CHAPTER 1

Vectors and the Space Geometry

Our space may be viewed as a collection of points. Every geometrical figure, such as a sphere, plane, or line, is a special subset of points in space. The main purpose of an algebraic description of various objects in space is to develop a systematic representation of these objects by numbers. Interestingly enough, our experience shows that so far real numbers and basic rules of their algebra appear to be sufficient to describe all fundamental laws of nature, model everyday phenomena, and even predict many of them. The evolution of the Universe, forces binding particles in atomic nuclei, and atomic nuclei and electrons forming atoms and molecules, star and planet formation, chemistry, DNA structures, and so on, all can be formulated as relations between quantities that are measured and expressed as real numbers. Perhaps, this is the most intriguing property of the Universe, which makes mathematics the main tool of our understanding of the Universe. The deeper our understanding of nature becomes, the more sophisticated are the mathematical concepts required to formulate the laws of nature. But they remain based on real numbers. In this course, basic mathematical concepts needed to describe various phenomena in a three-dimensional Euclidean space are studied. The very fact that the space in which we live is a three-dimensional Euclidean space should not be viewed as an absolute truth. All one can say is that this *mathematical model* of the physical space is sufficient to describe a rather large set of physical phenomena in everyday life. As a matter of fact, this model fails to describe phenomena on a large scale (e.g., our galaxy). It might also fail at tiny scales, but this has yet to be verified by experiments.

1. Rectangular Coordinates in Space

1.1. A system of real numbers. Recall the notion of a *real number system*. One starts with integers. Using the division, rational numbers are obtained: any rational number is a ratio of two integers. The set of rational numbers is closed with respect to all arithmetic operations (addition, subtraction, multiplication, and division), that is,the sum of two rational numbers is a rational number, and so are their product, ratio, and difference (in the ratio a/b, $b \neq 0$). For any two rational numbers x_1

and x_2, the *distance* between them is defined as $|x_1 - x_2|$. For example, the distance between -1 and $1/2$ is $|-1-1/2| = |-3/2| = 3/2$. One can always find two rational numbers with the distance between them being less than any preassigned positive rational number ϵ. Indeed, the distance between $x_1 = \epsilon/3$ and $x_2 = 2\epsilon/3$ is $|x_1 - x_2| = \epsilon/3 < \epsilon$. Clearly, two rational numbers are equal if and only if the distance between them is equal zero.

Having defined a distance on a set of rational numbers, one can study sequences of rational numbers. A sequence a_n, $n = 1, 2, ...$, is said to converge to a rational number a if all its terms, except possibly finitely many, stay arbitrary close to a. In other words, the distance $|a_n - a|$ does not exceed any preassigned positive rational number ϵ for all large enough n. Suppose a_n is monotonically increasing sequence, $a_{n+1} \geq a_n$ for all n. Suppose it converges to a rational number a. Then $a_n \leq a$ for all n. In other words, all terms of the monotonic convergent sequence are *bounded* by its limit from above. Moreover, the limit is *the least upper bound of all numbers* a_n because a_n can get arbitrary close to a, but never exceed it!

Suppose a_n is a monotonically increasing sequence that is bounded. Does it have a limit? This is a turning point in the development of a system of real numbers. One can prove that such a sequence does not necessarily have a limit in rational numbers or, alternatively, one can say that the least upper bound of all rational numbers a_n is not necessarily rational. It is not difficult to construct such a sequence. Let $a^2 = 2$. The positive root of this equation is denoted by $\sqrt{2}$. Then one can show that $a = \sqrt{2}$ is not a rational number (this is left to the reader an an exercise). On the other hand, the root of this equation can be approximated by terms of the decimal expansion of $\sqrt{2}$:

$$a_1 = 1\,,\ a_2 = 1.4\,,\ a_3 = 1.41\ \ a_4 = 1.414\,,\ a_5 = 1.4142\,, ...$$

The terms of this sequence are rational, they are increasing, and they can get arbitrary close and stay arbitrary close to the root of the equation $a^2 = 2$ in the sense that the distance $|a_n^2 - 2|$ is smaller than any preassigned rational number $\epsilon > 0$ for all large enough n:

$$|a_n^2 - 2| < \varepsilon\,.$$

Note well that the sequence a_n exists regardless of what is meant by the symbol $\sqrt{2}$. In fact, this symbol is *defined* as the limit of the above sequence. This example suggests that the system of rational numbers has some "gaps" and there are sequences of rational numbers that converge to these gaps. To fill out these gaps, an abstract notion of a number, called an *irrational number* is introduced. An irrational number is the non-rational limit of a convergent sequence of rational numbers. It cannot be obtained by arithmetic operations from rational numbers, but, by construction, it can be approximated with any desired accuracy by a rational number, that is, for any irrational number x one can find a rational number a that is arbitrary close to x (for any preassigned positive rational number ε, the distance $|x-a|$ is less than ε, no matter how small is ε). By basic laws of limits, all arithmetic operations can be extended to the irrational numbers.

