d^Ny = M
\end{aligned}$$

because $|\varphi(x,t)| \leq M$. A constant function is integrable with respect to t on any bounded interval. Since the support of φ is bounded (so that the integration with respect t is taken over a bounded interval), the limit $\epsilon \to 0^+$ exists. Thus,

$$G(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^N} e^{-\frac{|x|^2}{4a^2t}} \ \in\ \mathcal{D}'(\mathbb{R}^{N+1})$$

41.3. Solution to the generalized Cauchy problem. A solution to the generalized Cauchy problem $V = G * h$ will be called a *heat potential* (if the convolution exists). Suppose that the support of $h(x,t)$ lies in a half-cylinder,

$$\operatorname{supp} h \subset B_R \times [0, \infty)\,,$$

then the convolution $V = G * h$ exists. This follows from the definition of the convolution. Let $\eta_h(x,t)$ be a smooth function that takes unit value in a neighborhood of support of h. Then for any unit sequence η_n in $\mathbb{R}^{2N+2}$

$$(G * h, \varphi) = \lim_{n\to\infty} \Big(G(x,t) \cdot h(y,\tau), \eta_n(x,y,t,\tau)\psi(x,y,t,\tau) \Big)$$
$$\psi(x,y,t,\tau) = \eta(t)\eta_h(y,\tau)\varphi(x+y,t+\tau)$$

Let us show that the function ψ is compactly supported in $\mathbb{R}^{2N+2}$. Since $t \geq 0$ and $\tau \geq 0$ (owing to supports of G and h), ψ is not zero only a bounded region of spanned by t and τ because the test function φ is compactly supported and, hence, vanishes for $|t+\tau| > R$ for some R. Since $h(y,\tau) = 0$ for $|y| > R$ and φ vanishes if $|x+y| > R$ for some R, the function ψ has a bounded support and therefore is a test function of four variables. This implies that $\eta_n\psi = 0$ for all sufficiently large n, and the limit exists and is given by

$$\begin{aligned}(G * h, \varphi) &= \int_0^\infty \int \frac{e^{-\frac{|x|^2}{4a^2 t}}}{(2a\sqrt{\pi t})^N} \Big(h(y,\tau), \psi(y,x,t,\tau) \Big)\, d^N x\, dt \\ &= \Big(h(y,\tau), \int_0^\infty \int \frac{e^{-\frac{|x|^2}{4a^2 t}}}{(2a\sqrt{\pi t})^N} \varphi(x+y,t+\tau)\, d^N x\, dt \Big)\end{aligned}$$

The latter equality follows from commutativity of the direct product.

The action of h on ψ is a test function of two variables x and t. It has a bounded support. However, the latter is not necessary for integrals with respect x and t to converge because the Green's function falls off as a Gaussian distribution as $|x| \to \infty$. This suggests that the convolution can exist even if the support of h is not bounded in the space variable if (h, ψ) does not grow too fast as $|x| \to \infty$. This possibility will be further explored for regular distributions h below.

41.4. Limit properties of the Green's function. For any $t > 0$, the Green's function is a Gaussian distribution which is a regular distribution in the variable x. Since $G(x,t) > 0$ and

$$\int G(x,t)\, d^N x = 1\,, \quad t > 0\,.$$

The distributional limit of any such regular distribution is proved to be a delta function

$$\lim_{t\to 0^+} G(x,t) = \delta(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N)\,.$$

This property shows that the causal Green's for the heat operator is from class $C^\infty(0,\infty)\cap C^0([0,\infty))$ in the time variable t. Note that the time derivative of $G(x,t)$ cannot be continuously extended to $t=0$.

41.5. Surface heat potential. Let $h(x,t) = \delta(t)\cdot u_0(x)$ where a distribution $u_0 \in \mathcal{D}'(\mathbb{R}^N)$ is compactly supported. Then the heat potential reads

$$V_0(x,t) = G * \Big(\delta(t)\cdot u_0(x)\Big) = G(x,t) * u_0(x)$$

where the convolution is taken in the variable x (a proof of this assertion goes along a similar line of arguments as in the case of the transfer equation). Since the support of $h(x,t)$ lies in the hyperplane $t=0$, this special heat potential will be as a *surface heat potential*.

For every $t>0$, the distribution $V_0(x,t)$ can also be viewed as a distribution in the variable x. By continuity of the convolution of distributions one of which has a bounded support,

$$\lim_{t\to 0^+} V_0(x,t) = \delta(x) * u_0(x) = u_0(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N)\,.$$

For any test function $\varphi(x)$, the function

$$g(t) = \Big(V_0(x,t), \varphi(x)\Big) = \Big(G(x,t), \Big(u_0(y), \varphi(x+y)\Big)\Big)$$

is smooth for $t>0$ and has a continuous extension at $t=0$ because the Green's function $G(x,t)$ is from class $C^\infty(0,\infty)\cap C^0[0,\infty)$ in the variable t. Note that $(u_0(y), \varphi(x+y))$ is a test function in the variable x for any distribution u_0 with bounded support. Therefore the distribution v is a solution to the Cauchy problem for a homogeneous heat equation with distributional initial data:

$$\frac{\partial V_0(x,t)}{\partial t} - a^2\Delta_x V_0(x,t) = 0\,, \quad t>0\,,$$
$$V_0\Big|_{t=0} = u_0(x) \ \in \mathcal{D}'(\mathbb{R}^N)\,.$$

41.6. Regular surface heat potential. Supposed that u_0 is a regular distribution defined by a bounded function

$$M = \sup|u_0(x)| < \infty\,.$$

In this case, the classical convolution

$$G(x,t) * u_0(x) = \int G(x-y)u_0(y)\, d^N y$$

exists for $t > 0$ because $G(x,t) > 0$ and

$$\int |G(x-y,t)||u_0(y)|\, d^N y \le M \int G(x-y,t)\, d^N y = M < \infty$$

Since $G(x-y,t) = 0$ for $t < 0$, the classical convolution is locally integrable in $\mathbb{R}^{N+1}$ by Fubini's theorem (as it is bounded almost everywhere). The distributional convolution coincides with the classical one whenever the latter exists and is given by a locally integrable function. Therefore the surface heat potential reads

$$V_0(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^N} \int e^{-\frac{|x-y|^2}{4a^2 t}}\, u_0(y)\, d^N y$$

It is known as *the Poisson integral*.

Let us investigate smoothness properties of V_0. First, note that V_0 is bounded

$$|V_0(x,t)| \le \int G(x-y,t)|u_0(y)|\, d^N y \le M\,, \quad t > 0\,.$$

Next, let us show that

$$V_0(x,t) \in C^\infty(t > 0)\,.$$

It is sufficient to show that

$$\frac{d^p}{dt^p} \int e^{-\frac{|x-y|^2}{4a^2 t}}\, u_0(y)\, d^N y = \int \frac{\partial^p}{\partial t^p}\, e^{-\frac{|x-y|^2}{4a^2 t}}\, u_0(y)\, d^N y$$

for any $p > 0$. For $p = 1$, the integrand in the right-hand side has an integrable upper bound

$$\Big|\frac{\partial}{\partial t}\, e^{-\frac{|x-y|^2}{4a^2 t}}\, u_0(y)\Big| \le \frac{M|x-y|^2}{4a^2\epsilon^2}\, e^{-\frac{|x-y|^2}{4a^2\epsilon}}\,, \quad t \ge \epsilon > 0$$

By the theorem about differentiation of a function defined by an integral, the relation holds for $p = 1$. For $p > 1$, note that higher derivatives of the exponential with respect to t are polynomials in $|x-y|^2$ with coefficients proportional reciprocal powers of t multiplied by the exponential. Therefore a similar integrable bound exists for any derivative.

In order for V_0 to be a classical solution to the homogeneous Cauchy problem, it must have a continuous extension to the hyperplane $t = 0$. Let us show that this is the case if the initial data is given by a continuous and bounded function:

$$u_0 \in C^0\,, \quad M = \sup|u_0(x)| < \infty\,.$$

One has to show that

$$\lim_{t\to 0^+} V_0(x,t) = u_0(x)\,, \quad x \in \mathbb{R}^N\,.$$

One has for $t > 0$

$$\begin{aligned} V_0(x,t) - u_0(x) &= \int G(x-y,t)u_0(y)\, d^N y - u_0(x) \\ &= \int G(y,t)\Big(u_0(x+y) - u_0(x)\Big)\, d^N y \\ &= \pi^{-N/2} \int e^{-|z|^2} \Big(u_0(x - 2a\sqrt{t}z) - u_0(x)\Big)\, d^N z \end{aligned}$$

where, first, the integration variable was shifted by x, then the property that the integral of $G(y,t)$ is equal to one for $t > 0$ was used, and finally the integration variable was scaled $y = 2a\sqrt{t}z$. When $t \to 0^+$ the integrand vanishes for any z and x by continuity of u_0. On the other hand, the integrand has an integrable upper bound independent of x and t because

$$|u_0(x - 2a\sqrt{t}z) - u_0(x)| \le 2M$$

Therefore by the Lebesgue dominated convergence theorem

$$\lim_{t\to 0^+} \Big(V_0(x,t) - u_0(x)\Big) = 0\,.$$

and the surface heat potential is a classical solution

$$V_0 \in C^\infty(t > 0) \cap C^0(t \ge 0)$$

to the homogeneous Cauchy problem for the heat equation.

41.7. Regular heat potential. Let $h(x,t) = f(x,t)$ where the function $f(x,t)$ vanishes when $t < 0$ and is bounded on $\Omega_T = \mathbb{R}^N \times [0,T]$:

$$f(x,t) = 0\,,\ t < 0\,; \qquad \sup_{\Omega_T} |f(x,t)| = M_0(T) < \infty$$

The conditions imply that $f(x,t)$ is locally integrable in $\mathbb{R}^{N+1}$ and, hence, defines a regular distribution. The classical convolution of G and f exists by Fubini's theorem because $G(x,t) \ge 0$ and

$$\begin{aligned} \iint_0^\infty G(x-y,t-\tau)|f(y,\tau)|\, d^N y\, d\tau &\le M_0(t) \iint_0^t G(x-y,t-\tau)\, d^N y\, d\tau \\ &= M_0(t)t \end{aligned}$$

if $t > 0$ and vanishes if $t < 0$, and it is locally integrable (as it is bounded thanks to the above inequality). Therefore the heat potential is given by

$$V(x,t) = \int_0^t \int \frac{f(y,\tau)}{(2a\sqrt{\pi(t-\tau)})^N}\, e^{-\frac{|x-y|^2}{4a^2(t-\tau)}}\, d^N y\, d\tau$$

It follows that the heat potential is bounded because

$$|V(x,t)| \le tM_0(t)\,, \quad t > 0\,.$$

This bounds also shows that the heat potential fulfills the initial condition

$$\lim_{t\to 0^+} V(x,t) = 0\,.$$

In order for the heat potential to be a classical solution to the Cauchy problem with the zero initial condition, it should be from twice continuously differentiable for $t > 0$. Let us show that if f is twice continuously differentiable and all its partial derivatives are bounded in Ω_T,

$$f(x,t) \in C^2(t \ge 0)\,, \quad \sup_{\Omega_T} |D^\alpha f(x,t)| = M_\alpha(T) < \infty\,, \quad \alpha = 0,1,2\,.$$

for any $T > 0$, then

$$V(x,t) \in C^2(t > 0) \cap C^1(t \ge 0)\,.$$

In the integral representation of V, the integration variables are changed

$$y = x - 2a\sqrt{s}\, z\,, \quad \tau = t - s\,,$$

so that

$$V(x,t) = \pi^{-N/2} \int_0^t \int f(x - 2a\sqrt{s}z, t-s)\, e^{-|z|^2}\, d^N z\, ds$$

Partial derivatives of the integrand up to the second order are bounded by an integrable function independent of x and t

$$\left| D^\alpha f(x - 2a\sqrt{s}z, t-s)\, e^{-|z|^2} \right| \le M_\alpha(T) e^{-|z|^2}\,,$$

for all $(x,t) \in \Omega_T$. Therefore by the theorem about differentiation of functions defined by an integral

$$\frac{\partial V(x,t)}{\partial t} = \pi^{-N/2} \int f(x - 2a\sqrt{t}z, 0^+)\, e^{-|z|^2}\, d^N z$$
$$+\pi^{-N/2} \int_0^t \int \frac{\partial}{\partial t} f(x - 2a\sqrt{s}z, t-s)\, e^{-|z|^2}\, d^N z\, ds$$
$$\frac{\partial V(x,t)}{\partial x_j} = \pi^{-N/2} \int_0^t \int \frac{\partial}{\partial x_j} f(x - 2a\sqrt{s}z, t-s)\, e^{-|z|^2}\, d^N z\, ds$$

It follows from this representation that $V \in C^1(t \geq 0)$ owing to the continuity of the first partials of f for $t \geq 0$. The second partials are computed similarly for $t > 0$ and they are continuous for $t > 0$ by continuity of f and its partials. Note that $\frac{\partial^2 V}{\partial t^2}$ contains a singular factor $t^{-1/2}$ stemming from the differentiation of the first integral in $\partial V/\partial t$. For this reason $\frac{\partial^2 V}{\partial t^2}$ does not have a continuous extension to the hyperplane $t = 0$, whereas the other second partials $\frac{\partial^2 V}{\partial x_j \partial x_i}$ and $\frac{\partial^2 V}{\partial t \partial x_j}$ have continuous extensions to $t = 0$. Thus, $V \in C^2(t > 0)$.

It has been shown that the Cauchy problem for the heat equation has a unique classical solution given by the sum of the heat potentials

$$u(x,t) = V_0(x,t) + V(x,t)$$

if the initial data u_0 is a continuous bounded function and the inhomogeneity f is twice continuously differentiable function the half-space $t \geq 0$ with bounded partial derivatives in $\Omega_T = \mathbb{R}^N \times [0,T]$ for any $T > 0$.

41.8. Well-posedness of the Cauchy problem for the heat equation. Consider a class of distributions $h(x,t) = u_0(x) \cdot \delta(t) + f(x,t)$ where u_0 is a bounded function and f is bounded on Ω_T. It was shown that the convolution $G * h$ exists in this class and the regular distribution

$$\begin{aligned} u(x,t) &= (G * h)(x,t) = V_0(x,t) + V(x,t) \\ &= \frac{\theta(t)}{(2a\sqrt{\pi t})^N} \int e^{-\frac{|x-y|^2}{4a^2 t}}\, u_0(y)\, d^N y \\ &\quad + \int_0^t \int \frac{f(y,\tau)}{(2a\sqrt{\pi(t-\tau)})^N}\, e^{-\frac{|x-y|^2}{4a^2(t-\tau)}}\, d^N y\, d\tau \end{aligned}$$

is a solution to the generalized Cauchy problem. The solution is unique in the said class of distributions. Let us show that the solution changes continuously with initial data and inhomogeneity and, hence, the Cauchy problem is well posed in this class of initial data and inhomogeneity. Let u and $\tilde{u}$ be two solutions with the initial data u_0 and $\tilde{u}_0$ and inhomogeneities f and $\tilde{f}$, respectively. Suppose that

$$|u_0(x) - \tilde{u}_0(x)| \leq \varepsilon_0\,, \quad |f(x,t) - \tilde{f}(x,t)| \leq \varepsilon_1\,.$$

Then using the estimates of the heat potentials

$$\begin{aligned} |u(x,t) - \tilde{u}(x,t)| &\leq |V_0(x,t) - \tilde{V}_0(x,t)| + |V(x,t) - \tilde{V}(x,t)| \\ &\leq \varepsilon_0 + T\varepsilon_1 \end{aligned}$$

for all $(x,t) \in \Omega_T$. So, small variations of initial data and inhomogeneity yield a small change in the solution. The problem is well-posed.

42. The Schrödinder equation

Consider the following Cauchy problem

$$i\frac{\partial\psi(x,t)}{\partial t} = -\Delta_x\psi(x,t) + f(x,t)\,, \quad t > 0\,, \quad x \in \mathbb{R}^N$$
$$\psi\Big|_{t=0} = \psi_0(x)$$

where $\psi(x,t)$ is a complex-value function that is assumed to be smooth enough so that the problem is meaningful in the classical sense. It is necessary that ψ is from class $C^2(t > 0) \cap C^0(t \geq 0)$, and the inhomogeneity f must be continuous for $t \geq 0$. In quantum mechanics, a solution $\psi(x,t)$ is a probability amplitude for a particle in time $t > 0$ if the probability amplitude was $\psi_0(x)$ at time $t = 0$. The inhomogeneity $f(x,t)$ model effects of an environment in which the particle moves.

The problem is similar to the Cauchy problem for the heat equation and can be solved in the same way. However, there is an additional condition on solutions as well as on the initial data. In quantum mechanics, a solution is called *a wave function* of a quantum system (a free particle in this case). Its physical significance is that the squared absolute value $|\psi(x,t)|^2$ defines a probability density, meaning that, the probability $P(t)$ to find a particle in a spatial region Ω at time t is equal to

$$P(t) = \int_\Omega |\psi(x,t)|^2\, d^N x$$

Clearly, if $\Omega = \mathbb{R}^N$, then $P(t) = 1$ if the particle has no interaction with environment, $f = 0$, and $P(t) \leq 1$ for any Ω even if $f \neq 0$. This physical interpretation implies that any physical solution must be square integrable

$$\int |\psi(x,t)|^2\, d^N x < \infty$$

so that $\psi(x,t)$ can be normalized to have a unit $\mathcal{L}_2$ norm. So, in contrast to the heat equation, solutions that are not from the space $\mathcal{L}_2$ are rejected on the physical grounds. For this reason, here a different and simpler approach is adopted, which is based on the Fourier method and commonly used in quantum theory. It leads to the same result as a general method discussed earlier.

42.1. Generalized Cauchy problem. Any square integrable function defines a regular temperate distribution by the rule

$$(\psi, \varphi) = \int \psi(x)\varphi(x)\, d^N x$$

The integral exists because any test function $\varphi \in \mathcal{S}$ is square integrable, and the product of two square integrable function was shown to be integrable (see the section about the space $\mathcal{L}_2$). A solution to the Cauchy problem, if it exists, must be among distributional solutions to the Schrödinger equation

$$\psi(x,t) \in \mathcal{S}'(\mathbb{R}^{N+1}),$$

that are continuous in the variable $t \geq 0$. The latter is needed for the initial condition that must hold in the distributional sense:

$$\lim_{t\to 0^+} \Big(\psi(x,t), \varphi(x)\Big) = (\psi_0, \varphi), \quad \psi_0 \in \mathcal{S}'(\mathbb{R}^N),$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R}^N)$. If the solution to this generalized Cauchy problem exists and is unique, then a solution to the classical problem exists and is unique.

For a consistency of the problem it is required that $f \in \mathcal{S}'(\mathbb{R}^{N+1})$. By taking the Fourier transform of both sides of the Schrödinger equation in the variable x, one infers that

$$\Big(i\frac{d}{dt} - |k|^2\Big)\mathcal{F}_x[\psi](k,t) = \mathcal{F}_x[f](k,t)$$

which is a first-order linear differential equation in the space of temperate distributions. Its general solution can be found by the substitution

$$\mathcal{F}_x[\psi](k,t) = e^{-i|k|^2 t}\phi(k,t), \quad \phi(k,t) \in \mathcal{S}'(\mathbb{R}^{N+1})$$

so that ϕ satisfies the equation

$$i\frac{d}{dt}\,\phi(k,t) = e^{i|k|^2 t}\mathcal{F}_x[f](k,t)\Big).$$

It is worth noting that the exponential factor is a smooth temperate function from C_S. This method would not be justified for the heat equation because $e^{|k|^2 t}$ is not from C_S. Thus, a general solution reads

$$\psi(k,t) = -ie^{-i|k|^2 t}\Big(\phi_0(k) + D_t^{-1} e^{i|k|^2 t}\mathcal{F}_x[f](k,t)\Big),$$

where $\phi_0 \in \mathcal{S}'(\mathbb{R}^N)$ and D_t^{-1} denotes a distributional antiderivative with respect to t. Recall that a distributional antiderivative is unique up to an additive constant distribution. In the case considered, an additive constant is a distribution ϕ_0 independent of t.

Let a distribution $h(x,t) \in \mathcal{S}'(\mathbb{R}^{N+1})$ be continuous in t. Then its antiderivative $H(x,t) = D_t^{-1}h(x,t)$ is continuously differentiable in t. Let us select a particular antiderivative that vanishes at $t = 0$,

$$\lim_{t\to 0^+} \Big(H(x,t), \varphi(x)\Big) = 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^N).$$

This particular antiderivative will be denoted by a definite integral similarly to the classical antiderivative of a continuous function that vanishes at $t = 0$,

$$H(x,t) = \int_0^t h(x,\tau)\,d\tau\,.$$

With this choice of the antiderivative, $\mathcal{F}_x[\psi](k,t)$ is continuous in t and the initial condition $\mathcal{F}_x[\psi](x,t) \to \mathcal{F}[\psi_0](k)$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \to 0^+$ implies that $\phi_0 = i\mathcal{F}[\psi_0]$ so that

$$\mathcal{F}_x[\psi](k,t) = e^{-i|k|^2 t}\mathcal{F}[\psi_0](k) - ie^{-i|k|^2 t}\int_0^t e^{i|k|^2\tau}\mathcal{F}_x[f](k,\tau)\,d\tau\,.$$

Thus, the stated generalized Cauchy problem has a unique solution in $\mathcal{S}'(\mathbb{R}^{N+1})$ if the inhomogeneity $f(x,t)$ a temperate distribution that is continuous in the variable $t \geq 0$. The solution is given by the inverse Fourier transform of the above distribution with respect to k.

To compute the inverse Fourier transform, define a distribution

$$U_N(x,t) = \mathcal{F}_k^{-1}[e^{-i|k|^2 t}](x,t)\ \in \mathcal{S}'(\mathbb{R}^{N+1})\,.$$

It follows from the Lebesgue dominated convergence theorem that

$$\Big(\mathcal{F}_x[U_N],\varphi\Big) = \lim_{\epsilon\to 0^+}\int\int e^{-i|k|^2(t-i\epsilon)}\varphi(k,t)\,d^N k\,dt$$

for any test function φ. So, in the distributional sense

$$\mathcal{F}_x[U_N](k,t) = \lim_{\epsilon\to 0^+} e^{-i|k|^2(t-i\epsilon)} \stackrel{\text{def}}{=} e^{-i|k|^2(t-i0^+)}$$

Owing to the continuity of the Fourier transform and using the Gaussian integral computed earlier

$$\begin{aligned} U_N(x,t) &= \lim_{\epsilon\to 0^+}\mathcal{F}_k^{-1}\Big[e^{-i|k|^2(t-i\epsilon)}\Big](x,t) \\ &= \lim_{\epsilon\to 0^+}(2\pi)^{-N}\int e^{-i|k|^2(t-i\epsilon)-i(k,x)}\,d^N k \\ &= \lim_{\epsilon\to 0^+}\frac{1}{[4\pi(\epsilon+it)]^{\frac{N}{2}}}\exp\Big(-\frac{|x|^2}{4(\epsilon+it)}\Big) \\ &= \frac{e^{\frac{i|x|^2}{4(t-i0^+)}}}{[4\pi i(t-i0^+)]^{\frac{N}{2}}} \end{aligned}$$

Note that the singularity $|U_N(x,t)| \sim t^{-N/2}$ is not locally integrable if $N \neq 1$. Therefore a distributional extension (or regularization) is needed at $t = 0$ if $N \geq 2$. The regularization is achieved by the above limiting process for the Fourier transform of U_N. For $N = 1$, no regularization is needed, and

$$U_1(x,t) = \frac{e^{\frac{ix^2}{4t}}}{(4\pi i t)^{1/2}}$$

is a regular distribution.

