CHAPTER 7

Hilbert spaces

53. Inner product vector spaces

53.1. Inner product in a vector space. Let $\mathcal{X}$ be a vector space over complex numbers. A function

$$\langle \cdot, \cdot \rangle \ : \quad \mathcal{X} \times \mathcal{X} \to \mathbb{C}$$

that is, a rule that assigns a unique complex number to every pair of elements of $\mathcal{X}$, is called an *inner product* in $\mathcal{X}$ if it satisfies the following *inner product axioms* that hold for any u, v, and w from $\mathcal{X}$ and any complex number α:

$$\begin{array}{ll} \text{(i)} & \langle u, v\rangle = \overline{\langle v, u\rangle} \\ \text{(ii)} & \langle \alpha u, v\rangle = \alpha\langle u, v\rangle \\ \text{(iii)} & \langle \alpha u + w, v\rangle = \langle u, w\rangle + \langle w, v\rangle \\ \text{(iv)} & \langle u, u\rangle \geq 0\,, \quad \langle u, u, \rangle = 0 \quad \Leftrightarrow \quad u = 0 \end{array}$$

53.2. Properties of the inner product.

53.2.1. The Schwartz (or Cauchy-Bunyakowski) inequality.

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle$$

If $v = 0$, then the inequality holds. To prove the inequality in the case $v \neq 0$, consider a real function of two variables

$$f(t, s) = \langle u - \alpha v, u - \alpha v\rangle \geq 0\,, \quad \alpha = t + is$$

It is a quadratic polynomial

$$f(t, s) = \langle u, u\rangle + (s^2 + t^2)\langle v, v\rangle - 2t\mathrm{Re}\,\langle u, v\rangle - 2s\mathrm{Im}\,\langle u, v\rangle$$

that is bounded from below. Therefore it attains its absolute minimum at the only critical point:

$$f'_s = 2s\langle v, v\rangle - 2\mathrm{Im}\,\langle u, v\rangle = 0 \quad \Rightarrow \quad s = s^* = \frac{\mathrm{Im}\,\langle u, v\rangle}{\langle v, v\rangle}$$

$$f'_t = 2t\langle v, v\rangle - 2\mathrm{Re}\,\langle u, v\rangle = 0 \quad \Rightarrow \quad t = t^* = \frac{\mathrm{Re}\,\langle u, v\rangle}{\langle v, v\rangle}$$

It follows from the inequality $f(t^*, s^*) \geq 0$ and the inner product axioms that

$$0 \leq \langle u - \alpha^* v, u - \alpha^* v \rangle = \langle u, u \rangle - \frac{|\langle u, v \rangle|^2}{\langle v, v \rangle}$$

from the Schwartz inequality follows.

53.2.2. The Minkowski inequality.

$$\sqrt{\langle u + v, u + v \rangle} \leq \sqrt{\langle u, u \rangle} + \sqrt{\langle v, v \rangle}$$

Put $\|u\| = \sqrt{\langle u, u \rangle}$. Then the Schwartz inequality can be written as

$$\mathrm{Re}\,\langle u, v \rangle \leq |\langle u, v \rangle| \leq \|u\|\|v\|$$

It follows from this inequality that

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle = \|u\|^2 + \|v\|^2 + 2\mathrm{Re}\,\langle u, v \rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2 \end{aligned}$$

By taking the square root of this inequality, the Minkowski inequality is obtained:

$$\|u + v\| \leq \|u\| + \|v\|$$

53.3. Natural norm. The Minkowski inequality can be viewed as the triangle inequality for the norm

$$\|u\| = \sqrt{\langle u, u \rangle}$$

It is not difficult to see that the other norm axioms are also satisfied. It is called the *natural norm* of an inner product space. Thus, every inner product space is a normed space. The converse is not generally true. There is no "natural" inner product in a normed space (an inner product that can be defined via the norm).

The natural norm satisfies the *parallelogram law*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

which is similar to the parallelogram law for vectors in a real Euclidean space. A proof is simple

$$\begin{aligned} \|u + v\|^2 + \|u - v\|^2 &= \langle u + v, u + v \rangle + \langle u - v, u - v \rangle \\ &= \|u\|^2 + \|v\|^2 + 2\mathrm{Re}\,\langle u, v \rangle \\ &\quad + \|u\|^2 + \|v\|^2 - 2\mathrm{Re}\,\langle u, v \rangle \\ &= 2\|u\|^2 + 2\|v\|^2 \end{aligned}$$

Proposition **53.1**. *On order for a normed space $\mathcal{X}$ to be an inner product space, it is necessary and sufficient that the parallelogram law holds for the norm in $\mathcal{X}$. In this case, the inner product can be defined by the rule:*

$$\langle u, v\rangle = \frac{1}{4}\Big(\|u+v\|^2 - \|u-v\|^2\Big)$$

any u and v in $\mathcal{X}$.

Steps to prove this assertion are outlined in Exercises.

53.4. Continuity of the inner product. Since an inner product space is also a normed space, one can consider converging sequences. A sequence $\{u_n\} \subset \mathcal{X}$ is said to converge to a vector $u \in \mathcal{X}$ if it converges with respect to the natural norm:

$$\lim_{n\to\infty} \|u - u_n\| = 0$$

The inner product is *continuous with respect to its arguments*. Let $u_n \to u \in \mathcal{X}$ and $v_m \to v \in \mathcal{X}$. Then the numerical double sequence $\langle u_n, v_m\rangle$ converges to $\langle u, v\rangle$:

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle u_n, v_m\rangle = \lim_{m\to\infty}\lim_{n\to\infty}\langle u_n, v_m\rangle = \langle u, v\rangle$$

First, recall that any convergent sequence is bounded. Indeed, by convergence of u_n, there exists an integer N such that

$$\|u - u_n\| \leq 1$$

for all $n \geq N$. Hence

$$\|u_n\| = \|u_n - u + u\| \leq \|u_n - u\| + \|u\| \leq 1 + \|u\|\,, \quad n \geq N$$

Therefore

$$\|u_n\| \leq M = \max_{n<N}\{\|u_n\|, 1 + \|u\|\}$$

for all n. It follows that

$$\begin{aligned}
|\langle u_n, v_m\rangle - \langle u, v\rangle| &= |\langle u_n, v_m - v\rangle + \langle u_n, v\rangle - \langle u, v\rangle| \\
&\leq |\langle u_n, v_m - v\rangle| + |\langle u_n - u, v\rangle| \\
&\leq \|u_n\|\|v - v_m\| + \|u - u_n\|\|v\| \\
&\leq M\|v - v_m\| + \|v\|\|u - u_n\|
\end{aligned}$$

this shows that the sequence $\langle u_n, v_m\rangle$ converges to $\langle u, v\rangle$.

53.5. Hilbert space. An inner product space is called a *Hilbert space* if it is complete with respect to its natural norm. In other words, every Cauchy sequence in a Hilbert space has a limit in it. Every Hilbert space is a Banach space with respect to its natural norm. Every Banach space can be turned into a Hilbert space if the norm satisfies the parallelogram law.

For example, the Banach space of bounded functions with the supremum norm cannot be converted into a Hilbert space. Let $\Omega = (0, 1)$. Put $u(x) = 1$ and $v(x) = x$. Then $\|u + v\|_\infty = 2$ and $\|u - v\|_\infty = 1$, but $\|u\|_\infty = \|v\|_\infty = 1$ and the parallelogram law does not hold.

53.5.1. Complex Euclidean space. Let $\mathbb{C}^N$ be a linear space of $N-$vectors with complex components. Put

$$\langle u, v\rangle = \sum_{j=1}^{N} u_j \bar{v}_j$$

It is not difficult to see that the inner product axioms are satisfied. So, $\mathbb{C}^N$ is an inner product space. The natural norm is nothing but the Euclidean norm. The space $\mathbb{C}^N$ is a Banach space and, hence, is a Hilbert space.

53.5.2. Space $C_2^0(\bar{\Omega})$. Let Ω be bounded open set in a Euclidean space. Consider a linear space of continuous complex-valued functions on Ω that also have a continuous extension to the boundary $\partial\Omega$ that is assumed to be piecewise smooth. Put

$$\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}\, d^N x$$

The integral exists for any functions from $C^0(\bar{\Omega})$. The axioms are also satisfied. Clearly, $\|u\|^2 = 0$ implies that $u(x) = 0$. If $u(x) \neq 0$ at some $x = x_0$, then $u(x) \neq 0$ in a ball centered at x_0 by continuity of u. Therefore the integral of $|u(x)|^2$ cannot be zero.

This space is not complete with respect to its natural norm. For example, take $\Omega = (-1, 1)$. Put

$$u_n(x) = 1 - n|x|\,, \quad |x| < \frac{1}{n}\,, \quad u_n(x) = 0\,, \quad \frac{1}{n} < |x| < 1$$

It is not difficult to see that the integral

$$\|u_n - u_m\|^2 = \int_{-1}^{1} |u_n(x) - u_m(x)|^2 dx = \int_{-\frac{-1}{n}}^{\frac{1}{n}} |u_n(x) - u_m(x)|^2 dx$$

where $n \leq m$, vanishes with increasing n so that $\{u_n\}$ is a Cauchy sequence, but

$$\lim_{n\to\infty} u_n(x) = u(x) = \begin{cases} 0\ ,\ x \neq 0 \\ 1\ ,\ x = 0 \end{cases}$$

Evidently, $u(x)$ is not continuous on $[-1, 1]$.

53.5.3. Space of square summable complex sequences l_2. For any two square summable complex sequences $u = \{u_n\}$ and $v = \{v_n\}$, put

$$\langle u, v\rangle = \sum_n u_n \bar{v}_n$$

The series converges absolutely because

$$|u_n \bar{v}_n| \leq \frac{1}{2}(|u_n|^2 + |v_n|^2)\,, \qquad \sum_n |u_n|^2 < \infty\,, \quad \sum_n |v_n|^2 < \infty$$

Thus, the inner product exists. The axioms are straightforward to verify. Therefore l_2 is an inner product space. The space l_2 is complete with respect to the natural norm

$$\|u\| = \sqrt{\langle u, u\rangle} = \Big(\sum_n |u_n|^2\Big)^{1/2}$$

and, hence, l_2 is a Hilbert space.

54. Orthogonal sets in inner product spaces

54.1. Orthogonality and linear independence. Two elements of an inner product space are called *orthogonal* if their inner product vanishes:

$$u \perp v \quad \Leftrightarrow \quad \langle u, v\rangle = 0$$

Evidently, the zero element is orthogonal to any element. A collection of elements B is called an *orthogonal set* if all elements of B are orthogonal to one another:

$$\langle u, v\rangle = 0\,, \quad \forall u, v \in B$$

An orthogonal set is called *orthonormal* if, in addition, all its elements have unit natural norm, $\|u\| = 1$.

PROPOSITION **54.1**. *Elements of an orthogonal set are linearly independent*

Take n non-zero elements of an orthogonal elements, v_j, $j = 1, 2, ..., n$. Their linear independence means that the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. Let us show that this indeed so if

$$\langle v_j, v_k\rangle = 0\,, \quad j \neq k$$

By the properties of the inner product:

$$\begin{aligned} 0 &= \langle 0, v_k\rangle = \langle c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, v_k\rangle \\ &= c_1\langle v_1, v_k\rangle + c_2\langle v_2, v_k\rangle + \cdots + c_n\langle v_n, v_k\rangle \\ &= c_k\langle v_k, v_k\rangle = c_k\|v_k\|^2 \\ \Rightarrow \quad c_k &= 0 \end{aligned}$$

since $\|v_k\| \neq 0$. Thus, the said equation can have only the trivial solution.

54.2. Complete sets. A set B is a normed space $\mathcal{X}$ is *dense* in $\mathcal{X}$ if for any element u from $\mathcal{X}$ one can find an elements of B that is arbitrary close to u. Or, a more formal way, for any $u \in \mathcal{X}$ and any $\varepsilon > 0$ there exists $v \in B$ such that

$$\|u - v\| < \varepsilon$$

A set B is a normed space $\mathcal{X}$ is *complete* in $\mathcal{X}$ if the span of B is dense in $\mathcal{X}$.

For example, a set of monomials $B = \{x^n\}_0^\infty$ is complete in the space of continuous functions relative to the supremum norm. The assertion follows from the *Weierstrass theorem*:

THEOREM **54.1**. *For any continuous function $f(x)$ in an interval $[a,b]$, there exists a sequence of polynomials $p_n(x)$ that converges to f uniformly:*

$$\lim_{n\to\infty} \|f - p_n\|_\infty = 0$$

The set of all polynomials is the span of B. By the Weierstrass theorem Span B is dense in $C^0[a,b]$.

A complete linearly independent set $B = \{v_n\}$ in a normed space $\mathcal{X}$ is called a *basis* if for any element u in the space there exists a unique collection of complex numbers $\{\alpha_n\}$ such that

$$u = \sum_n \alpha_n v_n$$

where the convergence is understood with respect to the norm $\mathcal{X}$. In finite dimensional vector spaces, any vector can be uniquely expanded into a linear combination of N linearly independent vectors, where N is the dimension of the space. So, any N linearly independent vectors can serve as a basis. The situation is more subtle in infinite dimensional spaces. Some elements of a linearly independent set can be arbitrary close to linear combinations of the other elements and, for this reason, not every linearly independent complete set is a basis. For example, let

$$B = \{v_n\}_{-1}^{\infty}\,, \quad v_{-1}(x) = e^x\,, \quad v_n(x) = x^n\,, \quad n \geq 0\,.$$

Evidently, the set B is linearly independent because no element in any finite collection is a linear combination of the others. The set is also complete in the space of continuous functions on any interval because its subset $\{x^n\}_0^\infty$ is complete in $C^0[a,b]$. However, B cannot be a basis. Let $u(x) = e^x$. Then its expansion over B is not unique

$$e^x = v_{-1}(x)\,, \qquad e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, v_n(x)$$

where the series converges uniformly (because the power series for the exponential has infinite radius of convergence). Also e^x is linearly independent with any finite collection of monomials, there are linear combinations of monomials that are arbitrary close to e^x.

Complete orthogonal sets. In the space $C^2[a,b]$, the functions

$$B = \left\{1\,, \quad \cos\left(\frac{2\pi n x}{b-a}\right), \quad \sin\left(\frac{2\pi n x}{b-a}\right), \quad n = 1, 2, \ldots\right\}$$

are orthogonal. Linear combinations of them are called *trigonometric polynomials*:

$$p_n(x) = a_0 + \sum_{k=1}^{n} \left[a_n \cos\left(\frac{2\pi nx}{b-a}\right) + b_n \sin\left(\frac{2\pi nx}{b-a}\right) \right]$$

THEOREM **54.2**. (Weierstrass)
For any continuous function f on an interval $[a,b]$ that takes the equal values at the endpoints, $f(a) = f(b)$, and any $\varepsilon > 0$ there exists a trigonometric polynomial $p(x)$ such that

$$\|f - p\|_\infty < \varepsilon$$

It follows from the Weierstrass theorem for trigonometric polynomials that the said orthogonal functions is complete in $C_2^0[a,b]$. Let first $f(a) = f(b)$. Fix $\varepsilon > 0$ and find a trigonometric polynomial $p(x)$ as stated in the theorem. Then

$$\|f-p\|_2^2 = \int_a^b |f(x) - p(x)|^2 dx \leq \|f-p\|_\infty^2 \int_a^b dx$$
$$< \varepsilon^2 (b-a)$$

Therefore any continuous function with equal values at the endpoints of an interval can be approximated by a trigonometric polynomial with any desired accuracy relative to the natural norm in $C_2^0[a,b]$:

$$\|f-p\|_2 < \sqrt{b-a}\,\varepsilon$$

Suppose that $f(a) \neq f(b)$. Put $L = b - a$ and for any $x \in [a,b]$

$$f_n(x) = \begin{cases} f(x) \ , \ x \leq b - \frac{L}{n} \\ v_n(x) \ , \ x > b - \frac{L}{n} \end{cases}$$

where $n = 1, 2, ...$, and v_n is a linear function such that $v_n(b - \frac{L}{n}) = f(b - \frac{L}{n})$ and $v_n(b) = f(a)$, that is

$$v_n(x) = f(a) + \frac{f(a) - f(b - \frac{L}{n})}{\frac{L}{n}} (b - x)$$

The function $f_n(x)$ has equal values at the endpoints, $f_n(a) = f_n(b) = f(a)$ for any n, and it is continouus on $[a,b]$. Then f_n converges to f

in $C_2^0[a,b]$ because

$$\lim_{n\to\infty}\|f-f_n\|_2^2 = \lim_{n\to\infty}\int_a^b |f(x)-f_n(x)|^2dx$$
$$= \lim_{n\to\infty}\int_{b-L/n}^b |f(x)-v_n(x)|^2dx = 0$$

because the interval of integration shrinks to a point in the limit and the integrand is a bounded continuous function. Thus, given $\varepsilon > 0$, one can find f_n such that

$$\|f-f_n\|_2 < \varepsilon$$

Having found f_n, one can find a trigonometric polynomial p such that

$$\|f_n-p\|_\infty < \varepsilon$$

By the triangle inequality

$$\|f-p\|_2 \le \|f-f_n\|_2 + \|f_n-p\|_2 < \varepsilon + \sqrt{b-a}\,\varepsilon$$

which means that for any $f\in C_2^0[a,b]$ there exists a linear combination of orthogonal functions from B that is arbitrary close to f, or the orthogonal set B is complete in $C_2^0[a,b]$.

54.3. Gramm-Schmidt process. In an inner product space, every linearly independent set can be converted to an orthogonal set. The algorithm to do so is known as the *Gramm-Schmidt process.*

Let $S=\{u_n\}$ be a linearly independent set in an inner product space. Then there exists an orthonormal set $B=\{\varphi_n\}$ such that

$$\operatorname{Span} S = \operatorname{Span} B$$

and

$$\text{(i)}\qquad \langle\varphi_k,\varphi_n\rangle = \delta_{kn}$$
$$\text{(ii)}\qquad \varphi_n = \sum_{k=1}^n a_{nk}u_k$$
$$\text{(iii)}\qquad u_n = \sum_{k=1}^n b_{nk}\varphi_k$$

where each element φ_n is determined uniquely by the properties (i) $-$ (iii) up to a phase factor.

Let us prove the stated assertions. Put

$$\psi_1 = u_1\,, \quad \psi_2 = u_2 - \alpha_{21}\psi_1$$

where the number α_{21} is defined by the orthogonality condition

$$\langle \psi_2, \psi_1\rangle = 0 \quad \Rightarrow \quad \alpha_{21} = \frac{\langle u_2, u_1\rangle}{\|u_1\|^2}$$

so that

$$\psi_2 = u_2 - \frac{\langle u_2, u_1\rangle}{\|u_2\|^2}\, u_1$$

Then put

$$\varphi_1 = \frac{\psi_1}{\|\psi_1\|}\,, \quad \varphi_2 = \frac{\psi_2}{\|\psi_2\|}$$

It follows that the relations (i)-(iii) hold for $n = 1$ and $n = 2$ with our choice of $\varphi_{1,2}$.

Note that the vectors

$$\begin{aligned}
\psi_1 &= \langle u_1, \varphi_1\rangle\varphi_1\,,\\
\psi_2 &= u_2 - \langle u_2, \varphi_1\rangle\varphi_1\,,\\
\psi_3 &= u_3 - \langle u_3, \varphi_1\rangle\varphi_1 - \langle u_3, \varphi_2\rangle\varphi_2
\end{aligned}$$

are orthogonal, and the vectors $\varphi_j = \psi_j/\|\psi_j\|$ are orthonormal. Suppose that $n-1$ vectors φ_j, $j = 1, 2, ..., n-1$, constructed by this process are orthonormal. Let us show that n vectors constructed by this process are also orthonormal. Put

$$\psi_n = u_n - \sum_{j=1}^{n-1}\langle u_n, \varphi_j\rangle\varphi_j$$

Then it follows that

$$\langle \psi_n, \varphi_k\rangle = 0\,, \quad k = 1, 2, ..., n-1$$

Put $\varphi_n = \psi_n/\|\psi_n\|$. Then the system of n vectors is orthonormal if so is the system of $n-1$ vectors, and the latter is true by the assumption of mathematical induction. Thus, the process holds for any n and there exists a choice of φ_j for which relations (i)-(iii) holds.

Conversely, suppose that (i)-(iii) hold. Then it follows from (i) that (iii) must have the form

$$u_n = \sum_{k=1}^{n}\langle u_n, \varphi_k\rangle\varphi_k$$

for any n. For $n = 1$ it follows that φ_1 is determined up to a phase factor. Since φ_1 must be proportional to u_1, any other choice of φ_1 is just a scaling $\varphi_1 \to z\varphi_1$ for some complex z. But

$$\langle u_1, \varphi_1 \rangle \varphi_1 = \langle u_1, z\varphi_1 \rangle z\varphi_1 \quad \Rightarrow \quad |z|^2 = 1 \quad \Rightarrow \quad z = e^{i\theta_1}$$

Similarly,

$$u_2 - \langle u_2, \varphi_1 \rangle \varphi_1 = \langle u_2, \varphi_2 \rangle \varphi_2$$

implies that only scaling transformations are allowed for φ_2 for a given u_2 and φ_1. By the same arguments, the scaling factor must be a phase factor $e^{i\theta_2}$. Thus, relations (i)-(iii) imply that $\varphi_k \to e^{i\theta_k}\varphi_k$ is the only freedom to change φ_k. The Gramm-Schmidt process defines elements of an orthonormal set from a linearly independent set up to a phase factor. If the inner product space is real, then this uncertainty is reduced to the sign uncertainty.

Legendre polynomials. Consider the set of all monomials in $C_2^0[a,b]$:

$$S = \{v_n(x)\}_{n=0}^{\infty}\,, \quad v_n(x) = x^n$$

This is a linearly independent set. In particular, put $[a,b] = [-1,1]$. The Gramm-Schmidt process leads to the *Legendre polynomials*

$$P_n(x) = \frac{1}{2^n n!}\Big(\frac{d}{dx}\Big)^n (x^2-1)^n = \frac{1}{2^n}\sum_{k=0}^{n} (C_k^n)^2 (x-1)^{n-k}(x+1)^k$$

where C_k^n are binomial coefficients

$$C_k^n = \frac{n!}{(n-k)!k!}$$

and

$$\langle P_n, P_k \rangle = \int_{-1}^{1} P_n(x)P_k(x)\,dx = \frac{2}{2n+1}\,\delta_{nk}$$

55. Separable inner product spaces

55.1. Dense and countable sets. Recall that a set A is called *dense* in a set B in a normed (or metric) space if any element from B can be approximated by an element from A with any desired accuracy, or for any $\varepsilon > 0$ and any $u \in B$, there exist $v \in A$ such that

$$\|u - v\| < \varepsilon$$

If A is dense in B and B is dense in C, then A is also dense in C. Indeed, fix $\varepsilon > 0$. Then for any $u \in C$ there exists $v \in B$ such that $\|u - v\| < \varepsilon$. Having found v, one can also find $w \in A$ such that $\|w - v\| < \varepsilon$. By the triangle inequality

$$\|w - u\| \leq \|w - v\| + \|v - u\| < 2\varepsilon$$

A set A is called *countable* if there is one-to-one correspondence between elements of A and positive integers. In other words, A is countable if its elements can be counted. For example, a set of rational numbers $\mathbb{Q}$ is countable.

Proposition **55.1**. *A countable union of countable sets is countable*

Let

$$A = \bigcup_{n=1}^{\infty} A_n\,, \qquad A_n = \{a_{n,k}\}_{k=1}^{\infty}$$

So, all elements of A can be arranged into an infinite table in which the nth column is formed by elements of A_n. The counting of elements of A can carried out in the following way:

$$\begin{aligned} &a_{1,1} \to a_{2,1} \to a_{1,2} \to a_{1,3} \to a_{2,2} \to a_{3,1} \to \cdots \\ &\to a_{1,n} \to a_{2,n-1} \to a_{3,n-3} \to \cdots \to a_{n,1} \\ &\to a_{n+1,1} \to a_{n,2} \to \cdots \to a_{2,n} \to a_{1,n+1} \to \cdots \end{aligned}$$

A set is said to *at most countable* if it is either has finitely many or countably many distinct elements.

55.2. Separable spaces. A space (metric, normed, or inner product space) is called *separable* if it contains a countable dense subset. For example, a Euclidean space $\mathbb{R}^N$ is separable. A countable dense subset is the collection of all vectors with rational components, $\mathbb{Q}^N$. Indeed, for any real component x_j of a vector $x \in \mathbb{R}^N$ there is an arbitrary close rational number q_j:

$$|x_j - q_j| < \varepsilon\,, \quad j = 1, 2, ..., N$$

Therefore for any x, there is a vector $q \in \mathbb{Q}^N$ such that

$$\|x - q\|^2 = \sum_{j=1}^{N} |x_j - q_j|^2 < N\varepsilon^2$$

that is, q is arbitrary close to x, and, hence, $\mathbb{Q}$ is dense in $\mathbb{R}^N$. All elements of $\mathbb{Q}$ can be viewed as the union of N collections of rational numbers. But the union of finitely many countable sets is countable. So, $\mathbb{R}^N$ is separable.

PROPOSITION **55.2**. (Orthogonal systems in separable spaces) *Let $\mathcal{X}$ be a separable inner product space and B be an orthogonal system in $\mathcal{X}$. Then B is at most countable.*

Let $u_a \in B$ and the label a takes its values in a set M (whose nature is irrelevant). Since u_a and u_b are orthogonal if a and b are not the same element, one can always scale elements of B so that they all have unit length, $\|u_a\| = 1$. Then the distance between any two elements in B

$$\|u_a - u_b\| = \sqrt{2}$$

because

$$\|u_a - u_b\|^2 = \langle u_a - u_b, u_a - u_b\rangle = 2 - 2\mathrm{Re}\,\langle u_a, u_b\rangle = 2$$

Let $B(u_a; \frac{1}{2})$ be a ball of radius $\frac{1}{2}$ centered at u_a, that is, it contains all elements of $\mathcal{X}$ whose distance from u_a is less than $\frac{1}{2}$:

$$B(u_a; \tfrac{1}{2}) = \left\{ u \in \mathcal{X} \,\middle|\, \|u - u_a\| < \tfrac{1}{2} \right\}$$

Clearly, the ball centered at distinct elements of B, u_a and u_b, have no common elements

$$B(u_a; \tfrac{1}{2}) \bigcap B(u_b; \tfrac{1}{2}) = \emptyset\,, \quad a \neq b$$

By the hypothesis, $\mathcal{X}$ contains a countable dense subset A. Since A is dense in $\mathcal{X}$, there is an element of A that lies in $B(u_a; \frac{1}{2})$ for every a. Since A is countable, a collection of all elements of A that are found to be in each $B(u_a; \frac{1}{2})$ is at most countable.

Separable Hilbert space. A Hilbert space is *separable* if it contains a dense countable subset. Any orthogonal system in a separable Hilbert space is at most countable.

55.3. Exercises.

1. Consider a collection of all functions that square integrable on $(-R, R)$ for any $R > 0$, and satisfy the condition that the following limit exists:

$$\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} |u(x)|^2 \, dx < \infty$$

Define a space $\mathcal{X}$ whose elements are equivalence classes of the functions defined above: Two functions u and v belongs to the same equivalence class if

$$\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} |u(x) - v(x)|^2 \, dx = 0$$

In other words, $u = v$ in $\mathcal{X}$ if functions $u(x)$ and $v(x)$ representing equivalence classes u and v satisfy the above condition, similarly to $\mathcal{L}_2$: $u = v$ in $\mathcal{L}_2$ if $u(x) = v(x)$ *a.e.*

(i) Show that $\mathcal{X}$ is a linear space;

(ii) Define the inner product on $\mathcal{X}$ by

$$\langle u, v \rangle = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} u(x)\overline{v(x)} \, dx$$

Prove that it exists for any two functions from $\mathcal{X}$ and all axioms of the inner product are fulfilled.

(iii) Consider a set of elements of $\mathcal{X}$ represented by functions $u_\alpha(x) = \sin(\alpha x)$, $\alpha \in \mathbb{R}$. Show this set is orthogonal. Use this fact to prove that the inner product space $\mathcal{X}$ is not separable.

56. The space of square integrable functions

56.1. $\mathcal{L}_2(\Omega)$ is an inner product space. The inner product is defined by

$$\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}\, d^N x$$

First, note that it exists because

$$\left|u(x)\overline{v(x)}\right| \le \frac{1}{2}|u(x)|^2 + \frac{1}{2}|v(x)|^2$$

and u and v are square integrable. Recall that for any two complex numbers

$$|z_1 z_2| = |z_1||z_2| \le \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$$

because $(|z_1| - |z_2|)^2 \ge 0$. The first three axioms of the inner product

$$\langle u, v\rangle = \overline{\langle v, u\rangle}$$
$$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$$
$$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$

are obviously fulfilled. It is also required that

$$\langle u, u\rangle > 0\,, \quad u \ne 0 \quad \text{and} \quad \langle u, u\rangle = 0 \Leftrightarrow u = 0$$

If $u(x) = 0$ for all $x \in \Omega$, then $\langle u, u\rangle = 0$. However, the converse is not true:

$$\langle u, u\rangle = \int_\Omega |u(x)|^2\, d^N x = 0 \quad \Rightarrow \quad u(x) = 0\,,\ a.e.$$

In other words, any function that differs from the zero function on a set of measure zero would fulfill the condition $\langle u, u\rangle = 0$. To circumvent this difficulty and fulfill the fourth axiom of the inner product, each element of $\mathcal{L}_2(\Omega)$ is understood as the space of *equivalence classes*, each class consists of all square integrable functions that may differ from one another on a set of measure zero. In other words, the equality of two elements

$$u = v \quad \text{in } \mathcal{L}_2(\Omega)$$

means that

$$u(x) = v(x)\ a.e.$$

With this agreement, the fourth axiom of the inner product holds. Note that $\mathcal{L}_2(\Omega)$ is still a linear space even if its elements are now equivalence classes because the sum of two functions that are zero almost everywhere is a function that is zero almost everywhere.

The Cauchy-Bunyakowski inequality

$$|\langle u, v\rangle| \le \sqrt{\langle u, u\rangle}\sqrt{\langle v, v\rangle}$$

has the following form in $\mathcal{L}_2(\Omega)$:

$$\left|\int_\Omega u\bar{v}\, d^N x\right| \le \left(\int_\Omega |u|^2\, d^N x\right)^{1/2} \left(\int_\Omega |v|^2\, d^N x\right)^{1/2}$$

56.2. $\mathcal{L}_2(\Omega)$ is a normed space. The natural norm in $\mathcal{L}_2(\Omega)$ reads

$$\|u\| = \sqrt{\langle u, u\rangle} = \left(\int_\Omega |u|^2\, d^N x\right)^{1/2}$$

It satisfies all the norm axioms owing to the interpretation of $\mathcal{L}_2(\Omega)$ as a space of equivalence classes. Note $\|u\| = 0$ means that $u(x) = 0$ *a.e.* The triangle inequality in $\mathcal{L}_2(\Omega)$ is also know as *Minkowski inequality*:

$$\|u + v\| \le \|u\| + \|v\|$$

56.3. Properties of $\mathcal{L}_2(\Omega)$.

THEOREM **56.1**. ($\mathcal{L}_2(\Omega)$ vs $\mathcal{L}(\Omega)$)
Suppose that $\Omega \subset \mathbb{R}^N$ is bounded and $u \in \mathcal{L}_2(\Omega)$. Then $u \in \mathcal{L}(\Omega)$.

PROOF. The characteristic function χ_Ω of a bounded region Ω is square integrable on Ω:

$$\int_\Omega |\chi_\Omega|^2 d^N x = \int_\Omega d^N x = \mu_L(\Omega) < \infty$$

where $\mu_L(\Omega)$ is the Lebesgue measure (volume) of Ω. The measure is finite because Ω is contained in a ball and hence its measure is bounded by the volume of the ball. Since $|u|$ and χ_Ω are from $\mathcal{L}_2(\Omega)$, by the Cauchy-Bunyakowski inequality

$$\begin{aligned}\int_\Omega |u|\, d^N x &= \int_\Omega |u|\chi_\Omega\, d^N x = \langle |u|, \chi_\Omega\rangle \\ &\le \|u\|\|\chi_\Omega\| = \sqrt{\mu_L(\Omega)}\,\|u\| < \infty\end{aligned}$$

Therefore $|u| \in \mathcal{L}(\Omega)$ which implies that $u \in \mathcal{L}(\Omega)$ by the properties of the Lebesgue integral. □

56.4. $\mathcal{L}_2(\Omega)$ is a separable Hilbert space.

THEOREM **56.2**. (Riesz-Fisher, 1907)
$\mathcal{L}_2(\Omega)$ is complete and separable.

The completeness of the space $\mathcal{L}_2(\Omega)$ means that every Cauchy sequence of Lebesgue square integrable functions has a limit and the limit function is also Lebesgue square integrable integrable. This is contrast to the space $C_2^0(\Omega)$, an inner product space of continuous functions on a closed bounded set Ω with the same inner product. In other words, a completion of the space of continuous square integrable functions by adding the limits of all Cauchy sequences requires an extension of C_2^0 to $\mathcal{L}_2$ (an analog of the extension of all rational numbers $\mathbb{Q}$ to $\mathbb{R}$). A separability of $\mathcal{L}_2(\Omega)$ implies that $\mathcal{L}_2(\Omega)$ has a countable orthogonal basis, which is a foundation of the Fourier analysis. Thus, by the Riesz-Fisher theorem, *the space of Lebesgue square integrable functions is a separable Hilbert space*. It is also a Banach space with respect to its natural norm.

Completeness. A proof of completeness can be found in Sec. 28 of "Functional Analysis" by F. Riesz and B. Sz.-Nagy.

Separability. First, let us show separability of $\mathcal{L}_2(\mathbb{R}) \equiv \mathcal{L}_2$. One has to construct a countable subset $A \subset \mathcal{L}_2$ that is dense, that is, $\bar{A} = \mathcal{L}_2$.

Let $u \in \mathcal{L}_2$. Therefore $u \in \mathcal{L}_2(-n, n)$ because

$$\int_{-n}^{n} |u(x)|^2 dx \leq \int |u(x)|^2 dx < \infty\,.$$

By the definition of the Lebesgue integral, the integral of $|u|^2$ over $(-n, n)$ is the limit of Riemann integrals of some sequence of piecewise continuous functions on $[-n, n]$. Therefore the set of piecewise continuous functions on $[-n, n]$ is dense in the set $\mathcal{L}_2(-n, n)$. But the set $C_2^0([-n, n])$ is dense in the space of piecewise continuous functions. Indeed, a piecewise continuous function v has finitely many jump discontinuities in $[-n, n]$. Suppose v is not continuous at x_0, then define a continuous function $w(x) \in C_2^0([-n, n])$ that coincides with $v(x)$ except in $(x_0 - \delta, x_0 + \delta)$ in which $w(x)$ is a linear function such that $w(x_0 \pm \delta) = v(x_0 \pm \delta)$. It is clear that

$$\begin{aligned} \|v - w\|^2 &= \int_{-n}^{n} |v(x) - w(x)|^2 \, dx \\ &= \int_{x_0-\delta}^{x_0+\delta} |v(x) - w(x)|^2 \, dx \to 0 \quad \text{as} \quad \delta \to 0^+ \end{aligned}$$

The same construction can be extended to the case when v has finitely many jump discontinuities. Thus, $C_2^0([-n, n])$ is dense in $\mathcal{L}_2(-n, n)$. Now recall that the space $\mathcal{P}$ of polynomials is dense in $C_2^0([-n, n])$, and the countable set $\mathcal{P}_{\mathbb{Q}}$ of all polynomials with rational coefficients is dense in $\mathcal{P}$ and, hence, in $C_2^0([-n, n])$, and, hence, in $\mathcal{L}_2(-n, n)$.

It is not now difficult to construct a countable dense subset in $\mathcal{L}_2$. Let

$$v \in A_n : \quad v(x) = \chi_n(x) p(x)\,, \quad p \in \mathcal{P}_{\mathbb{Q}}$$

where $\chi_n(x)$ is the characteristic function of $[-n, n]$. Then the set A_n is countable. The union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is countable as the countable union of countable sets and so is the intersection $A \cap \mathcal{L}_2$. Thus, for every $u \in \mathcal{L}_2$ and any $\varepsilon > 0$ there exists $v \in A$ such that

$$\|u - v\|^2 = \int\limits_{|x|>n} |u(x)|^2 dx + \int_{-n}^{n} |u(x) - p(x)|^2 \, dx < \varepsilon$$

where p is a polynomial with rational coefficients. Note that the first term can be made arbitrary small because

$$\int\limits_{|x|>n} |u(x)|^2 dx = \int |u(x)|^2 dx - \int_{-n}^{n} |u(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty$$

whereas the second term can be made arbitrary small for any n because $\mathcal{P}_{\mathbb{Q}}$ is dense in $C_2^0([-n, n])$ and the latter is dense in $\mathcal{L}_2(-n, n)$.

An extension of this construction to $\mathbb{R}^N$ is left to the reader as an exercise.

56.5. Exercises.

1. Show that the Legendre polynomials form an orthogonal basis in the Hilbert space $\mathcal{H} = \mathcal{L}_2(-1, 1)$.

2. Prove that $\mathcal{L}_2(\Omega)$, $\Omega \subset \mathbb{R}^N$, is separable. *Hint*: Show first that $C_2^0(|x| \leq n)$ is dense in $\mathcal{L}_2(|x| < n)$, $n = 1, 2, ...$

3. Prove that the space of test functions $\mathcal{D}(\Omega)$ is dense in the Hilbert space $\mathcal{H} = \mathcal{L}_2(\Omega)$ for any bounded open $\Omega \subset \mathbb{R}^N$.

57. Linear manifolds in a Hilbert space

57.1. Orthogonal sets.

THEOREM **57.1**. *Each separable Hilbert space has a countable orthonormal set that is complete.*

PROOF. Let $A = \{v_n\}_1^\infty$ be a countable dense subset in a separable Hilbert space $\mathcal{H}$. Let us construct a linearly independent subset in A. Here is an algorithm:

$$\begin{aligned} \text{put } u_1 &= v_1 \\ u_2 &= v_2 \text{ if } v_1 \text{ and } v_2 \text{ are linearly independent} \\ \text{otherwise } u_2 &= v_3 \text{ if } v_1 \text{ and } v_3 \text{ are linearly independent} \\ \text{continue to get } u_2 &= v_{n_2} \\ \text{Put } u_k &= v_{n_k} \text{ if } \{v_{n_j}\}_{j=1}^k \text{ are linearly independent} \end{aligned}$$

By the linear independence of the set $\{u_k\}$,

$$\text{Span}A = \text{Span}\{u_k\}$$

and, hence, Span$\{u_k\}$ is dense in $\mathcal{H}$ which means that $\{u_k\}$ is complete in $\mathcal{H}$. By the Gram-Schmidt process, the set $\{u_k\}$ can be transformed into a complete orthonormal set in $\mathcal{H}$. □

Remark. A mere completeness and linear independence of the constructed countable set $\{u_k\}$ is not sufficient for $\{u_k\}$ to be a basis in $\mathcal{H}$ if $\mathcal{H}$ is infinite dimensional. Clearly, if the dimension is finite, then any complete linearly independent set is a basis. Recall that in order for $\{u_k\}$ to be a basis, every $u \in \mathcal{H}$ must have a unique expansion over it, that is, for any $v \in \mathcal{H}$ there should exist a unique sequence $\{\alpha_k\} \subset \mathbb{C}$ such that

$$v = \sum_k \alpha_k u_k$$

where the series converges in the norm in $\mathcal{H}$. Problem 1 in Exercises offers an example of a complete linearly independent set that is not a basis. A necessary and sufficient criterion for a complete orthogonal set to be an orthogonal basis in a separable Hilbert space will be established in Section **58**.

57.2. Projections on linear manifolds. Suppose $u \in \mathbb{R}^3$. Let $\mathcal{M}$ be a plane orthogonal to a vector v_1, that is, for any vector v in the plane $\langle v_1, v\rangle = 0$ where $\langle\cdot,\cdot\rangle$ stands for the dot product in $\mathbb{R}^3$. What is the best approximation of $u \in \mathbb{R}^3$ by a vector $\mathcal{M}$? The best approximation is a vector $v \in \mathcal{M}$ such that the distance $\|u - v\|$ is minimal. It is not difficult to find such a vector by the Pythagorean theorem:

$$v = u - \mathrm{Proj}_{v_1} u = u - \frac{\langle u, v_1 \rangle}{\|v_1\|^2}\, v_1$$

where $\mathrm{Proj}_{v_1} u$ stands for the vector projection of u onto v_1. It is also clear that such v is unique. If v_2 and v_3 are are orthogonal vectors in the plane, then v_j, $j = 1, 2, 3$, form an orthogonal basis in $\mathbb{R}^3$ and

$$v = \frac{\langle u, v_2 \rangle}{\|v_2\|^2}\, v_2 + \frac{\langle u, v_3 \rangle}{\|v_3\|^2}\, v_3$$

Can this picture be extended to an infinite dimensional Hilbert space? It turns out that the answer is affirmative.

THEOREM **57.2**. (Baby projection theorem)
Let $A_n = \{\varphi_k\}_1^n$ be an orthogonal set in an inner product space X and $\mathcal{M}_n = \mathrm{Span} A_n$. Then for any $u \in X$ there exists a unique element $v \in \mathcal{M}_n$ such that

$$\inf_{w \in \mathcal{M}_n} \|w - u\| = \|v - u\| \quad \text{and} \quad v = \sum_{k=1}^{n} \langle u, \varphi_k \rangle\, \varphi_k$$
$$\langle u - v, w \rangle = 0 \quad \forall w \in \mathcal{M}_n$$

PROOF. Put

$$w = \sum_{k=1}^{n} \alpha_k \varphi_k \;\in\; \mathrm{Span}\,\{\varphi_1, \varphi_2, ..., \varphi_n\} = \mathcal{M}_n\,,$$

where $\alpha_k \in \mathbb{C}$. Put

$$c_k = \langle u, \varphi_k \rangle\,.$$

Using the definition of c_k, the orthonormality relation $\langle \varphi_k, \varphi_n \rangle = \delta_{kn}$, and the properties of the inner product, one infers that

$$\begin{aligned}
\|u - w\|^2 &= \langle u - w, u - w \rangle \\
&= \langle u, u \rangle - \langle u, w \rangle - \langle w, u \rangle + \langle w, w \rangle \\
&= \|u\|^2 - \sum_{k=1}^{n} \left(\overline{\alpha_k} c_k + \overline{c_k} \alpha_k \right) + \sum_{k=1}^{n} \alpha_k \overline{\alpha_k} \\
&= \|u\|^2 - \sum_{k=1}^{n} |c_k|^2 + \sum_{k=1}^{n} |\alpha_k - c_k|^2
\end{aligned}$$

It follows from this representation that $\|u-w\|^2$ attains its absolute minimum if and only if $\alpha_k = c_k$, that is,

$$w = v = \sum_{k=1}^{n} c_k\varphi_k = \sum_{k=1}^{n} \langle u, \varphi_k\rangle\varphi_k$$

and in this case

$$\|u-v\|^2 = \|u\|^2 - \|v\|^2\,,$$
$$\langle u-v, w\rangle = \sum_{k=1}^{n}\left(c_k\overline{\alpha_k} - c_k\overline{\alpha_k}\right) = 0 \quad \forall w \in \mathcal{M}_n$$

which completes the proof. □

Note that it is crucial for the proof that the linear manifold $\mathcal{M}_n \subset X$ is finite dimensional, $\dim \mathcal{M}_n = n$, otherwise all finite sums in the expression for $\|u-w\|^2$ become series and the expression becomes meaningless because nothing is known about the convergence of the series.

57.3. Orthogonal complements of linear manifolds.

Definition **57.1**. (An orthogonal complement of a linear manifold) *An orthogonal complement $\mathcal{M}^\perp$ of a linear manifold $\mathcal{M}$ in a Hilbert space $\mathcal{H}$ is a collection of all elements $v \in \mathcal{H}$ that are orthogonal to any element from $\mathcal{M}$:*

$$v \in \mathcal{M}^\perp : \quad \langle v, u\rangle = 0\ \forall u \in \mathcal{M}\,.$$

First, note that $\mathcal{M}^\perp$ is a linear manifold, that is, if $v, w \in \mathcal{M}^\perp$, then a linear combination $\alpha v + \beta w \in \mathcal{M}^\perp$ belongs to it for any numbers α and β. The manifolds $\mathcal{M}$ and $\mathcal{M}^\perp$ have only one common element $u = 0$ (which is the only element in $\mathcal{H}$ that is orthogonal to all elements).

Properties of orthogonal complements. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a linear manifold in it. Then

(1) $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$
(2) $\mathcal{M} \subseteq (\mathcal{M}^\perp)^\perp = \overline{\mathcal{M}}$
(3) $\mathcal{M}$ is closed $\Rightarrow$ $\mathcal{M} = (\mathcal{M}^\perp)^\perp$
(4) $\mathcal{M}$ is closed $\Rightarrow$ $\mathcal{H} = \mathcal{M} \cup \mathcal{M}^\perp$
(5) $\mathcal{N} \subset \mathcal{M}$ $\Rightarrow$ $\mathcal{M}^\perp \subset \mathcal{N}^\perp$

All the properties, except (2), are familiar for finite dimensional Hilbert spaces, $\mathbb{R}^N$ or $\mathbb{C}^N$, because any finite dimensional linear manifold is closed so that its orthogonal complement is closed, too. This implies that (2) is automatically reduced to (3). The property (2) will be proved and discussed in more detail later in the theory of operators in Hilbert spaces. Here it is illustrated by an example.

A closed linear manifold in an inner product space is a Hilbert space. For example, the space of even functions in $\mathcal{L}_2(-1,1)$ form a Hilbert subspace. Even Legendre polynomials form an orthogonal basis in this Hilbert subspace of $\mathcal{L}_2(-1,1)$. The orthogonal complement of the space of even functions is the space of odd functions in $\mathcal{L}_2(-1,1)$ because

$$\langle u, v\rangle = \int_{-1}^{1} u(x)\overline{v(x)}\,dx = 0$$

for any $u(-x) = u(x)\,a.e.$ and $v(-x) = -v(x)\,a.e.$ Note that both the manifolds are closed; they are orthogonal Hilbert subspaces of $\mathcal{L}_2(-1,1)$. Let $\mathcal{M}$ be the linear manifold of all even continuous functions in $\mathcal{H} = \mathcal{L}_2(-1,1)$. The linear manifold $\mathcal{M}$ is not closed because, as noted before, the space of square integrable continuous function is not complete and, in particular, $C_2^0([-1,1])$ is not complete (its completion is $\mathcal{L}_2(-1,1)$). The closure $\overline{\mathcal{M}}$ is the space of all even functions in $\mathcal{L}_2(-1,1)$, that is, $\mathcal{M} \subset \overline{\mathcal{M}}$. The orthogonal complement $\mathcal{M}^\perp$ is the space of all odd functions in $\mathcal{L}_2(-1,1)$, and the orthogonal complement of $\mathcal{M}^\perp$ is the space of all even functions in $\mathcal{L}_2(-1,1)$. Thus, $\mathcal{M} \subset (\mathcal{M}^\perp)^\perp = \overline{\mathcal{M}}$.

57.4. Projection on a linear manifold. Keeping in mind the noted differences of linear manifolds in an infinite and finite dimensional Hilbert spaces, let us discuss the projection of a vector onto a linear manifold in a Hilbert space.

Theorem **57.3**. (Projection theorem)
Let $\mathcal{M}$ be a closed manifold (a Hilbert subspace) in a Hilbert space $\mathcal{H}$. Then

$$\forall v \in \mathcal{H} \quad \exists!\, u \in \mathcal{M} \text{ and } \exists!\, w \in \mathcal{M}^\perp \;:\quad v = u + w$$

and

$$\|v - u\| = \inf_{h\in\mathcal{M}} \|v - h\|\,.$$

Proof. First, let us prove that the existence of a solution to the best approximation problem:

$$\forall v \in \mathcal{H} \quad \exists\, u \in \mathcal{M} \;:\quad \inf_{h\in\mathcal{M}} \|v - h\| = \|v - u\|$$

that is, the minimum is actually attained in $\mathcal{M}$. The procedure from a finite dimensional case no longer applies. So, a different approach is invoked. Let

$$d = \inf_{h\in\mathcal{M}} \|v - h\|$$

Therefore there exists a sequence $\{h_n\} \subset \mathcal{M}$ such that $\|v - h_n\| \to d$ as $n \to \infty$. The natural norm in an inner product space satisfies the parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

A proof is left to the reader as an exercise. By the parallelogram law with $f = v - h_n$ and $g = v - h_m$, the sequence $\{h_n\}$ is shown to be a Cauchy sequence and, hence, it has a limit in $\mathcal{M}$ because $\mathcal{M}$ is closed. Indeed,

$$\begin{aligned} 0 \leq \|h_n - h_m\|^2 &\overset{(1)}{=} 2\|f-h_n\|^2 + 2\|f-h_m\|^2 - \|2v - (h_n + h_m)\|^2 \\ &\overset{(2)}{=} 2\|f-h_n\|^2 + 2\|f-h_m\|^2 - 2\|v - \tfrac{1}{2}(h_n + h_m)\|^2 \\ &\overset{(3)}{\leq} 2\|f-h_n\|^2 + 2\|f-h_m\|^2 - 2\|v - 4d^2 \\ &\overset{(4)}{\to} 2d^2 + 2d^2 - 4d^2 = 0 \quad \text{as } n, m \to \infty \end{aligned}$$

(1) by the parallelogram law;
(2) by factoring out a factor 2 in the last term;
(3) Since $\frac{1}{2}(h_n + h_m) \in \mathcal{M}$, $\|v - \frac{1}{2}(h_n + h_m)\| \geq d$ by definition of infimum;
(4) by the choice of the sequence $\{h_n\}$.
Thus, there exists $u \in \mathcal{M}$ such that $h_n \to u$ in $\mathcal{H}$ as $n \to \infty$.

Next, let us prove the existence of w in the decomposition of v. Having found $u \in \mathcal{M}$ with the said property for a given $v \in \mathcal{H}$, put $w = v - u$. The vector w is proved to be from $\mathcal{M}^\perp$. Indeed, given $u \in \mathcal{M}$, take another vector $h \in \mathcal{M}$. Then for any $\alpha \in \mathbb{C}$, $u + \alpha h \in \mathcal{M}$ and the following inequality holds:

$$\|(v-u) - \alpha h\|^2 = \|v - (u + \alpha h)\|^2 \geq \|v - u\|^2 = d^2$$

by the construction of u. This implies that the function

$$f(\alpha) = \|w - \alpha h\|^2 \geq \|w\|, \quad \forall \alpha \in \mathbb{C}, \quad \forall h \in \mathcal{M}$$

is bounded from below. Since f is a quadratic polynomial in two real variables (s,t), where $\alpha = s + it$, the function f has a minimum at $s = t = 0$. Since partial derivatives of f are continuous, they must vanish at $t = s = 0$:

$$\begin{aligned} f_s'(0,0) &= -2\mathrm{Re}\,\langle h, w\rangle = 0 \\ f_t'(0,0) &= -2\mathrm{Im}\,\langle h, w\rangle = 0 \end{aligned}$$

It follows from these relations that

$$\langle h, w\rangle = 0 \quad \Rightarrow \quad w \in \mathcal{M}^\perp$$

because h is an arbitrary vector from $\mathcal{M}$.

Finally, let us prove that the decomposition $v = u + w$ is unique. Suppose that $v = u + w = u' + w'$. Then

$$(u - u') + (w - w') = 0 \quad \Rightarrow \quad u = u' \quad \text{and} \quad w = w'$$

because $u - u' \in \mathcal{M}$ and $w - w' \in \mathcal{M}^{\perp}$. Thus, the said orthogonal decomposition is unique. □

57.5. On the best approximation in a Banach space. A Hilbert space is a Banach space with respect to the natural norm. The optimization problem to find a best approximation in a linear manifold to a given vector can also be posed in a Banach space. However the existence and uniqueness of the solution is not so obvious. A natural question to ask: Under what conditions on the norm $\|\cdot\|$ can a Banach space be converted into a Hilbert space with the natural norm being $\|\cdot\|$? If an affirmative answer exists, then the projection theorem proved above will be applicable to such Banach spaces. It turns out that the parallelogram law used in the proof of the projection theorem is essential for this question.

THEOREM 57.4. (normed vs inner product space)
In order for a normed space X to be an inner product space, it is necessary and sufficient that the norm in X satisfies the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 , \quad \forall u, v \in X$$

A proof of this assertion is left to the reader as an exercise. Guidelines for the proof are given below.

It is also interesting to note that the space $\mathcal{L}_p(\Omega)$, (the space of all functions u for which $|u|^p$ is Lebesgue integrable Ω) is a Banach space for any $p \geq 1$, including the case $p = \infty$ for which the norm is the supremum norm $\|u\|_\infty$. However, only $\mathcal{L}_2(\Omega)$ can be converted to a Hilbert space. This assertion can be established by checking the parallelogram law. Furthermore, in the framework of quantum mechanics, elements of the Hilbert space $\mathcal{L}_2(\Omega)$ describe all states of any material object in the universe.

57.6. Exercises.

1. Let $\phi_k(x) = x^k$, $k = 0, 1, ...$, and $\phi_{-1}(x) = e^x$.
(i) Show that $B = \{\phi_k\}_{-1}^{\infty}$ is linearly independent in $\mathcal{L}_2(0, 1)$;
(ii) Show that $B = \{\phi_k\}_{-1}^{\infty}$ is complete in $\mathcal{L}_2(0, 1)$, that is, Span B is dense in $\mathcal{L}_2(0, 1)$;

(iii) Show that $B = \{\phi_k\}_{-1}^{\infty}$ is not a basis in $\mathcal{L}_2(0,1)$ by demonstrating that for $u(x) = e^x$ there is no unique decomposition

$$u(x) = \sum_{k=-1}^{\infty} \alpha_k \phi_k(x)$$

that is, the choice of the expansion coefficients α_k is not unique (whereas for a basis this choice must be unique by definition). Thus, *the mere completeness and linear independence of a set in a Hilbert space is not sufficient for the set to be a basis.*

2. Prove the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

where $\|\cdot\|$ is the natural norm in an inner product space.

