Test 2 with solutions

1 (4 pts). Consider the linear first-order equation in the positive quadrant:

$$2xy\,\frac{\partial u}{\partial x} - (2x+y^2)\,\frac{\partial u}{\partial y} + c(x,y)u = f(x,y)\,, \quad x>0\,,\ y>0$$

(i) Find characteristics and a general solution if $c = f = 0$;
(ii) Find a general solution if $c(x,y) = y$ and $f(x,y) = 0$;
(iii) Find a general solution if $c(x,y) = y$ and $f(x,y) = 3xy(x^2+y^2x)$;
(iv) Find a solution with c and f as in Part (iii) that satisfies the following condition

$$u\Big(x, \frac{1}{\sqrt{x}}\Big) = x(x^2+1+\sqrt{x})$$

in other words, the solution has a given value on the hyperbola $y = \frac{1}{\sqrt{x}}$.

Solution: (i) The characteristics satisfy the exact equation

$$2xy\,dy + (2x+y^2)dx = 0 \quad \Rightarrow \quad d(x^2+y^2x) = 0 \quad \Rightarrow \quad \gamma(x,y) = x^2+y^2x = p$$

Therefore a general solution to PDE with $c = f = 0$ is

$$u(x,y) = g(p) = g(x^2+y^2x)$$

where g is any continuously differentiable function of a real variable.
(ii) The coefficient $a(x,y) = 2xy > 0$ does not vanish anywhere in the positive quadrant. Therefore in the new variables x and $p = \gamma(x,y)$,

$$\frac{c(x,y)}{a(x,y)} = \frac{y}{2xy} = \frac{1}{2x}$$

and the solution $u(x,y) = Z(x,p)$ in the new variables satisfies the ordinary differential equation

$$\frac{dZ}{dx} + \frac{c(x,y)}{a(x,y)}\,Z = 0 \quad \Rightarrow \quad \frac{dZ}{dx} + \frac{1}{2x}\,Z = 0\,.$$

Its general solution is obtained by separating variables $\frac{dZ}{Z} = -\frac{dx}{2x}$ so that

$$Z(t,p) = \frac{g(p)}{\sqrt{x}} \quad \Rightarrow \quad u(x,y) = Z(x,p)\Big|_{p=\gamma(x,y)} = \frac{g(x^2+y^2x)}{\sqrt{x}}$$

where g is any continuously differentiable function of a real variable.
(iii) In the new variables x and p

$$\frac{f(x,y)}{a(x,y)} = \frac{3xy(x^2+y^2x)}{2xy} = \frac{3}{2}(x^2+y^2x) = \frac{3}{2}\,p\,.$$

The solution is the new variables $u(x,y) = Z(x,p)$ is a general solution to the ordinary differential equation

$$\frac{dZ}{dx} + \frac{1}{2x} Z = \frac{3}{2} p$$

which is the sum of a general solution to the associated homogeneous equation that was found in Part (ii) and a particular solution. The latter is obtained by the method of variation of parameters $Z = V/\sqrt{x}$ where V satisfies the equation

$$\frac{dV}{dx}\frac{1}{\sqrt{x}} = p \quad \Rightarrow \quad V(x) = \frac{3}{2} p \int \sqrt{x} dx = p x^{3/2} \quad \Rightarrow \quad Z(x,p) = px \,.$$

Therefore

$$u(x,y) = \frac{g(x^2 + y^2 x)}{\sqrt{x}} + x(x^2 + y^2 x)$$

(iv) By setting $y = \frac{1}{\sqrt{x}}$ in the above general solution, an equation for the function g is obtained:

$$\frac{g(x^2+1)}{\sqrt{x}} + x(x^2+1) = x(x^2 + 1 + \sqrt{x}) \quad \Rightarrow \quad g(x^2+1) = x^2 \quad \Rightarrow \quad g(s) = s - 1$$

Therefore

$$u(x,y) = \frac{x^2 + y^2 x - 1}{\sqrt{x}} + x(x^2 + y^2 x) \,.$$

2 (3 pts). Consider the Cauchy problem for the linear equation

$$\frac{\partial u}{\partial t} + 2tx \frac{\partial u}{\partial x} + e^{-t^2} x u = f_0 \,, \quad u\Big|_{t=0} = \frac{1}{1+x^2}$$

where f_0 is a constant.
(i) Find characteristics $\gamma(t,x) = p$ for this equation such that $\gamma(0,p) = p$;
(ii) Find the solution to the Cauchy problem if $f_0 = 0$;
(iii) Find the solution to the Cauchy problem if $f_0 \neq 0$.