Limit properties of U_N. Let $k \in \mathbb{R}$ and $t > 0$. Consider a temperate distribution $g_t(x)$ in the variable $x \in \mathbb{R}$ whose Fourier transform is given by

$$\mathcal{F}[g_t(x)](k) = e^{-ik^2 t}$$

Then by the Lebesgue dominated convergence theorem its distributional limit as $t \to 0^+$ reads

$$\lim_{t\to 0^+} (\mathcal{F}[g_t(x)], \varphi) = \lim_{t\to 0^+} \int e^{-ik^2 t}\varphi(k)\, dk = \int \varphi(k)\, dk = (1, \varphi)$$

Owing to the continuity of the Fourier transform

$$\lim_{t\to 0^+} g_t(x) = \delta(x) \quad \text{in } \mathcal{D}'$$

Recall that convergence in $\mathcal{S}'$ implies convergence in $\mathcal{D}'$. Consider $\mathcal{F}_x[U_N](k,t)$ as a temperate distribution from $\mathcal{S}'(\mathbb{R}^N)$ for any $t > 0$. Its action on a test function is given by the rule

$$\Big(\mathcal{F}_x[U_N](k,t), \varphi(k)\Big) = \int e^{-i|k|^2 t}\varphi(k)\, d^N k$$

Since $e^{-i|k|^2 t}$ can be viewed as the direct product $e^{-ik_1^2 t} \cdots e^{-ik_N^2 t}$, it is concluded that

$$\lim_{t\to 0^+} U_N(x,t) = \delta(x_1)\cdots\delta(x_N) = \delta(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N)\,.$$

Solution to the generalized Cauchy problem. Let $f(x,t)$ be a temperate distribution. Then the product $\mathcal{F}_x[U_N](k,t)\mathcal{F}_x[f](k,t)$ is also a temperate distribution and its inverse Fourier transform in the variable k exists in $\mathcal{S}'(\mathbb{R}^{N+1})$. By analogy with the Fourier transform of the convolution of two distribution one of which has a bounded support, define a convolution in the variable x

$$\begin{aligned} U_N(x,t) * f(x,t) &= \mathcal{F}_k^{-1}\Big[\mathcal{F}_x[U_N](k,t)\mathcal{F}_x[f](k,t)\Big](x,t) \\ &= \mathcal{F}_k^{-1}\Big[e^{-i|k|^2 t}\mathcal{F}_x[f](k,t)\Big](x,t) \ \in \mathcal{S}'(\mathbb{R}^{N+1})\,. \end{aligned}$$

This definition of the convolution of two distributions with respect to one variable makes sense only if the Fourier transform of one of the distributions in that variable is a smooth temperate function of two variables so that the product of Fourier transforms is a temperate distribution of two variables.

Suppose that f is a regular temperate distribution such that the function $f(x,t)$ is Lebesgue integrable in x for any t. Then the convolution is a regular distribution and has the following integral representation:

$$\begin{aligned} U_N(x,t) * f(x,t) &= \mathcal{F}_k^{-1}\Big[e^{-i|k|^2 t}\mathcal{F}_x[f](k,t)\Big] \\ &= \int e^{-i|k|^2(t-i0^+)-i(k,x)} \int e^{i(k,y)} f(y,t)\, d^N y\, \frac{d^N k}{(2\pi)^N} \\ &= \int\int e^{-i|k|^2(t-i0^+)-i(k,x-y)} \frac{d^N k}{(2\pi)^N}\, f(y,t)\, d^N y \\ &= \int U_N(x-y,t) f(y,t)\, d^N y \end{aligned}$$

where the order of integration was reversed by Fubini's theorem (owing to the regularization of the integral with respect to k).

Define the conjugate distribution

$$U_N^\dagger(x,t) = U_N(x,-t) = \overline{U_N(x,t)}$$

Then

$$\mathcal{F}_x[U_N^\dagger](k,t) = e^{i|k|^2(t+i0^+)}$$

The conjugate distribution $U_N^\dagger$ has a convolution with any temperate distribution in the variable x, just like U_N:

$$\begin{aligned} U_N^\dagger(x,t) * f(x,t) &= \mathcal{F}_k^{-1}\Big[\mathcal{F}_x[U_N^\dagger](k,t)\mathcal{F}_x[f](k,t)\Big](x,t) \\ &= \mathcal{F}_k^{-1}\Big[e^{i|k|^2 t}\mathcal{F}_x[f](k,t)\Big](x,t) \ \in \mathcal{S}'(\mathbb{R}^{N+1}). \end{aligned}$$

Using the distribution U_N and its conjugate $U_N^\dagger$, the solution to the generalized Cauchy problem can be written as a convolution

$$\psi(x,t) = U_N(x,t) * \psi_0(x) - iU_N(x,t) * \int_0^t U_N^\dagger(x,\tau) * f(x,\tau)\, d\tau\,,$$

where all convolutions are taken with respect to x. It satisfies the the initial condition is the distributional sense in $\mathcal{S}'(\mathbb{R}^N)$ thanks to the limit properties of U_N and by the definition of the antiderivative with respect to t denoted by the definite integral.

42.2. Evolution operator. The distributions U_N and $U_N^\dagger$ can be viewed as temperate distributions in the variable x for any given real t. Then

$$\mathcal{F}[U_N](k,t_1)\mathcal{F}[U_N](k,t_2) = e^{-i|k|^2(t_1+t_2)} = \mathcal{F}[U_N](k,t_1+t_2)$$

Since the convolution of U_N with any temperate distribution exists in the variable x, it is concluded that

$$U_N(x,t_1) * U_N(x,t_2) = U_N(x,t_1+t_2)$$

and similarly

$$U_N(x,t_1) * U_N^\dagger(x,t_2) = U_N(x,t_1-t_2) = U_N^\dagger(x,t_2-t_1)$$

In particular

$$U_N(x,t) * U_N^\dagger(x,t) = \delta(x)$$

Consider an operator on the space of temperate distributions

$$\hat{U}_N(t) : \quad \mathcal{S}'(\mathbb{R}^N) \;\to\; \mathcal{S}'(\mathbb{R}^N)$$

defined by the rule

$$\hat{U}_N(t)\psi(x) = U_N(x,t) * \psi(x)\,, \quad \psi \in \mathcal{S}'(\mathbb{R}^N)$$

It is called *the evolution operator* for a quantum free particle. It has the property that

$$\hat{U}_N(t_1)\hat{U}_N(t_2) = \hat{U}_N(t_1+t_2)$$

The evolution operator is invertible. The inverse is given by

$$\hat{U}_N^{-1}(t)\psi(x) = U_N^\dagger(x,t) * \psi(x)$$

The evolution operator is continuous in parameter t, that is,

$$\lim_{t\to t_0} \hat{U}_N(t) * \psi(x) = \hat{U}_N(t_0) * \psi(x)\,, \quad \psi \in \mathcal{S}'(\mathbb{R}^N)$$

where the limit is understood in the distributional sense. If ψ is integrable, then

$$\hat{U}_N(t)\psi(x) = \int U_N(x-y,t)\psi(y)\,d^N y$$

and the distribution $U_N(x-y,t)$ is called *the evolution operator kernel* in quantum mechanics (by analogy with kernels of integral operators).

42.3. Causal Green's function. Consider the causal Green's function for the Schrödinger operator:

$$\Big(i\frac{\partial}{\partial t} + \Delta_x\Big)G_N(x,t) = \delta(t)\cdot\delta(x)\,, \quad G_N(x,t) = 0\,, \quad t < 0$$

Then it follows from the Fourier method that

$$G_N(x,t) = -i\theta(t)U_N(x,t) = -\frac{i\theta(t)}{[4\pi i(t-i0^+)]^{\frac{N}{2}}}\, e^{\frac{i|x|^2}{4(t-i0^+)}} \;\in \mathcal{D}'(\mathbb{R}^{N+1})$$

So, the method used for solving the transfer and heat equation would lead to the same answer as the above Fourier method (under a simplifying condition that a solution must be a temperate distribution which in turn was a consequence of the physical condition that any classical solution must have a finite $\mathcal{L}_2$ norm). The convolution of G_N exists with any distribution $h(x,t)$ such that $h(x,t) = 0$ for $t < 0$ and $\operatorname{supp} h \subset B_R \times [0,\infty)$. Its calculation is similar to the case of the heat equation. The Fourier method reveals a special feature of the Schrödinger problem. The convolution exists for any temperate distribution $h(x,t)$ that vanishes in the half-space $t < 0$. It is essential for quantum theory in which the space $\mathcal{L}_2$ plays a central role.

42.4. Regular solutions. Let $\psi_0(x) \in \mathcal{L}$ and $f(x,t)$ is also integrable in x and $f \in C^0(t \ge 0)$. By the established properties of the convolution of U_N with regular distribution in the variable x, the solution to the generalized Cauchy problem has the following integral representation

$$\begin{aligned}\psi(x,t) = &\int U_N(x-y,t)\psi_0(y)\,d^N y \\ &-i\int_0^t \int U_N(x-y,t-\tau)f(y,\tau)\,d^N y\,d\tau\end{aligned}$$

where it was used that the antiderivative D_t^{-1} commutes with taking the convolution with respect to x and $\hat{U}_N(t)\hat{U}_N^\dagger(\tau) = \hat{U}_N(t-\tau)$. For any $t > 0$

$$\begin{aligned}\psi(x,t) = &\frac{1}{(4\pi i t)^{\frac{N}{2}}}\int e^{\frac{i|x-y|^2}{4t}}\psi_0(y)\,d^N y \\ &-i\int_0^t\int \frac{e^{\frac{i|x-y|^2}{4(t-\tau-i0^+)}}}{[4\pi i(t-\tau-i0^+)]^{\frac{N}{2}}}\, f(y,\tau)\,d^N y\,d\tau\end{aligned}$$

Using this integral representation, one can find sufficient conditions on smoothness of the initial data and inhomogeneity under which $\psi(x,t)$ is a classical solution. For example, if $\psi_0 \in \mathcal{S}$ then $\psi(x,t) \in \mathcal{S}$ for all $t > 0$

and $\psi(x,t)$ converges to $\psi_0(x)$ uniformly as $t \to 0^+$ (see Exercises). However, for quantum mechanical applications, it is sufficient to show that the problem has a square integrable solution if the initial data and inhomogeneity are square integrable, and the solution is unique. To show this, some additional properties of the Fourier transform are needed.

Plancherel's theorem. The classical Fourier transform exists for any Lebesgue integrable function. However, not every Lebesgue integrable function is square integrable. For example, if $f(x) \sim x^{-1/2}$ near $x = 0$, then f is integrable if it has a bounded support, but $|f(x)|^2 \sim |x|^{-1}$ which is not an integrable singularity. So, f is not square integrable. Conversely, not every square integrable function is integrable. For example, if f is continuous and $|f(x)|^2 \sim |x|^{-2}$ as $|x| \to \infty$, then f is square integrable, but $|f(x)| \sim |x|^{-1}$ and f is not integrable. So, a classical solution to the Schrödinger equation must be from class

$$\psi(x,t) \in \mathcal{L} \cap \mathcal{L}_2$$

for any t in order to have the probabilistic interpretation as required by physics. The following theorem holds

THEOREM **42.1**. (Plancherel)
If $\psi \in \mathcal{L} \cap \mathcal{L}_2$, then its Fourier transform is square integrable, and

$$\mathcal{F}[\psi] \in \mathcal{L}_2, \qquad \int |\mathcal{F}[\psi](k)|^2 \, d^N k = (2\pi)^{-N} \int |\psi(x)|^2 \, d^N x$$

It is worth noting that the factor $(2\pi)^{-N}$ can be eliminated if the Fourier transform is *defined* with an extra factor $(2\pi)^{-N/2}$. With this definition, the relation between $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ does not contain the factor $(2\pi)^{-N}$, and the Fourier transform of ψ has the same $\mathcal{L}_2$ norm, $\|\psi\|_2 = \|\mathcal{F}[\psi]\|_2$.

Remark. Since the subspace $\mathcal{L} \cap \mathcal{L}_2$ is dense in $\mathcal{L}_2$ (recall that a smaller subspace $\mathcal{D}$ is dense in $\mathcal{L}_2$), one can show that there exists a unique extension of the Fourier transform from $\mathcal{L} \cap \mathcal{L}_2$ to the whole $\mathcal{L}_2$ that preserve the $\mathcal{L}_2$ norm. If $\{\psi_n\}$ is a sequence in $\mathcal{L} \cap \mathcal{L}_2$ that converges to $\psi \in \mathcal{L}_2$, that is, $\|\psi - \psi_n\|_2 \to 0$ as $n \to \infty$, then $\mathcal{F}[\psi]$ is defined as the limit of the sequence $\mathcal{F}[\psi_n]$. One can show that the limit exists and is unique in $\mathcal{L}_2$ and has the same norm as ψ. This extension, often called *Plancherel's transform*, will be discussed later in the framework of the operator theory.

Let us now show that the solution to the Schrödinger equation is square integrable in x for any $t > 0$.

Probability conservation. Let $f = 0$ in the Schrödinger equation. For any $\psi(x,t)$, the symbol $\|\psi(t)\|_2$ denotes the $\mathcal{L}_2$ norm with respect to x for any t. Then the time evolution of a quantum particle is *unitary*, meaning that, it preserve probability:

$$\begin{aligned}\|\psi(t)\|_2^2 &= \int |\psi(x,t)|^2 \, d^N x = (2\pi)^N \int |\mathcal{F}_x[\psi](k,t)|^2 \, d^N k \\ &= (2\pi)^N \int |e^{-i|k|^2 t} \mathcal{F}[\psi_0](k)|^2 \, d^N k = (2\pi)^N \int |\mathcal{F}[\psi_0](k)|^2 \, d^N k \\ &= \int |\psi_0(x)|^2 \, d^N x = \|\psi_0\|_2^2 \, .\end{aligned}$$

Note that one can always normalize the initial data so that $\|\psi_0\|_2 = 1$. Then the above relation means that a free particle cannot disappear in due course of evolution (the probability to find the particle in the whole space is always equal to 1).

Initial condition. Let us show that the initial condition is fulfilled in the $\mathcal{L}_2$ norm, that is,

$$\lim_{t\to 0^+} \|\psi(t) - \psi_0\|_2 = 0$$

Indeed, by Plancherel's theorem

$$\|\psi(t) - \psi_0\|_2 = (2\pi)^N \|\mathcal{F}_x[\psi(t) - \psi_0]\|_2 = (2\pi)^N \|(e^{-i|k|^2 t} - 1)\mathcal{F}[\psi_0]\|_2$$

Since for any $\psi_0 \in \mathcal{L}_2$

$$\left|(e^{-i|k|^2 t} - 1)\mathcal{F}[\psi_0](k)\right|^2 \leq 4|\mathcal{F}[\psi_0](k)|^2 \in \mathcal{L}$$

the assertion follows from the Lebesgue dominated convergence theorem. The initial condition is fulfilled *almost everywhere*,

$$\lim_{t\to 0^+} \psi(x,t) = \psi_0(x) \quad a.e. \, ,$$

which is consistent with the physical interpretation of the theory (alterations of a wave function on sets of measure zero do not change the probability to find a particle in a particular region of space).

Including an inhomogeneity. Put $\psi_0 = 0$ and let $f(x,t) \in C^0(t \geq 0)$ and $f(x,t)$ be square integrable in x for any $t \geq 0$. Let us show that the corresponding solution to the Cauchy problem $\psi(x,t)$ is square integrable in x for ant $t \geq 0$ and satisfies the zero initial condition almost everywhere. Recall the Schwartz inequality in $\mathcal{L}_2$

$$\left| \int \psi(x)\phi(x) \, d^N x \right| \leq \|\psi\|_2 \|\phi\|_2$$

which holds for any complex square integrable functions ϕ and ψ. It follows from this inequality for any continuous complex $h(x,t)$ which is square integrable in x for any $t \ge 0$, that

$$\begin{aligned}\Big\| \int_0^t h(x,\tau)\,d\tau\Big\|_2^2 &= \int\int_0^t\int_0^t \overline{h(x,\tau)}h(x,\tau')\,d\tau\,d\tau'\,d^Nx \\ &= \int_0^t\int_0^t\int \overline{h(x,\tau)}h(x,\tau')\,d^Nx\,d\tau\,d\tau' \\ &\le \int_0^t\int_0^t \|h(\tau)\|_2\|h(\tau')\|_2 d\tau\,d\tau' \\ &= \Big(\int_0^t \|h(\tau)\|_2\,d\tau\Big)^2 .\end{aligned}$$

The order of integration is changed by Fubini's theorem because

$$|\overline{h(x,\tau)}h(x,\tau')| \le \frac{1}{2}|h(x,\tau)|^2 + \frac{1}{2}|h(x,\tau')|^2 .$$

This inequality shows that the product is a Lebesgue integrable function of x, τ, and τ' in $\mathbb{R}^N \times (0,T)\times(0,T)$ for any $T>0$ owing to the continuity and square integrability of h. Then by Plancherel's theorem

$$\begin{aligned}\|\psi(t)\|_2^2 = (2\pi)^N\|\mathcal{F}_x[\psi(t)]\|_2^2 &= (2\pi)^N\Big\|\int_0^t e^{i|k|^2\tau}\mathcal{F}_x[f(\tau)]d\tau\Big\|_2^2 \\ &\le (2\pi)^N\Big(\int_0^t \|\mathcal{F}_x[f(\tau)]\|_2 d\tau\Big)^2 = \Big(\int_0^t \|f(\tau)\|_2 d\tau\Big)^2 < \infty\end{aligned}$$

because $f \in C^0(t\ge 0)$ and $\|f(t)\|_2 < \infty$ for all $t \ge 0$. Furthermore, this equation also implies that $\|\psi(t)\|_2 \to 0$ as $t \to 0^+$ or

$$\lim_{t\to 0^+}\psi(x,t) = 0 \quad a.e.$$

Since the sum of two square integrable functions is square integrable (recall that $\mathcal{L}_2$ is a a linear space), the solution to the Cauchy problem is square integrable for any $t \ge 0$ and satisfies the initial condition almost everywhere if $\psi_0 \in \mathcal{L}_2$ and $f(x,t) \in C^0(t\ge 0)$ and is square integrable in x for any $t \ge 0$. It remains to check if the Cauchy problem is well posed.

Well-posedness of the Cauchy problem. In quantum physics, two wave functions are different only if the $\mathcal{L}_2$ distance between them is not zero. So, the well-posedness of the Cauchy problem will be verified in the same sense. Consider two initial data and inhomogeneities such that

$$\|\psi_0 - \tilde{\psi}_0\|_2 \le \varepsilon_0\,, \quad \|f(t) - \tilde{f}(t)\|_2 \le \varepsilon_1\,, \quad t \ge 0\,.$$

Let $\psi(x,t)$ and $\tilde{\psi}(x,t)$ be the classical solutions corresponding to the initial data ψ_0 and $\tilde{\psi}_0$, respectively, and $f = \tilde{f} = 0$. Then by unitarity of the time evolution

$$\|\psi(t) - \tilde{\psi}(t)\|_2 = \|\hat{U}_N(t)(\psi_0 - \tilde{\psi}_0)\|_2 = \|\psi_0 - \tilde{\psi}_0\|_2 \leq \varepsilon_0$$

Let $\psi(x,t)$ and $\tilde{\psi}(x,t)$ be the solutions corresponding to the inhomogeneities $f(x,t)$ and $\tilde{f}(x,t)$, respectively, and $\psi_0 = \tilde{\psi}_0 = 0$. Replacing f by $f - \tilde{f}$ in the equations for $\|\psi(t)\|_2$ in the previous section, one infers that

$$\|\psi(t) - \tilde{\psi}(t)\|_2 = \int_0^t \|f(\tau) - \tilde{f}(\tau)\|_2 d\tau \leq \varepsilon_1 t$$

Since the solution is the sum of the considered two solutions, by the triangle inequality in $\mathcal{L}_2$,

$$\|\psi(t) - \tilde{\psi}(t)\|_2 \leq \varepsilon_0 + \varepsilon_1 t$$

Thus, the Cauchy problem for the Schrödinger equation is well posed in the sense required by a probabilistic physical interpretation.

42.5. Exercises.

1. The function $e^{ia|y|^2}$ where $a \neq 0$ and $\operatorname{Im} a \leq 0$ is a regular temperate distribution of $y \in \mathbb{R}^N$. For any test function φ, put

$$\phi(x) = \left(e^{ia|y|^2}, \varphi(x+y)\right) = \int e^{ia|y|^2} \varphi(x+y)\, d^N y$$

(i) Show that $\phi \in C^\infty$ and

$$D^\alpha \phi(x) = \int e^{ia|y-x|^2} D^\alpha \varphi(y)\, d^N y$$

(ii) Use the identity

$$x e^{ia|y-x|^2} = \left(y - \frac{1}{2ia}\nabla_y\right) e^{ia|y-x|^2}$$

and integration by parts to show that

$$x^\beta D^\alpha \phi(x) = \int e^{ia|y-x|^2} \left(y + \frac{1}{2ia}\nabla\right)^\beta D^\alpha \varphi(y)\, d^N y$$

and that

$$|x^\beta D^\alpha \phi(x)| \leq M < \infty$$

holds for any $\beta > 0$ and any $x \in \mathbb{R}^N$.
(iii) Show that ϕ is a test function

$$\phi \in \mathcal{S}(\mathbb{R}^N)$$

(iv) Show that for any temperate distribution $\psi_0(x)$, the convolution $\psi(x,t) = iG_N(x,t) * (\delta(t)\cdot\psi_0(x))$ exists in $\mathcal{D}(\mathbb{R}^{N+1})$ and

$$\Big(\psi(x,t),\varphi(x,t)\Big) = i\Big(G_N(x,t),\Big(\psi_0(y),\varphi(x+y,t)\Big)\Big)$$

for any test function $\varphi(x,t)\in\mathcal{D}$.
(v) If $\psi_0\in\mathcal{L}_1$, show that

$$\psi(x,t) = \frac{1}{(4\pi i t)^{\frac{N}{2}}}\int e^{\frac{i|x-y|^2}{4t}}\psi_0(y)\,d^N y\,,\quad t>0$$

and if $\psi_0\in\mathcal{S}$, then $\psi(x,t)\in\mathcal{S}$ and $\psi(x,t)\to\psi_0(x)$ in $\mathcal{S}$ as $t\to 0^+$.