3. Prove that in order for a normed real space X to be an inner product space, it is necessary and sufficient that the norm in X satisfies the parallelogram law

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2\,, \quad \forall u,v \in X$$

Consider the inner product defined by

$$\langle u,v\rangle = \frac{1}{4}\Big(\|u+v\|^2 - \|u-v\|^2\Big)$$

and prove that the inner product axioms

$$\begin{aligned}
&\langle u,u\rangle \geq 0 \quad \text{and} \quad u=0 \;\Leftrightarrow\; \langle u,u\rangle = 0\\
&\langle u,v\rangle = \langle v,u\rangle\\
&\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle\\
&\langle \alpha u,v\rangle = \alpha\langle u,v\rangle
\end{aligned}$$

follow from the norm axioms.
(i) To show that the third axiom holds, put

$$F(u,v,w) = 4\Big(\langle u+v,w\rangle - \langle u,w\rangle - \langle v,w\rangle\Big)$$

and show using the parallelogram law that

$$\begin{aligned}
F(u,v,w) = {}& -\|u+w-v\|^2 + \|u-w-v\|^2\\
&+\|u+w\|^2 - \|u-w\|^2 - \|v+w\|^2 + \|v-w\|^2
\end{aligned}$$

Then use the identity $F = \frac{1}{2}F + \frac{1}{2}F$ where F in the first term is taken from its definition, whereas F in the second term is taken from the

above representation, to show that $F = 0$ by means of the parallelogram law.
(ii) To prove the last axiom, put

$$f(\alpha) = \langle \alpha u, v\rangle - \alpha\langle u, v\rangle$$

show that $f \in C^0$ and $f(0) = f(-1) = 0$. If α is a positive integer n, use the identity

$$\langle nu, v\rangle = \langle u + u + \cdots + u, v\rangle$$

to show that $f(n) = 0$. Let α be a positive rational p/q. Use the relation (explain why it is true!)

$$\langle u, v\rangle = \frac{q}{q}\,\langle u, v\rangle = \frac{1}{q}\langle qu, v\rangle$$

for any positive integer q, to show that $f(p/q) = 0$. Use continuity to show that $f(\alpha) = 0$ for any $\alpha \geq 0$. Finally, show that $f(-\alpha) = 0$.

4. Show that the Banach space $\mathbb{R}_p^N$ of all finite real sequences $\{x_j\}_1^N$ cannot be turned into a Hilbert space in which the natural norm is

$$\|x\|_p = \left(\sum_{n=1}^{N} |x_j|^p\right)^{1/p}, \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max_j\{|x_j|\} = \lim_{p\to\infty} \|x\|_p$$

unless $p = 2$.

5. Show that the Banach space l_p of all real sequences $u = \{u_j\}_1^\infty$ cannot be turned into a Hilbert space in which the natural norm is

$$\|u\|_p = \left(\sum_{n=1}^{N} |u_j|^p\right)^{1/p} < \infty\,, \quad 1 \leq p < \infty$$

and

$$\|u\|_\infty = \sup_j\{|u_j|\} = \lim_{p\to\infty} \|u\|_p < \infty$$

unless $p = 2$.

6. Show that, unless $p = 2$, the Banach space $\mathcal{L}_p(\Omega)$ of all real valued functions $u: \ \Omega \to \mathbb{R}$ cannot be turned into a Hilbert space in which the natural norm is

$$\|u\|_p = \left(\int_\Omega |u(x)|^p d^N x \right)^{1/p} < \infty \,, \quad 1 \leq p < \infty$$

and

$$\|u\|_\infty = \sup_\Omega \{|u(x)|\} = \lim_{p\to\infty} \|u\|_p < \infty$$

7. Let $\mathcal{P}_3$ be the linear manifold of all polynomial of degree at most 3 in the Hilbert space $\mathcal{H} = \mathcal{L}_1(-1,1)$. Let $f(x) = e^x \in \mathcal{H}$.
(i) Find the best approximation $g(x)$ of $f(x)$ in $\mathcal{P}_3$ in the sense of the $\mathcal{L}_2$ norm:

$$\|f - g\|_2 = \inf_{h\in\mathcal{P}_3} \|f - h\|_2$$

(ii) Is this approximation better than the Taylor polynomial approximation about $x = 0$ of the same degree relative to the $\mathcal{L}_2$ norm?

58. Fourier series in a separable Hilbert space

58.1. Fourier series. Let X be an inner product space and $\{\varphi_k\}_1^\infty$ be a countable orthonormal set in X:

$$\langle \varphi_k, \varphi_n \rangle = \delta_{kn} \, .$$

For any $u \in X$, define a sequence of complex numbers

$$c_k = \langle f, \varphi_k \rangle$$

A formal series

$$u \sim \sum_{k=1}^{\infty} c_k \varphi_k$$

is called the Fourier series of u with respect to the set $\{\varphi_k\}_1^\infty$, and the numbers c_k are called the Fourier coefficients of u with respect to the set $\{\varphi_k\}_1^\infty$. The series is formal because nothing is said about its convergence. Yet, even if the series converges, does it converge to f?

If $\dim X = n < \infty$, then all these questions are easy to answer. In this case, any orthogonal set is finite and the series is just a finite sum and, hence, it exists for any u. If, in addition, the orthogonal set has exactly $n = \dim X$ elements, then the Fourier sum is nothing but a decomposition of u into a linear combination of elements from the orthogonal set.

If X is infinite dimensional, then answers to these questions are not so obvious. It turns out that they are still affirmative if X is a separable Hilbert space.

58.2. Bessel inequality.

THEOREM **58.1**. (Bessel inequality)
Let X be an inner product space and $\{\varphi_k\}_1^\infty$ be an orthonormal set in X. If $c_k = \langle u, \varphi_k \rangle$ are the Fourier coefficients of $f \in X$, then

$$\sum_{k=0}^{\infty} |c_k|^2 \le \|u\|^2$$

PROOF. By Theorem **57.2**, the partial sums of the Fourier series

$$v_n = \sum_{k=1}^{n} c_k \varphi_k \; \in \; \mathrm{Span}\,\{\varphi_1, \varphi_2, ..., \varphi_n\} = \mathcal{M}_n \, ,$$

provide the best approximation of u by vectors in $\mathcal{M}_n$ for every $n = 1, 2, ...$, and

$$\|u - v_n\|^2 = \|u\|^2 - \|v_n\|^2 \geq 0\,,$$
$$u - v_n = u - \sum_{k=1}^{n} \langle u, \varphi_k \rangle \varphi_k \ \in \ \mathcal{M}_n^{\perp}$$

The Bessel equality follows from the first of the above relations:

$$\|v_n\|^2 = \sum_{k=1}^{n} |c_k|^2 \leq \|f\|^2\,, \qquad n = 1, 2, ...$$

The sequence $\|v_n\|^2$ is monotonically increasing and bounded. Therefore it converges, which implies that

$$\sum_{k=1}^{\infty} |c_k|^2 \leq \|f\|^2\,.$$

□

It should be pointed out that the equality is not generally possible. For example, let $X = \mathcal{L}_2(-\pi, \pi)$ and

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}}\,, \quad \varphi_k(x) = \frac{1}{\sqrt{\pi}} \cos(kx)\,, \quad k = 1, 2, ...$$

The set $\{\varphi_k\}_0^\infty$ is orthonormal in $\mathcal{L}_2(-\pi, \pi)$:

$$\langle \varphi_k, \varphi_m \rangle = \int_{-\pi}^{\pi} \varphi_k(x) \overline{\varphi_m(x)} = \delta_{km}$$

If $u(-x) = -u(x)$ is an odd function in $\mathcal{L}_2(-1, 1)$, then its Fourier coefficients with respect to $\{\varphi_k\}_0^\infty$ vanish because φ_k are even:

$$c_k = \int_{-\pi}^{\pi} u(x)\varphi_k(x)dx = \int_{-\pi}^{\pi} u(x)\varphi_k(-x)dx$$
$$= \int_{-\pi}^{\pi} u(-y)\varphi_k(y)dy = -c_k \quad \Rightarrow \quad c_k = 0\,,$$

where $y = -x$. Clearly, the reason for this to happen is that $\{\varphi_k\}_0^\infty$ is not a complete set. Any linear combination of orthogonal functions $\sin(kx) \in \mathcal{L}_2(-1, 1)$, $k = 1, 2, ...$, is orthogonal to $\{\varphi_k\}_0^\infty$. This is indeed so in general.

THEOREM **58.2**. (Parseval-Steklov equality)
Let X be a separable inner product space and $B = \{\varphi_k\}_1^\infty$ be an orthonormal set in X, $\langle \varphi_k, \varphi_m \rangle = \delta_{km}$. Then B is complete if and only

if

$$\sum_{k=1}^{\infty} |\langle u, \varphi_k \rangle|^2 = \|u\|^2 \qquad \forall u \in X$$

PROOF. Suppose the Parseval-Steklov equality holds. Let us show that B is a basis. All partial sums of the Fourier series of u belong to the linear manifold Span B:

$$s_n = \sum_{k=1}^{n} c_k \varphi_k \in \operatorname{Span} B \,, \qquad c_k = \langle u, \varphi_k \rangle \,.$$

By Theorem **57.2**

$$\|u - s_n\|^2 = \|u\|^2 - \|s_n\|^2 = \|u\|^2 - \sum_{k=1}^{n} |c_k|^2$$

The right side of this equality converges to 0 as $n \to \infty$ because, by assumption, the Parseval-Steklov equality holds. Therefore for any $u \in X$ one can find an element v from Span B that is arbitrary close to u:

$$\forall \varepsilon > 0 \quad \forall u \in X \quad \exists v \in \operatorname{Span} B \;:\quad \|u - v\| < \varepsilon$$

which means that Span B is dense in X and, hence, B is complete in X.

Conversely, suppose that B is complete. Let us show that the Parseval-Steklov equality holds. If B is complete, then for any $u \in X$ one can find an element $v \in \operatorname{Span} B$ that is arbitrary close to u:

$$\forall \varepsilon > 0 \quad \forall u \in X \quad \exists w \in \operatorname{Span} B \;:\quad \|u - w\| < \varepsilon$$

Since $w \in \operatorname{Span} B$, it is a linear combination

$$w = \sum_{k=1}^{n} \alpha_k \varphi_k \in \operatorname{Span} \{\varphi_1, \varphi_2, \cdots, \varphi_n\} = B_n$$

By Theorem **57.2** there exists the best approximation $v \in B_n$ of u so that

$$\|u - v\| \le \|u - h\| \qquad \forall h \in B_n$$

and the equality is reached if and only if

$$h = v = \sum_{k=1}^{n} \langle u, \varphi_k \rangle \varphi_k$$

This implies that v is arbitrary close to u:

$$\|u - v\| \le \|u - w\| < \varepsilon$$

or, again by Theorem **57.2**,

$$\|u-v\|^2 = \|u\|^2 - \|v\|^2 = \|u\|^2 - \sum_{k=1}^{n} |\langle u, \varphi\rangle|^2 < \varepsilon$$

The sequence in the left side of the equality is positive and monotonically decreasing. Therefore if it can be made arbitrary small for some $n = N$, then it remains arbitrary small for all $n > N$. This means that the series $\sum_{k=1}^{n} |\langle u, \varphi\rangle|^2$ converges to $\|u\|^2$ and the Parseval-Steklov equality holds. □

58.3. Convergence of Fourier series. The following theorem due to Riesz and Fisher answers the question about convergence of Fourier series in a Hilbert space.

THEOREM **58.3**. (Riezs-Fisher theorem)
Suppose that

(1) $\mathcal{H}$ *is a Hilbert space*
(2) $B = \{\varphi_k\}_1^\infty$ *is an orthonormal set in* X
(3) $\{c_k\}_1^\infty \subset \mathbb{C} \;:\; \sum_{k=1}^{\infty} |c_k|^2 < \infty$

Then there exists a unique element $u \in X$ *such that*

$$c_k = \langle u, \varphi_k\rangle \quad \text{and} \quad \sum_{k=1}^{\infty} |c_k|^2 = \langle u, u\rangle = \|u\|^2$$

PROOF. Consider a sequence $\{u_n\}_1^\infty \subset \mathcal{H}$ where

$$u_n = \sum_{k=1}^{n} c_k \varphi_k$$

First, it is proved that this sequence is a Cauchy sequence and, hence, by completeness of a Hilbert space, it has a (unique) limit in it. One has

$$\begin{aligned} \|u_{n+m} - u_n\|^2 &= \|c_{n+1}\varphi_{n+1} + \cdots + c_{n+m}\varphi_{n+m}\|^2 \\ &= |c_{n+1}|^2 + |c_{n+2}|^2 + \cdots + |c_{n+m}|^2 \\ &= S_{n+m} - S_n \\ S_n &= \sum_{k=1}^{n} |c_k|^2 \end{aligned}$$

Since the series $\sum |c_k|^2 < \infty$ converges, the sequence of its partial $\{S_n\}$ sum also converges and, hence, is a Cauchy sequence, and so is the sequence $\{u_n\}$. Thus, there exists a unique element $u \in \mathcal{H}$ such that

$$\lim_{n\to 0} \|u - u_n\| = 0$$

Next, one has

$$\langle u, \varphi_k \rangle = \langle u_n, \varphi_k \rangle - \langle u - u_n, \varphi_k \rangle = c_k - \langle u - u_n, \varphi_k \rangle$$

It follows from this relation that

$$|\langle u, \varphi_k \rangle - c_k| = |\langle u - u_n, \varphi_k \rangle| \le \|u - u_n\|$$

by the Cauchy-Bunyakowski inequality. The right side of the above inequality can be made smaller than any preassigned positive number for any $k = 1, 2, ...$ by taking n large enough. This means that

$$c_k = \langle u, \varphi_k \rangle \,.$$

Furthermore by Theorem **57.2**

$$\|u - u_n\|^2 = \|u\|^2 - \|u\|_n^2 = \|u\|^2 - S_n$$

for all n. Taking the limit $n \to \infty$, it is concluded that

$$\|u\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \,.$$

□

It should be noted that the separability of a Hilbert is not required in the Riezs-Fisher theorem. The orthogonal set $\{\varphi_k\}$ is countable, but may not be a basis in $\mathcal{H}$. The Riesz-Fisher theorem allows us to answer the question about convergence of the Fourier series in a separable Hilbert space.

COROLLARY **58.1**. *Let $\{\varphi_k\}_1^\infty$ be an orthonormal set in a Hilbert space $\mathcal{H}$ for which the Parceval-Steklov equality holds. Then $\{\varphi_k\}_1^\infty$ is an orthonormal basis and for any $u \in \mathcal{H}$ the Fourier series*

$$\sum_{k=1}^{\infty} \langle u, \varphi_k \rangle \, \varphi_k = u$$

converges to u in $\mathcal{H}$

Indeed, by the Parseval-Steklov equality $\sum_k |c_k|^2 = \|u\|^2$ for any $u \in \mathcal{H}$, where $c_k = \langle u, \varphi_k \rangle$. By the Riesz-Fisher theorem, there exists

$v \in \mathcal{H}$ such that

$$v = \sum_{k=1}^{\infty} c_k \varphi_k \quad \text{and} \quad \langle v, \varphi_k \rangle = c_k = \langle u, \varphi_k \rangle$$

$$\Rightarrow \quad \|v\|^2 = \|u\|^2 \quad \text{and} \quad \langle u, v \rangle = \sum_{k=1}^{\infty} |c_k|^2 = \|u\|^2$$

by continuity of the inner product, $\langle u, s_n \rangle \to \langle u, s \rangle$ if $s_n \to s$ in $\mathcal{H}$ as $n \to \infty$. Therefore

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 + \|v\|^2 - \langle u, v \rangle - \langle v, u \rangle = 0$$

and, hence, $v = u$. If $u = \sum_k \alpha_k \varphi_k$ for some choice of complex α_k, then

$$0 = \|u - u\|^2 = \sum_{k=1}^{\infty} |\langle u, \varphi_k \rangle - \alpha_k|^2 \quad \Rightarrow \quad \alpha_k = \langle u, \varphi_k \rangle$$

Thus, the expansion coefficients are unique and $\{\varphi_k\}_1^\infty$ is an orthonormal basis.

58.4. Examples of bases in $\mathcal{L}_2$.

Legendre polynomials in $\mathcal{L}_2(-1,1)$. By applying the Gram-Schmidt process to the set of monomials $S = \{x^n\}_0^\infty$ in $\mathcal{L}_2(-1,1)$, the orthogonal set of Legendre polynomials $B = \{P_n(x)\}_0^\infty$ is obtained:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!}\Big(\frac{d}{dx}\Big)^n (x^2-1)^n \\ &= \frac{1}{2^n} \sum_{k=0}^{n} \frac{(n!)^2}{(k!(n-k)!)^2} (x-1)^{n-k}(x+1)^k \\ \langle P_n, P_m \rangle &= \int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2\delta_{nm}}{2n+1} \end{aligned}$$

The completeness follows from the Weierstrass theorem that asserts that for any continuous function $v(x)$ in a bounded interval and any $\varepsilon > 0$, one can find a polynomial $p(x)$ such that

$$\|v - p\|_\infty < \varepsilon$$

Fix $\varepsilon > 0$. Since the space of continuous functions is dense in $\mathcal{L}_2(-1,1)$, for any $u \in \mathcal{L}_2(-1,1)$, one can find a continuous function $v(x)$ such that

$$\|u - v\|_2 < \varepsilon$$

Having found v, one can find a polynomial p by the Weierstrass theorem. It also follows that

$$\|v-p\|_2^2 = \int_{-1}^{1} |v(x)-p(x)|^2\,dx \le \|v-p\|_\infty^2 \int_{-1}^{1} < 2\varepsilon^2$$

Therefore

$$\|u-p\|_2 \le \|u-v\|_2 + \|v-p\|_2 < \varepsilon(1+\sqrt{2})$$

This shows that the span of the set of Legendre polynomials is complete in $\mathcal{L}_2(-1,1)$ and, hence, is an orthogonal basis. So, in this case, the Parseval-Steklov equality follows from the Weierstrass theorem.

Trigonometric basis in $\mathcal{L}_2(a,b)$. Consider the set

$$\phi_0^c(x) = 1\,, \quad \phi_n^c(x) = \cos\left(\frac{2\pi n x}{b-a}\right), \quad \phi_n^s(x) = \sin\left(\frac{2\pi n x}{b-a}\right), \quad n = 1, 2, ...$$

It is an orthogonal set in $\mathcal{L}_2(a,b)$. Its completeness follows from the Weierstrass theorem for trigonometric polynomials, which asserts that for any $\varepsilon > 0$ and any continuous function $v(x)$ such that $v(a) = v(b)$, there exists a trigonometric polynomial

$$p_n(x) = \sum_{k=0}^{n} \alpha_k \phi_k^c(x) + \sum_{k=1}^{n} \beta_n \phi_k^s(x)$$

such that

$$\|v-p\|_\infty < \varepsilon$$

Fix $\varepsilon > 0$ and $u \in \mathcal{L}_2(a,b)$. The space of continuous functions is dense in $\mathcal{L}_2(a,b)$ and, hence, one can find a continuous function $w(x)$ such that

$$\|u-w\|_2 < \varepsilon$$

but in general $w(a) \ne w(b)$. Having found w, one can find a continuous function $v(x)$ that differs from $w(x)$ only in the interval $[b-\delta, b]$ and $v(a) = v(b)$. For example, this can be achieved by defining $v(x)$ in this interval as a linear function whose graph connects the points $(b-\delta, w(b-\delta))$ and $(b, w(a))$. It is easy to show that the parameter δ can be made small enough so that

$$\|w-v\|_2 < \varepsilon$$

Having found such a v, a trigonometric polynomial p is found by the Weierstrass theorem so that

$$\|v-p\|_2^2 = \int_a^b |v(x)-p(x)|^2 dx \le \|v-p\|_\infty^2 \int_a^b dx < (b-a)\varepsilon^2$$

The completeness of the span of the trigonometric set follows:

$$\|u-p\|_2 \le \|u-w\|_2 + \|w-v\|_2 + \|v-p\|_2 < \varepsilon(2-\sqrt{b-a})$$

In this case, the Parseval-Steklov equality follows from the Weierstrass theorem and that the convergence in the supremum norm implies the convergence in the $\mathcal{L}_2$ norm for any *bounded region*. This argument can no longer be used for orthogonal sets in unbounded intervals.

As a consequence, the set

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{inx}\,, \quad n = 0, \pm 1, \pm 2, ...$$

is an orthonormal basis in $\mathcal{L}_2(-\pi,\pi)$, known as the trigonometric Fourier basis.

Hermite polynomials in $\mathcal{L}_2(\mathbb{R})$. Consider the set of weighted monomials:

$$S = \left\{x^n e^{-x^2/2}\right\}_0^\infty \subset \mathcal{L}_2(\mathbb{R})$$

Its linear independence follows from the linear independence of monomials. The Gram-Schmidt process for this set leads to an orthogonal set

$$B = \left\{H_n(x) e^{-x^2/2}\right\}_0^\infty \subset \mathcal{L}_2(\mathbb{R})$$

where H_n are known as Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\, e^{-x^2} = \Big(2x - \frac{d}{dx}\Big)^n 1\,, \quad n = 0, 1, ...$$

Note that Gram-Schmidt process makes orthogonal functions by taking linear combinations of functions from S and any first n functions in B are linear combinations of the n first functions in S and vice versa. Therefore the weight $e^{-x^2/2}$ does not change and H_n must be polynomials. The set B is an orthogonal basis in $\mathcal{L}_2(\mathbb{R})$. A proof of this assertion is outlined in Exercise 3 below and technical details are left to the reader.

58.5. Smoothness of basis functions in $\mathcal{L}_2(\Omega)$. The the Parseval-Steklov equality provides the necessary and sufficient condition for an orthogonal set in an inner product space to be a basis. This criterion holds even if the inner product space is not complete. In particular, if X is a dense linear manifold in a Hilbert space $\mathcal{H}$, $X \subset \overline{X} = \mathcal{H}$, and $B = \{\varphi\}_1^\infty$ is an orthonormal basis in X, then $\{\varphi\}_1^\infty$ is an orthonormal basis in $\mathcal{H}$. Indeed, if $u \in \mathcal{H}$, $v \in X$, and $w \in \operatorname{Span} B$, then choosing

v arbitrary close to u (X is dense in $\mathcal{H}$) and w arbitrary close to v (Span B is dense in X), it is concluded from the triangle inequality

$$\|u - w\| \leq \|u - v\| + \|v - w\|$$

that Span B is dense in $\mathcal{H}$. Therefore the Parseval-Steklov equality holds for any element $u \in \mathcal{H}$, hence, B is a basis in $\mathcal{H}$ by Corollary **58.3**.

For example, the space of test functions $\mathcal{D}(\Omega)$, $\Omega \subseteq \mathbb{R}^N$, is dense in $\mathcal{L}_2(\Omega)$ (see Section **56.5**) and for any bounded Ω the following inclusion holds

$$\mathcal{D}(\Omega) \subset C^\infty(\bar{\Omega}) \subset C^{p+1}(\bar{\Omega}) \subset C^p(\bar{\Omega}) \subset \mathcal{L}_2(\Omega)$$

for any $p = 1, 2, ...$ This means that a basis in $\mathcal{L}_2(\Omega)$ can be found in any of the above spaces of smooth functions. Recall, for example, the Legendre polynomials. They are from the class $C^\infty([-1, 1])$ and form a basis in $\mathcal{L}_2(-1, 1)$.

This observation is essential in applications of the Fourier series in the theory of partial differential equations and quantum theory because bases in the Hilbert space $\mathcal{L}_2(\Omega)$ are formed by eigenfunctions of some differential operators, and the domain of such operators lie in spaces of smooth functions.

For example, consider the eigenvalue problem for the second-derivative operator

$$-u''(x) = \lambda u(x)\,, \quad x \in (-\pi, \pi)\,, \quad u \in C^2([-\pi, \pi])\,,$$
$$u(-\pi) = u(\pi)$$

where $\lambda \in \mathbb{C}$. It is not difficult to obtain a general solution for any complex λ and show that the boundary condition is satisfied only by solutions with real non-negative λ, namely, $\lambda = \lambda_n = n^2$, $n = 0, 1, 2, ...$ For $n = 0$ there is only one linearly independent solution $u(x) = u_0(x) = 1$ and for every $n > 0$, there are two linearly independent solutions $u(x) = u_n^\pm(x) = e^{\pm inx}$. By the Weierstrass theorem about trigonometric Fourier series, the C^∞ functions

$$\{1,\ e^{\pm ix},\ e^{\pm 2ix},\ e^{\pm 3ix}, \cdots\}$$

form an orthogonal basis in $C_2^0([-\pi, \pi])$. The Fourier series converges uniformly to any continuous function that has the same values at the end points of the interval, $u(-\pi) = u(\pi)$. Since $C_2^0([-\pi, \pi])$ is dense in $\mathcal{L}_2(-\pi, \pi)$, the above functions also form a basis in $\mathcal{L}_2(-\pi, \pi)$. However, the uniform convergence is lost for the trigonometric Fourier series of functions from $\mathcal{L}_2(-\pi, \pi)$, but the Fourier series still converges in the $\mathcal{L}_2$ norm.

Similarly, the space of test functions for temperate distributions $\mathcal{S} \subset \mathcal{L}_2(\mathbb{R}^N) = \mathcal{L}_2$ is dense in the space of square integrable functions and

$$\mathcal{S} \subset C^p \cap \mathcal{L}_2 \subset \mathcal{L}_2$$

Therefore bases in $\mathcal{L}_2$ can also be sought in spaces of square integrable smooth functions. An example of an orthogonal basis in $\mathcal{L}_2(\mathbb{R})$ that is formed by functions from $\mathcal{S}(\mathbb{R})$ will be given in Exercises.

One should not get an impression that all bases in $\mathcal{L}_2$ are formed by smooth functions. In fact, it is not difficult to construct an orthogonal basis that contains only piecewise constant functions. They are known as *Haar wavelets.*

58.6. Convergence in the mean. The Fourier series in a separable Hilbert space $\mathcal{L}_2(\Omega)$, $\Omega \subseteq \mathbb{R}^N$ does not generally converge pointwise because the convergence in the $\mathcal{L}_2$ norm means that the sum of the Fourier series coincides with the function *almost everywhere*

$$u(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x) \ a.e.\,, \quad c_k = \langle u, \varphi \rangle = \int_{\Omega} u(x)\overline{\varphi_k(x)}\, d^N x$$

that is, they may differ on a set of measure zero (this set may also depend on the choice of a basis).

Definition **58.1**. *The convergence of Fourier series in the $\mathcal{L}_2(\Omega)$ norm is called the* convergence in the mean.

For example, $u(x)$ be a continuous, $2\pi-$periodic function. Then its trigonometric Fourier series converges to u uniformly on any interval (Weierstrass theorem). Now suppose that that u is not continuous at x_0 (it is still periodic) where it has a jump discontinuity. Evidently, $u \in \mathcal{L}_2(-\pi, \pi)$ So its trigonometric Fourier series converges to u almost everywhere. In fact, it converges to $u(x)$ everywhere except possibly at $x = x_0$ at which the Fourier series converges to the midpoint of the jump discontinuity:

$$\lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx} = \frac{1}{2} \lim_{x\to x_0^+} u(x) + \frac{1}{2} \lim_{x\to x_0^-} u(x)$$

where c_k are the trigonometric Fourier coefficients of u. They do not depend on the value of $u(x_0)$. The above relation holds for any choice of $u(x_0)$. All such functions represent the same element of $u \in \mathcal{L}_2(-\pi, \pi)$ as they differ on a set of measure zero.

58.7. Resolution of unity. Let $\{\varphi_k\}_0^\infty$ be an orthonormal basis in $\mathcal{L}_2(\mathbb{R}^N)$. Then for every $y \in \mathbb{R}^N$, the sequence

$$\delta_n(x;y) = \sum_{k=1}^{n} \varphi_k(y)\overline{\varphi_k(x)}$$

converges in the distributional sense. Let us find its limit. Since φ_k is square integrable for any k, δ_n is a temperate distribution:

$$\Big(\delta_n(x;y), \phi(x)\Big) = \sum_{k=1}^{n} \varphi_k(y) \int \phi(x)\overline{\varphi_k(x)}d^N x$$

for any $\phi \in \mathcal{S}$. By the completeness of the space of temperate distribution $\mathcal{S}'$, the limit of δ_n exists for any y and defines a temperate distribution $\delta(x;y)$ that acts on any $\phi \in \mathcal{S}$ by the rule

$$\begin{aligned}\Big(\delta(x;y), \phi(x)\Big) &= \lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k(y) \int \phi(x)\overline{\varphi_k(x)}d^N x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k(y)\langle \phi, \varphi_k\rangle = \phi(y)\end{aligned}$$

This proves that

$$\delta(x;y) = \sum_{k=1}^{\infty} \varphi_k(y)\overline{\varphi_k(x)} = \delta(x-y) \quad \text{in} \quad \mathcal{S}'$$

The latter relation is called a resolution of unity in $\mathcal{L}_2$. In particular, using the trigonometric Fourier basis in $\mathcal{L}_2(-\pi,\pi)$

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(x-y)} = \delta(x-y)\,, \qquad x, y \in (-\pi,\pi)\,.$$

This is the Poisson summation formula *restricted to the interval* $(-\pi,\pi)$ in the distributional sense.

58.8. Exercises.

1. Find the Fourier series of $u(x) = e^x$ with respect to the basis of Legendre polynomials in $\mathcal{L}_2(-1,1)$:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n}\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(x-1)^{n-k}(x+1)^k$$

$$\int_{-1}^{1} P_n(x)P_m(x) = \frac{2}{2n+1}$$

Compare four first terms of the Fourier series with the four first terms of the power series representation of e^x.

2. An orthogonal basis in $\mathcal{L}_2(\mathbb{R})$.
The Hermite polynomials are defined by the relation

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\, e^{-x^2} = \Big(2x - \frac{d}{dx}\Big)^n 1\,, \quad n = 0, 1, ...$$

(i) Show that the Hermite polynomials satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

(ii) and the identity

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x)$$

Hint: One can use, for example, mathematical induction or the definition of the Hermite polynomials

Put

$$\varphi_n(x) = H_n(x)e^{-x^2/2}$$

(iii) Show that $\varphi_n \in \mathcal{S}(\mathbb{R})$ (they are test functions for temperate distributions) and
(iv) they are eigenfunctions of a differential operator:

$$L\varphi_n(x) = -\varphi_n''(x) + x^2\varphi_n(x) = \lambda_n\varphi_n(x)\,, \quad \lambda_n = 2n+1\,, \quad x \in \mathbb{R}$$

(v) For $n \neq m$, use integration by part to prove

$$\lambda_n\langle\varphi_n, \varphi_m\rangle = \langle L\varphi_n, \varphi_m\rangle = \langle\varphi_n, L\varphi_m\rangle = \lambda_m\langle\varphi_n, \varphi_m\rangle\,,$$

where

$$\langle u, v\rangle = \int_{-\infty}^{\infty} u(x)\overline{v(x)}\, dx$$

is the inner product in $\mathcal{L}_2(\mathbb{R})$, and
(vi) conclude that, if $n \neq m$, then

$$\langle\varphi_n, \varphi_m\rangle = \int_{-\infty}^{\infty} \varphi_n(x)\varphi_m(x)\, dx = \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\, dx = 0\,,$$

(vii) Use, e.g., mathematical induction and the recurrence relation for the Hermite polynomials to show that

$$\|\varphi_n\|^2 = \int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}\, dx = 2^n n!\sqrt{\pi}$$

Thus, $B = \{\varphi_n\}_0^\infty$ is an orthogonal set in $\mathcal{L}_2(\mathbb{R})$.

Prove that B is an orthogonal basis in $\mathcal{L}_2(\mathbb{R})$.
(viii) Show, first, for any $v \in \mathcal{L}_2(\mathbb{R})$, the Fourier series

$$\sum_{n=0}^{\infty} \frac{\langle v, \varphi_n \rangle}{\|\varphi_n\|^2}\,\varphi_n = u \in \mathcal{L}_2(\mathbb{R})$$

converges in the mean and

$$\|u\|^2 \leq \|v\|^2$$

If the equality holds, then B is a basis. Suppose that $\|u\|^2 < \|v\|^2$.
(ix) Show that this implies that $\mathcal{M}^\perp \neq \{0\}$ where $\mathcal{M} = \operatorname{Span} B$. Let $v \neq 0$ and $v \in \mathcal{M}^\perp$.
(x) Prove that

$$F(z) = \int_{-\infty}^{\infty} v(x) e^{zx - x^2/2}\, dx = \sum_{n=0}^{\infty} \frac{z^n}{n!} \langle v, x^n e^{-x^2/2} \rangle = 0$$

for any $z \in \mathbb{C}$. Note that this requires changing the order of integration and summation which must be justified. Set $z = ik$, $k \in \mathbb{R}$, and conclude that

$$F(ik) = \mathcal{F}\Big[v(x)e^{-x^2/2}\Big](k) = 0$$

where $\mathcal{F}$ stands for the Fourier transform.
(xi) Show that $v(x)e^{-x^2/2}$ is a regular temperate distribution. Use the inverse Fourier transform to show that

$$v(x)e^{-x^2/2} = 0$$

in the distributional sense, and conclude that $v = 0$ using the Du Bois-Reymond lemma. A contradiction. Hence, B is an orthogonal basis in $\mathcal{L}_2(\mathbb{R})$. In quantum mechanics, this basis is associated with stationary states of a harmonic oscillator of unit mass and unit frequency.

3. A generalization of the above construction is as follows. Let $A = \{x^n w(x)\}_0^\infty$ where $w(x) \in C^0$, $w(x) > 0$, and is such that $x^n w(x) \in \mathcal{L}_2(\Omega)$ where Ω can be any interval in $\mathbb{R}$, either bounded or not.
(i) Prove that A is linearly independent.
(ii) Prove that $\operatorname{Span} A$ is dense in $C_2^0(\bar{\Omega})$ and, hence, in $\mathcal{L}_2(\Omega)$.
(iii) Show that for any such $w(x)$, one can find a complete orthonormal set in $\mathcal{L}_2(\Omega)$ of the form

$$\varphi_n(x) = p_n(x)w(x)\,, \qquad \int_\Omega p_n(x)p_m(x)|w(x)|^2 dx = \delta_{nm}$$

where $p_n(x)$ is a polynomial of degree n. The above orthogonality relation is called an orthogonality with weight $|w(x)|^2 \geq 0$. For the Hermite polynomials the weight is $|w(x)|^2 = e^{-x^2}$.

4. Consider the space of square integrable functions with a weight $\rho(x) \geq 0$ and $\rho \in C^0$

$$u \in \mathcal{L}_2(\Omega; \rho) \quad \text{if} \quad \int_\Omega |u(x)|^2 \rho(x)\, dx < \infty$$

Put

$$\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}\, \rho(x)\, d^N x$$

Show that $\mathcal{L}_2(\Omega; \rho)$ is a separable Hilbert space. *Hint*: Use the results of Section **56.4**

5. Isomorphism of separable Hilbert spaces

DEFINITION **58.2**. *Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^*$ are* isomorphic*, which is denoted as $\mathcal{H} \sim \mathcal{H}^*$, if there exists a one-to-one mapping*

$$f: \quad \mathcal{H} \to \mathcal{H}^*$$

such that for any $u, v \in \mathcal{H}$ and any $\alpha \in \mathbb{C}$ the following properties hold

$$\begin{aligned}
&f(u + v) = f(u) + f(v) = u^* + v^* \\
&f(\alpha u) = \alpha f(u) = \alpha u^* \\
&\langle u, v\rangle_{\mathcal{H}} = \langle f(u), f(v)\rangle_{\mathcal{H}^*} = \langle u^*, v^*\rangle_{\mathcal{H}^*}
\end{aligned}$$

In other words, the isomorphism of two Hilbert spaces is a one-to-one correspondence between their elements that preserves linear transformations in them and the inner product.

THEOREM **58.4**. *Any two separable Hilbert spaces are isomorphic*

Prove the theorem by following the indicated steps:
(i) l_2 is a separable Hilbert space;
(ii) The Riesz-Fisher theorem implies that any separable Hilbert space is isomorphic to l_2;
(iii) Any two separable Hilbert spaces are isomorphic

This theorem shows that there is, in fact, only one separable Hilbert space as all separable Hilbert spaces are isomorphic to it. In this sense, $\mathcal{L}_2(\Omega)$ is unique. For some (unknown) reasons, the nature is such that all material objects in the universe can be viewed as elements of this space! This assertion is based on quantum theory and so far no evidence against it has been found.

Furthermore, since every separable separable Hilbert space has a countable orthonormal basis B, the distance between any two elements

u and v is given by

$$\|u - v\|^2 = \sum_{k=1}^{\infty} |\alpha_k - \beta_k|^2$$

where α_k and β_k are the Fourier coefficients of u and v, respectively, in the basis B. The above Parseval-Steklov equality looks like an infinite dimensional analog of the Pythagorean theorem. For this reason all separable Hilbert spaces are often denoted as $\mathbb{R}^\infty$ (real space) or $\mathbb{C}^\infty$ (complex space), an infinite dimensional Euclidean space.

CHAPTER 8

Operators in Hilbert spaces

53. Definition of an operator in Banach and Hilbert spaces

DEFINITION **53.1**. (Operators)
Let $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two Banach spaces with norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$*, respectively. A map*

$$A\ :\quad D_A \subseteq \mathcal{B}_1 \ \to\ \mathcal{B}_2$$

is called an operator*, the set* D_A *is called the* domain *of* A*, the set* $A(D_A) = R_A \subseteq \mathcal{B}_2$ *is called the* range *of* A*, and the set*

$$\mathcal{N}_A = \left\{\ u \in D_A \ \middle|\ Au = 0\ \right\}$$

is called the null space *of* A*.*

Example 1. Let $\mathcal{B}_1 = \mathbb{C}^N$ and $\mathcal{B}_2 = \mathbb{C}^M$. Let A be an $N \times M$ matrix with complex elements. Then A defines a map $A\ :\ \mathbb{C}^N \to \mathbb{C}^M$ by $Ax = y$, where $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$ (here Ax is interpreted as a matrix product). The null space of the operator A is a collection of all vectors annihilated by the matrix A.

Example 2. Let $K(x, y, z)$ be a continuous functions on $\Omega \times \Omega$ for every $z \in \mathbb{R}$, where Ω is a closed bounded subset in $\mathbb{R}^N$ with a piecewise smooth boundary, and, in addition, be Lipschitz continuous with respect to z

$$|K(x, y, z_1) - K(x, y, z_2)| \le m_0|z_1 - z_2|\,, \quad (x, y) \in \Omega \times \Omega$$

and $K(x, y, 0) = 0$. For any continuous function u on Ω, put

$$Au(x) = \int_\Omega K(x, y, u(y))\, d^N y$$

Then A is an operator with $D_A = C^0(\Omega) = \mathcal{B}_1$ (the norm here is the supremum norm, $\|\cdot\|_\infty$) and its range also lies in the space $C^0(\Omega) = \mathcal{B}_2$. The continuity of $Au(x)$ follows from the Lebesgue dominated convergence theorem and boundedness of Ω.

Example 3. Let L be a differential operator of order p:

$$L = \sum_{q=0}^{p} a_q(x) D^q$$

Then, under suitable restrictions on the coefficients $a_q(x)$, it can be viewed as an operator in the Hilbert space $\mathcal{L}_2$

$$L: \; C^p(\Omega) \cap \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$$

where $\Omega \subseteq \mathbb{R}^N$. In this case, $D_L = C^p(\Omega) \cap \mathcal{L}_2(\Omega)$ and the range is a subset in the space of square integrable functions. Note that if Ω is not closed or not bounded or both, then not every function from $C^p(\Omega)$ is square integrable. The null space is the set of all solutions to the homogeneous partial differential equation $Lu = 0$.

Example 4. A linear functional l on a Hilbert space is an operator $A: \; \mathcal{H} \to \mathbb{C}$ so that $Au = l(u)$. In this case, $\mathcal{B}_1 = \mathcal{H} = D_A$ and $\|\cdot\|_1$ is the natural norm on $\mathcal{H}$. The space $\mathcal{B}_2 = \mathbb{C}$ and $\|\cdot\|_2$ is the distance between two complex numbers.

DEFINITION **53.2.** (Linear operator)
An operator A is linear *if the image of a linear combination of elements in its domain is the same linear combination of images in its range*

$$A(\alpha u + \beta v) = \alpha A u + \beta A v \,, \quad \forall u, v \in D_A \quad \forall \alpha, \beta \in \mathbb{C}$$

Note that the domain and range of a linear operator must be linear manifolds. In particular, $A\,0 = 0$ (where 0 in the left side is the zero element in D_A, while 0 in the right side is the zero element in R_A). Operators in Examples 1, 3, and 4 are linear. The integral operator in Example 2 is not linear.

53.1. Bounded operators.

DEFINITION **53.3.** (Bounded operator)
An operator A is called bounded *if there exists a positive constant C such that*

$$\|Au\|_2 \le C \|u\|_1 \,, \quad \forall u \in D_A$$

and C is independent of $u \in D_A \subset \mathcal{B}_1$.

The integral operator in Example 2 is bounded. Indeed, for any $x \in \Omega$

$$|Au(x)| \le \int_\Omega |K(x, y, u(y))| \, d^N y \le m_0 \int_\Omega |u(y)| \, d^N y$$
$$\le m_0 \|u\|_\infty \int_\Omega d^N y = m_0 \mu_L(\Omega) \|u\|_\infty$$

where $\mu_L(\Omega)$ is the Lebesgue measure (volume) of Ω. Since Ω is bounded and has piecewise smooth boundaries, its volume is finite. The inequality holds for all $x \in \Omega$, by taking the supremum in the left side (which exists because $Au \in C^0(\Omega)$ and Ω is closed and bounded), one infers that

$$\|Au\|_\infty \leq C\|u\|_\infty \,, \quad C = m_0 \mu_L(\Omega) \,.$$

The norm of a bounded operator.

DEFINITION **53.4**. *Let $A: \ D_A \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded operator. The least number C for which*

$$\|Au\|_2 \leq C\|u\|_1$$

is called the norm *of A and denoted $\|A\|$ so that*

$$\|A\| = \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_1}$$

If,in addition, A is linear, then

$$\|A\| = \sup_{\|u\|_1 = 1} \|Au\|_2$$

By definition of the norm of the operator

$$\|Au\|_2 \leq \|A\|\|u\|_1 \,.$$

In Example 4, the operator A is bounded if, in addition, the linear functional l is continuous. In this case, by the Riesz representation theorem and the Cauchy-Bunyakowski inequality

$$Au = \langle u, v\rangle \quad \Rightarrow \quad |Au| \leq \|u\|\|v\| \quad \Rightarrow \quad \|A\| = \sup_{\|u\|=1} |\langle u, v\rangle| = \|v\|$$

because the Cauchy-Bunyakowski inequality becomes equality if and inly if u and v are proportional. The supremum is attained on $u = \frac{1}{\|v\|}\, v \in D_A$.

Remark. It is important to note that the supremum cannot always be attained on the domain of the operator because the domain of a bounded operator may not be closed, $D_A \subset \overline{D}_A \subseteq \mathcal{B}_1$. A situation is similar to the supremum of a bounded and continuous functions on a set that is not closed. If $\|A\| < \infty$, then put $C_n = \|A\|(1 + \frac{1}{n})$. Then for each $n = 1, 2, ...$, there is an element $u_n \in D_A$ such that

$$\|Au_n\|_2 \leq C_n\|u_n\|_1 \,, \quad n = 1, 2, ...$$

because $\|A\| = \inf\{C_n\}$. Even if $\{u_n\}$ is a Cauchy sequence and, hence, has a limit $u_0 \in \mathcal{B}_1$ in the Banach space, the limit point may not belong to the domain, $u_0 \notin D_A$ if D_A is not closed.

Furthermore, even if the domain of a bounded operator is closed, there cannot always exist $u_0 \in D_A$ such that $\|A\| = \|Au_0\|/\|u_0\|$. To understand this, note that the supremum is taken over all *non − zero* elements in D_A. When the zero element is removed from a closed set D_A, the resulting set is no longer closed and $u = 0$ is its limit point that is not in the set! A bounded operator maps every null sequence in the domain, $u_n \to 0$ as $n \to \infty$, to a null sequence in the range, $Au_n \to 0$. The limit of a positive sequence $\|Au_n\|_2/\|u\|_1$ becomes an indeterminate form $\frac{0}{0}$ and, if it exists, it can be any number between 0 and $\|A\|$. In particular, if D_A is closed and there is no $u_0 \in D_A$ for which $\|A\| = \|Au_0\|/\|u_0\|$, the limit is equal to $\|A\|$. In other words, the supremum in the definition of $\|A\|$ can occur for elements arbitrary close to the zero element.

These observations about the norm of a bounded operators can be summarized as follows.

Properties of bounded operators.

PROPOSITION **53.1**. *Suppose that A is a bounded operator. Then*

$$\textit{either} \quad \exists u_0 \in D_A \ : \quad \|A\| = \frac{\|Au_0\|_2}{\|u_0\|_1}$$
$$\textit{or} \quad \exists \{u_n\}_1^\infty \in D_A \ : \quad \lim_{n\to\infty} \frac{\|Au_n\|_2}{\|u_n\|_1} = \|A\|$$

PROPOSITION **53.2**. *If an operator A is not bounded, then there exists a sequence $\{u_n\}_1^\infty$ in its domain D_A such that*

$$\lim_{n\to\infty} \frac{\|Au_n\|_2}{\|u_n\|_1} = \infty \,, \quad u_n \neq 0 \,.$$

53.2. Continuous operators.

DEFINITION **53.5**. (Continuous operator)
An operator is said to be continuous *at $u \in D_A$ if for any sequence $\{u_n\}$ that converges to u in the domain D_A, its image $\{Au_n\}$ converges to Au in the range R_A:*

$$\forall \{u_n\} \subset D_A \ : \quad \lim_{n\to\infty} \|u - u_n\|_1 = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|Au - Au_n\|_2 = 0$$

An operator is said be continuous on a set if it is continuous at every element of the set.

For example, the integral operator in Example 2 is continuous because

$$\|Au - Au_n\|_\infty \le m_0 \mu_L(\Omega) \|u - u_n\|_\infty$$

and, hence, a uniform convergence of u_n to u implies the uniform convergence of Au_n to Au.

It turns out that the properties of linear continuous operators are similar to the properties of linear continuous functionals.

Properties of continuous linear operators.

PROPOSITION **53.3**. *If A is a linear operator that is continuous at $u = 0$, then A is continuous on its domain D_A.*

PROPOSITION **53.4**. *Suppose that A is a linear operator. Then A is continuous if and only if it is bounded:*

$$A \text{ is continuous} \quad \Leftrightarrow \quad A \text{ is bounded}$$

PROOF. If A is bounded, then for any sequence $\{u_n\} \subset D_A$ that converges to $u \in D_A$,

$$\|Au - Au_n\|_2 \le \|A\| \|u - u_n\|_1 \ \to \ 0$$

as $n \to \infty$. Therefore $\{Au_n\} \in R_A$ converges to $Au \in R_A$.

Conversely, let A be continuous. This means that for any sequence $\{u_n\}$ that converges to 0 in D_A, the sequence $\{Au_n\}$ converges to 0 in the range R_A. Suppose that A with the stated property is not bounded. Then for any positive integer n, one can find $v_n \in D_A$ such that

$$\|Av_n\|_2 \ge n \|v_n\|_1$$

or by linearity of A

$$n \le \|Aw_n\|_2 \,, \quad w_n = \frac{1}{\|v_n\|} v_n \,, \quad \|w_n\| = 1$$

The sequence $u_n = \frac{1}{n} w_n$ converges to $u = 0$ because

$$\|u_n\|_1 = \frac{\|w_n\|_1}{n} = \frac{1}{n} \ \to \ 0 \quad \text{as } n \to \infty$$

But

$$\|Au_n\|_2 = \frac{\|Aw_n\|_2}{n} \ge 1$$

and, hence, the sequence $\{Au_n\}$ cannot converge to 0, which contradicts the hypothesis that A is continuous. Thus, A must be bounded. $\square$

53.3. Hilbert-Schmidt operators. Let

$$K: \ \Omega \times \Omega \ \to \ \mathbb{C}, \quad \Omega \subseteq \mathbb{R}^N$$

be a square integrable function of two variables:

$$M = \int_\Omega \int_\Omega |K(x,y)|^2 \, d^N y \, d^N x \ < \ \infty$$

PROPOSITION **53.5**. *The operator*

$$\begin{aligned} A \ : \quad & \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega) \\ Au(x) = v(x) = & \int_\Omega K(x,y) u(y) \, d^N y \end{aligned}$$

is linear, bounded, and, hence, continuous on $\mathcal{L}_2(\Omega)$.

First, let us show that $D_A = \mathcal{L}_2(\Omega)$. By Schwartz inequality, for every $x \in \Omega$

$$\begin{aligned} |v(x)|^2 &= \left| \int_\Omega K(x,y) u(y) \, d^N y \right|^2 \\ &\le \int_\Omega |K(x,y)|^2 \, d^N y \int_\Omega |u(y)|^2 \, d^N y \\ &= \|u\|^2 \int_\Omega |K(x,y)|^2 \, d^N y \end{aligned}$$

It follows from this inequality that

$$\|Au\|^2 = \int_\Omega |v(x)|^2 \, d^N x \le M \|u\|^2$$

Therefore $Au \in \mathcal{L}_2(\Omega)$ for any $u \in \mathcal{L}_2(\Omega)$.

Furthermore

$$\|Au\| \le \sqrt{M} \|u\| \quad \Rightarrow \quad \|A\| \le \sqrt{M}$$

Linear integral operators with square integrable kernels are called *Hilbert-Schmidt operators.*

53.4. Differentiation operators. Put

$$Au(x) = u'(x) \,, \quad u \in C^1(0,1) \cap \mathcal{L}_2(0,1) \,, \quad u' \in \mathcal{L}_2(0,1)$$

The domain of this linear operator consists of all differentiable square integrable functions on $(0,1)$ whose derivative is also square integrable on $(0,1)$.

PROPOSITION **53.6**. *The differentiation operator is not bounded in* $\mathcal{L}_2(0,1)$

Let us find a sequence $\{u_n\}$ for which

$$\lim_{n\to\infty} \frac{\|Au_n\|}{\|u_n\|} = \infty$$

This would mean that A is unbounded. Put

$$u_n(x) = \sin(\pi n x)\,, \quad Au_n(x) = \pi n \cos(\pi n x)$$

Therefore

$$\|u_n\|^2 = \int_0^1 \sin^2(\pi n x)\,dx = \frac{1}{2}$$

$$\|Au_n\|^2 = \pi^2 n^2 \int_0^1 \cos^2(\pi n x)\,dx = \frac{1}{2}\,\pi^2 n^2$$

$$\lim_{n\to\infty} \frac{\|Au_n\|}{\|u_n\|} = \lim_{n\to\infty} \pi n = \infty$$

Thus, the differentiation operator is not bounded and, hence, not continuous.

Alternatively, one can take the null sequence

$$v_n(x) = \frac{e^{inx}}{n} \quad \Rightarrow \quad \|v_n\| = \frac{1}{n} \quad \Rightarrow \quad v_n \to 0 \text{ in } \mathcal{L}_2(0,1)$$

However, Av_n does not converge to 0 in $\mathcal{L}_2(0,1)$ because

$$Av_n(x) = v_n'(x) = ie^{inx} \quad \Rightarrow \quad \|Av_n\| = 1 > 0$$

So, the differentiation operator is not continuous and, hence, not bounded.

It is straightforward to extend this analysis to operators of partial derivatives of any order in $\mathcal{L}_2(\Omega)$ where $\Omega \subset \mathbb{R}^N$ with the same conclusion. Take a rectangular box $R = \prod_{j=1}^N \times(a_j, b_j) \subset \Omega$. Put

$$u_n(x) = \prod_{j=1}^N \sin\left(\frac{\pi n(x_j - a_j)}{b_j - a_j}\right), \quad a_j < x_j < b_j$$

and $u_n(x) = 0$ otherwise. If $A_j u = \frac{\partial u}{\partial x_j}$, then $\|A_j u_n\|/\|u_n\| = \pi n \to \infty$ as $n \to \infty$. By the same line of arguments, the operators of higher order partial derivatives are shown to be not bounded and, hence, not continuous, too in $\mathcal{L}_2(\Omega)$.

The norm in the domain and boundedness of an operator. It should be noted that boundedness of an operator depends on the norm in the Banach space which contains the domain of the operator. In particular, there are Banach spaces in which differentiation operators are bounded. An example is provided in Exercises.

53.5. Multiplication operators. Put

$$Au(x) = xu(x)\,, \quad u \in \mathcal{L}_2(0,1) = D_A$$

Then A is a linear operator in $\mathcal{L}_2(0,1)$ because

$$\|Au\|^2 = \int_0^1 x^2|u(x)|^2\,dx \le \int_0^1 |u(x)|^2\,dx = \|u\|^2 < \infty$$

so that $Au \in \mathcal{L}_2(0,1)$ for any $u \in \mathcal{L}_2(0,1)$. It is a bounded operator because

$$\|Au\| \le \|u\| \quad \Rightarrow \quad \|A\| \le 1$$

The domain of the multiplication operator is the whole Hilbert space $\mathcal{L}_2(0,1)$ and, hence, is closed. Despite this, there exists no function $u_0 \in \mathcal{L}_2(0,1)$ for which $\|A\| = \|Au_0\|/\|u_0\|$. Define a sequence of functions

$$u_n(x) = \begin{cases} 0\,, & 0 < x < 1 - \frac{1}{n} \\ 1\,, & 1 - \frac{1}{n} < x < 1 \end{cases}$$

Then

$$\|u_n\|^2 = \frac{1}{n} \quad \Rightarrow \quad u_n \to 0 \text{ in } \mathcal{L}_2(0,1)$$

as $n \to \infty$. So, $\{u_n\}$ is a null sequence. One also infers that

$$\|Au_n\|^2 = \int_{1-\frac{1}{n}}^1 x^2\,dx = \frac{1}{n} - \frac{1}{n^2} + \frac{1}{3n^3}$$

$$\lim_{n\to\infty} \frac{\|Au_n\|}{\|u_n\|} = \lim_{n\to\infty} \Big(1 + O(\tfrac{1}{n})\Big) = 1$$

Thus,

$$\|A\| = 1$$

because $\|A\| \le 1$. However the supremum in the definition of $\|A\|$ is reached on elements arbitrary close to the zero function.

This analysis can easily be extended to the multiplication operators $Au(x) = x_j u(x)$ where x_j is the jth component of $x \in \Omega \subseteq \mathbb{R}^N$ and $u \in \mathcal{L}_2(\Omega)$. If Ω is bounded, then $\|A\| < \infty$, otherwise A is unbounded.

