Test 4 with solutions

1 (2 pts). Consider the equation

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = f(x,t)\,, \quad t>0\,,\ -\infty<x<\infty$$

(i) Find a solution if

$$u\Big|_{t=0} = \sin(x)\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \cos(3x)\,, \quad f(x,t)=0\,.$$

(ii) Find a solution with the same conditions as in Part (i) but

$$f(x,t) = \sin(x-t)$$

Solution: (i) Using d'Alembert's formula for a homogeneous wave equation (with $c=2$, $u_0 = \sin(x)$, and $u_1 = \cos(3x)$)

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(u_0(x-ct)+u_0(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy \\ &= \frac{1}{2}\Big(\sin(x-2t)+\sin(x+2t)\Big) + \frac{1}{12}\Big(\sin(3x+6t)-\sin(3x-6t)\Big) \end{aligned}$$

(ii) A particular solution to the wave equation with the zero initial data $u_0 = u_1 = 0$ is given by d'Alembert's formula

$$\begin{aligned} w(x,t) &= \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)\,dy\,d\tau = -\frac{1}{4}\int_0^t \cos(y-\tau)\Big|_{x-2(t-\tau)}^{x+2(t-\tau)} d\tau \\ &= \frac{1}{4}\int_0^t \Big(\cos(x-2t+\tau)-\cos(x+2t-3\tau)\Big)\,d\tau \\ &= \frac{1}{4}\sin(x-2t+\tau)\Big|_0^t + \frac{1}{12}\sin(x+2t-3\tau)\Big|_0^t \\ &= \frac{1}{3}\sin(x-t) - \frac{1}{4}\sin(x-2t) - \frac{1}{12}\sin(x+2t) \end{aligned}$$

The solution to the problem is the sum of the solution in Part (i) and $w(x,t)$.

2 (4 pt). Use the reflection principle to analyze the following initial and boundary value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - 9\,\frac{\partial^2 u}{\partial x^2} = 0\,, \qquad t>0\,,\ -1<x<3\,, \\ &u\Big|_{x=-1} = u\Big|_{x=3} = 0\,, \quad t\ge 0\,, \\ &u\Big|_{t=0} = \begin{cases} \sin(\pi x/2) & , \quad 0\le x\le 2 \\ 0 & , \quad -1\le x\le 0 \text{ and } 2\le x\le 3 \end{cases} \\ &\frac{\partial u}{\partial t}\Big|_{t=0} = (x+1)(x-3)\,, \quad -1\le x\le 3\,. \end{aligned}$$

(i) Sketch the graphs of the extended initial data for $-5\le x\le 7$;
(ii) Find $u(1,\frac{4}{3})$ if $u(x,t)$ is the solution to the problem.
Suppose that the boundary conditions are changed to

$$\frac{\partial u}{\partial x}\Big|_{x=-1} = \frac{\partial u}{\partial x}\Big|_{x=3} = 0\,, \quad t\ge 0\,,$$

(iii) Sketch the graphs of the extended initial data for $-5 \le x \le 7$;
(iv) Find $u(1, \frac{4}{3})$ if $u(x,t)$ is the solution to the problem.

SOLUTION: (i) The graphs of the extended initial data are shown in Figure 1.

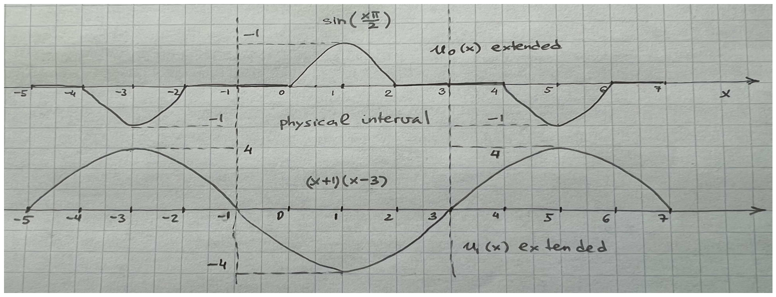

Figure 1: Top panel: Extended $u_0(x)$; Bottom panel: Extended $u_1(x)$. The extensions are obtained by reflections of the graphs in the physical interval about the endpoints of the interval. The extended data are skew-symmetric under these reflections.