A system of real numbers, by definition, consists of all rational and irrational numbers. The system is complete in the sense that it no longer has “gaps” in contrast to the system of rational numbers. This is known as the *completeness axiom* of real numbers.

The distance between two real numbers x_1 and x_2 is $|x_1 - x_2|$. If x_2 and x_2 are not rational, then one should take suitable sequences of rational numbers converging to x_1 and x_2, and the distance is computed as the limit of the absolute value of the difference of the sequences:

$$a_n \to x_1\,, \quad b_n \to x_2 \quad \Rightarrow \quad |x_1 - x_2| = \lim_{n\to\infty} |a_n - b_n|$$

If three distinct numbers are taken, x_1, x_2, and x_3 so that the distance $|x_1-x_3|$ is the largest amongst the distances between pairs of numbers, then

$$|x_1 - x_3| = |x_1 - x_2| + |x_2 - x_3|\,.$$

1.2. Basic objects in space. Let us describe in a more formal way some basic objects in space such as lines and planes.

Points. A *point* is an elementary object of the space in the sense that any other object is a collection of points. In what follows any such collection will be called *a point set*. In other words, a point cannot be “divided” any further into “more elementary objects”. A good analogy of a point is a number in a real number system. An interval $(1, 2)$ is a collection of all numbers between 1 and 2.

Distance between two points. Consider two points in space. They can be connected by a path. Among all the continuous paths that connect two points, there is a distinct one, namely, the one that has the smallest length. This path is called a *straight line segment*. This definition implies that one has a procedure how to measure the length of a path. Not to mention, a path as a point set also needs a definition. In practice, the problem is solved by defining a physical process that allows us to measure the distance between two points in space.

For example, one can postulate that light (e.g., a laser beam) always propagates along the shortest path between two points with a constant speed c. Then a unit of length can be defined as the length traveled by light in one unit of time. If the time is measured in seconds, then the unit of length is $c = 299,792,458\,\mathrm{m/s}\cdot 1\,\mathrm{s} \approx 3\cdot 10^8$ meters.

Regardless of physical processes that could be used to define the distance between two points, it is *postulated* that in our space there is a rule that assigns a unique real number $|AB|$ to any two points A and B. This rule is required to have the following properties, called *the distance axioms*:

$$\begin{aligned}
&|AB| > 0 \text{ if } A \neq B, \quad |AA| = 0\,,\\
&|AB| = |BA|\,,\\
&|AB| \leq |AC| + |CB|\,,
\end{aligned}$$

for any points A, B, and C. The first property means that the distance is always positive and vanishes if and only if $A = B$ (the points coincide). The second property states that the distance function is symmetric (whatever process is used to measure the distance, it should not matter whether the distance is measured from A to B or from B to A). The third property is called *the triangle inequality*. These properties comply with our every day experience. So they are put into foundations of a mathematical model of the space as *postulates* or *axioms*, the properties that are always assumed to be true. With the distance defined, our space is said to be a *metric* space.

Straight lines. Let us fix two points, A and B, and connect them by a shortest path. By our everyday experience, this path is a straight line segment connecting A and B, and, for any point C on this path, the triangle inequality in the distance axioms becomes the equality. So, it is reasonable to define *a straight line segment with endpoints A and B* as the collection of all points C such that $|AB| = |AC| + |CB|$.

Intuitively, a straight line contains segments of any length and has no "holes". that is, it looks like a segment of an infinite length. A straight line extends unboundedly in two directions from any point like positive and negative numbers relative to zero. So for any three points A, B, and C the triangle inequality becomes equality:

$$|AB| = |AC| + |CB|\,,$$

where $|AB|$ is the largest distance among $|AB|$, $|AC|$, and $|CB|$. Therefore a line is uniquely defined by any two distinct points in it. Fix two points in space and then find all points for which the triangle inequality becomes equality. The obtained point set is, by definition, is a line through two given points:

Given two points A and B, the line through them is the collection of all points for which the triangle inequality is saturated (becomes the equality).

