Test 3 with solutions

1 (5 pts). Consider a linear second-order PDE with constant coefficients:

$$u''_{xx} + 2u''_{xy} + cu''_{yy} + pu'_x + qu'_y + mu = f(x, y)$$

where parameters c, p, q, and m are real.

(i) Classify the equation, that is, specify the values of parameters for which the equation is either elliptic, or hyperbolic, or parabolic.

(ii) For each set of parameters found in Part (i), find a change of variable such that the linear combination of second partials assumes one of the standard forms.

(iii) Let $c = 1$, $p = q = 4$, $m = 5$, and $f(x, y) = 4(x - y)^2$. Find a general solution.

(iv) Let $c = -8$, $p = 1$, $q = 4$, $m = 0$, and $f(x, y) = 0$. Find the most general solution $u(x, y)$ that satisfies the condition $u(x, x) = 0$

(v) Let $c = 10$, $p = q = m = 0$, and $f(x, y) = 0$. Find all homogeneous polynomial solutions, that is, solutions that are linear combinations of monomials of degree n, $x^n y^{n-k}$, where $k = 0, 1, ..., n$, for each given positive integer n. In particular, give an explicit form of such a solution for $n = 3$.

SOLUTION: (i) The characteristics satisfy the equations

$$(dy)^2 - 2dxdy + c(dx)^2 = (dy - dx)^2 + (c - 1)(dx)^2 = 0$$

Therefore there are two real characteristics if $c < 1$, only one real characteristic if $c = 1$, and complex characteristics if $c > 1$. These cases correspond to hyperbolic, parabolic, and elliptic PDE, respectively:

$$\alpha = y - x + \sqrt{1-c}\,x\,, \quad \beta = y - x - \sqrt{1-c}\,x\,, \quad c < 1 \quad \Rightarrow \quad \text{PDE} = \text{hyperbolic}$$
$$\alpha = y - x\,, \quad c = 1 \quad \Rightarrow \quad \text{PDE} = \text{parabolic}$$
$$\xi = y - x + i\sqrt{c-1}\,x\,, \quad \bar{\xi} = y - x - i\sqrt{c-1}\,x\,, \quad c > 1 \quad \Rightarrow \quad \text{PDE} = \text{elliptic}$$

(ii) As shown in class, a change of variables that reduces a linear combination of second partials to one of the standards forms is obtained by taking new variables to be real characteristics in the case of hyperbolic PDEs,

$$\alpha = y - x + \sqrt{1-c}\,x\,, \quad \beta = y - x - \sqrt{1-c}\,x\,, \quad c < 1\,.$$

For a parabolic equation, one of the new variables is defined by the only characteristic and the other can be chosen arbitrary with one condition that the transformation is invertible. For example, one can take

$$\alpha = y - x\,, \quad \beta = y + x\,, \qquad c = 1 \quad \Rightarrow \quad x = \frac{1}{2}(\beta - \alpha)\,, \quad y = \frac{1}{2}(\beta + \alpha)\,.$$

In the case of elliptic PDEs, the change of variables is defined by the real and imaginary parts of the complex characteristic:

$$\alpha = \mathrm{Re}\,\xi = y - x\,, \quad \beta = \mathrm{Im}\,\xi = \sqrt{c-1}\,x\,, \quad c > 1\,.$$

(iii) Setting $c = 1$ and using the change of variables suggested in Part (ii) for the parabolic PDE, one infers the rule for the partials in the new variables

$$\partial_x = \alpha'_x\partial_\alpha + \beta'_x\partial_\beta = \partial_\beta - \partial_\alpha\,, \quad \partial_y = \alpha'_y\partial_\alpha + \beta'_y\partial_\beta = \partial_\beta + \partial_\alpha\,.$$

Therefore the PDE in question becomes

$$\begin{aligned} 4\alpha^2 &= [(\partial_\beta - \partial_\alpha)^2 + 2(\partial_\beta - \partial_\alpha)(\partial_\beta + \partial_\alpha) + (\partial_\beta + \partial_\alpha)^2]u + 4(\partial_\beta - \partial_\alpha)u + 4(\partial_\beta + \partial_\alpha)u + 5u \\ &= [1 - 2 + 1]u''_{\alpha\alpha} + [-2 + 2]u''_{\alpha\beta} + [1 + 2 + 1]u''_{\beta\beta} + 8u'_\beta + 5u \\ &= 4u''_{\beta\beta} + 8u'_\beta + 5u \end{aligned}$$

This is an ODE in the variable β and α plays the role of a numerical parameter. To solve the ODE, put $u = e^{k\beta}v$ and fix a parameter k so that the equation for v has no term with v'_β:

$$4u''_{\beta\beta} + 8u'_\beta + 5u = e^{k\beta}\Big(v''_{\beta\beta} + [8k+8]v'_\beta + [4k^2 + 8k + 5]v\Big)$$

Therefore $k = -1$ and the equation for v reads

$$v''_{\beta\beta} + \frac{1}{4}\,v = \alpha^2 e^\beta$$

A general solution to the associated homogeneous equation is a linear combination of $\cos(\frac{1}{2}\beta)$ and $\sin(\frac{1}{2}\beta)$. A particular solution is obtained by the method of undermined coefficients. Put $v = Ce^\beta$. Then $C + \frac{1}{4}C = \alpha^2$. Thus,

$$u = e^{-\beta}v = e^{-\beta}\Big[A(\alpha)\cos\Big(\frac{1}{2}\beta\Big) + B(\alpha)\sin\Big(\frac{1}{2}\beta\Big)\Big] + \frac{4}{5}\,\alpha^2$$

where A and B are twice continuously differentiable functions of a real variable.
(iv) If $c = -8$, then the change of variables found in Part (ii) for the hyperbolic PDE becomes $\alpha = y + 2x$ and $\beta = y - 4x$. Then the rules for partials in the new variables read

$$\partial_x = \alpha'_x\partial_\alpha + \beta'_x\partial_\beta = 2\partial_\alpha - 4\partial_\beta\,, \quad \partial_y = \alpha'_y\partial_\alpha + \beta'_y\partial_\beta = \partial_\beta + \partial_\alpha\,.$$