43. Cauchy problem for a wave equation

Consider the Cauchy problem for an equation that is second-order in the evoloution variable t:

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} + L(D)u = f(x,t)\,, \quad t > 0\,, \quad x \in \mathbb{R}^N$$
$$u\Big|_{t=0} = \lim_{t\to 0^+} u(x,t) = u_0(x)\,, \quad x \in \mathbb{R}^N\,,$$
$$\frac{\partial u}{\partial t}\Big|_{t=0} = \lim_{t\to 0^+} \frac{\partial u(x,t)}{\partial t} = u_1(x)\,, \quad x \in \mathbb{R}^N\,,$$
$$u \in C^2(t>0) \cap C^1 t \geq 0\,,$$

Let $u(x,t)$ be a solution. Define regular distributions from $\mathcal{D}'(\mathbb{R}^{N+1})$ by

$$v(x,t) = \theta(t)u(x,t)\,, \quad g(x,t) = \theta(t)f(x,t)$$

The distribution $v(x)$ is a smooth function that has a jump discontinuity in the variable t. Using the relation between the distributional and classical derivatives, one infers that

$$\frac{\partial v}{\partial t} = u_0(x)\cdot\delta(t) + \theta(t)\,\frac{\partial u}{\partial t}$$
$$\frac{\partial^2 v}{\partial t^2} = u_0(x)\cdot\delta'(t) + u_1(x)\cdot\delta(t) + \theta(t)\,\frac{\partial^2 u}{\partial t^2}$$

Similarly, any solution to the second problem is a distributional solution in $\mathcal{D}'(t>0)$:

$$\frac{\partial^2 v}{\partial t^2} + 2\gamma\frac{\partial v}{\partial t} + L(D)v = g(x,t) + (u_1(x) + 2\gamma u_0(x))\cdot\delta(t) + u_0(x)\cdot\delta'(t)\,.$$

If $\mathcal{E}(x,t) \in \mathcal{D}'(t>0)$ is a fundamental solution for the operator in the problem, then a solution to the distributional problem is given by

$$v(x,t) = (\mathcal{E} * g)(x,t) + \mathcal{E}(x,t) * (u_1(x) + 2\gamma u_0(x)) + \frac{\partial \mathcal{E}}{\partial t} * u_0(x)$$

where the two latter convolutions are taken in the variable x.

Suppose that the distributions in the right-hand side of the distributional equation are from the subset of distributions in $\mathcal{D}(t>0)$ whose elements have the convolution with $\mathcal{E}$. Then the distributional solution exists and is unique in this class. This implies that the corresponding classical Cauchy problem also has a unique solution if the initial data u_0 and u_1 and the inhomogeneity f are sufficiently smooth. One the convolutions are calculated, one has to verify that the obtained regular distribution $v(x,t)$ is has sufficiently many continuous partial derivatives. This task depends on $L(D)$ and, hence, should be studied for each specific $L(D)$.

43.1. Generalized Cauchy problem. A generalized Cauchy problem is to find a distribution $u(x,t) \in \mathcal{D}'$ that satisfies the corresponding classical equation,

$$\frac{\partial u}{\partial t} + L(D)u(x,t) = h(x,t)$$

or

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\,\frac{\partial u}{\partial t} + L(D)u(x,t) = h(x,t)$$

such that

$$u(x,t) = 0\,, \quad t < 0\,, \quad x \in \mathbb{R}^N$$

where the inhomogeneity $h(x,t) \in \mathcal{D}'$ is a distribution that also vanishes in the half-space

$$h(x,t) = 0\,, \quad t < 0\,, \quad x \in \mathbb{R}^N\,.$$

If $G(x,t)$ is a fundamental solution for the operator in question such that

$$G(x,t) = 0\,, \quad t < 0\,, \quad x \in \mathbb{R}^N$$

and the convolution $G * h$ exists, then

$$u(x,t) = (G * h)(x,t)$$

Indeed, for any test function $\varphi(x,t)$ whose support lies in the negative half-space, $t < 0$, and $\eta_n(x,y,t,\tau)$ is a unit sequence in $\mathbb{R}^{2N+2}$, then

$$\psi_n(x,y,t,\tau) = \eta(t)\eta(\tau)\eta_n(x,y,t,\tau)\varphi(x+y,t+\tau) = 0$$

where $\eta(t) \in C^\infty$, $\eta(t) = 1$ if $t \geq -a$ for some $a > 0$ and $\eta(t) = 0$ if $t \leq 2a$. This implies, by the definition of convolution, that

$$(u,\varphi) = (G * h,\varphi) = \lim_{n\to\infty}\Big(G(x,t)\cdot h(y,\tau),\psi_n(x,y,t,\tau)\Big) = 0$$

or that the distribution u vanishes in the negative half-space $t < 0$. It follows from the differentiation properties of the convolution that u is a solution to the generalized Cauchy problem. This solution is unique in the class of distributions $\mathcal{D}'_G$ for which the convolution with G exists. Using this fact one can investigate conditions on $h(x,t)$ under which a solution to the classical Cauchy problem exists and is unique. Moreover, by calculating the convolution $G * h$ an explicit form of the classical Cauchy problem can be found.

A fundamental solution that vanishes in the negative half-space will be called a *causal Green's function*.

43.2. Causal Green's function for a wave operator. Let us try to find a causal Green function, denoted by G_N, for a wave operator in the space of temperate distributions. Let us take the Fourier transform with respect to x of both sides of the equation

$$\frac{\partial^2 G_N}{\partial t^2} - c^2 \Delta_x G_N(x,t) = \delta(x)\cdot\delta(t)$$

One infers that

$$\Big(\frac{d^2}{dt^2} + c^2|k|^2\Big)\mathcal{F}_x[G_N](k,t) = \delta(t)\,, \quad \mathcal{F}_x[G_N](k,t) = 0\,, \quad t<0$$

A solution to this problem exists in the algebra $\mathcal{D}'_+$ for every $k \in \mathbb{R}^N$ and has the form

$$\mathcal{F}_x[G_N](k,t) = \theta(t)\frac{\sin(c|k|t)}{c|k|}$$

This is a smooth temperate function in k for any $t > 0$ and, hence, it is a regular temperate distribution. Therefore, the causal Green function for a wave operator is also a temperate distribution that is given by

$$G_N(x,t) = \frac{\theta(t)}{c}\mathcal{F}_k^{-1}\Big[\frac{\sin(c|k|t)}{|k|}\Big](x,t) = \frac{\theta(t)}{(2\pi)^N c}\mathcal{F}_k\Big[\frac{\sin(c|k|t)}{|k|}\Big](x,t)$$

Case $N = 3$**.** Recall the Fourier transform of the spherical delta function in $\mathbb{R}^3$:

$$\mathcal{F}[\delta_{S_a}(x)](k) = 4\pi a\,\frac{\sin(a|k|)}{|k|} \quad \Rightarrow \quad \mathcal{F}^{-1}\Big[\frac{\sin(a|k|)}{|k|}\Big](x) = \frac{1}{4\pi a}\,\delta_{S_a}(x)$$

Therefore

$$G_3(x,t) = \frac{\theta(t)}{4\pi c^2 t}\,\delta_{S_{ct}}(x) = \frac{\theta(t)}{4\pi c}\,\delta(c^2t^2 - |x|^2)$$

Its value on any test function is given by

$$(G_3,\varphi) = \frac{1}{4\pi c^2}\int_0^\infty \frac{1}{t}\int_{|x|=ct}\varphi(x,t)\,dS_x\,dt$$

Note that a test function is bounded, and hence the integral over the sphere is proportional to t^2 (the sphere radius) so that the integrand is not singular at $t = 0$.

Cases $N = 1$ and $N = 2$. If one recalls the Fourier transform of a window function $\theta(a - |x|)$ where $x \in \mathbb{R}$ and $a > 0$, then it is easy to infer that

$$G_1(x,t) = \frac{1}{2c}\,\theta(ct - |x|)$$

The case $N = 2$ requires the actual computation of the distributional Fourier transform. It leads to

$$G_2(x,t) = \frac{1}{2\pi c}\frac{\theta(ct - |x|)}{\sqrt{c^2t^2 - |x|^2}}$$

43.3. Convolution of distributions with support in a light cone. There exists a higher dimensional generalization of Theorem **31.2** which is essential for solving wave equations.

Let $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. The open set in $\mathbb{R}^{N+1}$

$$\Gamma^+ \,:\, ct - |x| > 0$$

where $c > 0$ is called *the future light cone*. Supports of fundamental solutions to the wave operator lie in the closure $\overline{\Gamma^+}$ of the future light cone.

THEOREM **43.1**. *Let f and g be distributions from $\mathcal{D}'(\mathbb{R}^{N+1})$ such that*

$$f(x,t) = 0\,,\quad t < 0\,,\qquad \operatorname{supp} g \subset \overline{\Gamma^+}$$

Then their convolution exists and can be computed by the rule

$$(\mathbf{43.1})\quad (f*g,\varphi) = \Big(f(x,t)\cdot g(y,\tau), \eta(t)\eta(\tau)\eta(c^2\tau^2-|y|^2)\varphi(x+y,t+\tau)\Big)$$

where η is a C^∞ function such that $\eta(t) = 1$ if $t > -\delta$ for some $\delta > 0$ and $\eta(t) = 0$ if $t < -a$ for some $a > \delta$. The convolution has the following properties:

$$(f * g)(x,t) = 0\,,\quad t < 0\,;$$

$f * g$ is continuous with respect to f and g, that is, for any sequence $f_n \to f$ in $\mathcal{D}'$ such that $f_n(x,t) = 0$ in $t < 0$,

$$f_n * g \to f * g \quad \text{in } \mathcal{D}'\,,\qquad \operatorname{supp} g \subset \overline{\Gamma^+}$$

and for any sequence $g_n \to g$ in $\mathcal{D}'$ such that $\operatorname{supp} g_n \subset \overline{\Gamma^+}$*,*

$$f * g_n \to f * g \quad \text{in } \mathcal{D}'\,,\qquad f(x,t) = 0\,,\ t < 0\,.$$

Let us first prove the rule (**43.1**). Let $\eta(t)$ be a C^∞ function such that $\eta(t) = 1$ if $t > -a$ and $\eta(t) = 0$ if $t < -b$ for some $b > a > 0$. For any test function $\varphi(t,x)$, the function of four variables

$$\psi(x,y,t,\tau) = \eta(t)\eta(\tau)\eta(c^2\tau^2 - y^2)\varphi(t+\tau, x+y)$$

is a test function. Indeed, it is smooth because it is the product of smooth functions. If the support of φ lies in a ball of radius R, then

$$\varphi(t+\tau, x+y) = 0\,, \quad |t+\tau| > R$$

Since the support of ψ lies in $t \geq -b$ and $\tau \geq -b$,

$$\psi(t, \tau, x, y) = 0\,, \quad (t,\tau) \notin [-b, R] \times [-b, R]$$

for all x and y. If the range of τ is bounded, the range of y is also bounded in the support of ψ because $\eta(c^2\tau^2 - y^2) = 0$ if $c^2\tau^2 - y^2 < -b$, that is, $\psi(t, \tau, x, y) = 0$ if $|y| > A > 0$ for all for any values of the other variables. Finally, note that

$$\varphi(t+\tau, x+y) = 0\,, \quad |x+y| > R$$

This implies that ψ vanishes for all $|x| > R + A$. Thus, the support of ψ is bounded, and it is a test function in $\mathbb{R}^{2N+2}$.

By the hypothesis about supports of distributions f and g,

$$f(t,x) = \eta(t) f(t,x)\,, \qquad g(\tau, y) = \eta(\tau)\eta(c^2\tau^2 - y^2) g(\eta, y)\,.$$

Fix a unit sequence $\eta_n(t, \tau, x, y)$. Then

$$\begin{aligned}
\Big(g * f, \varphi\Big) &= \lim_{n\to\infty} \Big(f(x,t), \Big(g(\tau, y), \eta_n(t,\tau,x,y)\varphi(t+\tau, x+y)\Big)\Big) \\
&= \lim_{n\to\infty} \Big(f(x,t), \Big(g(\tau, y), \eta_n(t,\tau,x,y)\psi(t,\tau,x,y)\Big)\Big) \\
&= \Big(f(x,t), \Big(g(\tau,y), \psi(t,\tau,x,y)\Big)\Big)\,.
\end{aligned}$$

The limit exists because ψ has a bounded support and $\eta_n = 1$ in the support of ψ for all large enough n. The rule (**43**.**1**) is established. A proof of continuity of the convolution is the same as for Theorem **31**.**1**.

43.3.1. d'Alembert's formula for a solution to a 2D wave equation. A distributional solution to a 2D wave equation can be obtained by the rule (**43**.**1**) where $g = \mathcal{E}$. Using the explicit form of $\mathcal{E}$, one infers from the rule (**43**.**1**) that

$$\begin{aligned}
\phi(y,\tau) = \Big(\mathcal{E}(t,x), \psi(\tau, t, y, x)\Big) &= \frac{\eta(\tau)}{2c}\int_0^\infty \int_{-ct}^{ct} \varphi(t+\tau, x+y)\,dx\,dt \\
= \frac{\eta(\tau)}{2c} \int_\tau^\infty \int_{y+c(\tau-t)}^{y-c(\tau-t)} \varphi(t,x)\,dx\,dt
\end{aligned}$$

Suppose that f is a regular distribution defined by a locally integrable function $f(t,x)$, $f(t,x)=0$ if $t<0$. Then

$$(\mathcal{E}*f,\varphi)=(f,\phi)=\frac{1}{2c}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{\tau}^{\infty}\int_{y+c(\tau-t)}^{y-c(\tau-t)} f(\tau,y)\varphi(t,x)\,dx\,dt\,dy\,d\tau$$

The iterated integral converges absolutely. Therefore by Fubini's theorem the order of integrations can be interchanged. To find an explicit form of $u=\mathcal{E}*f$, the integrations with respect to x and t should be carried out after the integrations with respect to y and τ. First the order of integrals with respect to y and t is swapped, then with respect to τ and t, followed by swapping the y and x integrations, and finally by the x and τ integrations. Put $\xi=c(\tau-t)$. Then the new integration limits are obtained from the shape of the integration region in the corresponding plane spanned by the two variables:

$$\int_0^{\infty}\int_{-\infty}^{\infty}\int_{\tau}^{\infty}\int_{y+\xi}^{y-\xi}\cdots\,dx\,dt\,dy\,d\tau=\int_0^{\infty}\int_{\tau}^{\infty}\int_{-\infty}^{\infty}\int_{y+\xi}^{y-\xi}\cdots\,dx\,dy\,dt\,d\tau$$
$$=\int_0^{\infty}\int_0^{t}\int_{-\infty}^{\infty}\int_{y+\xi}^{y-\xi}\cdots\,dx\,dy\,d\tau\,dt=\int_0^{\infty}\int_0^{t}\int_{-\infty}^{\infty}\int_{x+\xi}^{x-\xi}\cdots\,dy\,dx\,d\tau\,dt$$
$$=\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{t}\int_{x+\xi}^{x-\xi}\cdots\,dy\,d\tau\,dx\,dt$$

Thus, the function

$$u(t,x)=(\mathcal{E}*f)(t,x)=\theta(t)\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)\,dy\,d\tau$$

is a distributional solution to the wave equation

$$\square_c u(t,x)=f(t,x)\,,\quad t>0\,.$$

If f is from class $C^1(t>0)\cap C^0(t\geq 0)$, then $u(t,x)$ is twice continuously differentiable for $t>0$ and satisfies the initial conditions

$$u\Big|_{t=0}=\lim_{t\to 0^+}u(t,x)=0\,,$$
$$D_t u\Big|_{t=0}=\lim_{t\to 0^+}D_t u(t,x)=0\,.$$

Thus, it is a classical solution to the wave equation and its integral form is known as d'Alembert's formula. A verification of these initial conditions is left to the reader as an exercise.

A physical significance of this solution is that an elastic string was at rest at $t < 0$. Then a force $f(t, x)$ is applied at $t \geq 0$. The solution represents forced vibrations of the string under the force f. If the force acts only during a finite interval of time $f(t, x) = 0$, $0 \leq t \leq T$, then for $t > T$, the integration in time in d'Alembert's formula is limited to a fixed interval $[0, T]$ so that $u(t, x)$ becomes a superposition of waves propagating in the opposite directions with speed c.

It will be shown in the next chapter that the initial conditions can also be distributions, and the Cauchy problem for the wave equation in any number of dimensions can be solved in the distributional sense.

43.4. Wave potentials. As one can see, the causal Green function of a wave operator has support in the future light cone $\overline{\Gamma}^+$ where $ct \geq |x| \geq 0$. By evaluating the inverse Fourier transform of $\mathcal{F}_x[G_n]$ one can show that this holds in any dimension N. Therefore the generalized Cauchy problem has a solution

$$u(x, t) = (G_N * h)(x, t)$$

for any distribution $h(x, t)$ with support in the positive half-space, $t \geq 0$, and the solution is unique (the homogeneous equation has only the trivial solution in the set of distributions vanishing in the negative half-space, $t < 0$).

Let $g(x, t) \in \mathcal{D}'(\mathbb{R}^{N+1})$ be a distribution with support in $\overline{\Gamma^+}$ and $u(x) \in \mathcal{D}'(\mathbb{R}^N)$. Then it follows from the theorem about the convolution of distribution with supports in $\overline{\Gamma^+}$ and in $t \geq 0$ and from the commutativity and associativity of the direct product that

$$g * (u(x) \cdot \delta(t)) = g(x, t) * u(x) \in \mathcal{D}'(\mathbb{R}^{N+1}$$

where the distribution in the right-hand side is defined by the rule

$$\Big(g(x, t) * u(x), \varphi(x, t)\Big) = \Big(g(x, t) \cdot u(y), \eta(c^2t^2 - |x|^2)\varphi(x + y, t)\Big)\,.$$

Indeed, for any test function $\varphi(x,t) \in \mathcal{D}$ one has by associativity of the direct product

$$\begin{aligned}
&\Big(g * (u(x)\cdot\delta(t)), \varphi(x,t)\Big)\\
&= \Big(g(x,t)\cdot(u(y)\cdot\delta(\tau)), \eta(\tau)\eta(t)\eta(c^2t^2-|x|^2)\varphi(x+y,t+\tau))\Big)\\
&= \Big((g(x,t)\cdot u(y))\cdot\delta(\tau), \eta(\tau)\eta(t)\eta(c^2t^2-|x|^2)\varphi(x+y,t+\tau))\Big)\\
&= \Big(g(x,t)\cdot u(y), \eta(t)\eta(c^2t^2-|x|^2)\varphi(x+y,t))\Big)\\
&= \Big(g(x,t)\cdot u(y), \eta(c^2t^2-|x|^2)\varphi(x+y,t))\Big)
\end{aligned}$$

because $\eta(t)g(x,t) = g(x,t)$ as $\eta(t) = 1$ in the support of g. By differentiating the convolution p times with respect to t , one also infers that

$$g * (u(x)\cdot\delta^{(p)}(t)) = \frac{\partial^p g(x,t)}{\partial t^p} * u(x)$$

A solution to the generalized Cauchy problem for a wave equation can written as

$$u(x,t) = V_N(x,t) + V_N^{(0)}(x,t) + V_N^{(1)}(x,t)$$

where the distributions

$$\begin{aligned}
V_N(x,t) &= (G_N * f)(x,t)\,, \qquad f(x,t)=0\,, \quad t<0\,,\\
V_N^{(0)}(x,t) &= G_N * (u_1(x)\cdot\delta(t)) = G_N(x,t) * u_1(x)\,,\\
V_N^{(1)}(x,t) &= G_N * (u_0(x)\cdot\delta'(t)) = \frac{\partial G_N(x,t)}{\partial t} * u_1(x)\\
&= \frac{\partial}{\partial t}[G_N(x,t) * u_0(x)]\,,
\end{aligned}$$

are called the *wave potential* and *surface wave potentials* with densities f, u_1, and u_0, respectively.

Properties of the wave potential. Suppose that $f(x,t)$ and $u_{0,1}(x)$ are regular distributions. Then an integral representation for the wave potential in the dimensions $N \le 3$ can be deduced by calculating the convolution:

$$V_1(x,t) = \frac{\theta(t)}{2c}\int_0^t \int\limits_{|y-x|<c(t-\tau)} f(y,\tau)\,dy\,d\tau\,,$$

$$V_2(x,t) = \frac{\theta(t)}{2\pi c}\int_0^t \int\limits_{|y-x|<c(t-\tau)} \frac{f(y,\tau)\,d^2y\,d\tau}{\sqrt{c^2(t-\tau)^2-|x-y|^2}}\,,$$

$$V_3(x,t) = \frac{\theta(t)}{4\pi c^2}\int\limits_{|y-x|<ct} \frac{f(y,t-\frac{|x-y|}{c})}{|x-y|}\,d^3y$$

In all integrals, the integration region is a part of the past light cone with vertex at (x,t):

$$\overline{\Gamma^-}_0(x,t) = \{(y,\tau)\in\mathbb{R}^{N+1}|c^2(t-\tau)^2-|y-x|^2\ge 0\,,\ 0\le\tau\le t\}$$

In particular, the wave potential $V_3(x,t)$ is determined by the source values $f(y,\tau)$ at all points y in the ball $|y-x|\le ct$ and taken at retarded time moments $\tau = t-|y-x|/c$ where $|y-x|/c$ is the time needed for any perturbation at a point y to reach point x. In other words, $V_3(x,t)$ depends only on the values of $f(y,\tau)$ on the conic boundary $C_0^-(x,t)$ of $\overline{\Gamma^-}_0(x,t)$. For this reason, the wave potential V_3 is also called a *retarded wave potential*, and the Green's function G_3 is also called a retarded Green's function.

Let us prove the integral representation for V_3. By the theorem about a convolution of distributions with supports in $\overline{\Gamma^+}$ and in the half-space $t\ge 0$, one has for any test function $\varphi\in\mathcal{D}$

$$\begin{aligned}(V_3,\varphi) &= (G_3 * f,\varphi)\\ &= \Big(G_3(z,\tau)\cdot f(x,t), \eta(t)\eta(\tau)\eta(c^2\tau^2-|z|^2)\varphi(x+z),t+\tau)\Big)\\ &= \frac{1}{4\pi c^2}\int\limits_0^\infty \frac{1}{\tau}\int\limits_{|z|=c\tau}\int\limits_0^\infty\int f(x,t)\varphi(x+z,t+\tau)\,d^3x dt dS_z d\tau\end{aligned}$$

where $\eta(s)$ is a C^∞ function such that $\eta(s)=1$ if $s\ge -a$ and $\eta(s)=0$ if $s<-b$ for some $0<a<b$. First, note that owing to a bounded support of a test function, integrand is not zero only if $|t+\tau|<R$ for some R. Since t and τ are non-negative, the integrand is not zero only for a finite range $[0,R]$ of both t and τ. This implies that the range of y is bounded in $\mathbb{R}^3$, $|y|\le cR$. Owing to a bounded support of φ, the integrand is not zero only if $|x+y|<R$, which implies that x must range of a bounded region when the integrand is not zero. Therefore, the integrand is an integrable function of four variables x, y, t, and τ,

and Fubini's theorem applies to change the order of integration in any variables. Let us first shift the variable x by z and t by τ (at given z and τ) and then change the order of integration:

$$(V_3,\varphi) = \frac{1}{4\pi c^2}\int_0^\infty \frac{1}{\tau}\int_{|z|=c\tau}\int_\tau^\infty\int f(x-y,t-\tau)\varphi(x,t)\,d^3x dt dS_z d\tau$$

$$= \frac{1}{4\pi c^2}\int_0^\infty\int \varphi(x,t)\int_0^t\int_{|z|=ct}\frac{f(x-z,t-\tau)}{\tau}\,dS_z d\tau\, d^3x\, dt$$

Define a variable $y\in\mathbb{R}^3$ such that in spherical coordinates $d^3y = r^2 dSdr$ where dS is the measure on the unit sphere $|y|=1$. Setting $r=c\tau$, so that

$$dS_y d\tau = (c\tau)^2 dSd\tau = \frac{1}{c}r^2 dSdr = \frac{1}{c}\,d^3y\,,\quad r=|y|<ct$$

the integrals over τ and the sphere are converted into the integral over a ball:

$$(V_3,\varphi) = \frac{1}{4\pi c^2}\int_0^\infty\int \varphi(x,t)\int_{|y|<ct}\frac{f(x-y,t-\frac{|y|}{c})}{|y|}\,d^3y\,d^3x\,dt$$

$$= \frac{1}{4\pi c^2}\int_0^\infty\int \varphi(x,t)\int_{|y|<ct}\frac{f(y,t-\frac{|x-y|}{c})}{|x-y|}\,d^3y\,d^3x\,dt$$

which completes the proof.