(ii) Setting in d'Alembert's formula $x = 1$, $t = \frac{4}{3}$, and $c = 3$ so that $x - ct = 1 - 4 = -3$ and $x + ct = 1 + 4 = 5$, one infers that

$$u(1,4/3) = \frac{1}{2}\Big(u_0(5) + u_0(-3)\Big) + \frac{1}{6}\int_{-3}^{5} u_1(y)dy = \frac{1}{2}(-1-1) + 0 = -1$$

The integral of the extended u_1 vanishes by symmetry: the integral over $[-3,-1]$ cancels the integral over $[-1,1]$, and, similarly, the integrals over $[1,3]$ and $[3,5]$ cancel each other.

(iii) The graphs of the extended initial data are shown in Figure 2.

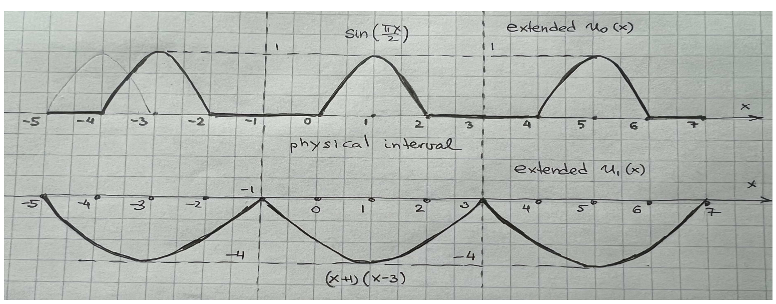

Figure 2: Top panel: Extended $u_0(x)$; Bottom panel: Extended $u_1(x)$. The extensions are obtained by reflections of the graphs in the physical interval about the vertical lines through the endpoints of the interval. The extended data are symmetric under these reflections.

(iv) The same as in Part (ii) but with the extended data shown in Figure 2:

$$\begin{aligned} u(1,4/3) &= \frac{1}{2}\Big(u_0(5) + u_0(-3)\Big) + \frac{1}{6}\int_{-3}^{5} u_1(y)dy = \frac{1}{2}(1+1) + \frac{2}{6}\int_{-1}^{3} u_1(y)dy \\ &= 1 + \frac{1}{3}\int_{-1}^{3}(y+1)(y-3)dy = 1 + \frac{1}{3}\int_{0}^{4} z(z-4)dz = 1 + \frac{1}{3}\Big(\frac{4^3}{3} - \frac{4^3}{2}\Big) \\ &= 1 - \frac{32}{9} = -\frac{23}{9} \end{aligned}$$

where the substitution $z = y + 1$ has been made to simplify the evaluation of the integral. Note that by symmetry of the extended u_1, the integral over $[-3, -1]$ and $[3, 5]$ is equal to the integral over the physical interval $[-1, 3]$, which explains a factor of 2 in the second equality.

3 (1 pts). Consider the equation

$$4\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \quad t > 0\,,\ -\infty < x < \infty$$

Find a solution if

$$u\Big|_{t=0} = \begin{cases} 0 & , \quad x \geq 1 \\ T_1 & , \quad 0 < x < 1 \\ T_2 & , \quad x \leq 0 \end{cases}$$

where T_1 and T_2 are positive non-equal constants. Express the answer in terms of the error function

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\, dx$$

SOLUTION: The solution is given by the Poisson integral (where the parameter $a = \frac{1}{2}$):

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{t}} u_0(y)\, dy = \frac{T_2}{\sqrt{\pi t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{t}}\, dy + \frac{T_1}{\sqrt{\pi t}} \int_0^1 e^{-\frac{(x-y)^2}{t}}\, dy \\ &= \frac{T_2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-s^2}\, ds - \frac{T_1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{t}}}^{\frac{x-1}{\sqrt{t}}} e^{-s^2}\, ds \\ &= \frac{T_2}{2}\Big[\Phi(\infty) - \Phi\Big(\frac{x}{\sqrt{t}}\Big)\Big] - \frac{T_1}{2}\Big[\Phi\Big(\frac{x-1}{\sqrt{t}}\Big) - \Phi\Big(\frac{x}{\sqrt{t}}\Big)\Big] \\ &= \frac{T_2}{2} + \frac{T_1 - T_2}{2}\,\Phi\Big(\frac{x}{\sqrt{t}}\Big) - \frac{T_1}{2}\,\Phi\Big(\frac{x-1}{\sqrt{t}}\Big) \end{aligned}$$

where the substitution $s = (x - y)/\sqrt{t}$ and the property $\Phi(\infty) = 1$ have been used.