53.6. Operators in a separable Hilbert space. Let

$$A\ :\quad D_A \subseteq \mathcal{H} \ \to\ \mathcal{H}$$

where $\mathcal{H}$ is a separable Hilbert space. Suppose that $\dim \mathcal{H} = N < \infty$. If $\{\varphi_n\}_1^N$ is an orthonormal basis in $\mathcal{H}$, then any vector u is uniquely defined by its components relative to this basis

$$u = \sum_{n=1}^N \alpha_n \varphi_n\,, \quad \alpha_n = \langle u, \varphi_n\rangle\,,$$

and, conversely, any vector in $\mathbb{C}^N$ with components α_n defines an element of $\mathcal{H}$ by the above relation. So, $\mathcal{H} \sim \mathbb{C}^N$.

Furthermore, *any linear operator* $A: \mathcal{H} \to \mathcal{H}$ *is uniquely defined by the matrix* $A_{nk} = \langle A\varphi_k, \varphi_n\rangle$ *relative to an orthonormal basis* $\{\varphi_n\}$ *and, conversely, any matrix* A_{nk} *defines a linear operator in* $\mathcal{H}$:

$$Au = \sum_{n=1}^{N} \beta_n \varphi_n\,, \quad \beta_n = \sum_{k=1}^{N} A_{nk}\alpha_k\,.$$

Indeed, for any A and any $u \in \mathcal{H}$ one has

$$Au = \sum_{n=1}^{N} \beta_n \varphi_n\,, \quad \beta_n = \langle Au, \varphi\rangle\,.$$

If $u = \sum_{k=1}^{N} \alpha_k \varphi_k$, then by linearity of A and the inner product,

$$\beta_n = \Big\langle A\Big(\sum_{k=1}^{N} \alpha_k\varphi_k\Big), \varphi_n\Big\rangle = \Big\langle \sum_{k=1}^{N} \alpha_k A\varphi_k, \varphi_n\Big\rangle$$

$$= \sum_{k=1}^{N} \langle A\varphi_k, \varphi_n\rangle \alpha_k = \sum_{k=1}^{N} A_{nk}\alpha_k\,. \tag{53.1}$$

The converse is obvious.

A natural question to ask whether this property can be extended to infinite dimensional separable Hilbert spaces. Riesz-Fisher theorem **58.3** allows us to claim that $\mathcal{H}$ is isomorphic to the space of square summable complex sequences, l_2. However, the second and third equalities in (**53.1**) require justification if $N = \infty$ as the argument of linearity of A and the inner product does not apply if $N = \infty$. Intuitively it looks plausible to interchange the order of A and summation (with $N = \infty$) if A is continuous. Owing to linearity of A, a continuous A is bounded, $\|A|\| < \infty$. The idea is indeed correct.

For any $u \in \mathcal{H}$ the sequence $u_{\scriptscriptstyle N} = \sum_{k=1}^{N} \alpha_k\varphi_k$, where α_k are Fourier coefficients of u, converges to u as $N \to \infty$. By continuity of A, the sequence $Au_{\scriptscriptstyle N}$ converges to Au as $N \to \infty$. Then by continuity of the inner product

$$\beta_n = \langle Au, \varphi_n\rangle = \langle \lim_{N\to\infty} Au_{\scriptscriptstyle N}, \varphi_n\rangle = \lim_{N\to\infty} \langle Au_{\scriptscriptstyle N}, \varphi_n\rangle$$

and the rest of (**53.1**) follows for $N = \infty$. By the Parseval-Steklov equality

$$\|A\varphi_k\|^2 = \sum_{n=1}^{\infty} |\langle A\varphi_k, \varphi_n\rangle|^2 = \sum_{n=1}^{\infty} |A_{nk}|^2 < \infty \tag{53.2}$$

for any k. So, the columns of infinite matrix A_{nk} are square summable.

It will be shown later that any bounded operator has the adjoint and the adjoint has the same norm. Using this property it will be proved that the rows of the infinite matrix A_{nk} are also square summable:

$$\sum_{k=1}^{\infty} |\langle A\varphi_k, \varphi_n\rangle|^2 = \sum_{k=1}^{\infty} |A_{nk}|^2 < \infty \,. \tag{53.3}$$

In particular, this relation implies that the series

$$\beta_n = \sum_{k=1}^{\infty} A_{nk}\alpha_k \tag{53.4}$$

converges absolutely for any $u \in \mathcal{H}$ and any linear bounded operator A. Indeed, for any n, this series can be viewed as the inner product in the Hilbert space l_2. The assertion follows from the Cauchy-Bunyakowky inequality and (**53.3**):

$$\sum_{k=1}^{\infty} |A_{nk}|\,|\alpha_k| \le \Big(\sum_{k=1}^{\infty} |A_{nk}|^2\Big)^{1/2} \|u\| < \infty \,.$$

So, the terms in the series (**53.4**) for β_n can be rearranged in any way.

The difference with a finite dimensional case is that *the converse is not generally true, meaning that, not with every infinite dimensional matrix of complex numbers A_{kn} with square summable columns and rows one can associate a bounded linear operator*. First note that in order for the vector defined by the rule

$$Au = \sum_{n=1}^{\infty} \beta_n\varphi_n \,, \tag{53.5}$$

where β_n are given by the series (**53.4**), to be in $\mathcal{H}$ for any $u \in \mathcal{H}$, the sequence $\{\beta_n\}$ must be square summable according to Riesz-Fisher theorem **58.3**. A *sufficient* (not necessary) condition for this is not difficult to obtain by using the Cauchy-Bunyakowky inequality in l_2 as in the proof of absolute convergence of series (**53.4**)

$$\|Au\|^2 = \sum_{n=1}^{\infty} |\beta_n|^2 \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |A_{nk}|^2\|u\|^2$$

This implies that A is bounded if all matrix elements are square summable and, in this case,

$$\|A\| \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |A_{nk}|^2 < \infty \,.$$

Evidently, not all matrices with square summable columns and rows have this property. If A_{kn} has just finitely many elements in each row and column, e.g., a matrix with finitely many non-zero super-diagonals and other elements being zeros, then the rows and columns are square summable in such a matrix, but all A_{nk} are not square summable if they grow unboundedly with increasing n and k. Matrices with this property can define a linear operator that is not bounded. For example, let the matrix be diagonal, $A_{nk} = c_n \delta_{nk}$ where $|c_n| \to \infty$ as $n \to \infty$. Every row and column has just one element. Then the operator A is defined by the rule (**53.5**) and its domain D_A is defined by the condition that the series $\|Au\|^2 = \sum_n |c_n|^2 |\alpha_n|^2$ converges for any $u \in D_A$ for a given sequence $\{c_n\}$. However this operator is not bounded because $\|A\varphi_n\| = |c_n| \to \infty$ as $n \to \infty$.

53.6.1. On a matrix representation of unbounded operators. A matrix representation for an unbounded operator does not generally exists because the sequence $Au_{_N}$ does not generally converge to Au, where $u_{_N}$ is a truncated Fourier series of u, due to the lack of continuity of an unbounded operator. In fact, the sequence $Au_{_N}$ may have no limit. For example, let $A = -(\frac{d}{dx})^2$ in $\mathcal{H} = \mathcal{L}_2(-\pi, \pi)$. One can take the domain of A to be $D_A = C^2([-\pi, \pi])$. Let $\{\varphi_n\}$ be the trigonometric Fourier basis. It consists of orthogonal functions 1, $\cos(nx)$, and $\sin(nx)$ where $n = 1, 2, ...$ Then $u(x) = \frac{1}{4}(\pi^2 - x^2)$ is from the domain of A, and $Au = -\frac{1}{2}$. The trigonometric Fourier series reads

$$u(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(nx) \, .$$

It follows that the sequence

$$Au_{_N} = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} A\cos(nx) = \sum_{n=1}^{N} (-1)^{n+1} \cos(nx)$$

has no limit in $\mathcal{L}_2(-\pi, \pi)$, not to mention convergence to Au. So, the matrix representation is not possible for all $u \in D_A$. However, it is possible for functions from a subspace of D_A.

Let u be a test function from $\mathcal{D}(-\pi, \pi)$. Evidently, $\mathcal{D}(-\pi, \pi) \subset D_A$. If α_n are the trigonometric Fourier coefficients of a test function with support in $(-\pi, \pi)$, then for any $p > 0$, $n^p |\alpha_n| \to 0$ as $n \to \infty$. So, the Fourier coefficients are decreasing faster then any power function. This implies that the series $\sum \alpha_n A\varphi_n = \sum \alpha_n n^2 \varphi_n$ converges uniformly on $[-\pi, \pi]$ as $|\varphi_n(x)| \le 1$ and, hence, $Au_{_N} \to Au$ in $\mathcal{L}_2(-\pi, \pi)$ if $u \in \mathcal{D}(-\pi, \pi)$. Therefore A has a matrix representation

$A_{nk} = \frac{1}{\pi}\langle A\varphi_k, \varphi_n\rangle = n^2\delta_{nk}$ because $\|\varphi_n\|^2 = \pi$, $n > 0$. In other words, if $\{\alpha_n\}$ are Fourier coefficients of u and $\{\beta_n\}$ are Fourier coefficients of Au, then $\beta_n = \sum_k A_{nk}\alpha_k$ if $u \in \mathcal{D}(-\pi, \pi)$, and this relation does not hold for all $u \in D_A$. So, *the matrix representation can be used for linear unbounded operators only with an appropriate reduction of the domain.* In particular, any differentiation operator (derivative of any order) has a matrix representation in $\mathcal{L}_2(-\pi, \pi)$ relative to the trigonometric Fourier basis if the domain of the operator is reduced to the space of test functions with support in $(-\pi, \pi)$.

It is noteworthy that the said reduction of the domain of an unbounded operator depends on the basis relative to which the matrix representation is sought. For example, let $P_n(x)$ be Legendre polynomials. Then $\varphi_n(x) = P_n(x/\pi)$ is an orthogonal basis in $\mathcal{L}_2(-\pi, \pi)$. In the above example, any polynomial is from the domain of the second derivative operator A and it is a *linear combination* of the basis functions. So, if u is a polynomial and $\varphi_n(x) = P_n(x/\pi)$, then $Au_{_N} \to Au$ because $u_{_N} = u$ for all large enough N. Furthermore, let u be represented by a power series $u = \sum c_n x^n$ with radius of convergence $R > \pi$. Since P_n are obtained by the Gram-Schmidt process for the set of monomials $\{x^n\}_0^\infty$, a truncated power series $u_{_N}$ for u is nothing but a linear combination of φ_n for $n \leq N$. So, a truncated power series $u_{_N}$ converges to u in $\mathcal{L}_2(-\pi, \pi)$. A convergent power series can be differentiated term-by-term any number of times to get the corresponding derivative of the sum in the interval $|x| < R$ and, hence, $Au_{_N} \to Au$. Thus, the matrix representation of A exists relative to the basis of Legendre polynomials if D_A is restricted to analytic functions. This is not true for the trigonometric Fourier basis (as shown with the above example).

53.7. Exercises.

1. Let $Au(x) = xu(x)$ where $u \in \mathcal{L}_2(a, b)$.

(i) If $-\infty < a < b < \infty$, find $\|A\|$.

(ii) If (a, b) is not bounded, specify the domain of A in $\mathcal{L}_2(a, b)$ and show that $\|A\| = \infty$.

2. Let

$$Au(x) = \frac{u(x)}{x}, \quad u \in \mathcal{L}_2(0, 1)$$

(i) Find D_A, R_A, and its null space $\mathcal{N}_A$

(ii) Show that A is not bounded. *Hint*: Take a continuous positive function v on $[0,1]$. Put $u_n(x) = \chi_n(x)v(x)$ where χ_n is the characteristic functions of the interval $[0, \frac{1}{n}]$. Investigate $\|Au_n\|$.

3. Differentiation operator in $C^1([a,b])$.
On the linear space $\mathcal{B} = C^1([a,b])$ define the norm

$$\|u\| = \sup_{[a,b]} |u(x)| + \sup_{[a,b]} |u'(x)|$$

(i) Show that the norm axioms are fulfilled and $\mathcal{B}$ is a normed space.

(ii) Show that $\mathcal{B}$ is a Banach space. To prove the completeness, use a relation between uniform convergence and differentiability, that is, if $u_n'(x)$ converges uniformly to $v(x)$ and $u_n \subset C^1$ converges to u, then $u \in C^1$ and $u' = v$.

(iii) Consider

$$A \ : \quad C^1([a,b]) = \mathcal{B}_1 \ \to \ \mathcal{B}_2 = C^0([a,b])\,, \qquad Au(x) = u'(x)\,,$$

where

$$\|u\|_1 = \sup|u(x)| + \sup|u'(x)|\,, \quad \|v\|_2 = \sup|v(x)|$$

Show that $\|A\| \le 1$, that is, the differentiation operator is bounded.

54. Operator algebra

54.1. The sum of operators.

DEFINITION 54.1. (Sum of operators)
Let A and B be linear operators

$$\begin{aligned} A &: \quad D_A \subset X \;\to\; Y \\ B &: \quad D_B \subset X \;\to\; Y \end{aligned}$$

where X and Y are linear manifolds. Then the sum *of A and B is an operator*

$$C = A + B \;:\quad D_C \subset X \;\to\; Y$$

with the domain $D_C = D_A \cap D_B$ that acts on it by the rule

$$Cu = Au + Bu\,,\quad \forall u \in D_C$$

In other words, the sum of operators makes sense only on common elements of the domains of the operators in the sum. Naturally, this can be extended to the sum of finitely many operators, provided their domains have common elements. The sum is commutative and distributive

$$A + B = B + A\,,\quad (A + B) + C = A + (B + C)$$

So, the sum of several operators will be written without specifying the order of summation as the latter is irrelevant:

$$A + B + \cdots + C\,.$$

54.2. Properties of the sum of operators in a Banach space. In this case, one discuss continuity and boundedness of operators. Clearly, the sum of two linear operators is linear

$$A\,,\; B \text{ are linear} \quad \Rightarrow \quad C = A + B \text{ is linear}$$

Next, it is assumed that X and Y are either Banach spaces or linear manifolds in them. In this case, one discuss continuity and boundedness of operators.

Simplified notation for the norm. In what follows, the norm in the domain $D \subset \mathcal{B}_1$ of an operator and the norm in its range $R \subset \mathcal{B}_2$ will not be indicated by the subscripts 1 and 2. This should not cause any confusion because $\|Au\|$ clearly means $\|Au\|_2$ because $Au \in \mathcal{B}_2$ and, similarly, $\|u\|$ means $\|u\|_1$ because A acts on u and, hence, $u \in \mathcal{B}_1$.

PROPOSITION **54.1**. (Continuity of the sum)
The sum of continuous operators is continuous:

$$A\,,\; B \text{ are continuous} \quad \Rightarrow \quad C = A + B \text{ is continuous}$$

Let $\{u_n\} \subset D_C$ be a sequence that converges to $u \in D_C$. If R_C denotes the range of C, then then by continuity of A and B, the sequences $\{Au_n\} \subset R_C$ and $\{Bu_n\} \subset R_C$ converge to $Au \in R_C$ and $Bu \in R_C$, respectively. By the basic laws of limits

$$Cu_n = Au_n + Bu_n \to Au + Bu = Cu \quad \text{as} \quad n \to \infty$$

which means that C is continuous at any $u \in D_C$.

PROPOSITION **54.2**. (Boundedness of the sum)
The sum of two bounded operators is bounded:

$$A\,,\; B \text{ are bounded} \quad \Rightarrow \quad C = A + B \text{ is bounded}$$

and

$$\|A + B\| \le \|A\| + \|B\|\,.$$

For any $u \in D_{A+B}$, one has

$$\|(A+B)u\| = \|Au + Bu\| \overset{(1)}{\le} \|Au\| + \|Bu\| \overset{(2)}{\le} \|A\|\|u\| + \|B\|\|u\|$$

here (1) is by the triangle inequality for the norm; (2) by the definition of the norm of an operator and that D_{A+B} is a subset in D_A and D_B (the supremum cannot decrease with enlarging the set on which it is taken). Therefore

$$\frac{\|(A+B)u\|}{\|u\|} \le \|A\| + \|B\|\,, \qquad \forall u \in D_{A+B}\,,\; u \neq 0$$

Taking the supremum in the left side of this inequality, it is concluded that

$$\|A + B\| \le \|A + \|B\|\,.$$

Therefore $A + B$ is bounded if A and B are bounded.

54.3. The product of operators.

DEFINITION **54.2**. *The* product *of operators A and B,*

$$A \;:\; D_A \subset X \to Y\,, \quad B \;:\; D_B \subset Y \to Z$$

is an operator

$$C = BA \;:\; D_C \subset D_A \subset X \;\to\; Z$$

that acts by the rule

$$Cu = B(Au) \in Z\,, \quad \forall u \in D_C$$

and whose domain

$$D_C = \{\, u \in D_A \,|\, Au \in D_B \,\}$$

consists of all elements of D_A *whose image under the action of* A *lies in the domain of* D_B.

Note that the description of the domain of the product is just a necessity in order for the rule $(BA)u = B(Au)$ to make sense, that is, u must be from D_A in order for Au to exist, and, in turn, Au must be from D_B in order $B(Au)$ to exist. As a consequence, the *product of operators is sensitive to the order*. First,

$$D_{AB} \neq D_{BA}$$

Second,

$$AB \neq BA \quad \text{even if } D_{AB} = D_{BA}$$

For example,

$$Bu = \int_a^b K(x,y)u(y)\,dy\,, \quad K \in \mathcal{L}_2((a,b)\times(a,b))\,, \quad u \in \mathcal{L}_2(a,b)$$

that is, $B : \mathcal{L}_2(a,b) \to \mathcal{L}_2(a,b)$ (a *Hilbert-Schmidt operator* or an integral operator with a square integrable kernel discussed above). Let

$$A \ : \quad C^1([a,b]) \subset \mathcal{L}_2 \to C^0([a,b]) \subset \mathcal{L}_2(a,b)\,, \quad Au(x) = u'(x)\,.$$

Then

$$BA \ : \quad C^1([a,b]) \subset \mathcal{L}_2 \to \mathcal{L}_2(a,b)$$

and

$$BAu(x) = \int_a^b K(x,y)u'(y)\,dy$$

However the operator AB does not even exist in general because the function $Bu(x)$ is not from $D_A = C^1([a,b])$ if $K(x,y)$ is not smooth enough.

To illustrate the *non-commutativity* of the product of operators, consider

$$\begin{aligned} Au(x) &= xu(x)\,, & D_A &= \mathcal{L}_2(a,b)\,, \\ Bu(x) &= u'(x)\,, & D_B &= C^1([a,b]) \subset \mathcal{L}_2(a,b) \end{aligned}$$

Then the rules

$$\begin{aligned} BAu(x) &= (xu(x))' = u(x) + xu'(x)\,, \\ ABu(x) &= xu'(x) \end{aligned}$$

both make sense if

$$D_{AB} = D_{BA} = C^1([a,b])$$

and in this case

$$(BA - AB)u(x) = u(x) \qquad \forall u \in C^1([a,b])$$

If $D_{AB} = D_{BA}$, then the operator

$$[A,B] = AB - BA$$

is called the commutator of A and B. In particular

$$\left[\frac{d}{dx} \, , \; x \right] = I$$

where I is the unit or identity operator $Iu(x) = u(x)$.

54.4. Properties of the product of operators.

PROPOSITION **54.3**. (Properties of AB)
Let A and B be operators with domains D_A and D_B such that AB exists. Then

(1) *D_A , D_B are linear manifolds $\Rightarrow$ D_{AB} is a linear manifold*
(2) *A , B are linear $\Rightarrow$ AB is linear*
(3) *A , B are continuous $\Rightarrow$ AB is continuous*

A proof of these assertions is left to the reader as an exercise.

PROPOSITION **54.4**. (Norm of the product of operators)
Let A and B be bounded operators with domains D_A and D_B such that AB exists. Then the product AB is bounded

$$A\,,\,B \text{ are bounded} \quad \Rightarrow \quad AB \text{ is bounded}$$

and in this case

$$\|AB\| \le \|A\|\,\|B\|$$

Let $u \in D_{AB}$. The following chain of inequalities hold for bounded operators A and B and any elements in their domains:

$$\|ABu\| = \|A(Bu)\| \le \|A\|\,\|Bu\| \le \|A\|\,\|B\|\,\|u\|$$

Therefore for any $u \neq 0$ from D_{AB}

$$\frac{\|ABu\|}{\|u\|} \le \|A\|\,\|B\|$$

Taking the supremum in the left side, it is concluded that

$$\|AB\| \le \|A\|\,\|B\|$$

Therefore the boundedness of A and B implies the boundedness of their product, $\|AB\| < \infty$.

It is worth noting that the equality is not always possible. Let $n > 0$ and $m >$). Put

$$\begin{aligned} & Au(x) = x^n u(x)\,, \quad Bu(x) = x^m u(x)\,, \quad D_A = D_B = \mathcal{L}_2(0,a) \\ \Rightarrow \quad & ABu(x) = x^{n+m} u(x) \\ \Rightarrow \quad & \|A\| = a^n\,, \quad \|B\| = a^m\,, \quad \|AB\| = a^{n+m} \\ \Rightarrow \quad & \|AB\| = \|A\|\,\|B\| \end{aligned}$$

However, put

$$\begin{aligned} & Au(x) = \int_0^x u(y)\,dy\,, \quad D_A = C^0([0,1]) \\ & Bu(x) = u'(x)\,, \quad D_B = \{\, u \in C^1([0,1]) \,|\, u(0) = 0 \,\} \\ \Rightarrow \quad & ABu(x) = \int_0^x u'(y)dy = u(x) - u(0) = u(x) \\ \Rightarrow \quad & \|AB\| = 1\,, \end{aligned}$$

but the differentiation operator is not bounded $\|B\| = \infty$ (the boundary condition $u(0) = 0$ does not affect this conclusion; take, for example, $u_n(x) = \sin(n\pi x)$ so that $u_n(0) = 0$, but $\|Bu_n\| \to \infty$ as $n \to \infty$). The operator of antiderivative A is bounded, $\|A\| < \infty$:

$$\begin{aligned} \|Au\|^2 &= \int_0^1 \int_0^x \int_0^x u(y)\overline{u(z)}\,dzdydx \le \int_0^1 \int_0^x \int_0^x |u(y)|\,|u(z)|\,dzdydx \\ &\le \frac{1}{2}\int_0^1 \int_0^x \int_0^x \Big(|u(y)|^2 + |u(z)|^2\Big)\,dzdydx \\ &= \int_0^1 x \int_0^x |u(y)|^2 dydx \le \int_0^1 x \int_0^1 |u(y)|^2 dydx \\ &= \|u\|^2 \int_0^1 xdx = \frac{1}{2}\|u\|^2 \quad \Rightarrow \quad \|A\| \le \frac{1}{\sqrt{2}}\,, \end{aligned}$$

where the inequality $ab \le \frac{1}{2}(a^2 + b^2)$ was used (note $(a-b)^2 \ge 0$). So, in this case $\|AB\| \le \|A\|\|B\|$ yields a trivial statement $1 < \infty$.

54.5. Linear space of operators. Let $L(X,Y)$ be the set of all operators whose domain is X and whose range is Y. Then

$$L(X,Y) \textit{ is a linear space}$$

Indeed, $L(X,Y)$ is closed under the operator addition:

$$D_A = D_B = X \quad \Rightarrow \quad D_{A+B} = X \quad \Rightarrow \quad A + B \in L(X,Y)$$

The multiplication by a number is defined by

$$(\alpha A)u \stackrel{\text{def}}{=} \alpha(Au)\,, \quad \forall u \in X$$

Note that this requires that $\alpha v \in Y$ for all $v \in Y$ and any $\alpha \in \mathbb{C}$. Therefore

$$D_{\alpha A} = X \quad \Rightarrow \quad \alpha A \in L(X,Y)$$

The null operator $A = 0$ is defined by

$$Au = 0 \quad \forall u \in X$$

Then $\alpha A = 0$ if $\alpha = 0$ and $A + (-1)A = 0$ for any operator A. Thus, $L(X,Y)$ is a linear space.

The operator norm was proved to satisfy the norm axioms

$$\begin{aligned}
&\|A\| = 0 \quad \Leftrightarrow \quad A = 0\\
&\|A\| > 0 \quad \forall A \neq 0\\
&\|\alpha A\| = |\alpha|\|A\|\\
&\|A + B\| \leq \|A\| + \|B\|
\end{aligned}$$

Therefore

$L(X,Y)$ is a normed linear space of bounded operators

THEOREM **54.1.** (Banach space of bounded operators)
Suppose that the domain X is a linear manifold in a Banach space, and the range $Y = \mathcal{B}$ is a Banach space $\mathcal{B}$. Then

$$(1) \qquad L(X,Y) \textit{ is a Banach space w.r.t. the operator norm}$$

and for any operator sequence $\{A_n\}_1^\infty \subset L(X,Y)$ such that the series $\sum \|A_n\| < \infty$ converges, there exists $A \in L(X,Y)$ such that

$$(2) \qquad A = \sum_{n=1}^{\infty} A_n \quad \textit{and} \quad \|A\| \leq \sum_{n=1}^{\infty} \|A_n\|$$

PROOF. Consider a sequence of partial sums

$$B_n = \sum_{k=1}^{n} A_k$$

Then $\{B_n\}_1^\infty$ is a Cauchy sequence in $L(X,Y)$. Indeed, for $n > m$

$$\|B_n - B_m\| = \Big\| \sum_{k=m+1}^{n} A_k \Big\| \leq \sum_{k=m+1}^{n} \|A_k\|$$

can be made arbitrary small for all large enough m because $\sum_k \|A_k\| < \infty$ and the partial sums of a convergent series form a Cauchy sequence in $\mathbb{R}$. Next, for any $u \in X$ the sequence $\{B_n u\}_1^\infty$ is a Cauchy sequence in Y because

$$\|B_n u - B_m u\| \le \|B_n - B_m\| \|u\| \to 0 \quad \text{as} \quad n, m \to \infty$$

By the hypothesis, $Y = \mathcal{B}$ is a Banach space (a complete normed linear space). Therefore there exists $v \in Y$ such that

$$\lim_{n\to\infty} B_n u = v\,, \quad \forall u \in X$$

Define an operator A with domain being X by

$$Au = v = \lim_{n\to\infty} B_n u = \sum_{k=1}^{\infty} A_k u\,, \quad u \in X$$

By the limit laws, A is a linear operator that maps X to Y.

Let us investigate the convergence of the sequence B_n to A. One has

$$\|Au - B_n u\| = \left\| Au - \sum_{k=1}^{n} A_k u \right\| = \left\| \sum_{k=n+1}^{\infty} A_k u \right\| \le \sum_{k=n+1}^{\infty} \|A_k\| \, \|u\| \,.$$

From which it follows that for any $u \in X$, $u \neq 0$,

$$\frac{\|Au - B_n u\|}{\|u\|} \le \sum_{k=n+1}^{\infty} \|A_k\|$$

Taking the supremum in the right side

$$\left\| A - B_n \right\| \le \sum_{k=n+1}^{\infty} \|A_k\|$$

In the limit $n \to \infty$ the right side of this inequality vanishes. This implies that the operator sequence of partial sums B_n converges to the constructed operator A with respect to the operator norm:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} A_k = \sum_{k=1}^{\infty} A_k \;\in\; L(X, Y)$$

and A is bounded because

$$\|A\| = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} A_k \right\| \le \lim_{n\to\infty} \sum_{k=1}^{n} \|A_k\| = \sum_{k=1}^{\infty} \|A_k\| < \infty\,.$$

□

54.6. Convergence for operator sequences in a Hilbert space. Let $\{A_n\}_1^\infty$ be a sequence of operators on a Hilbert space $\mathcal{H}$.

DEFINITION **54.3**. (Weak convergence)
An operator sequence $\{A_n\}_1^\infty$ is said to converge weakly *to an operator A if the following numerical sequence converges*

$$\lim_{n\to\infty} \langle A_n u, v\rangle = \langle Au, v\rangle\,, \qquad \forall u, v \in \mathcal{H}$$

DEFINITION **54.4**. (Convergence in the norm)
An operator sequence $\{A_n\}_1^\infty$ is said to converge to an operator A in the norm *if the following numerical sequence converges*

$$\lim_{n\to\infty} \|A_n u - Au\| = 0\,, \qquad \forall u \in \mathcal{H}$$

DEFINITION **54.5**. (Strong convergence)
An operator sequence $\{A_n\}_1^\infty$ is said to converge strongly *to an operator A if it converges in the operator norm*

$$\lim_{n\to\infty} \|A_n - A\| = 0\,.$$

It follows from the the Cauchy-Bunyakowski inequality

$$|\langle (A - A_n)u, v\rangle| \le \|v\|\|Au - A_n u\| \le \|v\|\|u\|\|A - A_n\|$$

that *the strong convergence implies the convergence in the norm and the latter implies the weak convergence.* This the reason why the weak convergence is called "weak" (as it the weakest of the three).

54.7. Exercises.

1. Suppose that

$$A_n u(x) = x^n u(x)\,, \quad u \in C^0([a,b])$$

(i) Show that $\|A_n\| < \infty$ for all n in the Banach space $C^0([a,b])$ (the norm is $\|\cdot\|_\infty$) and find $\|A_n\|$.

(ii) Investigate convergence of the series

$$A = \sum_{n=0}^{\infty} c_n A_n\,, \qquad c_n \in \mathbb{R}\,,$$

with respect to the operator norm and formulate a sufficient condition on the numerical coefficients c_n under which the series is strongly convergent.

(iii). Suppose that the above series is strongly convergent. Find the norm $\|A\|$ (express the answer via the coefficients c_n).

2. Geometric series of operators.
Let $A: X \to \mathcal{B}$ be a bounded operator such that $\|A\| < 1$, where $\mathcal{B}$ is a Banach space. Put

$$B = \sum_{n=0}^{\infty} A^n$$

(i) Show that the series is strongly convergent and

$$\|B\| \leq \frac{1}{1 - \|A\|}$$

(ii) Prove that the operator B has the following property

$$B(I-A)u = u\,, \quad \forall u \in D_A \quad \text{or} \quad \sum_{n=0}^{\infty} A^n = (I-A)^{-1}\,, \quad \|A\| < 1\,,$$

where I is the unit operator, $Iu = u$. *Hint*: Let B_n be a sequence of partial sums so that $B_n \to B$ strongly as $n \to \infty$. Investigate the convergence of the operator sequence $B_n(I-A)$.

3. von Neumann series for Fredholm equations.
Consider an integral equation in $\mathcal{L}_2(\Omega)$

$$u = \lambda K u + f\,, \quad u \in \mathcal{L}_2(\Omega)\,, \quad f \in \mathcal{L}_2(\Omega)\,, \quad \lambda \in \mathbb{C}$$

$$Ku(x) = \int_{\Omega} K(x,y) u(y)\, d^N y\,, \qquad K(x,y) \in \mathcal{L}_2(\Omega \times \Omega)$$

(i) Find a condition on the complex parameter λ such that there is a unique solution u for every $f \neq 0$ which is given by the von Neumann series

$$u = \sum_{n=0}^{\infty} \lambda^n K^n f$$

Express this condition in terms of the Hilbert-Schmidt kernel $K(x,y)$.

55. The inverse operator

In applications one often deals with the problem to find an element u from the domain of an operator A for a given f from the range of A such that:

$$Au = f$$

In a linear algebra, this problem has a unique solution if the matrix A is invertible.

DEFINITION **55.1**. (invertible operator)
An operator $A : \; D_A \to R_A$ is said to be invertible *if the equation $Au = f$ has a unique solution for any f from the range R_A. In this case, the solution is denoted as $u = A^{-1}f$, where the operator A^{-1} is called* the inverse *of A:*

$$A^{-1} \; : \quad R_A \; \to \; D_A \quad \text{and} \quad A(A^{-1}f) = f \quad \forall f \in R_A$$

By definition, the domain of the inverse is the range of the operator:

$$D_{A^{-1}} = R_A \, .$$

The main criterion for a linear operator to have the inverse is similar to the matrix theory.

THEOREM **55.1**. (Existence of the inverse for a linear operator)
Let A be a linear operator. In order for the inverse operator A^{-1} to exist, it is necessary and sufficient that the equation $Au = 0$ has only a trivial solution $u = 0$, or the null space of A has only the zero element

$$\exists \, A^{-1} \quad \Leftrightarrow \quad N_A = \{0\}$$

A proof of this criterion is identical to a proof of the case when A is a matrix. Recall from linear algebra that a matrix A has an inverse if and only if the equation $Au = 0$ has only a trivial solution $u = 0$). So, the proof is left to the reader as an exercise.

An important difference with the matrix theory is that operators in general are defined by its action on elements of its domain and by the domain itself. The domain of a linear operator is a *linear manifold*. This latter property is always the case in the matrix theory, but has to be verified for general operators, when applying the criterion given in Theorem **55.1**. This subtlety is illustrated with an example of differentiation operator.

Example: the differentiation operator. Let

$$D : \quad C^1([0,1]) \subset \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1) \, , \quad Du(x) = u'(x) \, .$$

In this case, the range of the operator is $R_D = C^0([0,1])$. The operator D is linear. Indeed, its domain is a *linear manifold* and the derivative of a linear combination is a linear combination of derivatives. So, Theorem **55.1** allows to conclude that D is not invertible because the equation

$$Du(x) = u'(x) = 0 \quad \Rightarrow \quad u(x) = u_0 = \text{const} \ \in C^1([0,1])$$

has non-trivial (constant) solutions. Alternatively, a solution to the equation $u'(x) = f(x)$ is given by an antiderivative of f, which is not unique. This merely reflects the well known fact from calculus that an antiderivative of a continuous function is determined up to an additive constant.

Let us keep the rule (differentiation), but change the domain to get a new operator A:

$$\begin{aligned} A \ : \quad & D_A \subset \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = u'(x)\,, \\ D_A = & \{\, u \in C^1([0,1]) \,|\, u(0) = 0\,\} \end{aligned}$$

In comparison with the operator D, the domain is reduced by imposing the condition $u(0) = 0$. Note that $u'(0)$ can have any value if $u(0) = 0$. So, $R_A = C^0([0,1])$. The domain D_A is still a *linear manifold* as the boundary condition is fulfilled for a linear combination of functions that satisfy the boundary condition. So, the operator A is linear. It is also invertible by Theorem **55.1**

$$\begin{cases} Au(x) = u'(x) = 0 \\ u(0) = 0 \end{cases} \quad \Leftrightarrow \quad u(x) = 0$$

One can find the inverse by solving the initial value problem:

$$\begin{cases} Au(x) = u'(x) = f(x) \\ u(0) = 0 \end{cases} \quad \Rightarrow \quad u(x) = A^{-1}f(x) = \int_0^x f(y)\,dy$$

and the solution is unique for any $f \in R_A = C^0([0,1])$.

On the other hand, consider the operator

$$\begin{aligned} D_B &= \{\, u \in C^1([0,1]) \,|\, u(0) = 1\,\}\,, \\ Bu(x) &= u'(x) \end{aligned}$$

A similar initial value problem also has a unique solution

$$\begin{cases} Bu(x) = u'(x) = f(x) \\ u(0) = 1 \end{cases} \quad \Rightarrow \quad u(x) = B^{-1}f(x) = 1 + \int_0^x f(y)\,dy$$

for any $f \in R_B = C^0([0,1])$. So, B is invertible. However, the homogeneous problem

$$\begin{cases} Bu(x) = u'(x) = 0 \\ u(0) = 1 \end{cases} \quad \Leftrightarrow \quad u(x) = 1$$

has a non-trivial solution $u(x) = 1$ in D_B. The domain of B is *not a linear manifold* because $u(x) = 0$ does not belong to it. The criterion given in Theorem **55.1** does not apply to this operator. It is also interesting to note that the operator A^{-1} is linear, whereas the operator B^{-1} is not linear!

The lesson here is that an operator is defined not only by the rule it acts, but also by its domain. Properties of operators defined by the same rule but acting on different domains might be quite different.

55.1. Natural domain. In what follows, if an operator is defined only by its action (without specifying its domain), then it is assumed that the domain of the operator is its *natural domain* which is a collection of all elements for which the said rule makes sense. Thus, unless the domain is explicitly specified, a rule

$$A \; : \quad D_A \subset X \to Y$$

defines an operator whose domain is the natural domain:

$$D_A = \{ \, u \in X \, | \, Au \in Y \, \} \, .$$

For example, if

$$A \; : \quad D_A \subset \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega) \, , \quad Au(x) = u'(x)$$

The natural domain D_A consists of all functions that are square integrable on Ω, differentiable almost everywhere in Ω, and whose derivative is square integrable on Ω:

$$D_A = \{ \, u \in \mathcal{L}_2(\Omega) \, | \, u' \in \mathcal{L}_2(\Omega) \, \} \, .$$

55.2. Properties of the inverse operator.

PROPOSITION **55.1**. *If A is a linear operator, then its inverse is also linear if A is invertible.*

PROOF. Let f and g be from the range $R_A = D_{A^{-1}}$. One has to verify that

$$\begin{aligned} &\alpha f + \beta g \; \in \; R_A \, , \quad \alpha, \beta \in \mathbb{C} \\ &A^{-1}(\alpha f + \beta g) = \alpha A^{-1} f + \beta A^{-1} g \end{aligned}$$

Since f and g are from the range R_A, there are u and v from D_A such that

$$Au = f\,, \quad Av = g$$

By linearity of A, D_A is a linear manifold so that $\alpha u + \beta v \in D_A$ and

$$A(\alpha u + \beta v) = \alpha Au + \beta Av = \alpha f + \beta g \in R_A$$

Therefore

$$A^{-1}(\alpha f + \beta g) = \alpha u + \beta v = \alpha A^{-1} f + \beta A^{-1} g$$

as required. □

THEOREM **55.2.** (Banach theorem about the inverse operator) *Let A be an operator with the domain and range being Banach spaces. Suppose that A is linear, bounded, and invertible operator. Then the inverse operator is bounded, $\|A^{-1}\| < \infty$.*

In other words, if the domain and range of a linear bounded (or continuous) operator are complete linear manifolds, then the inverse operator is bounded, too (provided it exists)[1]. A simpler version of this theorem will be proved later.

Although the Banach theorem about the inverse operator plays a significant role in the operator theory, it should be noted that the hypothesis about the completeness of the domain is only a *sufficient condition* for the conclusion of the theorem. This hypothesis is not true for differential operators that are often studied in applications (their domains are not complete linear manifolds). Nevertheless there are invertible differential operators whose inverses are either bounded or unbounded. This subtlety is illustrated with an example of the second derivative operator.

Example: The inverse of the second derivative operator. Define the operator

$$\begin{aligned} A \quad &: \quad D_A \subset \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = -u''(x)\,, \\ D_A &= \Big\{ u \in C^2[0,1] \,\Big|\, u(0) = u(1) = 0 \Big\}\,. \end{aligned}$$

The operator A is linear because the boundary condition is fulfilled for a linear combination of functions each of which satisfies it, that is,

[1] A proof of this theorem can be found, e.g., in the textbook by A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter IV, Section 5.4

D_A is a linear manifold. By Theorem **55.1**, A is invertible because the boundary value problem

$$\begin{cases} Au(x) = -u''(x) = 0\,, \\ u(0) = u(1) = 0 \end{cases} \quad \Rightarrow \quad u(x) = 0$$

has only trivial solution. The domain of A is not complete because $D_A \subset \overline{D_A} = \mathcal{L}_2(0,1)$. However, the inverse A^{-1} is bounded.

The domain of A^{-1} is the range of A:

$$D_{A^{-1}} = R_A = C^0[0,1] \subset \mathcal{L}_2(0,1)\,.$$

In order find A^{-1}, one has to solve the boundary value problem

$$\begin{cases} Au(x) = -u''(x) = f(x)\,, \quad f \in R_A\,, \\ u(0) = u(1) = 0 \end{cases}$$

It can be done by means of the Green's function of A (a fundamental solution that satisfies the boundary conditions):

$$G''(x,y) = \delta(x-y)\,, \quad y \in (0,1)\,; \quad G(0,y) = G(1,y) = 0$$

As was shown earlier

$$\begin{aligned} u(x) \,&=\, A^{-1}f(x) = -\int_0^1 G(x,y)\, f(y)\, dy \\ &= (1-x)\int_0^x yf(y)\, dy + x\int_x^1 (1-y)f(y)\, dy \end{aligned}$$

The inverse operator is an integral Hilbert-Schmidt operator because its kernel $G(x,y)$ is square integrable on $(0,1)\times(0,1)$. The reader is advised to calculate

$$M = \left(\int_0^1\int_0^1 |G(x,y)|^2 dxdy\right)^{1/2} < \infty$$

and show that $\|A^{-1}\| \le M$.

Here a slightly different avenue is adopted to demonstrate the boundedness of A^{-1}. One infers that

$$\begin{aligned} \left|\int_0^x yf(y)\, dy\right| &\le \int_0^x y|f(y)|\, dy \le \int_0^1 y|f(y)|\, dy \\ &\le \|y\|\,\|f\| = \frac{1}{\sqrt{3}}\,\|f\| \end{aligned}$$

where the Cauchy-Bunyakowski inequality has been used for $\langle y, |f|\rangle$. Similarly,

$$\left|\int_x^1 (1-y)f(y)\, dy\right| \le \int_0^1 (1-y)|f(y)|\, dy \le \|(1-y)\|\,\|f\| = \frac{1}{\sqrt{3}}\,\|f\|$$

so that

$$|A^{-1}f(x)| \le \frac{1}{\sqrt{3}}\,\|f\|\,(1-x) + \frac{1}{\sqrt{3}}\,\|f\|\,x = \frac{1}{\sqrt{3}}\,\|f\|$$

Therefore, the inverse operator is bounded

$$\|A^{-1}f\| \le \frac{1}{\sqrt{3}}\,\|f\| \quad \Rightarrow \quad \|A^{-1}\| \le \frac{1}{\sqrt{3}}$$

despite that the domain of A is not complete.

Let us define a new operator by keeping the rule (the second derivative) but changing the domain:

$$B \;:\quad D_B \subset \mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R})\,, \quad Bu(x) = -u''(x)\,,$$
$$D_B = \Big\{ u \in C^2(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R}) \,\Big|\, u'' \in \mathcal{L}_2(\mathbb{R}) \Big\}\,.$$

Clearly, B is a linear operator. The operator B is invertible because

$$\begin{aligned} Bu(x) = u''(x) = 0 \quad &\Rightarrow \quad u(x) = ax + b \;\in D_B \\ &\Rightarrow \quad a = b = 0 \quad \text{or} \quad u(x) = 0 \end{aligned}$$

Note that a linear function is not square integrable on $\mathbb{R}$, unless it is equal to zero. Thus, B^{-1} exists. As in the previous case, the domain of B is also not complete because $D_B \subset \overline{D}_B = \mathcal{L}_2(\mathbb{R})$.

In contrast to the previous case, the inverse B^{-1} is not bounded, $\|B^{-1}\| = \infty$. Recall that a linear operator is bounded if and only if it is continuous. So, it maps every null sequence to a null sequence. Take a positive numerical sequence $a_n > 0$, $n = 1, 2, \ldots$. Put

$$u_n = \sqrt{\frac{a_n}{\pi}}\, e^{-a_n^2 x^2/2}\,, \quad \|u_n\| = 1\,, \quad a_n > 0$$

Suppose that $a_n \to 0$ as $n \to \infty$. Then the sequence $f_n = Bu_n \in R_B$ is a null sequence because

$$\|f_n\| = \|Bu_n\| = \|u_n''\| = O(a_n) \to 0\,, \quad \text{as} \quad n \to \infty$$

Assume that $\|B^{-1}\| < \infty$. Then the operator B^{-1} should map the null sequence $f_n = Bu_n$ to a null sequence in D_B because

$$\|B^{-1}f_n\| \le \|B^{-1}\|\|f_n\|$$

However this cannot be true because $B^{-1}f_n = u_n$ so that

$$\|B^{-1}f_n\| = \|u_n\| = 1 \le \|B^{-1}\|\|f_n\| \quad \Rightarrow \quad \frac{1}{\|f_n\|} \le \|B^{-1}\|$$

where $\|f_n\| \to 0$ as $n \to \infty$. A contradiction. Thus, $\|B^{-1}\| = \infty$.

On the range of differential operators. If B is the second derivative operator in $\mathcal{L}_2$, then it is proved to be invertible. The domain of the inverse is the range of B:

$$D_{B^{-1}} = R_B \subset C^0(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$$

Note well that the range is not the whole set of continuous square integrable functions. For any $u \in D_B$, $u'' \in C^0(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$, this implies that

$$\langle Bu, u\rangle = -\int_{-\infty}^{\infty} u''(x)\overline{u(x)}\, dx = \int_{-\infty}^{\infty} |u'(x)|^2\, dx = \|u'\|^2$$

after integration by parts ($u(x) \to 0$ as $|x| \to \infty$ if $u \in D_A$). This means that the derivative u' is also square integrable because the inner product $\langle Bu, u\rangle$ exists. Hence, $u'(x) \to 0$ as $|x| \to \infty$ because $u' \in \mathcal{L}_2$. By continuity of u'',

$$f(x) = -u''(x) \quad \Rightarrow \quad \int_{-a}^{x} f(y)dy = -\int_{-a}^{x} u''(y)dy = -u'(x) + u'(-a)$$

It follows from $u'(-a) \to 0$ as $a \to \infty$ that for any $f \in R_B$

$$u'(x) = -\int_{-\infty}^{x} f(y)dy \in \mathcal{L}_2(\mathbb{R})\,, \quad \forall f \in R_B$$

In other words, an antiderivative of f must be square integrable. In particular, this implies that

$$\lim_{x\to\infty} u'(x) = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(y)dy = 0$$

Clearly, not every continuous square integrable function would satisfy the above conditions. For example, $f(x) = e^{-x^2}$ is not from the range of B because it fails to satisfy the above condition:

$$\int_{-\infty}^{\infty} f(x)dx = \sqrt{\pi} \neq 0 \quad \Rightarrow \quad f \notin R_D$$

but $f \in C^0 \cap \mathcal{L}_2$. Therefore the equation $Bu = f$ has no solution in $D_B \subset \mathcal{L}_2$ because $f \notin R_B$. On the other hand, the differential equation

$$-u''(x) = e^{-x^2} \quad \Rightarrow \quad u(x) = a + bx + \int_0^x \int_0^y e^{-z^2} dzdy$$

has many solutions in the class $C^2(\mathbb{R})$, where a and b are constants. The point is that none of these solutions is square integrable.

55.3. Perturbation theory for an invertible operator. Suppose that the linear problem

$$A_0 u = f$$

can be solved for some operator A_0. The following question is of interest in applications: Can it be solved for $A = A_0 + \Delta A$ where ΔA is a "small perturbation" of A_0? The following theorem address this important problem.

THEOREM **55.3**. (Perturbations of an invertible operator)
Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Suppose that a linear operator A_0 is invertible and bounded:

$$A_0 \ : \quad D_{A_0} = \mathcal{B}_1 \to R_{A_0} = \mathcal{B}_2 \,, \quad \|A_0\| < \infty \,, \quad \exists\, A_0^{-1} \,.$$

Let ΔA be a linear bounded operator such that

$$\Delta A \ : \quad \mathcal{B}_1 \to R_{\Delta A} \subseteq \mathcal{B}_2 \,, \qquad \|\Delta A\| < \frac{1}{\|A_0^{-1}\|}$$

Then the operator $A = A_0 + \Delta A$ is invertible and its inverse is bounded, $\|A^{-1}\| < \infty$.

PROOF. Note that by Theorem **55.2**, the inverse A_0^{-1} is bounded, $\|A_0^{-1}\| < \infty$. Fix $f \in \mathcal{B}_2$. Consider the operator

$$B \ : \quad \mathcal{B}_1 \to \mathcal{B}_2 \,, \qquad Bu = A_0^{-1} f - A_0^{-1} \Delta A u$$

Then the operator B is a contraction on $\mathcal{B}_1$. Indeed, for any $u,\ v \in \mathcal{B}_1$

$$\begin{aligned} \|Bu - Bv\| &\overset{(1)}{=} \|A_0^{-1}\Delta A u - A_0^{-1}\Delta A v\| \overset{(2)}{=} \|A_0^{-1}\Delta A(u-v)\| \\ &\overset{(3)}{\leq} \|A_0^{-1}\Delta A\|\,\|u-v\| \overset{(4)}{\leq} \|A_0^{-1}\|\,\|\Delta A\|\,\|u-v\| \\ &= C\|u-v\| \,, \quad C = \|A_0^{-1}\|\,\|\Delta A\| < 1 \end{aligned}$$

where (1) is by the definition of B, (2) by linearity of the product of linear operators, (3) by the property of the norm of an operator, and (4) by the property of the norm of the product of bounded operators. By the contraction principle, the operator B has a unique fixed point in $D_B = \mathcal{B}_1$:

$$\forall f \in \mathcal{B}_2 \quad \exists!\, u \in \mathcal{B}_1 \ : \quad u = Bu$$

Let u be the fixed point of B for a given $f \in \mathcal{B}_2$. Then it follows that

$$\begin{aligned} & u = A_0^{-1} f - A_0^{-1}\Delta A u \\ \Rightarrow\quad & A_0 u = f - \Delta A u \\ \Rightarrow\quad & Au = (A_0 + \Delta A)u = f \end{aligned}$$

The latter equality implies that for any $f \in \mathcal{B}_2$ there exists a unique $u \in \mathcal{B}_1$ that satisfies the equation $Au = f$, that is, A is invertible. Since A is bounded:

$$\|A\| = \|A_0 + \Delta A\| \leq \|A_0\| + \|\Delta A_0\| < \infty$$

its inverse is bounded, too, $\|A^{-1}\| < \infty$, by the Banach theorem **55.2**. □

The theorem gives sufficient conditions under which the problem

$$Au = (A_0 + \Delta A)u = f$$

has a unique solution if $A_0 u = f$ has a unique solution. The solution can be found by means of the operator geometric series using the following algorithm

$$\begin{aligned} & A_0 u = f \quad \Rightarrow \quad u = A_0^{-1} f = v \\ & (A_0 + \Delta A)u = f \\ \Rightarrow \quad & A_0^{-1}(A_0 + \Delta A)u = A_0^{-1} f = v \\ \Rightarrow \quad & (I - B)u = v\,, \quad B = -A_0^{-1}\Delta A\,, \quad \|B\| < 1 \\ \Rightarrow \quad & u = (I - B)^{-1} v\,, \qquad (I - B)^{-1} = \sum_{n=0}^{\infty} B^n \end{aligned}$$

where the series converges strongly (in the operator norm)[2] and

$$\|(I - B)^{-1}\| \leq \sum_{n=0}^{\infty} \|B\|^n = \frac{1}{1 - \|B\|}$$

Since

$$\|B^n v\| \leq \|B\|^n \, \|v\| \to 0 \qquad n \to \infty$$

the terms in the series for the solution are successively smaller. For this reason the series is often called a perturbative expansion in powers of ΔA. If

$$u_n = \sum_{k=0}^{n} B^k v$$

is a approximation of the solution u by a *perturbative solution of order* n. Then the accuracy of the perturbative approximation is

$$\|u - u_n\| = \Big\| \sum_{k=n+1}^{\infty} B^k v \Big\| \leq \sum_{k=n+1}^{\infty} \|B\|^k \|v\| = \frac{\|B\|^{n+1}}{1 - \|B\|} \, \|v\|$$

It is decreasing to zero with increasing n because $\|B\| < 1$.

[2] See Exercise 2 in Section 10.7

In applications, a perturbation of A_0 often contains a numerical parameter:

$$A = A_0 - \lambda \Delta A\,, \quad \lambda \in \mathbb{C}$$

Then the perturbation theory

$$Au = f \quad \Rightarrow \quad u = \sum_{n=0}^{\infty} \lambda^n (A_0^{-1}\Delta A)^n A_0^{-1} f$$

is always valid, provided $A_0^{-1}\Delta A$ is bounded and

$$|\lambda| < \|A_0^{-1}\Delta A\|$$

If A_0^{-1} happens to be an integral operator, then the perturbative expansion of the solution is nothing by the von Neumann series for a solution to the Fredholm problem.

55.4. Operators bounded away from zero. A linear problem $Au = f$ is said to be *well-posed* if it has a unique solution that depends continuously on f. If A is invertible, then the problem has a unique solution $u = A^{-1}f$. If a sequence f_n converges to f in the range of A, then the problem is well-posed if the sequence $u_n = A^{-1}f_n$ converges to $u = A^{-1}u$ in the domain of A. This is true if the inverse is bounded, $\|A^{-1}\| < \infty$. So, small variations of f produce small variations of the solution $u = A^{-1}f$. If A^{-1} is not bounded, then the latter is not generally true. When analyzing a well-posedness of a linear problem, it is therefore important to establish a criterion for an operator A to have a bounded inverse.

DEFINITION **55.2**. *An operator A is called* bounded away from zero *if for any element u in the domain of A there is a constant $C > 0$ independent of u such that*

$$\|Au\| \geq C\|u\|\,, \quad \forall u \in D_A$$

For example, a multiplication operator

$$Au(x) = xu(x)\,, \quad u \in D_A = \mathcal{L}_2(a,b)\,,$$

is bounded away from zero if $0 < a < b$:

$$\|Au\|^2 = \int_a^b x^2|u(x)|^2\,dx \geq a^2 \int_a^b |u(x)|^2\,dx \quad \Rightarrow \quad \|Au\| \geq a\|u\|\,.$$

The differentiation operator in $\mathcal{L}_1(0,1)$ is not bounded from zero. Let $\{a_n\}$ be a numerical sequence that converges to zero, e.g., $a_n = \frac{1}{n}$.

Consider the sequence $u_n(x) = e^{ia_n x}$ in $\mathcal{L}_2(0,1)$. Then

$$\|u_n\|^2 = \int_0^1 |u_n(x)|^2\, dx = 1$$

Similarly

$$\|Au_n\|^2 = \|u_n'\|^2 = \int_0^1 |u_n'(x)|^2 dx = a_n^2$$

Therefore

$$\lim_{n\to\infty} \frac{\|Au_n\|}{\|u_n\|} = \lim_{n\to\infty} |a_n| = 0$$

So, there are functions for which the ratio $\|Au\|/\|u\|$ can be made arbitrary close to zero and, hence, A cannot be bounded away from zero.

Criterion for an operator to be not bounded away from zero. The technique used to show that the differentiation operator is not bounded away from zero in $\mathcal{L}_2$ is rather general. Suppose that there is a unit sequence $\|u_n\| = 1$ in the domain of an operator A such that its image is a null sequence in the range:

$$\|u_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|Au_n\| = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\|Au_n\|}{\|u_n\|} = 0$$

The latter implies that there are elements in the domain of A for which $\|Au\|/\|u\|$ can be arbitrary close to 0 and, hence, the operator is not bounded away from zero

Criterion for an operator to have a bounded inverse. Operators bounded away from zero have a remarkable property.