Angles. Any three points in space A, B, and C can be connected by straight line segments forming a *triangle*. If the angle at one of the vertices is the right angle, say at the vertex A, then the Pythagorean theorem holds:

$$|AB|^2 + |AC|^2 = |BC|^2 .$$

However, in our mathematical model of the space, the very notion of an angle between two straight line segments has not yet been defined. Since our experience shows that the Pythagorean theorem is an exclusive property of right-angled triangles, one can define the right angle by turning the theorem into a *postulate*. Two straight line segments AB and AC are said to be *perpendicular* if $|AB|^2 + |AC|^2 = |BC|^2$:

$$AB \perp AC \quad \Leftrightarrow \quad |AB|^2 + |AC|^2 = |BC|^2 .$$

Similarly, two intersecting straight lines are said to be perpendicular if segments of them originating from the point of intersection are perpendicular.

Consider two straight line segments AB and AC. Let $\mathcal{L}$ denote the straight line containing the segment AC. Construct a line through B that is perpendicular to $\mathcal{L}$ and intersect it at a point D. If the segments AD and AC lie on the same side from the point A (or the point A is not in the interval CD), then the angle θ between the segments is *defined* by

$$\cos\theta = \frac{|AD|}{|AB|}, \qquad 0 \le \theta \le \frac{\pi}{2}$$

and, if AD and AC are on the opposite sides (A lies in CD), then

$$\cos\theta = -\frac{|AD|}{|AB|}, \qquad \frac{\pi}{2} < \theta \le \pi .$$

Note that the points A, B, and D are vertices of a right-angled triangle and the segment AB is its hypotenuse. Therefore by the triangle inequality the ratio $|AD|/|AB|$ can only take values between 0 and 1. The cosine function $\cos\theta$ is defines a one-to-one correspondence between the intervals $[-1, 1]$ and $[0, \pi]$. This property makes our definition consistent. In the first case, θ is the angle at the vertex A. In the second case, the angle at the vertex A is $\pi - \theta$, which explains the minus sign in the definition. In other words, $0 \leq \theta \leq \pi$ is the "smallest" angle between the segments.

It should be noted that this construction defines only $\cos\theta$, not the angle θ itself. For a logical consistency the cosine function has to be defined in some algebraic way without any reference to geometry. For example, the cosine and sine functions can be defined by their *power series*:

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2n}}{(2n)!}\,, \quad \sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!}\,,$$

The series converge for all real θ (for example, by the ratio test) and, hence, defined functions called the cosine and sine functions, respectively. Then the angle between two line segments is the root of the equation $\cos\theta = a$ that lies in the interval $\theta \in [-\pi, \pi]$ where the number a is calculated via the lengths of the interval as given above. The left side of the equation is given by the cosine power series.

Three familiar consequences of the above definition of the angle between straight line segments can be proved. If θ is an angle between segments AB and AC, then

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\theta\,.$$

This relation is known as the *cosine theorem*. If α, β, and γ are angles at the vertices A, B, and C, respectively, of a triangle ABC, then

$$\alpha + \beta + \gamma = \pi\,,$$
$$\frac{\sin\alpha}{|BC|} = \frac{\sin\beta}{|CA|} = \frac{\sin\gamma}{|AB|}\,.$$

The last relation is known as the *law of sines* for triangles. These assertions will be proved later when our model of the space is complete. Here they are just noted to emphasize that so far our hypotheses about the space lead to familiar results and, hence, our progress in modeling the space is on a right track.

Planes. Take a straight line $\mathcal{L}$ and fix a point A in it. A point set that is the union of all straight lines through A and perpendicular to $\mathcal{L}$ is called a *plane*. In other words, *a plane $\mathcal{P}$ is uniquely determined by a point A in it and a straight line segment AB perpendicular to it* so that a point C belongs to the plane whenever the segments AC and

AB are perpendicular:

$$C \in \mathcal{P} \quad \Leftrightarrow \quad AC \perp AB \quad \Leftrightarrow \quad |AC|^2 + |AB|^2 = |BC|^2 .$$

Later it will be proved that the vertices of any triangle uniquely define a plane through them. Two lines are said to be *parallel* if they do not intersect and lie in a plane. Two planes are said to be *parallel* if they do not intersect. Clearly, if two planes are perpendicular to the same line, then they are parallel.

Rigid transformations in space. A *transformation* in space is a rule that assigns a unique point to every point of the space. In other words, given a point A, a transformation T maps A to a unique point $T(A)$ of the space. For example, a *parallel transport* is a transformation when all points are moved along parallel lines by the same distance so that any straight line segment preserves its length under the transformation and is parallel to its image. A *rotation* in space is transformation under which at least one point remains fixed and the distance between any two points does not change. In general, a transformation that preserve the distance between any two points is called a *rigid transformation* or an *isometry*. In addition to parallel transports and rotations, reflections about a plane or a point in space are rigid transformations. A composition of rigid transformations is a rigid transformation.

