Test 1 with solutions

1 (1 pts). For the Laplace equation in three variables

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

find all solutions of the form

$$u(x, y) = f(s)\,, \quad s = x^2 + y^2 + z^2$$

where f is twice continuously differentiable function of one real variable. Indicate the largest region of space where such solutions exist.

SOLUTION: By the chain rule

$$\frac{\partial u}{\partial x} = f'(s)\,\frac{\partial s}{\partial x} = 2xf'(s)\,, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\big(2xf'(s)\big) = 2f'(s) + 4x^2 f''(s)$$

The corresponding derivatives with respect to y and z are obtained by replacing x by y and z in the above equations. Therefore

$$\Delta u = 6f'(s) + 4sf''(s) = 0$$

Put $g(s) = f'(s)$. Then by separating variables in the first-order equation for g

$$\frac{g'}{g} = -\frac{3}{2s} \quad \Rightarrow \quad g(s) = \frac{A}{s^{3/2}} \quad \Rightarrow \quad f(s) = \int g(s)\,ds = \frac{B}{\sqrt{s}} + C$$

where A, B, and C are arbitrary constants. The solution is valid in any region that does not contain the origin $s \neq 0$ if $B \neq 0$, and a constant solution C is valid in the whole space ($B = 0$).

2 (1 pt). Find the most general solution to the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}\,, \quad a^2 > 0\,,$$

that can be obtained by separating variables t and x that is bounded

$$|u(x, t)| \leq M\,, \quad t \geq 0\,, \quad -\infty < x < \infty$$

for all non-negative $t \geq 0$ and all real values of x.

SOLUTION: Let $u(x, t) = T(t)X(x)$. Then the heat equation is satisfied if

$$X''(x) = \lambda X(x)\,, \quad T'(t) = a^2 \lambda T(t)$$

where λ is a constant of separation of variables. The latter equation has the following general solution

$$T(t) = Ae^{\lambda a^2 t}$$

where A is a constant. This solution is bounded for $t \geq 0$ only if $\lambda = 0$ or $\lambda = -\nu^2 < 0$ If $\lambda = 0$, then $X(x) = Cx + B$. This function is bounded for all x only if $C = 0$. If $\lambda = -\nu^2$, then

$X(x) = C\cos(\nu x) + B\sin(\nu x)$ so that $|X(x)| \le |C| + |B|$ for all x because cos and sin function take values between -1 and 1. Thus,

$$u(x,t) = C_0 + \sum_{n=1}^{N} e^{-a^2\nu_n^2 t}\Big(C_n\cos(\nu_n x) + B_n\sin(\nu_n x)\Big)$$

is a bounded solution for any choice of real constants C_n, B_n, and ν_n.

3 (1 pt). Find the most general solution to the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + (2x - y)u = 0$$

that can be obtained by separation of variables x and y.

SOLUTION: Let $u(x,y) = X(x)Y(y)$. Substituting this function into the equation and dividing the result by u, one infers that

$$\frac{X'(x)}{X(x)} + 2x + \frac{Y'(y)}{Y(y)} - y = 0$$

for all x and y, which is only possibly if

$$\frac{X'(x)}{X(x)} + 2x = \lambda\,, \qquad \frac{Y'(y)}{Y(y)} - y = -\lambda$$

Integrating these equations,

$$\ln\Big(X(x)\Big) + x^2 = \lambda x + A\,, \quad \ln\Big(Y(x)\Big) - \frac{1}{2}y^2 = -\lambda y + B\,,$$

Therefore by taking the exponential of these relations

$$X(x)Y(y) = Ce^{-x^2+\frac{1}{2}y^2+\lambda(x-y)}\,, \quad u(x,y) = \sum_{n=1}^{N} C_n e^{-x^2+\frac{1}{2}y^2+\lambda_n(x-y)}$$

for any choice of constants C_n and λ_n.

