Test 6 with solutions

1 (2 pt). (i) Solve the eigenvalue problem

$$-X''(x) = \lambda X(x)\,, \quad -a < x < a\,, \quad X'(-a) = X'(a) = 0$$

(ii) Expand the function $g(x) = a^2 - x^2$ over the basis of eigenfunctions from Part (i).

SOLUTIONS: (i) This is a Sturm-Liouville problem with Neumann boundary conditions. So, $\lambda = 0$ is an eigenvalue, and the corresponding eigenfunction is $X_0(x) = 1$. Other eigenvalues are positive. So, put $\lambda = \nu^2$, $\nu > 0$. It is convenient to change the variable $y = x + a$ so that $0 \le y \le 2a$ if $-a \le x \le a$. Since $\frac{d}{dx} = \frac{d}{dy}$, if $Y(y)$ is a solution to the problem

$$-Y''(y) = \lambda Y(y)\,, \quad 0 < y < 2a\,, \quad Y'(0) = Y'(2a) = 0$$

Then $X(x) = Y(x + a)$ is the solution to the problem in question. A solution $Y(y)$ that satisfies the boundary condition at $y = 0$ is $Y = Y(y, \nu) = \cos(\nu y)$. The boundary condition at $y = 2a$ defines the eigenvalues:

$$Y'(2a, \nu) = 0 \quad \Rightarrow \quad \sin(2\nu a) = 0 \quad \Rightarrow \quad \nu = \nu_n = \frac{\pi n}{2a}\,,\ n = 1, 2, ...$$

and the corresponding eigenfunctions are

$$X_n(x) = Y(x + a, \nu_n) = \cos[\nu_n(x + a)]\,, \quad n = 1, 2, ...$$

(ii) The expansion reads

$$\begin{aligned}
g(x) &= g_0 X_0(x) + \sum_{n=1}^{\infty} g_n X_n(x) = g_0 + \sum_{n=1}^{\infty} g_n \cos[\nu_n(x + a)] \\
&= \frac{2a^2}{3} - \frac{4a^2}{\pi^2} \sum_{m=1}^{\infty} \frac{\cos[\nu_{2m}(x + a)]}{m^2} \\
\|X_0\|^2 &= \langle X_0, X_0 \rangle = \int_{-a}^{a} X_0^2(x)dx = \int_{-a}^{a} dx = 2a \\
\|X_n\|^2 &= \langle X_n, X_n \rangle = \int_{-a}^{a} X_n^2(x)dx = \int_0^{2a} X_n^2(y - a)\,dy = \int_0^{2a} \cos^2(\nu_n y)\,dy \\
&= \frac{1}{2} \int_0^{2a} (1 + \cos(2\nu_n y))\,dy = a \\
g_0 &= \frac{\langle g, X_0 \rangle}{\|X_0\|^2} = \frac{1}{2a} \int_{-a}^{a} (a^2 - x^2)dx = \frac{1}{a}\Big(a^3 - \frac{a^3}{3}\Big) = \frac{2a^2}{3} \\
g_n &= \frac{\langle g, X_n \rangle}{\|X_n\|^2} = \frac{1}{a} \int_{-a}^{a} g(x) X_n(x)\,dx = \frac{1}{a} \int_0^{2a} g(y - a) X_n(y - a)\,dy \\
&= \frac{1}{a} \int_0^{2a} (2a - y) y \cos(\nu_n y)\,dy = \frac{1}{a\nu_n} \int_0^{2a} (2a - y) y d\sin(\nu_n y)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a\nu_n}(2a-y)y\sin(\nu_n y)\Big|_0^{2a} - \frac{1}{a\nu_n}\int_0^{2a}(2a-2y)\sin(\nu_n y)\,dy \\
&= 0 + \frac{1}{a\nu_n^2}(2a-2y)\cos(\nu_n y)\Big|_0^{2a} + \frac{2}{a\nu_n^2}\int_0^{2a}\cos(\nu_n y)\,dy \\
&= -\frac{2[(-1)^n+1]}{\nu_n^2} + \frac{2}{a\nu_n^2}\langle X_0, X_n\rangle = -\frac{2[(-1)^n+1]}{\nu_n^2}
\end{aligned}
$$

because $\cos(2a\nu_n) = (-1)^n$ and $X_0 = 1$ is orthogonal to all other eigenfunctions X_n, $n > 0$. So, only $g_{2m} \neq 0$ for $m = 1, \ldots$.