4 (1 pt) Consider the initial and boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = 4\,\frac{\partial^2 u}{\partial x^2}\,, \quad t > 0\,,\ 0 < x < L\,, \\ &u\Big|_{t=0} = T_0\,, \quad 0 \leq x \leq L \\ &\frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=L} = 0\,, \quad t \geq 0\,. \end{aligned}$$

where T_0 and L are positive constants. If $u(x,t)$ is the solution to this problem, find $u(\frac{L}{2}, \frac{4}{L})$. Express the answer via parameters T_0 and L.

SOLUTION: The solution is given by the Poisson integral with the extended initial data. The Neumann boundary conditions require a symmetric extension of the data in the physical interval under the reflections about the endpoints of the interval. Since u_0 is a constant function, its symmetric extension is also a constant function. Thus, the extended $u_0(x) = T_0$. A constant function solves the Cauchy problem for the heat equation, that is, $u(x,t) = T_0$. By the uniqueness of the solution, no other solution exists. Therefore

$$u\Big(\frac{L}{2}, \frac{4}{L}\Big) = u(x,t) = T_0$$

Of course, the constant solution is also readily obtained from the Poisson integral with the constant extended data:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)u_0(y)\,dy = T_0 \int_{-\infty}^{\infty} G(x-y,t)dy = T_0 \int_{-\infty}^{\infty} G(z,t)\,dz = T_0$$

by the normalization property of the Green's function $G(z,t)$ for the heat operator.

5 (Extra credit, 1 pt). Suppose $u(x,t)$ is an amplitude of some "signal" propagating along a line. The variable x defines a position on the line and t is time. Suppose that $u(x,0)$ is a smooth function of x that vanishes for all $|x| > a$. What is the smallest time $t_b \geq 0$ for which $u(b,t) \neq 0$ for some $b > a$ and $t > t_b$ if the propagation of the signal is described either by $u'_t = c^2 u''_{xx}$ or by $u''_{tt} = c^2 u''_{xx}$ (in the latter case, $u'_t(x,0) = 0$)? Explain your answer!

Solution: In the case of the wave equation, d'Alembert's formula for the solution to the Cauchy problem gives $u(x,t) = \frac{1}{2}[u_0(x+ct) + u_0(x-ct)]$. So, the signal propagates in the directions of positive and negative x without changing the shape of the initial signal with speed c. The front of the initial signal at $x = a$ will reach the point $x = b > a$ at the time $t_b = (b-a)/c$. For $t = t_b + \Delta t$, the signal at $x = b$ is equal to $u(b,t) = \frac{1}{2}u_0(b-ct) = \frac{1}{2}u_0(a - c\Delta t)$ which is not zero for all $\Delta < 2a/c$. When $\Delta t = 2a/c$, the rear end of the signal (located at $x = -a$ at $t = 0$) reaches $x = b$, and after that no signal can be detected at $x = b$. In the case of the heat equation, by the integral mean value theorem there exists a point y_a in $[-a,a]$ such that

$$u(x,t) = \int_{-a}^{a} G(x-y,t)u_0(y)dy = 2aG(x-y_a)u_0(y_a) = \frac{2a}{2c\sqrt{\pi t}} e^{\frac{(x-y_a)^2}{4c^2 t}} u_0(y_a)$$

Note that y_a can depend on t. Since u_0 is arbitrary (e.g., non-negative), $u_0(y_a) \neq 0$. This shows that $u(x,t) \neq 0$ for any $t > 0$ and any x. So, $t_b = 0$.