Areas and volumes. Consider two pairs of perpendicular straight line segments of length a and b. By the Pythagorean theorem, it is not difficult to see that they can be arranged in a plane to form a *rectangle* whose characteristic property is that any two adjacent sides (segments) are perpendicular. The number

$$S = ab$$

is called the *area* of the rectangle. The diagonal of the rectangle cuts it into two right-angled triangles that can be transformed to one another by a suitable rotation and a parallel transport in the plane. The area does not change under such transformations because the latter preserve the distance between points. Therefore the area of a right-angled triangle is $ab/2$. Using the definition of the angle θ between two segments AB and AC, one can prove a familiar result that the area of the triangle ABC is

$$S = \frac{1}{2}|AB||AC|\sin\theta .$$

Consider a rectangle with adjacent sides of length a and b. At each vertex, construct a segment of length c that is perpendicular to the plane in which the rectangle lies so that the segments are piercing in the same direction from the plane. The solid bounded by parallel planes that are perpendicular to one segment and contain the other two is called a *rectangular box*. Its volume is, by definition,

$$V = abc\,.$$

The volume does not change under rigid transformations.

Limitations of mathematical modeling. The above geometrical model of our space is based on the Pythagorean theorem which was *postulated* as a fundamental property of the space. A space in which the stated properties hold is known as a *Euclidean* space. However, the validity of this postulate for the space we live in has yet to be verified by observations. In fact, Einstein's general relativity asserts that our space can be viewed as a Euclidean space only in sufficiently small regions of space in which the gravitational pull from nearby masses (planets and stars) does not vary much. Even at distances comparable with the radius of the Earth deviations from the Euclidean geometry were observed. In particular, the sum of angles in a triangle is not equal to π, but its deviation from π is hardly noticeable in our everyday life.

Furthermore our ability to verify postulates of a Euclidean space at small distances is limited by the shortest interval that can be measured. At present, the shortest measurable distance is about $10^{-18}\,cm$ (achieved at the Large Hadron Supercollider at CERN). Any object with dimensions smaller than that would appear as a point without any structure. As of now, there is no reason to believe that the Pythagorean theorem holds at subatomic scales of our space. Yet, the very idea of "continuity" of space at subatomic scales may also be questioned. It may well be that the physical space has a fundamental length, meaning that the length can only be changed by an elementary quantum, just like a penny is the smallest amount of money by which a price of an item can be changed.

1.3. Rectangular coordinate systems. Now it is time to restate our geometrical postulates of a Euclidean space in terms of real numbers and thereby to complete a mathematical model of our space. In doing so, geometrical properties of the space will be formulated in a pure algebraic fashion. In turns, using only basic rules of algebra, novel geometrical facts about various objects in space can be established.

Fix a point O in space. It will be called the *origin*. Consider three mutually perpendicular lines through it. A real number is associated with every point of each line in the following way. The origin corresponds to 0. Distances to points on one side of the line from the origin are marked by positive real numbers, while distances to points on the other half of the line are marked by negative numbers (the absolute value of a negative number is the distance). Recall that each line has exactly two points equidistant from a given point. The half-line with the grid of positive numbers will be indicated by an arrow pointing from the origin to distinguish it from the half-line with the grid of negative numbers. In this way, the real number system is identified with each line in space through the origin O. Note that this identification implies that a particular unit of length is used to measure distances in space, e.g., the number 1 is associated with a point in space that lies on the line at a distance of 1 unit of length (e.g., 1 meter) so that the distances to all other points on the line are *measured in these units* . The described system of lines is called a *rectangular coordinate system* at the origin O. The lines with the constructed grid of real numbers are called *coordinate axes*.

1.4. Points in Space as Ordered Triples of Real Numbers. The position of any point in space can be *uniquely* specified as an *ordered triple of real numbers* relative to a given rectangular coordinate system. Consider a rectangular box with mutually perpendicular sides at every vertex, whose two opposite vertices (the endpoints of the largest diagonal) are the origin and a point P, while its sides that are adjacent at the origin lie on the axes of the coordinate system. The construction is illustrated in Figure **1.1**. For every point P there is only one such rectangular box. It is uniquely determined by its three sides adjacent at the origin. Let the number x mark the position of one such side that lies on the first axis, the numbers y and z do so for the second and third sides, respectively. Note that, depending on the position of P, the numbers x, y, and z may be negative, positive, or even 0. In other words, any point P in space is associated with a unique *ordered triple* of real numbers (x, y, z) determined relative to a rectangular coordinate system. This geometrical fact is written as

$$P = (x, y, z)\,.$$

The ordered triple of numbers (x, y, z) is called *rectangular coordinates* of a point P relative to a given coordinate system.