2 (5pts) Use the results of Problem 1 to find formal solutions to the following Cauchy (initial value) problems for the wave equation

$$
\begin{aligned}
\frac{\partial^2 u}{\partial^2 t} &= 9\frac{\partial^2 u}{\partial^2 x} + f(x,t)\,, \quad t>0\,, \quad -a<x<a \\
u\Big|_{t=0} &= 0\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = v_1(x)\,, \quad \frac{\partial u}{\partial x}\Big|_{x=\pm a} = 0
\end{aligned}
$$

and the heat equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial^2 x} + g(x,t)\,, \quad t>0\,, \quad -a<x<a \\
u\Big|_{t=0} &= u_0(x)\,, \quad \frac{\partial u}{\partial x}\Big|_{x=\pm a} = 0
\end{aligned}
$$

in the following cases:
(i) $f(x,t) = 0$ and $v_1(x) = a^2 - x^2$;
(ii) $f(x,t) = \sin(\omega t)x$ and $v_1(x) = 0$ where $\omega = \frac{3\pi}{a}$;
(iii) $g(x,t) = 0$ and $u_0(x) = x$;
(iv) $g(x,t) = te^{-\beta t}(a^2 - x^2)$ and $u_0(x) = 0$ where $\beta = \frac{9\pi^2}{a^2}$.
(v) Find the formal solution to the above wave equation if f is given in Part (ii) and v_1 is given in Part (i), and the formal solution to the heat equation if if g is given in Part (iv) and u_0 is given in Part (iii)

Solution: The formal solution to any of (i)-(v) has the form

$$
u(x,t) = T_0(t)X_0(x) + \sum_{n=1}^{\infty} T_n(t)X_n(x)\,.
$$

where $X_0(x) = 1$ and $X_n(x)$ are given in the solution to Problem 1(i).
(i) The Fourier coefficients are solutions to the initial value problem

$$
T_n''(t) + 9\nu_n^2 T_n(t) = 0\,, \quad T_n(0) = 0\,, \quad T_n'(0) = g_n\,,
$$

where g_n are the Fourier coefficients of $v_1(x)$ found in Problem 1(ii). For $n > 1$, a general solution is $T_n(t) = A_n\cos(3\nu_n t) + B_n\sin(3\nu_n t)$. The first initial condition yields $A_n = 0$ and the second one gives $2\nu_n B_n = g_n$. For $n = 0$, $\nu_0 = 0$, so that $T_0(t) = A_0 + B_0 t$ and the initial conditions yield $A_0 = 0$ and $B_0 = g_0$. Thus

$$T_0(t) = g_0 t\,, \quad T_n(t) = \frac{g_n}{3\nu_n}\sin(3\nu_n t)\,.$$

(ii) The Fourier coefficients satisfy the initial value problem

$$T_n''(t) + 9\nu_n^2 T_n(t) = f_n \sin(\omega t)\,, \quad T_n(0) = 0\,, \quad T_n'(0) = 0\,,$$

where f_n are the Fourier coefficients of the function $f(x) = x$ in the basis given in Problem 1 (ii).

$$\begin{aligned} f_0 &= \frac{\langle f, X_0\rangle}{\|X_0\|^2} = \frac{1}{2a}\int_{-a}^{a} x\,dx = 0\,,\\ f_n &= \frac{\langle f, X_n\rangle}{\|X_n\|^2} = \frac{1}{a}\int_{-a}^{a} x\cos[\nu_n(x+a)]\,dx = \frac{1}{a}\int_0^{2a}(y-a)\cos(\nu_n y)\,dy\\ &= -\frac{1}{a\nu_n}\int_0^{2a}\sin(\nu_n y)\,dy = \frac{(-1)^n - 1}{a\nu_n^2} \end{aligned}$$

Thus, the Fourier coefficients for all even n vanish and so do $T_n(t) = 0$ because the initial conditions are zero. Thus, only $T_n(t)$ for odd n are to be found. To find a particular solution by the method of undetermined coefficients, one must check if ω matches any of eigen-frequencies $3\nu_n$ for odd n (recall the resonance phenomenon). The condition $\omega = 3\nu_n$ or $1 = \frac{n}{2}$ is satisfied only of $n = 2$ which is even. Therefore, for any odd n a particular solution must have the form $T_{np}(t) = C_n \sin(\omega t)$ (no resonance). A substitution to the equation yields $C_n = f_n/(9\nu_n^2 - \omega^2)$. A general solution reads

$$T_n(t) = A_n\cos(3\nu_n t) + B_n \sin(3\nu_n t) + \frac{f_n}{9\nu_n^2 - \omega^2}\sin(\omega t)$$