As note before, the wave potentials are unique classical solutions to the corresponding wave equations with zero initial conditions if the inhomogeneity f is sufficiently smooth. The following theorem can be established

THEOREM **43.2**. (Classical wave potential)
If $f\in C^2(t\geq 0)$ when $N=2,3$ and $f\in C^1(t\geq 0)$ when $N=1$, then the wave potential V_N is from class $C^2(t\geq 0)$ and satisfies the following conditions

$$|V_N(x,t)| \leq \frac{t^2}{2}\max_{\overline{\Gamma_0^-}(x,t)}|f(y,\tau)|\,,\quad N=1,2$$

$$|V_3(x,t)| \leq \frac{t^2}{2}\max_{C_0^-(x,t)}|f(y,\tau)|\,,$$

and the initial conditions

$$V_N\Big|_{t=0} = \frac{\partial V_N}{\partial t}\Big|_{t=0} = 0\,.$$

The proof is limited to the case $N = 3$. The cases $N = 1, 2$ can be proved in a similar manner. Put $y = x + ctz$. Then

$$V_3(x, y) = \frac{t^2}{4\pi}\int_{|z|<1} \frac{f(x + ctz, t(1 - |z|))}{|z|}\, d^3z$$

Since $f \in C^2(t \geq 0)$, f and its partial derivatives up to order 2 are bounded on any bounded region. The singularity of the integrand at $z = 0$ is integrable. Therefore $V_3 \in C^2(t \geq 0)$ by the theorem about differentiation of functions defined by an integral. Put

$$M = \max_{|z|\leq 1} |f(x + ctz, t(1 - |z|))| = \max_{C_0^-(x,t)} |f(y, \tau)|$$

The upper bound also follows from the above representation:

$$|V_3(x, t) \leq \frac{t^2 M}{4\pi}\int_{|z|<1} \frac{d^3z}{|z|} = \frac{t^2 M}{2}\,.$$

Properties of the surface potentials. For every $t > 0$, $G_N(x, t)$ is a distribution from $\mathcal{D}'(\mathbb{R}^N)$. Let us investigate the limit properties of G_N in $\mathcal{D}'(\mathbb{R}^N)$ and its partial derivatives with respect to t as $t \to 0^+$. The following relations can be established

$$\lim_{t\to 0^+} G_N(x, t) = 0\,, \quad \lim_{t\to 0^+} \frac{\partial G_N(x, t)}{\partial t} = \delta(x)\,, \quad \lim_{t\to 0^+} \frac{\partial^2 G_N(x, t)}{\partial t^2} = 0$$

where the limits are understood in the distributional sense in $\mathcal{D}'(\mathbb{R}^N)$.

Consider the case $N = 3$. The other dimensions can be treated in a similar way. For any $\varphi(x) \in \mathcal{D}'(\mathbb{R}^3)$, one has

$$\Big(G_3(x, t), \varphi(x)\Big) = \frac{\theta(t)}{4\pi c^2 t}\int_{|x|=ct} \varphi(x)\, dS_x = \frac{\theta(t)t}{4\pi}\int_{|z|=1} \varphi(ctz)\, dS_z$$
$$= \frac{\theta(t)t}{4\pi}\, h(t)\,,$$

where the function

$$h(t) = \int_{|z|=1} \varphi(ctz)\, dS_z$$

is from class C^∞. Indeed, $|\frac{d^p}{dt^p}\varphi(ctz)| = c|(z, \nabla)^p\varphi)| \leq c|D^p\varphi|$ on the sphere $|z| = 1$. Since all partials $D^p\varphi$ are bounded, $h(t) \in C^\infty$ by the theorem about differentiation of a function defined by an integral.

Since h is even, $h(-t) = h(t)$, all its odd derivatives vanish at $t = 0$, that is, $h'(0) = h'''(0) = 0$, etc. It is concluded that

$$\lim_{t\to 0^+} \Big(G_3(x,t), \varphi(x)\Big) = 0$$

for any test function. Similarly, for $t > 0$

$$\lim_{t\to 0^+} \Big(\frac{\partial G_3(x,t)}{\partial t}, \varphi(x)\Big) = \lim_{t\to 0^+} \frac{d}{dt}\frac{th(t)}{4\pi} = \frac{h(0)}{4\pi} = \varphi(0) = (\delta, \varphi)$$

Finally,

$$\lim_{t\to 0^+} \Big(\frac{\partial^2 G_3(x,t)}{\partial t^2}, \varphi(x)\Big) = \lim_{t\to 0^+} \frac{d^2}{dt^2}\frac{th(t)}{4\pi} = \lim_{t\to 0^+} \Big(\frac{h'(t)}{2\pi} + \frac{th''(t)}{4\pi}\Big) = 0\,.$$

For every $t > 0$, the support of $G_N(x,t)$ (as a distribution in $\mathcal{D}'(\mathbb{R}^N)$) is in the ball $|x| \le ct$. Therefore for all $0 < t \le T$, the supports of $G_N(x,t)$ are in one ball $|x| \le cT$. Owing to the theorem about continuity of the convolution of distributions one of which has a bounded support, it is concluded that

$$\begin{aligned}
\lim_{t\to 0^+} V_N^{(0)}(x,t) &= \lim_{t\to 0^+} G_N(x,t) * u_1(x) = 0\,,\\
\lim_{t\to 0^+} \frac{\partial V_N^{(0)}(x,t)}{\partial t} &= \lim_{t\to 0^+} \frac{\partial G_N(x,t)}{\partial t} * u_1(x) = (\delta * u_1)(x) = u_1(x)\,,\\
\lim_{t\to 0^+} V_N^{(1)}(x,t) &= \lim_{t\to 0^+} \frac{\partial G_N(x,t)}{\partial t} * u_0(x) = (\delta * u_0)(x) = u_0(x)\,,\\
\lim_{t\to 0^+} \frac{\partial V_N^{(1)}(x,t)}{\partial t} &= \lim_{t\to 0^+} \frac{\partial^2 G_N(x,t)}{\partial t^2} * u_0(x) = 0\,.
\end{aligned}$$

Thus, the surface wave potentials are distributional solutions in the spatial variable x to the homogeneous wave equation

$$\Big(\Box_c V^{(0,1)}(x,t), \varphi(x)\Big) = 0\,, \quad \varphi \in \mathcal{D}(\mathbb{R}^N)$$

that satisfy *distributional initial conditions* for any choice of $u_{0,1} \in \mathcal{D}'(\mathbb{R}^N)$.

For example, consider the generalized initial value problem

$$\begin{aligned}
&\Box_c u(x,t) = 0\,, \quad u(x,t) = 0\,, \quad t < 0\\
&u\Big|_{t=0} = 0\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \delta(x)
\end{aligned}$$

Its solution is unique and given by the surface potential

$$u(x,t) = V_N^{(0)} = G_N(x,t) * \delta(x) = G_N(x,t)$$

Let $N = 3$. Then the wave equation can be used to describe sound waves where $u(x,t)$ is a deviation of the pressure from a constant background pressure (e.g., atmospheric pressure) at a point x and time $t > 0$. An explosion at $x = 0$ creates an instant change of the pressure at $x = 0$ with respect to time, modeled here by $\delta(x)$. The sound wave created by a point explosion is a spherical wave propagating outward from $x = 0$ with the speed c and whose amplitude is decreasing inversely proportional to the distance traveled:

$$u(x,t) = \frac{1}{4\pi c^2 t}\,\delta_{S_{ct}}(x)\,.$$

In reality, an explosion cannot occur in a point. So, a classical (smooth) solution can be obtained by a regularization of $\delta(x)$, e.g., by a hat function $\omega_a(x)$. In this case, the solution

$$u_a(x,t) = \frac{1}{4\pi c^2 t}\int_{|x-y|=ct} \omega_a(y)\,dS_y$$

has a support in a spherical layer of width $2a$ with the central sphere of radius ct. The solution is from $C^\infty(t > 0)$. Clearly, $u_a(x,t) \to u(x,t)$ in $\mathcal{D}'(\mathbb{R}^N)$ as $a \to 0^+$. So, characteristic physical properties of the distributional and physical solutions are similar, even though the former does not have values (it is a singular distribution).

Let u_0 and u_1 be regular distributions defined by locally integrable functions. By evaluating the convolution, one infers that

$$\begin{aligned}
V_1^{(0)}(x,t) &= \frac{\theta(t)}{2c}\int_{|y-x|<ct} u_1(y)\,dy \\
V_2^{(0)}(x,t) &= \frac{\theta(t)}{2\pi c}\int_{|y-x|<ct} \frac{u_1(y)\,d^2y}{\sqrt{c^2t^2 - |y-x|^2}}\,, \\
V_3^{(0)}(x,t) &= \frac{\theta(t)}{4\pi c^2 t}\int_{|y-x|=ct} u_1(y)\,dS_y\,.
\end{aligned}$$

The surface potential $V_N^{(0)}$ is obtained by replacing $u_1(y)$ by $u_0(y)$ with the subsequent differentiation of the right-hand sides with respect to t in the distributional sense. The following theorem establishes conditions on $u_{0,1}$ under which the surface potentials are classical solutions.

THEOREM **43.3**. (Classical surface wave potentials)
If $u_0 \in C^3$ and $u_1 \in C^2$ when $N = 2, 3$, and $u_0 \in C^2$ and $u_1 \in C^1$ when $N = 1$, the surface wave potentials are from class $C^2(t \geq 0)$, satisfy

the homogeneous wave equation in $t > 0$ and the initial conditions

$$\lim_{t\to 0^+} V_N^{(0)}(x,t) = 0\,, \quad \lim_{t\to 0^+} \frac{\partial V_N^{(0)}(x,t)}{\partial t} = u_1(x)\,,$$
$$\lim_{t\to 0^+} V_N^{(1)}(x,t) = u_0(x)\,, \quad \lim_{t\to 0^+} \frac{\partial V_N^{(1)}(x,t)}{\partial t} = 0\,.$$

Let us prove the result for $V_3^{(0)}$. It is convenient to rewrite it in the following form by shifting and scaling the integration variable so that for $t > 0$

$$\begin{aligned} V_3^{(0)}(x,t) &= \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} u_1(y)\,dS_y = \frac{1}{4\pi c^2 t}\int_{|y|=ct} u_1(x+y)\,dS_y \\ &= \frac{t}{4\pi}\int_{|z|=1} u_1(x+ctz)\,dS_z \end{aligned}$$

If $u_1 \in C^2$, then it is bounded on any bounded set, and by the Lebesgue dominated convergence theorem, $V_3^{(0)}(x,t) \to 0$ as $t \to 0^+$. Furthermore, its partial derivatives Du_1 and D^2u_1 are also bounded on any bounded set. Therefore by the theorem about differentiation of a function defined by an integral, $u \in C^2(t > 0)$ and

$$\begin{aligned} D_x^\alpha V_3^{(0)} &= \frac{t}{4\pi}\int_{|z|=1} D^\alpha u_1(x+ctz)\,dS_z\,, \quad \alpha = 1,2\,, \\ \frac{\partial V_3^{(0)}}{\partial t} &= \frac{1}{4\pi}\int_{|z|=1} u_1(x+ctz)\,dS_z + \frac{ct}{4\pi}\int_{|z|=1} (z,\nabla_x)u_1(x+ctz)\,dS_z\,, \\ \frac{\partial^2 V_3^{(0)}}{\partial t^2} &= \frac{c}{2\pi}\int_{|z|=1} (z,\nabla_x)u_1(x+ctz)\,dS_z \\ &\quad + \frac{c^2t^2}{4\pi}\int_{|z|=1} (z,\nabla_x)^2 u_1(x+ctz)\,dS_z \end{aligned}$$

By the Lebesgue dominated convergence theorem, the integrals of $u_1(x+ctz)$, $(z,\nabla_x)u_1(x+ctz)$, and $(z,\nabla_x)^2u_1(x+ctz)$ are continuous in $t \ge 0$ because $u \in C^2$ and the integration region is compact. Therefore $\frac{\partial V_3^{(0)}}{\partial t} \to u_1(x)$ as $t \to 0^+$. Furthermore, the integral of $(z,\nabla_x)u_1(x)$ over the sphere $|z| = 1$ vanishes by symmetry (the sphere is symmetric under $z \to -z$, while the integrand is skew-symmetric). Therefore $\frac{\partial^2 V_3^{(0)}}{\partial t^2} \to 0$ as $t \to 0^+$ as required. The latter also shows that $V_3^{(0)} \in C^2(t \ge 0)$.

43.5. Solution to the classical Cauchy problem for a wave equation. Owing to the properties of the wave and surface wave potentials, a solution to the classical Cauchy problem is obtained by taking the sum of wave potentials for $t > 0$. For $N = 1$, the solution is given by the *d'Alembert formula*

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)\,dy\,d\tau + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy + \frac{1}{2}\Big(u_0(x+ct) + u_0(x-ct)\Big)$$

by the *Poisson formula* for $N = 2$

$$u(x,t) = \frac{1}{2\pi c}\int_0^t \int_{|y-x|<c(t-\tau)} \frac{f(y,\tau)\,d^2y\,d\tau}{\sqrt{c^2(t-\tau)^2 - |x-y|^2}} + \frac{1}{2\pi c}\int_{|y-x|<ct} \frac{u_1(y)\,d^2y}{\sqrt{c^2t^2 - |x-y|^2}} + \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{|y-x|<ct} \frac{u_0(y)\,d^2y}{\sqrt{c^2t^2 - |x-y|^2}}$$

and for $N = 3$ by the *Kirchhoff formula*

$$u(x,t) = \frac{1}{4\pi c^2}\int_{|y-x|<ct} \frac{f(y, t - \frac{|x-y|}{c})}{|x-y|}\,d^3y + \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} u_1(y)\,dS_y + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\frac{1}{t}\int_{|y-x|=ct} u_0(y)\,dS_y$$

If $u_0 \in C^2$, $u_1 \in C^1$, and $f \in C^1(t \geq 0)$, then $u(x,t)$ for $N = 1$ is twice continuously differentiable as one infers from the fundamental theorem of calculus. It solves the wave equation by construction (as a solution of the generalized Cauchy problem) and satisfies the initial conditions by the limit properties of the wave potentials.

43.6. Hyugens principle. Suppose a wave process was initiated by an external source $f(t,x)$ and the source was terminated after some time, $f(t,x) = 0$, $t > t_0$. If the source is localized in space (that is, the support of $f(t,x)$) is bounded, then the solution given by Kirchhoff's formula also has a bounded support in space:

$$\text{supp} V_3(t,x) \subset B_R$$

for any finite t. The wave propagation for $t > t_0$ can be described by a homogeneous wave equation

$$\Box_c u(t,x) = 0\,, \quad t > t_0\,,$$

with the initial data

$$u\Big|_{t=t_0} = V_3(t_0,x) = u_0(x)\,, \quad u'_t\Big|_{t=t_0} = V'_{3t}(t_0,x) = u_1(x)$$

The boundary of the support of $u(t,x)$ for a given t is called the *wave front*. The question arises, given the wave front at a time t, can one infer the shape of the wave front a later time?

In what follows the time is counted from t_0 so that the wave propagates freely (no sources) for $t > 0$ and has the initial data specified above at $t = 0$. A free wave propagation is described by the surface potentials. Fix a point x in space that is not in the support of the initial data u_1. Then the surface integral in $V_3^{(0)}$ vanish if the sphere of radius ct centered at x is not intersecting the support of u_1:

$$\int\limits_{|x-y|=ct} u_1(y)\,dS_y = 0 \quad \text{if} \quad \operatorname{supp} u_1 \cap S_{ct}(x) = \emptyset$$

In turn, this implies that the support of $V_3^{(0)}$ for a given moment of time t is given by the union of spheres of radius ct with centers being points of the support of u_1 at a time $t > 0$:

$$\operatorname{supp} V_3^{(0)} = \bigcup_{y \in \operatorname{supp} u_1} S_{ct}(y)$$

This shows that every point reached by the wave by $t = 0$ becomes a source of a spherical wave expanding with speed c, and the region occupied by the wave process in time $t > 0$ is a "superposition" of such waves. A similar conclusion holds for the other surface potential (by the same reasons).

Thus, if a wave process was occupying a region Ω in space at a time $t = 0$, then at a time $t > 0$, the process will occupy the region

$$\Omega_t = \bigcup_{y\in\Omega} S_{ct}(y)$$

being the union of all spheres of radius ct that are centered at all points of Ω as if each point of Ω emits a spherical wave that expands with the speed c. This is known as the *Hyugens principle* for wave propagation. It allows to reconstruct the wave front at any later time if it was known at some initial time.

43.7. Exercises.

1. Define a contour delta function by the rule

$$(\delta_C, \varphi) = \int_C \varphi(x)\, ds\,, \quad \varphi \in \mathcal{D}(\mathbb{R}^N)$$

where C is a smooth curve in $\mathbb{R}^N$. Consider the generalized Cauchy problem

$$\begin{aligned} &\Box_c u(x,t) = 0\,, \quad u(x,t) = 0\,,\ t < 0\,,\ x \in \mathbb{R}^3\,, \\ &u\Big|_{t=0} = 0\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \delta_C(x) \end{aligned}$$

(i) Let C be a straight line segment connecting points A and B of length l. Sketch and/or describe the support of the distribution $u(x,t)$ for some $0 < t < l/(2c)$, $l/(2c) < t < l/c$, and t much greater than l/c. (ii) Let C be a circle of radius R. Sketch and/or describe the support of the distribution $u(x,t)$ for some $0 < t < R/c$, $R/c < t < 2R/c$, and t much greater than $2R/c$.

2. Fourier method for the wave equation. In acoustics, a solution $u(x,t)$ to a wave equation defines the acoustic pressure (the local deviation from the ambient pressure) $p(x,t)$ and the velocity vector field $v(x,t)$ (a velocity of a particle in the medium at a point x and time t) by the relations

$$p = -\rho\frac{\partial u}{\partial t}\,, \quad v = \nabla_x u$$

where ρ is the mass density. The energy of an acoustic disturbance in a spatial region Ω is defined as the sum of the potential (compression) and kinetic energy densities integrated over that region, respectively,

$$E_\Omega(t) = \int_\Omega \frac{p^2}{2\rho_0 c^2}\, d^3x + \frac{1}{2}\int_\Omega \rho\, v^2\, d^3x$$

where ρ_0 is the ambient mass density (without any acoustic disturbance). A total energy of any acoustic disturbance should be finite. This implies that partial derivatives of the solution $u(x,t)$ must be square integrable (possibly with some weight). So, if a solution to the classical Cauchy problem exists, that is physically acceptable, then it should be a temperate distribution. This distribution can be found by the Fourier method similarly to the Cauchy problem for the Schrödinger equation.

In the Cauchy problem for the wave equation, assume that the inhomogeneity and the initial data are temperate distribution:

$$f(x,t) \in \mathcal{S}'(t>0)\,, \quad u_{0,1}(x) \in \mathcal{S}'(\mathbb{R}^3)$$

Suppose that the distribution $f(x,t)$ is continuous in the variable $t \geq 0$, that is, for any test function $\varphi(x)$, the distribution

$$g(t) = \Big(f(x,t), \varphi(x)\Big) \in C^0(t \geq 0)$$

is a continuous function in $[0,\infty)$. A solution to the generalized Cauchy problem is sought as a temperate distribution that is continuously differentiable in the time variable $t > 0$,

$$u(x,t) \in \mathcal{S}'(t>0)\,, \quad \Big(u(x,t), \varphi(x)\Big) \in C^1(t \geq 0)\,, \quad \varphi \in \mathcal{S}(\mathbb{R}^3)$$

that satisfies the wave equation and initial conditions in the distributional sense:

$$\begin{aligned} &u''_{tt}(x,t) - c^2 \Delta_x u(x,t) = f(x,t)\,, \quad t>0\,, \\ &\lim_{t\to 0^+} u(x,t) = u_0(x)\,, \quad \lim_{t\to 0^+} u'_t(x,t) = u_1(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^3) \end{aligned}$$

(i) Use the Fourier transform in the variable x to show that if such a solution exists, then it satisfies the initial value problem

$$\begin{aligned} &\left(\frac{d^2}{dt^2} + c^2|k|^2\right) \mathcal{F}_x[u](k,t) = \mathcal{F}_x[f](k,t)\,, \quad t>0\,, \\ &\mathcal{F}_x[u]\Big|_{t=0} = \mathcal{F}[u_0](k)\,, \quad \frac{d}{dt}\mathcal{F}_x[u]\Big|_{t=0} = \mathcal{F}[u_1](k) \quad \text{in } \mathcal{S}'(\mathbb{R}^3) \end{aligned}$$

(ii) Show that under the said assumptions about the distribution $f(x,t)$, the problem has a unique solution given by

$$\begin{aligned} \mathcal{F}_x[u](k,t) = &\int_0^t \frac{\sin[c|k|(t-\tau)]}{c|k|}\, \mathcal{F}[f](k,\tau)\, d\tau \\ &+ \cos(c|k|t)\, \mathcal{F}[u_0](k) + \frac{\sin(c|k|t)}{c|k|}\, \mathcal{F}[u_1](k)\,, \quad t>0\,. \end{aligned}$$

where the integral denotes a particular distributional antiderivative with respect to the variable t that vanishes as $t \to 0^+$ in the distributional sense. Show that $u(x,t) \in \mathcal{S}'(t>0)$ and it is continuously differentiable in the variable $t \geq 0$ and satisfies the initial conditions in the distributional sense.

