Test 5 with solutions

1 (2 pts). Consider the following boundary value problem

$$\Delta u(x,y) = \frac{ax^2}{[x^2+y^2]^{5/2}}\,, \qquad \Omega:\ x^2+y^2>4\,,$$
$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = y^2\,.$$

where $\mathbf{n}$ is the outward unit normal for the boundary $\partial\Omega$.
(i) Find all possible values of a parameter a for which the problem has a bounded solution;
(ii) Find all bounded solutions for each value of a found in Part (i).

Solution: (i) By the solvability condition for the external Neumann problem, the following double integral and line integral must be equal:

$$\begin{aligned}
\iint_\Omega \frac{ax^2}{[x^2+y^2]^{5/2}}\,dxdy &= a\int_0^{2\pi}\cos^2(\theta)\,d\theta\int_2^\infty \frac{dr}{r^2} = \frac{a}{2}\int_0^{2\pi}[1+\cos(2\theta)]\,d\theta\int_2^\infty\frac{dr}{r^2} = a\cdot\pi\cdot\frac{1}{2}\\
\oint_{\partial\Omega} y^2\,ds &= \int_0^{2\pi} 4\sin^2(\theta)\,2d\theta = 4\int_0^{2\pi}[1-\cos(2\theta)]\,d\theta = 8\pi\\
\frac{\pi a}{2} &= 8\pi \quad\Rightarrow\quad a=16
\end{aligned}$$

where the double integral is evaluated in polar coordinates, $dxdy = rdrd\theta$, $x = r\cos(\theta)$, $y = r\sin(\theta)$, and in the line integral $ds = 2d\theta$, $x = 2\cos(\theta)$, and $y = 2\sin(\theta$ with $0\le\theta\le 2\pi$ for the parametric equations of the circle of radius 2.
(ii) The inhomogeneity (with $a=16$) and the boundary data have the following representation in polar coordinates

$$\begin{aligned}
\frac{8x^2}{[x^2+y^2]^{5/2}} &= \frac{16r^2\cos^2(\theta)}{r^5} = \frac{8}{r^3}+\frac{8\cos(2\theta)}{r^3}\\
y^2\Big|_{\partial\Omega} &= r^2\sin^2(\theta)\Big|_{r=2} = 2-2\cos(2\theta)
\end{aligned}$$

This shows that the solution must have the form

$$u = A_0(r) + A_2(r)\cos(2\theta)$$

where A_0 and A_2 are solutions to the associated boundary value problems for the Cauchy-Euler equation in the interval $2<r<\infty$. Since the outward normal on the circle $r=2$ points toward the origin, the normal derivative is equal to $-\frac{\partial u}{\partial r}$ at $r=2$. Therefore,

$$r^2A_0''+rA_0' = \frac{8}{r}\,,\quad -A_0'(2)=2\,,\quad |A_0(r)|\le M<\infty \quad\Rightarrow\quad A_0(r) = C_1+\frac{8}{r}\,.$$

A general solution is $A_0 = C_1 + C_2\ln(r) + A_p$ where a particular solution A_p can be found in the form $A_p = \frac{C}{r}$ (the method of undetermined coefficients). A substitution of A_p into the equation yields $C=8$. The boundedness of the solution requires that $C_2=0$, and the boundary condition is satisfied for any C_1:

$$-A_0'\Big|_{r=2} = \frac{8}{r^2}\Big|_{r=2} = 2$$

Similarly,

$$r^2 A_2'' + rA_2' - 4A_0 = \frac{8}{r}, \quad -A_2'(2) = -2, \quad |A_2(r)| \le M < \infty \quad \Rightarrow \quad A_2(r) = -\frac{16}{3r^2} - \frac{8}{3r}.$$