THEOREM **55.4**. (Simplified Banach theorem)
A linear operator A has a bounded inverse if and only if it is bounded away from zero.

PROOF. Suppose A is bounded away from zero:

$$\|Au\| \geq C\|u\|\,, \quad \forall u \in D_A\,, \quad C > 0\,.$$

Then the equation $Au = 0$ can have only the trivial solution $u = 0$ because

$$0 = \|Au\| \geq C\|u\| \quad \Rightarrow \quad \|u\| = 0 \quad \Rightarrow \quad u = 0$$

This means that the inverse A^{-1} exists (Theorem **55.1**). Let us show that the inverse is bounded. Put $v = Au$. Then

$$\|v\| = \|Au\| \geq C\|u\| \quad \Rightarrow \quad \|u\| \leq \frac{1}{C}\,\|v\|$$

It follows from $u = A^{-1}v$ and $v \neq 0$ that

$$\|u\| = \|A^{-1}v\| \leq \frac{1}{C}\,\|v\| \quad \Rightarrow \quad \frac{\|A^{-1}v\|}{\|v\|} \leq \frac{1}{C} \quad \Rightarrow \quad \|A^{-1}\| \leq \frac{1}{C}\,.$$

Thus, the inverse is bounded because $C > 0$.

Conversely, suppose that A has a bounded inverse, $\|A^{-1}\| \leq M < \infty$. Then for any v in the range of A, there is a unique $u = A^{-1}v$ in the domain of A and, boundedness of the inverse,

$$\|A^{-1}v\| \leq \|A^{-1}\|\|v\| \leq M\|v\|$$

or, restating this inequality in terms of u,

$$\|u\| \leq M\|Au\| \quad \Rightarrow \quad \|Au\| \geq \frac{1}{M}\,\|u\|$$

for all u in the domain of A. Thus, A is bounded away from zero. □

Remark. An operator A that is not bounded away from zero can still have the inverse. In this case, the Banach theorem implies that the inverse is not bounded. An example is provided by the second derivative operator in the Hilbert space $\mathcal{L}_2(\mathbb{R})$ discussed in Section **55.2**.

55.5. Exercises.

1. Suppose that A and B are invertible operators. Show that the product AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}\,.$$

2. Consider the differentiation operator in $\mathcal{L}_2(0,\infty)$:

$$A \;:\quad D_A \subset \mathcal{L}_2(0,\infty) \to \mathcal{L}_2(0,\infty)\,, \quad Au(x) = u'(x)\,,$$
$$D_A = \Big\{ u \in C^1([0,\infty)) \cap \mathcal{L}_2(0,\infty) \,\Big|\, u(0) = 0 \Big\}$$

(i) Show that A is invertible.

(ii) Find the explicit form of $A^{-1}f$.

(iii) Show that A is not bounded away from zero. *Hint*: Consider $u(x) = xe^{-kx}$, $k > 0$. Calculate $\|Au\|/\|u\|$.

(iv) Show that neither A nor its inverse A^{-1} is bounded.

(v) Show that the range of A is a proper subset of $C^0([0,\infty)) \cap \mathcal{L}_2(0,\infty)$. In particular, solve the equation $Au = f$, $u \in D_A$, where $f(x) = e^{-x} \in C^0([0,\infty)) \cap \mathcal{L}_2(0,\infty)$ or show that no solution exists.

3. Prove that the multiplication operator $Au(x) = xu(x)$ is not bounded away from zero in $\mathcal{L}_2(a, b)$ if the closed interval $[a, b]$ contains $x = 0$. *Hint*: Construct a unit sequence $\|u_n\| = 1$ such that $\{Au_n\}$ is a null sequence.

4. Let $\{\lambda_n\}_1^\infty \subset \mathbb{C}$ be a sequence such that $|\lambda_n| \geq 1$ for all n. Let $\{\varphi_n\}_1^\infty$ be an orthonormal set in a Hilbert space. Consider a sequence of operators:

$$A_n u = \sum_{k=1}^{n} \frac{\langle u, \varphi_k \rangle}{\lambda_k} \varphi_k \,, \quad n = 1, 2, \ldots$$

(i) Use the Riesz-Fisher theorem to prove that the operator sequence converges in the norm, that is, there exists a unique v such that

$$\lim_{n\to\infty} \|A_n u - v\| = 0$$

and one can define

$$Au = \lim_{n\to\infty} A_n u$$

for any u in the Hilbert space.

(ii) Show that, if the series $\sum |\lambda_n|^{-1} < \infty$ converges, then the above operator sequence converges strongly (in the operator norm) and

$$\|A\| \leq \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|}$$

Hint: Show that

$$P_n u = \langle u, \varphi_n \rangle \varphi_n \quad \Rightarrow \quad \|P_n\| = 1$$

(iii) Determine the null space of A. Formulate the condition on the set $\{\varphi_n\}_1^\infty$ under which the null space contains only the zero element $u = 0$ and, hence, A has an inverse. In the latter case, find the inverse A^{-1}. Give an explicit form $A^{-1}f$ for any f from the range of A in terms of φ_n and λ_n. Show that the range of A consists of the elements

$$f \in R_A \;:\quad f = \sum_{n=1}^{\infty} f_n \varphi_n \,, \quad \sum_{n=1}^{\infty} |f_n|^2 \leq \sum_{n=1}^{\infty} |f_n|^2 |\lambda_n|^2 < \infty \,.$$

(iv) Investigate if A is bounded away from zero or not. In particular, consider the cases when $\{\lambda_n\}$ is bounded or not bounded, and when $\{\varphi_n\}$ is a basis or not a basis.

(v) Use the results of parts (iii) and (iv), to determine the conditions on $\{\lambda_n\}$ under which the inverse of A is bounded or unbounded.

5. Recall from Section **58.8** that the functions $\phi_n(x) = H_n(x)e^{-x^2/2}$, where $H_n(x)$ are the Hermit polynomials, $n = 0, 1, ...$, form an orthogonal basis in $\mathcal{L}_2(\mathbb{R})$, and

$$A\phi_n(x) = -\phi_n''(x) + x^2\phi_n(x) = \lambda_n\phi_n(x)\,, \quad \lambda_n = 2n + 1$$

Consider the linear problem

$$Au(x) = -u''(x) + x^2u(x) = f(x)\,, \quad u \in D_A \subset \mathcal{L}_2(\mathbb{R})\,, \quad f \in R_A$$

(i) Use the definition of an invertible operator to show that A is invertible. *Hint*: Prove that $\langle u, \phi_n\rangle = 0$ if $Au = 0$.

(ii) Show that

$$Au = f \quad \Rightarrow \quad u = A^{-1}f = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n\rangle}{\lambda_n\|\phi_n\|}\,\phi_n$$

Hint: Prove that $\langle u, \phi_n\rangle = \langle f, \phi_n\rangle/\lambda_n$ if $Au = f$.

(iii) Show that A^{-1} is a continuous operator. Show that A is an unbounded operator that is bounded away from zero.

56. An extension of an operator

DEFINITION **56.1**. (An extension of an operator)
An operator B is an extension *of an operator A if the domain of B includes the domain of A and the actions of B and A coincide on the domain of A:*

$$(1) \quad D_A \subset D_B\,,$$
$$(2) \quad Bu = Au\,, \quad \forall u \in D_A$$

56.1. An extension of a bounded operator in a Hilbert space. A linear bounded operator in a Hilbert is continuous. It turns out that any such operator can be extended to the whole Hilbert space. In this sense, linear bounded operators have properties identical to matrices in Euclidean spaces. One might even think of bounded operators and infinite dimensional matrices.

THEOREM **56.1**. (Extension of a bounded operator)
Let A be a linear bounded operator in a Hilbert space. Then A can be extended to the whole Hilbert space and the extension has the same norm:

$$A \;:\quad D_A \subset \mathcal{H} \to \mathcal{H} \quad \textit{linear}\,, \quad \|A\| < \infty$$
$$\Rightarrow \quad \exists B \;:\quad D_B = \mathcal{H} \to \mathcal{H}\,, \quad Bu = Au\,, \quad u \in D_A\,, \quad \|B\| = \|A\|$$

PROOF. Suppose first that D_A is a closed linear manifold:

$$\overline{D_A} = D_A\,.$$

Put

$$Bu = Au\,, \quad \forall u \in D_A\,, \quad Bu = 0\,, \quad u \in D_A^{\perp}$$

Clearly, B is linear if A is linear. Since D_A is a closed linear manifold, by the projection theorem (Theorem **57.3**), any $u \in \mathcal{H}$ can be uniquely represented as the sum of $v \in D_A$ and $w \in D_A^{\perp}$:

$$u = v + w\,, \quad v \in D_A\,, \quad w \in D_A^{\perp}$$

Then by linearity of B:

$$Bu = B(v+w) = Bv + Bw = Au + 0 = Au\,, \quad \forall u \in \mathcal{H}$$

It follows from this relation that

$$\|B\| = \|A\|$$

as required.

Suppose D_A is not closed,

$$D_A \subset \overline{D_A}\,.$$

In this case, let us fist extend to the closure $D_B = \overline{D_A}$, and then use the above construction to extend B to the whole Hilbert space. Let $\{u_n\}$ be a Cauchy sequence in D_A. By completeness of $\mathcal{H}$ there exists $u \in \overline{D_A}$ to which this sequence converges. Since $\|A\| < \infty$, the operator A is continuous and, hence, it maps any Cauchy sequence to a Cauchy sequence:

$$\|Au_n - Au_m\| = \|A(u_n - u_m)\| \leq \|A\|\|u_n - u_m\|$$

so that the right side can be made arbitrary small for all large enough n in m. Therefore there exists $f \in \mathcal{H}$ to which the sequence $\{Au_n\}$ converges. Put

$$Bu = \lim_{n\to\infty} Au_n\,, \quad \forall u \in \overline{D_A}$$

This definition does not depend on the choice of the Cauchy sequence converging to a given $u \in \overline{D_A}$. Indeed, if $\{u_n\}$ and $\{v_n\}$ are two sequences in D_A that converges to the same $u \in \overline{D_A}$. The the sequence $w_n = u_n - v_n$ is a null sequence, and $Aw_n \to 0$ as $n \to \infty$ by continuity of A. This implies that $\{Au_n\}$ and $\{Av_n\}$ have the same limit.

The extension B is bounded. Indeed, since the domain of B is closed, $D_B = \overline{D_A}$, by construction, for any $u \in D_B$ there exists a sequence $u_n \in D_A$ that converges to u. Therefore

$$\|Bu\| = \lim \|Bu_n\| = \lim \|Au_n\| \leq \|A\| \lim \|u_n\| = \|A\|\|u\|$$

Since this true for any $u \in D_B$,

$$\|B\| = \sup_{u\neq 0} \frac{\|Bu\|}{\|u\|} \leq \|A\|$$

Recall that for a bounded operator whose domain is closed, the supremum of $\|Bu\|/\|u\|$ can either be reached for some $u_0 \in D_B$ or occur on elements arbitrary close to zero. In the latter case, $\|B\| = \|A\|$ because $D_A \subset D_B$ and $0 \in D_A$ because D_A is a linear manifold. In the former case, if the supremum is reached on $u_0 \in D_A$, then again $\|A\| = \|B\|$. If the supremum reached on $u_0 \in \overline{D_A}$, but $u_0 \notin D_A$, then there exists a sequence $\{u_n\} \in D_A$, $\|u_n\| \neq 0$, that converges to u_0, one has

$$\|B\| = \lim_{n\to\infty} \frac{\|Bu_n\|}{\|u_n\|} = \lim_{n\to\infty} \frac{\|Au_n\|}{\|u_n\|} = \|A\|\,.$$

An extension to the whole $\mathcal{H}$ is obtained by setting $Bu = 0$ for any u from the orthogonal complement of $D_B^\perp = (D_A^\perp)^\perp = \overline{D_A}$. □

56.2. An extension of an unbounded operator. If an operator is not bounded, then it is not continuous. The continuity was necessary for an extension of a bounded operator. Note that if an operator A is not bounded, then the image $\{Au_n\}$ of a null sequence $\{u_n\} \in D_A$ is not necessarily a null sequence so that an extension to $\overline{D_A}$ used for bounded operators is no longer valid. If two sequences in D_A converge to the same element in $\overline{D_A}$, then the limits of their images under the action of A may not exist or, even if they both exist, they may not be the same.

Differential operators in $\mathcal{L}_2$ are the most common examples of unbounded operators. They are of fundamental significance for quantum physics and Fourier analysis.

Let us divide all unbounded operators into three classes by the properties of the images of null sequences.

$$\begin{aligned}
&\textsf{Class 1} : \quad \lim_{n\to\infty} u_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} Au_n = 0 \\
&\textsf{Class 2} : \quad \lim_{n\to\infty} u_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} Au_n \text{ does not exist} \\
&\textsf{Class 3} : \quad \lim_{n\to\infty} u_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} Au_n = f \neq 0
\end{aligned}$$

Operators from Class 3 will not be considered as they are somewhat "pathological" and are not common in applications. Note that if the above classification is adopted to all linear operators, then all bounded operators would be in Class 1.

Definition **56.2**. (Closable operator)
A linear operator is called closable *if for any null sequence $\{u_n\}$, the image sequence $\{Au_n\}$ either is a null sequence or has no limit.*

Naturally, all bounded operators are closable.

Differentiation operator is closable. An investigation if a particular unbounded operator is closable might be quite technical. The procedure is illustrated with the simplest example of the differentiation operator in $\mathcal{L}_2(0,1)$. Put

$$A \;:\quad D_A = C^1(0,1) \cap \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = u'(x)$$

The range of this operator is $R_A = C^0(0,1) \cap \mathcal{L}_2(0,1)$.

Proposition **56.1**. *A continuously differentiable and square integrable function on an open interval is bounded if its derivative is square integrable on the interval:*

$$\left.\begin{array}{l} u \in C^1(a,b) \cap \mathcal{L}_2(a,b) \\ u' \in \mathcal{L}_2(a,b) \end{array}\right\} \quad \Rightarrow \quad |u(x)| \leq M\,, \quad a < x < b$$

The conclusion of this proposition is not obvious because there are functions from $C^1(a,b) \cap \mathcal{L}_2(a,b)$ that are not bounded. For example, $u(x) = x^\nu$ is not bounded on $(0,1)$ if $0 > \nu > -\frac{1}{2}$ but it is continuously differentiable and square integrable on $(0,1)$.

Let us prove the proposition. By the fundamental theorem of calculus (by continuity of u'):

$$u(x) = u(c) + \int_c^x u'(t)dt \equiv u(c) + v(x)\,, \quad \forall x, c \in (a,b)$$

Then u is bounded if and only if v is bounded. If v is complex-valued, then it is bounded if and only if its real and imaginary parts are bounded. So, it is sufficient to show the boundedness of a real-valued v. One has

$$\begin{aligned}
v^2(x) &= \left|\int_c^x \int_c^x u'(t)u'(y)dtdy\right| \le \int_c^x \int_c^x |u'(t)||u'(y)|\,dtdy \\
&\overset{(1)}{\le} \frac{1}{2}\int_c^x \int_c^x \Big(|u'(t)|^2 + |u'(y)|^2\Big)\,dtdy \\
&\overset{(2)}{\le} \frac{1}{2}\int_a^b \int_a^b \Big(|u'(t)|^2 + |u'(y)|^2\Big)\,dtdy \\
&\overset{(3)}{=} \frac{l}{2}\|u'\|^2 + \frac{l}{2}\|u'\|^2 = l\|u'\|^2\,, \quad l = b - a
\end{aligned}$$

where (1) is by $|pq| \le \frac{1}{2}(p^2 + q^2)$; (2) follows from the non-negativity of the integrand and that $u' \in \mathcal{L}_2(a,b)$; (3) by evaluating the integrals. Thus,

$$|v(x)| \le \sqrt{l}\,\|u'\| \quad \Rightarrow \quad |u(x)| \le |u(c)| + \sqrt{l}\,\|u'\| < \infty$$

as required.

Now with the help of the above proposition, one can show that A is closable. Suppose that the converse is true, that is, A is not closable. Then there exists a null sequence $\{u_n\} \subset D_A$ such that

$$\lim u_n = 0 \quad \Rightarrow \quad \lim Au_n = f \neq 0$$

By continuity of the inner product

$$\lim\langle Au_n, g\rangle = \langle f, g\rangle \quad \forall g \in \mathcal{L}_2(0,1)$$

In particular, let g be from D_A and in addition it satisfies the zero boundary conditions

$$g(0) = g(1) = 0$$

for example, $g \in \mathcal{D}(0,1)$ (the space of test functions with support in $(0,1)$). Then

$$\langle Au_n, g\rangle = \int_0^1 u_n'(x)\overline{g(x)}\,dx \stackrel{(1)}{=} u_n(x)\overline{g(x)}\Big|_0^1 - \int_0^1 u_n(x)\overline{g'(x)}\,dx$$
$$\stackrel{(1)}{=} 0 - \langle u_n, g'\rangle = -\langle u_n, g'\rangle$$

where (1) is by integration by parts; (2) the boundary term vanishes because by the proposition $|u_n(x)|$ is bounded and $g(0) = g(1) = 0$. Taking the limit in the above relation and using the continuity of the inner product again, it is concluded that

$$\lim\langle Au_n, g\rangle = -\lim\langle u_n, g'\rangle \quad \Rightarrow \quad \langle f, g\rangle = 0$$

Now recall that $\mathcal{D}(0,1) \subset C^0([0,1])$ (the space of test functions) is dense in $\mathcal{L}$. So, for any $\varepsilon > 0$ and any $u \in \mathcal{L}_2(0,1)$ there is a test function $g \in \mathcal{D}(0,1)$ such that

$$\|u - g\| \le \varepsilon$$

Therefore

$$|\langle f, u\rangle| = |\langle f, u-g\rangle + \langle f, g\rangle| = |\langle f, u-g\rangle| \le \|f\|\|u-g\| \le \|f\|\varepsilon$$

This means that absolute value of the inner product of f with *any* element of the Hilbert space is smaller than any preassigned positive number, which means that the inner product is equal to zero:

$$\langle f, u\rangle = 0\,, \quad \forall u \in \mathcal{L}_2(0,1) \quad \Rightarrow \quad f = 0$$

Thus, $f = 0$. A contradiction. So, the differentiation operator is closable.

The reader is advised to repeat the above line of arguments to show that the second derivative operator

$$A \;:\quad D_A = C^2(0,1) \cap \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = -u''(x)$$

is closable, too.

56.3. An extension of a closable operator. Suppose A is a linear closable operator in a Hilbert space. Take a Cauchy sequence $\{u_n\}$ in the domain D_A. Then there is an element $u \in \mathcal{H}$ to which the sequence converges. Note that the limit point u is not generally in D_A. Suppose that the image of the sequence under the action of A also has a limit point:

$$\lim u_n = u \in \mathcal{H} \quad \Rightarrow \quad \lim Au_n = f \in \mathcal{H}$$

Take another sequence $\{v_n\}$ in the domain D_A that converges to the same element u. Since A is closable, there are only two following possibilities:

$$\text{Case 1}: \quad \lim A(u_n - v_n) = 0 \quad \Rightarrow \quad \lim Au_n = \lim Av_n = f$$
$$\text{Case 2}: \quad \lim A(u_n - v_n) \text{ does not exist or is not zero}$$

In Case 1, the limit of Au_n does not depend on the choice of the sequence just like in the case of a bounded operator, one can extend A to such u by

$$Au = \lim Au_n = f\,.$$

In the second case, no extension of A to u is possible. The domain D_A is extended by adding all $u \in \mathcal{H}$ for which the sequence $\{Au_n\}$ has a limit whenever $\{u_n\} \in D_A$ and $\{u_n\}$ converges to u. Clearly,

$$D_A \subseteq D_B \subseteq \overline{D_A}$$

because there are limit points of D_A for which no extension is possible.

Definition **56.3**. (The closure of an operator)
The closure $\overline{A}$ of a closable linear operator A is an extension of A such that for any $u \in D_{\overline{A}}$ there exists a sequence $\{u_n\}$ in the domain of A that converges to u and the sequence Au_n converges to $f \in \mathcal{H}$ in which case $\overline{A}u = f$.

Proposition **56.2**. (Properties of the closure)
If an operator is bounded, then the domain of the closure is closed:

$$\|A\| < \infty \quad \Rightarrow \quad D_{\bar{A}} = \overline{D_A}$$

If an operator is unbounded, then then the domain of the closure is not generally closed:

$$\|A\| = \infty \quad \Rightarrow \quad D_{\bar{A}} \subseteq \overline{D_A}$$

Note that any closed linear manifold in a Hilbert space is also a Hilbert space so that $\overline{D_A}$ is a Hilbert space. The second assertion follows from the fact that an unbounded operator cannot be generally extended to all limit points of D_A.

56.4. Closed operators. Evidently, no further extension of the closure of an operator can be made by the procedure given in the definition of the closure, that is,

$$\bar{\bar{A}} = \bar{A}$$

the closure of the closure of A is the closure of A. An operator whose closure does not change the operator is called *closed*.

DEFINITION **56.4.** (Closed operator)
A linear operator $A: \ D_A \subset \mathcal{H} \to \mathcal{H}$ *in a Hilbert space* $\mathcal{H}$ *is called* closed *if*

$$\begin{cases} u \in D_A \\ Au = f \in R_A \end{cases} \quad \textit{whenever} \quad \begin{cases} \{u_n\} \in D_A \\ \lim u_n = u \in \mathcal{H} \\ \lim Au_n = f \in \mathcal{H} \end{cases}$$

or

$$A \textit{ is closed} \quad \Leftrightarrow \quad \bar{A} = A$$

56.4.1. Properties of closed operators. In this section any operator is assumed to be linear, unless stated otherwise.

PROPOSITION **56.3**. *A closed linear operator with a closed domain is bounded:*

$$\left.\begin{matrix} A = \bar{A} \\ D_A = \overline{D_A} \end{matrix}\right\} \quad \Rightarrow \quad \|A\| < \infty$$

Let us show that A is continuous. Then the conclusion would follow from the linearity of A. For any u in the domain of $\overline{A}$, there exists a sequence $\{u_n\}$ in D_A such that $u_n \to u$ and $Au_n \to f$ (in which case $\overline{A}u = f$). Since the domain D_A is closed by the hypothesis, the limit point u belongs to D_A as well. The operator is closed. This means that $Au_n \to f = \overline{A}u = Au$ and, hence, A is continuous on D_A.

PROPOSITION **56.4**. *The inverse operator of a closed operator is closed:*

$$\left.\begin{matrix} A = \bar{A} \\ \exists\, A^{-1} \end{matrix}\right\} \quad \Rightarrow \quad \overline{A^{-1}} = A^{-1}$$

Let $\{f_n\}$ be a sequence in the domain of the inverse A^{-1}, which is the range of A, $D_{A^{-1}} = R_A$. Suppose that $f_n \to f \in \mathcal{H}$ and $A^{-1}f_n \to u \in \mathcal{H}$. One has to show that $f \in D_{A^{-1}} = R_A$ and $u \in R_{A^{-1}} = D_A$. Put $u_n = A^{-1}f_n$. Then $u_n \to u \in \mathcal{H}$ and $Au_n = f_n \to f \in \mathcal{H}$. The operator A is closed. This means that $u \in D_A$ and $f \in R_A$ (here $Au = f$) as required.

PROPOSITION **56.5**. *The inverse of a closed operator is bounded if and only if the range is closed:*

$$\left.\begin{matrix} A = \bar{A} \\ \exists\, A^{-1} \end{matrix}\right\} \quad \Rightarrow \quad R_A = \overline{R_A} \quad \Leftrightarrow \quad \|A^{-1}\| < \infty$$

Suppose the range is closed, $R_A = \overline{R_A}$. Under the hypotheses, the inverse is closed because A is closed (Proposition **56.4**). By the assumption, the domain of the inverse is closed because it is the range,

$D_{A^{-1}} = R_A$. A linear closed operator with closed domain is bounded (Proposition **56.3**). Therefore the inverse is bounded:

$$\left.\begin{array}{l} A^{-1} = \overline{A^{-1}} \\ R_A = \overline{R_{A^{-1}}} \end{array}\right\} \quad \Rightarrow \quad \|A^{-1}\| < \infty$$

Conversely, suppose that the inverse is bounded, $\|A^{-1}\| < \infty$. One has to show that every Cauchy sequence in the range R_A has a limit in it. Take a Cauchy sequence $\{f_n\} \subset R_A$. Then its image $\{A^{-1}f_n\}$ is a Cauchy sequence in the domain D_A because, by the hypothesis, the inverse is bounded and, hence, continuous. By the completeness of the Hilbert space, there exist $u \in \mathcal{H}$ and $f \in \mathcal{H}$ to which the above sequences converge:

$$u_n \to u \in \mathcal{H}\,, \quad Au_n = f_n \to f \in \mathcal{H}$$

The operator A is closed. This means that $u \in D_A$ and $f \in R_A$ as required.

56.5. The closure of differentiation operators in $\mathcal{L}_2$. Let $\mathcal{H} = \mathcal{L}_2(I)$ where I is any open interval in $\mathbb{R}$ (bounded or not). A natural domain of a differentiation operator is a subset of smooth functions C^p, $p \geq 1$, that are square integrable on I:

$$D_A \subset C^p(I) \cap \mathcal{L}_2(I)\,, \quad Au(x) = u'(x)\,, \quad p \geq 1\,.$$

Note that for different p, the operators A are different because they have different domains. Even for the same p, the domain can be further restricted by some boundary conditions at the endpoints of I leading to different operators. For example, for any differential operator one can always choose the "smallest" domain $D_A = \mathcal{D}(I)$ (test functions on I). The space of test functions is dense in any $C^p(I)$. If I is not the whole $\mathbb{R}$, then all derivatives $u^{(n)}$ vanish at the endpoints of I. Similarly, one can take D_A in $C^p(I)$ and the values of u and some of its derivatives are restricted by some linear boundary conditions.

Here no distinction between all differentiation operators will be made. The main question here is to investigate a space that contains $D_A \subseteq C^1(I) \cap \mathcal{L}_2(I)$ *and the domain of the closure of* A,

$$D_A \subset D_{\overline{A}} \subseteq \mathcal{L}_2(I)\,.$$

For example, the derivative of a piecewise smooth function does not exists everywhere but nonetheless can be square integrable. So, any such function is not from $C^1(I)$ but it can be in $D_{\overline{A}}$ because even the smallest domain $D_A = \mathcal{D}(I)$ is dense in $\mathcal{L}_2(I)$. The objective here is to find the largest space of functions in which the domain of the closure of a differentiation operator lies. The question about boundary conditions will be studied later.

For any $u \in D_A$, the Fundamental theorem of calculus holds:

$$u(x) = u(a) + \int_a^x Au(y)dy\,, \quad a, x \in I\,.$$

The closure of A is an extension of A such that if $u \in D_{\overline{A}}$, then there exists a sequence $\{u_n\} \subset D_A$ that converges to u and $Au_n \to f \in \mathcal{L}_2(I)$. So, to construct the closure, let us take $u \in \mathcal{L}_2(I)$. Since D_A is dense in $\mathcal{L}_2(I)$, there exists a sequence $u_n \in D_A$ that converges to u. If, in addition, this sequence has the property that $Au_n = u_n' \to f \in \mathcal{L}_2(I)$, then $u \in D_{\overline{A}}$ and $\overline{A}u = f$. Therefore

$$u_n(x) = u_n(a) + \int_a^x u_n'(y)\,dy \equiv u_n(a) + Bu_n'(x)$$

Let us assume for a moment (and for the sake of an argument) that the action of B and taking the limit $n \to \infty$ can be interchanged. Then by taking the limit one would infer that

$$u(x) = C + Bg(x) = C + \int_a^x g(y)\,dy = u(a) + \int_a^x f(y)\,dy$$

where $u_n(a) \to C$ and the constant C is fixed by setting $x = a$. It follows from this integral representation that $u \in C^0(I) \cap \mathcal{L}_2(I)$ by continuity of the Lebesgue integral. Note no restrictions on f is imposed but f must be locally integrable on I in order for the integral representation of u to hold. This is indeed true for any $f \in \mathcal{L}_2(I)$.

Proposition **56.6**. *Let Ω be any open set in $\mathbb{R}^N$. If $f \in \mathcal{L}_2(\Omega)$, then f is locally integrable on Ω*

One has to show that integral of $|f(x)|$ over any bounded open subset $\Omega_b \subset \Omega$ converges. If $\langle\cdot,\cdot\rangle_b$ denotes the inner product in $\mathcal{L}_2(\Omega_b)$ and $\|\cdot\|_b$ denotes the natural norm, then by the Cauchy-Schwartz inequality

$$\int_{\Omega_b} |v(x)|\,d^N x = \langle 1, |v|\rangle_b \le \|1\|_b \|v\|_b = \sqrt{\mu(\Omega_b)}\|v\|_b < \infty$$

where $\mu(\Omega_b) < \infty$ is the Lebesgue measure of Ω_b, and

$$\|v\|_b^2 = \int_{\Omega_b} |v(x)|^2\,d^N x \le \int_{\Omega} |v(x)|^2\,d^N x = \|v\|^2 < \infty\,.$$

Let us investigate the limit of Bv_n'. It follows from Proposition **56.6** that the antiderivative operator B is defined for any $f \in \mathcal{L}_2(I)$ and, hence, for any $f \in \mathcal{L}_2(b,c)$ where $(b,c) \subset I$ and $a \in (b,c)$:

$$B:\ D_B = \mathcal{L}_2(b,c) \to \mathcal{L}_2(b,c)\,, \quad Bv(x) = \int_a^x v(y)\,dy\,, \quad a \in (b,c)\,.$$

If $v_n' \to f$ in $\mathcal{L}_2(I)$, then $v_n' \to f$ in $\mathcal{L}_2(b,c)$. So the sequence of derivatives v_n' converges to f in the mean on any bounded interval (b,c). Next, let us show that B is a bounded operator on $\mathcal{L}_2(b,c)$ and, hence, is continuous. This implies that for any $u \in D_{\overline{A}}$

$$u(x) = u(a) + \int_a^x f(y)\,dy\,, \quad x, a \in (b,c) \tag{56.1}$$

for any bounded interval $(b,c) \subset I$ where $f \in \mathcal{L}_2(I)$. One has by the Cauchy-Schwartz inequality in $\mathcal{L}_2(b,c)$ that

$$\begin{aligned} |Bv(x)| &\le \int_a^x |v(y)|\,dy \le \int_b^c |v(x)|\,dx = \langle 1, |v|\rangle \\ &\le \|1\|\|v\| = \sqrt{c-b}\|v\| \\ \Rightarrow \quad \|Bv\|^2 &= \int_b^c |Bv(x)|^2 dx \le (c-b)^2\|v\|^2 \\ \Rightarrow \quad \|B\| &\le c - b < \infty\,. \end{aligned}$$

Therefore B is bounded and, hence, continuous on $\mathcal{L}_2(0,1)$.

Functions that have a characteristic property (**56**.1) where f is any locally integrable function on I have some remarkable properties. Let us investigate them.

56.5.1. Absolutely continuous functions. Suppose that a function f is monotonic and its derivative exists almost everywhere and is Lebesgue integrable. Owing to a possible lack of continuity of f', the fundamental theorem of calculus does not hold:

$$\int_a^b f'(x)dx \le f(b) - f(a) \quad \forall a < b$$

For example, take a monotonically increasing piece-wise constant function. Then its derivative is zero almost everywhere and its integral vanishes on any interval, whereas $f(b) - f(a)$ is either equal zero if a and b lie in the same interval of continuity or it is some positive number if a and b are in different intervals of continuity.

It is interesting to note that there are functions for which the inequality is *strict* for any choice of a and b. The most famous example of this sort is the so called *Cantor ladder*, a function that is continuous, monotonically increasing, and whose derivative vanishes almost everywhere[3]

$$f(x) \in C^0 ; \quad f'(x) = 0 \ a.e. ; \quad f(x) < f(y) , \ x < y$$

One can always set $f(0) = 0$ and, in this case, by monotonicity

$$0 = \int_0^x f'(y)dy < f(x)$$

that is, the inequality is strict.

It is therefore natural to ask about the largest set of functions for which the fundamental theorem of calculus holds if the integral in it is understood in the Lebesgue sense. The answer is well known [4]. Here only some basic facts, that are necessary for what follows, about these functions are given without proofs.

DEFINITION **56.5**. (Absolutely continuous functions)
A function u is called absolutely continuous on an interval I if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of non-overlapping intervals $I_j = (a_j, b_j) \subset I$, $j = 1, 2, ..., n$, $I_j \cap I_k = \emptyset$, $j \neq k$, the total absolute variation of u on these intervals does exceed ε whenever the total length of the intervals does not exceed δ:

$$\sum_{j=1}^{n} |u(b_j) - u(a_j)| < \varepsilon \quad \text{whenever} \quad \sum_{j=1}^{n} |b_j - a_j| < \delta$$

Clearly every absolutely continuous function is continuous. It turns out that one can prove that absolutely continuous functions are differentiable almost everywhere, the derivative is locally integrable in the Lebesgue sense, and the Fundamental theorem of calculus holds for them. The converse is also true

THEOREM **56.2**. (Absolutely continuous functions)
For every absolutely continuous function u on $[a, b]$, there exists a Lebesgue integrable function $f \in \mathcal{L}(a, b)$ such that

$$u(x) = u(a) + \int_a^x f(y)\, dy$$

and in this case

$$u'(x) = f(x) \quad a.e.$$

[3] A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter VI, Sec. 4

[4] A.N. Kolmogorov and S.V. Fomin, Elements of the theory of functions and functional analysis, Chapter VI

Owing to this theorem, in what follows it will be used as the definition of absolutely continuous functions. The set of absolutely continuous functions on an interval I will be denoted as $AC^0(I)$.

The set AC^0 is a linear space with respect the usual addition of functions and multiplication of a function by a number. It is larger than the set of Lipschitz continuous functions but smaller than the set of function differentiable almost everywhere, and it is also larger than C^1 but smaller than C^0:

$$\{\text{Lipschitz continuous}\} \subset AC^0 \subset \{\text{differentiable a.e.}\}$$
$$C^1 \subset AC^0 \subset C^0$$

56.5.2. Integration by parts. The integration by parts follows from the identity $(uv)' = u'v + uv'$ and the fundamental theorem of calculus which is applicable in the Riemann integration theory if u and v are from class C^1. In the Lebesgue theory, the fundamental theorem of calculus is extended to absolutely continuous functions. Therefore, the integration by part is valid for absolutely continuous functions:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int u'(x)v(x)\,dx\,, \quad u, v \in AC^0$$

56.5.3. The closure of differentiation operators (continued). It follows from the above analysis that the domain of the closure of a differentiation operator in $\mathcal{L}_2(I)$ lies in the space of absolutely continuous functions that are square integrable on I and whose derivatives are also square integrable on I:

$$D_{\bar{A}} \subseteq \{\, u \in AC^0(I) \cap \mathcal{L}_2(I) | u' \in \mathcal{L}_2(I)\,\} \subset \mathcal{L}_2(I)\,,$$
$$\bar{A}u(x) = u'(x) \ a.e.$$

If $I = (b, c)$ is bounded and no boundary conditions are imposed on absolutely continuous functions from D_A, then the range of the closure is the whole space $\mathcal{L}_2(b, c)$.

The operators of derivatives of higher-orders can be closed in a similar fashion. If $\{u_n\} \subset D_A$ and $Au(x) = u^{(p)}(x)$, then

$$u_n^{(q-1)}(x) = u_n^{(q-1)}(a) + \int_a^x u_n^{(q)}(y)\,dy\,, \quad q = 1, 2, ..., p\,.$$

If one demands that $u_n^{(p)} \to f_p \in \mathcal{L}_2(I)$, then $u_n^{(p-1)}$ must converge in the mean to $f_{p-1}(x) = g_{p-1}(a) + \int_a^x f_p(y)\,dy$ and so on

$$u_n^{p-q}(x) \to f_{p-q}(x) = f_{p-q}(a) + \int_a^x f_{p-q+1}(y)\,dy\,, \quad q = 1, 2, ..., p$$

on any interval (b, c). So, the domain of $\overline{A}$ lies in the of functions whose derivatives up to order p are square integrable and the derivative of order $p-1$ is absolutely continuous. If derivatives of a functions are absolutely continuous up to order p, then the function is said to be from class AC^p. So, the closure of a differential operator of order p lies in $AC^p(I) \cap \mathcal{L}_2(I)$.

The closure of a differential operator is not bounded. For example, if $Au = u^{(p)}$ in $\mathcal{L}_2(a, b)$, then for $u = e^{kx} \in D_{\bar{A}}$ so that

$$\frac{\|\bar{A}u\|}{\|u\|} = \frac{\|u'\|}{\|u\|} = |k|^p$$

Since k is arbitrary, $\|\bar{A}\| = \infty$. Note that the domain of the closure of a differential operator is not closed because it contains only functions from AC^0 class that is a proper subset of $\mathcal{L}_2$. So, the second hypothesis of Proposition **56.3** is not fulfilled.

56.5.4. Boundary conditions and the closure of a differential operator. If A is a differential operator in $\mathcal{L}_2(a, b)$, then functions from its domain can be required to satisfy some boundary condition at the endpoints of the interval. In this case, one can ask what happens to the boundary conditions upon closing the operator. Do the boundary conditions survive the closure? It turns out that the boundary condition may not survive the closure. Each boundary condition must be investigated when closing a differential operator to see if it survives or does not. The assertion is illustrated by a few examples.

Put

$$D_{A_1} = \{u \in C^1([0,1]) \,|\, u(0) = 0\}$$
$$A_1 : \; D_{A_1} \subset \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad A_1 u(x) = u'(x)$$

Let us construct the closure of A_1 and verify whether the boundary condition survives the closure. The domain of A_1 is dense in $\mathcal{L}_2(0, 1)$. Therefore for any $u \in \mathcal{L}_2(0, 1)$, there exists a sequence $\{u_n\} \subset D_{A_1}$ that converges to u. The difference with the previously studied case is that the sequence $\{u_n\}$ satisfies the boundary condition $u_n(0) = 0$. This implies that

$$u_n(x) = \int_0^x u_n'(x)\, dx = Bu_n'(x)$$

and owing to continuity of B in $\mathcal{L}_2(0,1)$, the condition that $\lim u_n' = f \in \mathcal{L}_2(0,1)$ implies that the domain of the closure $\overline{A}$ consists of absolutely continuous functions on $[0,1]$ of the form

$$u(x) = \int_0^x f(y)\,dy\,.$$

By continuity of the Lebesgue integral, $u(0) = 0$. Therefore,

$$D_{\bar{A}_1} = \{\, u \in AC^0[0,1] \,|\, u(0) = 0\,, u' \in \mathcal{L}_2(0,1)\}\,, \quad \bar{A}_1 u(x) = u'(x)\,a.e.$$

Thus, the only boundary condition survives the closure.

Now put

$$D_{A_2} = \{u \in C^1([0,1]) \,|\, u(0) = 0 \;\; u'(0) = 0\}\,, \quad A_2 u(x) = u'(x)$$

The domain is dense in $\mathcal{L}_2(0,1)$ so that for any $u \in \mathcal{L}_2(0,1)$ there is a sequence $u_n \in D_{A_2}$ that converges to u. Since $u_n(0) = 0$, the domain of the closure of A_2 consists of absolutely continuous functions that have the same integral representation as in the previous case. Therefore the boundary condition $u(0) = 0$ survives the closure. However the second boundary condition does not survive the closure because

$$u'(x) = f(x) \quad a.e. \qquad \Rightarrow \qquad u'(0) = f(0) \neq 0$$

as f is a general square integrable function. Thus, $D_{\bar{A}_2} = D_{\bar{A}_1}$.

Let

$$D_{A_3} = \mathcal{D}(0,1)\,, \quad A_3 u(x) = u'(x)$$

Any function from the domain vanishes at the endpoints together with any derivative, $u^{(p)}(0) = u^{(p)}(1) = 0$ for any non-negative integer p. Let us investigate what happens to these boundary conditions upon the closure of A_3. The domain is dense in $\mathcal{L}_2(0,1)$ and, hence, for any $u \in \mathcal{L}_2(0,1)$, there exists a sequence $\{u_n\}$ of test functions that converges to u. Since $u_n(0) = u_n(1) = 0$, there are two equivalent integral representations

$$u_n(x) = \int_0^x u_n'(y)\,dy = \int_1^x u_n'(y)\,dy$$

Each of these integrals can be viewed as the action of a bounded operator in $\mathcal{L}_2(0,1)$ on the sequence $\{u_n'\}$. Therefore the domain of the closure of A_3 consists of absolutely continuous functions that have two integral representations

$$u(x) = \int_0^x f(y)\,dy = \int_1^x f(y)\,dy\,.$$

This shows two things. First, the boundary conditions $u(0) = u(1) = 0$ survives the closure, while the zero conditions on the derivatives do not survive the closure because $u'(x) = f(x)$ a.e. but $f(0) \neq 0$ and $f(1) \neq 0$. The higher order derivatives do not even exist at the and points for a generic $f \in \mathcal{L}_2(0,1)$. Second, the range of the closure is a proper subset of $\mathcal{L}_2(0,1)$ because f must be orthogonal to the unit function, in contrast to the two previous cases in which the range of the closure is the whole Hilbert space $\mathcal{L}_2(0,1)$. Thus,

$$D_{\bar{A}_3} = \{\, u \in AC^0[0,1] \,|\, u(0) = u(1) = 0\,, u' \in \mathcal{L}_2(0,1)\}\,,$$
$$\bar{A}_3 u(x) = u'(x)\ a.e.$$

56.6. Classification of operators for solvability $Au = f$. Let A be a linear operator. A linear problem

$$Au = f\,, \quad A\,:\, D_A \to \mathcal{H}\,,$$

of finding $u \in D_A$ for a given $f \in \mathcal{H}$ often appears in applications. The following questions are significant for solving the problem and properties of the solution:

- Does the inverse of A exist?
- Does a solution exist for any $f \in \mathcal{H}$?
- Does the solution depend continuously on f?

The first question is about the existence and uniqueness of the solution. The second equation is about the range of A. If $R_A = \mathcal{H}$, then the answer is affirmative. The third question is about stability of the solution under small variations of f. Its practical significance is the following. Let $\{f_n\}$ be a sequence of successive approximations to f, $\|f - f_n\| \to 0$ as $n \to \infty$. Suppose A is invertible. Does the sequence of the solutions $u_n = A^{-1}f_n$ converge to $u = A^{-1}f$? If the inverse is bounded, then the answer is affirmative, otherwise no conclusion can be given. In addition, if $\{f_n\}$ is a Cauchy sequence in the range R_A and $\|A^{-1}\| < \infty$, then $\{A^{-1}f_n\}$ is also a Cauchy sequence and, hence, has a limit in $\mathcal{H}$. Is the limit point a solution to the problem? If the range is not closed, then $\{f_n\}$ can converge to some f that is not in the range of A and, hence, $\{A^{-1}f_n\}$ converges to some u that is not in the domain of A and therefore cannot be a solution to the problem. In applications, f can depend on various parameters and the behavior of the solution under variations of these parameters is of great significance. From these perspectives, all operators can be classified by two sets of properties, properties of A^{-1} and properties of the range R_A.

Properties of A^{-1}.

(I)	$\|A^{-1}\| < \infty$	(there exists a bounded inverse)
(II)	$\|A^{-1}\| = \infty$	(there exists a unbounded inverse)
(II)	A is not invertible	

Properties of R_A.

(1c)	$R_A = \overline{R_A} = \mathcal{H}$	(the range is dense and closed)
(1n)	$R_A \subset \overline{R_A} = \mathcal{H}$	(the range is dense, but not closed)
(2c)	$R_A = \overline{R_A} \subset \mathcal{H}$	(the range is closed, but not dense)
(2n)	$R_A \subset \overline{R_A} \subset \mathcal{H}$	(the range is not dense and not closed)

The numbers 1 and 2 indicate whether the range is dense (1) or not (2), and the letter c or n stands for a closed or not closed range, respectively. So, all operators are divided into classes (μ, ν) where μ indicate the property of A^{-1} (that is, I, or II, or III), while ν indicates the property of the range in a similar fashion.

The existence of A^{-1} can be studied by investigating solutions to the homogeneous problem $Au = 0$. If the null space of A contains only the zero element, then A^{-1} exists. It is bounded if and only if A is bounded away from zero. These two criteria help to find the first index without an explicit solution to the linear problem.

Class (I,1c). Operators from this class are called regular or well-posed. The linear problem has a unique solution for any $f \in \mathcal{H}$ and the solution depends continuously on f:

$$\lim f_n = f \quad \Rightarrow \quad \lim A^{-1} f_n = A^{-1} f$$

because A^{-1} is continuous (it is bounded) so that its maps every convergent sequence to a convergent sequence and $D_{A^{-1}} = R_A = \mathcal{H}$ so that $A^{-1} f \in D_A$. This is the most desirable class of operators for linear problems.

Class (II,1c). The linear problem still has a unique solution for any $f \in \mathcal{H}$. However the solution does not depend continuously on variations of f. If one takes a sequence of successive approximations to f, $\lim f_n = f$, then the corresponding sequence of solution would not generally converge to the solution. In fact, the sequence of approximate solutions $\{A^{-1} f_n\}$ might have no limit at all. So, depending on f and the choice of $\{f_n\}$ there are three possibilities

$$\lim A^{-1}f_n = A^{-1}f$$
$$\lim A^{-1}f_n = g \neq A^{-1}f$$
$$\lim A^{-1}f_n \text{ does not exist}$$

In other words, any approximation methods for solving the linear problem with operators from this class require additional investigations to show that a sequence of approximate solutions converges to the solution that is sought-after.

Class (I,1n). The linear problem does not have a solution for any $f \in \mathcal{H}$, but it does have a unique solution if $f \in R_A$ and the solution depends continuously on f. The difference with the class (I,1c) is that one has to be careful with the choice of f. If $\{f_n\}$ is a Cauchy sequence in the range, then the sequence of solutions $A^{-1}f_n$ has a limit in $\mathcal{H}$ (by boundedness of A^{-1}), but the limit point might not be from the domain of A because $\{f_n\}$ might converge to some f not from R_A (because R_A is not closed).

Class (II,1n). The linear problem does not have a solution for any $f \in \mathcal{H}$, but it does have a unique solution if $f \in R_A$. However, the solution does not depends continuously on f.

Other classes. The linear problem for an operator from class (III,ν) may have no solution, and, if a solution u exists, then it is not unique because $u+u_0$ is also a solution where $Au_0 = 0$ and $u_0 \neq 0$ (a non-trivial u_0 always exists because A is not invertible).

56.6.1. The closure of an operator and well-posedness of a linear problem. An extension of an operator enlarges the domain and, possibly, the range. In particular, this is the case for the closure of an operator. So, if the operator in the linear problem is replaced by its closure then the well-posedness of the linear problem can be improved especially in the case of differential operators.

To illustrate the point, $\mathcal{H} = \mathcal{L}_2(0,1)$ and consider a simple problem $Au = f$ where A is the derivative operator. Let

$$D_A = C^1[0,1], \quad Au(x) = u'(x)$$

so that $R_A = C^0[0,1]$. The operator does not have the inverse because $Au(x) = 0$ has a non-zero constant solution that belongs to D_A. The range is not closed. So, this problem is from class (II,1n). The range of the closure $\bar{A}$ is the whole Hilbert space. So, the problem $\bar{A}u = f$ has a better class (II,1c). A solution is now required to satisfy the equation a.e., that is, $u'(x) = f(x)$ a.e. for any $f \in \mathcal{L}_2(0,1)$.

The non-existence of the inverse is obviously due to a lack of boundary conditions. However, imposing boundary conditions implies changing the domain of the operator and some boundary conditions may not survive the closure. Nonetheless, closing the operator in the linear problem promotes the problem to a better class.

Consider the linear problem for the operator A_1 from Sec. **56.5.4**. The range $R_{A_1} = C^0[0,1]$ is a proper subset of $\mathcal{L}_2(0,1)$ but it is closed in $\mathcal{L}_2(0,1)$. The operator is invertible because it follows from $u'(x) = 0$ that $u(x) = 0$ if $u(0) = 0$. The explicit form of the inverse reads

$$u(x) = A^{-1}f(x) = \int_0^x f(y)\,dy\,, \quad f \in R_{A_1}\,.$$

The inverse is bounded as shown in Sec. **56.5.4**. This operator from class (I,1n). The boundary condition $u(0) = 0$ survives the closure and the range becomes the whole $\mathcal{L}_2(0,1)$. Thus, after closing A_1, the linear problem becomes well-posed.

For the operator A_2 in Sec. **56.5.4**, the range R_{A_2} consists of continuous functions on $[0,1]$ that vanish at $x = 0$. It is a subset of R_{A_1}. The operator is invertible by the same reason as A_1 and the inverse is bounded

$$u(x) = A_2^{-1}f(x) = \int_0^x f(y)\,dy\,, \quad f \in C^0[0,1]\,,\ f(0) = 0\,.$$

So, the linear problem is from class $(I,1n)$. The boundary condition $u'(0) = 0$ does not survive the closure and $\bar{A}_2 = \bar{A}_1$. Closing A_2 promotes the problem to well-posed one. This also shows that the boundary condition $u'(0) = 0$ should be discarded in the first place to obtain a well-posed problem.

For the operator A_3 in Sec. **56.5.4**, the range R_{A_3} consists of test functions from $\mathcal{D}(0,1)$ whose integral over $(0,1)$ vanishes. The equation $u'(x) = 0$ has only the trivial solution in $\mathcal{D}(0,1)$ and, hence, A_3 is invertible

$$u(x) = A_3^{-1}f(x) = \int_0^x f(y)\,dy\,, \quad f \in \mathcal{D}(0,1)\,,\ \int_0^1 f(x)\,dx = 0\,.$$

This problem is from class (I,2n) because the range is in the orthogonal complement to all constant functions in $\mathcal{L}_2(0,1)$ so that $\overline{R_{A_3}} \subset \mathcal{L}_2(0,1)$. By closing A_3, the problem is promoted to a better class $(I,2c)$. The range of the closure is the whole orthogonal complement of the space of constant functions. It is closed by the projection theorem.

56.7. Exercises.

1. The closure of the second derivative operator. Let

$$A\ :\quad D_A = C^2([0,1]) \subset \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,,\quad Au(x) = u''(x)$$

Then the operator A is closable and its closure is

$$\bar{A}\ :\quad D_{\bar{A}} = AC^1[0,1] \to \mathcal{L}_2(0,1)\,,\quad Au(x) = u''(x)$$

where AC^p, $p = 0, 1, ..$, denotes a space of functions whose derivatives up to order p are absolutely continuous, $u^{(p)} \in AC^0$. Prove this assertion.

(i) Show that for any $u \in \mathcal{L}_2(0,1)$ there is a sequence $\{u_n\} \subset D_A$ that converges to u.

(ii) Suppose that $\lim Au_n = g \in \mathcal{L}_2(0,1)$. Show that for any $v \in D_A$

$$w(x) = w(0) + w'(0)x + \int_0^x \int_0^y w''(z)\,dz\,dy \equiv w(0) + w'(0)x + Bw''(x)\,.$$

(iii) Show that

$$\lim BAu_n = Bg$$

Use that

$$\frac{d}{dx}\int_0^x f(y)dy = f(x)\,,\quad \forall f \in AC^0[0,1] \subset C^0([0,1])$$

to show that $Bg \in AC^1[0,1]$, that is, $(Bg)' \in AC^0[0,1]$.

(iv) Show that the sequence of linear functions $v_n(x) = u_n(0) + u_n'(0)x$ converges to a linear function such that

$$u(x) = u(0) + u'(0)x + Bg(x) \in AC^1[0,1]$$

which proves the assertion.

2. Let $A :\ D_A \subset \mathcal{H} \to \mathcal{H}$, and $\mathcal{H} = \mathcal{L}_2(0,1)$ and

$$D_A = \{\, u \in C^1([0,1])\,|\, u(0) = au(1)\,, a \in \mathbb{C}\,\}\,,\quad Au(x) = u'(x)$$

(i) Show that the boundary condition survives the closure:

$$D_{\bar{A}} = \{\, u \in AC^0[0,1]\,|\, u(0) = au(1)\,\}\,,\quad \bar{A}u(x) = u'(x)\,a.e.$$

and the range of the closure is $R_{\bar{A}} = \mathcal{L}_2(0,1)$.

(ii) Show that the closure is invertible if $a \neq 1$ and

$$\bar{A}u = f \in R_{\bar{A}} \quad \Rightarrow \quad u(x) = \bar{A}^{-1}f(x) = \int_0^1 G_a(x,y)f(y)\,dy$$
$$G(x,y) = \frac{a}{1-a}\,\theta(y-x) + \frac{1}{1-a}\,\theta(x-y)$$
$$\frac{d}{dx}G(x,y) = \delta(x-y)\,, \quad G(0,y) = aG(1,y)\,, \quad y \in (0,1)$$

(iii) Show that the range of the closure is closed and that the inverse of the closure is bounded, $\|\bar{A}^{-1}\| < \infty$.

(iv) Show that $A \in (I, 1n)$ and $\bar{A} \in (I, 1c)$.

3. Let $A: \ D_A \subset \mathcal{H} \to \mathcal{H}$, and $\mathcal{H} = \mathcal{L}_2(0,1)$ and

$$D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = u(1) = 0 \,\}\,, \quad Au(x) = u'(x)$$

(i) Show that the boundary conditions survive the closure and

$$D_{\bar{A}} = \{\, u \in AC^0[0,1] \,|\, u(0) = u(1) = 0 \,\}\,, \quad \bar{A}u(x) = u'(x)\, a.e.$$

(ii) Show that the closure is invertible if $a \neq 1$ and

$$\bar{A}u = f \in R_{\bar{A}} \quad \Rightarrow \quad u(x) = \bar{A}^{-1}f(x) = \int_0^x f(y)\,dy$$

(iii) Show that the inverse of the closure is bounded, $\|\bar{A}^{-1}\| < \infty$, and that the range of the closure is closed, $R_{\bar{A}} = \overline{R_{\bar{A}}}$ using the properties of closed operators.

(iv) Show that the range of the closure is the proper subset in the Hilbert space, $R_{\bar{A}} \subset \mathcal{L}_2(0,1)$. In particular, $\langle 1, f\rangle = 0$ for any $f \in R_{\bar{A}}$, that is, f is orthogonal to any constant function.

(v) Show that $A \in (I, 2n)$ and $\bar{A} \in (I, 2c)$.