(iii) Use the Fourier transform of convolution to show that the solution can also be written in as the sum wave and surface wave potentials

$$\begin{aligned} u(x,t) &= V_3(x,t) + V_3^{(0)}(x,t) + V_3^{(1)}(x,t) \\ &= \frac{1}{4\pi c^2}\int_0^t \delta_{S_{c(t-\tau)}}(x) * f(x,\tau)\,\frac{d\tau}{t-\tau} \\ &\quad + \frac{1}{4\pi c^2 t}\,\delta_{S_{ct}}(x) * u_1(x) + \frac{\partial}{\partial t}\frac{1}{4\pi c^2}\,\delta_{S_{ct}}(x) * u_0(x) \end{aligned}$$

where all the convolutions are taken in the variable x, and the integral denotes a particular distributional antiderivative with respect to t that vanishes as $t \to 0^+$ in the distributional sense.

(iv) Show that if f, u_0, and u_1 are regular temperate distributions, then the solution is given by the Kirchhoff formula.

3. Cauchy problem for the telegraph equation. Consider the following Cauchy problem:

$$u''_{tt}(x,t) + 2\gamma u'_t(x,t) - c^2 u''_{xx}(x,t) = f(x,t)\,, \quad t > 0\,,$$
$$u\Big|_{t=0} = u_0(x)\,, \quad u'_t\Big|_{t=0} = u_1(x)\,, \quad x \in \mathbb{R}$$

where γ is a positive constant. It describes a propagation of an electric signal in a conducting wire. A position along the wire is defined by the variable x. The term with a parameter γ models Ohmic losses in the wire. For this reason, this equation is often called a *telegraph equation*.
(i). Find the generalized Cauchy problem in $\mathcal{D}'(\mathbb{R}^2)$.
(ii). Use the Fourier method to find the causal Green's function for the telegraph operator

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial t}{\partial t} - c^2\frac{\partial^2}{\partial x^2}\Big)G(x,t) = \delta(t)\cdot\delta(x)\,, \quad G(x,t) = 0\,,\ t < 0\,.$$

(iii). Show that the solution to the generalized Cauchy problem exists and is unique. Express the solution as the sum of wave and surface wave potentials (by analogy with the solution to the wave equation). Show that the necessary convolutions exist in $\mathcal{D}'(\mathbb{R}^2)$.
(iv). Find integral representation of the wave and surface wave potentials if f, u_0, and u_1 are regular distributions.
(v). Find smoothness conditions on the functions $u_{0,1}(x)$ and $f(x,t)$ under which the solution is from class $C^2(t > 0) \cap C^1(t \geq 0)$.

44. Cauchy problem for Maxwell's equations

In this section, vectors in $\mathbb{R}^3$ will be denoted by boldface letters, unless stated otherwise. For example, a vector field that depends on space-time variables is denoted by $\mathbf{F}(\mathbf{x},t)$, the spatial gradient of a function $f(\mathbf{x})$ is $\boldsymbol{\nabla} f(\mathbf{x})$, the dot and cross products of two vectors are denoted by $(\mathbf{a},\mathbf{b})$ and $\mathbf{a}\times\mathbf{b}$, respectively, the divergence of a vector field is $\operatorname{div}\mathbf{F} = (\boldsymbol{\nabla},\mathbf{F})$, and the curl of a vector field is $\operatorname{curl}\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F}$.

Let $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ be electric and magnetic fields, respectively, $\mathbf{J}(\mathbf{x},t)$ be the electric current density, and $\rho(\mathbf{x},t)$ be the electric charge density. The total electric charge in a spatial region Ω is given by

$$Q(t) = \int_\Omega \rho(\mathbf{x},t)\, d^3x$$

The rate of change of the total charge is determined by the outward flux of the electric current field

$$\frac{dQ}{dt} = \int_{\partial\Omega} (\mathbf{J}, d\boldsymbol{\Sigma}) = \int_{\partial\Omega} (\mathbf{J},\mathbf{n})\, dS$$

where $\mathbf{n}$ is the unit outward normal on the boundary of Ω. If ρ and $\mathbf{J}$ are from the class C^1, then it follows from the divergence theorem that *the charge conservation law* can also be written in the local form

$$\frac{\partial\rho}{\partial t} + (\boldsymbol{\nabla},\mathbf{J}) = 0\,.$$

Maxwell's equations define the electromagnetic fields in the presence of external electric charges whose motion obeys the charge conservation law. In Gaussian units convention, they read

$$\begin{aligned}
\frac{\partial\mathbf{E}}{\partial t} - c\boldsymbol{\nabla}\times\mathbf{B} &= -4\pi\mathbf{J}\,,\\
\frac{\partial\mathbf{B}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{E} &= \mathbf{0}\,,\\
(\boldsymbol{\nabla},\mathbf{E}) &= 4\pi\rho\,,\\
(\boldsymbol{\nabla},\mathbf{B}) &= 0
\end{aligned}$$

where c is the speed of light in the vacuum. Note that the divergence of the electric field is determined by the density of electric charges. Electric charges are sources of the electric field as its outward flux across the boundary of a spatial region is determined by the total electric charge in that region (by the divergence theorem). This is known as the *Gauss law*. The magnetic field has no sources, meaning that, there are no magnetic charges (magnetic monopoles) in nature. The last two equations are *constraints* on the electromagnetic fields that must be satisfied at any moment of time t. This implies that the time derivatives of the constraints must vanish for any solution. This is indeed the case. For example,

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}, \mathbf{E}) = \Big(\boldsymbol{\nabla}, (c\boldsymbol{\nabla} \times \mathbf{B} - 4\pi \mathbf{J})\Big) = -4\pi(\boldsymbol{\nabla}, \mathbf{J}) = 4\pi \frac{\partial \rho}{\partial t}$$

where the first equality is obtained from the first Maxwell's equation and the second follows from the charge conservation law. Thus, the charge conservation is necessary for the consistency of Maxwell's equations.

Energy conservation. Consider electromagnetic fields occupying a bounded spatial region Ω that has no sources. The integral

$$E_\Omega(t) = \frac{1}{8\pi}\int_\Omega \Big(|\mathbf{E}|^2 + |\mathbf{B}|^2\Big)\, d^3x$$

is called the energy of electromagnetic fields. It changes with time. Let us find the rate of change. Assuming that the field are from class C^1 and using Maxwell's equations, one has

$$\begin{aligned}\frac{d}{dt}E_\Omega(t) &= \frac{1}{4\pi}\int_\Omega \Big[\Big(\mathbf{E}, \frac{\partial \mathbf{E}}{\partial t}\Big) + \Big(\mathbf{B}, \frac{\partial \mathbf{B}}{\partial t}\Big)\Big]\, d^3x \\ &= \frac{c}{4\pi}\int_\Omega \Big[\Big(\mathbf{E}, \boldsymbol{\nabla} \times \mathbf{B}\Big) + \Big(\mathbf{B}, \boldsymbol{\nabla} \times \mathbf{E}\Big)\Big]\, d^3x \\ &= -\frac{c}{4\pi}\int_\Omega \Big(\boldsymbol{\nabla}, \mathbf{E} \times \mathbf{B}\Big)\, d^3x \\ &= -\frac{c}{4\pi}\int_{\partial\Omega} \Big(\mathbf{E} \times \mathbf{B}, d\boldsymbol{\Sigma}\Big)\end{aligned}$$

The vector field $\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$ is called the *Poynting vector*. Its outward flux across the boundary $\partial\Omega$ defines the rate at which the electromagnetic energy is decreasing in Ω.

If sources have bounded support and $\partial\Omega$ is a sphere enclosing the sources and its radius is much larger than a diameter of the support of sources, then the field should fall off inversely proportional to the distance from the sources in order to create a constant flow of electromagnetic energy. Such fields are called *radiation or far fields* generated by the sources. The fields whose strength falls off faster are called *near fields*. The near fields do not create any energy flow across a sphere of an arbitrary large radius.

44.1. The Cauchy problem for Maxwell's equations. The Cauchy problem for Maxwell's equations is to find the electromagnetic fields that satisfy Maxwell's equations for $t > 0$ and the initial conditions

$$\mathbf{E}\Big|_{t=0} = \mathbf{E}_0(\mathbf{x})\,, \quad \mathbf{B}\Big|_{t=0} = \mathbf{B}_0(\mathbf{x})\,,$$

for given a given external sources $\mathbf{J}$ and ρ that satisfy the charge conservation law and are smooth enough in order for a classical solution to exist in the class $C^1(t > 0) \cap C^0(t \geq 0)$. The Cauchy problem must also be well-posed, that is, it should have a unique solution and depend continuously on the initial data and sources.

44.2. Vector-valued distributions. A vector-valued distribution is a linear continuous functional of the space of test function whose values are vectors. Any vector field F in $\mathbb{R}^N$ whose components $F_j(y)$ are locally integrable functions in $\mathbb{R}^M$ and are defines a regular vector-valued distribution by the rule

$$(F_j, \varphi) = \int F_j(y)\varphi(y)\, d^M y\,, \quad \varphi \in \mathcal{D}(\mathbb{R}^M)\,, \quad j = 1, 2, ..., N\,.$$

Partial derivatives of a vector-valued distribution are defined in the same way as for scalar-valued distributions:

$$\Big(\frac{\partial F_j}{\partial y_n}, \varphi\Big) = -\Big(F_j, \frac{\partial \varphi}{\partial y_n}\Big)\,.$$

Other operations like the direct product or the Fourier transform or convolution are defined in the same component-wise fashion.

44.3. The generalized Cauchy problem. Let $\mathbf{E}$ and $\mathbf{B}$ be solutions to the classical Cauchy problem. Define regular distributions from $\mathcal{D}'(\mathbb{R}^4)$ by extending the solutions to the half-space $t < 0$ by zeros:

$$\mathbf{E}(\mathbf{x}, t) = 0\,, \quad \mathbf{B}(\mathbf{x}, t) = 0\,, \quad t < 0\,,\ \mathbf{x} \in \mathbb{R}^3$$

Similarly, the electric current density and the charge density are also extended by zeros for $t < 0$ thus becoming regular distributions too

$$\mathbf{J}(\mathbf{x}, t) = 0\,, \quad \rho(\mathbf{x}, t) = 0\,, \quad t < 0$$

The distributional densities satisfy the distributional charge conservation law that follows from classical one

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\boldsymbol{\nabla}, \mathbf{J}) &= \delta(t) \cdot \rho_0(\mathbf{x}) + \Big\{\frac{\partial \rho}{\partial t} + (\boldsymbol{\nabla}, \mathbf{J})\Big\} \\ &= \frac{1}{4\pi}\, \delta(t) \cdot (\boldsymbol{\nabla}, \mathbf{E}_0)\,. \end{aligned} \tag{44.1}$$

It follows from the relation between the distributional and classical derivatives that

$$\frac{\partial \mathbf{E}}{\partial t} = \mathbf{E}_0(\mathbf{x}) \cdot \delta(t) + \left\{ \frac{\partial \mathbf{E}}{\partial t} \right\},$$
$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B}_0(\mathbf{x}) \cdot \delta(t) + \left\{ \frac{\partial \mathbf{B}}{\partial t} \right\},$$

The spatial partial distributional derivatives are equal to the corresponding classical ones. Therefore any classical solution is also a solution to the distributional problem:

$$\begin{aligned}
&\frac{\partial \mathbf{E}}{\partial t} - c\boldsymbol{\nabla} \times \mathbf{B} = -4\pi\mathbf{J} + \delta(t) \cdot \mathbf{E}_0(\mathbf{x})\,,\\
&\frac{\partial \mathbf{B}}{\partial t} + c\boldsymbol{\nabla} \times \mathbf{E} = \delta(t) \cdot \mathbf{B}_0(\mathbf{x})\,,\\
&(\boldsymbol{\nabla}, \mathbf{E}) = 4\pi\rho\,, \quad (\boldsymbol{\nabla}, \mathbf{B}) = 0\,,\\
&\mathbf{E}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}, t) = \mathbf{0}\,, \quad t < 0\,.
\end{aligned}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are vector-valued distributions from $\mathcal{D}'(\mathbb{R}^3)$ such that $(\boldsymbol{\nabla}, \mathbf{B}_0) = 0$, the distributions $\mathbf{J}(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ are from $\mathcal{D}'(\mathbb{R}^4)$. They vanish in the open half-space $t < 0$ and satisfy the distributional charge conservation law (**44.1**) The problem can be further generalized.

A generalized Cauchy problem for Maxwell's equations is to find vector-valued distributions $\mathbf{E}$ and $\mathbf{B}$ that vanish in the half-space $t < 0$ and satisfy the equations:

$$\begin{aligned}
&\frac{\partial \mathbf{E}}{\partial t} - c\boldsymbol{\nabla} \times \mathbf{B} = -4\pi\mathbf{J}_e\,, \quad \frac{\partial \mathbf{B}}{\partial t} + c\boldsymbol{\nabla} \times \mathbf{E} = -\mathbf{J}_m\,,\\
&(\boldsymbol{\nabla}, \mathbf{E}) = 4\pi\rho_e\,, \quad (\boldsymbol{\nabla}, \mathbf{B}) = \rho_m,\,,
\end{aligned}$$

for any vector-valued distributions $\mathbf{J}_e$ and $\mathbf{J}_m$ and any scalar distributions ρ_e and ρ_m that vanish in the half-space $t < 0$ and satisfy the electric and magnetic charge conservation laws:

$$\frac{\partial \rho_e}{\partial t} + (\boldsymbol{\nabla}, \mathbf{J}_e) = 0\,, \quad \frac{\partial \rho_m}{\partial t} + (\boldsymbol{\nabla}, \mathbf{J}_m) = 0$$

The conservation laws are needed for the consistency of the problem. The last pair of equations in the generalized Cauchy problem should hold for any t so they are distributional constraints, and hence the time derivatives of their left- and right-hand sides must be equal for any distributional solution, which is guaranteed by the conservation laws.

It should be emphasized that in the generalized Cauchy problem, the distributions $\rho_{e,m}$ are uniquely determined by the conservation laws because $\rho_{e,m}(\mathbf{x}, t) = 0$ for $t < 0$. A general solution to the conservation law is given by $\rho_{e,m} = -D_t^{-1}(\boldsymbol{\nabla}, \mathbf{J}_{e,m})$. A distributional antiderivative is unique up to an additive constant in t which is a distribution of $\mathbf{x}$. This constant distribution must be chosen so that the corresponding particular time antiderivative vanishes for $t < 0$. Let us denote this particular time antiderivative by using definite integral notations:

$$\rho_{e,m}(\mathbf{x}, t) = -\int_0^t (\boldsymbol{\nabla}, \mathbf{J}_{e,m}(\mathbf{x}, \tau))\, d\tau$$

So, the electric and magnetic currents, $\mathbf{J}_{e,m}$, are the only independent inhomogeneities in the generalized Cauchy problem for Maxwell's equations.

A classical solution is contained among solutions when the electric and magnetic currents are

$$\mathbf{J}_e = \mathbf{J}(\mathbf{x}, t) - \frac{1}{4\pi}\delta(t) \cdot \mathbf{E}_0(\mathbf{x})\,, \quad \rho_e = \rho\,, \tag{44.2}$$

$$\mathbf{J}_m = -\delta(t) \cdot \mathbf{B}_0(\mathbf{x})\,, \quad \rho_m = 0 \tag{44.3}$$

with distributions $\mathbf{J}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ being smooth enough. In this case,

$$\rho(\mathbf{x}, t) = -\int_0^t (\boldsymbol{\nabla}, \mathbf{J}(\mathbf{x}, \tau))\, d\tau + \frac{1}{4\pi}\,\theta(t) \cdot (\boldsymbol{\nabla}, \mathbf{E}_0(\mathbf{x}))\,.$$

The last term describes a possibility that the initial electric field has sources that are unrelated to external electric current $\mathbf{J}$ that was turn on at $t = 0$.

44.4. Solving the generalized Cauchy problem. Distributions can be differentiated any number of times and partial derivatives obey Clairaut's theorem. Let us differentiate the first pair of equations with respect to time and combine them to get the second-order equations separately for $\mathbf{E}$ and $\mathbf{B}$. For example,

$$\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} - 4\pi \frac{\partial \mathbf{J}_e}{\partial t} \\
\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} &= -\boldsymbol{\nabla} \times \Big(c\boldsymbol{\nabla} \times \mathbf{E} + \mathbf{J}_m\Big) \\
&= c\Delta \mathbf{E} - c\boldsymbol{\nabla}(\boldsymbol{\nabla}, \mathbf{E}) - c\boldsymbol{\nabla} \times \mathbf{J}_m \\
&= c\Delta \mathbf{E} - 4\pi c \boldsymbol{\nabla}\rho_e - c\boldsymbol{\nabla} \times \mathbf{J}_m
\end{aligned}$$

and similarly for the second time derivative of $\mathbf{B}$. In doing so, it is concluded that the distribution $\mathbf{E}$ and $\mathbf{B}$ are solutions to the generalized

Cauchy problem for the wave equations:

$$\Big(\frac{\partial^2}{\partial t^2} - c^2\Delta\Big)\mathbf{E} = -4\pi c^2\boldsymbol{\nabla}\rho_e - c\boldsymbol{\nabla}\times\mathbf{J}_m - 4\pi\frac{\partial \mathbf{J}_e}{\partial t} \equiv \mathbf{H}_e\,,$$
$$\Big(\frac{\partial^2}{\partial t^2} - c^2\Delta\Big)\mathbf{B} = -c^2\boldsymbol{\nabla}\rho_m + 4\pi c\boldsymbol{\nabla}\times\mathbf{J}_e - \frac{\partial \mathbf{J}_m}{\partial t} \equiv \mathbf{H}_m\,.$$

Note that the vector-valued distributions $\mathbf{H}_e$ and $\mathbf{H}_m$ vanish for $t<0$. Furthermore, the distributional wave equation holds for each component independently. So, these are equations are nothing but six scalar generalized Cauchy problems for the wave equation. Its solution exists and is unique and given by the convolution

$$\mathbf{E}(x,t) = (G_3 * \mathbf{H}_e)(\mathbf{x},t)\,,\quad \mathbf{B}(x,t) = (G_3 * \mathbf{H}_m)(\mathbf{x},t)\,,$$
$$G_3(\mathbf{x},t) = \frac{\theta(t)}{4\pi c^2 t}\delta_{S_{ct}}(\mathbf{x})\,.$$

A substitution of $\mathbf{H}_{e,m}$ with the electric and magnetic currents given in (**44.2**) and (**44.2**) yields the following distributional solutions

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t} - \boldsymbol{\nabla}\times\mathbf{A}_b\,, \tag{44.4}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\times\mathbf{A}_e - \frac{1}{c}\frac{\partial \mathbf{A}_b}{\partial t} \tag{44.5}$$

where the distributions Φ, $\mathbf{A}$, $\mathbf{A}_e$, and $\mathbf{A}_b$ are called the *scalar potential, vector potential, surface (electric and magnetic) vector potentials*, respectively. They are given by the convolutions

$$\Phi(\mathbf{x},t) = 4\pi c^2 (G_3 * \rho)(\mathbf{x},t)\,, \tag{44.6}$$

$$\mathbf{A}(\mathbf{x},t) = 4\pi c (G_3 * \mathbf{J})(\mathbf{x},t)\,, \tag{44.7}$$

$$\mathbf{A}_f(\mathbf{x},t) = -cG_3(\mathbf{x},t) * \mathbf{F}_0(\mathbf{x}) \tag{44.8}$$

where f is either e or b when $\mathbf{F}_0$ is either $\mathbf{E}_0$ or $\mathbf{B}_0$, respectively. Before studying classical solutions, let us investigate properties of some special distributional solutions that are often used in applications of Maxwell's equations.

44.5. Waves generated by initial distributions of electromagnetic fields. Suppose that

$$\mathbf{J}(\mathbf{x},t) = \mathbf{0}$$

Then the electric charge conservation (**44.1**) requires that

$$\rho(\mathbf{x},t) = \frac{1}{4\pi}\,\theta(t)\cdot(\boldsymbol{\nabla},\mathbf{E}_0(\mathbf{x}))$$

Note that this is the only distributional solution to (**44.1**) that vanishes for $t < 0$. In this case, the electromagnetic fields are uniquely determined by their initial configurations. Let us evaluate the convolution $G_3 * (\theta(t) \cdot h(\mathbf{x}))$ for any $h \in \mathcal{D}'(\mathbb{R}^3)$. Using the theorem about the convolution of a distribution with supports in the future light cone,

$$
\begin{aligned}
&\Big(G_3 * (\theta(t) \cdot h(\mathbf{x})), \varphi(\mathbf{x}, t)\Big) \\
&= \int_0^\infty \frac{1}{4\pi c^2 t} \int_{|\mathbf{x}|=ct} \int_0^\infty \Big(h(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y}, t+\tau)\Big)\, d\tau\, dS_x\, dt \\
&= \int_0^\infty \frac{1}{4\pi c^2 t} \int_t^\infty \int_{|\mathbf{x}|=ct} \Big(h(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y}, s)\Big)\, dS_x\, ds\, dt \\
&= \int_0^\infty \int_0^s \frac{1}{4\pi c^2 t} \int_{|\mathbf{x}|=ct} \Big(h(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y}, s)\Big)\, dS_x\, dt\, ds \\
&= \int_0^\infty \int_0^{sc} \frac{1}{4\pi c^2 r} \int_{|\mathbf{x}|=r} \Big(h(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y}, s)\Big)\, dS_x\, dr\, ds \\
&= \int_0^\infty \int_{|\mathbf{x}|<tc} \Big(h(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y}, t)\Big) \frac{d^3x}{4\pi c^2 |\mathbf{x}|}\, dt \\
&= \Big(\frac{\theta(ct - |\mathbf{x}|)}{4\pi c^2 |\mathbf{x}|} \cdot h(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y}, t)\Big)
\end{aligned}
$$

The second equality follows from a change of the integration variable $s = t + \tau$. The third equality is obtained by reversing the order of integration. The fourth one is deduced by setting $r = ct$. The fifth equality follows from that $dS_x dr = r^2 dS dr = d^3x$ where dS is the measure on the unit sphere. The final equality is by definition of the direct product of distributions. It also shows that

$$
G_3 * (\theta(t) \cdot h(\mathbf{x})) = g_3(\mathbf{x}, t) * h(\mathbf{x}), \quad g_3(\mathbf{x}, t) = \frac{\theta(ct - |\mathbf{x}|)}{4\pi c^2 |\mathbf{x}|}
$$

where the convolution in the right-hand side is taken with respect to $\mathbf{x}$. It is interesting to note that the third equality shows that the distribution g_3 is the time antiderivative of G_3 that vanishes for $t < 0$:

$$
\frac{\partial}{\partial t} g_3(\mathbf{x}, t) = G_3(\mathbf{x}, t), \quad g_3(\mathbf{x}, t) = 0, \quad t < 0.
$$

so that using the integral notation, the only solution to this equation reads

$$
g_3(\mathbf{x}, t) = \int_0^t G_3(\mathbf{x}, \tau)\, d\tau = \frac{\theta(ct - |\mathbf{x}|)}{4\pi c^2 |\mathbf{x}|}
$$

By construction g_3 is continuous in t (because G_3 is continuous in t as was shown earlier), and

$$\lim_{t\to 0^+} g_3(\mathbf{x},t) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3)\,.$$

Thus, the solution to the generalized Cauchy problem has the form

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= -c^2\boldsymbol{\nabla} g_3(\mathbf{x},t) * (\boldsymbol{\nabla},\mathbf{E}_0(\mathbf{x})) + \frac{\partial}{\partial t}G_3(\mathbf{x},t) * \mathbf{E}_0(\mathbf{x}) \\ &\quad + c\boldsymbol{\nabla}\times(G_3(\mathbf{x},t) * \mathbf{B}_0(\mathbf{x})) \\ \mathbf{B}(\mathbf{x},t) &= \frac{\partial}{\partial t}G_3(\mathbf{x},t) * \mathbf{B}_0(\mathbf{x}) - c\boldsymbol{\nabla}\times(G_3(\mathbf{x},t) * \mathbf{E}_0(\mathbf{x}))\end{aligned}$$

Owing to the continuity of g_3 and G_3 in the variable t, the distributional solutions are also continuous in t. The initial condition holds in the distributional sense in $\mathcal{D}'(\mathbb{R}^3)$

$$\lim_{t\to 0^+} \mathbf{E}(\mathbf{x},t) = \mathbf{E}_0(\mathbf{x})\,, \quad \lim_{t\to 0^+} \mathbf{B}(\mathbf{x},t) = \mathbf{B}_0(\mathbf{x})\,,$$

thanks to the limit properties $G_3 \to 0$, $\frac{\partial}{\partial t}G_3 \to \delta$, and $g_3 \to 0$ in $\mathcal{D}'$ as $t \to 0^+$ (that were established in the previous section), and to the continuity of the convolution of distributions of this type.

