Test 1 with solutions

1 (2 pts). Consider the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = 0$$

There is a reason to believe that this equation has a solution of the form

$$u(x,t) = 4\arctan(e^{w(x,t)}), \quad w(x,t) = \gamma(x - ct),$$

for any real x and t, where γ and c are numerical parameters.
(i) Suppose that a solution has the above form. Assume that $\gamma > 0$. Sketch the graphs $u(x,0)$ and $u(x,t)$ for some $t > 0$. Explain how the two graphs are related.
(ii). Find a relation between numerical parameters γ and c so that the above function is a solution. Hint: Use the trigonometric identity

$$\sin(4\phi) = \frac{4\tan(\phi)[1 - \tan^2(\phi)]}{[1 + \tan^2(\phi)]^2}$$

where $\phi = \arctan(e^w)$, to calculate $\sin(4u)$ in terms of w.

SOLUTION[1]: (i) Since $w(x,0) = \gamma x$, $e^{w(x,0)}$ is increasing monotonically from 0 to $+\infty$ as x is increasing from $-\infty$ to $+\infty$. Therefore $u(x,0)$ is monotonically increasing from 0 to 2π. The horizontal lines $y = 0$ and $y = 2\pi$ are horizontal asymptotes of $y = u(x,0)$ and the graph lies between these lines. The graph intersects the y axis at $y = \pi$ and has the inflection point at this intercept. The graph $y = u(x,t)$ has the same shape but shifted along the x axis by ct. One can say that the graph is moving with rate c to the right with increasing t.
(ii) By the hint

$$\sin(u) = \sin(4\arctan(e^w)) = \frac{4e^w(1 - e^{2w})}{[1 + e^{2w}]^2}$$

Let us calculate the needed partials:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{4e^w}{1 + e^{2w}}\frac{\partial w}{\partial t} = -c\gamma\frac{4e^w}{1 + e^{2w}} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\frac{\partial u}{\partial t} = -c\gamma\frac{\partial w}{\partial t}\frac{d}{w}\frac{4e^w}{1 + e^{2w}} = c^2\gamma^2\frac{4e^w(1 - e^{2w})}{[1 + e^{2w}]^2} = c^2\gamma^2\sin(u) \\
\frac{\partial^2 u}{\partial x^2} &= \gamma^2\frac{4e^w(1 - e^{2w})}{[1 + e^{2w}]^2} = \gamma^2\sin(u)
\end{aligned}$$

where the hint has been used. The last equation is obtained from the previous one by replacing $\frac{\partial w}{\partial t}$ by $\frac{\partial w}{\partial x} = \gamma$ in it (because the chain rule is identical for partials w.r.t. x and t). The equation is satisfied if

$$[c^2\gamma^2 - \gamma^2 + 1]\sin(u) = 0 \quad \Rightarrow \quad \gamma^2 = \frac{1}{1 - c^2}$$

[1]The equation is the celebrated *sine-Gordon equation*. It was first discovered by a French engineer, Edmond Bour, in 1862. In 1970s this equation was extensively studied because it has *soliton* solutions. The solution in Problem 1 is the simplest soliton solution. A soliton is a solution to a non-linear PDE that propagates like a particle without changing its shape. The sine-Gorgon equation provides an example of a non-linear PDE that is *integrable* (all its solutions are known).

2 (3 pt). Consider the following PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + (3x + y)u = 0 \,.$$

(i) Show that this equation can be reduced to an ordinary differential equation (ODE) in the new variables $\xi = 3x + y$ and $\eta = x - y$;
(ii) Find a general solution to the PDE by integrating the ODE;
(iii) Suppose that $u(x, 0) = \sin(x)e^{-9x^2/8}$. What is the solution to the PDE that satisfies this condition at $y = 0$?