4. Let $A: \ D_A \subset \mathcal{H} \to \mathcal{H}$, and $\mathcal{H} = \mathcal{L}_2(0,1)$ and

$$D_A = \{\, u \in C^1([0,1]) \,|\, u'(0) = 0 \,\}\,, \quad Au(x) = u'(x)$$

(i) Show that the boundary condition does not survive the closure and

$$D_{\bar{A}} = AC^0[0,1]\,, \quad \bar{A}u(x) = u'(x)\, a.e.$$

Hint: Follow the procedure for constructing the closure of the derivative operator on $D_A = C^1([0,1])$ (no boundary condition). Does the stated boundary condition affects the procedure?

(ii) Show that the closure is not invertible.

(iii) Show that the range of the closure is closed and

$$R_{\bar{A}} = \overline{R_{\bar{A}}} = \mathcal{L}_2(0,1)\,.$$

(iv) Show that $A \in (III, 1n)$ and $\bar{A} \in (III, 1c)$.

5. Show that the multiplication operator

$$Au(x) = xu(x)\,, \quad D_A = \mathcal{L}_2(0,1)$$

is from class (II,1n).

6. The shift operator A in a separable Hilbert space is defined by the property

$$A\phi_k = \phi_{k+1}\,, \quad k = 1, 2, ...$$

where $\{\phi_k\}_1^\infty$ is an orthonormal basis in the Hilbert space. Show that $\|A\| \le 1$ and $A \in (II, 1n)$.

6. A modified shift operator A in a separable Hilbert space is defined by the property

$$A\phi_k = \frac{1}{k^2}\,\phi_{k+1}\,, \quad k = 1, 2, ...$$

where $\{\phi_k\}_1^\infty$ is an orthonormal basis in the Hilbert space. Show that A is bounded and invertible. In particular,

$$\|A\| = 1\,, \quad N_A = \{0\}$$

and its range is

$$R_A = \left\{ \sum_k \alpha_k \varphi_k \,\middle|\, \alpha_1 = 0\,,\ \sum_k k^4 |\alpha_k|^2 < \infty \right\}$$

and deduce from the above properties that $A \in (II, 2n)$.

7. Let

$$A:\ D_A = \mathcal{D}(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = u'(x)$$

(i) What is the range of A?
(ii) Find the closure $\bar{A}$.
(iii) Find the range of the closure. Hint: If $u \in D_A$, then $u(0) = u(1) = 0$. Do these boundary conditions survive the closure?
(iv) Classify the closure.

57. Linear functionals in a Hilbert space

DEFINITION **57.1**. *Let $M \subset \mathcal{H}$ be a linear manifold in a Hilbert space. A* linear functional *on M is a function*

$$l: \quad M \to \mathbb{F}$$

satisfying the linearity property

$$l(\alpha u + \beta v) = \alpha l(u) + \beta l(v)\,, \quad \forall u, v \in M\,, \quad \forall \alpha, \beta \in \mathcal{F}$$

where $\mathbb{F}$ is either $\mathbb{R}$ (a real functional) or $\mathbb{C}$ (a complex functional)

Note that any linear manifold contains the zero element because $0 \cdot u = 0$. Therefore by the linearity property

$$l(0) = 0$$

For example, if $M = \mathbb{R}^N$ then any linear functional can be written in the form

$$l(x) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N = \langle \alpha, x \rangle\,, \quad \forall x \in \mathbb{R}^N$$

and some $\alpha \in \mathbb{R}^N$. So, a linear functional on a Hilbert space is, loosely speaking, a linear homogeneous function of *infinitely many variables.* It is worth noting that it is far from obvious that every linear functional can be written in the form of an inner product with some element of a Hilbert space.

Consider the following three functionals defined for some $v \in \mathcal{H}$ and some number $\alpha \in \mathbb{F}$:

$$\begin{aligned} &(1) \quad l(u) = \langle u, v \rangle\,, \\ &(2) \quad l(u) = \langle v, u \rangle + \alpha\,, \\ &(3) \quad l(u) = \langle v, u \rangle\,, \end{aligned}$$

If $\mathbb{F} = \mathbb{C}$, then only (1) is a linear functional. If $\mathbb{F} = \mathbb{R}$, then only (2) is not a linear functional. The reader is asked to explain why.

Recall also that distributions are linear functionals on the space $\mathcal{D}$ of test functions which is a linear manifold. However, a liner functional is required to be continuous on $\mathcal{D}$ in order to define a distribution.

57.1. Continuous linear functionals.

DEFINITION **57.2**. (Continuous linear functional)
A linear functional l is continuous at an element *$v \in M$ of a linear manifold M if for any positive number $\varepsilon > 0$ there exists a positive number $\delta > 0$ such that*

$$|l(u) - l(v)| < \varepsilon \quad \textit{whenever} \quad \|u - v\| < \delta$$

A linear functional l is said to be continuous on M if it is continuous at any element of M.

As in the case of distributions, this definition means that a continuous functional maps a convergent sequence in M onto a convergent numerical sequence. The latter property is necessary and sufficient for a linear functional to be continuous.

THEOREM **57.1**. (Continuous linear functional)
A liner function l is continuous on a linear manifold M if and only if for any convergent sequence, $u_n \to u$ in M as $n \to \infty$, the numerical sequence $l(u_n)$ converges to $l(u)$.

A proof of this assertion is left to the reader as an exercise.

Since l is linear, a continuity at a particular element of M implies continuity at any point.

THEOREM **57.2**. *If a linear functional on M is continuous at $v = 0$, then it is continuous at any $v \in M$.*

Indeed, suppose $u_n \to u \in M$ as $n \to \infty$. Put $v_n = u_n - u$. Then $v_n \to 0$ in M. By continuity at $v = 0$ and linearity of l it follows that

$$\lim_{n\to\infty} l(v_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \Big(l(u_n) - l(u) \Big) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} l(u_n) = l(u) \,.$$

which is true for any $u \in M$.

Continuity of a linear functional and topology in its domain. Consider the following functional on a linear manifold $M = C_2^0([-1,1])$ defined by

$$l(u) = u(0) \,, \quad \forall u \in C_2^0([-1,1])$$

This functional is similar to the Dirac delta-function. It is obviously linear. Let us investigate its continuity. Note that $C_2^0([-1,1]) \subset \mathcal{L}_2(-1,1)$ so that the convergence is understood with respect to the natural norm of $\mathcal{L}_2(-1,1)$. Let f be continuous on $\mathbb{R}$ such that $f(x) = 0$ if $|x| > 1$. Put $u_n(x) = f(nx) \in C_2^0([-1,1])$ and, hence, $u_n(x) = 0$ if $|x| > \frac{1}{n}$. Therefore the sequence $\{u_n(x)\}$ converges pointwise:

$$\lim_{n\to 0} u_n(x) = u(x) = \begin{cases} 0, & x \neq 0 \\ f(0), & x = 0 \end{cases}$$

The limit function $u(x) = 0$ *a.e.* and, hence,

$$\lim_{n\to\infty} u_n = 0 \quad \text{in } \mathcal{L}_2(-1,1) \,.$$

Alternatively, by a direct calculation

$$\|u_n\|^2 = \int_{-1}^{1} |f(nx)|^2 dx = \frac{1}{n} \int_{-n}^{n} |f(y)|^2 dy = \frac{1}{n} \int_{-1}^{1} |f(y)|^2 dy = \frac{\|f\|^2}{n}$$

because $f(x) = 0$ if $|x| > 1$. This shows that $\|u_n\| \to 0$ as $n \to \infty$ and, hence, $u_n \to 0$ in $\mathcal{L}_2(-1,1)$ as $n \to \infty$. However,

$$l(u_n) = u_n(0) = f(0) \neq 0$$

Thus, this linear functional is not continuous.

Recall that the Dirac delta function defines a linear continuous functional on the space of test functions $\mathcal{D}$. The difference is that the convergence (or topology) in $\mathcal{L}_2$ (or in any inner product space, in general) is not the same as in the linear space $\mathcal{D}$. The convergence of u_n to u in $\mathcal{D}$ means that partial derivatives $D^\alpha u_n$ converge to $D^\alpha u$ uniformly for all $\alpha = 0, 1, ...$ Take, for example the hat function $f(x) = \omega_1(x) \in \mathcal{D}$. Its support is the interval $[-1, 1]$. Then the support of $u_n(x) = \omega_1(nx) = \omega_{1/n}(x)$ is the interval $[-\frac{1}{n}, \frac{1}{n}]$. Therefore

$$(\delta, u_n) = u_n(0) = \omega_1(0) \neq 0$$

which *does not* imply that δ is not continuous on $\mathcal{D}$ because the sequence $\{u_n\}$ has no limit in $\mathcal{D}$ and, hence, is not suitable to verify continuity of δ on $\mathcal{D}$. Thus, the linear functional $l = \delta$ is continuous on $\mathcal{D}$ (viewed as a linear space with *its topology*), but not continuous on C_2^0 (with topology defined by the natural norm in $\mathcal{L}_2$).

57.2. Bounded linear functionals.

DEFINITION **57.3**. (Bounded linear functional)
A linear functional is called bounded *on a linear manifold M if there is a constant $C > 0$ such that*

$$|l(u)| \leq C\|u\|\,, \quad \forall u \in M$$

For example, for some $v \in \mathcal{H}$ in a Hilbert space $\mathcal{H}$, put

$$l(u) = \langle u, v\rangle\,, \quad \forall u \in \mathcal{H}$$

Then l is a complex linear and bounded functional on $\mathcal{H}$. Indeed, by linearity of the inner product

$$l(\alpha u + \beta w) = \langle \alpha u + \beta w, v\rangle = \alpha\langle u, v\rangle + \beta\langle w, v\rangle = \alpha l(u) + \beta l(w)$$

and by the Cauchy-Bunyakowski inequality

$$|l(u)| = |\langle u, v\rangle| \leq \|v\|\|u\| \quad \Rightarrow \quad C = \|v\|$$

The Dirac delta function is not continuous in the topology of $\mathcal{L}_2$. Let us investigate if it is bounded or not on $M = C_2^0([-1,1])$. Suppose it is bounded. Then there should exist a constant $C > 0$ such that

$$|l(u)| \leq C\|u\| \quad \forall u \in C_2^0([-1,1])$$

Let $v_n = \frac{1}{\|u_n\|}\, u_n$, where $u_n(x) = f(nx)$ that is introduced above so that $\|v_n\| = 1$. It follows that

$$|l(v_n)| = \frac{|l(u_n)|}{\|u_n\|} = \frac{|u_n(0)|}{\|u_n\|} = \frac{|f(0)|}{\|f\|}\,\sqrt{n}$$

But the continuity of l implies that $|l(v_n)| \leq C\|v_n\| = C$ for all n, which is impossible. Thus, l is not bounded.

The method by which the non-boundedness of $l = \delta$ was established is noteworthy. It can be used as a general principle to investigate whether a given functional is not bounded.

PROPOSITION **57.1**. *A linear functional on M is not bounded if and only if there exists a unit sequence $\{u_n\} \subset M$, $\|u_n\| = 1$, such that*

$$C_n \leq |l(u_n)| \quad \textit{and} \quad \lim_{n\to\infty} C_n = \infty$$

Indeed, suppose that l is not bounded. Let us find a unit sequence with the required property. Negating the definition of a bounded functional, for any $C > 0$ there exists $v \in M$ such that $|l(v)| \geq C\|v\|$. In particular, for each $C = C_n = n$, $n = 1, 2, ...$, there is $v_n \in M$ such that

$$C_n = n \leq \frac{|l(v_n)|}{\|v_n\|} = |l(u_n)|\,, \quad u_n = \frac{1}{\|v_n\|}\, v_n\,, \quad \|u_n\| = 1$$

The converse is obvious because $l(u_n)$ is not a bounded sequence if $C_n \to \infty$ as $n \to \infty$.

57.3. Continuity and boundedness. It appears that the continuity and boundedness are equivalent properties of a linear functional.

THEOREM **57.3**. (Linearity vs boundedness)
A linear functional is continuous on a linear manifold M if and only if it is bounded on M.

PROOF. Suppose a linear functional l is continuous. One has to show that it is bounded. Assume that the converse is true. This implies that there exists a unit sequence $\{u_n\} \subset M$ such that $C_n \leq |l(u_n)|$ and $C_n \to \infty$ as $n \to \infty$. Put

$$v_n = \frac{1}{C_n}\, u_n \quad \Rightarrow \quad \lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \frac{1}{C_n} = 0$$

Therefore $v_n \to 0$ in M as $n \to \infty$. By continuity of l

$$\lim_{n\to\infty} l(v_n) = 0$$

But

$$|l(v_n)| = \frac{1}{C_n}\,|l(u_n)| \geq \frac{C_n}{C_n} = 1$$

A contradiction. Therefore l is bounded.

Conversely, suppose l is bounded. Let a sequence $\{u_n\}$ converge to $u \in M$. Then

$$|l(u_n) - l(u)| = |l(u_n - u)| \leq C\|u_n - u\| \to 0 \quad \text{as} \quad n \to \infty$$

by the linearity and boundedness of l. It follows from this inequality that the convergence of $\{u_n\}$ to u implies the convergence of $\{l(u_n)\}$ to $l(u)$. Hence, l is continuous on M because u is arbitrary. □

57.4. Null space of a linear continuous functional. The null space of a linear functional l on a Hilbert space $\mathcal{H}$ is a collection of all elements of $\mathcal{H}$ on which l vanishes:

$$\mathcal{N} = \left\{ u \in \mathcal{H} \,\middle|\, l(u) = 0 \right\}$$

By linearity of l, its null space is a linear manifold. For any $u, v \in \mathcal{N}$ and any $\alpha, \beta \in \mathbb{C}$,

$$l(\alpha u + \beta v) = \alpha l(u) + \beta l(v) = 0 = 0 \quad \Rightarrow \quad \alpha u + \beta v \in \mathcal{N}$$

If, in addition, l is continuous on $\mathcal{H}$, then the null space is closed. Indeed, Let $\{u_n\} \subset \mathcal{N}$ be a Cauchy sequence. Since $\mathcal{N} \subset \mathcal{H}$, there exists $u \in \mathcal{H}$ to which the sequence $\{u_n\}$ converges. Since every continuous functional is bounded and $l(u_n) = 0$,

$$0 \leq |l(u)| = |l(u) - l(u_n)| = |l(u - u_n)| \leq C\|u - u_n\| \quad \forall n$$

and some $C > 0$. By taking the limit $n \to \infty$ in the right side of this equality, it is concluded that $l(u) = 0$ which means that $u \in \mathcal{N}$. So, every Cauchy sequence in $\mathcal{N}$ has a limit in $\mathcal{N}$ and, therefore, $\mathcal{N}$ is closed:

$$l \text{ is continuous} \quad \Rightarrow \quad \mathcal{N} = \overline{\mathcal{N}}\,.$$

If $\mathcal{H} = \mathbb{R}^N$, then, as noted above, $l(u) = \langle u, v\rangle$ for some $v \in \mathbb{R}^N$ and the null space consists of all vectors orthogonal to v so that $\mathcal{N} \sim \mathbb{R}^{N-1}$ if $v \neq 0$.

57.5. The Riesz representation theorem. As noted before any linear functional on a finite dimensional Hilbert space can be written in the form $l(u) = \langle u, v\rangle$. Can this be extended to infinite dimensional Hilbert spaces? It turns out that such a generalization is only possible for continuous (or bounded) functionals. The result is know as the Riesz representation theorem. The theorem plays a fundamental role in the theory of self-adjoint operators in Hilbert spaces.

THEOREM **57.4.** (Riesz representation theorem)
Let l be a linear continuous functional on a Hilbert space $\mathcal{H}$. Then there exists a unique $v \in \mathcal{H}$ such that

$$\exists!\, v \in \mathcal{H} \;:\quad l(u) = \langle u, v\rangle \quad \forall u \in \mathcal{H}$$

PROOF. Let $\mathcal{N}$ be the null space of l. If $\mathcal{N} = \mathcal{H}$, then $v = 0$ satisfies the required property: $l(u) = \langle u, v\rangle = 0$ for all u. If $v \neq 0$ has the same property, then $l(v) = \|v\|^2 = 0$ so that $v = 0$.

Suppose now that $\mathcal{N} \subset \mathcal{H}$, a proper subset.
Existence. Let us show first that v with required property exists. Since l is continuous, its null space is closed and

$$\mathcal{N} = \overline{\mathcal{N}} \quad \Rightarrow \quad \mathcal{N}^{\perp} \neq \{0\}$$

by the properties of the orthogonal complement. Therefore the orthogonal complement $\mathcal{N}^{\perp}$ must contain non-zero elements:

$$\mathcal{N} \subset \mathcal{H} \quad \Rightarrow \quad \exists w \neq 0\,, \quad w \in \mathcal{N}^{\perp}$$

Put

$$h = \alpha w + \beta u\,, \quad u \in \mathcal{H}$$

and demand that

$$h \in \mathcal{N} \quad \Leftrightarrow \quad l(h) = 0\,.$$

It follows from the linearity of l that the relation

$$l(h) = \alpha l(w) + \beta l(u) = 0$$

is fulfilled if $\alpha = l(u)$ and $\beta = -l(w)$. It is concluded that

$$h = l(u)w - l(w)u \in \mathcal{N}\,, \quad \forall w \in \mathcal{N}^{\perp}\,, \quad \forall u \in \mathcal{H}$$

Using this property it is now not difficult to find $v \in \mathcal{H}$ such that $l(u) = \langle u, v\rangle$. Since $h \in \mathcal{N}$ and $w \in \mathcal{N}^{\perp}$, they are orthogonal

$$0 = \langle h, w\rangle = l(u)\langle w, w\rangle - l(w)\langle u, w\rangle \quad \Rightarrow \quad l(u) = \frac{l(w)}{\|w\|^2}\langle u, w\rangle$$

The above relation holds for any element $u \in \mathcal{H}$. Therefore

$$l(u) = \langle u, v\rangle\,, \quad v = \frac{l(w)}{\|w\|^2}\, w$$

Uniqueness. Suppose that $l(u) = \langle u, v_1\rangle = \langle u, v_2\rangle$ for all $u \in \mathcal{H}$. Hence

$$\langle u, v_1 - v_2\rangle = l(u) - l(u) = 0 \quad \forall u \in \mathcal{H} \quad \Rightarrow \quad v_1 - v_2 = 0$$

because only the zero element is orthogonal to all elements in a Hilbert space. Thus, $v_1 = v_2$ and v in the representation of $l(u)$ is unique. □

COROLLARY **57.1**. *Let l be a linear continuous functional on $\mathcal{L}_2(\Omega)$. Then there exists a unique square integrable function v such that*

$$l(u) = \langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}\, d^N x \qquad \forall u \in \mathcal{L}_2(\Omega)$$

Remark. The above equality is a reason to use the notation (f, φ) for the action of a distribution $f \in \mathcal{D}'$ on a test function $\varphi \in \mathcal{D}$.

57.6. Exercises. Recall that a complete linearly independent set $\{\phi_k\}$ is a basis in a Hilbert space $\mathcal{H}$ if for any $u \in \mathcal{H}$ there exists a unique numerical sequence $\{\alpha_k\}$ such that

$$u = \sum_{k=1}^{\infty} \alpha_k \phi_k$$

and the series converges with respect to the natural norm in $\mathcal{H}$.

1. Basis representation of a linear functional. Prove

COROLLARY **57.2**. *A linear continuous functional on a Hilbert space is uniquely determined by its values on elements of any orthonormal basis.*

Hint: If $\{\varphi_k\}$ is an orthonormal basis in $\mathcal{H}$, use the Riesz representation theorem to show that

$$l(u) = \langle u, v\rangle\,, \quad v = \sum_{k=1}^{\infty} \overline{l(\varphi_k)}\, \varphi_k$$

for any $u \in \mathcal{H}$.

2. Dual basis in a finite dimensional Hilbert space. Let ϕ_1 and ϕ_2 be two linearly independent vectors in $\mathcal{H} = \mathbb{R}^2$. Clearly, they form a basis. Use vector algebra to show that there exist two unique vectors ϕ_1^* and ϕ_2^* such that

$$\langle \phi_k, \phi_n^*\rangle = \delta_{kn}\,, \quad k, n = 1, 2$$

The vectors ϕ_k^* form the so-called dual basis. The concept can be extended to any finite dimensional Hilbert space. Dual bases are used in crystallography.

3. Dual basis in a Hilbert space. Prove

THEOREM **57.5**. *Let $\{\phi_k\}_1^\infty$ be a basis in a (separable) Hilbert space $\mathcal{H}$. Then there exists a set $\{\phi_k^*\}_1^\infty \subset \mathcal{H}$ such that*

$$\langle \phi_k, \phi_n^* \rangle = \delta_{kn}$$

$$u = \sum_{k=1}^{\infty} \alpha_k \phi_k \,, \quad \alpha_k = \langle u, \phi_k^* \rangle \,, \quad \forall u \in \mathcal{H}$$

$$u = \sum_{k=1}^{\infty} \beta_k \phi_k^* \,, \quad \beta_k = \langle u, \phi_k \rangle \,, \quad \forall u \in \mathcal{H}$$

that is, $\{\phi_k^*\}_1^\infty$ is a basis in $\mathcal{H}$.

(i) Extend the result of Corollary in Problem 1 to an arbitrary basis, that is, any linear continuous functional is uniquely determined by its values on basis vectors, $l(\phi_k)$. To do so, note that for any $u \in \mathcal{H}$, there are unique numbers α_k such that

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k \phi_k$$

Use the continuity of l to show that $l(u)$ is defined by numbers $l(\phi_k)$. Assume that $l(\phi_k) = l'(\phi_k)$ for any two linear functionals. Show that $l(u) = l'(u)$ for all $u \in \mathcal{H}$ and conclude that $l = l'$.

(ii) Define linear continuous functionals $l_k : \mathcal{H} \to \mathbb{C}$ by $l_k(\varphi_n) = \delta_{kn}$ for all n. Use the Riesz representation theorem to show the existence and uniqueness of the dual basis.

(iii) Put

$$v = \sum_{k=1}^{\infty} \langle u, \phi_k \rangle \, \phi_k^*$$

Use that Span $\{\phi_k\}_1^\infty$ is dense in $\mathcal{H}$ to show that $u = v$. Similarly, for the expansion of u over $\{\phi_k\}_1^\infty$.

4. Let $\{\phi_k^*\}_0^\infty$ be the basis dual to the basis of monomials $\phi_k(x) = x^k$, $k = 0, 1, ...,$ in $\mathcal{L}_2(0, 1)$. Find the expansions of $u(x) = e^x$ over $\{\phi_k\}$ and over $\{\phi_k^*\}$.

5. Let $\phi_k(x) = x^k$, $k = 0, 1, ...,$ and $\phi_{-1}(x) = e^x$. The set $\{\phi_k\}_{-1}^\infty$ is linearly independent and complete. Recall that this set is not a basis in $\mathcal{L}_2(0, 1)$ (see Problem 1 in Sec. **57.6**). Show that there is no dual linearly independent and complete set $\{\phi_k\}_{-1}^\infty$ with the property

$$\langle \phi_k, \phi_n^* \rangle = \delta_{kn} \,, \quad k, n = -1, 0, 1, 2, ...$$

Hint: Show that $\langle \phi_k, \phi_{-1}^* \rangle = \delta_{k,-1}$ implies $\langle \phi_{-1}, \phi_{-1}^* \rangle = 0$. A contradiction.

58. The adjoint of an operator

Recall from linear algebra that the adjoint A^* of a square matrix A is defined as

$$\langle Au, v\rangle = \langle u, A^* v\rangle\,, \quad \forall u, v \in \mathbb{C}^N$$

If $\{e_j\}_1^N$ is an orthonormal basis in $\mathbb{C}^N$, then the matrix elements of the adjoint can be expressed via the matrix elements of A as

$$a^*_{jk} = \langle A^* e_j, e_k\rangle = \overline{a_{kj}}\,, \quad a_{jk} = \langle Ae_j, e_k\rangle$$

They are obtained by the transposition and subsequent complex conjugation of the matrix elements a_{jk}.

The symmetric or Hermitian matrices

$$A^* = A$$

have remarkable properties:

(1) The eigenvalues are real: $Au = \lambda u$, $\lambda \in \mathbb{R}$;
(2) Among all linearly independent eigenvectors, one can select N orthonormal vectors φ_k, $k = 1, 2, ..., N$, that form a basis in $\mathbb{C}^N$, $\langle \varphi_j, \varphi_k\rangle = \delta_{jk}$;
(3) If none of the eigenvalue is equal to zero, $\lambda \neq 0$, then the symmetric operator A belongs to the class (I,1c) and the solution to the linear problem $Au = f$ is given by

$$Au = f \quad \Rightarrow \quad u = \sum_{k=1}^{N} \frac{\langle f, \varphi_k\rangle}{\lambda_k}\,\varphi_k\,, \quad A\varphi_k = \lambda_k \varphi_k\,, \quad \lambda_k \neq 0$$

It is natural to ask: Can the concepts of the adjoint and symmetric matrices be extended to linear operators in a Hilbert space? Can eigenvectors of symmetric linear operators be used as bases in a (separable) Hilbert space?

Matrices provide a simple example of bounded operators. Yet, bounded operators in an infinite dimensional Hilbert space are also determined by their matrix elements in an orthonormal basis. So, it is natural to start the discussion with bounded operators.

58.1. The adjoint of a bounded operator. Any linear bounded operator can be extended to the whole Hilbert space and the extension has the same norm. Therefore without loss of generality, it will always be assumed that

$$A\ :\quad D_A = \mathcal{H} \to \mathcal{H}\,, \quad \|A\| < \infty$$

One has to show the existence of the adjoint. Note that an operator is defined by its action and its domain. The latter is a main concern in the infinite dimensional case.

Fix $v \in \mathcal{H}$. Consider a linear functional associated with a bounded operator A and the element v:

$$l_A \;:\quad \mathcal{H} \to \mathbb{C}\,, \quad l_A(u) = \langle Au, v\rangle\,, \quad \forall u \in \mathcal{H}\,.$$

This linear functional is bounded because the operator A is bounded:

$$|l_A(u)| = |\langle Au, v\rangle| \le \|Au\|\|v\| \le \|A\|\|u\|\|v\|\,,$$
$$\Rightarrow \quad \|l_A\| = \sup_{u\neq 0} \frac{|l_A(u)|}{\|u\|} \le \|A\|\|v\| < \infty$$

A bounded linear functional is continuous. Therefore by the Riesz representation theorem for linear functionals on a Hilbert space, there exists a unique $g \in \mathcal{H}$ such that

$$\exists!\, g \in \mathcal{H} \;:\quad l_A(u) = \langle u, g\rangle\,, \quad \forall u \in \mathcal{H}$$

Of course, the element $g = g(v)$ depends on v. In fact, g is a linear function of v. Indeed, let $v = v_1 + v_2$. For three elements v, v_1, and v_2 there are unique elements $g = g(v)$, $g_1 = g(v_1)$, and $g_2 = g(v_2)$ such that

$$\langle Au, v\rangle = \langle u, g\rangle\,, \quad \langle Au, v_1\rangle = \langle u, g_1\rangle\,, \quad \langle Au, v_2\rangle = \langle u, g_2\rangle\,, \quad \forall u \in \mathcal{H}$$

By linearity of the inner product:

$$\langle u, g\rangle = \langle Au, v_1 + v_2\rangle = \langle u, g_1 + g_2\rangle\,, \quad \forall u \in \mathcal{H}$$
$$\Rightarrow \quad g = g_1 + g_2 \quad \Rightarrow \quad g(v_1 + v_2) = g(v_1) + g(v_2)$$

This implies, in particular, that

$$g(v) = 0 \quad \Leftrightarrow \quad v = 0$$

In other words, g is the result of an action of a linear operator:

$$g(v) = A^* v\,; \qquad A^* \;:\quad D_{A^*} = \mathcal{H} \to \mathcal{H}$$

because $g(v)$ is uniquely defined for any $v \in \mathcal{H}$. This operator is called the adjoint of A.

Definition **58.1**. (Adjoint of a bounded operator)
Let A be a linear bounded operator on a Hilbert space $\mathcal{H}$. The adjoint A^* of A is an operator on $\mathcal{H}$ for which

$$\langle Au, v\rangle = \langle u, A^* v\rangle\,, \quad \forall u, v \in \mathcal{H}$$

holds for all u and v in $\mathcal{H}$

Properties of the adjoint of a bounded operator. The adjoint operator is bounded. Indeed,

$$|\langle u, A^*v\rangle| = |\langle Au, v\rangle| \le \|u\|\|v\|\|A\|\,, \quad \forall u \in \mathcal{H}$$

In particular, take $u = A^*v$. The above inequality is reduced to

$$\frac{\|A^*v\|}{\|v\|} \le \|A\| \quad \Rightarrow \quad \|A^*\| \le \|A\| < \infty$$

Therefore one can construct the double adjoint $A^{**} = (A^*)^*$.

PROPOSITION **58.1**. *The double adjoint of a linear bounded operator is the operator itself:*

$$\|A\| < \infty \quad \Rightarrow \quad A^{**} = (A^*)^* = A$$

A proof of this assertion follows from the definition of the adjoint:

$$\begin{aligned}\langle Au, v\rangle &= \langle u, A^*v\rangle = \overline{\langle A^*v, u\rangle} = \overline{\langle v, A^{**}u\rangle}\\ &= \langle A^{**}u, v\rangle\,, \quad \forall u, v \in \mathcal{H}\\ &\Rightarrow \quad Au = A^{**}u\,, \quad \forall u \in \mathcal{H}\\ &\Rightarrow \quad A = A^{**}\end{aligned}$$

PROPOSITION **58.2**. *The adjoint of a bounded operator has the same norm:*

$$\|A^*\| = \|A\|$$

It has been already shown that $\|A^*\| \le \|A\|$. Let us prove that the converse is also true:

$$\begin{aligned}&\|Au\|^2 = \langle Au, Au\rangle| = \langle u, A^*Au\rangle \le \|u\|^2\|A^*A\| \le \|u\|^2\|A\|\|A^*\|\\ \Rightarrow \quad &\frac{\|Au\|^2}{\|u\|^2} \le \|A\|\|A^*\|\\ \Rightarrow \quad &\|A\|^2 \le \|A\|\|A^*\| \quad \Rightarrow \quad \|A\| \le \|A^*\|\end{aligned}$$

Therefore $\|A\| = \|A^*\|$.

The adjoint of a Hilbert-Schmidt operator. Recall that a Hilbert-Schmidt operator is an integral operator with a square integrable kernel:

$$Au(x) = \int_\Omega K(x,y)u(y)\,d^Ny\,, \quad \|A\|^2 \le \int_\Omega\int_\Omega |K(x,y)|^2 d^Nx d^Ny < \infty$$

Let us find the kernel $K^*(x,y)$ of the adjoint of A. Using Fubini's theorem:

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega \overline{v(x)}\int_\Omega K(x,y)u(y)d^Nyd^Nx = \int_\Omega u(y)\int_\Omega K(x,y)\overline{v(x)}d^Nxd^Ny \\ \langle u, A^*v\rangle &= \int_\Omega u(y)\int_\Omega \overline{K^*(x,y)v(x)}d^Nxd^Ny \\ &\Rightarrow \quad K^*(x,y) = \overline{K(y,x)}\end{aligned}$$

The rule is similar to matrices: taking the adjoint means a combination of the transposition (of arguments) and complex conjugation.

Matrix elements of the adjoint. Recall that a bounded operator in a separable Hilbert space is uniquely defined by its matrix elements. It is not difficult to show that the matrix elements of a bounded operator and its adjoint are related exactly in the same way as a matrix and its adjoint:

$$a_{kj} = \langle A\varphi_k, \varphi_j\rangle\,, \quad a^*_{kj} = \langle A^*\varphi_k, \varphi_j\rangle \quad \Rightarrow \quad a^*_{kj} = \overline{a_{jk}}$$

58.2. The adjoint of an unbounded operator. The Riesz representation theorem can no longer be used to construct the adjoint if the operator is not bounded. Here it will be assumed that domain of an unbounded operator is dense in the Hilbert space:

$$A \;:\quad D_A \subset \mathcal{H} \to \mathcal{H}\,, \quad \overline{D_A} = \mathcal{H}\,, \quad \|A\| = \infty$$

Domains of differential operators in the space of square integrable functions are dense: $\mathcal{D} \subset \mathcal{S} \subset C^p \subset \mathcal{L}_2$ and $\overline{\mathcal{D}} = \mathcal{L}_2$ The key difference with the case of bounded operators is that there are elements g in the Hilbert space for which the equality $\langle Au, v\rangle = \langle u, g\rangle$ does not hold for all $u \in D_A$.

DEFINITION 58.2. (The adjoint operator)
Let the domain of an operator A be dense in a Hilbert space $\mathcal{H}$. Let D_{A^*} be a collection of all elements $v \in \mathcal{H}$ for which there exists a unique element $g \in \mathcal{H}$ such that $\langle Au, v\rangle = \langle u, g\rangle$:

$$D_{A^*} = \{v \in \mathcal{H} \,|\, \exists!\, g \in \mathcal{H} \;:\; \langle Au, v\rangle = \langle u, g\rangle\,,\ \forall u \in D_A\}$$

Put

$$A^* \;:\quad D_{A^*} \subset \mathcal{H} \to \mathcal{H}\,, \quad A^*v = g$$

The operator A^* is called the adjoint *of A.*

In other words, to construct the adjoint, one has to find all pairs (v, g) for which $\langle Au, v\rangle = \langle u, g\rangle$ for any u from the domain of A. The first element in all such pairs form the domain of the adjoint, A^*, while the second elements form the range R_{A^*}, provided g is unique for each v.

- Why is it required that D_A is dense in $\mathcal{H}$? *This conditions ensures that the adjoint is a linear operator if it exists.*

As in the case of bounded operators the element g in the pair $(v, g) \in D_{A^*} \times R_{A^*}$ depends on v, $g = g(v) = A^*v$. If $g(v)$ is a function, then for any v there should be a *unique* element g such that $g = g(v)$. By construction, if such $g(v)$ exists, then $g(\alpha v) = \alpha g(v)$ for any complex α. Therefore, $g(v) = A^*v$ is linear if $g(0) = A^*0 = 0$. Let us show that if $v = 0$, then there is a unique $g(0)$ and $g(0) = 0$, provided D_A is dense in $\mathcal{H}$:

$$\begin{aligned} A^*0 = g(0) = 0 \quad &\Leftrightarrow \quad \langle u, g\rangle = \langle Au, 0\rangle = 0 \quad \forall u \in D_A \\ &\Leftrightarrow \quad g = 0 \ \text{ if and only if } \ \overline{D_A} = \mathcal{H} \end{aligned}$$

Note if $\overline{D_A} \subset \mathcal{H}$ (a proper subset), then the orthogonal complement $D_A^\perp$ contains non-zero elements (recall the properties of orthogonal complements). This implies that $g(0) \neq 0$ and A^* is not linear:

$$\overline{D_A} \subset \mathcal{H} \quad \Rightarrow \quad D_A^\perp \neq \{0\} \quad \Rightarrow \quad \exists\, g \neq 0 \ :\ \langle u, g\rangle = 0\,,\ \forall u \in D_A$$

- It is important to understand that *the domain of the adjoint of an unbounded operator is not necessarily dense in the Hilbert space*:

$$\overline{D_A} = \mathcal{H} \quad \Rightarrow \quad \overline{D_{A^*}} \subseteq \mathcal{H}$$

In other words, there are unbounded operators with a dense domain for which $\overline{D_{A^*}}$ is a proper subset of $\mathcal{H}$. This is a drastic difference with the case of bounded operators. An example is given below.

Example: A "pathological" operator. When the closure of unbounded operators was discussed, it was noted that there are operators which map a null sequence to a sequence converging to a non-zero element. Consider $\mathcal{H} = \mathcal{L}_2(\mathbb{R}) = \mathcal{L}_2$. Let f be a bounded function that is not square integrable:

$$|f(x)| \leq M \ a.e.\,, \quad f \notin \mathcal{L}_2$$

For example $f(x) = 1$. Fix $u_0 \in \mathcal{L}_2$, e.g., $u_0(x) = e^{-x^2}$. Define a linear operator

$$A \ :\ D_A = \{\, u \in \mathcal{L}_2 \,|\, |\langle f, u\rangle| < \infty \,\} \to \mathcal{L}_2\,; \quad Au = \langle u, f\rangle\, u_0$$

The domain of the operator consists of square integrable functions u for which the integral

$$|\langle u, f\rangle| = \left|\int u(x)\overline{f(x)}\,dx\right| < \infty$$

exists, or $u\bar{f} \in \mathcal{L}$. Since the domain of integration is not bounded, not every square integrable function is also Lebesgue integrable. For this reason $D_A \subset \mathcal{L}_2$.

The operator A is not bounded and, hence, it is not continuous. Indeed, suppose $f \in \mathcal{L}_{loc}$ is locally integrable but $f \notin \mathcal{L}_2$. Then the numerical sequence

$$c_n = \int_{-n}^{n} |f(x)|^2\,dx \to \infty \text{ as } n \to \infty$$

diverges. Consider the following sequence in D_A

$$u_n(x) = \begin{cases} 0 & , |x| > n \\ \frac{1}{c_n}\overline{f(x)} & , |x| < n \end{cases} \quad \Rightarrow \quad \|u_n\|^2 = \frac{1}{c_n^2}\int_{-n}^{n} |f(x)|^2 dx = \frac{1}{c_n}$$

So, $\{u_n\}$ is a null sequence in the domain D_A. Consider the image of the sequence under the action of A:

$$Au_n = \langle u_n, f\rangle u_0 = u_0\frac{1}{c_n}\int_{-n}^{n} |f(x)|^2 dx = u_0$$

Therefore

$$\lim_{n\to\infty} \|Au_n - u_0\| = 0 \quad \Leftrightarrow \quad \lim Au_n = u_0$$

This implies that A is not continuous and, hence, it is unbounded. It also provides an example of a pathological operator for which the closure $\bar{A}$ does not exist (A is not closable).

The domain of this operator is dense in $\mathcal{L}_2$. This assertion follows from the relation between functional spaces:

$$\mathcal{S} \subset D_A \subset \mathcal{L}_2 \quad \Rightarrow \quad \overline{D_A} = \mathcal{L}_2$$

Since the space of test functions for temperate distributions is dense in $\mathcal{L}_2$, so is D_A.

Let us find the adjoint. One has to find all pairs (v, g) for which

$$\langle Au, v\rangle = \langle u, g\rangle \quad \forall u \in D_A$$

Then all such v form the domain D_{A^*} and all such g form the range $A^*v = g \in R_{A^*}$ by definition of the adjoint. Fix $v \in \mathcal{L}_2$. One has

$$\begin{aligned} \langle Au, v\rangle &= \langle\langle u, f\rangle\langle u_0, v\rangle = \langle u, f\rangle\langle u_0, v\rangle = \langle u, \overline{\langle u_0, v\rangle} f\rangle = \langle u, A^*v\rangle \\ &\Rightarrow \quad \langle u, A^*v - \overline{\langle u_0, v\rangle} f\rangle = 0 \\ &\Rightarrow \quad A^*v = \langle v, u_0\rangle f \quad \text{because} \quad \overline{D_A} = \mathcal{L}_2 \end{aligned}$$

However $f \notin \mathcal{L}_2$ and, hence, the latter equality is only possible if $\langle v, u_0\rangle = 0$. Thus, the domain of the adjoint consists of all functions that are orthogonal to u_0:

$$D_{A^*} = \{ v \in \mathcal{L}_2 \,|\, \langle v, u_0\rangle = 0 \}$$

This means that the orthogonal complement $D_{A^*}^{\perp}$ contains $u_0 \neq 0$ and, hence, the closure of D_{A^*} is a proper subset of the Hilbert space

$$\overline{D_{A^*}} \subset \mathcal{L}_2 \,, \qquad A^* v = 0 \,.$$

by the orthogonal projection theorem.

58.3. Properties of the adjoint. The adjoint has the following properties

PROPOSITION **58.3**. (Properties of the adjoint)

(I) *If an operator B is an extension of an operator A, then the adjoint A^* is an extension of B^**

$$A \subset B \quad \Rightarrow \quad B^* \subset A^*$$

(II) *The adjoint is closed:*

$$\overline{A^*} = A^*$$

(III) *The adjoint operators of an operator and its closure are the same:*

$$(\overline{A})^* = A^*$$

(IV) *An operator is closable if and only if the domain of the adjoint is dense in the Hilbert space:*

$$\exists\, \overline{A} \quad \Leftrightarrow \quad \overline{D_{A^*}} = \mathcal{H}$$

and in this case the closure is the double adjoint:

$$\overline{A} = A^{**}$$

To establish (I), one has to show that $v \in D_{B^*}$ implies $v \in D_{A^*}$ and $A^*v = B^*v$. Let v be from the domain of B^*. This means that there exists a unique $g \in \mathcal{H}$ such that $\langle Bu, v\rangle = \langle u, g\rangle$ for all u in the domain of B. Since $A \subset B$, $Au = Bu$ for any u in the domain of A and, hence,

$$\langle Bu, v\rangle = \langle Au, v\rangle = \langle u, g\rangle \,, \quad u \in D_A$$

This means that $v \in D_{A^*}$ and $g = A^*v = B^*v$ for any $v \in B^*$ or $B^* \subset A^*$.

(II). Let v be in the domain of the closure of the adjoint. This means that there exists a sequence $\{v_n\}$ in the domain of A^* such that

$v_n \to v$ and $A^*v_n \to g$ (in which case $\overline{A^*}v = g$). For any $u \in D_A$, one has

$$\langle u, g\rangle = \lim\langle u, A^*v_n\rangle = \lim\langle Au, v_n\rangle = \langle Au, v\rangle$$

by continuity of the inner product. This means that $v \in D_{A^*}$ and $g = A^*v$ or $\overline{A^*} = A^*$.

(III). Let w be from the domain of the closure $\overline{A}$. This means that there exists a sequence $\{u_n\} \subset D_A$ such that $u_n \to w$ and $Au_n \to f$ (in which case, $\overline{A}w = f$). For any v in the domain of the adjoint $(\overline{A})^*$, there exists a unique g such that $\langle \overline{A}w, v\rangle = \langle w, g\rangle$ for all $w \in D_{\overline{A}}$ (in which case, $(\overline{A})^*v = g$). By Property (I) $(\overline{A})^* \subset A^*$ because $A \subset \overline{A}$. So, $v \in D_{A^*}$. Therefore

$$\langle w, g\rangle = \langle \overline{A}w, v\rangle = \lim\langle Au_n, v\rangle = \lim\langle u_n, A^*v\rangle = \langle w, A^*v\rangle$$

for all w in the domain of $\overline{A}$. Since the domain of $\overline{A}$ is dense in the Hilbert space (recall that A must be densely defined in order to have the adjoint), it follows that $g = A^*v$ or $(\overline{A})^* = A^*$.

(IV). Suppose that the adjoint A^* has a dense domain. Let us show that A is closable. If one assumes that the latter is not true, then there should exist a null sequence $\{u_n\}$ in the domain of A such that $Au_n \to f \neq 0$. Let v belong to the domain of the adjoint A^*. This means that there exists a unique g such that $\langle Au, v\rangle = \langle u, g\rangle$ and, hence

$$\langle f, v\rangle = \lim\langle Au_n, v\rangle = \lim\langle u_n, A^*v\rangle = 0$$

for all $v \in D_{A^*}$, by continuity of the inner product. Therefore $f \neq 0$ must be from the orthogonal complement of D_{A^*} which consists of the zero element because D_{A^*} is dense in the Hilbert space. Therefore $f = 0$ and A must be closable. A proof of the converse as well as the relation $\overline{A} = A^{**}$ is more difficult and long, and will be omitted here[5].

It is worth noting this property states that only "pathological" densely defined operators would have the adjoint that is not densely defined. If D_{A^*} is dense, then the closure $\bar{A}$ can be constructed by taking the double adjoint $A^{**} = \bar{A}$, which is often technically simpler than finding the closure using the definition.

58.3.1. Example. Second derivative operators. Let $Au(x) = u''(x)$ in $\mathcal{H} = \mathcal{L}_2(0,1)$ and $D_A = \mathcal{D}(0,1)$. Note that $u^{(p)}(0) = u^{(p)}(1) = 0$ for any $p \geq 0$. Let B be an extension of A such that $D_B = C^2[0,1]$. In fact, one take $C^p[0,1]$ as D_B for any $p \geq 2$.

[5]see a proof, e.g.,in: M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol 1. Chapter VIII

The adjoint operator A^*. One has to find all pairs (v, g) in $\mathcal{L}_2(0,1)$ for which $\langle Au, v\rangle = \langle u, g\rangle$ for all test functions $u \in \mathcal{D}(0,1)$ and g is uniquely defined by v. First, let us show that $D_{A^*} \subseteq AC^1(0,1)$. For any $v \in \mathcal{L}_2(0,1)$, there exists a sequence $\{v_n\}$ in a dense subset of smooth functions, for which the integration by parts is permitted, that converges to v, $v_n \to v$ in $\mathcal{L}_2(0,1)$. For example, let us take $\mathcal{D}(0,1)$ as such a dense subset. Then

$$\begin{aligned}\langle Au, v\rangle &= \lim\langle Au, v_n\rangle = \lim \int_0^1 u''(x)\overline{v_n(x)}\,dx \\ &= \lim\left(u'(x)\overline{v_n(x)}\Big|_0^1 - u(x)\overline{v_n'(x)}\Big|_0^1 + \langle u, v_n''\rangle\right) = \lim\langle u, v_n''\rangle\end{aligned}$$

So, if $v \in D_{A^*}$, then one should demand that $v_n'' \to g \in \mathcal{L}_2(0,1)$. By the fundamental theorem of calculus

$$\begin{aligned}v_n'(x) &= v_n'(a) + \int_a^x v_n''(y)dy = v_n(a) + Kv_n''(x)\,, \\ v_n(x) &= v_n(a) + \int_a^x v_n'(y)\,dz\,dy = v_n'(a) + Kv_n'(x)\end{aligned}$$

for any $0 < a < 1$. The operator K maps $\mathcal{L}_2(0,1)$ into $\mathcal{L}_2(0,1)$ (recall that any square integrable function is locally integrable). The operator K is bounded (as shown earlier). Therefore K is continuous and, hence,

$$\begin{aligned}v_n' &\to f(x) = c_1 + Kg(x) = f(a) + \int_a^x g(y)\,dy\,, \\ v_n &\to v(x) = c_0 + Kf(x) = v(a) + f(a)(x-a) + \int_a^x f(y)\,dy\end{aligned}$$

where $v_n'(a) \to c_1$, $v_n(a) \to c_0$ and the constants are fixed by setting $x = a$. This shows that $f \in AC^0(0,1)$ and $f'(x) = g(x)$ a.e., and, hence, $v \in AC^1(0,1)$ and $v''(x) = g(x)$ a.e. Thus, $D_{A^*} \subseteq AC^1(0,1)$. The domain D_{A^*} might still be further restricted by some boundary condition (depending on D_A). Let us check if this is the case or not. For any $v \in AC^1(0,1)$ the integration by part is permitted twice because u'' is locally integrable, and one has

$$\begin{aligned}\langle Au, v\rangle &= \int_0^1 u''(x)\overline{v(x)}\,dx = u'(x)\overline{v(x)}\Big|_0^1 - u(x)\overline{v'(x)}\Big|_0^1 + \langle u, v''\rangle \\ &= \langle u, v''\rangle = \langle u, A^*v\rangle\end{aligned}$$

because the boundary terms vanish for any $u \in \mathcal{D}(0,1)$. This means that

$$D_{A^*} = AC^1[0,1]\,,\quad A^*v(x) = v''(x) \in \mathcal{L}(0,1)\,.$$

This method can be used for finding the adjoint of any differential operator.

The adjoint B^*. Using the same line of argument, it is shown that $D_{B^*} \subseteq AC^1[0,1]$. Then for any $u \in C^2[0,1]$ and $v \in D_{A^*}$ one must have

$$\begin{aligned}\langle Bu, v\rangle &= \int_0^1 u''(x)\overline{v(x)}\,dx = u'(x)\overline{v(x)}\Big|_0^1 - u(x)\overline{v'(x)}\Big|_0^1 + \langle u, v''\rangle \\ &= \langle u, B^*v\rangle\end{aligned}$$

which is possible only if the boundary terms vanish. Since the values $u(0)$, $u'(0)$, $u(1)$, and $u'(1)$ are unrestricted, this implies that the corresponding boundary values of v must vanish. Therefore

$$\begin{aligned}D_{B^*} &= \{v \in AC^1[0,1] \,|\, v(0) = v'(0) = v(1) = v'(1) = 0\}\,,\\ B^*v(x) &= v''(x) \in \mathcal{L}_2(0,1)\,.\end{aligned}$$

This example illustrates Property (I) of the adjoint operators. Clearly $B^* \subset A^*$ because $D_{B^*} \subset D_{A^*}$ whereas $D_A \subset D_B$.

The closure of A. The domain of A is dense in $\mathcal{L}_2(0,1)$. For any sequence u_n of test functions that converges to $u \in \mathcal{L}_2(0,1)$, one has by the fundamental theorem of calculus

$$\begin{aligned}u_n'(x) &= \int_0^x u_n''(y)\,dy = \int_1^x u_n''(y)\,dy\\ u_n(x) &= \int_0^x u_n'(y)\,dy = \int_1^x u_n'(y)\,dy\,.\end{aligned}$$

because $u_n(x)$ and $u_n'(x)$ vanish at $x = 0$ and $x = 1$. If one demands that $u \in D_{\bar{A}}$ then there exists $f \in \mathcal{L}_2(0,1)$ such that $u_n'' \to f$. Integrations in the above relations can be viewed as actions of a bounded operator on v_n'' and v_n'. Therefore the convergence of u_n'' to f implies the convergence of v_n' to some $g \in AC^0[0,1]$ and

$$\begin{aligned}g(x) &= \int_0^x f(y)\,dy = \int_1^x f(y)\,dy\,, \quad g'(x) = f(x) \text{ a.e.}\,,\\ u(x) &= \int_0^x g(y)\,dy = \int_1^x g(y)\,dy\,, \quad u'(x) = g(x)\,, \quad u''(x) = f(x) \text{ a.e.}.\end{aligned}$$

It follows from this representations that $u(0) = u(1) = 0$ and $g(0) = g(1) = 0$. This shows that the domain of the closure consists of functions from $AC^1[0,1]$ whose second derivative is square integrable that satisfy the stated boundary conditions. Therefore

$$\bar{A} = B^*$$

The double adjoint A^{**}. Let us calculate A^{**} as the adjoint of A^* and verify the property $A^{**} = \bar{A}$. Since $A^*u = u''$, the domain of A^{**} lies in class $AC^1[0,1]$. One has to verify only for possible boundary conditions. If $v \in D_{A^{**}}$, then there exists a unique $g \in \mathcal{L}_2(0,1)$ such that $\langle A^*u, v\rangle = \langle u, g\rangle$ for any $u \in D_{A^*}$. For functions from $AC^1[0,1]$, the integration by parts is permitted twice so that

$$\langle A^*u, v\rangle = u'(x)\overline{v(x)}\Big|_0^1 - u(x)\overline{v'(x)}\big|_0^1 + \langle u, v''\rangle = \langle u, g\rangle$$

Therefore the boundary terms must vanish and $A^{**}v(x) = g(x) = u''(x)$ a.e. The values of $u(x)$ and $u'(x)$ at $x = 0$ and $x = 1$ are unrestricted. Therefore one should demand that $v(0) = v(1) = v'(0) = v'(1) = 0$ so that

$$A^{**} = B^* = \bar{A}$$

The closure of the adjoint A^* If $u_n \in D_{A^*} = AC^1[0,1]$ that converges to some $u \in \mathcal{L}_2(0,1)$, then

$$u_n'(x) = u_n'(0) + \int_0^x u_n''(y)\,dy\,,$$
$$u_n(x) = u_n(0) + \int_0^x u_n'(y)\,dy\,,$$

If u belongs to the domain of the closure $\overline{A^*}$, then there exists $f \in \mathcal{L}_2(0,1)$ such that $A^*u = u_n'' \to f$. By continuity of the antiderivative operator in the above relations, this implies that $u_n' \to g \in AC^0[0,1]$ and, hence,

$$g(x) = g(0) + \int_0^x f(y)\,dy\,, \quad u(x) = u(0) + \int_0^x g(y)\,dy$$

Therefore $u \in AC^1[0,1] = D_{\overline{A^*}}$ and $\overline{A^*}u(x) = u''(x)$ a.e. Thus, as expected, the adjoint is closed,

$$\overline{A^*} = A^*\,.$$

The adjoint of the closure $\bar{A}$. If v belongs to the domain of the adjoint of $\bar{A}$, there exists a unique $g \in \mathcal{L}_2(0,1)$ such that $\langle \bar{A}u, v\rangle = \langle u, g\rangle$ for all $u \in D_{\bar{A}}$ and in this case $g = (\bar{A})^*v$. It follows from the above analysis that v must be from class $AC^1[0,1]$ and v'' is square integrable on $(0,1)$. Therefore the integration by parts is permitted twice so that

$$\langle \bar{A}u, v\rangle = u'(x)\overline{v(x)}\Big|_0^1 - u(x)\overline{v'(x)}\Big|_0^1 + \langle u, v''\rangle = \langle u, g\rangle$$

This shows that $(\bar{A})^*v = v'' = A^*v$ and the domain of $(\bar{A})^*$ coincides with D_{A^*} because the boundary terms vanish for any $v \in AC^0[0,1]$

owing to the boundary conditions for $u \in D_{\bar{A}}$. Thus, as expected

$$(\bar{A})^* = A^* .$$

58.4. Symmetric (hermitian) operators.

DEFINITION **58.3**. (Symmetric (or hermitian) operator)
An operator A in a Hilbert space is called symmetric (or hermitian) *if its domain is dense in the Hilbert space and the adjoint A^* is an extension of A:*

$$\overline{D_A} = \mathcal{H}, \quad A \subseteq A^*$$

that is,

$$A^*u = Au, \quad \forall u \in D_A, \quad D_A \subseteq D_{A^*}$$

The criterion for an operator to be symmetric is similar to the finite dimensional case with one additional requirement that the domain must be dense.

THEOREM **58.1**. *An operator A in a Hilbert space is symmetric if and only if*

$$\langle Au, v\rangle = \langle u, Av\rangle, \quad \forall u, v \in D_A, \quad \overline{D_A} = \mathcal{H}$$

Note that the above condition implies that

$$\langle u, A^*v\rangle = \langle u, Av\rangle \quad \Rightarrow \quad A^*v = Av, \quad \forall v \in D_A$$

because D_A is dense (its orthogonal complement $D_A^\perp = \{0\}$ contains only zero element). Since $\langle Au, v\rangle$ exists for any v, it is concluded that $D_A \subseteq D_{A^*}$. So, A^* is an extension of A.