The first initial condition yields $A_n = 0$ and B_n is found from $T_n'(0) = 0$ so that

$$T_n(t) = \frac{f_n}{3\nu_n}\cdot\frac{3\nu_n\sin(\omega t) - \omega\sin(3\nu_n t)}{9\nu_n^2 - \omega^2}\,, \qquad n = 2m-1\,,\ m = 1, 2, \ldots\,, \qquad T_{2m}(t) = 0\,.$$

(iii) The Fourier coefficients satisfy the initial value problem

$$T_n'(t) = 3\nu_n^2 T_n(t)\,, \quad T_n(0) = a_n$$

where a_n are the Fourier coefficients of the function $u_0(x) = x$ in the basis from Problem 1(i). They were found in Part (ii), $a_n = f_n$. Therefore

$$T_n(t) = f_n e^{-3\nu_n^2 t}\,, \quad f_0 = 0\,, \quad f_n = \frac{(-1)^n - 1}{a\nu_n^2}\,.$$

(iv) The Fourier coefficients satisfy the initial value problem

$$T_n'(t) + 3\nu_n^2 T_n(t) = g_n t e^{-\beta t}\,, \quad T_n(0) = 0\,.$$

where g_n are the Fourier coefficients of $a^2 - x^2$ because $\langle g(x,t), X_n\rangle = te^{-\omega t}\langle a^2 - x^2, X_n\rangle$. They were found in Problem 1(ii). The condition $\beta = 3\nu_n^2$ or $3 = \frac{n^2}{4}$ cannot be satisfied for any integer n. Therefore, according to the method of undetermined coefficients, a particular solution must have the form $T_{np}(t) = (a_n t + b_n)e^{-\beta t}$. A substitution to the equation yields

$$a_n e^{-\beta t} - (a_n t + b_n)\beta e^{-\beta t} + 3\nu_n^2(a_n t + b_n)e^{-\beta t} = g_n te^{-\beta t} \quad \Rightarrow \quad b_n = \frac{a_n}{3\nu_n^2 - \beta}\,, \quad a_n = \frac{g_n}{3\nu_n^2 - \beta}$$

When $n = 0$, $\nu_n = 0$ in the above equations. A general solution is $T_n(t) = A_n e^{-3\nu_n^2 t} + T_{pn}(t)$. The initial condition gives $A_n + T_{pn}(0) = 0$ or $A_n = -a_n$ (also holds for $n = 0$). Thus,

$$T_n(t) = \frac{g_n}{3\nu_n^2 - \beta}\Big(e^{-\beta t} - e^{-3\nu_n^2 t}\Big) + \frac{g_n t}{(3\nu_n^2 - \beta)^2}\, e^{-\beta t}$$

where $\nu_n = 0$ if $n = 0$.

(v) By linearity of the problems, the said problem for the wave equation is solved by the sum of solutions from Parts (i) and (ii) and, similarly, the said problem for the heat equation is solved by the sum of solutions from Parts (iii) and (iv).

3 (3 pts) Consider the following boundary value problem

$$\Delta u(x,y) = f(x,y)\,, \quad 0 < x < 2\,,\ 0 < y < 1$$
$$u(0,y) = 0\,, \quad u(2,y) = y(1-y)\,, \quad u(x,0) = x(2-x)\,, \quad u(x,1) = 0$$

(i) Find the eigenvalues and eigenfunctions for the associated vertical and horizontal Sturm-Liouville operators. The horizontal eigenfunctions form an orthogonal basis in the interval $0 < x < 2$, while the vertical eigenfunctions form an orthogonal basis in $0 < y < 1$.
(ii) Find the formal solution to the homogeneous problem $f(x,y) = 0$.
(iii) Let $f(x,y) = yx(2-x)$. Expand $f(x,y)$ over the horizontal basis and find the formal solution to the problem.