A decay of an electric string. Define a line delta function δ_C that is supported on a smooth curve C of length L which connects points $\mathbf{x}_1$ and $\mathbf{x}_2$ by the rule

$$\Big(\delta_C(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2),\varphi(\mathbf{x})\Big) = \int_C \varphi(\mathbf{x})\,ds$$

Let $\mathbf{e}$ be a unit tangent vector to the curve C. Put

$$\mathbf{B}_0(\mathbf{x}) = \mathbf{0}\,, \quad \mathbf{E}_0(\mathbf{x}) = \mathbf{e}\delta_C(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2)$$

If $\mathbf{x} = \mathbf{x}(s)$ is a natural parameterization, then $\mathbf{e} = \mathbf{x}'(s)$, and

$$\Big(\mathbf{E}_0(\mathbf{x}),\varphi(\mathbf{x})\Big) = \int_0^L \mathbf{x}'(s)\varphi(\mathbf{x}(s))\,ds\,, \quad \varphi\in\mathcal{D}\,,$$

For example, if C is a line segment from $\mathbf{x}_1$ to $\mathbf{x}_2$, then

$$\mathbf{x}(s) = \frac{s}{L}\mathbf{x}_1 + \frac{L-s}{L}\mathbf{x}_2\,, \quad 0\le s\le L\,, \quad \mathbf{e} = \mathbf{x}'(s) = \frac{\mathbf{x}_2-\mathbf{x}_1}{L}$$

So, at the initial moment of time there is no magnetic field and the electric field is confined into a smooth curve, it is tangent to the curve and has a unit magnitude. A single flow line of the initial vector field originates from the point $\mathbf{x}_1 = \mathbf{x}(0)$ and terminates at $\mathbf{x}_2 = \mathbf{x}(L)$.

Therefore this field should have sources (charges). To find their density, let us compute the distributional divergence of $\mathbf{E}_0$:

$$\Big((\boldsymbol{\nabla}, \mathbf{E}_0), \varphi\Big) = -\Big(\mathbf{E}_0, \boldsymbol{\nabla}\varphi\Big) = -\int_0^L (\mathbf{x}'(s), \boldsymbol{\nabla})\varphi(\mathbf{x}(s))\, ds$$
$$= -\int_0^L \frac{d}{ds}\varphi(\mathbf{x}(s))\, ds = \varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2)$$

This means that

$$(\boldsymbol{\nabla}, \mathbf{E}_0) = \delta(\mathbf{x} - \mathbf{x}_1) - \delta(\mathbf{x} - \mathbf{x}_2)$$

So, the sources are two point-like opposite charges located at the end-point of the electric string. Define the surface scalar potential by

$$\Phi_0(\mathbf{x}, t) = c^2 g_3(\mathbf{x}, t) * (\boldsymbol{\nabla}, \mathbf{E}_0(\mathbf{x}))$$
$$= \frac{\theta(ct - |\mathbf{x} - \mathbf{x}_1|)}{4\pi|\mathbf{x} - \mathbf{x}_1|} - \frac{\theta(t - |\mathbf{x} - \mathbf{x}_2|)}{4\pi|\mathbf{x} - \mathbf{x}_2|}$$

Then it follows from (**44.4**) and (**44.5**) that the electric string decays according to the distributional solution

$$\mathbf{E}(\mathbf{x}, t) = -\boldsymbol{\nabla}\Phi_0(\mathbf{x}, t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_e(\mathbf{x}, t)\,, \quad \mathbf{B}(\mathbf{x}, t) = \boldsymbol{\nabla} \times \mathbf{A}_e(\mathbf{x}, t)\,.$$

Recall the properties of surface wave potentials discussed in the previous section (the Huygens-Fresnel principle). The support of the terms containing the distribution $\mathbf{A}_e$ is the union of spheres of radius ct and centered at all points of support of $\mathbf{E}_0$. If C is a line segment, then the union of the spheres form a solid without any cavity for $ct < L/2$. For $L/2 < ct < L$, a cavity appears. It is symmetric under rotations about the line C and has the largest radius in the plane perpendicular to the line C and passing through its midpoint. It is the intersection of two balls of radius ct that are centered at $\mathbf{x}_{1,2}$. For $ct > L$, the cavity contains the whole line segment C. Eventually, the cavity expands to the whole space for $ct \gg L$. The cavity is never empty. It contains the Coulomb field of two opposite charges at $\mathbf{x}_1$ and $\mathbf{x}_2$. Note that Φ is a Coulomb potential of two such point charges, but the support of Φ is the union of two balls of radius ct centered at $\mathbf{x}_1$ and $\mathbf{x}_2$. So, the Coulomb field occupies only these balls. As ct becomes larger than $L/2$, the balls have a common region that is exactly the cavity in the support of the other terms in the solution. So, the cavity is always filled with the Coulomb field of two opposite charges. As the cavity expands, the Coulomb field fills out the whole space, which is the asymptotic stationary state of the electromagnetic fields as $t \to \infty$.

44.6. Wave generated by external electric currents. Let $\mathbf{E}_0 = \mathbf{B}_0 = \mathbf{0}$. In this case, the surface vector potentials vanish and the electromagnetic fields are determined only by the scalar and vector potentials and given in (**44.4**) and (**44.5**) with $\mathbf{A}_e = \mathbf{A}_b = \mathbf{0}$. This solution describes an electromagnetic radiation generated by external electric currents. As an example, let us solve the radiation problem for an electric dipole that is often used to model the electromagnetic radiation by simplest antennas.

An electric dipole radiation. Let $\mathbf{p}(t)$ be a vector-valued distribution of the time variable that vanishes for $t < 0$. Put

$$\mathbf{J}(\mathbf{x}, t) = -\dot{\mathbf{p}}(t)\delta(\mathbf{x})\,, \quad \rho(\mathbf{x}, t) = (\mathbf{p}(t), \boldsymbol{\nabla})\delta(\mathbf{x})\,, \quad \mathbf{p}(t) = 0\,,\ t < 0\,.$$

Here a common convention to use a dot to denote the time derivative is used for distributions that depends only on time, that is, $\dot{\mathbf{p}}(t) = \frac{d}{dt}\mathbf{p}(t)$. The product of distributions is understood as the direct product (the dot notation is omitted for brevity). The charge conservation law is satisfied for any $\mathbf{p}(t)$.

Let us calculate the potentials. For any test function $\varphi(\mathbf{x}, t)$

$$\begin{aligned}(\mathbf{A}, \varphi) &= -4\pi c \int_0^\infty \frac{1}{4\pi c^2 \tau} \int_{|x|=c\tau} \Big(\dot{\mathbf{p}}(t), \varphi(\mathbf{x}, t+\tau)\Big)\, dS_x\, d\tau \\ &= -\frac{1}{c} \int \frac{1}{|\mathbf{x}|} \Big(\dot{\mathbf{p}}(t), \varphi(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c})\Big)\, d^3x \\ &\stackrel{\text{def}}{=} -\frac{1}{c} \Big(\frac{\dot{\mathbf{p}}(t - \frac{|\mathbf{x}|}{c})}{|\mathbf{x}|}, \varphi(\mathbf{x}, t) \Big)\end{aligned}$$

The first equality readily follows from the theorem about the convolution of distributions supported in the future light cone and the explicit form of Green's function G_3. The second equality is obtained by setting $r = c\tau$ and using the volume measure $dS_x dr = r^2 dS dr = d^3x$ where dS is the area measure on a unit sphere. The last equality serves as a definition. Recall the definition of a shifted distribution. Here $\mathbf{p}(t)$ is a distribution of a single real variable t, whereas a "shifted" distribution $\mathbf{p}(t - \frac{|\mathbf{x}|}{c})$ becomes a distribution of four variables and, hence, its action on a test function of four variables should be defined. If $\mathbf{p}(t)$ is a regular distribution, then the function $\mathbf{p}(t - \frac{|\mathbf{x}|}{c})$ is a regular distribution of four variables. So, the last equality is a definition only if $\mathbf{p}(t)$ is a singular distribution. The scalar potential Φ is calculated in a similar fashion. Since the convolution exists, the gradient operator $\boldsymbol{\nabla}$ can be applied after calculating the convolution of G_3 with $\mathbf{p}(t)\delta(\mathbf{x})$, leading to

$$\Phi(\mathbf{x},t) = \Big(\boldsymbol{\nabla}, \frac{\mathbf{p}(t-\frac{|\mathbf{x}|}{c})}{|\mathbf{x}|}\Big), \quad \mathbf{A}(\mathbf{x},t) = -\frac{\dot{\mathbf{p}}(t-\frac{|\mathbf{x}|}{c})}{c|\mathbf{x}|}$$

To find an explicit form of Φ and the electromagnetic fields, one has to express spatial derivatives of the shifted distribution $\mathbf{p}$ in terms of its time derivatives, and for that a few properties of the shifted distribution $\mathbf{p}$ need to be established.

Additional properties of $\mathbf{p}(t-\frac{|\mathbf{x}|}{c})$. Let us investigate smoothness properties of the function

$$P(x) = \Big(p(t), \varphi(x, t+|x|)\Big), \quad p \in \mathcal{D}'_+(\mathbb{R}), \quad \varphi \in \mathcal{D}(\mathbb{R}^{N+1})$$

that is, the distribution $p(t)$ vanishes for $t < 0$. One can see that $P(x) = g(x, |x|)$ where

$$g(x,s) = \Big(p(t), \varphi(x, t+s)\Big) = \Big(p(t), \eta(t)\varphi(x, t+s)\Big)$$

and $\eta(t) = 1$ for $t > -a$, $\eta(t) = 0$ for $t < -b$ for any $0 < a < b$, and $\eta \in C^\infty$. By the consistency theorem, $g(x,s)$ is a test function in the variable x for any real s. It is also continuous in s because p is a continuous functional. If $s_n \to s$ as $n \to \infty$, then the sequence of test functions converges in $\mathcal{D}$: $\varphi_n(x,t) = \varphi(x, t+s_n) \to \varphi(x, t+s)$ and, hence, $g(x, s_n) \to g(x,s)$. Similarly, g is differentiable in s:

$$\begin{aligned} \frac{\partial g(x,s)}{\partial s} &= \lim_{\epsilon\to 0}\Big(p(t), \frac{\varphi(x,t+s+\epsilon)-\varphi(x,t+s)}{\epsilon}\Big) \\ &= \Big(p(t), \frac{\partial}{\partial s}\varphi(x,t+s)\Big) \end{aligned}$$

The partial derivative is continuous by the continuity of the functional p because all derivatives of a test function are tests functions. This argument holds for partial derivatives of any order. Thus, $g(x,s)$ is from class $C^\infty(\mathbb{R}^{N+1}$ with bounded support in the variable x and

$$D_x^\alpha D_s^\beta g(x,s) = \Big(p(t), D_x^\alpha D_t^\beta \varphi(x, t+s)\Big) = \Big(D_t^\beta p(t), D_x^\alpha \varphi(x,t+s)\Big).$$

Therefore

$$P(x) = g(x, |x|) \in C^\infty(x \neq 0) \cap C^0, \quad P(x) = 0, \quad |x| > R$$

for some $R > 0$. The derivatives of $P(x)$ are calculated by the chain rule for $x \neq 0$

$$\begin{aligned}\frac{\partial P}{\partial x_j} &= \frac{\partial g(x,s)}{\partial x_j}\Big|_{s=|x|} + \frac{\partial g(x,s)}{\partial s}\Big|_{s=|x|}\frac{x_j}{|x|} \\ &= \Big(p(t), \frac{\partial \varphi(x,s)}{\partial x_j}\Big|_{s=t+|x|} + \frac{x_j}{|x|}\frac{\partial \varphi(x,s)}{\partial s}\Big|_{s=t+|x|}\Big) \\ &= \Big(p(t), \frac{\partial \varphi(x,s)}{\partial x_j}\Big|_{s=t+|x|}\Big) - \Big(\dot{p}(t), \varphi(x, t+|x|)\Big)\frac{x_j}{|x|}\end{aligned}$$

where the latter equality follows from the definition of the derivative $\dot{p}(t)$. So, $P(x)$ and its first partial are bounded functions. Classical higher order derivatives are obtained similarly. But it should be noted that they are not bounded in a neighborhood of $x = 0$ and, hence, not locally integrable in general.

For small $|x|$, P has the following asymptotic behavior

$$P(x) = g(0,0) + x_j \frac{\partial g}{\partial x_j}\Big|_{x=0,s=0} + |x|\frac{\partial g}{\partial s}\Big|_{x=0,s=0} + O(|x|^2)$$

It follows from this consideration that the vector fields

$$\mathbf{P}_\alpha(\mathbf{x}) = \Big(D_t^\alpha \mathbf{p}(t), \varphi(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c})\Big)$$

are continuous and have bounded supports for any test function φ. Their asymptotic behavior near the origin has the form

$$\mathbf{P}_\alpha(\mathbf{x}) = \mathbf{P}_\alpha(\mathbf{0}) + M_\alpha \mathbf{x} + b_\alpha|\mathbf{x}| + O(|\mathbf{x}|^2)$$

for some 3×3 matrices M_α and constants b_α and for any $\alpha \geq 0$.

Let us calculate first partial derivatives of $\mathbf{P}_\alpha$. Put $\partial_j = \frac{\partial}{\partial x_j}$ for brevity, and for any test function $\varphi(\mathbf{x}, t)$

$$\varphi_j'(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c}) \stackrel{\text{def}}{=} \partial_j \varphi(\mathbf{x}, \tau)\Big|_{\tau = t + \frac{|\mathbf{x}|}{c}},$$

that is, first the partial derivative is computed and then the time variable is shifted. Define

$$\mathbf{P}_{\alpha j}(\mathbf{x}) = \Big(D_t^\alpha \mathbf{p}(t), \varphi_j'(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c})\Big)$$

These vector fields have the same smoothness properties as $\mathbf{P}_\alpha$. The following identity follows from the chain rule:

$$\partial_j \varphi(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c}) = \varphi_j'(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c}) + \varphi_t'(\mathbf{x}, t + \tfrac{|\mathbf{x}|}{c})\frac{x_j}{c|\mathbf{x}|}$$

where φ'_t denotes the derivative of φ with respect to t. Using this identity to express φ'_j in the definition of $\mathbf{P}_{\alpha j}$ and the chain rule for derivatives of $\mathbf{P}_\alpha$ established above, one infers that

$$\partial_j \mathbf{P}_\alpha \mathbf{x} = \mathbf{P}_{\alpha j}(\mathbf{x}) - \frac{x_j}{c|\mathbf{x}|}\,\mathbf{P}_{\alpha+1}(\mathbf{x})$$

This identity is helpful for calculating the distributional electromagnetic fields.

Calculation of the electromagnetic fields. Then

$$\begin{aligned}
&\Big(\partial_j \frac{\mathbf{p}(t-\frac{|\mathbf{x}|}{c})}{|\mathbf{x}|}, \varphi(\mathbf{x},t)\Big) = -\Big(\frac{\mathbf{p}(t-\frac{|\mathbf{x}|}{c})}{|\mathbf{x}|}, \partial_j\varphi(\mathbf{x},t)\Big)\\
&= -\int \frac{1}{|\mathbf{x}|}\Big(\mathbf{p}(t), \varphi'_j(\mathbf{x}, t+\tfrac{|\mathbf{x}|}{c})\Big)\, d^3x = -\int \frac{\mathbf{P}_{0j}(\mathbf{x})}{|\mathbf{x}|}\, d^3x\\
&= -\lim_{a\to 0^+}\int_{|\mathbf{x}|>a} \Big(\frac{1}{|\mathbf{x}|}\,\partial_j\mathbf{P}_0(\mathbf{x}) + \frac{x_j}{c|\mathbf{x}|^2}\,\mathbf{P}_1(\mathbf{x})\Big)\, d^3x\\
&= -\int \Big(\frac{x_j}{|\mathbf{x}|^3}\mathbf{P}_0(\mathbf{x}) + \frac{x_j}{c|\mathbf{x}|^2}\,\mathbf{P}_1(\mathbf{x})\Big)\, d^3x
\end{aligned}$$

The first equality is by the definition of a distributional derivative. The second one is by the definition of the shifted distribution $\mathbf{p}$. Then the field $\mathbf{P}_{\alpha j}$ was used to obtain the third equality. The fourth equality follows from the continuity of the Lebesgue integral and the equation for derivatives $\partial_j \mathbf{P}_\alpha$. The last equality is obtained by integration by parts. Since the functions $\mathbf{P}_\alpha$ were shown to have a bounded support, the integration region is limited to a ball of a sufficiently large radius. The integrand is from class $C^\infty(|\mathbf{x}| > a)$ for any $a > 0$. However it has a singularity at $\mathbf{x} = \mathbf{0}$ so that the divergence theorem does not apply in the whole integration region. The singularity is locally integrable. So, it was necessary to use the continuity of the Lebesgue integral by removing a ball B_a from the integration region and then taking the limit $a \to 0^+$ after integrating by parts. One can show that the surface integral over the sphere $|\mathbf{x}| = a$ is proportional to a and, hence, vanishes in the said limit. This technical detail is left to the reader to verify. The integrand is locally integrable after integration by parts because its absolute value behaves as $|\mathbf{x}|^{-2}$ near the origin due to the asymptotic properties of $\mathbf{P}_\alpha$. Therefore the regularization can be removed, thus leading to the final result:

$$\partial_j \frac{\mathbf{p}(t-\frac{|\mathbf{x}|}{c})}{|\mathbf{x}|} = -\frac{x_j\mathbf{p}(t-\frac{|\mathbf{x}|}{c})}{|\mathbf{x}|^3} - \frac{x_j\dot{\mathbf{p}}(t-\frac{|\mathbf{x}|}{c})}{c|\mathbf{x}|^2}\,.$$

If $\mathbf{p}(t)$ is a smooth function, then this relation is nothing but the chain rule. It follows that

$$\Phi(\mathbf{x},t) = -\frac{(\mathbf{x},\mathbf{p}(t_r))}{|\mathbf{x}|^3} - \frac{(\mathbf{x},\dot{\mathbf{p}}(t_r))}{c|\mathbf{x}|^2}$$

$$\mathbf{B}(\mathbf{x},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t) = \frac{\mathbf{x}\times\dot{\mathbf{p}}(t_r)}{c|\mathbf{x}|^3} + \frac{\mathbf{x}\times\ddot{\mathbf{p}}(t_r)}{c^2|\mathbf{x}|^2}\,,$$

where

$$t_r = t - \tfrac{|\mathbf{x}|}{c}$$

is called the retarded time.

To compute $\mathbf{E}$, note that the scalar potential Φ has an integrable singularity $|\Phi| \sim |\mathbf{x}|^{-2}$ if $\mathbf{p}(t)$ is a bounded function. Therefore its distributional gradient is no longer a locally integrable function and, hence, it is a singular distribution. This can already be anticipated because the electric field satisfies the Gauss law:

$$(\boldsymbol{\nabla},\mathbf{E}) = 4\pi\rho = 4\pi(\mathbf{p}(t),\boldsymbol{\nabla})\delta(\mathbf{x})\,.$$

The gradient of the first term in Φ produces a singular distribution.

Using the vectors $\mathbf{P}_\alpha$, the distributional gradient of Φ can be written as follows

$$\Big(\partial_j\Phi,\varphi\Big) = -\Big(\Phi,\partial_j\varphi\Big) = \lim_{a\to 0^+}\int_{|\mathbf{x}|>a}\Big(\frac{(\mathbf{x},\mathbf{P}_{0j})}{|\mathbf{x}|^3} + \frac{(\mathbf{x},\mathbf{P}_{1j})}{c|\mathbf{x}|^2}\Big)\,d^3x$$

$$= \lim_{a\to 0^+}\int_{|\mathbf{x}|>a}\Big(\frac{(\mathbf{x},\partial_j\mathbf{P}_0)}{|\mathbf{x}|^3} + \frac{(\mathbf{x},\partial_j\mathbf{P}_1)}{c|\mathbf{x}|^2} + \frac{x_j(\mathbf{x},\mathbf{P}_1)}{c|\mathbf{x}|^4} + \frac{x_j(\mathbf{x},\mathbf{P}_2)}{c^2|\mathbf{x}|^3}\Big)\,d^3x$$

One has to express this integral only in terms of $\mathbf{P}_\alpha$ in order to find the distributional derivative via the shifted $\mathbf{p}$ and its time derivatives. Therefore the first two terms are transformed by integration by parts. The surface integral arising from integration by parts in the second is proportional to a and vanishes in the limit. This technical detail is left to the reader to verify. The surface integral arising from integration by parts in the first integral has the form (after the scaling transformation $\mathbf{x} = a\mathbf{y}$ where $|\mathbf{y}| = 1$)

$$\int\limits_{|\mathbf{y}|=1} n_j(\mathbf{y},\mathbf{P}_0(a\mathbf{y}))\,dS = -\int\limits_{|\mathbf{y}|=1} y_j(\mathbf{y},\mathbf{b})\,dS - \int\limits_{|\mathbf{y}|=1} y_j(\mathbf{y},\mathbf{P}_0(a\mathbf{y})-\mathbf{b})\,dS$$

where $\mathbf{b} = \mathbf{P}_0(\mathbf{0})$ is a constant vector and $n_j = -y_j$ is the outward normal on the unit sphere. The second integral is proportional to a in the leading order for small a thanks to the asymptotic properties of $\mathbf{P}_0$

and, hence, vanishes in the limit. Using a parameterization of the unit unit sphere by spherical angles, it is not difficult to infer that

$$\int_{|\mathbf{y}|=1} y_j y_n \, dS = \frac{4\pi}{3}\delta_{jn}$$

Therefore

$$\lim_{a\to 0^+}\int_{|\mathbf{y}|=1} y_j(\mathbf{y},\mathbf{P}_0(a\mathbf{y}))\, dS = \frac{4\pi}{3}\, b_j = \frac{4\pi}{3}\Big(p_j(t)\cdot\delta(\mathbf{x}),\varphi(\mathbf{x},t)\Big)$$

Finally, one should see if the limit exist for the all four volume integrals after integration by parts. Owing to the asymptotic properties of $\mathbf{P}_\alpha$, the integrands in the last three volume integrals are locally integrable and their limits exist. Let $(P_0)_n$ stands for the nth component of $\mathbf{P}_0$. Then the first integral reads

$$\lim_{a\to 0^+}\int_{a<|\mathbf{x}|<R} \frac{3|\mathbf{x}|^2\delta_{jn} - x_j x_n}{|\mathbf{x}|^5}\,(P_0)_n(\mathbf{x})\, d^3x$$

where a ball of radius R contains support of $\mathbf{P}_0$. If one adds and subtract b_n, then the integrand with the factor $(P_0)_n - b_n$ is locally integrable owing to the asymptotic properties of $\mathbf{P}_0$ and, hence, the regularization can be removed for it. It turns out that the most singular part vanishes for any $a > 0$. One has in spherical coordinates

$$\int_{a<|\mathbf{x}|<R} \frac{3|\mathbf{x}|^2\delta_{jn} - x_j x_n}{|\mathbf{x}|^5}\, d^3x = \int_a^R \frac{dr}{r}\int_{|\mathbf{y}|=1}(3\delta_{jn} - y_j y_n)\, dS = 0$$

because the surface integral vanishes. Thus, the limit exists for all four volume integrals. The limit offers a distributional extension of the most singular term to $\mathbf{x} = \mathbf{0}$. This extension will be denoted by $\mathcal{P}$ (the spherical principal value).