For any linear operator A and any u and v from its domain, it is not difficult to verify the *polarization identity*

$$\begin{aligned} 4\langle u, Av\rangle = &\langle u+v, A(u+v)\rangle - \langle u-v, A(u-v)\rangle \\ &+ i\langle u+iv, A(u+iv)\rangle - i\langle(u-iv, A(u-iv)\rangle \end{aligned}$$

It follows from this identity and Theorem **58.1** that *an operator A is symmetric if and only if the quadratic form $\langle u, Au\rangle$ is real for all* $u \in D_A$:

$$A \subset A^* \quad \Leftrightarrow \quad \langle u, Au\rangle \in \mathbb{R}, \quad \forall u \in D_A .$$

As a consequence, all eigenvalues of a symmetric operator are real. Indeed, if $Au = \lambda u$ for some non-zero $u \in D_A$ and some complex λ, then $\bar{\lambda}\|u\|^2 = \bar{\lambda}\langle u, u\rangle = \langle u, Au\rangle$ is real and, hence, λ must be real.

58.4.1. The second derivative operator as a symmetric operator. Consider the operator

$$A \ : \quad \mathcal{D}(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = u''(x)$$

This operator is densely defined and symmetric:

$$\langle Au, v\rangle = \int_0^1 u''(x)\overline{v(x)}\,dx = \int_0^1 u(x)\overline{v''(x)}\,dx = \langle u, Av\rangle$$

because the boundary terms arising after integrating by parts twice vanish for test functions from $\mathcal{D}(0,1)$.

One can enlarge the domain of the second derivative operator so that the rule still makes sense, $u'' \in \mathcal{L}_2(0,1)$. The operator remains symmetric if functions from its domain obey suitable boundary conditions. For example, let $Bu(x) = u''(x)$ and

$$D_B = \{u \in C^2[0,1]\,|\,u(0) = u(1) = 0\}$$

Then B is symmetric, which can readily be verified by integration by parts. Since $D_A \subset D_B$, B is a *symmetric extension* of A. From the properties of the adjoint it follows that

$$A \subset B \subset B^* \subset A^*$$

This relation holds for any symmetric extension of any symmetric operator. So, the domain of the adjoint of a symmetric operator is shrinking upon a symmetric extension of the operator. It is therefore possible that for some symmetric extension B, the domains of B and B^* match. Operators with this characteristic property are called *self-adjoint*. Their spectral properties are similar to those of symmetric matrices in a finite dimensional Hilbert space. In particular, spectra of self-adjoint operators define values of physical quantities, like energy or momentum, in quantum theory. Eigenfunctions of some of such operators form orthogonal bases in a Hilbert space, which is important for Fourier methods for solving linear problems in a Hilbert space.

58.5. Self-adjoint operators.

Definition 58.4. (Self-adjoint operator)
An operator A in a Hilbert space is called self-adjoint *it is symmetric and its adjoint has the same domain, $D_A = D_{A^*}$.*

The difference between symmetric (or hermitian) and self-adjoint operators is in the domain of the adjoint:

$$\text{Symmetric (or hermitian)}: \quad A \subseteq A^*$$
$$\text{Self-adjoint}: \quad A = A^*$$

Every self-adjoint operator is symmetric, but the converse is not true.

58.5.1. Symmetric operators versus self-adjoint operators. Suppose that A is symmetric, $A \subseteq A^*$. Then the domain of the adjoint is dense in the Hilbert space:

$$D_A \subseteq D_{A^*} \quad \Rightarrow \quad \overline{D_{A^*}} = \mathcal{H}$$

By Property (I) of the adjoint, $\overline{A^*} = A^*$, which implies that

$$D_A \subseteq D_{A^*} = D_{\overline{A^*}} \subseteq \overline{D_{A^*}} = \mathcal{H}$$

Since the double adjoint exists and by Property (II), $\bar{A} = A^{**}$, it is concluded that

$$A \subseteq \bar{A} = A^{**} \qquad \text{(by Property (II))}$$
$$A \subseteq A^* = \overline{A^*} \qquad \text{(by Property (I))}$$

Recall that $\bar{A}$ is the smallest closed extension of A. Therefore *any self-adjoint operator must be closed*:

$$A = A^* \quad \Rightarrow \quad \bar{A} = A$$

Let us summarize these observations about the domains of a symmetric operator and its adjoint:

$$\begin{aligned} \text{symmetric operator}: &\quad A \subseteq \bar{A} = A^{**} \subseteq A^* \\ \text{closed symmetric operator}: &\quad A = \bar{A} = A^{**} \subseteq A^* \\ \text{self-adjoint operator}: &\quad A = \bar{A} = A^{**} = A^* \end{aligned}$$

Since self-adjoint operators are important for applications due to their unique priperties, two natural questions arise:

(i) How to verify if a symmetric operator is self-adjoint?
(ii) How to construct a self-adjoint extension of a symmetric operator?

58.5.2. Criteria for self-adjointness. There are several ways to answer the first question. Given an operator A, one can always check if it is symmetric or not using the criterion in Theorem **58.1** as it uses only the domain D_A. If A happens to be symmetric, then one can construct the adjoint A^* and check if $D_{A^*} = D_A$. However the construction of the adjoint can be technically involved. Here is a criterion based on the range of a symmetric operator.

THEOREM **58.2**. *Suppose that the range of a symmetric operator A coincides with the whole Hilbert space $\mathcal{H}$. Then A is self-adjoint:*

$$A \subset A^*, \quad R_A = \mathcal{H} \quad \Rightarrow \quad A = A^*.$$

Since by the hypothesis $D_A \subset D_{A^*}$, to prove the assertion, one has to show that $v \in D_{A^*}$ implies $v \in D_A$. If $v \in D_{A^*}$, then there exists a unique $g \in \mathcal{H}$ such that $\langle Au, v\rangle = \langle u, g\rangle$ and $g = A^*v$. Since $R_A = \mathcal{H}$, there exists $w \in D_A$ such that $g = Aw$ and the following equalities hold for a symmetric A

$$\langle Au, v\rangle = \langle u, g\rangle = \langle u, Aw\rangle = \langle Au, w\rangle \,,$$

for all $u \in D_A$. Since $R_A = A(D_A) = \mathcal{H}$, this implies that $\langle h, v\rangle = \langle h, w\rangle$ for any $h \in \mathcal{H}$ or $v = w \in \mathcal{D}_A$. Thus, $D_{A^*} = D_A$ and $A^* = A$.

Unfortunately, this criterion can hardly be applied to *symmetric* differential operators in an $\mathcal{L}_2$ space. For example, let $Bu(x) = u''(x)$ and the domain of B consists of all square integrable function such that $u'' \in \mathcal{L}_2(0,1)$. It follows from the integral representation $u'(x) = u(a) + \int_a^x Bu(y)dy$ that u' is from $AC^0[0,1]$ and $u' \in \mathcal{L}_2(0,1)$. By repeating the argument, D_B is found to consists of function from class $AC^1[0,1]$. Comparing B with the adjoint A^* is constructed in Sec. **58.3.1**, it is concluded that $S = A^*$. Therefore $B^* = A^{**} = \bar{A} \subset A^*$ or $B^* \subset B$ and, hence, B is not symmetric.

Loosely speaking, if the range of a differential operator B is the whole $\mathcal{L}_2(I)$, where I is an interval, then by applying an antiderivative operator to an $\mathcal{L}_2$ function (which is permitted as any such function is locally integrable on I), one can anticipate that the domain of the operator consists of functions from class $AC^p(I)$, where p is the order of the differential operator, that are not restricted by any boundary conditions at the endpoints of I. A verification of the condition $\langle Bu, v\rangle = \langle u, Bv\rangle$ requires integration by parts and all boundary terms arising from this procedure must necessarily vanish. The latter is not possible because u and v do not obey any boundary conditions. So, B is not symmetric. In fact, $B^* \subset B$ because if B^*v is defined by the same rule as Bu, the relation $\langle Bu, v\rangle = \langle u, B^*v\rangle$ holds only if v satisfies boundary conditions under which all boundary terms arising from integration by parts vanish, or $D_{B^*} \subset D_B$.

So, with some exceptions, a construction of A^* is necessary to verify that $A = A^*$ and if $A \subset A^*$, the second question becomes significant.

58.5.3. Essentially self-adjoint operators. Let A be symmetric. Then its eigenvalues are real. Indeed if $Au = \lambda u$

Suppose that A is self-adjoint. Then A^* (and, hence, A) can only have real eigenvalues. Indeed, let v be a non-zero solution to the eigenvalue problem

$$A^*v = \lambda v\,, \quad v \in D_{A^*}$$

Since $A^* = A$, v is also a non-zero solution to the eigenvalue problem for A:

$$Au = \lambda u\,, \quad u \in D_A = D_{A^*}$$

Note that the condition that $D_A = D_{A^*}$ is crucial for this conclusion. If A is merely symmetric, then it is possible that v is not from $D_A \subset D_{A^*}$ and λ is not an eigenvalue of A. It follows that

$$\begin{aligned}\lambda\langle u, u\rangle &= \langle \lambda u, u\rangle = \langle Au, u\rangle = \langle u, A^*u\rangle = \langle u, Au\rangle = \langle u, \lambda u\rangle \\ &= \bar{\lambda}\langle u, u\rangle \quad \Rightarrow \quad \lambda = \bar{\lambda}\end{aligned}$$

because $\|u\| \neq 0$. Thus, a self-adjoint operator cannot have complex eigenvalues. All eigenvalues of a symmetric operator

How about the converse? It appears that the converse can also be made true if, in addition, a symmetric operator is closed.

THEOREM **58.3**. (Criterion for self-adjointness)
Suppose that A is symmetric. Then A is self-adjoint if and only if A is closed and the null space of the operators $A^* \pm iI$ consists only of the zero vector ($\pm i$ are not eigenvalues of the adjoint):

$$\begin{aligned} A \subseteq A^* \quad \Rightarrow \quad &(a)\ \textit{is true} \quad \Leftrightarrow \quad (b)\ \textit{is true} \\ &(a): \quad A = A^* \\ &(b): \quad A = \bar{A}\,, \quad N_{A^* \pm iI} = \{0\}\end{aligned}$$

The use of this theorem requires that A has to be closed. If it is not closed, then A cannot be self-adjoint. Therefore if one wants to find a *self-adjoint extension* of a symmetric operator, one has to find all its closures first and then apply the above criterion. This is not always a simple task.

Essentially self-adjoint operators. Differential operators have domains that are not closed in $\mathcal{L}_2(\Omega)$ However, they are commonly closable and, hence, may have self-adjoint extension.

DEFINITION **58.5**. (Essentially self-adjoint operator)
An operator is essentially self-adjoint *if its closure is self-adjoint:*

$$(\bar{A})^* = \bar{A}$$

Typically energy operators in quantum mechanics are essentially self-adjoint. Since $(\bar{A})^* = A^*$ (Property (III)), the criterion for essential self-adjointness is proved to be similar.

THEOREM **58.4**. (Criterion for essential self-adjointness)
Suppose that A is symmetric. Then A is essentially self-adjoint if and

only if A is closed and the null space of the operators $A^* \pm iI$ consists only of the zero vector ($\pm i$ are not eigenvalues of the adjoint):

$$A \subseteq A^* \quad \Rightarrow \quad (a)\text{ is true} \quad \Leftrightarrow \quad (b)\text{ is true}$$
$$(a): \quad A \text{ is essentially self-adjoint}$$
$$(b): \quad N_{A^*\pm iI} = \{0\}$$

Example: Differentiation operator on a half-line. Let

$$Au(x) = -iu'(x)$$
$$A: \; D_A = \{C^1[0,\infty) \cap \mathcal{L}_2(0,\infty) \,|\, u(0) = 0\} \;\rightarrow\; \mathcal{L}_2(0,\infty)$$

This operator is symmetric, $A \subseteq A^*$, because

$$\langle Au, v\rangle = -i\int_0^\infty u'(x)\overline{v}(x)\,dx = -iu(x)\overline{v(x)}\Big|_0^\infty + \langle u, Av\rangle = \langle u, Av\rangle$$

The boundary term vanishes for any u and v from the domain of the operator A. If one wants to find out if this operator has any self-adjoint extension by using Theorem **58.3**, then all closures of A must be constructed first. In this case, it is not difficult to do. But Theorem **58.4** saves the effort. It is sufficient to find A^* and investigate solutions to the equation $A^*v = \pm iv$.

Let us find A^*. For any $v \in D_{A^*}$ there exists a sequence $\{v_n\} \subset D_A$ that converges to v and $Av_n = A^*v_n \rightarrow g$ (so that $A^*v = g$). It was shown earlier that $D_{A^*} \subseteq AC^0 \cap \mathcal{L}_2$ and $A^*v = -iv'$ where v' is square integrable on $(0,\infty)$. So, one has to check only if any boundary conditions are required for functions from D_{A^*}. Let $u \in D_A$ and v be absolutely continuous and square integrable in $[0,\infty)$. Then

$$\begin{aligned}\langle Au, v\rangle &= -i\lim_{R\to\infty}\int_0^R u'(x)\overline{v(x)} = -i\lim_{R\to\infty} u(x)\overline{v(x)}\Big|_0^R + \langle u, A^*v\rangle \\ &= iu(0)\overline{v(0)} + \langle u, A^*v\rangle = \langle u, A^*v\rangle\end{aligned}$$

The boundary term vanishes for any value $v(0)$. Therefore

$$D_{A^*} = AC^0[0,\infty)\mathcal{L}_2(0,\infty)$$

Let us know show that A is not essentially self-adjoint, or it does not have any self-adjoint extension. A general solution to the equation

$$A^*v(x) = -iv'(x) = \pm iv(x)$$

reads

$$v(x) = Ce^{\pm x}$$

where C is a constant. The exponentially growing solution should be discarded because it is not square integrable. Thus, the equation has a non-zero solution from the domain of D_{A^*}, which is $v(x) = e^{-x}$. By Theorem **58.4**, A has no self-adjoint extension.

It is worth noting that $A^{**} = \bar{A}$ and the domain of the closure is a subset of the domain of the adjoint A^* that consists of all absolutely continuous functions that vanish at $x = 0$ so that

$$A \subset \bar{A} = A^{**} \subset A^*$$

By Theorem **58.3**, the closure is not self-adjoint.

Self-adjoint extensions of differential operators. In particular, if one wants to find a self-adjoint extension of a differential operator, one should find its symmetric extensions and compare them with the closure. The latter might be a challenging task. Fortunately, there is a simple criterion to do so.

Any linear differential operator with smooth coefficients can be defined on $\mathcal{D}(\Omega)$ which is dense in $\mathcal{L}_2(\Omega)$. An extension is obtained by enlarging the domain to some subspace of $C^p(\Omega)$. However an extension is not generally symmetric even the operator on $\mathcal{D}(\Omega)$ was symmetric. A symmetric extension is obtained if certain boundary conditions are imposed at the boundary $\partial\Omega$. Clearly there can be many symmetric extension of a differential operator. For example, for the second derivative operator on an interval, these conditions were shown to be characterized by two real parameters (β_1/α_1 and β_0/α_0). But, generally, not for any set of parameters, the obtained symmetric operator is essentially self-adjoint, that is, it has a self-adjoint closure. Thus, a differential operator on $\mathcal{D}(\Omega)$ can have many self-adjoint extensions and can also have none. In application, a choice of a specific self-adjoint extension is determined by additional conditions (e.g., by experiments because the spectrum of a self-adjoint operator (a collection of all eigenvalues), like the energy operator, is observable, and different self-adjoint extensions may have different spectra).

How to construct a self-adjoint extension. Theorem **58.4** offers useful guide lines for constructing self-adjoint extensions of an unbounded operator.

(1) Verify if a given operator A is symmetric:

$$\overline{D_A} = \mathcal{H}\,, \quad \langle Au, v\rangle = \langle u, Av\rangle\,, \quad \forall u, v \in D_A$$

(2) Construct the adjoint, that is, find all pairs (v, g) for which $\langle Au, v\rangle = \langle u, g\rangle$ for all $u \in D_A$. Then $v \in D_{A^*}$ and $A^*v = g \in R_{A^*}$.

(3) Verify if the A is essentially self-adjoint:

$$A^*v = \pm iv\,, \quad v \in D_{A^*} \quad \Rightarrow \quad v = 0$$

If the latter holds, then the closure $\bar{A}$ is self-adjoint and $(\bar{A})^* = \bar{A} = A^*$ (Property (III)). In other words, A^* is the self-adjoint extension of A.

58.6. Self-adjoint extensions of the differentiation operator in $\mathcal{L}_2(0,1)$. Let

$$A \;:\quad D_A = \mathcal{D}(0,1) \subset \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = -iu'(x)$$

This operator is obviously symmetric because by integration by parts

$$\langle Au, v\rangle = -i\int_0^1 u'(x)\overline{v(x)}\,dx = i\int_0^1 u(x)\overline{v'(x)}\,dx = \langle u, Av\rangle$$

The boundary term vanishes as any function from $\mathcal{D}(0,1)$ vanishes at the endpoints.

The domain is dense in the Hilbert space, so the operator admits symmetric extensions and some of which can be essentially self-adjoint. Let us find them using the procedure outlined above.

Symmetric extensions of the second derivative. Let us construct all symmetric extensions of A with the domain in $C^2([0,1])$. Put

$$A \;:\; D_A \subseteq C^2([0,1]) \subset \mathcal{L}_1(0,1) \to \mathcal{L}_2\,, \quad Au(x) = u''(x)\,.$$

The domain is dense in $\mathcal{L}_2(0,1)$ because its subset $\mathcal{D}(0,1) \subseteq D_A$ is dense in $\mathcal{L}_2(0,1)$. Then for any two functions from D_A, using the integration by parts twice (which is valid for functions from D_A)

$$\begin{aligned}\langle Au, v\rangle &= \int_0^1 u''(x)\overline{v(x)}\,dx \\ &= u'(x)\overline{v(x)}\Big|_0^1 - u(x)\overline{v'(x)}\Big|_0^1 + \int_0^1 u(x)\overline{v''(x)}\,dx \\ &= u'(1)\overline{v(1)} - u(1)\overline{v'(1)} - u'(0)\overline{v(0)} + u(0)\overline{v'(0)} + \langle u, Av\rangle\end{aligned}$$

If one assumes that the local behaviors of the functions u and v near $x = 0$ and $x = 1$ are unrelated to one another, the operator of the second derivative becomes symmetric if its domain is restricted by two conditions

$$\begin{aligned}u'(1)\overline{v(1)} &= u(1)\overline{v'(1)} \\ u'(0)\overline{v(0)} &= u(0)\overline{v'(0)}\end{aligned}$$

If $v(1)$ and $v'(1)$ do not vanish simultaneously, then owing to that these numbers are unrelated to $u(1)$ and $u'(1)$, it is concluded that

$$\frac{u'(1)}{u(1)} = \frac{\overline{v'(1)}}{\overline{v(1)}} = \text{const} \quad \Rightarrow \quad \alpha_1 u(1) + \beta_1 u'(1) = 0$$

for some complex α_1 and β_1. The function v must also obey the same condition as one can always swap u and v in the argument. Similarly,

$$\alpha_0 u(0) + \beta_0 u'(0) = 0$$

One can always choose $\alpha_{0,1}$ to be real by dividing the above conditions by the phase factor of the complex $\alpha_{0,1}$. These conditions are *necessary* for A to be symmetric but not yet sufficient because the boundary terms must vanish. If $\alpha_1 = 0$, then $u'(0) = 0$ and $v'(0) = 0$ and the boundary term vanishes at $x = 1$. If $\alpha_1 \neq 0$, then

$$\begin{aligned} u'(1)\overline{v(1)} - u(1)\overline{v'(1)} &= \left(v(1) - \frac{\beta_1}{\alpha_1} v'(1)\right) u'(1) \\ &= \left(\frac{\bar{\beta}_1}{\alpha_1} - \frac{\beta_1}{\alpha_1} v'(1)\right) u'(1) v'(1) = 0 \\ &\Rightarrow \quad \bar{\beta}_1 = \beta_1 \end{aligned}$$

because α_1 is real. Similarly

$$u'(0)\overline{v(0)} - u(0)\overline{v'(0)} = 0 \quad \Rightarrow \quad \bar{\beta}_0 = \beta_0$$

Thus, the second derivative operator becomes symmetric if its domain is reduced to

$$D_A = \left\{ u \in C^2([0,1]) \,\middle|\, \alpha_1 u(1) + \beta_1 u'(1) = 0\,, \alpha_0 u(1) + \beta_0 u'(1) = 0 \right\}$$

where $\alpha_{0,1}$ and $\beta_{0,1}$ are real numbers. Note that imposing the boundary conditions does not affect that D_A is dense in $\mathcal{L}_2(0,1)$. These boundary conditions are known as the Sturm-Liouville type boundary conditions. The construction of the adjoint will be discussed later.

The discussed approach to construct symmetric extensions of the second derivative is not the only possible. Think of an example when the endpoints of an interval are identified (the interval becomes topologically equivalent to a circle). In this case, it is natural to admit that values of $u(0)$ and $u(1)$ are related and so are $u'(0)$ and $u'(1)$, e.g., any smooth function on a circle may be viewed as a periodic function for which $u(0) = u(1)$ and $u'(0) = u'(1)$. The periodic boundary conditions also lead to a symmetric extension of the second derivative operator (see also Exercises for all symmetric extensions of this type).

Symmetric extensions. Let us try to extend the domain as much as possible so that the integration by parts would still be applicable to all functions from it:

$$A \ : \quad D_A \subset C^1([0,1]) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = -iu'(x)$$

One has for any u and v from $C^1([0,1])$

$$\begin{aligned}\langle Au, v\rangle &= -i\int_0^1 u'(x)\overline{v(x)}\,dx = iu(x)\overline{v(x)}\Big|_0^1 + i\int_0^1 u(x)\overline{v'(x)}\,dx \\ &= iu(1)\overline{v(1)} - iu(0)\overline{v(0)} + \langle u, Av\rangle\end{aligned}$$

The extension is symmetric if the boundary term vanishes:

$$\frac{u(0)}{u(1)} = \frac{\overline{v(1)}}{\overline{v(0)}}$$

Since u and v are arbitrary, any function from D_A necessarily satisfies the condition that

$$A \subseteq A^* \quad \Rightarrow \quad u(0) = zu(1)\,, \quad z \in \mathbb{C}\,, \quad \forall u \in D_A$$

The boundary term vanishes:

$$\frac{u(0)}{u(1)} = \frac{\overline{v(1)}}{\overline{v(0)}} \quad \Rightarrow \quad z = \frac{1}{\bar{z}} \quad \Rightarrow \quad |z| = 1 \quad \Rightarrow \quad z = e^{i\theta}$$

So, there are infinitely many symmetric extensions

$$A_\theta \subseteq A_\theta^*$$

labeled by a real parameter $\theta \in [0, 2\pi)$:

$$A_\theta \ : D_{A_\theta} = \{u \in C^1([0,1]) \,|\, u(0) = e^{i\theta}u(1)\,\}\,, \quad A_\theta u(x) = -iu(x)$$

The closure of symmetric extensions. The differentiation operator A_θ is not closed. Recall that D_{A_θ} can be further extended to a subspace of $AC^0[0,1]$. The question is: *Does the boundary condition survives the closure?*

Let us use the definition to construct the closure $\overline{A}_\theta$. So, take $\{u_n\} \subset D_{A_\theta}$ such that

$$\lim u_n = u \in \mathcal{L}_2(0,1)\,, \quad \lim A_\theta u_n = -i\lim u_n' = f \in \mathcal{L}_2(0,1)$$

one has to find all such u and f. Then $\overline{A}_\theta\, u = f$. Using the fundamental theorem of calculus

$$u_n(x) = u_n(0) + i\int_0^x A_\theta u_n(y)\,dy = u_n(0) + BA_\theta u_n(x)$$

The integral operator B is bounded on $\mathcal{L}_2(0,1)$ and, hence, continuous. So, by taking the limit one infers that

$$u(x) = u(0) + i\int_0^x f(y)dy \quad \Rightarrow \quad u \in AC^0[0,1]\,, \quad \forall f \in \mathcal{L}_2(0,1)$$

The limit of the constant functions $v_n(x) = u_n(0)$ exists by the limit laws because $v_n = u_n - BA_\theta u_n$ and $\lim v_n = v$ where $v(x) = u(0)$. Applying a similar line of arguments, one can obtain another representation

$$u(x) = u(1) + i\int_x^1 f(y)dy$$

from which it is then established that the sequence $w_n(x) = u_n(1)$ converges to a constant function $\lim w_n = w$, $w(x) = u(1)$. Since $v_n(x) = u_n(0) = e^{i\theta}u_n(1) = e^{i\theta}w_n(x)$, it is concluded that the boundary condition survives the closure:

$$u(0) = e^{i\theta}u(1)$$

Thus,

$$D_{\overline{A}_\theta} = \{\, u \in AC^0[0,1] \,|\, u(0) = e^{i\theta}u(1)\,\}\,, \quad R_{\overline{A}_\theta} = \mathcal{L}_2(0,1)\,.$$

The adjoint A^*_θ. One has to find all pairs (v, g) such that

$$\langle A_\theta u, v\rangle = \langle u, g\rangle\,, \quad \forall u, v \in D_{A_\theta}\,, \quad \Leftrightarrow \quad v \in D_{A^*_\theta}\,,\ A^*_\theta v = g\,.$$

Since D_{A_θ} is dense in $\mathcal{L}_2(0,1)$ and $D_A \subseteq D_{A^*_\theta}$, for any $v \in D_{A^*_\theta}$ there exists a sequence $\{v_n\} \in D_{A_\theta}$ that converges to v. Then

$$\lim\langle A_\theta u, v_n\rangle = \langle u, g\rangle\,, \quad \forall u \in D_{A_\theta}$$

Since A_θ is symmetric and D_{A_θ} is dense in the Hilbert space

$$\langle A_\theta u, v_n\rangle = \langle u, A_\theta v_n\rangle \quad \Rightarrow \quad \lim A_\theta v_n = -i\lim v_n' = g$$

By the fundamental theorem of calculus

$$\begin{aligned} v_n(x) &= v_n(0) + \int_0^x v_n'(y)dy = v_n(0) + i\int_0^x A_\theta v_n(y)dy \\ &= v_n(0) + BA_\theta v_n(x) \end{aligned}$$

Using the boundedness of the integral operator B and the convergence of sequences $\{v_n\}$ and $\{A_\theta v_n\}$ in $\mathcal{L}_2(0,1)$, it is concluded that

$$v(x) = v(0) + \int_0^x g(y)\,dy \quad \Rightarrow \quad D_{A^*_\theta} \subseteq AC^0[0,1]\,, \quad A^*_\theta v(x) = -iv'(x)$$

because $g \in \mathcal{L}_2(0,1)$ and, hence, $g \in \mathcal{L}(0,1)$ by boundedness of the interval $(0,1)$.

Let us check if the boundary condition still holds in $D_{A_\theta^*}$. It is demanded that

$$\langle A_\theta u, v\rangle = \langle u, A_\theta^* v\rangle\,, \quad \forall u \in D_{A_\theta}\,, \quad \forall v \in D_{A_\theta^*} \subseteq AC^0[0,1]$$

Since $u \in C^1([0,1])$, the product $u\bar{v} \in AC^0[0,1]$ and the fundamental theorem of calculus holds for the derivative of the product $(u\bar{v})'$. Therefore the integration by part is valid:

$$\langle u, A_\theta^* v\rangle = i\int_0^1 u(x)\overline{v'(x)}\,dx = iu(x)\overline{v(x)}\Big|_0^1 + \langle A_\theta u, v\rangle$$

Using the same arguments as when constructing symmetric extensions of the derivative operator from $\mathcal{D}(0,1)$ to $C^1([0,1])$, it is concluded that the boundary condition also holds for functions from $D_{A_\theta^*}$ because it holds for $u \in D_{A_\theta}$:

$$u(x)\overline{v(x)}\Big|_0^1 = 0 \quad \Rightarrow \quad u(1)\overline{v(1)} = u(0)\overline{v(0)} \quad \Rightarrow \quad \overline{v(1)} = e^{i\theta}\overline{v(0)}$$
$$\Rightarrow \quad D_{A_\theta^*} = \{\, v \in AC^0[0,1] \,|\, v(0) = e^{i\theta}v(1)\,\}$$

Self-adjoint extensions. Since A_θ and its closure $\overline{A}_\theta$ share the same adjoint, the closure $\overline{A}_\theta$ is a self-adjoint operator if and only if the null spaces of $A_\theta^* \pm iI$ contain only the zero element. One has to solve the eigenvalue problem:

$$\begin{cases} A_\theta^* v = \pm iv \\ v \in D_{A_\theta^*} \end{cases} \quad \Rightarrow \quad \begin{cases} -iv'(x) = \pm iv(x) \\ v(0) = e^{i\theta}v(1) \end{cases} \quad \Rightarrow \quad \begin{cases} v(x) = Ce^{\pm x} \\ C = Ce^{\pm 1} \end{cases}$$

The boundary condition yields $C = 0$ and, hence, $v = 0$. It is concluded that

$$(\overline{A}_\theta)^* = \overline{A}_\theta$$

that is, all symmetric extensions are essentially self-adjoint operators. All self-adjoint extensions of the derivative operator are labeled by a real parameter $\theta \in [0, 2\pi)$.

Remark. Note that the eigenvalues of the self-adjoint extensions depend on the extension parameter θ

$$\begin{cases} \overline{A}_\theta v = \lambda v \\ v \in D_{\overline{A}_\theta} \end{cases} \quad \Rightarrow \quad \begin{cases} -iv'(x) = \lambda v(x) \\ v(0) = e^{i\theta}v(1) \end{cases} \quad \Rightarrow \quad \begin{cases} v(x) = v_n(x) = e^{i\lambda_n x} \\ \lambda = \lambda_n = 2\pi n - \theta \end{cases}$$

where $n = 0, \pm 1, \pm 2, \ldots$. In physics, the boundary condition $\psi(0) = \psi(1)e^{i\theta}$ are often used to model *anions*, a generalization of spin states, $\theta = 0$ for bosons and $\theta = \pi$ for fermions.

58.7. Extensions of the differentiation operator in $\mathcal{L}_2(0,\infty)$. Put

$$A\ :\quad D_A = \mathcal{D}(0,\infty) \to \mathcal{L}_2(0,\infty)\,,\quad Au(x) = -iu'(x)$$

The operator is symmetric because

$$\langle Au, v\rangle = -i\int_0^\infty u'(x)\overline{v(x)}\,dx = i\int_0^\infty u(x)\overline{v'(x)}\,dx = \langle u, Av\rangle$$

by integration by parts (the boundary term vanishes).

Symmetric extensions. Let us take the largest extension of D_A in the class $C^1\cap\mathcal{L}_2$. Then by integration by parts

$$\begin{aligned}
&\langle Au, v\rangle = \langle u, Av\rangle\,,\quad \forall u, v \in D_A \subseteq C^1([0,\infty))\cap\mathcal{L}_2(0,\infty)\\
\Rightarrow\quad & u(x)\overline{v(x)}\Big|_0^\infty = 0 \quad\Rightarrow\quad u(0)\overline{v(0)} = 0\\
\Rightarrow\quad & D_A = \{\, u\in C^1\cap\mathcal{L}_2\,|\, u(0) = 0\,\}\,,\quad Au(x) = -iu'(x)
\end{aligned}$$

Thus, there is only one symmetric extension of A.

The adjoint A^*. One has to find all pairs (v, g) such that

$$\langle A_\theta u, v\rangle = \langle u, g\rangle\,,\quad \forall u, v\in D_{A_\theta}\,,\quad\Leftrightarrow\quad v\in D_{A_\theta^*}\,,\ A_\theta^* v = g\,.$$

Since D_{A_θ} is dense in $\mathcal{L}_2(0,\infty)$ and $D_A\subseteq D_{A_\theta^*}$, for any $v\in D_{A_\theta^*}$ there exists a sequence $\{v_n\}\in D_{A_\theta}$ that converges to v. Then

$$\lim\langle A_\theta u, v_n\rangle = \langle u, g\rangle\,,\quad \forall u\in D_{A_\theta}$$

Since A_θ is symmetric and D_{A_θ} is dense in the Hilbert space

$$\langle A_\theta u, v_n\rangle = \langle u, A_\theta v_n\rangle \quad\Rightarrow\quad \lim A_\theta v_n = -i\lim v_n' = g$$

Since $g\in\mathcal{L}_2(0,\infty)$, it is locally integrable:

$$\int_a^b |g(x)|\,dx \le \left(\int_a^b 1dx\right)^{1/2}\left(\int_a^b |g(x)|^2\right)^{1/2} \le (b-a)^{1/2}\|g\| < \infty$$

for any $(a, b)\subset(0,\infty)$. Therefore for any such interval

$$\left|\int_a^b (Av_n(x) - g(x))\,dx\right| \le (b-a)^{1/2}\|Av_n - g\| \to 0 \quad\text{as}\quad n\to\infty$$

which implies that

$$\lim_{n\to\infty}\left|-i\Big(v_n(b) - v_n(a)\Big) - \int_a^b g(x)dx\right| = 0$$

for any $(a,b) \subset (0,\infty)$. Since $\lim v_n = v$ in $\mathcal{L}_2(0,\infty)$, it is concluded that for any $a > 0$

$$v(x) = v(a) + i\int_a^x g(x)dx \quad \Rightarrow \quad v \in AC^0[0,\infty)$$
$$\Rightarrow \quad D_{A^*} \subset AC^0[0,\infty) \cap \mathcal{L}_2(0,\infty)\,, \quad A^*v(x) = -iv'(x)\,,$$

because $g \in \mathcal{L}_{loc}$. Note that the value $v(0)$ is not known.

Let us now investigate if the boundary condition holds in the domain of the adjoint. It is demanded that

$$\langle Au, v\rangle = \langle u, A^*v\rangle\,, \quad \forall u \in D_A\,, \quad \forall v \in D_{A^*}$$

Since $u \in C^1$ and $v \in AC^0$ on any interval $[0,b]$, the derivative of the product $u\bar{v}$ belongs to AC^0 and, hence, the fundamental theorem of calculus holds and so does the integration by parts:

$$\int_0^b \bar{v}du = \int_0^b d(u\bar{v}) - \int_0^b ud\bar{v}$$

Therefore for any $v \in D_{A^*}$

$$\lim_{b\to\infty} u(x)\overline{v(x)}\Big|_0^b = 0 \quad \Rightarrow \quad u(0)\overline{v(0)} = 0$$

This condition imposes no restriction on $v(0)$ because $u(0) = 0$ for any function from D_A. Thus, the boundary condition does not survives the symmetric extension and

$$D_{A^*} = AC^0[0,\infty) \cap \mathcal{L}_2(0,\infty)\,, \quad A^*v(x) = -iv(x)$$

Essential self-adjointness. Let us verify if the closure of A is self-adjoint or merely symmetric (recall that $(\overline{A})^* = A^*$). One has to investigate the eigenvalue problem:

$$\begin{cases} A^*_\theta v = \pm iv \\ v \in D_{A^*_\theta} \end{cases} \quad \Rightarrow \quad \begin{cases} -iv'(x) = \pm iv(x) \\ v \in \mathcal{L}_2(0,\infty) \end{cases} \quad \Rightarrow \quad v(x) = e^{-x} \neq 0$$

Thus, A is not essentially self-adjoint and has no self-adjoint extension. The reader is advised to construct the closure of A and show that

$$A \subset \bar{A} = A^{**} \subset A^*$$

where all the inclusions are proper.

Remark. The operator $A = -i\frac{d}{dx}$ is the momentum operator of a quantum particle moving on a half-line. The motion is restricted by an "infinite potential wall" at the endpoint. The momentum operator has no self-adjoint extension and is merely symmetric (hermitian). In contrast, the energy operator, which is proportional to the second derivative, does have a unique self-adjoint extension (see Exercises).

58.8. Exercises.

1. Self-adjoint extensions of the second derivative.
Consider

$$A \ : \quad D_A = \mathcal{D}(0,\infty) \to \mathcal{L}_2(0,\infty)\,, \quad Au(x) = u''(x)\,.$$

(i) Show that A is symmetric on D_A.
(ii) Show that A has infinitely many symmetric extensions

$$D_{A_\alpha} = \{\, C^2([0,\infty)) \cap \mathcal{L}_2(0,\infty) \,|\, u'(0) = \alpha u(0)\,\}\,, \quad A_\alpha u(x) = u''(x)\,.$$

where α is any element from the extended real system $\mathbb{R} \cup \{\infty\}$, where the case $\alpha = \infty$ corresponds to $u(0) = 0$.

(iii) Constructing the adjoint A^*_α (part 1). Show first that for any pair (v, g) such that

$$\langle A_\alpha u, v\rangle = \langle u, g\rangle\,, \quad \forall u \in D_{A_\alpha}$$

the following representation holds

$$v(x) = v(a) + v'(a)x + \int_a^x \int_a^y g(z)\,dz\,, \quad 0 < a < x < b$$

and any $b > 0$, and conclude that $D_{A^*_\alpha} \subset AC^1[0,\infty) \cap \mathcal{L}_2(0,\infty)$. To do so, show first that for any such v there is a sequence $\{v_n\}$ in D_{A_α} that converges to v. Next show that $g \in \mathcal{L}_{loc}$ and

$$\lim_{n\to\infty} \left| \int_a^b A_\alpha v_n(x)\,dx - \int_a^b g(x)dx \right| = 0$$

for any $(a, b) \subset (0,\infty)$. Deduce from this relation that

$$\lim_{n\to\infty} \Big(v_n'(b) - v_n'(a) \Big) = f(b) - f(a)\,,$$

$$f(x) = f(a) + \int_a^x g(y)dy \in AC^0[0,\infty)$$

for any $(a,b) \subset (0,\infty)$. Next, show that

$$\lim_{n\to\infty}\Big(v_n(b) - v_n(a)\Big) = h(b) - h(a)\,,$$
$$h(x) = h(a) + \int_a^x f(y)dy \in AC^0[0,\infty)$$

for any $(a,b) \subset (0,\infty)$. Finally, use the above two relations to deduce the required relation between v and g.

(iv) Constructing the adjoint A^*_α (part 2). Verify that the boundary condition survives the symmetric extension for any α and

$$D_{A^*_\alpha} = \{\, v \in AC^1[0,\infty)\mathcal{L}_2(0,\infty) \,|\, v(0) = \alpha v'(0)\,\}\,, \quad A^*v(x) = v''(x)$$

(v) Show that A_α is essentially self-adjoint for all α by verifying the criterion for essential self-adjointness and conclude that A^*_α is the self-adjoint extension of A for all α:

$$A \subset A_\alpha \subset \overline{A}_\alpha = A^*_\alpha = (\overline{A}_\alpha)^*$$

2. Self-adjoint extensions of the second derivative in $\mathcal{L}_2(0,1)$. Consider

$$A \;:\quad D_A = \mathcal{D}(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = u''(x)\,.$$

(i) Show that A is symmetric on D_A.

(ii) Show that A has infinitely many symmetric extensions $A \subset A_{\alpha,\beta}$ labeled by complex numbers α and β:

$$D_{A_{\alpha,\beta}} = \{\, u \in C^2([0,1]) \,|\, u(0) = \alpha u(1)\,,\ u'(0) = \beta u'(1)\ \ \alpha\bar{\beta} = 1\,\}\,,$$

(iii) Construct the adjoint $A^*_{\alpha,\beta}$. First show that for any pair (v,g) such that

$$\langle A_\alpha u, v\rangle = \langle u, g\rangle\,, \quad \forall u \in D_{A_\alpha}$$

the following representation holds

$$v(x) = v(a) + v'(a)x + \int_a^x \int_a^y g(z)\,dz\,, \quad 0 \le a < x < b \le 1$$

by following the steps of Part (iii) of Problem 1. Conclude that $D_{A^*_{\alpha,\beta}} \subset AC^1[0,1]$. Next, show that the boundary conditions holds for functions from $D_{A^*_{\alpha,\beta}}$ by verifying the condition

$$\langle A_{\alpha,\beta} u, v\rangle = \langle u, A^*_{\alpha,\beta}\rangle\,, \quad \forall u \in D_{A_{\alpha,\beta}}\,, \quad \forall v \in D_{A^*_{\alpha,\beta}}\,.$$

(iv) Find all α and β for which $A_{\alpha,\beta}$ is essentially self-adjoint and, hence, A has self-adjoint extensions.

3. (i) Find all self-adjoint extensions for the Sturm-Liouville operator

$$A\ :\quad D_A \to \mathcal{L}_2(0,1)\,,\quad Au(x) = u''(x)\,,$$
$$D_A = \Big\{\, u \in C^2([0,1]) \,\Big|\, \alpha_1 u(1) + \beta_1 u'(1) = 0\,, \alpha_0 u(1) - \beta_0 u'(1) = 0 \,\Big\}$$

where $\alpha_{0,1}$ and $\beta_{0,1}$ are real.

(ii). Find the eigenvalues of all self-adjoint extensions of A if $\alpha_{0,1} \geq 0$, $\beta_{0,1} \geq 0$, and $\alpha_1 + \beta_1 > 0$, $\alpha_0 + \beta_0 > 0$.
Hint: The eigenvalues satisfy a transcendental equation. Solve it graphically indicating interval in which each eigenvalue lies.

(iii) Construct the closure $\bar{A}$ and compare it with A^*

4. Construct the closure of A if

$$A\ :\quad D_A \to \mathcal{L}_2(0,\infty)\,,\quad Au(x) = -iu'(x)\,,$$
$$D_A = \{\, u \in C^1([0,\infty)) \cap \mathcal{L}_2(0,\infty) \,|\, u(0) = 0 \,\}\,,$$

by taking the double adjoint $A^{**} = \bar{A}$. Compare $\bar{A}$ with A^*.

59. The spectrum of an operator

59.1. The spectrum of a square matrix. Consider a finite dimensional Hilbert space $\mathcal{H}$. Then any linear operator on it, $A\,:\ \mathcal{H} \to \mathcal{H}$, is uniquely determined by its matrix elements in an orthonormal basis. So, without loss of generality, let A be an $n \times n$ complex matrix:

$$A\ :\quad \mathbb{C}^N \to \mathbb{C}^N$$

Consider the eigenvalue problem

$$Au = \lambda u$$

it has a non-zero solution if and only if

$$\det(A - \lambda I) = 0$$

Therefore λ is a root of a polynomial of degree N. By the fundamental theorem of algebra, any such polynomial has N complex roots if the roots are counted with their multiplicity, that is, a root of multiplicity 2 is counted twice, etc. The collection of distinct roots

$$\sigma(A) = \{\lambda_j\}\,,\quad j = 1, 2, ..., n \leq N$$

is called the spectrum of the matrix A.

This definition of the spectrum is impossible to extend to the infinite dimensional case because, first, the determinant of an infinite matrix has to be defined, second, only bounded operators are uniquely defined by their matrix elements. So, let us try to defined the spactrum of a matrix without the use of the determinant. Consider the linear problem

$$(A - \lambda I)u = f \,.$$

It has a unique solution for any f if and only if $\lambda \notin \sigma(A)$, or the matrix $(A - \lambda I)$ is invertible in this case:

$$\lambda \notin \sigma(A) \subset \mathbb{C} \quad \Rightarrow \quad \exists\,!u \ : \quad (A - \lambda I)u = f\,, \quad \forall f \in \mathbb{C}^N$$

The inverse of $(A - \lambda I)$ is called the resolvent of the matrix A:

$$\mathcal{R}_A(\lambda) = (A - \lambda I)^{-1}\,, \quad \lambda \notin \sigma(A)$$

In other words, $\lambda \in \sigma(A)$ if and only if the resolvent exists. Let $\rho(A)$ be a set of all complex λ for which the resolvent exists:

$$\rho(A) = \{\, \lambda \in \mathbb{C} \,|\, \exists\, \mathcal{R}_A(\lambda) \,\}$$

Then, the spectrum of A can be found as its complement:

$$\sigma(A) = \mathbb{C} \setminus \rho(A) \,.$$

This definition of a spectrum can be extended to the infinite dimensional case because it involves only the question about sovability of the linear problem which can be addressed for any operator in a Hilbert space (Recall the classification of operator by the properties of the inverse and the range).

59.2. The resolvent of an operator in a Hilbert space. The resolvent of an operator in a Hilbert space $\mathcal{H}$ is defined via a unique solution to the linear problem

$$(A - \lambda I)u = f\,, \quad u \in D_A\,, \quad \Rightarrow \quad u = \mathcal{R}_A(\lambda) f$$

There are two conditions for solvability of this problem. First, the operator $A - \lambda I$ should be invertible. Second, the vector f must be from the range of $A - \lambda I$. If one follows the analogy with the finite dimensional case, then those λ's for which the operator $A - \lambda I$ is not invertible would form the spectrum $\sigma(A)$. The second issue does not even arise in the finite dimensional case because By definition the operator $A - \lambda I$ is invertible if this linear problem has only the trivial solution $u = 0$. By analogy with the finite dimensional case, the set of complex λ for which the linear problem has a non-trivial solution should be the spectrum of A.

The spectrum of a matrix is always a non-empty finite collection of complex numbers. Furthermore, recall from the linear algebra that if the matrix is symmetric $A = A^*$, then the spectrum is real, and there exists a set of orthonormal eigenvectors that is a basis in $\mathbb{C}^N$. Let us study a few examples to see how many of these features survive in the infinite dimensional case.

Example 1: An empty spectrum. Let

$$\begin{aligned} A \ : \quad & D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = u(1) = 0 \,\} \to \mathcal{L}_2(0,1)\,, \\ & Au(x) = -iu'(x) \end{aligned}$$

This operator is symmetric $A \subseteq A^*$ because by integration by part

$$\begin{aligned} \langle Au, v\rangle &= -i\int_0^1 u'(x)\overline{v(x)}\,dx = i\int_0^1 u(x)\overline{v'(x)}\,dx \\ &= \langle u, Av\rangle\,, \quad \forall u, v \in D_A \end{aligned}$$

and the boundary term vanishes. It is not difficult to find the adjoint. One has to find all pairs (v, g) in the Hilbert space for which

$$\langle Au, v\rangle = \langle u, g\rangle\,, \quad \forall u \in D_A$$

Note that $\mathcal{D}(0,1) \subset D_A$ is dense in $\mathcal{L}_2$ and so is D_A. It was shown in the previous section that $D_{A^*} \subset AC^0[0,1]$ and $A^*v(x) = -iv(x)$ for any derivative operator (for any boundary conditions). It is important to investigate if the boundary conditions survive the symmetric extension. Using the integration by parts it is concluded that

$$\begin{aligned} \langle Au, v\rangle &= -i\int_0^1 u'(x)\overline{v(x)}\,dx = i\int_0^2 u(x)\overline{v'(x)}\,dx \\ &= \langle u, A^*v\rangle\,, \quad u \in D_A\,, \quad \forall v \in AC^0[0,1] \end{aligned}$$

The boundary terms vanish for any values of $v(1)$ and $v(0)$. Therefore, the boundary conditions do not survive the symmetric extension and

$$D_{A^*} = AC^0[0,1]\,, \quad R_{A^*} = \mathcal{L}_2(0,1)\,, \quad A^*v(x) = -iv(x)$$

The operator is not essentially self adjoint because the null space $N_{A^* \pm iI}$ contains non-zero elements:

$$-iv'(x) = \pm iv(x) \quad \Rightarrow \quad v(x) = Ce^{\pm x} \in AC^0[0,1]$$

Thus,

$$A \subset \bar{A} = A^{**} \subset A^* = (\bar{A})^*$$

The reader is advised to show that

$$D_{\bar{A}} = \{\, u \in AC^0[0,1] \,|\, u(0) = u(1) = 0 \,\}\,, \quad \bar{A}u(x) = -iu(x)\,,$$

that is, the boundary conditions survive the closure. So, the closure is symmetric but not self-adjoint.

Let is find eigenvalues of the closure

$$\begin{cases} \bar{A}u = \lambda u \\ u \in D_{\bar{A}} \end{cases} \Rightarrow \begin{cases} -iu'(x) = \lambda u(x) \\ u(0) = u(1) = 0 \end{cases} \Rightarrow \begin{cases} u(x) = Ce^{i\lambda x} = 0 \\ C = 0 \end{cases}$$

Thus, the resolvent $\mathcal{R}_A(\lambda) = (A - \lambda I)^{-1}$ exists for all $\lambda \in \mathbb{C}$ and, hence, the spectrum $\sigma(A)$ is empty, even though the operator in question is symmetric (or hermitian) in full contrast with the finite dimensional case.

It is not difficult to find the resolvent. One has to solve the boundary value problem:

$$(A - \lambda I)u = f\,, \quad u \in D_A\,,$$

where f must be from the range of $A - \lambda I$ which is a subset of $\mathcal{L}_2(0,1)$. It is not difficult to find the general solution to the differential equation

$$-iu'(x) - \lambda u(x) = f(x) \quad \Rightarrow \quad u(x) = Ce^{i\lambda x} + ie^{i\lambda x}\int_0^x e^{-i\lambda y} f(y)dy\,.$$

for any $f \in \mathcal{L}_2(0,1)$ so that $u \in AC^0[0,1]$. The solution must be from the domain $D_A \subset AC^0[0,1]$, that is, it has to satisfy the boundary conditions:

$$u(0) = 0 \quad \Rightarrow \quad u(x)C = 0 \quad \Rightarrow \quad u(x) = ie^{i\lambda x}\int_0^x e^{-i\lambda y} f(x)$$

$$u(1) = 0 \quad \Rightarrow \quad \int_0^1 e^{-i\lambda y} f(y)dy = 0 \quad \Rightarrow \quad \langle f, v_\lambda\rangle = 0\,,\ v_\lambda(x) = e^{i\lambda x}$$

The latter condition shows that the range of $A - \lambda I$ is not dense in the Hilbert space $\mathcal{L}_2(0,1)$ because f must be orthogonal to v_λ. Consequently, the domain of the resolvent is not dense in the Hilbert space:

$$\mathcal{R}_A(\lambda)f(x) = ie^{i\lambda x}\int_0^x e^{-i\lambda y} f(y)dy\,, \quad \langle f, v_\lambda\rangle = 0$$

The resolvent here is a bounded operator

$$|\mathcal{R}_A(\lambda)f(x)| \le \int_0^1 |f(x)|\,dx = \langle 1, |f|\rangle \le \|1\|\|f\| = \|f\|$$

Therefore

$$\|\mathcal{R}_A(\lambda)\| \le 1\,, \quad \forall \lambda \in \mathbb{C}$$

So, a symmetric operator can have no eigenvectors in the infinite dimensional case and its resolvent exists on the entire complex plane, in full contrast with the matrix theory.

Example 2: A discrete spectrum. Let

$$A \;:\quad D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = u(1) = 0 \,\} \to \mathcal{L}_2(0,1)\,,$$
$$Au(x) = -u''(x)$$

It was shown in the previous section that this operator has a symmetric extension

$$D_{A*} = \{\, v \in AC^1[0,1] \,|\, u(0) = u(1) = 0 \,\} \to \mathcal{L}_2(0,1)\,, \quad A^*u(x) = -v''(x)$$

Furthermore, A is essentially self-adjoint because the null space $N_{A^*\pm iI}$ contains only the trivial element $v = 0$. Thus,

$$A \subset \bar{A} = A^{**} = A^* = (\bar{A})^*$$

Let us investigate the resolvent of A (or its self-adjoint closure $\bar{A}$). Since

$$\lambda\langle u, u\rangle = \langle Au, u\rangle = \int_0^1 |u'(x)|^2 dx \geq 0,\,, \quad \forall u \in D_A\,,$$

any eigenvalue λ is strictly positive because a non-zero linear function does not does not satisfy the boundary conditions. The general solution to the equation

$$-u''(x) = \lambda u(x) \quad \Rightarrow \quad u(x) = A\cos(\nu x) + B\sin(\nu x)\,, \quad \nu = \sqrt{\lambda} > 0\,,$$

satisfies the boundary condition if and inly if

$$\nu = \nu_n = \pi n\,, \quad n = 1, 2, ...$$

Thus, the spectrum is discrete

$$\sigma(A) = \{\pi^2 n^2\}_1^\infty$$

so that the resolvent $\mathcal{R}_A(\lambda)$ exists if $\lambda \neq \pi^2 n^2$, $n = 1, 2, ...$ The corresponding eigenfunctions are

$$Au{=}\lambda_n u_n \quad \Rightarrow \quad u_n(x) = \sin(\pi n x)\,, \quad \langle u_n, u_m\rangle = 0\,, n \neq m$$

They are orthogonal and form an orthogonal basis in $\mathcal{L}_2(0,1)$ by the Steklov theorem[6] This example resembles the finite dimensional case with one difference that the spectrum and the associated basis contain countably many elements.

Let us construct the resolvent. For any complex λ

$$-u''(x) - \lambda u(x) = f(x)$$
$$\Rightarrow \quad u(x) = C_1\cos(\nu x) + C_2\,\frac{\sin(\nu x)}{\nu} - \int_0^x \frac{\sin(\nu(x-y))}{\nu}\, f(y)dy$$

[6]see, e.g., V.S. Vladimirov, Basic equations of mathematical physics, Chapter V, Sec. 22.3. This and related theorems will be discussed in the framework of the operator theory later.

where $\nu^2 = \lambda$. Note that the general solution depends analytically on $\nu^2 = \lambda$ (which follows from the power series representation of the cosine and sine functions). The resolvent is analytic in complex λ. The solution lies in the domain of the operator if the boundary conditions are fulfilled:

$$\begin{aligned} u(0) = 0 \quad &\Rightarrow \quad C_1 = 0 \\ u(1) = 0 \quad &\Rightarrow \quad C_2 = \int_0^1 \frac{\sin(\nu(1-y))}{\sin(\nu)}\, f(y)dy \\ \mathcal{R}_A(\lambda) f(x) = \frac{\sin(\nu x)}{\nu} &\int_0^1 \frac{\sin(\nu(1-y))}{\sin(\nu)}\, f(y)dy \\ &- \int_0^x \frac{\sin(\nu(x-y))}{\nu}\, f(y)dy \end{aligned}$$

Note that the integration constant C_2 exists if and only if $\nu \neq \pi n$ and the resolvent is symmetric under $\nu \to -\nu$. So, the resolvent does not exists only if λ is an eigenvalue of A, $\lambda = \pi n$, $n = 1, 2, ...$ The resolvent is a bounded operator for any λ for which it exists:

$$\exists\, \mathcal{R}_A(\lambda) \quad \Rightarrow \quad \|\mathcal{R}_A(\lambda)\| < \infty\,.$$

A verification of this assertion is left to the reader as an exercise.