Solution: (i) The horizontal eigenvalue problem reads

$$-X''(x) = \lambda X(x)\,, \quad 0 < x < 2\,, \quad X(0) = X(2) = 0$$

This problem was shown in class to have positive simple eigenvalues and the following eigenfunctions

$$\lambda = \lambda_n = \nu_n^2\,, \quad \nu_n = \frac{\pi n}{2}\,, \quad X(x) = X_n(x) = \sin(\nu_n x)\,, \quad n = 1, 2, ...$$

The horizontal basis functions are normalized by

$$\|X_n\|^2 = \int_0^2 \sin^2(\nu_n x)\, dx = \frac{1}{2}\int_0^2 (1 - \cos(2\nu_n x))\, dx = 1\,.$$

The vertical eigenvalue problem is similar

$$-Y''(y) = \mu Y(y)\,, \quad 0 < y < 1\,, \quad Y(0) = Y(1) = 0$$

Therefore

$$\mu = \mu_n = \xi_n^2\,, \quad \xi_n = \pi n\,, \quad Y(y) = Y_n(y) = \sin(\xi_n y)\,, \quad n = 1, 2, ...$$

The vertical basis functions are normalized by

$$\|Y_n\|^2 = \int_0^1 \sin^2(\xi_n y)\, dy = \frac{1}{2}\int_0^1 (1 - \cos(2\xi_n y))\, dy = \frac{1}{2}\,.$$

(ii) The formal solution is the sum of two Fourier series over the vertical and horizontal bases:

$$u(x, y) = \sum_{n=1}^{\infty} \tilde{Y}_n(y) X_n(x) + \sum_{n=1}^{\infty} \tilde{X}_n(x) Y_n(y)$$

The Fourier coefficient satisfy the following boundary value problems

$$\begin{array}{l} \tilde{Y}_n''(y) - \nu_n^2 \tilde{Y}_n(y) = 0\,, \quad \tilde{Y}_n(0) = a_n\,, \quad \tilde{Y}_n(1) = 0 \\ \tilde{X}''(x) - \xi_n^2 \tilde{X}_n(x) = 0\,, \quad \tilde{X}_n(0) = 0\,, \quad \tilde{X}_n(2) = b_n \end{array}$$

where a_n and b_n are the Fourier coefficients of the boundary data in the horizontal and vertical bases:

$$\begin{aligned} a_n &= \frac{\langle x(2-x), X_n\rangle}{\|X_n\|^2} = \int_0^2 x(2-x)\sin(\nu_n x)\, dx = \frac{1}{\nu_n}\int_0^2 (2-2x)\cos(\nu_n x)\, dx \\ &= \frac{2}{\nu_n^2}\int_0^2 \sin(\nu_n x)\, dx = -\frac{2}{\nu_n^3}(1 - \cos(2\nu_n)) = \frac{2((-1)^n - 1)}{\nu_n^3} \\ b_n &= \frac{\langle y(1-y), Y_n\rangle}{\|Y_n\|^2} = 2\int_0^1 y(1-y)\sin(\xi_n y)\, dy = \frac{2}{\xi_n}\int_0^1 (1-2y)\cos(\xi_n y)\, dy \\ &= \frac{4}{\xi_n^2}\int_0^1 \sin(\xi_n y)\, dy = -\frac{4}{\xi_n^3}(1 - \cos(\xi_n)) = \frac{4((-1)^n - 1)}{\xi_n^3} \end{aligned}$$

A general solution for $\tilde{Y}_n$ reads

$$\tilde{Y}_n(y) = A_n \sinh(\nu_n y) + B_n \cosh(\nu_n y)$$

The boundary condition at $y = 0$ requires that $B_n = a_n$, and the boundary condition at $y = 1$ requires that $A_n = -B_n \cosh(\nu_n)/\sinh(\nu_n)$. Therefore

$$\tilde{Y}_n(y) = \frac{a_n}{\sinh(\nu_n)}\Big(\sinh(\nu_n)\cosh(\nu_n y) - \cosh(\nu_n)\sinh(\nu_n y)\Big)$$