Therefore the PDE in question becomes

$$\begin{aligned} 0 &= [(2\partial_\alpha - 4\partial_\beta)^2 + 2(2\partial_\alpha - 4\partial_\beta)(\partial_\beta + \partial_\alpha) - 8(\partial_\beta + \partial_\alpha)^2]u + (2\partial_\alpha - 4\partial_\beta)u + 4(\partial_\beta + \partial_\alpha)u \\ &= [4 + 4 - 8]u''_{\alpha\alpha} + [-16 + 4 - 8 - 16]u''_{\alpha\beta} + [16 - 8 - 8]u''_{\beta\beta} + 6u'_\alpha \\ &= -36u''_{\alpha\beta} + 6u'_\alpha \end{aligned}$$

A substitution $u = e^{k\beta}v$ reduces the equation to

$$0 = e^{\beta k}\Big(-36v''_{\alpha\beta} + (6 - 36k)v'_\alpha\Big) \quad \Rightarrow \quad v''_{\alpha\beta} = 0$$

if $k = \frac{1}{6}$. Therefore a general solution reads

$$u(x,y) = e^{\frac{1}{6}\beta}\Big(f(\alpha) + g(\beta)\Big)$$

where f and g are twice continuously differentiable functions of a real variables. To implement the condition $u(x,x) = 0$, note that $\alpha = 3x$ and $\beta = -3x$ if $y = x$. Therfore the said condition yields $f(3x) = -g(-3x)$ which must hold for any x. Thus

$$u(x,y) = e^{\frac{1}{6}\beta}\Big(g(\beta) - g(-\alpha)\Big)$$

(v) If $c = 10$, the change of variables found in Part (ii) for the elliptic PDE becomes $\alpha = y - x$ and $\beta = 3x$ so that

$$\partial_x = \alpha'_x\partial_\alpha + \beta'_x\partial_\beta = 3\partial_\beta - \partial_\alpha\,, \quad \partial_y = \alpha'_y\partial_\alpha + \beta'_y\partial_\beta = \partial_\alpha\,.$$

The equation becomes

$$\begin{aligned} 0 &= [(3\partial_\beta - \partial_\alpha)^2 + 2(3\partial_\beta - \partial_\alpha)\partial_\alpha + 10\partial_\alpha^2]u = [1 - 2 + 10]u''_{\alpha\alpha} + [-6 + 6]u''_{\alpha\beta} + 9u''_{\beta\beta} \\ &= 9\Big(u''_{\alpha\alpha} + u''_{\beta\beta}) \end{aligned}$$

So the polynomial solutions are given by harmonic polynomials. According to the lecture notes, all such real polynomials are linear combinations of monomials $\operatorname{Re} z^n$ and $\operatorname{Im} z^n$ where $z = \alpha + i\beta$. In particular for $n = 3$,

$$\begin{aligned} (\alpha + i\beta)^3 &= \alpha^3 - 3\beta^2\alpha + i(3\beta\alpha^2 - \beta^3) \quad \Rightarrow \\ u(x,y) &= A(\alpha^3 - 3\beta^2) + B(3\beta\alpha^2 - \beta^3)\,, \quad \alpha = y - x\,, \quad \beta = 3x \end{aligned}$$

where A and B are constants.

6 Extra credit. Find a general solution $u(x,y,z)$ to the equation

$$a^2u''_{xx} + 2abu''_{xy} + b^2u''_{yy} - u''_{zz} = 0$$

where a and b are positive constants.

$$\partial_x = \alpha'_x\partial_\alpha + \beta'_x\partial_\beta = \partial_\beta - \partial_\alpha\,, \quad \partial_y = \alpha'_y\partial_\alpha + \beta'_y\partial_\beta = \partial_\beta + \partial_\alpha\,.$$

Solution: Let us reduce the first three terms in the equation to the standard form by changing variables in the xy plane. The characteristics satisfy the equation

$$a^2(dy)^2 - 2abdydx + b^2(dx)^2 = (ady - bdx)^2 = 0 \quad \Rightarrow \quad \alpha = ay - bx$$

The equation is parabolic in the xy plane as there is only one characteristic. Put $\beta = ay + bx$. The Jacobian of the transformation $(x,y) \to (\alpha,\beta)$ is not zero, $J = -b^2 - a^2 \neq 0$. Then

$$\partial_x = b(-\partial_\alpha + \partial_\beta)\,, \quad \partial_y = a(\partial_\alpha + \partial_\beta)$$

so that

$$a^2u''_{xx} + 2abu''_{xy} + b^2u''_{yy} = (a\partial_x + b\partial_y)^2u = (2ab\partial_\beta)^2u = 4a^2b^2u''_{\beta\beta}$$

In the new variables, the equation becomes a 2D wave equation in the variables β and z

$$u''_{\beta\beta} - c^2u''_{zz} = 0\,, \quad c = \frac{1}{2ab}$$

Its general solution reads

$$u(x, y, z) = f(\alpha, z - c\beta) + g(\alpha, z + c\beta)\,, \quad \beta = ay + bx\,, \quad \alpha = ay - bx$$

where f and g are functions from class C^2 of two real variables.