A general solution is $A_2 = \frac{C_1}{r^2} + C_2 r^2 + A_p$ where a particular solution A_p can be found in the form $A_p = \frac{C}{r}$ (the method of undetermined coefficients). A substitution of A_p into the equation yields $-3C = 8$ or $C = -\frac{8}{3}$. The boundedness of the solution requires that $C_2 = 0$, and the boundary condition is satisfied if

$$-A_2'\Big|_{r=2} = \frac{2C_1}{r^3} - \frac{8}{3r^2}\Big|_{r=2} = \frac{C_1}{4} - \frac{2}{3} = -2 \quad \Rightarrow \quad C_1 = -\frac{16}{3}$$

2 (3 pts). Consider the following boundary value problem

$$\Delta u(x,y) = f(x,y), \quad \Omega: \ 1 < x^2 + y^2 < 4,$$
$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{x^2+y^2=1} = v_1(x,y)\Big|_{x^2+y^2=1}, \quad \Big(u + \frac{\partial u}{\partial \mathbf{n}}\Big)\Big|_{x^2+y^2=4} = v_2(x,y)\Big|_{x^2+y^2=4},$$

where $\mathbf{n}$ is the outward unit normal for the boundary $\partial\Omega$.
(i) Find all bounded solutions to the problem if

$$f(x,y) = 0, \quad v_1(x,y) = 1 + x, \quad v_2(x,y) = y$$

or show that no solution exists.
(ii) Find all bounded solutions to the problem if

$$f(x,y) = 6xy, \quad v_1(x,y) = 0, \quad v_2(x,y) = 0$$

or show that no solution exists.
(iii) Explain (give reasons!) how solutions to Parts (i) and (ii) should be used to find all bounded solutions to the problem if f is given in Part (ii) and the boundary data $v_{1,2}$ are given in Part (i).

Solution: This is a mixed problem for the Poisson equation in an annulus, and it was shown in class to have a unique solution. The inhomogeneity and boundary data have the following expansion in polar coordinates

$$f(x,y) = 6r^2\cos(\theta)\sin(\theta) = 3r^2\sin(2\theta), \quad v_1 = 1 + \cos(\theta), \quad v_2 = 2\sin(\theta)$$

The normal derivative on the circle $r = 1$ is equal to $-\frac{\partial u}{\partial r}$ at $r = 1$ because $\mathbf{n}$ points toward the origin, and the normal derivative on the circle $r = 2$ is equal to $\frac{\partial u}{\partial r}$ at $r = 2$ because $\mathbf{n}$ points from the origin.
(i) In this case the solution must have the form

$$u = A_0(r) + A_1(r)\cos(\theta) + B_1(r)\sin(\theta)$$

where

$$r^2 A_0'' + rA_0' = 0, \quad -A_0'(1) = 1, \quad A_0(2) + A_0'(2) = 0 \quad \Rightarrow \quad A_0(r) = -\frac{1}{2} - \ln\Big(\frac{r}{2}\Big)$$

A general solution reads $A_0 = C_1 + C_2\ln(r)$ so that $A_0' = \frac{C_2}{r}$ and $C_2 = -1$ by the boundary condition at $r = 1$. The second boundary condition yields $C_1 - \ln(2) + \frac{1}{2} = 0$ or $C_1 = -\frac{1}{2} + \ln 2$. Similarly,

$$r^2 A_1'' + rA_1' - A_1 = 0\,, \quad -A_1'(1) = 1\,, \quad A_1(2) + A_1'(2) = 0 \quad \Rightarrow \quad A_1(r) = -\frac{1}{13}\Big(r - \frac{12}{r}\Big)$$

A general solution reads $A_1 = C_1 r + \frac{C_2}{r}$ so that $A_1' = C_1 - \frac{C_2}{r^2}$. The boundary condition at $r = 1$ gives $-C_1 + C_2 = 1$, and the second boundary condition gives $2C_1 + \frac{C_2}{2} + C_1 - \frac{C_2}{4} = 0$ or $C_2 = -12C_1$. Therefore $C_1 = -\frac{1}{13}$ and $C_2 = \frac{12}{13}$. Finally,

$$r^2 B_1'' + rB_1' - B_1 = 0\,, \quad -B_1'(1) = 0\,, \quad B_1(2) + B_1'(2) = 2 \quad \Rightarrow \quad B_1(r) = \frac{8}{13}\Big(r + \frac{1}{r}\Big)$$