Solution: (i) The characteristic equation

$$\frac{dx}{dt} = 2tx \,, \quad x(0) = p$$

is solved by separating variables, $\frac{dx}{x} = 2tdt$ so that $\ln(x) = t^2 + C$ and, hence,

$$x = X(t) = x(0) e^{t^2} = p e^{t^2} \quad \Rightarrow \quad p = \gamma(t,x) = x e^{-t^2} \,.$$

(ii) In the new variables t and p, $e^{-t^2} x = p$ and the solution to the initial value problem

$$\frac{dZ}{dt} + pZ = 0 \,, \quad Z(0) = \frac{1}{1+p^2}$$

is given by

$$Z(t) = Z(0)e^{-pt} = \frac{e^{-pt}}{1+p^2}$$

The solution to the Cauchy problem is obtained by substituting $p = \gamma(t,x)$ into $Z(t)$:

$$u_0(t,x) = Z(t)\Big|_{p=\gamma(t,x)} = \frac{\exp(-txe^{-t^2})}{1+x^2e^{-2t^2}}$$

(iii) The solution has the form $u(t,x) = u_0(t,x) + u_p(t,x)$ where $u_0(t,x)$ is obtained in Part (ii), and a particular solution in the new variables $u_p(t, X(t,p)) = Z_p(t)$ is the solution to the initial value problem

$$\frac{dZ_p}{dt} + pZ_p = f_0\,, \quad Z_p(0) = 0\,.$$

It is found by the method of variation of parameters, $Z_p(t) = V(t)e^{-pt}$, $V(0) = 0$. The substitution into the above equation yields

$$V'(t) = f_0e^{pt}\,, \quad V(0) = 0 \quad \Rightarrow \quad V(t) = f_0\int_0^t e^{p\tau}d\tau = \frac{f_0}{p}\Big(e^{pt}-1\Big) \quad \Rightarrow \quad Z_p(t) = \frac{f_0}{p}\Big(1-e^{-pt}\Big)$$

Therefore

$$u(t,x) = u_0(t,x) + u_p(t,x)\,, \quad u_p(t,x) = Z_p(t)\Big|_{p=\gamma(t,x)} = \frac{f_0e^{t^2}}{x}\Big[1-\exp(-xte^{-t^2})\Big]\,.$$

3 (2 pt). Consider the Cauchy problem

$$\frac{\partial u}{\partial t} + 2t\,\frac{\partial u}{\partial x} + 3xu^2 = 0 \quad t > 0\,; \quad u\Big|_{t=0} = u_0(x)$$

(i) Find parametric characteristics for this equation by solving the associated autonomous system with general initial conditions $X(0) = x_0$, $Z(0) = z_0$.
(ii) Solve the Cauchy problem for an arbitrary function $u_0(x)$ from class C^1. Express the answer in terms of the function u_0.

SOLUTION: (i) The parametric characteristic $x = X(t)$, $z = Z(t)$, passing through the point $X(0) = x_0$, $Z(0) = z_0$ is the solution to the initial value problem for the following autonomous system of ordinary differential equations:

$$\begin{array}{ll} \frac{dX}{dt} = 2t\,, & X(0) = x_0 \\ \frac{dZ}{dt} = -3XZ^2\,, & Z(0) = z_0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x &=& X(t) = t^2 + x_0 \\ -\frac{dZ}{Z^2} &=& 3(t^2+p)dt \end{array}$$

The integration of the latter equation gives

$$\frac{1}{Z} = t^3 + 3pt + \frac{1}{z_0} \quad \Rightarrow \quad z = Z(t) = \frac{z_0}{z_0(t^3+3tp)+1}$$

(ii) For the initial data curve $x_0 = x_0(p) = p$, $z_0 = z_0(p) = u_0(p)$, the function $Z(t)$ give the solution to the Cauchy problem in the variables t and p. To obtain the solution in the original variables t and x, one should solve $x = X(t)$ for p (with $x_0 = p$) and substitute $p = \gamma(t, x)$ into $Z(t)$. One has $x = t^2 + p$ or $p = \gamma(x, t) = x - t^2$. Therefore setting $z_0 = u_0(p)$ in $Z(t)$ and substituting p, one infers that

$$u(x,t) = Z\Big|_{p=x-t^2} = \frac{u_0(x-t^2)}{u_0(x-t^2)(t^3+3tp)+1}\,.$$

4 Extra credit (1 pt) Let f be a continuously differentiable function of a real variable and the derivative f' does not vanish anywhere. Solve the Cauchy problem

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{1}{f'(u)}\,, \quad t > 0\,, \quad u\Big|_{t=0} = u_0(x)$$

where c is a constant and $u_0(x)$ is a continuously differentiable function. Express the answer in terms of f, the inverse function f^{-1}, and the function u_0. Give reasons that the inverse function f^{-1} exists and is continuously differentiable.

Solution: The parametric characteristic $x = X(t)$, $z = Z(t)$ passing through the point $x_0 = X(0)$, $z_0 = Z(0)$ is the solution to the system

$$\begin{aligned}
&X' = c\,, \quad X(0) = x_0 \quad \Rightarrow \quad x = X(t) = ct + x_0\,,\\
&Z' = \frac{1}{f'(Z)}\,, \quad Z(0) = z_0 \quad \Rightarrow \quad \int_0^{z_0} f'(Z)dZ = \int_0^t d\tau \quad \Rightarrow \quad f(Z(t)) - f(z_0) = t\\
&\quad \Rightarrow \quad Z(t) = f^{-1}\big(t + f(z_0)\big)
\end{aligned}$$

Put $x_0 = p$ and $z_0 = u_0(p)$. The solution $u(t, x)$ in the new variables t and $p = x - ct$ (defined by $x = X(t)$) is given by the function $Z(t)$ where $p = x - ct$. Therefore

$$u(t,x) = f^{-1}\Big(t + f(u_0(x-ct))\Big)$$

Since f' does not vanish anywhere, f is a monotonic function, and, by the inverse function theorem, f is invertible and the derivative of f^{-1} is the reciprocal of f'. So, the derivative of f^{-1} is continuous.