SOLUTION: (i) The partials in the new variables have the form

$$\frac{\partial}{\partial x} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \,, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$$

Therefore the PDE in the new variables has the form

$$4\frac{\partial u}{\partial \xi} + \xi u = 0$$

which is a first-order ODE in the variable ξ.
(ii) It is solved by separating variables

$$\int \frac{du}{u} = -\frac{1}{4}\int \xi d\xi \quad \Rightarrow \quad u = g(\eta)e^{-\xi^2/8} = g(x - y)e^{-\frac{1}{8}(3x+y)^2}$$

where the integration constant g can be any function of η.
(iii) By setting $y = 0$ in the general solution, one infers that

$$g(x)e^{-9x^2/8} = \sin(x)e^{-9x^2/8} \quad \Rightarrow \quad g(x) = \sin(x) \quad \Rightarrow \quad u(x, y) = \sin(x - y)e^{-\frac{1}{8}(3x+y)^2}$$

3 (1 pt). (i) Use a general solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \,,$$

to find all solutions that satisfy the following initial conditions

$$u(x, 0) = u_0(x) \,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \,,$$

where $u_0(x)$ is a twice continuously differentiable function that vanishes when $|x| \geq 1$. Express the solutions $u(x, t)$ in terms of the function u_0.
(ii) Explain how the graph of the solution $u(x, t)$ for $t > 0$ is related to the graph of $u_0(x)$. If necessary, sketch an example of $u_0(x)$ and $u(x, t)$.

SOLUTION: (i) A general solution to the 2D wave equation has the form

$$u(x, t) = f(x - ct) + g(x + ct)$$

where f and g are twice continuously differentiable functions of a single real variable. The initial conditions yield

$$\begin{aligned} u(x, 0) &= f(x) + g(x) = u_0(x) \\ \frac{\partial u}{\partial t}\Big|_{t=0} &= -cf'(x - ct) + cg'(x + ct)\Big|_{t=0} = c[g'(x) - f'(x)] = 0 \end{aligned}$$

It follows from the last equation that $g(x) = f(x) + A$ where A is an arbitrary constant. A substitution of the latter into the first equation gives $2f(x) + A = u_0(x)$. Therefore

$$\begin{aligned} f(x) &= \frac{1}{2}u_0(x) - \frac{A}{2}, \qquad g(x) = \frac{1}{2}u_0(x) + \frac{A}{2} \\ u(x,t) &= \frac{1}{2}u_0(x-ct) + \frac{1}{2}u_0(x+ct). \end{aligned}$$

(ii) The graph of u_0 is a bump supported on the interval $[-1, 1]$. The solution $u(x, t)$ can be viewed as a half-bump (by its amplitude) is moving to the left at a rate c and the other half to the right with the same rate. So, the solution represents two bumps of width 2 having the same shape and traveling in the opposite directions with the same speed.

4 (1 pt). Find the most general solution to the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a^2 > 0,$$

that can be obtained by separating variables t and x that is bounded

$$|u(x,t)| \le M, \quad t \ge 0, \quad -\infty < x < \infty$$

for all non-negative $t \ge 0$ and all real values of x.

SOLUTION: It has been shown in class that if $u(x, t) = T(t)X(x)$ then

$$T'(t) = a^2\lambda T(t) \quad \Rightarrow \quad T(t) = Ae^{a^2\lambda t}$$

where λ is the separation constant. This solution is bounded for all $t \ge 0$ only if $\lambda \le 0$. Let $\lambda = -\nu^2$. Then, as shown in class,

$$X(x) = B\cos(\nu x) + C\sin(\nu x)$$

Evidently $|X(x)| \le |B| + |C| < \infty$ for all x. If $\lambda = 0$, then $X(x) = B + Cx$. Since x is not bounded, one has to set $C = 0$. So, for $\lambda = 0$, the solution is constant. A general bounded solution of this type is a finite sum of

$$u_0(x,t) = A_0, \quad u_\nu(x,t) = e^{-a^2\nu^2 t}\Big(A_\nu \cos(\nu x) + B_\nu \sin(\nu x)\Big)$$

for some distinct values of $\nu > 0$.

