Test 2 with solutions

1 (2pts). Consider the equation

$$2xyu'_x - (x^2+y^2)u'_y = x^2y^3\,, \quad x>0\,,\; y>0$$

(i) Find the characteristics;
(ii) Find a general solution to the equation.

Solution: (i) In the standard notations used in class, here $a = 2xy$, $b = -(x^2+y^2)$, $c=0$, and $f = x^2y^3$. The characteristics are given by solutions of the exact equation:

$$\begin{aligned} ady - bdx = 2xydy + (x^2+y^2)dx = d\left(xy^2 + \frac{1}{3}x^3\right) = 0 \;&\Rightarrow\; \gamma(x,y) = xy^2 + \frac{1}{3}x^3 = p \\ &\Rightarrow\; y^2 = \frac{1}{x}\Big(p - \frac{1}{3}x^3\Big) = Y^2(x,p) \end{aligned}$$

(ii) The associated ODE problem is easy to solve

$$\begin{aligned} Z'_x(x,p) &= \frac{f}{a}\Big|_{y=Y(x,p)} = \frac{x^2y^3}{2xy}\Big|_{y=Y(x,p)} = \frac{1}{2}xy^2\Big|_{y=Y(x,p)} = \frac{1}{2}\Big(p - \frac{1}{3}x^3\Big) \\ Z(x,p) &= \frac{px}{2} - \frac{x^4}{24} + Z_0(p) \end{aligned}$$

Therefore the general solution is

$$u(x,y) = Z(x,p)\Big|_{p=\gamma(x,y)} = \frac{1}{2}\,x^2y^2 + \frac{1}{8}\,x^4 + Z_0\Big(xy^2 + \frac{1}{3}\,x^3\Big)$$

where Z_0 is any function from the class C^1.

2 (3pts). Consider the equation

$$(x^2+1)u'_x - (2xy-3)u'_y + u = f(x,y)$$

in some region of the plane.
(i) Find its characteristics;
(ii) Find a general solution if $f(x,y)=0$;
(iii) Find a general solution if $f(x,y) = y(x^2+1) - 3x$.

Solution: (i) In the standard notations used in class, here $a = x^2+1$, $b = -(2xy-3)$, and $c = 1$. The characteristic equation is exact so that

$$ady - bdx = (x^2+1)dy + (2xy-3)dx = 0 \quad\Rightarrow\quad d\Big(x^2y + y - 3x\Big) = 0$$

and the characteristics are

$$\gamma(x,y) = (x^2+1)y - 3x = p \quad\Rightarrow\quad y = \frac{p+3x}{x^2+1} = Y(x,p)$$

The associated ODE problem is

$$\begin{aligned} Z'_x(x,p) + \frac{c}{a}\Big|_{y=Y(x,p)} Z(x,p) &= \frac{f}{a}\Big|_{y=Y(x,p)} \\ Z'_x(x,p) + \frac{1}{x^2+1} Z(x,p) &= \frac{p}{x^2+1} \end{aligned}$$

Note that $f(x,y) = \gamma(x,y)$ so it is constant $f(x,y) = p$ on any characteristic.
(ii) A general solution to the homogeneous problem is

$$Z'_x(x,p) + \frac{1}{x^2+1} Z(x,p) = 0 \quad \Rightarrow \quad Z = Z_h(x,p) = Z_0(p)e^{-\arctan(x)}$$

where $Z_0(p)$ is an arbitrary C^1 function so that the solution to the PDE with $f = 0$ reads

$$u(x,y) = Z_h(x,p)\Big|_{p=\gamma(x,y)} = Z_0(\gamma(x,y))e^{-\arctan(x)}$$

(iii) A particular solution to the non-homogeneous problem is sought in the form $Z_p = Ve^{-\arctan(x)}$ so that

$$V'e^{-\arctan(x)} = \frac{p}{x^2+1} \quad \Rightarrow \quad V(x,p) = p\int \frac{e^{\arctan(x)}}{x^2+1}\,dx = pe^{\arctan(x)}$$

so that $Z(x,p) = Z_h(x,p) + Z_p(x,p)$ is the general solution to the associated ODE problem. Thus, a general solution to the said PDE reads

$$u(x,y) = Z(x,p)\Big|_{p=\gamma(x,y)} = Z_0(\gamma(x,y))e^{-\arctan(x)} + \gamma(x,y)$$

3 (1 pt). Solve the initial value (Cauchy) problem

$$u'_t + 2tu'_x + xu = 0\,, \quad t > 0\,; \quad u|_{t=0} = u_0(x)$$

Express the answer in terms of the function $u_0(x)$.