Similarly, a general solution for $\tilde{X}_n$ reads

$$\tilde{X}_n(x) = A_n \sinh(\xi_n x) + B_n \cosh(\xi_n x)$$

The boundary condition at $x = 0$ requires that $B_n = 0$, and the boundary condition at $x = 2$ requires that $A_n = b_n/\sinh(2\xi_n)$. Therefore

$$\tilde{X}_n(x) = \frac{b_n}{\sinh(2\xi_n)}\, \sinh(\xi_n x)$$

(iii) Let $\Delta v(x,y) = f(x,y)$ and $v(x,y)$ vanishes on the boundary of the rectangle $[0,2]\times[0,1]$. Then the solution to the said Poisson equation is the sum of the solution in Part (ii) and $v(x,y)$. Let us use the horizontal basis to find $v(x,y)$. The advantage is that the Fourier coefficients of f are already calculated in Part (ii):

$$F_n(y) = \frac{\langle f, X_n\rangle}{\|X_n\|^2} = y\frac{\langle x(2-x), X_n\rangle}{\|X_n\|^2} = a_n y$$

Therefore

$$v(x,y) = \sum_{n=1}^{\infty} \tilde{Y}_n(y) X_n(x)$$

where the Fourier coefficients are solutions to the boundary value problem

$$\tilde{Y}_n''(y) - \nu_n^2 \tilde{Y}_n(y) = F_n(y) = a_n y\,, \quad \tilde{Y}_n(0) = 0\,, \quad \tilde{Y}_n(1) = 0$$

A particular solution to the equation can be found in the form $\tilde{Y}_p = Cy$. A substitution to the equation yields $C = -a_n/\nu_n^2$. So a general solution is

$$\tilde{Y}_n(y) = A_n \sinh(\nu_n y) + B_n \cosh(\nu_n y) - \frac{a_n y}{\nu_n^2}$$

The boundary condition at $y = 0$ requires that $B_n = 0$ and the boundary condition at $y = 1$ requires that $A_n = a_n/\nu_n^2$. Thus,

$$\tilde{Y}_n(y) = \frac{a_n}{\nu_n^2}\Big(\sinh(\nu_n y) - y\Big)$$

4 EC (1 pts). Consider the following boundary value problem

$$\begin{aligned}
&\Delta u(x,y) = 6x^2 y\,, \quad 0 < x < 2\,,\ 0 < y < 1 \\
&u_x'(0,y) = -3\,, \quad u_x'(2,y) = -2y\,, \quad u_y'(x,0) = -2x\,, \quad u_y'(x,1) = 1
\end{aligned}$$

Is it correct that a solution to this problem exists? Is it true that if a solution exits, then it can be written as the sum $u(x,y) = v(x,y) + h(x,y)$ where v and h are solutions to the following problems:

$$\begin{aligned}
&\Delta v(x,y) = 6x^2 y\,, \quad 0 < x < 2\,,\ 0 < y < 1 \\
&v_x'(0,y) = 0\,, \quad v_x'(2,y) = 0\,, \quad v_y'(x,0) = -2x\,, \quad v_y'(x,1) = 1 \\
&\Delta h(x,y) = 0\,, \quad 0 < x < 2\,,\ 0 < y < 1 \\
&h_x'(0,y) = -3\,, \quad h_x'(2,y) = -2y\,, \quad h_y'(x,0) = 0\,, \quad h_y'(x,1) = 0\,?
\end{aligned}$$

SOLUTION: This is a Neumann problem. So, it has a solution only if the solvability condition is fulfilled. If $\frac{\partial u}{\partial \mathbf{n}} = w(x,y)$ on the boundary of $\Omega = [0,2]\times[0,1]$, then

$$\oint_{\partial\Omega} w(x,y)\,ds = \iint_{\Omega} f(x,y)dxdy = 6\int_0^2 x^2 dx \int_0^1 y dy = 8$$

To calculate the line integral, note that the normals derivative coincides with $-u'_x$ on $x = 0$, u'_x on $x = 2$, $-u'_y$ on $y = 0$, and u'_y on $y = 1$. Therefore

$$\oint_{\partial\Omega} w(x,y)\,ds = \int_0^1 3\,dy + \int_0^1 (-2y)\,dy + \int_0^2 2x\,dx + \int_0^2 1\,dx = 3 - 1 + 4 + 2 = 8$$

Thus, a solution exists. However it cannot be written as the said sum because $v(x,y)$ and $h(x,y)$ do not exist. For example, the solvability condition for h is not fulfilled:

$$\oint_{\partial\Omega} \frac{\partial h}{\partial \mathbf{n}}\,ds = \int_0^1 3dy + \int_0^1 (-2y)\,dy = 3 - 1 = 2 \neq 0 = \iint_{\Omega} 0\,dxdy\,.$$