A general solution reads $B_1 = C_1 r + \frac{C_2}{r}$ so that $B_1' = C_1 - \frac{C_2}{r^2}$. The boundary condition at $r = 1$ gives $C_1 = C_2$, and the second boundary condition gives $2C_1 + \frac{C_1}{2} + C_1 - \frac{C_1}{4} = 2$ or $C_1 = \frac{8}{13}$.
(ii) In this case, the solution must have the form

$$u = B_2(r)\sin(2\theta)$$

where

$$r^2 B_2'' + rB_2' - 4B_2 = 3r^4\,, \quad \begin{cases} -B_2'(1) &= 0 \\ B_2(2) + B_2'(2) &= 0 \end{cases} \quad \Rightarrow \quad B_2(r) = -\frac{3r^2}{2} - \frac{1}{r^2} + \frac{r^4}{4}\,.$$

A general solution reads $B_2 = C_1 r^2 + \frac{C_2}{r^2} + B_p$. A particular solution B_p can be found in the form $B_p = Cr^4$. A substitution into the equation yields $12C + 4C - 4C = 3$ or $C = \frac{1}{4}$. Therefore $B_2' = 2C_1 r - \frac{2C_2}{r^3} + r^3$. The coefficients are chosen to satisfy the boundary conditions

$$\begin{cases} -B_2'(1) = -2C_1 + 2C_2 - 1 &= 0 \\ B_2(2) + B_2'(2) = 8C_1 + 12 &= 0 \end{cases} \quad \Rightarrow \quad C_1 = -\frac{3}{2}\,, \quad C_2 = -1$$

(iii) Let u_1 and u_2 be solutions found in parts (i) and (ii), respectively. The sought-after solution is $u = u_1 + u_2$. Indeed, $\Delta u = \Delta u_1 + \Delta u_2 = 0 + f = f$ so it satisfies the equation. The boundary conditions $\mathrm{BC}[u]$ (the said combination of u and its normal derivative) are also linear and $\mathrm{BC}[u_1] = v$ and $\mathrm{BC}[u_2] = 0$ so that $\mathrm{BC}[u_1 + u_2] = \mathrm{BC}[u_1] + \mathrm{BC}[u_2] = v + 0 = v$ as required.

3 (2 pts). Let

$$f(x) = 1 - \frac{|x|}{\pi}\,, \quad |x| \le \pi\,, \quad f(x + 2\pi) = f(x)\,, \quad -\infty < x < \infty$$

(i) Expand $f(x)$ into a trigonometric Fourier series.
(ii) Find all points x for which the trigonometric Fourier series converges to $f(x)$. In particular, by studying the convergence of the Fourier series at $x = 0$, show that

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$$

SOLUTION: (i) The Fourier coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}\int_{-1}^{1}(1-|y|)\,dy = \int_0^1 (1-y)dy = \frac{1}{2} \\
a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{-1}^{1}(1-|y|)\cos(\pi m y)\,dy = \frac{2}{\pi m}\int_0^1 (1-y)d\sin(\pi m y) \\
&= \frac{2}{\pi m}\int_0^1 \sin(\pi m y)\,dy = \frac{2[1-(-1)^m]}{(\pi m)^2} \\
&\Rightarrow\ a_{2m-1} = \frac{4}{\pi^2(2m-1)^2}\,, \quad a_{2m} = 0\,, \quad m = 1, 2, \ldots \\
b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = 0
\end{aligned}
$$

where $x = y/\pi$ and the latter integral vanishes by symmetry (as f is even). Therefore

$$
f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)x]}{(2m-1)^2}
$$

(ii) The function f is continuous and 2π periodic. Therefore its trigonometric series converges to $f(x)$ for any real x. In particular, by setting $x = 0$ so that $f(0) = 1$ yields

$$
1 = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} \quad \Rightarrow \quad \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \frac{\pi^2}{8}
$$