Collecting all terms, one infers that

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= -\boldsymbol{\nabla}\Phi(\mathbf{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\,\mathbf{A}(\mathbf{x},t)\\ &= \frac{4\pi}{3}\mathbf{p}(t)\delta(\mathbf{x}) - \mathcal{P}\frac{3\mathbf{x}(\mathbf{x},\mathbf{p}(t_r)) - |\mathbf{x}|^2\mathbf{p}(t_r)}{|\mathbf{x}|^5}\\ &\quad - \frac{3\mathbf{x}(\mathbf{x},\dot{\mathbf{p}}(t_r)) - |\mathbf{x}|^2\dot{\mathbf{p}}(t_r)}{c|\mathbf{x}|^4} - \frac{\mathbf{x}(\mathbf{x},\ddot{\mathbf{p}}(t_r)) - |\mathbf{x}|^2\ddot{\mathbf{p}}(t_r)}{c^2|\mathbf{x}|^3}\end{aligned}$$

where t_r is the retarded time introduced earlier. Since $\mathbf{p}(t) = \mathbf{0}$ for $t < 0$, the electromagnetic fields are supported in a ball $|\mathbf{x}| \le ct$, and its boundary is expanded with the speed of light c. Any perturbation of the source $\mathbf{p}$ at a time t can be observed with the delay of $|\mathbf{x}|/c$ at a point $\mathbf{x}$ which is the time needed for the wave to travel the distance from the point source to the point $\mathbf{x}$. This explains the name "retarded time".

Radiation far-fields. In the asymptotic region, $|\mathbf{x}| \to \infty$, the leading contribution comes from

$$\mathbf{E}(\mathbf{x},t) = \frac{|\mathbf{x}|^2\ddot{\mathbf{p}}(t_r) - \mathbf{x}(\mathbf{x},\ddot{\mathbf{p}}(t_r))}{c^2|\mathbf{x}|^3} + O\Big(\frac{1}{|\mathbf{x}|^2}\Big)$$

$$\mathbf{B}(\mathbf{x},\mathbf{t}) = \frac{\mathbf{x}\times\ddot{\mathbf{p}}(t_r)}{c^2|\mathbf{x}|^2} + O\Big(\frac{1}{|\mathbf{x}|^2}\Big)$$

If $\mathbf{p}(t)$ is a smooth function, then the far-fields are smooth despite that the sources are singular distributions in space. Note also that the far fields at a point $\mathbf{x}$ are orthogonal to the position vector of the point $\mathbf{x}$ relative to the source. Therefore, the leading contribution to the Poynting vector is parallel to $\mathbf{x}$ and has the form

$$\mathbf{S} = \frac{|\ddot{\mathbf{p}}(t_r)|^2 - (\hat{\mathbf{x}},\ddot{\mathbf{p}}(t_r))^2}{4\pi c^3|\mathbf{x}|^2}\,\hat{\mathbf{x}} + O\Big(\frac{1}{|\mathbf{x}|^3}\Big)\,,\quad \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}\,.$$

Let us find the rate at which the dipole radiates electromagnetic energy to th. Its outward flow across the sphere $|\mathbf{x}| = R$ so that $d\mathbf{\Sigma} = \hat{\mathbf{x}}R^2dS$, where dS is the area measure on a unit sphere, reads

$$\int_{\partial\Omega}(\mathbf{S},d\mathbf{\Sigma}) = \frac{1}{4\pi c^3}\int_{|\mathbf{y}|=1}\Big(|\ddot{\mathbf{p}}(t-\tfrac{R}{c})|^2 - (\hat{\mathbf{x}},\ddot{\mathbf{p}}(t-\tfrac{R}{c}))^2\Big)\,dS + O\Big(\frac{1}{R}\Big)\,.$$

Using spherical angles to parameterize the unit sphere so that the zenith angle ϕ is counted from $\ddot{\mathbf{p}}(t_r)$, the integral is easy to evaluate:

$$\int_{\partial\Omega}(\mathbf{S},d\mathbf{\Sigma}) = \frac{2\pi|\ddot{\mathbf{p}}(t-\frac{R}{c})|^2}{4\pi c^3}\int_0^\pi \sin^3(\phi)\,d\phi = \frac{2|\ddot{\mathbf{p}}(t-\frac{R}{c})|^2}{3c^3}$$

The flow is positive so that the dipole generates a steady flow of electromagnetic energy carried by electromagnetic waves. The rate at which the dipole sends electromagnetic energy is positive but depends on time. In applications, for a monochromatic source, $\mathbf{p}(t) = \theta(t)\mathbf{p}_0\cos(\omega t)$, one is often interested in the average rate per one cycle $T = 2\pi/\omega$:

$$\Big\langle \int_{\partial\Omega}(\mathbf{S},d\mathbf{\Sigma})\Big\rangle_T = \frac{1}{T}\int_0^T\int_{\partial\Omega}(\mathbf{S},d\mathbf{\Sigma})\,dt = \frac{\omega^2}{3c^3}\,|\mathbf{p}_0|^2$$

Remark. If $\rho = 0$, then the charge conservation law requires that the electric current density is divergence free. The simplest system of this type is a magnetic dipole:

$$\rho = 0\,,\quad \mathbf{J} = \mathbf{\nabla}\times\boldsymbol{\mu}(t)\delta(\mathbf{x})$$

where $\boldsymbol{\mu}(t)$ is a vector-valued distribution of time that vanishes for $t < 0$. It is called a magnetic dipole moment.

44.7. Regular solutions. Let $\mathbf{J}$, ρ, $\mathbf{E}_0$, and $\mathbf{B}_0$ be regular distributions. Then their convolutions with the causal Green's function G_3 are computed in the same as in the case of a scalar wave equation:

$$\Phi = -4\pi c^2 (G_3 * \rho)(\mathbf{x}, t) = -\int\limits_{|y-x|<ct} \frac{\rho(y, t - \frac{|x-y|}{c})}{|x-y|}\, d^3y\,,$$

$$\mathbf{A} = 4\pi c (G_3 * \mathbf{J})(\mathbf{x}, t) = -\frac{1}{c}\int\limits_{|y-x|<ct} \frac{\mathbf{J}(y, t - \frac{|x-y|}{c})}{|x-y|}\, d^3y\,,$$

$$\mathbf{A}_f = -cG_3(\mathbf{x}, t) * \mathbf{F}_0(\mathbf{x}) = -\frac{1}{4\pi ct}\int\limits_{|y-x|=ct} \mathbf{F}_0(y)\, dS_y$$

where $t > 0$ and the index f is either e or b when $\mathbf{F}_0$ is equal to either $\mathbf{E}_0$ or $\mathbf{B}_0$, respectively. If the sources and the initial data are smooth enough so that the above convolution are from class $C^2(t > 0)$, then the fields are from class $C^1(t > 0)$ and solve Maxwell's equations:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t} - \boldsymbol{\nabla} \times \mathbf{A}_b\,,$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \mathbf{A}_e - \frac{1}{c}\frac{\partial \mathbf{A}_b}{\partial t}$$

It follows from Theorem **43.3** that if the initial data $\mathbf{E}_0$ and $\mathbf{B}_0$ are from class C^2, then the surface vector potentials $\mathbf{A}_{e,b}$ are from class $C^2(t \geq 0)$ and

$$\lim_{t\to 0^+} \Big(-\frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t} - \boldsymbol{\nabla} \times \mathbf{A}_b \Big) = \mathbf{E}_0\,,$$

$$\lim_{t\to 0^+} \Big(\boldsymbol{\nabla} \times \mathbf{A}_e - \frac{1}{c}\frac{\partial \mathbf{A}_b}{\partial t} \Big) = \mathbf{B}_0$$

It follows from Theorem (**43.2**) that, if $\mathbf{J}$ and ρ are from class $C^2(t \geq 0)$, then the scalar and vector potentials Φ and $\mathbf{A}$ are also from class $C^2(t \geq 0)$ and

$$\lim_{t\to 0^+} \Big(-\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \Big) = \mathbf{0}\,,$$

$$\lim_{t\to 0^+} \Big(\boldsymbol{\nabla} \times \mathbf{A} \Big) = \mathbf{0}$$

This analysis shows that the electromagnetic fields are from class $C^1(t \geq 0)$ and satisfy Maxwell's equation and the initial conditions. They give a classical solution to the Cauchy problem. The problem is well-posed because the solution is unique and Theorems (**43.3**) and (**43.2**) show that the solution depends continuously on the sources and initial data.

44.8. Helmholtz decomposition of a vector field. A vector field that is sufficiently smooth and falls off sufficiently fast in the asymptotic region can be represented as a sum of divergence-free and curl-free vector fields in $\mathbb{R}^3$. The curl-free part is also a conservative vector field (it is the gradient of a scalar function called a potential):

$$\mathbf{F} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla} \times \mathbf{A}\,.$$

PROPOSITION **44.1**. *Let Ω be open and bounded region and a vector field $\mathbf{F}$ be from class $C^2(\bar{\Omega})$. Then the Helmholtz decomposition holds in Ω and*

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi}\int_\Omega \frac{(\boldsymbol{\nabla}, \mathbf{F}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|}\, d^3y - \frac{1}{4\pi}\int_{\partial\Omega} \frac{(\mathbf{n}_y, \mathbf{F}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|}\, dS_y\,,\\ \mathbf{A}(\mathbf{x}) &= \frac{1}{4\pi}\int_\Omega \frac{\boldsymbol{\nabla}\times \mathbf{F}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d^3y - \frac{1}{4\pi}\int_{\partial\Omega} \frac{\mathbf{n}_y\times \mathbf{F}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, dS_y\,,\end{aligned}$$

where $\mathbf{n}_y$ is the outward unit normal at a point $\mathbf{y}$ on the boundary $\partial\Omega$.

To prove this proposition, consider an extension of $\mathbf{F}$ to the whole $\mathbb{R}^3$ by zero. Then $\mathbf{F}$ is a vector-valued regular distribution with a bounded support being $\bar{\Omega}$. Therefore its convolution with the regular distribution $\frac{1}{|\mathbf{x}|}$ exists in $\mathcal{D}'$, and the following chain of equalities holds in the distributional sense

$$\begin{aligned}\mathbf{F} &\overset{(1)}{=} \delta * \mathbf{F} \overset{(2)}{=} -\frac{1}{4\pi}\Big(\Delta\,\frac{1}{|\mathbf{x}|}\Big) * \mathbf{F} \overset{(3)}{=} -\frac{1}{4\pi}\Delta\left(\frac{1}{|\mathbf{x}|} * \mathbf{F}\right)\\ &\overset{(4)}{=} -\frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}|} * \Delta\mathbf{F}\right) \overset{(5)}{=} \frac{1}{4\pi}\,\frac{1}{|\mathbf{x}|} * \Big(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) - \boldsymbol{\nabla}(\boldsymbol{\nabla}, \mathbf{F})\Big)\\ &\overset{(6)}{=} \frac{1}{4\pi}\boldsymbol{\nabla}\times\Big(\frac{1}{|\mathbf{x}|} * (\boldsymbol{\nabla}\times\mathbf{F})\Big) - \frac{1}{4\pi}\boldsymbol{\nabla}\Big(\frac{1}{|\mathbf{x}|} * (\boldsymbol{\nabla}, \mathbf{F})\Big)\end{aligned}$$

Here (1) holds by the properties of the delta-function; (2) follows from an explicit form of the fundamental solution of the Laplace operator; (3), (4), and (6) hold because the convolution of any distribution with a distribution having a bounded support exists; (5) follows from the identity

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) = -\Delta\mathbf{F} + \boldsymbol{\nabla}(\boldsymbol{\nabla}, \mathbf{F})\,.$$

that is valid for any vector-valued distribution (it is established by the *"bac-cab" rule* for the double cross product). This calculation shows that the Helmholtz decomposition holds in the distributional sense with the scalar and vector potential given by the following distributions

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}|} * (\boldsymbol{\nabla}, \mathbf{F})\,, \quad \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}|} * (\boldsymbol{\nabla} \times \mathbf{F})\,.$$

To find the classical Helmholtz decomposition in Ω, one should calculate the distributional divergence and curl of $\mathbf{F}$ in terms of the classical divergence and curl of $\mathbf{F}$ in Ω. Let D_j denote the partial derivative with respect to x_j, $j = 1, 2, 3$. It was shown in Section **21.5.1** that a distributional derivative of a piecewise smooth function is given by

$$D_j\mathbf{F} = \{D_j\mathbf{F}\} - n_j\mathbf{F}\,\delta_{\partial\Omega}\,,$$

where n_j is the jth component of the unit outward normal on the boundary $\partial\Omega$ and $\nu\delta_{\partial\Omega}$ is a simple layer distribution with density ν which is a continuous function on $\partial\Omega$. Therefore the distributional divergence and curl of $\mathbf{F}$ are related to the classical ones as

$$(\boldsymbol{\nabla}, \mathbf{F}) = \{(\boldsymbol{\nabla}, \mathbf{F})\} - \nu\,\delta_{\partial\Omega}\,, \quad \nu = (\mathbf{n}, \mathbf{F})\,,$$
$$\boldsymbol{\nabla} \times \mathbf{F} = \{\boldsymbol{\nabla} \times \mathbf{F}\} - \boldsymbol{\mu}\,\delta_{\partial\Omega}\,, \quad \boldsymbol{\mu} = \mathbf{n} \times \mathbf{F}$$

Since the classical divergence and curl are continuous in $\bar{\Omega}$ and vanish in the complement of $\bar{\Omega}$, and the densities ν and $\boldsymbol{\mu}$ are continuous on $\partial\Omega$ (or piece-wise continuous for a piece-wise smooth $\partial\Omega$), the needed convolutions have the following integral representations

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi}\frac{1}{|\mathbf{x}|} * \Big(\{(\boldsymbol{\nabla}, \mathbf{F})\} - \nu\,\delta_{\partial\Omega}\Big) \\
&= \frac{1}{4\pi}\int_\Omega \frac{(\boldsymbol{\nabla}, \mathbf{F}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|}\,d^3y - \frac{1}{4\pi}\int_{\partial\Omega} \frac{(\mathbf{n}_y, \mathbf{F}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|}\,dS_y\,, \\
\mathbf{A}(\mathbf{x}) &= \frac{1}{4\pi}\frac{1}{|\mathbf{x}|} * \Big(\{\boldsymbol{\nabla} \times \mathbf{F}\} - \boldsymbol{\mu}\,\delta_{\partial\Omega}\Big) \\
&= \frac{1}{4\pi}\int_\Omega \frac{\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d^3y - \frac{1}{4\pi}\int_{\partial\Omega} \frac{\mathbf{n}_y \times \mathbf{F}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,dS_y\,.
\end{aligned}$$

Owing to the theorem about differentiation of potential-like integrals, the integral representation of the vector and scalar potentials define functions from class $C^1(\bar{\Omega})$ and, hence, their distributional derivatives in Ω are equal to the classical ones so that the Helmholtz decomposition holds in the classical sense in Ω.

Extension to the whole space. Let Ω be a ball of radius R, then in the limit $R \to \infty$, the surface terms vanish in the Helmholtz decomposition in Ω if $|\mathbf{F}|$ tends to zero faster than $\frac{1}{|\mathbf{x}|}$ as $|\mathbf{x}| \to \infty$. This observation leads to the conclusion that if $\mathbf{F}$ is from class C^2 and falls off sufficiently fast as $|\mathbf{x}| \to \infty$, then the Helmholtz decomposition holds in the whole space and the potentials have the form

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{(\boldsymbol{\nabla}, \mathbf{F}(\mathbf{y}))}{|\mathbf{x} - \mathbf{y}|}\, d^3y\,, \quad \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\, d^3y$$

A proof of this extension would be similar to the previous case if one makes a simplifying assumption that the classical convolution

$$\frac{1}{|\mathbf{x}|} * \mathbf{F} = \int \frac{\mathbf{F}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\, d^3y$$

exists. Clearly this is true if $\mathbf{F}$ has a bounded support. This is also true if the following limit exists

$$\lim_{R\to\infty} \int_{|\mathbf{y}|<R} \frac{|\mathbf{F}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|}\, d^3y < \infty$$

A sufficient condition for the latter reads

$$|\mathbf{F}(\mathbf{y})| \leq \frac{M}{1 + |\mathbf{y}|^p}\,, \quad p > 2$$

However, this condition is too restrictive. One might notice that it is sufficient for the assertion to hold if the classical convolution of the divergence and curl of $\mathbf{F}$ with $\frac{1}{|\mathbf{x}|}$ exists. A sufficient condition for the latter has a similar form but it is less restrictive: $p > 1$. However, the proof used in the previous case is false if the convolution $\frac{1}{|\mathbf{x}|} * \mathbf{F}$ does not exist.

An alternative proof is based on the Fourier transform. Since $\mathbf{F}$ is continuous and falls off to zero at infinity, it is a bounded vector field and, hence, defines a regular temperate distribution whose Fourier transform exists in the distributional sense. A vector $\mathcal{F}[\mathbf{F}]$ has a unique orthogonal decomposition into the sum of a vector parallel to $\mathbf{k}$ and a vector perpendicular to $\mathbf{k}$. Recall that $\mathbf{k} \times \mathbf{a}$ is perpendicular to $\mathbf{k}$ for any choice of $\mathbf{a}$. In particular, one can always chose $\mathbf{a}$ to be orthogonal to $\mathbf{k}$, that is, $(\mathbf{k}, \mathbf{a}) = 0$. So, by analogy with the vector algebra, put

$$\mathcal{F}[\mathbf{F}](\mathbf{k}) = -i\mathbf{k}\, g(\mathbf{k}) + i\mathbf{k} \times \mathbf{G}(\mathbf{k})\,, \quad (\mathbf{k}, \mathbf{G}) = 0\,.$$

If $\mathbf{G}$ and g exists in $\mathcal{S}'$, then the Helmholtz decomposition holds in the space of temperate distributions with $\Phi = \mathcal{F}^{-1}[g]$ and $\mathbf{A} = \mathcal{F}^{-1}[\mathbf{G}]$. If the potentials are continuously differentiable, then the classical decomposition holds as well.

Taking the dot and cross products of the decomposition equation with $\mathbf{k}$, it is concluded that the problem is equivalent to finding conditions on $\mathbf{F}$ under which the equations

$$|\mathbf{k}|^2 g = i(\mathbf{k}, \mathcal{F}[\mathbf{F}])\,, \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{G}) = -i\mathbf{k}\times\mathcal{F}[\mathbf{F}]$$

have solutions in the space of temperate distributions. The first equation is nothing but the Fourier transform of the Poisson equation in $\mathcal{S}'$ with the inhomogeneity being the divergence $(\boldsymbol{\nabla}, \mathbf{F})$. It has a unique solution in the class of temperate distributions which have a convolution with the fundamental solution for the Laplace operator:

$$\Delta\mathcal{E}_3(\mathbf{x}) = \delta(\mathbf{x}) \quad \Rightarrow \quad \mathcal{E}_3(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}$$

Therefore, if the convolution $\mathcal{E}_3(\mathbf{x}) * (\boldsymbol{\nabla}, \mathbf{F})$ exists, then g exists, and the scalar potential is $\Phi = \mathcal{F}^{-1}[g] = -\mathcal{E}_3(\mathbf{x}) * (\boldsymbol{\nabla}, \mathbf{F})$. The convolution is a classical one if $|\mathbf{F}| \sim |\mathbf{x}|^{-p}$, $p > 1$, in the asymptotic region. It defines a continuously differentiable function owing to that $\mathbf{F} \in C^2$ and $(\boldsymbol{\nabla}, \mathbf{F}) \sim |\mathbf{x}|^{-p-1}$ in the asymptotic region.

The second equation is also reduced to the vector Poisson equation by expanding the double cross product and using the orthogonality of $\mathbf{G}$ and $\mathbf{k}$ so that

$$|\mathbf{k}|^2\mathbf{G} = i\mathbf{k}\times\mathcal{F}[\mathbf{F}] \quad \Rightarrow \quad \mathbf{A} = \mathcal{F}^{-1}[\mathbf{G}] = -\mathcal{E}_3 * (\boldsymbol{\nabla}\times\mathbf{F})$$

provided the convolution exists in $\mathcal{S}'$ which is the case under the stated condition on the asymptotic behavior of $\mathbf{F}$. The convolution defines a continuously differentiable vector potential under the stated smoothness and asymptotic conditions on $\mathbf{F}$.

Helmholtz decomposition of a vector field with bounded support. Let $\mathbf{F}$ be a vector field with bounded support D. Then its divergence and curl, $(\boldsymbol{\nabla}, \mathbf{F})$ and $\boldsymbol{\nabla}\times\mathbf{F}$, are also supported in D. However the scalar and vector potential in the Helmholtz decomposition do not vanish in the complement $\Omega = D^c$ of D as they are given by the convolution integral. Since $\mathbf{F}$ vanishes in Ω, one has

$$\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\Phi\,, \quad x\in\Omega\,. \tag{44.9}$$

There is no contradiction in this equation. By taking the curl and divergence of this equation and using that $(\boldsymbol{\nabla}, \mathbf{A}) = 0$, it is concluded that the components of the vector potential and the scalar potential are harmonic functions in Ω

$$\Delta\mathbf{A}(x) = \mathbf{0}\,, \quad \Delta\Phi(x) = 0\,, \quad x\in\Omega\,.$$

A harmonic function in an open region Ω is uniquely determined by its values on the boundary $\partial\Omega = \partial D$. Clearly, the convolutions that define $\mathbf{A}$ and Φ do not vanish on ∂D. Therefore the external Dirichlet problem for the said harmonic functions has a non-trivial solution.