4 (4 pts). Consider the initial and boundary value problem for a 2D wave equation

$$\text{(PDE)}: \quad \frac{\partial^2 u}{\partial^2 t} - 4\frac{\partial^2 u}{\partial x^2} = f(x,t)\,, \quad t > 0\,, \quad 0 < x < 2\,,$$

$$\text{(IC)}: \quad u\Big|_{t=0} = u_0(x)\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = u_1(x)\,, \quad 0 \le x \le 2\,,$$

$$\text{(BC)}: \quad u\Big|_{x=0} = u\Big|_{x=2} = 0\,, \quad t \ge 0\,.$$

(i) Let $f = 0$. Find the most general solution to (PDE) by separating variables t and x that satisfies the boundary conditions (BC).
(ii) Let $f = 0$ and the initial conditions (IC) be

$$u_0(x) = 3\sin(\pi x)\,, \quad u_1(x) = \sin(2\pi x) - 2\sin(\pi x/2)$$

Find a solution obtained in Part (i) that also satisfies these initial conditions or show that no such solution exists.
(iii) Use the form of a solution obtained in Part (i) to solve the problem if

$$f(x,t) = \sin(2t)\sin(\pi x)\,, \quad u_0(x) = u_1(x) = 0\,.$$

(iv) Find a solution to the problem using the form obtained in Part (i) if the initial conditions are given in Part (ii) and the inhomogeneity f is given in Part (iii) or show that no such solution exists.

SOLUTION: (i) In the wave equation with boundary conditions (fixed ends) discussed in class, put $c = 2$ and $L = 2$. Then a solution to (PDE) satisfying (BC) reads

$$u(x,t) = \sum_{n=1}^{N} T_n(t)\sin(\nu_n x)\,, \quad \nu_n = \frac{\pi n}{L} = \frac{\pi n}{2}$$

where T_n are solution to $T_n''(t) + c^2\nu_n^2 T_n(t) = 0$ with $c = 2$.
(ii) The initial condition (IC) contains linear combinations of $\sin(\nu_n x)$ for $n = 1$, $n = 2$, and $n = 4$. Therefore only T_n for these values of n contribute to the solution (others vanish as shown in class). A general solution reads

$$T_n(t) = A\cos(\pi n t) + B\sin(\pi n t)$$

Implementation of (IC):

$$\begin{aligned}
n = 1 \;&:\; T_1(0) = 0\,, \quad T_1'(0) = -2 \quad \Rightarrow \quad T_1(t) = -\frac{2}{\pi}\sin(\pi t)\,,\\
n = 2 \;&:\; T_2(0) = 3\,, \quad T_2'(0) = 0 \quad \Rightarrow \quad T_2(t) = 3\cos(2\pi t)\,,\\
n = 4 \;&:\; T_4(0) = 0\,, \quad T_4'(0) = 1 \quad \Rightarrow \quad T_4(t) = \frac{1}{4\pi}\sin(4\pi t)\,,\\
\Rightarrow \quad & u(x,t) = -\frac{2}{\pi}\sin(\pi t)\sin(\pi x/2) + 3\cos(2\pi t)\sin(\pi x) + \frac{1}{4\pi}\sin(4\pi t)\sin(2\pi x)
\end{aligned}$$

(iii) The inhomogeneity is proportional to $\sin(\nu_n x)$ for $n = 2$. Therefore only $T_2(t)$ is not zero and satisfies the initial value problem:

$$T_2''(t) + (2\pi)^2 T_2(t) = \sin(2t)\,, \quad T_2(0) = T_2'(0) = 0\,.$$

The problem is solved by the method of undetermined coefficients. A particular solution has the form $C\sin(2t)$. Its substitution to the equation yields $C = 1/(4\pi^2 - 4)$. Thus, a general solution reads

$$T_2(t) = A\cos(2\pi t) + B\sin(2\pi t) + \frac{\sin(2t)}{4(\pi^2 - 1)}$$

The condition $T_2(0) = 0$ implies that $A = 0$ and $T_2'(0) = 0$ implies that $2\pi B + \frac{1}{2(\pi^2-1)} = 0$. Therefore

$$u(x,t) = \frac{1}{4(\pi^2 - 1)}\Big(\sin(2t) - \frac{1}{\pi}\sin(2\pi t)\Big)\sin(\pi x)$$

(iv) Let u_2 and u_3 be solutions in Parts (ii) and (iii), respectively. Then the sum $u = u_2 + u_3$ is a solution to (PDE) that satisfies (BC). Since $u_3 = 0$ at $t = 0$, u satisfies the first (IC) because u_2 satisfies it. Since $\frac{\partial u_3}{\partial t} = 0$ at $t = 0$, u satisfies the second (IC) because $\frac{\partial u_2}{\partial t}$ satisfies it. Thus, the solution is the sum of solutions obtained in Parts (ii) and (iii).