4 (2 pts) Consider the following boundary value problem

$$
\begin{aligned}
&\Delta u(x,y) = 0\,, \quad (x,y) \in \Omega\,, \\
&u\Big|_{\partial\Omega} = v(x,y)\Big|_{\partial\Omega}\,, \qquad v(x,y) = 1 - \frac{1}{\pi}\Big|\arctan\Big(\frac{y}{x}\Big)\Big|\,,
\end{aligned}
$$

where the branches of the arctan function is chosen so that on any simple curve (no self intersections) encircling the origin it is increasing from 0 at $y = 0$, $x > 0$ to π at $y = 0$, $x < 0$ along the part of the curve for which $y > 0$ and it is decreasing from 0 at $y = 0$, $x > 0$ to $-\pi$ at $y = 0$, $x < 0$ along the part of the curve for which $y < 0$. Assuming that $u(x,y)$ is bounded in Ω, use the method of trigonometric Fourier series to find a formal solution to the problem if

(i) $\Omega : x^2 + y^2 < 1\,;$

(ii) $\Omega : x^2 + y^2 > 1\,.$

Hint: Find a relation between the boundary data and the function $f(x)$ in Problem **3**.

SOLUTION: In polar coordinates, let us choose the interval of the polar angle to be $[-\pi, \pi]$ and $y/x = \tan(\theta)$. Therefore the boundary data are

$$
v(\cos(\theta), \sin(\theta)) = 1 - \frac{|\theta|}{\pi}\,, \quad -\pi \le \theta \le \pi
$$

Therefore its trigonometric Fourier series is the same as the one found in Problem **3** and, hence, the formal solution must have the form

$$
u = A_0(r) + \sum_{m=1}^{\infty} A_{2m-1}(r)\cos[(2m-1)\theta]
$$

(i) The coefficients are regular solutions to the boundary value problem for the Cauchy-Euler equation in the interval $0 < r < 1$:

$$r^2 A_0'' + rA_0' = 0\,, \quad A_0(1) = a_0\,,\ |A_0(r)| \le M < \infty \quad \Rightarrow \quad A_0(r) = a_0 = \frac{1}{2}$$

because a general bounded solution is constant in this case. Similarly

$$r^2 A_{2m-1}'' + rA_{2m-1}' - (2m-1)^2 A_{2m-1} = 0\,, \quad A_{2m-1}(1) = a_{2m-1}\,,\ |A_{2m-1}(r)| \le M < \infty$$
$$\Rightarrow \quad A_{2m-1}(r) = a_{2m-1} r^{2m-1} = \frac{4r^{2m-1}}{\pi^2(2m-1)^2}$$

because a general solution is a linear combination of r^{2m-1} and $\frac{1}{r^{2m-1}}$ but the second one is not bounded.

(ii) The solution must have the same form as in Part (i) but boundedness requires that solutions proportional to r^{2m-1} (or $\ln(r)$ for A_0) must be discarded so that

$$A_0(r) = \frac{1}{2}\,, \quad A_{2m-1}(r) = \frac{a_{2m-1}}{r^{2m-1}} = \frac{4}{\pi^2(2m-1)^2 r^{2m-1}}$$

5 Extra credit (1 pt). Solve the boundary value problem

$$\Delta u(x,y) = 1\,, \quad (x,y) \in \Omega: \quad x^2 + y^2 < 4\,,$$
$$\frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = v(x,y)\Big|_{\partial\Omega}\,,$$

where $\mathbf{n}$ is the unit outward normal for the boundary $\partial\Omega$ and $v(x,y)$ is given in Problem **4**.

SOLUTION: The problem has no solution because the solvability condition for the Neumann problem is not fulfilled:

$$\iint_\Omega 1\, dxdy = \text{Area}(\Omega) = \pi 2^2 = 4\pi$$
$$\oint_{\partial\Omega} v ds = 2\int_{-\pi}^{\pi} \Big(1 - \frac{|\theta|}{\pi}\Big)\, d\theta = 4\pi a_0 = 2\pi \ne 4\pi$$

Note that $ds = 2d\theta$ as the radius is equal to 2.