5 (2 pts). Solve the following boundary value problems for the 2D Laplace equation by separating variables in polar coordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in \Omega, \qquad u(x,y)\Big|_{\partial\Omega} = x^2 - 4xy\Big|_{\partial\Omega}$$

if
(i) Ω is a disk $x^2 + y^2 < 4$;
(ii) Ω is a complement of a disk, $x^2 + y^2 > 4$, and, in addition, the solution is required to be bounded in Ω, $|u(x, y)| \le M$ for all (x, y) in Ω.

SOLUTION: (i) The boundary condition can also be written as a trigonometric polynomial:

$$u(x,y)\Big|_{\partial\Omega} = x^2 - 4xy\Big|_{r=2} = 2\cos^2(\theta) - 16\cos(\theta)\sin(\theta) = 2 + 2\cos(2\theta) - 8\sin(2\theta)$$

where the double-angle equation has been used. It was shown in class that a general solution obtainable by separating variables in polar coordinates that is regular at the origin has the form

$$u = A_0 + \sum_{m=1}^{N} r^m \Big[A_m \cos(m\theta) + B_m \sin(m\theta)\Big]$$

By setting $r = 2$ and equating the result with the boundary function, one infers that

$$\begin{aligned} 1 &: \quad A_0 = 2\,, \\ \cos(2\theta) &: \quad 4A_2 = 2 \quad \Rightarrow \quad A_2 = \frac{1}{2}\,, \\ \sin(2\theta) &: \quad 4B_2 = -8 \quad \Rightarrow \quad B_2 = -2\,, \\ u &= \quad 2 + r^2\Big[\frac{1}{2}\cos(2\theta) - 2\sin(2\theta)\Big] \end{aligned}$$

(ii) As shown in class, a general solution obtainable by separating variables in polar coordinates that is bounded in the complement of a disk has the form

$$u = C_0 + \sum_{m=1}^{N} r^{-m} \Big[C_m \cos(m\theta) + D_m \sin(m\theta)\Big]$$

By setting $r = 2$ and equating the result with the boundary function, one infers that

$$\begin{aligned} 1 &: \quad C_0 = 2\,, \\ \cos(2\theta) &: \quad \frac{C_2}{4} = 2 \quad \Rightarrow \quad C_2 = 8\,, \\ \sin(2\theta) &: \quad \frac{D_2}{4} = -8 \quad \Rightarrow \quad D_2 = -32\,, \\ u &= \quad 2 + \frac{1}{r^2}\Big[8\cos(2\theta) - 32\sin(2\theta)\Big] \end{aligned}$$

6 (Extra credit, 1pt). Someone claims that there exists a non-zero polynomial solution to the Laplace equation in an annulus centered at some point (x_0, y_0) in a plane that has different constant values on the outer and inner boundary circles of the annulus. Should you believe this person? Your answer must be supported by reasonings! If the solution is not required to be a polynomial, can such a solution exist?

SOLUTION: The answer does not depend on the center of the annulus (x_0, y_0) because the Laplace equation has the same form in the shifted variables $\xi = x - x_0$ and $\eta = y - y_0$. So, without loss of generality, $(x_0, y_0) = (0, 0)$. If a solution is a polynomial in x and y, then on any circle centered at the origin, the solution is a trigonometric polynomial, and any such solution can be obtained by separating variables in polar coordinates. It is the sum of general solutions used in Part (i) and (ii) in Problem 5. As shown in class, the coefficients A_m, B_m, C_m and D_m are uniquely determined by boundary data for $r = a$ and $r = b$, and since the boundary data are zeros for any $m \geq 1$ (the solution is a constant for $r = a$ and $r = b$), $A_m = C_m = 0$ and $B_m = D_m = 0$. So, the solution must be a constant, $u = A_0$. But a constant solution cannot have different values for $r = a$ and $r = b$. Hence, no such solution exists. If u is not polynomial, then $u = A_0 + C_0 \ln(r)$ is a general solution independent of θ. It can have two different values for $r = a$ and $r = b$.