It is worth noting that A (or better its closure) is proportional to the energy operator (Hamiltonian) of a particle in an infinite well. The eigenvalues of the Hamiltonian are admissible values of the energy of the particle. The operator $A = -id/dx$ is the momentum operator of the particle. The particle in a well has no states in which the momentum has a specific value as was found in Example 1, despite that the momentum operator is hermitian.

Example 3: Discrete spectrum in $\mathcal{L}_2(\mathbb{R})$. Now recall that the functions

$$\varphi_n(x) = H_n(x) e^{-x^2/2}\,, \quad n = 0, 1, ...$$

where $H_n(x)$ are Hermite polynomials, were proved to form and orthonormal basis in $\mathcal{L}_2(\mathbb{R})$ and

$$A\varphi_n(x) = -\varphi_n''(x) + x^2\varphi_n(x) = (2n+1)\varphi_n(x)$$

where A is defined on $D_A = C^2 \cap \mathcal{L}_2$. So, they are eigenfunctions of the operator A corresponding to the eigenvalues $\lambda_n = 2n+1$. The operator A can be symmetrically extended to

$$D_{A^*} = \{\, v \in AC^1 \cap \mathcal{L}_2 \,|\, A^* v = -v'' + x^2 v \in \mathcal{L}_2 \}\,, \quad A \subseteq A^*\,.$$

In the theory of ordinary differential equations, it is proved that the equation

$$-u''(x) + (x^2 - \lambda)u(x) = 0$$

is a particular case for the so called *hyper geometric* differential equation. Its general solution is a hyper-geometric function. If, in addition, it is demanded that the solution is square integrable, that is, necessarily $u(x) \to 0$ as $|x| \to \infty$, then such a hyper-geometric function exists if and only if $\lambda = 2n + 1$. This proves two things. First, the operator A is essentially self-adjoint ($\lambda = \pm i$ is not an eigenvalue of A^*):

$$A \subseteq \bar{A} = A^{**} = A^* = (\bar{A})^*$$

Second, the numbers $\lambda_n = 2n + 1$ form the spectrum $\sigma(A) = \sigma(\bar{A})$ as the resolvent $\mathcal{R}_A(\lambda)$ exists for all complex $\lambda \neq \lambda_n$. The resolvent has the form (see Problem 5 in Section **55.5**)

$$\mathcal{R}_A(\lambda)f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \varphi_n \rangle}{(\lambda_n - \lambda)\|\varphi_n\|^2}\, \varphi_n(x)$$

The resolvent is bounded for any λ for which it exists. Indeed, if $\lambda \neq \lambda_n$, then

$$|\lambda - \lambda_n| \geq a > 0\,, \quad \forall n$$

and some positive a because the sequence $\{\lambda_n\}$ has no limit point. Therefore

$$\|\mathcal{R}_A(\lambda)f\|^2 = \sum_{n=0}^{\infty} \frac{1}{|\lambda_n - \lambda|^2} \frac{|\langle f, \varphi_n \rangle|^2}{\|\varphi_n\|^2} \geq \frac{1}{a^2} \sum_{n=0}^{\infty} \frac{|\langle f, \varphi_n \rangle|^2}{\|\varphi_n\|^2} = \frac{\|f\|^2}{a^2}$$

and, hence,

$$\exists\, \mathcal{R}_A(\lambda) \quad \Rightarrow \quad \|\mathcal{R}_A(\lambda)f\| \leq \frac{1}{a} < \infty\,.$$

This example again resembles the finite dimensional case. In quantum theory, the operator A is proportional to the energy operator of a quantum harmonic oscillator.

Example 4: Continuous spectrum. The operator

$$A \ : \quad \mathcal{S} \to \mathcal{L}_2(\mathbb{R})\,, \quad Au(x) = -iu'(x)$$

is also symmetric on its domain (the space of test functions of temperate distributions) and it can be symmetrically extended $A \subseteq A^*$ where

$$A^* \ : \quad D_{A^*} = AC^0 \cap \mathcal{L}_2 \to \mathcal{L}_2\,, \quad A^*v(x) = -iv'(x)$$

The reader is advised to construct the adjoint A^*. The operator A is essentially self-adjoint because the equation

$$A^*v(x) = -iv'(x) = \pm iv(x)\,, \quad v \in \mathcal{L}_2 \quad \Rightarrow \quad v(x) = 0$$

has no square integrable solutions that are not identically zero. Therefore just like in Examples 2 and 3:

$$A \subseteq \bar{A} = A^{**} = A^* = (\bar{A})^*$$

In quantum mechanics, the operator A is the momentum operator of a particle moving on a line.

Let us investigate the resolvent. The equation

$$-iu'(x) = \lambda u(x)\,, \quad u \in \mathcal{L}_2 \quad \Rightarrow \quad u(x) = 0\,, \quad \forall \lambda \in \mathbb{C}$$

has no square integrable solutions for any complex λ. Therefore the operator $A - \lambda I$ is invertible for any $\lambda \in \mathbb{C}$.

Let us construct the resolvent. The problem

$$-iu'(x) - \lambda u(x) = f(x)\,, \quad u, f \in \mathcal{S}$$

can easily be solved by means of the Fourier transform if $\operatorname{Im}\lambda = b \neq 0$. Indeed, taking the Fourier transform of both sides of the equation, one infers that

$$\begin{aligned} & (k-\lambda)\mathcal{F}[u](k) = \mathcal{F}[f](k) \\ \Rightarrow \quad & u(x) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[f](k)}{k-\lambda}\right](x) = \mathcal{R}_A(\lambda)f(x)\,, \quad \operatorname{Im}\lambda = b \neq 0 \end{aligned}$$

This equation can be proved to hold for the closure of A because $\mathcal{S}$ is dense in $\mathcal{L}_2$ and the resolvent operator is bounded if $b \neq 0$. The boundedness of the resolvent on $\mathcal{S}$ can be established directly using its above explicit form. Here a more general result is invoked to prove the assertion.

THEOREM **59.1**. *For any $u \in \mathcal{L}_2$, its Fourier transform $\mathcal{F}[u]$ also belongs to $\mathcal{L}_2$, and the Fourier transform $\mathcal{F}: \mathcal{L}_2 \to \mathcal{L}_2$ is an isometry*

$$\|u\|^2 = 2\pi\|\mathcal{F}[u]\|^2\,, \quad \forall u \in \mathcal{L}_2$$

If $u \in \mathcal{S}$ (a test function for temperate distributions), then the assertion follows from $\mathcal{F}^{-1}[u](k) = (2\pi)^{-1}\mathcal{F}[u](-k)$ in $\mathbb{R}$. By this theorem

$$\begin{aligned} \|\mathcal{R}_A(\lambda)f\|^2 &= \left\|\mathcal{F}^{-1}\Big[\frac{\mathcal{F}[f](k)}{k-\lambda}\Big]\right\|^2 = \frac{1}{2\pi}\left\|\frac{\mathcal{F}[f](k)}{k-\lambda}\right\|^2 \\ &\leq \frac{1}{2\pi b^2}\Big\|\mathcal{F}[f]\Big\|^2 = \frac{1}{b^2}\,\|f\|^2 \end{aligned}$$

Therefore

$$\|\mathcal{R}_A(\lambda)f\| \leq \frac{1}{|b|}\,,\|f\|\,, \quad \forall f \in \mathcal{L}_2 \quad \Rightarrow \quad \|\mathcal{R}_A(\lambda)\| \leq \frac{1}{|\operatorname{Im}\lambda|}$$

If $\lambda \in \mathbb{R}$, the resolvent is unbounded. Let $u \in D_A$. Then $v(x) = e^{i\lambda x}u(x)$ is also from D_A if $\lambda \in \mathbb{R}$ and moreover

$$|u(x)| = |v(x)| \quad \Rightarrow \quad \|u\| = \|v\|$$

Therefore

$$|(A - \lambda I)v(x)| = |e^{i\lambda x}Au(x)| = |Au(x)| \quad \Rightarrow \quad \frac{\|(A-\lambda I)v\|}{\|v\|} = \frac{\|Au\|}{\|u\|}$$

Since the derivative operator is not bounded away from from zero in $\mathcal{L}_2(\mathbb{R})$ so is the operator $A - \lambda I$ for real λ. Therefore this operator cannot have a bounded inverse:

$$\|\mathcal{R}_A(\lambda)\| = \infty\,, \quad \lambda \in \mathbb{R}$$

The explicit form of the resolvent acting on $f \in \mathcal{S}$ can also be obtained by a suitable regularization of $(k - \lambda)^{-1}$ in the Fourier transform:

$$\mathcal{R}_A(\lambda)f(x) = i\int_x^\infty e^{i\lambda(x-y)}f(y)\,dy\,, \quad \lambda \in \mathbb{R}$$

Examples 1 and 4 have one thing in common, the operators have no eigenvalues. The difference is that the resolvent in Example 1 is bounded for all complex λ, while the resolvent in Example 4 is not bounded only for real λ.

59.3. The spectrum of an operator.

Definition **59.1**. (Regular value of an operator)
A complex number λ is called a regular value *of an operator A on a Hilbert space $\mathcal{H}$ if all of the following three properties hold:*

(R1) $\exists\, \mathcal{R}_A(\lambda) = (A - \lambda I)^{-1}$ *(the resolvent exists)*
(R2) $\|\mathcal{R}_A(\lambda)\| \leq \infty$ *(the resolvent is bounded)*
(R3) $\mathcal{R}_A(\lambda)$ *is defined on a dense set in $\mathcal{H}$*

The operator in Example 1 has no regular values because the property (R3) fails to hold. In examples 2 and 3 any complex λ that is not equal to an eigenvalue is a regular value. All non-real λ are regular values of the operator in Example 4.

Definition **59.2**. (Resolvent set)
The collection of all regular values of an operator A is called the resolvent set *of A*

$$\rho(A) = \{\, \lambda \in \mathbb{C} \,|\, \lambda = \text{ regular value}\,\}$$

Definition 59.3. (Spectrum of an operator)
The complement

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

is called the spectrum *of the operator A and a complex number $\lambda \in \sigma(A)$ from the spectrum is called a* spectral value *of A.*

The spectrum of the operator from Example 1 is the whole complex plane $\sigma(A) = \mathbb{C}$ because the operator has no regular values. The spectrum of operators in Examples 2 and 3 is a real sequence $\{\lambda_n\} \subset \mathbb{R}$ that has no limit point. The spectrum of the operator in Example 4 consists of all real numbers, $\sigma(A) = \mathbb{R}$.

59.4. Properties of the spectrum.

Theorem 59.2. *The spectrum of an operator A in a Hilbert space is the union of three disjoint sets*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$$

where $\sigma_p(A)$ is called the point (or discrete) spectrum *and consists of all spectral values for which the resolvent does not exist; $\sigma_c(A)$ is called the* continuum spectrum *and consists of all spectral values for which the resolvent exists but it is not bounded; and $\sigma_r(A)$ is called the* residual spectrum *and consists of all spectral values for which the resolvent exists but it is not defined on a dense set in a Hilbert space.*

The spectra of the operators from the examples considered above have the following classification:

$$\begin{aligned} \text{Example 1}: &\quad \sigma(A) = \sigma_r(A) = \mathbb{C} \\ \text{Examples 2 and 3}: &\quad \sigma(A) = \sigma_p(A) = \{\lambda_n\} \subset \mathbb{R} \\ \text{Example 4}: &\quad \sigma(A) = \sigma_c(A) = \mathbb{R} \end{aligned}$$

Definition 59.4. (Approximate and compression spectra)
The union

$$\sigma_a(A) = \sigma_c(A) \cup \sigma_p(A)$$

is called the approximate spectrum *of the operator A. The union*

$$\sigma_{\rm com}(A) = \sigma_p(A) \cup \sigma_r(A)$$

is called the compression spectrum *of the operator A.*

The name "compression" stems from that the range of the operator $A - \lambda I$ is compressed so that its closure is a proper subset in the Hilbert space $R_{A-\lambda I} \subseteq \overline{R_{A-\lambda I}} \subset \mathcal{H}$.

The following table defines the type of a spectrum using regular values of the operator.

Properties satisfied	Properties failed	$\lambda \in$
(R1), (R2), (R3)	$--$	$\rho(A)$
$--$	(R1)	$\sigma_p(A)$
(R1), (R3)	(R2)	$\sigma_c(A)$
(R1)	(R3)	$\sigma_r(A)$

The next table defines the type of a spectrum using the properties of the resolvent

Properties of the resolvent	$\lambda \in$
$\exists\,\mathcal{R}_A(\lambda)\,,\ \Vert\mathcal{R}_A(\lambda)\Vert < \infty\,,\ \overline{R_{A-\lambda I}} = \mathcal{H}$	$\rho(A)$
$\mathcal{R}_A(\lambda)$ does not exist	$\sigma_p(A)$
$\exists\,\mathcal{R}_A(\lambda)\,,\ \Vert\mathcal{R}_A(\lambda)\Vert = \infty\,,\ \overline{R_{A-\lambda I}} = \mathcal{H}$	$\sigma_c(A)$
$\exists\,\mathcal{R}_A(\lambda)\,,\ \overline{R_{A-\lambda I}} \subset \mathcal{H}$	$\sigma_r(A)$

Finally, recall the classification of operators from the point of view of solvability of the linear problem $(A - \lambda I)u = f$ given in Section **56.6**. The following table defines the type of a spectrum using the class of the operator $A - \lambda I$.

Class of $A - \lambda I$	$\lambda \in$
(I,1*)	$\rho(A)$
(III,*)	$\sigma_p(A)$
(II,*)	$\sigma_c(A)$
(I,2*), (II,2*)	$\sigma_r(A)$

Here any suitable classification index can put in place of the star. For example, (I,1*) means that either (I,1c) or (I,1n) (in this case suitable classification indices are “c” and “n”). The characteristic property of the compression spectrum is

$$\overline{R_{A-\lambda I}} \subset \mathcal{H} \quad \Leftrightarrow \quad \lambda \in \sigma_{\rm com}(A)$$

59.5. The spectrum of a symmetric operator.

THEOREM **59.3**. (Spectrum of a symmetric operator)
If A is a symmetric operator, $A \subseteq A^*$, then

$$\langle Au, u\rangle \in \mathbb{R}\,, \quad \forall u \in D_A$$

the approximate spectrum is real

$$\sigma_a(A) = \sigma_p(A) \cup \sigma_c(A) \subseteq \mathbb{R}$$

and two eigenvectors corresponding to different eigenvalues are orthogonal

$$\langle u_1, u_2\rangle = 0\,, \quad \forall \lambda_1 \neq \lambda_2 \in \sigma_p(A)\,, \quad Au_j = \lambda_j u_j\,,\ j = 1, 2$$

PROOF. The first property follows from the definition of the inner product and the adjoint:

$$\langle Au, u\rangle = \overline{\langle u, Au\rangle} = \overline{\langle Au, u\rangle} \quad \Rightarrow \quad \langle Au, u\rangle \in \mathbb{R}$$

The approximate spectrum contains the point spectrum which is the collection of all eigenvalues of the operator. Let $\lambda \in \sigma_p(A)$ be an eigenvalue. Then there exists a non-zero element $u \in D_A$ such that

$$\lambda \in \sigma_p(A) \quad \Rightarrow \quad Au = \lambda u \quad \Rightarrow \quad \lambda = \frac{\langle Au, u\rangle}{\|u\|^2} \in \mathbb{R}$$

by the first property. Suppose that $\lambda \in \sigma_c(A)$. In this case the resolvent is not bounded and, hence, the operator $A - \lambda I$ is not bounded away from zero:

$$\lambda \in \sigma_c(A) \quad \Rightarrow \quad \|\mathcal{R}_A(\lambda)\| = \infty \quad \Rightarrow \quad \inf_{u \neq 0} \frac{\|(A-\lambda I)u\|}{\|u\|} = 0$$

Put $\lambda = \xi + i\eta$. Suppose the contrary is true, that is, $\eta \neq 0$. Using the basic properties of the inner product and that A is symmetric

$$\begin{aligned}\|Au - \lambda u\|^2 &= \langle Au - \xi u - i\eta u, Au - \xi u - i\eta u\rangle = \|Au - \xi u\|^2 + \eta^2\|u\| \\ &\geq \eta^2\|u\|\end{aligned}$$

which shows that $A - \lambda I$ is bounded away from zero and, hence, its inverse $\mathcal{R}_A(\lambda)$ must be bounded. A contradiction. Thus, $\eta = 0$ and $\lambda_a(A) \subseteq \mathbb{R}$.

The third property follows from the equality

$$\begin{aligned}&\lambda_1\langle u_1, u_2\rangle = \langle Au_1, u_2\rangle = \langle u_1, Au_2\rangle = \lambda_2\langle u_1, u_2\rangle \\ \Rightarrow \quad &(\lambda_1 - \lambda_2)\langle u_1, u_2\rangle = 0 \quad \Rightarrow \quad \langle u_1, u_2\rangle = 0\end{aligned}$$

because A is symmetric, its eigenvalues are real, and $\lambda_1 \neq \lambda_2$. □

DEFINITION **59.5**. (Deficiency of a spectral value)
A spectral value λ is said to have the deficiency *m if the orthogonal complement of the range of $A - \lambda I$ has the dimension m:*

$$m = \dim R^{\perp}_{A-\lambda I} \,, \quad \lambda \in \sigma_{\rm com}(A)$$

The deficiency of a spectral value show how much the range of the operator $A - \lambda I$ is smaller than the whole Hilbert space. Recall from the linear algebra that if A is a symmetric matrix, then m is the multiplicity of an eigenvalue λ. A similar result holds in general.

THEOREM **59.4**. (Compression spectrum)
A spectral value $\lambda \in \sigma_{\rm com}(A)$ of an operator A has the deficiency m if and only if the complex conjugate $\bar{\lambda}$ is an eigenvalue of the adjoint A^* with multiplicity m:

$$m = \dim R^{\perp}_{A-\lambda I} = \dim N_{A^*-\bar{\lambda} I} \,, \quad \bar{\lambda} \in \sigma_p(A^*) \,.$$

PROOF. Consider an operator B. Then $Bu \in R_B$ for any u in the domain of B. The the orthogonal complement $R_B^{\perp}$ is defined by

$$v \in R_B^{\perp} \quad \Leftrightarrow \quad \langle Bu, v\rangle = 0 \,, \quad \forall u \in D_B$$

Consider the adjoint B^*. There are pairs (v, g) such that

$$\langle Bu, v\rangle = \langle u, g\rangle \,, \quad \forall u \in D_B \quad \Rightarrow \quad B^* v = g \,, \quad v \in D_{B^*}$$

If $v \in R_B^{\perp}$, then a suitable pair is $(v, 0)$ so that $B^* v = 0$. Therefore $R_B^{\perp} \subseteq N_{B^*}$ (the null space of B^*). Conversely, if $v \in N_{B^*}$, then

$$0 = \langle u, B^* v\rangle = \langle Bu, v\rangle \,, \quad \forall u \in D_B \quad \Rightarrow \quad v \in R_B^{\perp}$$

Thus,

$$R_B^{\perp} = N_{B^*}$$

Put $B = A - \lambda I$. Then

$$R^{\perp}_{A-\lambda I} = N_{A^*-\bar{\lambda} I}$$

The null space $N_{A^*-\bar{\lambda} I}$ is the span of all eigenvectors

$$A^* v = \bar{\lambda} v \,, \quad \bar{\lambda} \in \sigma_p(A^*) \,.$$

Suppose that $\lambda \in \sigma_{\rm com}(A)$ has the deficiency m. Then

$$m = \dim R^{\perp}_{A-\lambda I} = \dim N_{A^*-\bar{\lambda} I}$$

is the multiplicity of the eigenvalue $\bar{\lambda}$ of the adjoint and $\bar{\lambda} \in \sigma_p(A^*)$. Conversely, suppose that $\bar{\lambda} \in \sigma_p A^*$ is and eigenvalue of A^* with multiplicity m. This implies that $R^{\perp}_{A-\lambda I}$ contains non-zero vectors and λ has the deficiency m:

$$m = \dim N_{A^*-\bar{\lambda}I} = \dim R^{\perp}_{A-\lambda I}$$

□

THEOREM **59.5**. (Spectrum of a self-adjoint operator)
If A is a self-adjoint operator, $A = A^*$, then its spectrum is real:

$$\sigma(A) \subseteq \mathbb{R}$$

its residual spectrum is empty:

$$\sigma_r(A) = \emptyset \quad \textit{or} \quad \sigma_{\rm com}(A) = \sigma_p(A)$$

and multiplicity of any eigenvalue $\lambda \in \sigma_p(A)$ is the deficiency of λ:

$$m = \dim N_{A-\lambda I} = \dim R^{\perp}_{A-\lambda I}$$

PROOF. Since A is symmetric, its approximate spectrum is real, $\sigma_a(A) \subseteq \mathbb{R}$, and, hence, any complex λ that is not real must be either in the resolvent set $\rho(A)$ or in the the residual spectrum $\sigma_r(A)$. Suppose that the residual spectrum is not empty, $\sigma_r(A) \neq \emptyset$. Then for any $\lambda \in \sigma_r(A)$, the complex conjugate $\bar{\lambda}$ must be in the point spectrum of the adjoint A^* (by the compression spectrum theorem) and hence be also in the point spectrum of A because $A^* = A$. Since the point spectrum is real, so must be the residual spectrum. But the residual and point spectra are disjoint sets. A contradiction.

$$\lambda \in \sigma_r(A) \quad \Rightarrow \quad \bar{\lambda} \in \sigma_p(A^*) = \sigma_p(A) \stackrel{\sigma_r(A)\cap\sigma_p(A)=\emptyset}{\Rightarrow} \sigma_r(A) = \emptyset$$

Since the residual spectrum is empty, the compression and point spectra coincide, $\sigma_{\rm com}(A) = \sigma_p(A) = \sigma_p(A^*)$, which means that any spectral value with non-zero deficiency is also an eigenvalue whose multiplicity is equal to the deficiency. □

59.6. How to find the spectrum of an operator. The properties of spectral values of an operator in a Hilbert space allows us to develop a procedure to find the spectrum of an operator. The domain of the operator is assumed to be dense in the Hilbert space in order to define the adjoint.

Step 1: Point spectrum, $\sigma_p(A)$. Solve the eigenvalue problem

$$Au = \lambda u\,, \quad u \in D_A \quad \Rightarrow \quad \lambda \in \sigma_p(A)$$

to find the point spectrum

Step 2: Compression spectrum, $\sigma_{\text{com}}(A)$. Construct the adjoint A^* and solve the eigenvalue problem for the adjoint

$$A^* v = \lambda v\,, \quad v \in D_{A^*} \quad \Rightarrow \quad \bar{\lambda} \in \sigma_{\text{com}}(A)$$

to find the compression spectrum of A as a collection of the complex conjugated eigenvalues of the adjoint.

Step 3: Residual spectrum, $\sigma_r(A)$. Find the residual spectrum using the decomposition

$$\sigma_{\text{com}}(A) = \sigma_r(A) \cup \sigma_p(A) \quad \Rightarrow \quad \sigma_r(A) = \sigma_{\text{com}}(A) \setminus \sigma_p(A)$$

because the residual and point spectra are disjoint sets.

Step 4: Continuum spectrum, $\sigma_c(A)$. For any $\lambda \notin \sigma_{\text{com}}(A)$, the resolvent $\mathcal{R}_A(\lambda)$ exists. If the resolvent is bounded, then $\lambda \in \rho(\lambda)$ (the resolvent set), otherwise $\lambda \in \sigma_c(A)$:

$$\|\mathcal{R}_A(\lambda)\| = \infty\,, \quad \lambda \notin \sigma_{\text{com}}(A) \quad \Rightarrow \quad \lambda \in \sigma_c(A)$$

This can either be accomplished by evaluating the norm of the resolvent if its explicit form is found or by finding all λ for which the operator $A - \lambda I$ is not bounded away from zero.

Remark. If the operator in question is self-adjoint, then $\sigma(A) = \sigma_a(A) = \sigma_p(A) \cup \sigma_c(A) \subset \mathbb{R}$ and Steps 2 and 3 are not needed.

Example 5. Consider the operator

$$A \;:\quad \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Au(x) = xu(x) + \int_0^1 u(x)dx$$

Let us find its spectrum. The operator is symmetric on its domain. Since the domain is the whole Hilbert space, the operator is self-adjoint, $A = A^*$. Let us find the point spectrum.

$$Au(x) = \lambda u(x)\,, \quad \lambda \in \mathbb{R} \quad \Rightarrow \quad (x-\lambda)u(x) = -\int_0^1 u(x)\,dx = C$$

If an eigenfunction exists, it should have the form

$$u(x) = \frac{C}{x-\lambda} \in \mathcal{L}_2(0,1) \quad \Rightarrow \quad \lambda \notin [0,1]$$

for some constant $C \neq 0$. The eigenvalues, if any, must lie outside the interval $[0,1]$. Substituting $u(x)$ into the eigenvalue equation and using that

$$\int_0^1 u(x) = C\int_0^1 \frac{dx}{x-\lambda} = C(\ln|1-\lambda| - \ln|\lambda|)$$

the eigenvalue problem is reduced to the equation

$$1 = \ln|\lambda| - \ln|1-\lambda| \quad \Rightarrow \quad \lambda = \lambda_0 = \frac{e}{e-1} \notin [0,1]$$

Thus, the point spectrum contains a single number

$$Au = \lambda u \quad \Rightarrow \quad \lambda \in \sigma_p(A) = \{\lambda_0\}\,, \quad u(x) = \frac{1}{\lambda_0 - x}$$

Let us find the continuum spectrum. If $\lambda \notin [0,1]$ and $\lambda \neq \lambda_0$, then the resolvent is easy to find:

$$Au(x) - \lambda u(x) = f(x) \quad \Rightarrow \quad u(x) = \frac{f(x)}{x-\lambda} - \frac{C}{x-\lambda}$$

where the constant C is chosen so that $u(x)$ is the solution:

$$\begin{aligned} C &= \int_0^1 u(x)dx = \int_0^1 \frac{f(x)}{x-\lambda} - C\int_0^1 \frac{dx}{x-\lambda} \\ \Rightarrow \quad C &= \frac{1}{1+\ln\left|\frac{1-\lambda}{\lambda}\right|}\int_0^1 \frac{f(y)}{y-\lambda}\,dy \\ \Rightarrow \quad \mathcal{R}_A f(x) &= \frac{1}{x-\lambda}\left(f(x) - \frac{1}{1+\ln\left|\frac{1-\lambda}{\lambda}\right|}\int_0^1 \frac{f(y)}{y-\lambda}\,dy\right) \end{aligned}$$

The resolvent is bounded. Indeed, if $\lambda \notin [0,1]$, then there is a positive number a such that

$$0 < a \leq |x-\lambda|\,, \quad x \in [0,1]$$

Therefore

$$\|\mathcal{R}_A f\|^2 \leq \frac{1}{a^2}\|f - C\|^2 \leq \frac{1}{a^2}\Big(\|f\| + |C|\Big)^2$$

where the triangle inequality was used $\|f - C\| \leq \|f\| + |C|$ because the norm of a unit function is equal to one, $\|1\| = 1$, is this case. The constant $|C|$ can be estimated as follows:

$$|C| \leq \left|1+\ln\left|\frac{1-\lambda}{\lambda}\right|\right|^{-1}\frac{1}{a}\int_0^1 |f(y)|dy \leq \left|1+\ln\left|\frac{1-\lambda}{\lambda}\right|\right|^{-1}\frac{\|f\|}{a}$$

because $\langle 1, |f|\rangle \leq \|f\|$. Thus,

$$\|\mathcal{R}_A f\| \leq M\|f\|\,, \quad \forall f \in \mathcal{L}_2(0,1) \quad \Rightarrow \quad \|\mathcal{R}_A\| < \infty$$

Note that the bound M increases as $\lambda \to \lambda_0$. So, the continuum spectrum, if any, must be in the interval $\sigma_c(A) \subseteq [0,1]$.

The resolvent exists in this interval because

$$Au(x) - \lambda u(x) = 0 \quad \leftrightarrow \quad u(x) = \frac{\int_0^1 u(y)dy}{x - \lambda}$$

The right side of the equation cannot be square integrable for any $\lambda \in [0,1]$ for any $u(x) \neq 0$. Therefore only the trivial solution $u(x) = 0$ is possible, which means that the operator $A - \lambda I$ is invertible. It is not straightforward to find an explicit form of the inverse. However, the question about its boundedness can be answered without it. The operator $A - \lambda I$ $\lambda \in [0,1]$, is proved be not bounded away from zero and, hence, its inverse is not bounded. Indeed, for any $\lambda \in (0,1)$, consider the sequence

$$u_n(x) = \begin{cases} 1 \ , & x \in (\lambda, \lambda + \frac{1}{n}) \\ -1 \ , & x \in (\lambda - \frac{1}{n}, \lambda) \\ 0 \ , & \text{otherwise} \end{cases}$$

Then for all n for which $(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}) \subset (0,1)$,

$$\|(A - \lambda I)u_n\|^2 = \frac{2}{3n^2}\,, \quad \|u_n\|^2 = \frac{2}{n}$$

so that

$$\frac{\|(A - \lambda I)u_n\|^2}{\|u_n\|^2} = \frac{1}{3n} \to 0 \quad \text{as } n \to \infty$$

This implies that $A - \lambda I$ is not bounded away from zero. In a similar fashion, it is not difficult to construct such sequences for $\lambda = 0$ and $\lambda = 1$. Thus, the continuum spectrum is $\sigma_c = [0,1]$ and

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) = [0,1] \cup \{\lambda_0\}\,, \quad \rho(A) = \mathbb{C} \setminus \sigma(A)$$

59.7. Exercises.

1. Spectrum of a projection operator

Let $\{v_n\}$ be an orthonormal set that is not complete in a Hilbert space $\mathcal{H}$. Define the operator:

$$P \ : \quad \mathcal{H} \to \mathcal{H}\,, \quad Pu = \sum_n \langle u, v_n\rangle v_n$$

(i) Show that

$$P^2 = P \quad \text{and} \quad P^* = P$$

(ii) Show that

$$\sigma_p(P) = \{1, 0\}$$

(iii) Determine the range of P. Show that for any $\lambda \notin \sigma_p(P)$, the resolvent is

$$\mathcal{R}_P(\lambda)f = (1-\lambda)^{-1}Pf - \lambda^{-1}(f - Pf)$$

(iv) Show that the resolvent is bounded and that

$$\sigma(P) = \sigma_p(P) = \{1, 0\}$$

2. The derivative operator on a circle.
Define the operator

$$A \ : \quad D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = u(1)\,\}\,, \quad Au(x) = -iu'(x)$$

(i) Show that

$$\sigma_p(A) = \{2\pi n\}_{-\infty}^{\infty}$$

(ii) Construct the adjoint A^*. Show that A has a self-adjoint extension and

$$\sigma_r(A) = \emptyset$$

(iii) Show that the resolvent is

$$\mathcal{R}_A f(x) = C_f e^{i\lambda x} + i\int_0^x e^{i\lambda(x-y)} f(y)\,dy$$
$$C_f = \frac{ie^{i\lambda}}{1-e^{i\lambda}} \int_0^1 e^{-i\lambda y} f(y)\,dy$$

for any complex $\lambda \notin \sigma_p(A)$.
(iv) Prove that $\|\mathcal{R}_A\| < \infty$ and find the spectrum of the operator.

3. Repeat the analysis of Problem 2 for the operator

$$A \ : \quad D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = e^{i\theta}u(1)\,\}\,, \quad Au(x) = -iu'(x)$$

where $\theta \in [0, 2\pi)$. Find the spectrum $\sigma(A)$

4. Repeat the analysis of Problem 2 for the operator

$$A \ : \quad D_A = \{\, u \in C^1([0,1]) \,|\, u(0) = zu(1)\,\}\,, \quad Au(x) = -iu'(x)$$

where $z \in \mathbb{C}$. Find the spectrum $\sigma(A)$. Note that the operator is not longer symmetric if $|z| \neq 1$.

5. Quantum particle on a half-axis
Consider the operator

$$A \;:\quad D_A = \{u \in C^2([0,\infty)) \cap \mathcal{L}_2(0,\infty)\,|\,u(0)=0\} \to \mathcal{L}_2(0,\infty)$$
$$Au(x) = -u''(x)$$

(i) Show that A is symmetric, $A \subset A^*$, and show that

$$D_{A^*} = \{u \in AC^1[0,\infty) \cap \mathcal{L}_2(0,\infty)\,|\,u(0)=0\}$$

Show that A is essentially self-adjoint, and, conclude that the self-adjoint extension is $\bar{A} = A^* = (\bar{A})^*$.
(ii) Show that $\sigma_p(A) = \emptyset$.
(iii) For all complex $\lambda \notin [0,\infty)$ find the resolvent by solving the boundary value problem:

$$-u''(x) - \lambda u(x) = f(x)\,, \quad u \in D_A$$

To do so, find the Green's function

$$-G''_{xx}(x,y) - \lambda G(x,y) = \delta(x-y)\,,$$
$$G(0,y) = 0\,, \quad \lim_{x\to\infty} G(x,y) = 0\,, \quad \forall y \in (0,\infty)$$

A particular solution can be found by taking the Fourier transform of the equation so that the general solution is

$$G(x,y) = C_1\cos(\sqrt{\lambda}x) + C_2\,\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + \mathcal{F}^{-1}\left[\frac{e^{iky}}{k^2-\lambda}\right](x)$$

Find the constants $C_{1,2}$ to satisfy the boundary condition and show that

$$G(x,y) = \theta(x-y)\frac{\sin(\sqrt{\lambda}y)}{\sqrt{\lambda}}\,e^{i\sqrt{\lambda}x} + \theta(y-x)\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\,e^{i\sqrt{\lambda}y}$$

and that $G(x,y) \in \mathcal{L}_2(0,\infty)$ for any fixed $x \in (0,\infty)$ so that

$$\mathcal{R}_A(\lambda) = \int_0^\infty G(x,y)f(y)\,dy$$

where the cut of the complex plane to define $\sqrt{\lambda}$ is taken over the half-axis $\operatorname{Re}\lambda \ge 0$ so that $\lambda = |\lambda|e^{i\theta}$ and $\sqrt{\lambda} = \sqrt{|\lambda|}e^{i\theta/2}$, where $\theta \in [0,2\pi)$.
(iv) Show that $\sigma(A) = \sigma_c(A) \subset \mathbb{R}$. Show that if $\lambda < 0$

$$\|\mathcal{R}_A(\lambda)\| < \infty\,, \quad \sqrt{\lambda} = i\nu\,, \quad \nu > 0\,.$$

This could be done either by showing $\|\mathcal{R}_A(\lambda)f\| \le M\|f\|$ or by showing that $A - \lambda I$ is bounded away from zero (in the latter case, Step (iii) can be omitted if one is interested only in finding the spectrum $\sigma(A)$).

Conclude that $\sigma_c(A) \subseteq [0, \infty)$.
(v) Show that $A - \lambda I$ is invertible if $\lambda \geq 0$. Use the sequence

$$v_n = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} e^{-x/n} \in D_A$$

to show that $A - \lambda I$ is not not bounded away from zero. The case $\lambda = 0$ is understood as the limit $\lambda \to 0^+$ in v_n. Conclude that

$$\sigma(A) = \sigma_c(A) = [0, \infty)$$

which is the energy spectrum of a quantum particle on a half-line.

6. A justification of the large box approximation in quantum mechanics
Consider the operator

$$\begin{aligned} &A \; : \quad D_A = \{\, u \in C^2([0,b]) \,|\, u(0) = u(b) = 0 \,\} \to \mathcal{L}_2(0,b)\,, \\ &Au(x) = -u''(x) \end{aligned}$$

(i) Show that A is symmetric. Construct the adjoint:

$$D_{A^*} = \{\, u \in AC^1[0,b] \,|\, u(0) = u(b) = 0 \,\}$$

Show that A is essentially self-adjoint. Conclude that $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$
(ii) Show that

$$\sigma_p(A) = \left\{ \left(\frac{\pi n}{b}\right)^2 \,,\; n = 1, 2, ... \right\}$$

(iii) Show that the distribution

$$G_b(x,y) = \frac{\sin\left(\sqrt{\lambda}\, x_<\right) \sin\left(\sqrt{\lambda}\,(b - x_>)\right)}{\sqrt{\lambda}\sin(l\sqrt{\lambda})}\,, \quad \lambda \notin \sigma_p(A)$$

where $x_< = \min(x,y)$, $x_> = \max(x,y)$, and $x, y \in (0,b)$, is the Green function:

$$\begin{aligned} &\Big(-\frac{d^2}{dx^2} - \lambda\Big) G_b(x,y) = \delta(x-y)\,, \quad 0 < y < b\,, \\ &G_b(0,y) = G(b,y) = 0\,, \quad 0 < y < b\,, \\ &G_b(x,y) \in \mathcal{L}_2(0,b)\,, \quad 0 < x < b \end{aligned}$$

Conclude that

$$\mathcal{R}_A(\lambda) f(x) = \int_0^b G_b(x,y) f(y)\, dy$$

(iv) Show that

$$\|\mathcal{R}_A(\lambda)\|^2 \leq \int_0^b \int_0^b |G_b(x,y)|^2 dy dy < \infty\,,$$

and conclude that

$$\sigma_c(A) = \emptyset\,, \quad \sigma(A) = \sigma_p(A)$$

(v) If $\lambda \notin [0, \infty)$, find the pointwise limit

$$\lim_{b\to\infty} G_b(x, y)$$

Compare the resolvent set of the operator in this problem in the limit $b \to \infty$ with the resolvent set of the operator in Problem 5. The outlined limiting procedure is often used in quantum mechanics to interpret a continuum spectrum.

60. Compact operators

There is a particular class of operators in a Hilbert space whose properties are close to those of matrices.

DEFINITION **60.1**. (Compact operator)
An operator K is compact *in a Hilbert space if the image $\{Ku_n\}$ of any bounded sequence $\{u_n\}$, $\|u_n\| \le M$, in the domain of K contains a convergent subsequence:*

$$\exists\,\{n_k\}\ :\quad \{v_k\} = \{Ku_{n_k}\}\ \textit{is a Cauchy sequence}$$

A compact operator has several equivalent definition. Here is another one that is often used in applications.

THEOREM **60.1**. *An operator is compact* if and only *if it maps every weakly convergent sequence to a convergent sequence in the norm:*

$$\forall\{u_n\}\ :\ \lim\langle u_n - u, v\rangle = 0\,,\quad \forall v\in\mathcal{H}\quad\Rightarrow\quad \lim\|Ku_n - Ku\| = 0$$

This theorem offers a necessary and sufficient condition for an operator to be compact. So, it can be used as an alternative definition of a compact operator. The compact operators are also called completely continuous operators because, as is shown below, they form a special subset of bounded operators.

60.1. Properties of compact operators.

PROPOSITION **60.1**. *A compact operator is bounded:*

$$K\ \textit{is compact}\quad\Rightarrow\quad \|K\| < \infty \tag{60.1}$$

This implies in particular that any compact operator can be extended to the whole Hilbert space. So, in what follows

$$K\ :\quad D_K = \mathcal{H}\to\mathcal{H}$$

PROOF. Suppose that the converse is true, $\|K\| = \infty$. This implies that there is a unit sequence $\{u_n\}$, $\|u_n\| = 1$, whose image diverges:

$$\|K\| = \infty\quad\Rightarrow\quad \exists\,\{u_n\}\,,\ \|u_n\| = 1\ :\quad \lim_{n\to\infty}\|Ku_n\| = \infty$$

Therefore one can always select a monotonically increasing subsequence:

$$\{Ku_m\}\subset\{Ku_n\}\ :\quad \|Ku_{m_1}\| < \|Ku_{m_2}\|\,,\ \forall m_1 < m_2$$

Then the sequence $\{u_m\}$ is bounded because $\|u_m\| = 1$, but its image has no convergent subsequence. A contradiction. Thus, the operator is bounded, $\|K\| < \infty$. □

PROPOSITION **60.2**. *Not every bounded operator is compact:*

$$(\mathbf{60.2}) \qquad \text{the converse of } (\mathbf{60.1}) \text{ is false}$$

Let $A = I$. It is a bounded operator $\|A\| = \|I\| = 1$. Take an orthonormal basis in an infinite dimensional Hilbert space $\{u_n\}_1^\infty$. Then $Au_n = u_n$. It is impossible to fund any convergent subsequence in an orthonormal set because all elements are mutually orthogonal. So, the identity operator is bounded, but not compact.

PROPOSITION **60.3**. *A compact operator maps every infinite orthonormal set to a null sequence:*

$$(\mathbf{60.3}) \qquad \left.\begin{array}{l} K \text{ is compact} \\ \{u_n\}_1^\infty \;:\; \langle u_n, u_m\rangle = \delta_{mn} \end{array}\right\} \quad\Rightarrow\quad \lim_{n\to\infty} \|Ku_n\| = 0$$

PROOF. Suppose the converse is true, that is, $\{Ku_n\}$ is not a null sequence. This means that $\{Ku_n\}$ has infinitely many terms outside any neighborhood of the zero element. Therefore for any $\varepsilon > 0$ there exists a subsequence such that

$$\exists\,\{v_n\} \subset \{u_n\} \;:\; \quad \|Kv_n\| > \varepsilon > 0$$

Since K is compact, there is a subsequence of $\{v_n\}$ whose image converges to some element u:

$$K \text{ is compact} \quad\Rightarrow\quad \exists\,\{w_n\} \subset \{v_n\} \;:\; \quad \lim Kw_n = u \neq 0$$

because $\|Kw_n\| > \varepsilon > 0$. By continuity of the inner product:

$$\lim_{n\to\infty} \langle Kw_n, u\rangle = \|u\|^2 \neq 0$$

By (**60.1**) K is bounded and, hence, its domain can always be extended to the whole Hilbert space, while preserving its norm. Therefore a compact operator has the adjoint defined on the whole Hilbert space and

$$\langle Kw_n, u\rangle = \langle w_n, K^*u\rangle$$

The sequence $\{w_n\}$ is an orthonormal set by construction, $\langle w_n, w_m\rangle = \delta_{nm}$, By the Bessel inequality

$$\sum_n |\langle w_n, v\rangle|^2 \leq \sum_n |\langle u_n, v\rangle|^2 \leq \|v\|^2\,, \quad \forall v \in \mathcal{H}$$

The convergence of the series implies that its terms tend to zero:

$$\lim_{n\to\infty} |\langle w_n, v\rangle| = 0\,, \quad \forall v \in \mathcal{H}$$

In particular,

$$v = K^*u \quad\Rightarrow\quad \lim_{n\to\infty} |\langle w_n, K^*u\rangle| = 0$$

But the sequence $\langle w_n, K^*u\rangle = \langle Kw_n, u\rangle$ was shown to converge to $\|u\|^2 > 0$. A contradiction. Thus, $\{Ku_n\}$ is a null sequence. □

PROPOSITION **60.4**. *The inverse of a compact operator in an infinite dimensional Hilbert space is unbounded:*

$$\left.\begin{array}{l} K \textit{ is compact} \\ \exists\, K^{-1} \\ \dim \mathcal{H} = \infty \end{array}\right\} \quad \Rightarrow \quad \|K^{-1}\| = \infty \tag{60.4}$$

or, K is not bounded away from zero.

PROOF. Since $\dim \mathcal{H} = \infty$, there is an infinite orthonormal set $\{u_n\}_1^\infty$, $\langle u_n, u_m\rangle = \delta_{nm}$. By (**60.3**), its image is a null sequence $\lim \|Ku_n\| = 0$. This implies that K is not bounded away from zero and, hence, cannot have a bounded inverse:

$$\exists\, K^{-1}\,, \quad \|u_n\| = 1\,, \quad \lim_{n\to\infty} \|Ku_n\| = 0 \quad \Rightarrow \quad \|K^{-1}\| = \infty$$

□

PROPOSITION **60.5**. *The limit a sequence of compact operators that strongly converges(in the operator norm) is a compact operator:*

$$\left.\begin{array}{l} \{K_n\}_1^\infty\,,\ K_n \textit{ is compact} \\ \exists\, K \ :\ \lim_{n\to\infty} \|K - K_n\| = 0 \end{array}\right\} \quad \Rightarrow \quad K \textit{ is compact} \tag{60.5}$$

PROOF. One has to show that the image of a bounded sequence under the action of the limit operator K has a convergent subsequence. Take a bounded sequence $\{u_m\}_1^\infty$, $\|u_m\| \le M$. Since every K_n is compact, one can select subsequences with the following properties:

$$\begin{array}{lll} n = 1\,, & \exists \{u_m^{(1)}\}_1^\infty \subset \{u_m\}_1^\infty \ : & \{K_1 u_m^{(1)}\} \text{ converges} \\ n = 2\,, & \exists \{u_m^{(2)}\}_1^\infty \subset \{u_m^{(1)}\}_1^\infty \ : & \{K_2 u_m^{(2)}\} \text{ converges} \\ \vdots & & \\ n = 1, 2, \ldots\,, & \exists \{u_m^{(n+1)}\}_1^\infty \subset \{u_m^{(n)}\}_1^\infty \ : & \{K_{n+1} u_m^{(n+1)}\} \text{ converges} \end{array}$$

Let $\{v_m\}_1^\infty$ be the subsequence that consists of the "diagonal elements":

$$v_m = u_m^{(m)}$$

By construction the images of this sequence under the action of any K_n converge:

$$\lim_{m\to\infty} K_n v_m = w_n \in \mathcal{H}\,, \quad n = 1, 2, \ldots$$

Fix $\varepsilon > 0$. Then

$$\begin{array}{ll} \|K - K_n\| < \varepsilon\,, & \text{for all large enough } n\,, \\ \|K_n(v_m - v_j)\| < \varepsilon\,, & \text{for all large enough } m\,, j \end{array}$$

by the convergence of $\{K_n\}$ and convergence of $\{K_n v_m\}_{m=1}^{\infty}$ for any n. Then it follows that the sequence $\{Kv_m\}$ is a Cauchy sequence:

$$\begin{aligned}
\|Kv_m - Kv_j\| &= \|(K-K_n)v_m - (K-K_n)v_j + K_n(v_n - v_j)\| \\
&\le \|(K-K_n)v_m\| + \|(K-K_n)v_j\| + \|K_n(v_n - v_j)\| \\
&\le \|(K-K_n)\|\|v_m\| + \|(K-K_n)\|\|v_j\| + \|K_n(v_n - v_j)\| \\
&\le 2M\varepsilon + \varepsilon
\end{aligned}$$

for all large enough m and j. By the completeness of the Hilbert space, there exists $w \in \mathcal{H}$ such that $\lim Kv_m = w$. Thus, the image $\{Ku_m\}$ of any bounded sequence $\{u_m\}$ has a convergent subsequence $\{Kv_m\}$, which means that K is compact. □

PROPOSITION **60.6**. *Every bounded operator with a finite dimensional range is compact:*

$$\textbf{(60.6)}\qquad \left.\begin{array}{l} K\,:\,\mathcal{H} \to R_K \subset \mathcal{H} \\ \dim R_K < \infty \\ \|K\| < \infty \end{array}\right\} \quad \Rightarrow \quad K \text{ is compact}$$

PROOF. Let $N = \dim R_K$ be the dimension of the range and $\{\phi_j\}_1^N$ be an orthonormal basis in the range $R_K \subset \mathcal{H}$. Take a bounded sequence $\{u_n\} \subset \mathcal{H}$, $\|u_n\| \le M$. Then

$$Ku_n = \sum_{j=1}^{N} c_{nj}\phi_j\,, \quad c_{nj} = \langle Ku_n, \phi_j\rangle$$

Then $c_n = (c_{n1}, c_{n2}, ..., c_{nN}) \in \mathbb{C}^N \sim \mathbb{R}^{2N}$. Since K is bounded, the image sequence is bounded too:

$$\|u_n\| \le M \quad \Rightarrow \quad \|Ku_n\| \le \|K\|\|u_n\| \le M\|K\| \quad \Rightarrow \quad \|c_n\| \le M\|K\|$$

By the Bolzano-Weierstrass theorem, every bounded sequence in a Euclidean space has a convergent subsequence (see below). Therefore the sequence $\{c_n\} \subset \mathbb{C}^N$ has a convergent subsequence, which implies that $\{Ku_n\}$ has a convergent subsequence and, hence, K is compact. □

PROPOSITION **60.7**. *A Hilbert-Schmidt operator is compact:*

$$\textbf{(60.7)}\qquad \left.\begin{array}{l} Ku(x) = \int_\Omega K(x,y)u(y)d^Ny \\ K(x,y) \in \mathcal{L}_2(\Omega\times\Omega) \end{array}\right\} \quad \Rightarrow \quad K \text{ is compact}$$

PROOF. If $\{\phi_n\}$ is an orthonormal basis in $\mathcal{L}_2(\Omega)$, then

$$\psi_{nm}(x,y) = \phi_n(x)\overline{\phi_m(y)}$$

is an orthonormal basis in $\mathcal{L}_2(\Omega\times\Omega)$ (a proof is given below). Therefore using Fubini's theorem

$$\begin{aligned}
K(x,y) &= \sum_{n,m} K_{nm}\psi_{nm}(x,y) \\
K_{nm} &= \int_{\Omega\times\Omega} K(x,y)\overline{\psi_{nm}(x,y)}\,dxdy \\
&= \int_\Omega \int_\Omega K(x,y)\phi_m(y)\,dy\;\overline{\phi_n(x)}\,dx \\
&= \langle K\phi_m, \phi_n\rangle
\end{aligned}$$

By the Parseval-Steklov equality for $\mathcal{L}_2(\Omega \times \Omega)$:

$$\sum_{n,m} |K_{nm}|^2 = \int_\Omega \int_\Omega |K(x,y)|^2 dxdy < \infty$$

Consider the sequence of Hilbert-Schmidt operators K_N with kernels

$$K_N(x,y) = \sum_{n,m\le N} K_{nm}\psi_{nm}(x,y)$$

The range of K_N is finite dimensional (its dimension is N). It is also bounded

$$\|K_N\|^2 \le \int_{\Omega\times\Omega} |K_N(x,y)|^2 dxdy = \sum_{n,m\le N} |K_{nm}|^2 < \infty$$

By (**60**.**6**), K_N is compact for any $N = 1,2,\dots$ This sequence operators strongly converges to K:

$$\begin{aligned}
\|K - K_N\|^2 &\le \int_{\Omega\times\Omega} |K(x,y) - K_N(x,y)|^2 dxdy \\
&= \sum_{n,m>N} |K_{nm}|^2 \to 0 \quad \text{as} \quad N\to\infty
\end{aligned}$$

because the series $\sum |K_{nm}|^2 < \infty$ converges. By (**60**.**5**), the limit operator K is compact. □

THEOREM **60**.**2**. *Let* $\{\phi_n\}$ *be an orthonormal basis in* $\mathcal{L}_2(\Omega)$. *Then*

$\psi_{nm}(x,y) = \phi_n(x)\overline{\phi_m(y)}$ *is an orthonormal basis in* $\mathcal{L}_2(\Omega\times\Omega)$

The orthonormality of the set is easy to check

$$\begin{aligned}\langle \psi_{nm}, \psi_{n'm'}\rangle &= \int_{\Omega\times\Omega} \psi_{nm}(x,y)\overline{\psi_{n'm'}(x,y)}\, dxdy \\ &= \int_\Omega \phi_n(x)\overline{\phi_{n'}(x)}\, dx \int_\Omega \overline{\phi_m(y)}\phi_{m'}(y)\, dy \\ &= \delta_{nn'}\delta_{mm'}\end{aligned}$$

One still has to show the completeness of the set. Note that the set of complex conjugated functions $\{\overline{\phi_n}\}$ is also an orthonormal basis in $\mathcal{L}_2(\Omega)$ because the Parseval-Steklov criterion for completeness holds:

$$\sum_n |\langle g, \overline{\phi_n}\rangle|^2 = \sum_n |\langle \bar{g}, \phi_n\rangle|^2 = \|\bar{g}\|^2 = \|g\|^2$$

for any $g \in \mathcal{L}_2(\Omega)$. Suppose that $f \in \mathcal{L}_2(\Omega\times\Omega)$ that is orthogonal to all ψ_{nm}. Put

$$F_m(x) = \int_\Omega f(x,y)\phi_m(y)\, dy$$

By the Cauchy-Bunyakowski inequality, this function is square integrable

$$|F_m(x)|^2 \le \int_\Omega |f(x,y)|^2 dy \quad \Rightarrow \quad F_m \in \mathcal{L}_2(\Omega)$$

because $\|\phi_m\| = 1$ and $f \in \mathcal{L}_2(\Omega\times\Omega)$. Therefore one can take the inner product:

$$\langle F_m, \phi_n\rangle \overset{(1)}{=} \langle f, \psi_{nm}\rangle = 0\,, \quad \forall m$$

where (1) holds by Fubini's theorem. By completeness of $\{\phi_n\}$, $F_m(x) = 0$ and, hence, $f(x,y) = 0$ *a.e.* for any $x \in \Omega$. Similarly, one can show that

$$G_n(y) = \int_\Omega f(x,y)\overline{\phi_n(x)}\, dx \;\in\; \mathcal{L}_2(\Omega)$$

and $\langle G_n, \overline{\phi_n}\rangle = \langle f, \psi_{nm}\rangle = 0$ so that $G_m = 0$ and $f(x,y) = 0$ *a.e.* for any $y \in \Omega$. Thus, $f(x,y) = 0$ *a.e.* and $f = 0$ in $\mathcal{L}_2(\Omega\times\Omega)$.

PROPOSITION **60.8**. *The product of bounded and compact operators is a compact operator:*

$$(\mathbf{60.8}) \qquad \left.\begin{array}{l} K \text{ is compact} \\ \|A\| < \infty \end{array}\right\} \quad \Rightarrow \quad KA \text{ and } AK \text{ are compact}$$

PROOF. Take a bounded sequence $\{u_n\}$, $\|u_n\| \le M$. Then by compactness of K, the image sequence $\{Ku_n\}$ has a convergent subsequence, say, $\{Kv_n\}$. A bounded operator is continuous and, hence, the sequence $\{AKv_n\}$ also converges. So, AK is a compact operator.