5 (2 pts). Solve the following boundary value problems for the 2D Laplace equation by separating variables in polar coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \quad (x, y) \in \Omega\,, \qquad u(x,y)\Big|_{\partial\Omega} = x^2 - 4xy\Big|_{\partial\Omega}$$

if
(i) Ω is a disk $x^2 + y^2 < 4$;
(ii) Ω is a complement of a disk, $x^2 + y^2 > 4$, and, in addition, the solution is required to be bounded in Ω, $|u(x, y)| \leq M$ for all (x, y) in Ω.

SOLUTION: The most general solution obtainable by separating variables in polar coordinates in any region reads

$$u = A_0 + C_0 \ln(r) + \sum_{m=1}^{N} r^m \Big[A_m \cos(m\theta) + B_m \sin(m\theta)\Big] + \sum_{m=1}^{N} \frac{1}{r^m}\Big[C_m \cos(m\theta) + D_m \sin(m\theta)\Big]$$

The boundary condition can also be written as a linear combination of $\cos(m\theta)$ and $\sin(m\theta)$:

$$u(x,y)\Big|_{\partial\Omega} = x^2 - 4xy\Big|_{r=2} = 2\cos^2(\theta) - 16\cos(\theta)\sin(\theta) = 2 + 2\cos(2\theta) - 8\sin(2\theta)$$

where the double-angle equation has been used.
(i). In this case any solution must be regular at $r = 0$. This demands that $C_0 = 0$ and $C_m = D_m = 0$ for all m. Since the boundary data is a linear combination of 1, $\cos(2\theta)$, and $\sin(2\theta)$, the solution is also a linear combination of them, that is, only A_0, A_2, and B_2 do not vanish. Implementation of the boundary condition:

$$\begin{aligned} 1 &: \quad A_0 = 2\,, \\ \cos(2\theta) &: \quad 4A_2 = 2 \quad \Rightarrow \quad A_2 = \frac{1}{2}\,, \\ \sin(2\theta) &: \quad 4B_2 = -8 \quad \Rightarrow \quad B_2 = -2\,, \\ u &= \quad 2 + r^2\Big[\frac{1}{2}\cos(2\theta) - 2\sin(2\theta)\Big] \end{aligned}$$

(ii) Boundedness of the solution requires that $C_0 = 0$ and $A_m = b_m = 0$ for all m. By the same reason as in Part (i), only A_0, C_2, and D_2 are non-zero here. Implementation of the boundary condition:

$$\begin{aligned} 1 &: \quad A_0 = 2\,, \\ \cos(2\theta) &: \quad \frac{C_2}{4} = 2 \quad \Rightarrow \quad A_2 = 8\,, \\ \sin(2\theta) &: \quad \frac{D_2}{4} = -8 \quad \Rightarrow \quad D_2 = -32\,, \\ u &= \quad 2 + \frac{1}{r^2}\Big[8\cos(2\theta) - 32\sin(2\theta)\Big] \end{aligned}$$

6 (Extra credit, 1pt). Find the most general solution to the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x - y)u^2$$

SOLUTION: Let $\xi = x + y$ and $\eta = x - y$. Then by the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial \xi}$$

Therefore in the new variables, the equation becomes an ordinary differential equation in the variable ξ whereas η is a parameter. The equation is integrated by separating variables:

$$\frac{\partial u}{\partial \xi} = \eta u^2 \quad \Rightarrow \quad \int \frac{du}{u^2} = \eta \int d\xi + g(\eta) \quad \Rightarrow \quad u = -\frac{1}{\eta\xi + g(\eta)} = -\frac{1}{x^2 - y^2 + g(x-y)}$$

where g is any continuously differentiable function of a real variable.