Test 3 with solutions

1 (3 pts). (i) Classify the following equation, find its characteristics, reduce the equation to the standard form,

$$\frac{1}{25}\left(2u''_{xx} + u''_{xy} - 3u''_{yy}\right) + u'_x + u'_y + 2u = f(x,y)$$

(ii) use a suitable substitution to eliminate the terms linear in the first partials and find a general solution if $f(x,y) = 0$
(iii) Find a general solution if $f(x,y) = e^{3y-7x}$.

SOLUTION: (i) To simplify the linear combination of second partial derivatives in the parentheses, let us find the characteristics

$$2(dy)^2 - dydx - 3(dx)^2 = (2dy - 3dx)(dy + dx) = 0 \quad \Rightarrow \quad \begin{cases} \alpha &= 2y - 3x \\ \beta &= y + x \end{cases}$$

There are two real characteristics, and, hence, the equation is *hyperbolic*. Using the derivative transformations, the expression in the parentheses becomes

$$\begin{aligned} \frac{\partial}{\partial x} &= -3\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}, \quad \frac{\partial}{\partial x} = 2\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}, \\ 2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2} &= 2\Big(-3\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\Big)^2 + \Big(-3\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\Big)\Big(2\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\Big) - 3\Big(2\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\Big)^2 \\ &= (18 - 6 - 12)\frac{\partial^2}{\partial \alpha^2} + (2 + 1 - 3)\frac{\partial^2}{\partial \beta^2} + (-12 + 2 - 3 - 12)\frac{\partial^2}{\partial \alpha \partial \beta} \\ &= -25\frac{\partial^2}{\partial \alpha \partial \beta} \end{aligned}$$

Thus the equation in the new variables assumes the standard form

$$u''_{\alpha\beta} + u'_\alpha - 2u'_\beta - 2u = -f$$

(ii) The terms proportional to the first partials are eliminated by the substitution

$$\begin{aligned} u &= e^{a\alpha + b\beta}v, \quad u'_\alpha = (av + v'_\alpha)e^{a\alpha+b\beta}, \quad u'_\beta = (bv + v'_\beta)e^{a\alpha+b\beta}, \\ u''_{\alpha\beta} &= (v''_{\alpha\beta} + av'_\beta + bv'_\alpha + abv)e^{a\alpha+b\beta} \end{aligned}$$

so that the unknown function v satisfies the equation

$$v''_{\alpha\beta} + (1+b)v'_\alpha + (a-2)v_\beta + (-2 + ab + a - 2b)v = -fe^{-a\alpha - b\beta}$$

Put $a = 2$ and $b = -1$ so that the coefficients at the first partials vanish and $-2 + ab + a - 2b = 0$ so that the equation is reduced to

$$v''_{\alpha\beta} = -fe^{-2\alpha+\beta}$$

If $f = 0$, its general solution reads

$$v = h(\alpha) + g(\beta)$$

where f and g are any C^2 functions of a single variable, and

$$u(x,y) = e^{3y-7x}\Big(f(2y - 3x) + g(y + x)\Big).$$

(iii) If $f \neq 0$, then a general solution is the sum of a particular solution and a general solution the associated homogeneous problem given in Part (ii). A particular solution is obtained by integrating the equation

$$v''_{\alpha\beta} = -e^{3y-7x}e^{-2\alpha+\beta} = -1 \quad \Rightarrow \quad v'_\beta = -\alpha \quad \Rightarrow \quad v = -\alpha\beta = -(2y-3x)(y+x)$$
$$\Rightarrow \qquad u(x,y) = -(2y-3x)(y+x)e^{3y-7x} + e^{3y-7x}\Big(f(2y-3x) + g(y+x)\Big)$$

2 (4 pt). For the following initial and boundary value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - 9\,\frac{\partial^2 u}{\partial x^2} = 0\,, \qquad t>0\,,\ -1<x<3\,,\\ &u\Big|_{x=-1} = u\Big|_{x=3} = 0\,, \quad t \geq 0\,,\\ &u\Big|_{t=0} = \begin{cases} \sin(\pi x/2) & , \quad 0 \leq x \leq 2 \\ 0 & , \quad -1 \leq x \leq 0 \text{ and } 2 \leq x \leq 3 \end{cases}\\ &\frac{\partial u}{\partial t}\Big|_{t=0} = (x+1)(x-3)\,, \quad -1 \leq x \leq 3\,. \end{aligned}$$

(i) Sketch the graphs of the extended initial data for $-5 \leq x \leq 7$;
(ii) Find $u(1,\frac{4}{3})$ if $u(x,t)$ is the solution to the problem.
Suppose that the boundary conditions are changed to

$$\frac{\partial u}{\partial x}\Big|_{x=-1} = \frac{\partial u}{\partial x}\Big|_{x=3} = 0\,, \quad t \geq 0\,,$$

(iii) Sketch the graphs of the extended initial data for $-5 \leq x \leq 7$;
(iv) Find $u(1,\frac{4}{3})$ if $u(x,t)$ is the solution to the problem.

SOLUTION: (i) The graphs of the extended initial data are shown in Figure 1.