The sequence $\{Au_n\}$ is bounded because

$$\|Au_n\| \leq \|A\|\|u_n\| \leq M\|A\| < \infty$$

By compactness of K, the sequence $\{KAu_n\}$ has a convergent subsequence and, hence, KA is a compact operator. □

60.2. Spectral properties of a compact operator. [7]

PROPOSITION **60.9**. *A compact operator can have only finitely many linearly independent eigenvectors for all eigenvalues outside any disk centered at the origin:*

$$(\mathbf{60.9}) \qquad K \textit{ is compact} \quad \Rightarrow \quad \dim \bigcup_{|\lambda|>a} N_{K-\lambda I} < \infty\,, \quad \forall a > 0$$

PROPOSITION **60.10**. *A non-zero spectral value of a compact operator is in either the point spectrum or in the resolvent set:*

$$(\mathbf{60.10}) \qquad \left.\begin{array}{l} K \textit{ is compact} \\ \lambda \neq 0 \end{array}\right\} \quad \Rightarrow \quad \lambda \in \sigma_p(K) \cup \rho(K)$$

Recall that an isolated eigenvalue is a pole of the resolvent. It follows from the properties (**60.9**) and (**60.10**) that

- *the resolvent of a compact operator has finitely many poles in any open region of the complex plane that does not contain zero.*

PROPOSITION **60.11**. *The residual or continuum spectrum of a compact operator is either empty or consists of a single point λ:*

$$(\mathbf{60.11}) \qquad \left.\begin{array}{l} K \textit{ is compact} \\ \lambda \neq 0 \end{array}\right\} \quad \Rightarrow \quad \lambda \notin \sigma_c(K) \cup \sigma_r(K)$$

$$\Rightarrow \quad \sigma_c(K) \cup \sigma_r(K) = \left\{\begin{array}{l} \emptyset \\ \{0\} \end{array}\right.$$

PROPOSITION **60.12**. *If a complex number is an eigenvalue of a compact operator, then its complex conjugation is an eigenvalue of the adjoint, and these eigenvalues have an equal finite multiplicity:*

$$(\mathbf{60.12}) \qquad \left.\begin{array}{l} K \textit{ is compact} \\ \lambda \neq 0 \end{array}\right\} \quad \Rightarrow \quad \dim N_{K-\lambda I} = \dim N_{K^*-\bar{\lambda} I} < \infty$$

[7] Proofs of the following propositions (or equivalent to them) can be found in F. Riesz and B. Sz.-Nagy, Functional analysis, Sec. 93; I. Stackgold, Green's functions and boundary value problems, Chapter 5, Sec. 8; M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. 1, Chapter VII

PROPOSITION **60.13**. *Zero belongs to the approximate spectrum of a compact operator in an infinite dimensional Hilbert space:*

$$(\mathbf{60.13})\qquad \left.\begin{array}{l} K\textit{ is compact} \\ \dim\mathcal{H}=\infty \end{array}\right\} \quad\Rightarrow\quad \{\lambda=0\}\subset\sigma_a(K)=\sigma_p(K)\cup\sigma_c(K)$$

PROPOSITION **60.14**. *Zero belongs to the continuum spectrum of a compact operator in an infinite dimensional Hilbert space if the operator is invertible:*

$$(\mathbf{60.14})\qquad \left.\begin{array}{l} K\textit{ is compact} \\ \dim\mathcal{H}=\infty \\ \exists\, K^{-1} \end{array}\right\} \quad\Rightarrow\quad \sigma_c(K)=\{0\}\,,\quad \sigma_r(K)=\emptyset\,,$$

and, in this case, zero is not an eigenvalue, the residual spectrum of the operator is empty, while the continuum spectrum contains only one point, zero.

Summary of most important properties of the spectrum.

(K1) The spectrum $\sigma(K)$ of a compact operator K is at most countable:

$$\sigma(K)=\{\lambda_n\}\subset\mathbb{C}$$

(K2) Any non-zero spectral value belongs to the point spectrum:

$$\lambda\in\sigma(K)\,,\quad \lambda\neq 0\quad\Rightarrow\quad \lambda\in\sigma_p(K)$$

(K3) Every non-zero eigenvalue has a finite multiplicity:

$$\lambda\in\sigma_p(K)\,,\quad \lambda\neq 0\quad\Rightarrow\quad \dim N_{K-\lambda I}=m<\infty$$

(K4) The point spectrum can have only one limit point, zero.

(K5) Zero belongs to the approximate spectrum if a compact operator acts in an infinite dimensional Hilbert space. If, in addition, the operator is invertible, then zero is not an eigenvalue:

$$\begin{aligned} \dim\mathcal{H}=\infty \quad&\Rightarrow\quad 0\in\sigma_a(K)=\sigma_p(K)\cup\sigma_c(K)\\ &\Rightarrow\quad 0\in\sigma_c(K)\quad\text{if}\quad \exists\, K^{-1} \end{aligned}$$

Properties (K2) and (K3) are a restatement of (**60.10**). Property (K1) follows from (**60.9**) and (K2). Property (K4) follows from (**60.9**) and (**60.10**). By (**60.9**), it is possible to order the eigenvalues $\{\lambda_n\}$ of a compact operator into a monotonically decreasing sequence

$$|\lambda_1|\geq|\lambda_2|\geq\cdots\geq|\lambda_n|\geq|\lambda_{n+1}|\geq\cdots$$

where $|\lambda_1|$ is the largest eigenvalue. It exists because a compact operator is bounded. Then this sequence either converges to zero and, in this case, zero is the limit point of the spectrum, or it does not converge to zero. In the latter case, there is an open disk centered at the origin in which there is at most one spectral value, zero, and by (**60.9**) the spectrum is finite and, hence, cannot have any limit point. Property (K5) follows from (**60.11**)

It is also worth noting that if zero is an eigenvalue of a compact operator, then its multiplicity can be infinite. For example, consider an orthogonal projection onto a finite dimensional subspace of an infinite dimensional Hilbert space. Then this operator can be defined in a suitable orthonormal basis $\{\varphi_j\}$ in the Hilbert space:

$$K \ : \quad \mathcal{H} \to R_K \subset \mathcal{H}\,, \quad Ku = \sum_{j=1}^{N} \langle u, \varphi_j \rangle \varphi_j$$

This projection operator is compact. Indeed, take a weakly convergent sequence $\{u_n\}$:

$$\lim_{n\to\infty} \langle u_n - u, v\rangle = 0\,, \quad \forall v \in \mathcal{H}$$

Then

$$Ku_n - Ku = \sum_{j=1} \langle u_n - u, \varphi_j \rangle \varphi_j$$

and, hence,

$$\begin{aligned} \lim_{n\to\infty} \|Ku_n - Ku\|^2 &= \lim_{n\to\infty} \sum_{j=1}^{N} |\langle u_n - u, \varphi_j\rangle|^2 \\ &= \sum_{j=1}^{N} \lim_{n\to\infty} |\langle u_n - u, \varphi_j\rangle|^2 = 0 \end{aligned}$$

Thus, the image of any weakly convergent sequence is the sequence that converges in the norm. Therefore K is a compact operator. Note well that changing the order of summation and taking the limit always holds for a *finite* sum but might not be possible if the range is infinite dimensional.

The spectrum of the projection operator is

$$\sigma(K) = \sigma_p(K) = \{0, 1\}$$

The multiplicity of the unit eigenvalue is $N < \infty$ in accord with (K3), whereas the multiplicity of the zero eigenvalue is infinite:

$$N_K = R_K^{\perp} \quad \Rightarrow \quad \dim N_K = \dim R_K^{\perp} = \infty$$

60.3. The Fredholm alternative for compact operator. Here the linear problem

$$Au = (K - \lambda I)u = f$$

for a compact operator K is analyzed. It turns out that its solution has properties similar to a finite dimensional case.

THEOREM **60.3**. (Fredholm alternative)
Let K be a compact operator in a separable Hilbert space. Put

$$A = K - \lambda I\,, \quad A^* = K^* - \bar{\lambda} I\,, \quad \lambda \in \mathbb{C}$$

If $\lambda \neq 0$, then either the following alternatives holds

(a) *The equation $Au = 0$ has only the trivial solution ($\lambda \notin \sigma_p(K)$) and, in this case, the adjoint equation $A^*v = 0$ has also only the trivial solution ($\bar{\lambda} \notin \sigma_p(K^*)$) and the linear problem $Au = f$ has precisely one solution.*

(b) *The equation $Au = 0$ has finitely many linearly independent solutions ($\lambda \in \sigma_p(K)$, $\dim N_A = m < \infty$) and, in this case, the adjoint equation $A^*v = 0$ has the same number of linearly independent solutions ($\bar{\lambda} \in \sigma_p(K^*)$, $\dim N_A = \dim N_{A^*} = m < \infty$) and the linear problem $Au = f$ has solutions if an only if f lies in the orthogonal complement of the null space of the adjoint A^* ($f \in N_{A^*}^{\perp}$). The general solution has the form*

$$u = u_p + u_0 = u_p + \sum_{j=1}^{m} c_j u_j$$

where u_p is a particular solution and u_0 is a generic element from the null space of A; it can be expanded over any basis $\{u_j\}_1^m$ in N_A, $m = \dim N_A$.

A proof of the Fredholm alternative is based on three following facts[8]:

THEOREM **60.4**. *The orthogonal complement of the range of an operator A is the null space of the adjoint:*

$$R_A^{\perp} = N_{A^*}$$

THEOREM **60.5**. (Solvability for operators with closed range)
If an operator A has a closed range, then the linear problem $Au = f$

[8] I. Stakgold, Green's functions and boundary value problems, Chapter 5, Section 5

has a solution if and only if f belongs to the orthogonal complement of the null space of the adjoint:

$$\overline{R}_A = R_A \quad \Rightarrow \quad \exists\, u \ :\ Au = f \quad \Leftrightarrow \quad f \in N_{A^*}^{\perp}$$

THEOREM **60.6**. (Criterion for the range to be closed)
Let A be a closed operator, and let A be bounded away from zero on $N_A^{\perp} \cap D_A$. Then the range is closed.

Suppose $A = K - \lambda I$ where K is a compact operator. Any complex $\lambda \neq 0$ is either from the point spectrum $\sigma_p(K)$ or from the resolvent set $\rho(K)$ by (**60.10**). If $\lambda \in \rho(K)$, then the range $R_A = \mathcal{H}$ is the whole Hilbert space and A is invertible (the inverse is bounded). Therefore the linear problem $Au = f$ is well posed and has the unique solution ($A \in$ (I,1c)). Furthermore by Theorem **60.4**:

$$N_{A^*} = R_A^{\perp} = \mathcal{H}^{\perp} = \{0\}$$

so that the alternative (a) holds.

The alternative (b) would follow from (**60.12**) and Theorem **60.5** if the range R_A is proved to be closed. The later is established by mean of the criterion given in Theorem **60.6**. Let us show that $A = K - \lambda I$ is bounded away from zero on $N_A^{\perp}$:

$$\|Au\| \geq C\|u\|\,, \quad \forall u \in N_A^{\perp}$$

If this were not true, then there should exist a unit sequence $\{u_n\} \subset N_A^{\perp}$, $\|u_n\| = 1$, such that $\|Au_n\| \to 0$ as $n \to \infty$. Since K is compact, the image sequence $\{Ku_n\}$ should have a convergent subsequence. So, let us take a subsequence $\{v_n\} \subseteq \{u_n\}$ such that the sequence $\{Kv_n\}$ converges. Since $Av_n = Kv_n - \lambda v_n$ converges to zero and Kv_n also converges, by the limit laws, the sequence $\{v_n\}$ must have a limit, say, v because $\lambda \neq 0$. By continuity of a compact operator (a compact operator is bounded), it is concluded that

$$\lim_{n\to\infty} (Kv_n - \lambda v_n) = Kv - \lambda v = 0 \quad \Rightarrow \quad v \in N_A$$

Recall that the orthogonal complement of a linear manifold is closed so that $N_A^{\perp}$ is closed and, hence, the limit of the sequence $\{v_n\}$ must in $N_A^{\perp}$. There is only one element which belongs to $N_A^{\perp}$ and N_A. It is the zero element. Thus, $v = 0$. But v is the limit of the unit sequence and, hence, $\|v\| = 1$. A contradiction. So, A is bounded away from zero on $N_A^{\perp}$ and, hence, the range R_A is closed.

60.4. The spectrum of a symmetric compact operator. Since every compact operator is bounded, it can be extended to the whole Hilbert space. Therefore every symmetric compact operator is automatically self-adjoint:

$$K \text{ is compact and symmetric} \quad \Rightarrow \quad K = K^*$$

PROPOSITION **60.15**. *The spectrum of a compact self-adjoint operator is not empty:*

$$\left.\begin{array}{l} K \textit{ is compact} \\ K = K^* \\ \sigma(K) = \{0\} \end{array}\right\} \quad \Rightarrow \quad K = 0 \tag{60.15}$$

PROPOSITION **60.16**. *The non-zero eigenvalues of a compact self-adjoint operator are real and have finite multiplicity, and the corresponding linearly independent eigenvector can be chosen orthonormal so that*

$$\left.\begin{array}{l} K \textit{ is compact} \\ K = K^* \\ \{\lambda_j\} = \sigma(K) \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} Ku = \sum_j \lambda_j \langle u, e_j\rangle e_j \\ Ke_j = \lambda_j e_j\,, \quad \langle e_j, e_k\rangle = \delta_{jk} \\ \dim N_{K-\lambda_j I} < \infty \end{array} \tag{60.16}$$

Eigenvectors corresponding to distinct eigenvalues of a symmetric operator are orthogonal. Since the multiplicity of any non-zero eigenvalue of a compact operator is finite, the null space $N_{K-\lambda_j I}$ has finitely many linearly independent vectors. They can be turned into an orthonormal basis in $N_{K-\lambda_j I}$. If $\{e_j\}$ is the collection of orthonormal vectors for all eigenvalues $\lambda_j \neq 0$, then by Riesz-Fisher theorem the following series converges

$$v = \sum_j \langle u, e_j\rangle e_j \in N_K^\perp\,, \quad \forall u \in \mathcal{H}$$

and by continuity of K

$$Kv = \sum_j \lambda_j \langle u, e_j\rangle e_j$$

The null space N_K is closed (by continuity of a compact operator) and by the projection theorem there is a unique decomposition $u = v + w$ where $w \in N_K$ and $v \in N_A^\perp$. It follows that $Ku = Kv + Kw = Kv$.

PROPOSITION **60.17**. *Linearly independent eigenvectors of a symmetric, self-adjoint operator that is invertible can be chosen to form an* *orthonormal basis in the Hilbert space:*

$$(\mathbf{60.17}) \qquad \left.\begin{array}{l} K \ \textit{is compact} \\ K = K^* \\ \exists\, K^{-1} \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} u = \sum_j \langle u, e_j\rangle e_j\,, \quad \forall u \in \mathcal{H} \\ K e_j = \lambda_j e_j\,, \quad \langle e_j, e_k\rangle = \delta_{jk} \end{array}$$

If K is invertible, then $\lambda = 0$ is not an eigenvalue so that $N_K = \{0\}$ and the orthonormal set in (**60.17**) becomes an orthonormal basis. The operator K is invertible and

$$Ku = f \quad \Rightarrow \quad u = K^{-1} f = \sum_j \frac{1}{\lambda_j} \langle f, e_j\rangle e_j$$

Furthermore by (**60.14**), the inverse is not bounded in an infinite dimensional space:

$$\dim \mathcal{H} = \infty \quad \Rightarrow \quad 0 \in \sigma_c(K) \quad \Rightarrow \quad \|K^{-1}\| = \infty$$

60.5. The spectral theorem for compact self-adjoint operators. Let f and g be two vectors in a Hilbert space, define the operator

$$Au = \langle u, g\rangle f\,, \quad \forall u \in \mathcal{H}$$

This operator is called a tensor product of f and g and is denoted by

$$A = f \otimes g$$

In particular, for any unit vector $\|e\| = 1$, the tensor product operator

$$P = e \otimes e \quad \Rightarrow \quad Pu = \langle u, e\rangle e$$

is the orthogonal projection of any vector on e, $P^2 = P$ and $P^* = P$. If $\dim \mathcal{H} < \infty$, the property (**60.16**) can be restated as the sum of the projection operators

$$K = \sum_j \lambda_j e_j \otimes e_j$$

It turns out the this representation holds for any compact self-adjoint operator. There are only finitely many distinct eigenvalues outside any interval that does not contain zero for any compact operator, and every such eigenvalue has a finite multiplicity. Consider the sequence of operator defined by finite sums

$$K_n = \sum_{|\lambda_j| > \frac{1}{n}} \lambda_j e_j \otimes e_j$$

The sequence of operators $\{K_n\}$ is proved to strongly converge to K:

$$\lim_{n\to\infty} \|K - K_n\| = 0$$

The assertion is known as the spectral theorem for compact self-adjoint operators.

THEOREM **60.7**. (Spectral theorem for compact self-adjoint operators) *Let K be a compact and self-adjoint operator in a separable Hilbert space. Let P_j be the orthogonal projector on the null space $N_{K-\lambda_j I}$ (the space of all eigenvectors of K corresponding to the eigenvalue λ_j):*

$$\forall u \in \mathcal{H}\,, \quad P_j u \in N_{K-\lambda_j I}\,, \quad P_j^2 = P_j\,, \quad P_j^* = P_j\,.$$

Then the operator K and its resolvent can be written as

$$K = \sum_j \lambda_j P_j$$

$$\mathcal{R}_K(\lambda) = \sum_j \frac{P_j}{\lambda_j - \lambda}$$

where the series converges strongly (with respect to the operator norm).

Let $m = \dim N_{K-\lambda_j I}$, and let $\{e_{jn}\}_{n=1}^m$ be an orthonormal set of linearly independent eigenvectors corresponding to the eigenvalue $\lambda_j \neq 0$. Then

$$P_j = \sum_{n=1}^m e_{jn} \otimes e_{jn}$$

60.6. Foundations of the Fourier method. The spectral theorem for compact self-adjoint operators provides foundations for the Fourier method to solve partial differential equations. The basic idea can be formulated as follows. Consider the initial value problem

$$u_t'(x,t) = Lu(x,t)\,, \quad t > 0\,, \quad u|_{t=0} = u_0(x)$$

where L is a linear differential operator independent of the parameter t (e.g., time or any other "evolution" variable).

$$L \;:\quad D_L \subset C^p(\Omega) \subset \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$$

where Ω is open in a Euclidean space, and D_L is dense in $\mathcal{L}_2(\Omega)$. Consider the eigenvalue problem

$$Lu = \lambda u$$

Suppose that $\lambda = 0$ is not an eigenvalue. Then L is invertible. The inverse of L is an integral operator whose kernel is a suitable Green's function

$$LG(x,y) = \delta(x-y)\,, \quad y \in \Omega$$

where $G(x, y)$ is also required to satisfy some boundary conditions for functions in the domain D_L. Then

$$Ku(x) = L^{-1}u(x) = \int_\Omega G(x, y)u(y)\, d^N y$$

The eigenvalue problem can be restated for the integral operator K:

$$Lu = \lambda u \quad \Rightarrow \quad Ku = \xi u\,, \quad \xi = \frac{1}{\lambda}$$

Suppose that the Green's functions is such that the integral operator K is compact and symmetric, e.g., $G(x, y)$ is the kernel of a Hilbert-Schmidt operator such that

$$\overline{G(y, x)} = G(x, y) \in \mathcal{L}_2(\Omega \times \Omega) \quad \Rightarrow \quad K^* = K \text{ is compact}$$

Note that L is essentially self-adjoint. Then linearly independent unit eigenvectors form a basis in $\mathcal{L}_2(\Omega)$ which can be made orthonormal. The basis functions are also eigenfunctions of L with reciprocal eigenvalues $\lambda_j = 1/\xi_j$. This allows us to write the solution to the initial value problem using the expansion of the initial data into the Fourier series

$$Ke_j = \xi_j e_j\,, \quad \langle e_j, e_n\rangle = \delta_{jn} \quad \Rightarrow \quad Le_j = \lambda e_j\,, \quad \lambda_j = \frac{1}{\xi_j}$$

$$u_0 = \sum_j \langle u_0, e_j\rangle e_j \quad \Rightarrow \quad u = \sum_j e^{\lambda_j t}\langle u_0, e_j\rangle e_j$$

provided the series converges for all $t \geq 0$ (or at least in some interval $0 \leq t \leq t_0$). For example, if $\lambda_j < 0$ (this is a heat or diffusion type of partial differential equations, then the convergence in the mean is guaranteed

$$\|u\|^2 = \sum_j e^{2\lambda_j t}|\langle u_0, e_j\rangle|^2 \leq \sum_j |\langle u_0, e_j\rangle|^2 = \|u_0\|^2 < \infty$$

If the evolution equation of the Schroedinger type, then the norm is preserved in the evolution:

$$iu'_t = Lu \quad \Rightarrow \quad u = \sum_j e^{-i\lambda_j t}\langle u_0, e_j\rangle e_j \quad \Rightarrow \quad \|u\|^2 = \|u_0\|^2$$

The wave type equation

$$u''_{tt} = Lu\,, \quad t > 0\,, \quad u\Big|_{t=0} = u_0\,, \quad u'_t\Big|_{t=0} = u_1$$

can be treated similarly.

If Ω is bounded, then the eigenvalue problem for an essentially self-adjoint second-order differential operator

$$Lu(x) = -\mathrm{div}\left(p(x)\,\mathrm{grad}\,u(x)\right) + q(x)u(x)$$

where the parameter functions p and q and the boundary condition for $u \in D_L$ at $\partial\Omega$ are chosen so that

$$\langle Lu, u\rangle > 0\,, \quad \forall u \neq 0 \in D_L$$

is known as the Sturm-Liouville problem for elliptic equations[9]. It is solved exactly along the lines indicated above. Here the simplest example is considered just to illustrate the concept.

Heat and Schroedinger equations in an interval. Consider the second-derivative operator

$$A \;\; : \quad D_A = \{\, u \in C^2([0,1])\,|\, u(0) = u(1) = 0\,\} \to \mathcal{L}_2(0,1)$$
$$Au(x) = -u''(x)$$

This operator was shown to be essentially self-adjoint:

$$A \subset \bar{A} = A^* = (\bar{A})^*$$

So, one can consider the self-adjoint extension $\bar{A}$ instead of A. Its spectrum is discrete

$$\sigma(A) = \sigma_p(A) = \{(\pi n)^2\}_{n=1}^{\infty}$$

The normalized eigenfunctions for an orthonormal set

$$A\varphi_n = (\pi n)^2\varphi_n\,, \quad \varphi_n(x) = \sqrt{2}\sin(\pi n x)\,, \quad \langle\varphi_n, \varphi_j\rangle = \delta_{nj}$$

The key question: *Is the orthonormal set* $\{\varphi_n\}_1^{\infty}$ *an orthonormal basis in* $\mathcal{L}_2(0,1)$*?*

This question can be answered by means of reducing the eigenvalue problem for A to the eigenvalue problem for an integral operator which is a symmetric Hilbert-Schmidt operator and, hence, compact. Since $\lambda = 0$ is not an eigenvalue of A, A is invertible

$$-u''(x) = \lambda u(x) \quad \Rightarrow \quad u(x) = \lambda \int_0^1 G(x,y)u(y)\,dy$$
$$G(x,y) = x(1-y)\theta(y-x) + (1-x)y\theta(x-y)$$

It is not difficult to see that

$$G(x,y) = G(y,x)\,, \quad \int_0^1\int_0^1 |G(x,y)|^2 dyxy < \infty$$

[9]A further discussion can be found in: V.S. Vladimirov, Equations of Mathematical Physics, Chapter V.

Therefore the operator

$$K \;:\quad \mathcal{L}_2(0,1) \to \mathcal{L}_2(0,1)\,, \quad Ku(x) = \int_0^1 G(x,y)u(y)\,dy$$

is a compact self-adjoint operator on $\mathcal{L}_2(0,1)$ (as a symmetric Hilbert-Schmidt operator). Using the closure of A, it is not difficult to see that the eigenvalue problems for $\bar{A} = A^* = (\bar{A})^*$ and K are equivalent:

$$Ku = \xi u \quad \Leftrightarrow \quad \bar{A}u = \lambda u$$

where $\lambda = 1/\xi$. By the theorem spectral theorem for compact self-adjoint operators, it is concluded that:

$$u = \sum_{n=1}^{\infty} \langle u, \varphi_n\rangle \varphi_n\,, \quad \forall u \in \mathcal{L}_2(0,1)$$

$$K = (\bar{A})^{-1} = \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2}\, \varphi_n \otimes \varphi_n$$

$$\mathcal{R}_{\bar{A}}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2 - \lambda}\, \varphi_n \otimes \varphi_n$$

The first series converges in the mean, the operator series converge strongly (in the operator norm). Note that $\xi_n = 1/\lambda_n = (\pi n)^{-2}$. Zero is the limit point of the spectrum $\sigma(K)$ in full accord with the developed theory of compact operators.

Consider the initial and boundary value problem for the heat equation

$$u'_t(x,t) = -\bar{A}u(x,t)\,, \quad u\Big|_{t=0} = u_0(x)\,, \quad u \in D_{\bar{A}}$$

The eigenfunctions φ_n of A were proved to form an orthonormal basis in $\mathcal{L}_2(0,1)$, the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{-(\pi n)^2 t} \langle u_0, \varphi_n\rangle \varphi_n(x)$$

The series converges in the mean and

$$\|u\|^2 \le \|u_0\|\,, \quad t \ge 0$$

It should be noted, however, that this solution might not be from class C^2 as is required in the classical theory of partial differential equations. For this reason, the above solution is called a formal solution in the classical theory.

Consider the Schroedinger equation for a free particle in a box [10]:

$$iu'_t(x,t) = -\bar{A}u(x,t)\,, \quad u\Big|_{t=0} = u_0(x)\,, \quad u \in u \in D_{\bar{A}}$$

Then the solution to the initial value problem is

$$u(x,t) = \sum_{n=1}^{\infty} e^{i(\pi n)^2 t}\langle u_0, \varphi_n\rangle \varphi_n(x)$$

The series converges in the mean and

$$\|u\|^2 = \|u_0\|\,, \quad t \ge 0\,.$$

60.7. Exercises.

1. Consider the eigenvalue problem for the operator

$$A\ :\quad D_A = \{u \in C^2([0,1])\,|\,u(0) = u'(1) = 0\,\} \to \mathcal{L}_2(0,1)$$
$$Au(x) = -\Big(p(x)u'(x)\Big)'\,, \quad p(x) > 0\,, \quad p \in C^1([0,1])$$

(i). Prove that A is invertible and that the following two problems are equivalent

$$Au = \lambda u\,, \quad u \in D_A \quad \Leftrightarrow$$
$$Ku = \xi u\,, \quad u \in C^1([0,1])\,, \quad Ku(x) = \int_0^x \frac{1}{p(y)}\int_y^1 u(z)\,dz$$

that is, if $u \in D_A$ solves the differential equation, then it also solves the integral equation, and if $u \in C^1([0,1])$ solves the integral equation then $u \in D_A$ and solves the differential equation.

(ii). Prove that K is the Hilbert-Schmidt operator and, hence, bounded. Show that its extension to the whole $\mathcal{L}_2(0,1)$ is compact and self-adjoint.

(iii) Show that K is invertible and $\|K^{-1}\| = \infty$. Conclude that $\xi = 0$ is not in the point spectrum of K. Show that the operators A and K share the same eigenvectors and all eigenvalues are strictly positive by analyzing the quadratic form $\langle Au, u\rangle$.

(iv) Show that each eigenvalue has multiplicity $m = 1$. Assume that u_1 and u_2 from D_A are linearly independent eigenfunctions corresponding to the same eigenvalue λ, $Au_{1,2} = \lambda u_{1,2}$. Recall from basic theory of

[10] the units are chosen so that the Planck constants is equal to one, and the mass is equal to $\frac{1}{2}$.

differential equations that u_1 and u_2 are linearly independent on an interval if and only if their Wronskian is does not vanish in this interval. Use that $u_{1,2} \in D_A$ to show that the Wronskian has zero value in $[0,1]$ and conclude that u_1 and u_2 are not linearly independent.

(v) Show that $\sigma(A) = \sigma_p(A) = \{\lambda_j\}$ and the sequence is strictly positive, can be arranged to be strictly monotonic, and has no limit point. Prove that the corresponding unit eigenvectors of A form an orthonormal basis in $\mathcal{L}_2(0,1)$ by means of the spectral theorem for compact self-adjoint operators.

(vi) Show that A is essentially self-adjoint and find its self-adjoint extension $\bar{A}$. Find the spectrum of the self-adjoint extension of A and all its linearly independent eigenfunctions. Compare the spectra $\sigma(\bar{A})$, $\sigma(A)$, and $\sigma(K)$.

2. Let

$$A : \quad D_A = \{u \in C^2([0,1]) \,|\, u'(0) = u'(1) = 0\} \to \mathcal{L}_2(0,1)$$
$$Au(x) = -\Big(p(x)u'(x)\Big)', \quad p(x) > 0, \quad p \in C^1([0,1])$$

(i) Show that the operator is symmetric. Show that zero is its eigenvalue with multiplicity $m = 1$. Find the corresponding eigenfunction.

(ii) Prove that the linear problem

$$Au = f$$

has a solution if and only if f is orthogonal to a constant function $\langle f, 1\rangle = 0$ and, in this case,

$$u(x) = c + \int_0^1 G(x,y)f(y)\,dy\,, \quad \langle f, 1\rangle = \int_0^1 f(x)dx = 0\,,$$
$$G(x,y) = \begin{cases} v(y)\,, & y < x \\ v(x)\,, & y > x \end{cases}, \qquad v(x) = \int_0^x \frac{dy}{p(y)}$$

where c is an arbitrary constant.

(iii) Show that if $\lambda \neq 0$, then the following eigenvalue problems are equivalent:

$$
\begin{aligned}
&Au = \lambda u\,, \quad u \in D_A \quad \Leftrightarrow \\
&Ku = \xi u\,, \quad Ku(x) = \int_0^1 G(x,y)u(y)\,dy \\
&D_K = \{\, u \in C^1([0,1]) \,|\, \langle u, 1\rangle = 0\,\}
\end{aligned}
$$

that is, if $u \in D_A$ solves the differential equation with $\lambda \neq 0$, then it also solves the integral equation (invoke the result of part (ii) to do so), and if $u \in D_K$ solves the integral equation then $u \in D_A$ and solves the differential equation.

(iv) Show that K is bounded and the closure of its domain is a proper subset of the Hilbert space, $\overline{D_K} \subset \mathcal{L}_2(0,1)$; it is formed by all vectors orthogonal to the unit function. Show that the extension $\bar{K}$ of K to $\mathcal{L}_2(0,1)$ is compact self-adjoint operator (review the extension of a bounded operator). Show that $\xi = 0$ is an eigenvalue of the extension $\bar{K}$ and the corresponding eigenfunction is a constant function.

(v) Show that $\sigma(A) = \sigma_p(A) = \{\lambda_j\} \cup \{0\}$ and $\sigma(\bar{K}) = \{1/\lambda_j\} \cup \{0\}$, where the sequence λ_j is strictly positive, can be arranged to be strictly monotonic, and has no limit point. Prove that the corresponding unit eigenvectors of A form an orthonormal basis in $\mathcal{L}_2(0,1)$ by means of the spectral theorem for compact self-adjoint operators.

(vi) Show that A is essentially self-adjoint and find its self-adjoint extension $\bar{A}$. Find the spectrum of the self-adjoint extension of A and all its linearly independent eigenfunctions.

61. Spectral theorem for self-adjoint operators

A spectrum of a self-adjoint operator is non-empty. A compact self-adjoint operators have only point spectrum. By the spectral theorem for compact self-adjoint operators, the orthonormal eigenvectors form a basis in a Hilbert space. It was also shown that the orthonormal eigenvector of some self-adjoint differential operators (which are not compact because they are not bounded) also form a basis in the Hilbert space of square integrable functions. However, these operators were shown to have a compact inverse. There are self-adjoint operators that do not have a compact inverse and whose spectrum has a non-empty continuum part. For example the second derivative operator in $\mathcal{L}_2()$ is essentially self-adjoint, and its spectrum contains only the continuous spectrum $\sigma_p = [0, \infty)$. Is there any analog of the spectral theorem for such operators? The answer is affirmative. But in order formulate it, one has to make reformulate the Lebesgue integral in the framework of the measure theory.

61.1. Stielties integral. Let $\mu(x)$ be a monotonic function on $\mathbb{R}$. Let $\{x_j\}_0^N$ be a partition of $[a, b]$ such that $x_0 = a$ and $x_N = b$ and $\Delta x = x_j - x_{j-1} = (b-a)/N$. Let f be a bounded function. Put

$$\begin{aligned} \Delta\mu_j &= \mu(x_j) - \mu(x_{j-1}) \\ M_j &= \sup_{I_j} f(x)\,, \quad m_j = \inf_{I_j} f(x)\,, \quad I_j = [x_{j-1}, x_j]\,, \end{aligned}$$

$$L_N(f, \mu) = \sum_{j=1}^{N} m_j \Delta\mu_j\,, \quad U_N(f, \mu) = \sum_{j=1}^{N} M_j \Delta\mu_j$$

Suppose that the limits of L_N and U_N exist as $N \to \infty$ and are equal. In this case the limit is called the Stieltjes integral of f with respect to μ:

$$\int_a^b f(x) d\mu(x) = \lim_{N\to\infty} L_N(f, \mu) = \lim_{N\to\infty} U_N(f, \mu)$$

If $\mu(x) = x$, then the Stieltjes integral is nothing but the Riemann integral. However, it can exists even if μ is not differentiable!

For example, consider a piecewise constant μ:

$$\mu(x) = \begin{cases} \mu_1\ ,\ a \le x < c_1 \\ \mu_2\ ,\ c_1 \le x < c_2 \\ \mu_3\ ,\ c_2 \le x \le b \end{cases}, \quad \mu_1 < \mu_2 < \mu_3$$

This function is monotonic:

$$x \le y \quad \Rightarrow \quad \mu(x) \le \mu(y)$$

The value of $\Delta\mu_j$ is either 0, or $\mu_2 - \mu_1$, or $\mu_3 - \mu_2$. So, the integral is

$$\int_a^b f(x)d\mu(x) = f(c_1)(\mu_2 - \mu_1) + f(c_2)(\mu_3 - \mu_2)$$

Note that in the distributional sense

$$\mu(x) = \mu_1 + (\mu_2 - \mu_1)\theta(x - x_1) + (\mu_3 - \mu_2)\theta(x - x_2)$$

The distributional derivative is

$$\mu'(x) = (\mu_2 - \mu_1)\delta(x - x_1) + (\mu_3 - \mu_2)\delta(x - x_2)$$

If f is a test function, then

$$\int_a^b f(x)d\mu(x) = (\mu', f)$$

The idea of Stieltjes integral stems from the problem of finding the center of mass of an extended linear object. If $\rho(x)$ is the mass density of an object occupying the interval $[a, b]$, then the mass $\mu(x)$ of the part occupying $[a, x]$ is

$$\mu(x) = \int_a^x \rho(y)\, dy$$

assuming that ρ is locally integrable. If $\rho(x)$ is non-negative, then $\mu(x)$ is monotonic. If ρ is continuous, then $d\mu(x) = \rho(x)\, dx$ and the center of mass is

$$x_c = \frac{1}{M}\int_a^b x d\mu(x)\,, \quad M = \int_a^b d\mu(x)$$

The function $\mu(x)$ is the measure of mass, while its derivative is the mass density distribution. Suppose the object also include point-like masses. Then $\mu(x)$ will suffer jump discontinuities and will no longer be differentiable. However, the equation for the center of mass still remains valid, provided the integral is treated as the Stieltjes integral!

Another example is offered by the probability theory. Suppose that a random variable can take any value in $\mathbb{R}$ Let $P(x)$ be the probability that the random variable takes its value in $(-\infty, x]$. Then $P(x) \geq 0$ and $P(x)$ is monotonic. One can defined the Stieltjes integral with respect to $\mu(x) = P(x)$. Then

$$\int dP(x) = 1$$

The expectation value of a function f of a random variable is

$$\langle f \rangle = \int f(x)\, dP(x)$$

Note that the probability $P(x)$ is not required to be differentiable. It can, for example, be piecewise constant. In this case, the probability distribution or probability density does not exist in the classical sense and cannot be used to calculate expectation values using the Riemann integral (it is a sum of Dirac delta-functions (in the distributional sense)).

61.2. The spectral theorem for self-adjoint operators.

DEFINITION **61.1**. (Spectral family)
A family of symmetric operators E_λ, $\lambda \in \mathbb{R}$, in a Hilbert space $\mathcal{H}$ is called a spectral family *if the following properties hold:*

$$\begin{aligned}
&(\text{E1}): \qquad E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda\,, \quad \lambda < \mu\,,\\
&(\text{E2}): \qquad \lim_{\lambda\to-\infty} \|E_\lambda u\| = 0\,, \quad \forall u \in \mathcal{H}\\
&(\text{E3}): \qquad \lim_{\lambda\to\infty} \|u - E_\lambda u\| = 0\,, \quad \forall u \in \mathcal{H}\\
&(\text{E4}): \qquad \lim_{\mu\to\lambda^+} \|E_\mu u - E_\lambda u\| = 0\,, \quad \forall u \in \mathcal{H}
\end{aligned}$$

The property (E4) is the right continuity of the operator E_λ with respect to the parameter λ.

Example 1. Let $\mathcal{H} = \mathbb{C}^N$. Let $A = A^*$ be a symmetric matrix. Suppose its eigenvalues are distinct $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ Let $\{e_n\}_1^N$ be an orthonormal basis of the corresponding eigenvectors $Ae_n = \lambda_n e_n$ Put

$$E_\lambda = \sum_{\lambda \le \lambda} P_n\,, \quad P_n u = \langle u, e_n\rangle e_n$$

So, P_n is the orthogonal projection onto the null space $N_{A-\lambda_n I}$. Then

$$\begin{aligned}
&E_\lambda = 0\,, \quad \lambda < \lambda_1\\
&E_\lambda = P_1\,, \quad \lambda_1 \le \lambda < \lambda_2\\
&E_\lambda = P_1 + P_2\,, \quad \lambda_2 \le \lambda < \lambda_3\\
&\vdots\\
&E_\lambda = P_1 + P_2 + \cdots + P_N = I\,, \quad \lambda \ge \lambda_N
\end{aligned}$$

It is not difficult to see that all the properties (E1)–(E2) are fulfilled. The functions E_λ is a piecewise constant operator-valued function which is continuous from the right at each jump-discontinuity. This can easily be extended to the case when eigenvalues are not simple.

Example 2. The functions $e_n(x) = (2\pi)^{-1/2}e^{i\pi nx}$, $n = 0, \pm1, \pm2, ...$, form an orthonormal (trigonometric Fourier) basis in $\mathcal{L}_2(-1,1)$. They also can be viewed as eigenfunction of the operator $A = -i\frac{d}{dx}$ whose domain contains function satisfying $u(-1) = u(1)$. This operator is essentially self-adjoint. Put

$$P_N u(x) = \sum_{|n|\le N} \langle u, e_n\rangle e_n(x) = \int_{-1}^{1} D_N(x-y)u(y)\,dy$$

where

$$D_N(x-y) = \sum_{|n|\le N} e_n(x)\overline{e_n(y)} = \frac{\sin[\pi(N+\frac{1}{2})(x-y)]}{\sin[\frac{p}{i}2(x-y)]}$$

is known as the Dirichlet kernel. The operators

$$E_\lambda = P_N\,, \quad N \le \lambda < N+1$$

is a spectral family. If $\lambda < 0$, then $E_\lambda = 0$, and

$$\lim_{\lambda\to\infty} E_\lambda u = \sum_{n=-\infty}^{\infty} \langle u, e_n\rangle e_n = u$$

by the completeness of the set $\{e_n\}$ (by the Parseval-Steklov equality).

Example 3. Consider $\mathcal{H} = \mathcal{L}_2(\mathbb{R})$. If $\mathcal{F}$ denotes the Fourier transform, put

$$\begin{aligned} E_\lambda u(x) &= \mathcal{F}^{-1}\Big[\theta(|k|-\lambda)\mathcal{F}[u](k)\Big](x) \\ &= \frac{1}{2\pi}\int_{-\lambda}^{\lambda} e^{-ikx}\int_{\infty}^{\infty} e^{iky}u(y)\,dy\,dk \\ &= \int_{-\infty}^{\infty} \frac{\sin[\lambda(x-y)]}{\pi(x-y)}\,u(y)\,dy = \int_{-\infty}^{\infty} D_\lambda(x-y)\,u(y)\,dy \end{aligned}$$

Note that $E_\lambda = 0$ if $\lambda < 0$ and $E_\lambda u \to \mathcal{F}^{-1}[\mathcal{F}[u]] = u$ as $\lambda \to \infty$. One also has

$$\begin{aligned} E_\lambda E_\mu u &= E_\lambda \mathcal{F}^{-1}\Big[\theta(|k|-\mu)\mathcal{F}[u]\Big] \\ &= \mathcal{F}^{-1}\Big[\theta(|k|-\lambda)\theta(|k|-\mu)\mathcal{F}[u]\Big] = E_\mu E_\lambda u \\ &= \mathcal{F}^{-1}\Big[\theta(|k|-\lambda)\mathcal{F}[u]\Big] = E_\lambda u\,, \quad \lambda < \mu \end{aligned}$$

The spectral theorem. The function

$$\mu(\lambda) = \langle u, E_\lambda u \rangle$$

is real and monotonic for any choice of $u \in \mathcal{H}$. In particular for the above three examples, it is not difficult to show that

$$\mu(\lambda') - \mu(\lambda) = \|(E_{\lambda'} - E_\lambda)u\|^2 \geq 0\,, \quad \lambda' > \lambda$$

The technicalities are left to the reader as an exercise. Therefore it can be used to construct the Stieltjes integral with respect to it:

$$\int_{-\infty}^{\infty} f(\lambda)\, d\mu(\lambda)$$

It follows from

$$d\mu(\lambda) = \mu(\lambda) - \mu(\lambda - d\lambda) = \langle u, (E_\lambda - E_{\lambda - d\lambda})u \rangle\,, \quad d\lambda > 0$$

that

$$\int_{-\infty}^{\infty} d\mu(\lambda) = \|u\|^2 < \infty$$

because $\mu(-\infty) = 0$ and $\mu(\infty) = \|u\|^2$. So, $\|u\|^{-2}\mu(\lambda)$ is a probability measure.

THEOREM **61.1**. (Spectral theorem in terms of projection measures) *There exists a one-to-one correspondence between self-adjoint operators and spectral families such that*

$$\langle u, Au \rangle = \int \lambda\, d\mu(\lambda)\,, \quad \mu(\lambda) = \langle u, E_\lambda u \rangle\,, \quad \forall u \in D_A$$

If A is a compact self-adjoint operator, then $\sigma(A) = \sigma_p(A) = \{\lambda_n\}$ and

$$E_\lambda = \sum_{|\lambda_n^{-1}| \leq \lambda} P_{\lambda_n}$$

is the spectral family if P_{λ_n} is the projection operator onto the null space $N_{A-\lambda_n I}$. In particular, $E_\lambda = 0$ if $\lambda \leq 0$. For every $\lambda > 0$, E_λ is a projection onto a finite dimensional subspace. If $\dim \mathcal{H} = \infty$, then λ_n^{-1} has no limit point (because λ_n has one limit point which is zero). Then the sequence $|\lambda_n^{-1}|$ is strictly positive and can be ordered to be monotonically increasing to infinity. Then the function

$$\mu(\lambda) = \langle u, E_\lambda u \rangle\,, \quad u \in \mathcal{H}\,,$$

is monotonically increasing, piecewise constant, and continuous from the right at each jump discontinuity:

$$\mu(\lambda) = \sum_{n=1}^{\infty} c_n \theta\Big(\lambda - |\lambda_n^{-1}|\Big)\,, \quad c_n = \|P_{\lambda_n} u\|^2$$

In particular

$$\mu(\lambda) = 0\,, \quad \lambda \leq 0\,, \quad \lim_{\lambda\to\infty} \mu(\lambda) = \sum_{n=1}^{\infty} c_n = \|u\|^2$$

The spectral theorem for a compact self-adjoint operator can wriiten in the form of the Stieltjes integral

$$\langle u, Au\rangle = \int \lambda d\mu(\lambda)$$

Recall from linear algebra that a symmetric matrix has diagonal matrix elements in the basis of its eigenvectors

$$\langle e_j, Ae_n\rangle = \lambda_n \langle e_j, e_n\rangle = \lambda_n \delta_{nj}$$

The spectral theorem merely states that for compact self-adjoint operators there exists an orthonormal basis in the Hilbert space in which the matrix elements of the operator are diagonal just like in the finite dimensional case so that in this basis

$$\langle u, Au\rangle = \sum_{n=1}^{\infty} \lambda_n |\langle u, e_n\rangle|^2\,, \quad \forall u \in \mathcal{H}$$

If the spectrum of a self-adjoint has a non-empty continuum components, then there is no such basis. Nevertheless, The spectral theorem states that in this case one can find a projection measure such that the action of A is still a multiplication operator taking its values in the spectrum of A:

$$\langle u, Au\rangle = \int \lambda d\langle u, E_\lambda u\rangle\,, \quad \forall u \in \mathcal{H}$$

Furthermore, consider operator-valued projection measure defined by

$$dE_\lambda = E_\lambda - E_{\lambda - d\lambda}\,, \quad d\lambda > 0\,.$$

which means that for any u and v from the Hilbert space

$$\langle u, dE_\lambda v\rangle = \langle u, (E_\lambda - E_{\lambda-d\lambda})v\rangle$$

It is then proved that for any self-adjoint operator there exists a operator-valued projection measure such that

$$A = \int \lambda dE_\lambda$$

In particular, for a compact self-adjoint operators

$$E_\lambda = \sum_{n=1}^{\infty} \theta(\lambda - |\lambda_n^{-1}|) P_{\lambda_n}$$

so that

$$A = \int \lambda dE_\lambda = \sum_{n=1}^{\infty} \lambda_n P_{\lambda_n}$$

$$\langle v, Au\rangle = \sum_{n=1}^{\infty} \lambda_n \langle v, P_{\lambda_n} u\rangle\,, \quad \forall u, v \in \mathcal{H}$$

In other words, the operator acts as a multiplication operator which take its values in the spectrum of A. Since v is arbitrary,

$$Au = \int \lambda \; dE_\lambda u = \sum_{n=1}^{\infty} \lambda_n P_{\lambda_n} u\,, \quad \forall u \in \mathcal{H}$$

which is nothing but the expansion of Au over the basis of eigenvectors of a compact self-adjoint operators.

For a general self-adjoint operator, a similar representation holds even though the operator might have no eigenvectors (the spectrum is continuous, $\sigma(A) = \sigma_c(A)$):

$$\langle v, Au\rangle = \int \lambda d\langle u, E_\lambda v\rangle\,, \quad \forall u \in D_A\,, \quad \forall v \in \mathcal{H}$$

or alternatively

$$Au = \int \lambda \, dE_\lambda u\,, \quad u \in D_A$$

The operator acts as a multiplication operator that takes its values in the spectrum $\sigma(A)$. The projection measure dE_λ vanishes if $\lambda \notin \sigma(A)$, or it has support on the spectrum $\sigma(A)$. The spectral theorem states that such a measure exists for any self-adjoint operator.

One can defined a function of a self-adjoint operator by

$$f(A) = \int f(\lambda) \, dE_\lambda$$

provided the integral exists. In particular, if dE_λ is the operator-valued projection measure for a self-adjoint operator, then

$$I = \int dE_\lambda$$

This is known in quantum theory as the resolution of unity. In the case of compact self-adjoint operators, this is a familiar expression of the completeness of an orthonormal basis of eigenvectors:

$$I = \sum_{n=1}^{\infty} P_{\lambda_n} \quad \Rightarrow \quad \langle u, Iv\rangle = \langle u, v\rangle = \sum_{n=1}^{\infty} \langle u, e_n\rangle\langle e_n, v\rangle$$

The spectral theorem states that a similar resolution of unity exist for any self-adjoint operator whose spectrum has non-empty continuum part, with one difference that the summation over eigenvectors should be replaced by a suitable Stieltjes integral with the measure that has support on the spectrum of the operator:

$$\langle u, Iv\rangle = \langle u, v\rangle = \int d\langle u, E_\lambda v\rangle$$

Example. Consider the operators

$$A = -i\frac{d}{dx}\,, \quad B = A^2 = -\frac{d^2}{dx^2}$$

in the Hilbert space $\mathcal{H} = \mathcal{L}_2(\mathbb{R})$. In quantum theory, these are the momentum and energy operators, respectively, for a particle on a line. These operators are essentially self-adjoint and have been shown to have the self-adjoint extensions. So, the spectral theorem is applicable their self-adjoint extension (in fact, in quantum theory the self-adjoint extensions of A and B are the momentum and energy operators). Put

$$E_\lambda u(x) = \mathcal{F}^{-1}\Big[\theta(\lambda - k)\mathcal{F}[u](k)\Big](x)$$

Then evidently

$$\lim_{\lambda\to\infty} E_\lambda u = 0\,, \quad \lim_{\lambda\to\infty} E_\lambda u = u$$

in the norm of $\mathcal{L}_2$. Then using the isometry of the Fourier transform in $\mathcal{L}_2(\mathbb{R})$:

$$\begin{aligned} d\mu(\lambda) &= \langle u, (E_\lambda - E_{\lambda-d\lambda})u\rangle = \frac{1}{2\pi}\langle \mathcal{F}[u], \mathcal{F}[(E_\lambda - E_{\lambda-d\lambda})u]\rangle \\ &= \frac{1}{2\pi}\int_{\lambda-d\lambda}^{\lambda} \Big|\mathcal{F}[u](k)\Big|^2 dk = \frac{1}{2\pi}\Big|\mathcal{F}[u](\lambda)\Big|^2 d\lambda \end{aligned}$$

the latter equality is *formal* and adopts a convention that only terms linear in $d\lambda$ are kept. However this formal treatment of the integral implies some smoothness of the integrand. But the integrand is a generic function from $\mathcal{L}_2$ and such an expansion does not generally exist. Using the properties that

$$\mathcal{F}[Au](k) = k\mathcal{F}[u](k)\,, \quad \mathcal{F}[Bu](k) = k^2\mathcal{F}[u](k)$$

the spectral theorem for these self-adjoint operators is established:

$$\begin{aligned} \langle u, Au\rangle &= \int_{-\infty}^{\infty} \lambda d\mu(\lambda)\,, \quad u \in D_A \subset \mathcal{L}_2(\mathbb{R}) \\ \langle u, Bu\rangle &= \int_{-\infty}^{\infty} \lambda^2 d\mu(\lambda)\,, \quad u \in D_B \subset \mathcal{L}_2(\mathbb{R}) \end{aligned}$$

Note well that the above relations do not hold for arbitrary $u \in \mathcal{L}_2(\mathbb{R})$ even though $d\mu(\lambda)$ exists on the whole Hilbert space $\mathcal{L}_2(\mathbb{R})$. The Stieltjes integral can diverge (it would not exist) if u is not from the domain of the operator.

Let us find the operator-valued projection measure and its action on any any element of the Hilbert space:

$$\begin{aligned} dE_\lambda u(x) &= \mathcal{F}^{-1}\Big[\Big(\theta(\lambda - k) - \theta(\lambda - d\lambda - k)\Big)\mathcal{F}[u](k)\Big](x) \\ &= \frac{1}{2\pi}\int_{\lambda - d\lambda}^{\lambda} e^{-ikx}\int_{-\infty}^{\infty} e^{iky}u(y)\,dy\,dk \\ &= \int_{-\infty}^{\infty} W(x,\lambda,d\lambda;y)u(y)\,dy \end{aligned}$$

So the projection measure acts as an integral operator with the kernel

$$W(x,\lambda,d\lambda;y) = e^{-i(\lambda - \frac{1}{2}d\lambda)(x-y)}\,\frac{\sin[\frac{1}{2}d\lambda(x-y)]}{\pi(x-y)}$$

This kernel is square integrable in y for any x, λ, and $d\lambda$, because it is continuous and $|W|^2 \sim 1/y^2$ as $|y| \to \infty$. In particular,

$$\|W(x,\lambda,d\lambda)\|^2 = \frac{d\lambda}{4\pi^2}\int_{-\infty}^{\infty}\frac{\sin^2(z)}{z^2}\,dz < \infty$$

If the variables x, λ, and $d\lambda$ are viewed as parameters, then

$$dE_\lambda u(x) = \langle W(x,\lambda,d\lambda), u\rangle$$

This is an analog of the projection of u onto an eigenfunction of a compact self-adjoint operator. Let discuss its meaning in the framework of the probability (or measure) theory.

Recall that positive Stieltjes measures can be viewed as probability measures. In particular, let us find the value of A in the state W:

$$\frac{\langle W(x,\lambda,d\lambda), AW(x,\lambda,d\lambda)\rangle}{\|W(x,\lambda,d\lambda)\|^2} = \lambda - \frac{1}{2}\,d\lambda$$

where $AW(x,\lambda,d\lambda;y) = -iW'_y(x,\lambda,d\lambda;y)$. If one computes the standard deviation of A from its value:

$$\frac{\langle W, (A - \lambda + \frac{1}{2}d\lambda)^2 W\rangle}{\|W\|^2} = \frac{1}{4}(d\lambda)^2$$

It is concluded that the (momentum) operator A has the mean value which is the *midpoint* of the interval $[\lambda - d\lambda, \lambda] \subset \sigma(A) = \mathbb{R}$ with an uncertainty $\pm\frac{1}{2}d\lambda$. In other words, the operator-valued projection measure dE_λ project any state u to onto a state in which the operator value of A is $\lambda - \frac{1}{2}d\lambda$ with an uncertainty $\pm\frac{1}{2}d\lambda$. The operator A has no eigenstates. So the projection onto a states in which the operator has specific spectral value is impossible. However, the existence of the operator-valued projection measure guarantees that there are states in which the operator takes any value from its continuous spectrum, but with some *arbitrary small* uncertainty!

In quantum mechanics,

$$\frac{|W(x,\lambda,d\lambda;y|^2dy}{\|W\|^2}$$

gives the probability to find a particle in the interval $[y, y-dy]$. Sketching the graph of $|W|^2$ shows that it has maximum at $y = x$ and then it rapidly falls off to zero with increasing the distance $|x-y|$. So, in quantum mechanics, the state $W(x, \lambda, d\lambda)$ describes a particle that have a momentum $\lambda - \frac{1}{2}d\lambda$ and position x. The identity stemming from the spectral theorem

$$u(x) = \int dE_\lambda u(x) = \int \langle W(x,\lambda,d\lambda), u\rangle$$

is known in quantum mechanics as the wave packet decomposition. It states that any state can be expanded in to the sum (meaning the Stieltjes integral) of states in which the particle has a specific momentum λ with arbitrary small uncertainty $d\lambda$, and the mean position x.