Solution: In the standard notations used in class, here $b = 2t$, $c = x$, and $f = 0$. A characteristic passing through the point $(x,t) = (p,0)$ is the solution to the initial value problem:

$$X'_t(t,p) = 2t\,, \quad X(0,p) = p \quad \Rightarrow \quad X(t,p) = t^2 + p \quad \Rightarrow \quad x = X(t,p) \quad \Rightarrow \quad p = x - t^2$$

The associated initial value ODE problem reads

$$\begin{cases} Z'_t(t,p) + c\Big|_{x=X(t,p)} Z(t,p) = 0 \\ Z(0,p) = u_0(p) \end{cases} \quad \Rightarrow \quad \begin{cases} Z'_t(t,p) + (t^2+p)Z(t,p) = 0 \\ Z(0,p) = u_0(p) \end{cases}$$

$$\Rightarrow \quad Z(t,p) = u_0(p)e^{-\frac{1}{3}t^3 - pt}$$

The solution to the given Cauchy problem is

$$u(x,t) = Z(t,p)\Big|_{p=x-t^2} = u_0(x-t^2)e^{\frac{2}{3}t^3 - tx}$$

4 (1 pt). Solve the initial value (Cauchy) problem

$$u'_t + u'_x + u^3 = 0\,, \quad t > 0\,; \quad u\Big|_{t=0} = x^2$$

Solution: In the standard notations used in class, here $b = 1$, $f = -u^3$, and $u_0 = x^2$. Therefore the associated autonomous system that defines all the characteristics is

$$\begin{cases} X'_t(t,p) = 1 \\ Z'_t(t,p) = -Z^3(t,p) \end{cases}, \quad X(0,p) = p\,,\ Z(0,p) = p^2$$

The latter equation is integrated as

$$-\int \frac{dZ}{Z^3} = \int dt \quad \Rightarrow \quad \frac{1}{Z^2} = 2t + \frac{1}{p^4}$$

where the integration constant is set by the initial condition. The solution reads

$$X(t,p) = t + p\,, \quad Z(t,p) = \left(2t + \frac{1}{p^4}\right)^{-1/2}$$

so that $x = X(t,p)$ implies that $p = x - t$. Note that $Z(0,p) = p^2 > 0$ so that the negative solution should be discarded. Therefore the solution to the Cauchy problem is

$$u(x,t) = Z(t,p)\Big|_{p=x-t} = \left(2t + (x-t)^{-4}\right)^{-1/2}$$

5 Extra credit (1 pt) Find a general solution to the equation

$$yu'_x - a^2 x u'_y + f(y^2 + a^2x^2)u = 0\,, \quad x > 0\,,\ y > 0\,,$$

where a is a constant and $f(s)$ is a continuously differentiable function of real variable s. Express the answer in terms of the function f.

Solution: The characteristics are

$$ydy + a^2xdx = 0 \quad \Rightarrow \quad y^2 + a^2x^2 = p \quad \Rightarrow \quad y = Y(x,p) = \sqrt{p - a^2x^2}$$

because $y > 0$. The associated ordinary differential equation is

$$\frac{dZ}{dx} + \frac{f(p)}{Y(x,p)} Z = 0 \quad \Rightarrow \quad Z = g(p)e^{f(p)\int \frac{dx}{Y(x,p)}}$$

where g is any C^1 function. Evaluating the integral by substitution $s = ax/\sqrt{p}$

$$\int \frac{dx}{Y(x,p)} = \int \frac{dx}{\sqrt{p - a^2x^2}} = \frac{1}{a} \arcsin\left(\frac{ax}{\sqrt{p}}\right)$$

and substituting it into Z and then setting $p = y^2 + a^2x^2$, the solution is obtained

$$u(x,y) = g(y^2 + a^2x^2) \exp\left[\frac{f(y^2 + a^2x^2)}{a} \arctan\left(\frac{ax}{y}\right)\right].$$