Note however that equation (**44.9**) has trivial solution if $\Omega = \mathbb{R}^3$ (with suitable asymptotic conditions on $\mathbf{A}$ and Φ at infinity). In this case, $\mathbf{A}$ and Φ are temperate distributions and the Fourier transform of (**44.9**) leads to $\mathbf{k}\times\mathcal{F}[\mathbf{A}] = \mathbf{k}\mathcal{F}[\Phi]$ which is possible only if $\mathcal{F}[\mathbf{A}]$ and $\mathcal{F}[\Phi]$ have a point support $\mathbf{k} = \mathbf{0}$. By the theorem about distributions with point supports, the distributions $\mathbf{A}$ and Φ are polynomials. If $\mathbf{A}$ and Φ are required to vanish at infinity, then $\mathbf{A} = \mathbf{0}$ and $\Phi = 0$ are the only solutions.

The scalar and vector potentials of electromagnetic fields. Since the magnetic field is divergence-free, it is the curl of a vector potential $\mathbf{A}$. In general, the vector potential can have an additive potential part (as the gradient of some scalar function). The magnetic field is independent of this part as only the divergence free part of $\mathbf{A}$ contributes to the curl of $\mathbf{A}$. Put

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}\,, \quad \mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

The Helmholtz decomposition for $\mathbf{E}$ was chosen so that the second Maxwell's equation is fulfilled identically:

$$\frac{\partial\mathbf{B}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{E} = \boldsymbol{\nabla}\times\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\times\frac{\partial\mathbf{A}}{\partial t} = \mathbf{0}$$

So, the vector and scalar potentials are to be found by solving the first Maxwell's equation and the Gauss law. This representation allows one to reduce twice the number of equations. However, the scalar and vector electromagnetic potentials are not unique. The electromagnetic field do not change under the so-called *gauge transformations*

$$\Phi \to \Phi_\omega = \Phi - \frac{1}{c}\frac{\partial\omega}{\partial t}\,, \quad \mathbf{A}\to\mathbf{A}_\omega = \mathbf{A} + \boldsymbol{\nabla}\,\omega$$

For example, one can chose ω so that the transformed vector potential is divergence free. In other words, the gauge freedom can eliminated by imposing *a gauge condition* on the potentials. For example, the *Coulomb gauge* requires the vector potentials to be divergence free

$$(\boldsymbol{\nabla}, \mathbf{A}) = 0$$

In the relativity theory, one often uses the *Lorenz gauge*

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + (\boldsymbol{\nabla}, \mathbf{A}) = 0$$

The electromagnetic field are independent of the choice of a gauge, they are said to be *gauge invariant*.

44.9. Exercises.

1. Static solutions. Suppose that the sources ρ and $\mathbf{J}$ are static, that is, they are independent of time.
(i) Use the Helmholtz representation of the electromagnetic fields to solve Maxwell's equations if ρ and $\mathbf{J}$ are distributions with bounded support. Prove the uniqueness of the solution.
(ii) Assume that ρ and $\mathbf{J}$ are regular distributions. Find integral representations for the static electric and magnetic fields. Find conditions of the smoothness of the sources under which the solution is a classical one.

2. Monochromatic solutions. Assume that the electric current density is monochromatic

$$\mathbf{J}(\mathbf{x}, t) = e^{i\omega t}\mathbf{J}_0(\mathbf{x})\,.$$

and the distribution $\mathbf{J}_0$ has a bounded support.
(i) Find the charge density $\rho(\mathbf{x}, t)$ alternating in time for which the charge is conserved.
(ii) Assume that the electromagnetic fields are monochromatic and derive equations for the amplitudes of the fields.
(iii) Find a solution to Maxwell's equations for the amplitudes if the latter satisfy the Sommerfeld radiation conditions using a suitable Green's function for the Helmholtz operator.

3. Radiation of a magnetic dipole. (i) Formulate the generalized Cauchy problem for a point-like magnetic dipole (see Remark at the end of Section **44.6**).

(ii) Solve the Cauchy problem and calculate the explicit form of electromagnetic fields as vector-valued distributions in the same fashion as in Section **44.6** for the electric dipole.

(iii) Find the far fields, calculate their Poynting vector, and find its outward flux across a sphere of an arbitrary large radius. Put $\boldsymbol{\mu}(t) = \theta(t)\boldsymbol{\mu}_0\cos(\omega t)$ (a monochromatic magnetic dipole) and caculate the average rate per one cycle $T = 2\pi/\omega$ at which the dipole emits electromagnetic energy.

4. Liénard-Wiechert potentials. A charged particle moving on a smooth trajectory $\mathbf{x} = \mathbf{r}(t)$ creates the following charge and current densities:

$$\rho(\mathbf{x}, t) = q\delta(\mathbf{x} - \mathbf{r}(t))\,, \quad \mathbf{J}(\mathbf{x}, t) = q\dot{\mathbf{r}}(t)\delta(\mathbf{x} - \mathbf{r}(t))$$

where q is the electric charge of the particle. These distributions are defined by the rule

$$(\rho, \varphi) = q \int \varphi(\mathbf{r}(t), t)\, dt\,, \quad (\mathbf{J}, \varphi) = q \int \dot{\mathbf{r}}(t)\varphi(\mathbf{r}(t), t)\, dt\,.$$

(i). Show that the electric current is conserved. Formulate and solve the generalized Cauchy problem with the zero initial initial conditions $\mathbf{E}_0 = \mathbf{B}_0 = \mathbf{0}$. Find the far fields.
(ii) Find the corresponding scalar and vector potentials. They are known as the Liénard-Wiechert potentials.
(iii) Suppose a particle moves with a constant speed on a circular trajectory. Find the far fields. The electromagnetic radiation generated by such a particle is called a *synchrotron radiation* (after the name of a device (synchrotron) in which a charged particle can move with a constant speed on a circular trajectory).

5. Solving Maxwell's equation using electromagnetic potentials.
(i) Formulate the Cauchy problem for Maxwell's equation using the scalar and vector electromagnetic potentials in the Coulomb gauge.
(ii) Formulate the generalized Cauchy problem for the potentials and solve it by Green's function method.
(iii) Calculate the corresponding electromagnetic fields and compare them with the solution of the Cauchy problem for Maxwell's equations in this section.
(iv) Repeat the analysis for the Lorenz gauge.

6. The Cauchy problem for the Klein-Gordon-Fock equation. Consider the Cauchy problem

$$\frac{\partial^2 u}{\partial x_0^2} - \Delta_x u + m^2 u = 0\,, \quad x_0 > 0\,,\ x \in \mathbb{R}^3\,,$$
$$u\Big|_{x_0=0} = u_0(x)\,, \quad \frac{\partial u}{\partial x_0}\Big|_{x_0=0} = u_1(x)$$

A solution to this equation is a wave function that describes a free relativistic scalar particle of mass m.
(i) Formulate the generalized Cauchy problem.

(ii) Let $G(x_0, x)$ be the retarded Green's function for the Klein-Gordon-Fock operator

$$\Big(\frac{\partial^2}{\partial x_0^2} - \Delta_x + m^2\Big)G(x, x_0) = \delta(x_0)\cdot\delta(x)\,, \quad G(x, x_0) = 0\,, \quad x_0 < 0\,,$$

Show that if it exists as a temperate distribution, then for any $\varepsilon > 0$

$$G_\varepsilon(x_0, x) = e^{-\varepsilon x_0}G(x_0, x) \in \mathcal{S}' \quad \text{and} \quad G_\varepsilon \to G \quad \text{in } \mathcal{S}'$$

as $\varepsilon \to 0^+$, and G_ε satisfies the equation

$$\Big[\Big(\frac{\partial}{\partial x_0} + \varepsilon\Big)^2 - \Delta_x + m^2\Big]G_\varepsilon(x, x_0) = \delta(x_0)\cdot\delta(x)\,.$$

(iii) Show that $\mathcal{F}[G_\varepsilon] \to \mathcal{F}[G]$ in $\mathcal{S}'$ as $\varepsilon \to 0^+$ and that

$$\mathcal{F}[G_\varepsilon(x_0, x)](k_0, k) = \frac{1}{|k|^2 + m^2 - (k_0 + i\varepsilon)^2} \in \mathcal{O}_M$$

that is, the Fourier transform of G_ε is a regular temperate distribution and for any test function $\varphi(x_0, x)$ from $\mathcal{S}$

$$(G_\varepsilon, \varphi) = \frac{1}{(2\pi)^4}\lim_{R\to\infty}\int\int\int_{-R}^{R}\frac{e^{-ik_0x_0}}{\nu^2 - (k_0 + i\varepsilon)^2}\,dk_0 e^{-i(k,x)}\varphi\,d^3k\,d^4x$$

where $d^4x = d^3x d_0^x$ for brevity and $\nu = (|k|^2 + m^2)^{1/2}$.
(iv) Use the residue theorem to evaluate the integral over k_0, calculate the limit $R \to \infty$ by justifying interchanging the order of integration and taking the limit. Next convert the integral over k to spherical coordinates so that $(k, x) = |k||x|\cos(\phi)$ where ϕ is the zenith angle and evaluate the integral over the spherical angles (justify changing the order of integration). Finally, show that

$$(G, \varphi) = \frac{\theta(x_0)}{16\pi^2}\lim_{a\to\infty}\int\frac{1}{r}\frac{\partial}{\partial r}\int_{-a}^{a}\Big(e^{ipr} - e^{-ipr}\Big)\Big(e^{i\nu x_0} - e^{-i\nu x_0}\Big)\frac{dp}{\nu}\varphi\,d^4x$$

where $r = |x|$, $p = |k|$, and $\nu = (p^2 + m^2)^{1/2}$.
(v) Let $s = x_0^2 - r^2$. Justify the following parameterization of x_0 and r

$$x_0 = \sqrt{s}\cosh(\xi_0)\,, \quad r = \sqrt{s}\sinh(\xi_0)\,, \quad s > 0$$
$$x_0 = \sqrt{-s}\sinh(\xi_0)\,, \quad r = \sqrt{-s}\cosh(\xi_0)\,, \quad s < 0$$

Reduce the integral over p to a standard form by means of the substitution

$$p = m\sinh(\xi)\,,$$

and evaluate the limit $a \to \infty$ using the following integral representation of cylindrical functions

$$\frac{1}{2\pi i} \lim_{b\to\infty} \int_{-b}^{b} e^{\pm im\sqrt{s}\,\cosh(\xi+\xi_0)} d\xi = \frac{1}{2}\Big(J_0(m\sqrt{s}) \pm iN_0(m\sqrt{s})\Big)$$
$$\frac{i}{2\pi} \lim_{b\to\infty} \int_{-b}^{b} e^{\pm im\sqrt{-s}\,\sinh(\xi+\xi_0)} d\xi = \frac{i}{\pi} K_0(m\sqrt{-s})$$

where J_μ, N_μ, and K_μ are Bessel, Neumann, and Bessel functions of the third kind of order μ, respectively. Note that integral converges conditionally and, hence, a justification for interchanging the order of integration with respect to x and taking the limit $a \to \infty$ is required.
(vi) Use express the derivatives of cylindrical function of order $\mu = 0$ via cylindrical functions of other orders and show that

$$G(x, x_0) = \frac{\theta(x_0)}{2\pi}\delta(x_0^2 - |x|^2) - \frac{m}{4\pi}\theta(x_0 - |x|)\frac{J_1(m\sqrt{x_0^2 - |x|^2})}{\sqrt{x_0^2 - |x|^2}}\,.$$

(vii) Show that for any test function $\varphi(x) \in \mathcal{D}'(\mathbb{R}^3)$, the function

$$g(x_0) = \Big(G(x, x_0), \varphi(x)\Big)\,,$$

is from class $C^2(x_0 > 0)$ and find the limits of $g(x_0)$, $g'(x_0)$, and $g''(x_0)$ as $x_0 \to 0^+$.
(viii) Show that the solution to the generalized Cauchy problem for arbitrary distributional initial data, $u_{0,1} \in \mathcal{D}'(\mathbb{R}^3)$ is given by

$$u(x, x_0) = G(x, x_0) * u_1(x) + \frac{\partial}{\partial x_0} G(x, x_0) * u_0(x)\,.$$

Use the result of Part (vii) to show that the solution $u(x, x_0)$ is a distribution from class $C^1(x_0 > 0)$ in the variable x_0, and $u(x, x_0)$ and its partial derivative $\frac{\partial u}{\partial x_0}$ converge to the distributions $u_0(x)$ and $u_1(x)$, respectively, in $\mathcal{D}'(\mathbb{R}^3)$ as $x_0 \to 0^+$.
(ix) Show that if u_0 and u_1 have a bounded support, then the solution to the generalized Cauchy problem has a bounded support in the variable x for any $x_0 > 0$.

7. The Cauchy problem for the Dirac equation. The Dirac equation describes a quantum relativistic free particle with spin $\frac{1}{2}$ and mass m. Its solution is a wave function $\psi(x, x_0)$ that has four complex components (it is from $\mathbb{C}^4$):

$$\Big(i\gamma^\mu \frac{\partial}{\partial x_\mu} - mI\Big)\psi = 0$$

where $\mu = 0, 1, 2, 3$, I is the 4×4 unit matrix, $I_{ij} = \delta_{ij}$, and γ^μ are Dirac matrices defined via Pauli matrices:

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \; \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \; \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \; \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

and the Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Formulate the generalized Cauchy problem for the Dirac equation

$$\Big(i\gamma^\mu \frac{\partial}{\partial x_\mu} - mI\Big)\psi = 0, \quad x_0 > 0, \quad \psi\Big|_{x_0=0} = \psi_0(x)$$

(ii) Use the multiplication property of Pauli matrices

$$\sigma_a \sigma_b = i\varepsilon_{abc}\sigma_c, \quad a, b, c = 1, 2, 3$$

where ε_{abc} is the Levi-Civita symbol, to show that

$$\Big(i\gamma^\mu \frac{\partial}{\partial x_\mu} - mI\Big)\Big(i\gamma^\mu \frac{\partial}{\partial x_\mu} + mI\Big) = -I\Big(\frac{\partial^2}{\partial x_0^2} - \Delta_x + m^2\Big)$$

(iii) Show that the matrix-valued distribution

$$G_D(x, x_0) = -\Big(i\gamma^\mu \frac{\partial}{\partial x_\mu} + mI\Big)G(x, x_0)$$

where $G(x, x_0)$ is the retarded Green's function for the Klein-Gordon-Fock operator, is the retarded (or causal) Green's function for the Dirac operator

$$\Big(i\gamma^\mu \frac{\partial}{\partial x_\mu} - mI\Big)G_D(x, x_0) = \delta(x_0) \cdot \delta(x)\, I, \quad G_D(x, x_0) = 0, \; x_0 < 0.$$

(iv) Show that for any test function $\varphi(x) \in \mathcal{D}'(\mathbb{R}^3)$, the matrix-valued function

$$g(x_0) = \Big(G_D(x, x_0), \varphi(x)\Big),$$

is from class $C^0(x_0 > 0)$ and find the limit of $g(x_0)$ as $x_0 \to 0^+$.

(v) Show that the solution to the generalized Cauchy problem for arbitrary distributional initial data, $\psi_0 \in \mathcal{D}'(\mathbb{R}^3)$ is given by

$$\psi(x, t) = -iG_D(x, x_0) * \gamma^0 \psi_0(x).$$

The convolution of matrix-valued and vector-valued distributions is defined by the usual linear algebra rules in which the multiplication of components is replaced by the convolution. Use the result of Part (iv) to show that the solution $\psi(x, x_0)$ is a distribution from class $C^0(x_0 > 0)$ in the variable x_0, and $\psi(x, x_0)$ converges to $\psi_0(x)$ in $\mathcal{D}'(\mathbb{R}^3)$ as

$x_0 \to 0^+$.
(v) Show that if ψ_0 has a bounded support, then the solution to the generalized Cauchy problem has a bounded support in the variable x for any $x_0 > 0$.

8. The Cauchy problem in elastodynamics. Elastic waves in an isotropic and homogeneous media are described by the Navier-Cauchy equation

$$\frac{\partial^2 u_a}{\partial t^2} - c_s^2 \Delta_x u_a - (c_l^2 - c_s^2)\frac{\partial^2 u_b}{\partial x_a \partial x_b} = f_a(x,t)\,,$$

where $a, b = 1, 2, 3$ enumerate components of the displacement vector field $u_a(x,t)$, and c_s^2 and c_l^2 are wave speeds for the shear and compression modes. The vector field $u_a(x,t)$ defines a displacement vector of a point x of the media when an elastic disturbance occurs at x and time t.
(i) Consider the Cauchy problem in elastodynamics. One has to find a solution to the Navier-Cauchy equation for $t > 0$ that satisfies the initial conditions:

$$u_a\Big|_{t=0} = v_a(x)\,, \qquad \frac{\partial u_a}{\partial t}\Big|_{t=0} = w_a(x)\,, \quad x \in \mathbb{R}^3$$

Formulate the generalized Cauchy problem.
(ii) The causal Green's function for the Navier-Cauchy operator is a matrix-valued distribution that solves the equation

$$\Big(\delta_{ab}\frac{\partial^2}{\partial t^2} - c_s^2\delta_{ab}\Delta_x - (c_l^2 - c_s^2)\frac{\partial^2}{\partial x_a \partial x_b}\Big)G_{bc}(x,t) = \delta_{ac}\delta(t)\cdot\delta(x)$$
$$G_{ab}(x,t) = 0\,, \quad t < 0\,.$$

Show that the Fourier transform of the Green's function in the variable x has the form

$$\mathcal{F}_x[G_{ab}](k,t) = g_s(k,t)P^{\perp}_{ab}(k) + g_l(k,t)P^{\parallel}_{ab}(k)$$

where the matrices $P^{\perp}$ and $P^{\parallel}$ are orthogonal projectors of any vector onto the plane orthogonal to the vector k and onto the vector k, respectively:

$$P^{\perp}_{ab}(k) = \delta_{ab} - \frac{k_a k_b}{|k|^2}\,, \quad P^{\parallel}_{ab} = \frac{k_a k_b}{|k|^2}$$

that is, they satisfy the relations $P^{\perp}P^{\perp} = P^{\perp}$, $P^{\parallel}P^{\parallel} = P^{\parallel}$, and $P^{\parallel}P^{\perp} = 0$, and $g_{s,l}$ are temperate distributions that satisfy the equations

$$\Big(\frac{d^2}{dt^2} + c_\beta^2|k|^2\Big)g_\beta(k,t) = \delta(t)\,, \quad g_\beta(k,t) = 0\,,\ t < 0\,, \quad \beta = s, l$$

(iii). Let $h_a(x)$ be a vector-valued temperate distribution. Then by the Helmholtz theorem it can uniquely be expanded into the sum of a divergence-free distribution $h_a^\perp$ and a conservative vector distribution $h_a^\parallel$ so that $h_a = h_a^\perp + h_a^\parallel$. Show that

$$\mathcal{F}[h_a^\perp] = P_{ab}^\perp \mathcal{F}[h_b]\,, \quad \mathcal{F}[h_a^\parallel] = P_{ab}^\parallel \mathcal{F}[h_b]\,,$$

and

$$h_a^\parallel = \hat{P}^\parallel h_a = \partial_a\partial_b\,\mathcal{E}_3 * h_b\,, \quad h_a^\perp = \hat{P}^\perp h_a = h_a - \partial_a\partial_b\,\mathcal{E}_3 * h_b\,, \quad \partial_a h_a^\perp = 0\,,$$

where $\partial/\partial x_a = \partial_a$ for brevity, and $\mathcal{E}_3(x)$ is the fundamental solution for the Laplace operator in $\mathbb{R}^3$ that vanishes in the asymptotic region $|x| \to \infty$. In other words, the operators $\hat{P}^\perp$ and $\hat{P}^\parallel$ project any vector field h_a (that vanishes fast enough as $|x| \to \infty$ to ensure the existence of the convolution $\mathcal{E}_3 * h_a$) onto its rotational and conservative parts. Use these relations to show that the following problem

$$\begin{aligned}
&\Big(\delta_{ab}\frac{\partial^2}{\partial t^2} - c_s^2\delta_{ab}\Delta_x - (c_l^2 - c_s^2)\frac{\partial^2}{\partial x_a \partial x_b}\Big)u_b(x,t) = h_a(t,x)\\
&u_a(x,t) = 0\,, \quad t < 0\,,
\end{aligned}$$

where $h_a(t,x)$ is a vector-valued temperate distribution that vanishes for $t < 0$, has a unique solution given by

$$u_a(t,x) = G_s(t,x) * h_a^\perp(t,x) + G_l(t,x) * h_a^\parallel(t,x)$$

where $G_{s,l}$ are causal Green's functions of the 4D wave operator with wave speeds $c_{s,l}$, respectively,

$$G_\beta(x,t) = \frac{\theta(t)}{4\pi c_\beta^2 t}\,\delta_{S_{c_\beta t}}(x)\,.$$

(iv) For brevity, put $\partial_t = \partial/\partial t$. Assume that the initial data v_a and w_a and the inhomogeneity f_a are decreasing fast enough at infinity to ensure the existence of their convolution with $\mathcal{E}_3$. Show that the solution to the generalized Cauchy problem is unique and can be written in the form

$$\begin{aligned}
u_a(x,t) &= W_a(x,t) + V_a(x,t) + U_a(x,t)\,,\\
W_a(x,t) &= G_s(x,t) * \hat{P}^\perp w_a(x) + G_l(x,t) * \hat{P}^\parallel w_a(x)\,,\\
V_a(x,t) &= \partial_t G_s(x,t) * \hat{P}^\perp v_a(x) + \partial_t G_l(x,t) * \hat{P}^\parallel v_a(x)\,,\\
U_a(x,t) &= G_s(x,t) * \hat{P}^\perp f_a(x,t) + G_l(x,t) * \hat{P}^\parallel f_a(x,t)\,,
\end{aligned}$$

where the convolution in the surface potentials W_a and V_a is taken with respect to x (for fixed t).

(iv) If v_a and w_a are vector-valued distributions from $\mathcal{D}'$ and their convolution with $\mathcal{E}_3$ exists, show that the surface wave potentials V_a and W_a are from class $C^1(t > 0)$ in the variable t and satisfy the limit properties in $\mathcal{D}'(\mathbb{R}^3)$:

$$\lim_{t\to 0^+} V_a(x,t) = v_a(x)\,, \quad \lim_{t\to 0^+} \frac{\partial V_a(x,t)}{\partial t} = 0\,,$$
$$\lim_{t\to 0^+} W_a(x,t) = 0\,, \quad \lim_{t\to 0^+} \frac{\partial W_a(x,t)}{\partial t} = w_a(x)\,.$$

(v) Suppose that v_a and w_b have a bounded support. Show that the surface wave potentials have a bounded support.

(vi) Use the Helmholtz decomposition theorem for the initial data v_a and w_a and for the inhomogeneity f_a to find out the speed at which elastic waves propagate if they are generated by divergence free fields v_a, w_a, and f_a and by conservative fields v_a, w_a, and f_a. Formulate the Huygens principle for elastic waves.