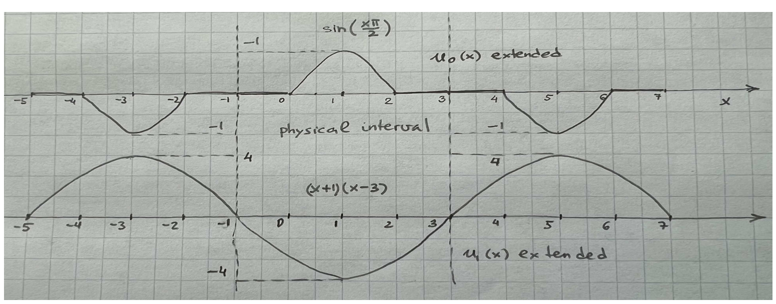

Figure 1: Top panel: The graph of $u_0^{\rm ex}(x)$ (an extension of $u_0(x)$); Bottom panel: The graph of $u_1^{\rm ex}(x)$ (an extension of $u_1(x)$). The extensions are obtained by reflections of the graphs in the physical interval about the endpoints of the interval. The extended data are skew-symmetric under these reflections.

(ii) Setting in d'Alembert's formula $x = 1$, $t = \frac{4}{3}$, and $c = 3$ so that $x - ct = 1 - 4 = -3$ and $x + ct = 1 + 4 = 5$, one infers that

$$u(1,4/3) = \frac{1}{2}\Big(u_0^{\rm ex}(5) + u_0^{\rm ex}(-3)\Big) + \frac{1}{6}\int_{-3}^{5} u_1^{\rm ex}(y)dy = \frac{1}{2}(-1-1) + 0 = -1$$

The integral of the extended u_1 vanishes by symmetry: the integral over $[-3,-1]$ cancels the integral over $[-1,1]$, and, similarly, the integrals over $[1,3]$ and $[3,5]$ cancel each other.

(iii) The graphs of the extended initial data are shown in Figure 2.

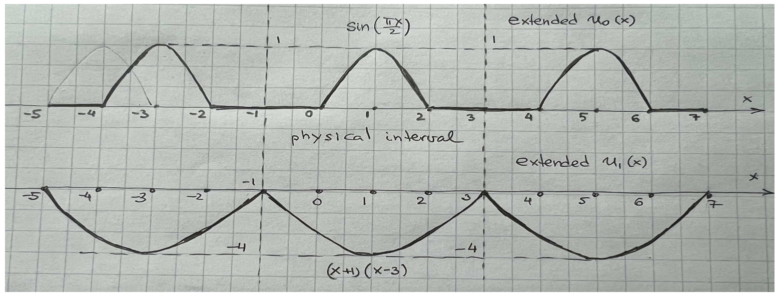

Figure 2: Top panel: The graph of $u_0^{\rm ex}(x)$ (an extension of $u_0(x)$); Bottom panel: The graph of $u_1^{\rm ex}(x)$ (an extension of $u_1(x)$). The extensions are obtained by reflections of the graphs in the physical interval about the vertical lines through the endpoints of the interval. The extended data are symmetric under these reflections.

(iv) The same as in Part (ii) but with the extended data shown in Figure 2:

$$\begin{aligned} u(1,4/3) &= \frac{1}{2}\Big(u_0^{\rm ex}(5)+u_0^{\rm ex}(-3)\Big)+\frac{1}{6}\int_{-3}^{5}u_1^{\rm ex}(y)dy = \frac{1}{2}(1+1)+\frac{2}{6}\int_{-1}^{3}u_1^{\rm ex}(y)dy \\ &= 1+\frac{1}{3}\int_{-1}^{3}(y+1)(y-3)dy = 1+\frac{1}{3}\int_0^4 z(z-4)dz = 1+\frac{1}{3}\Big(\frac{4^3}{3}-\frac{4^3}{2}\Big) \\ &= 1-\frac{32}{9} = -\frac{23}{9} \end{aligned}$$

where the substitution $z=y+1$ has been made to simplify the evaluation of the integral. Note that by symmetry of the extended u_1, the integral over $[-3,-1]$ and $[3,5]$ is equal to the integral over the physical interval $[-1,3]$, which explains a factor of 2 in the second equality.

4 Extra credit (1 pt). Find a general solution $u=u(x,y,z)$ to

$$a^2u''_{xx}+2abu''_{xy}+b^2u''_{yy}-u''_{zz}=0\,.$$

where $a>0$ and $b>0$

SOLUTION: Let us reduce the first three terms in the equation to the standard form by changing variables in the xy plane. The characteristics satisfy the equation

$$a^2(dy)^2-2abdydx+b^2(dx)^2=(ady-bdx)^2=0 \quad\Rightarrow\quad \alpha=ay-bx$$

The equation is parabolic in the xy plane as there is only one characteristic. To reduce the equation to the standard form, put $\beta=ay+bx$. The Jacobian of the transformation $(x,y)\to(\alpha,\beta)$ is not zero. Then

$$\frac{\partial}{\partial x}=b\Big(-\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\Big)\,,\quad \frac{\partial}{\partial y}=a\Big(-\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\Big)$$

so that

$$a^2u''_{xx}+2abu''_{xy}+b^2u''_{yy}=\Big(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\Big)^2u=\Big(2ab\frac{\partial}{\partial\beta}\Big)^2u$$

In the new variables, the equation becomes a 2D wave equation in the variables β and z

$$u''_{\beta\beta}-c^2u''_{zz}=0\,,\quad c=\frac{1}{2ab}$$

Its general solution reads

$$u(x,y,z)=f(\alpha,z-c\beta)+g(\alpha,z+c\beta)\,,\quad \beta=ay+bx\,,\quad \alpha=ay-bx$$

where f and g are C^2 functions of two real variables.