To reflect the order in (x, y, z), the axes of the coordinate system will be marked as x, y, and z axes. For example, to find a point in space with rectangular coordinates $(1, 2, -3)$, one has to construct a rectangular box with a vertex at the origin such that its sides adjacent at the origin occupy the intervals $[0, 1]$, $[0, 2]$, and $[-3, 0]$ along the x, y, and z axes, respectively. The point in question is the vertex opposite to the origin.

By construction, *two points in space coincide if and only if their corresponding coordinates are equal*:

$$A = B \qquad \Leftrightarrow \qquad x_a = x_b, \quad y_a = y_b, \quad z_a = z_b$$

where $A = (x_a, y_a, z_b)$ and $B = (x_b, y_b, z_b)$.

1.5. A Point as an Intersection of Coordinate Planes. Given a coordinate system with the origin at O, the plane containing the x and y axes is called the *xy plane*. It is perpendicular to the z axis. For all points in this plane, the z coordinate is 0. The *algebraic* condition that a point lies in the xy plane can therefore be stated as $z = 0$. The xz and yz planes can be defined similarly. The condition that a point lies in the xz or yz plane reads $y = 0$ or $x = 0$, respectively. The origin $(0, 0, 0)$ can be viewed as the *intersection* of three *coordinate planes* $x = 0$, $y = 0$, and $z = 0$.

By definition, the *intersection* of two point sets $\mathcal{S}_1$ and $\mathcal{S}_2$ in space is the collection of common points of these sets:

$$P \in \mathcal{S}_1 \cap \mathcal{S}_2 \quad \Leftrightarrow \quad P \in \mathcal{S}_1 \text{ and } P \in \mathcal{S}_2\,.$$

Consider all points in space whose z coordinate is fixed to a particular value $z = z_0$ (e.g., $z = 1$). It is a plane that consists of all lines through the point $(0, 0, z_0)$ perpendicular to the z axis. The planes $z = 0$ and $z = z_0 \neq 0$ do not have common points. Therefore they are parallel. All planes described by the condition that the z coordinate is fixed, $z = z_0$, are perpendicular to the z axis. *The distance between two parallel planes is defined as the length of the straight line segment between the points of intersection of a line perpendicular to the planes.* The plane $z = z_0$ lies $|z_0|$ units of length above the coordinate plane $z = 0$ if $z_0 > 0$ or below it if $z_0 < 0$. A point P with coordinates (x_0, y_0, z_0) can therefore be viewed as an intersection of three *coordinate planes* $x = x_0$, $y = y_0$, and $z = z_0$ as shown in Figure **1.1**. The faces of the rectangular box introduced to specify the position of P relative to a rectangular coordinate system lie in the coordinate planes. The coordinate planes are perpendicular to the corresponding coordinate axes: the plane $x = x_0$ is perpendicular to the x axis, and so on.

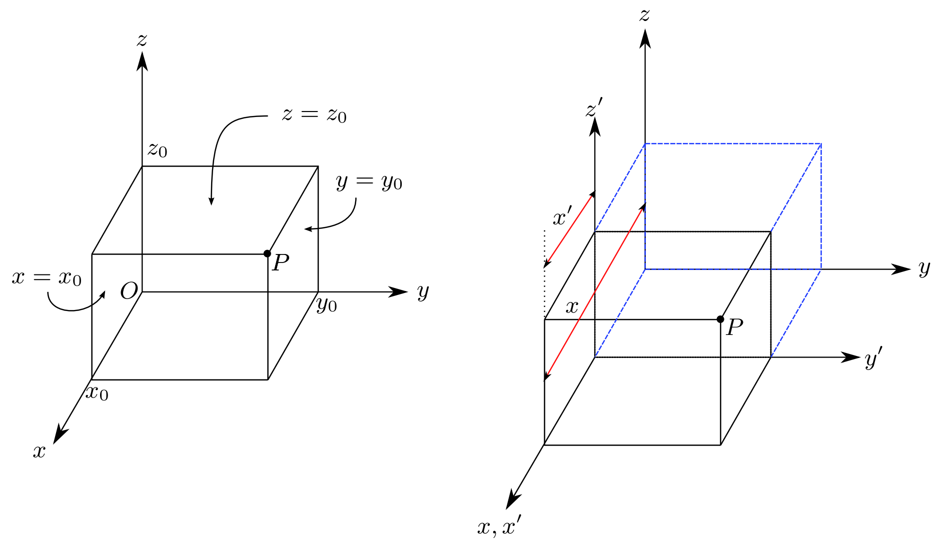


FIGURE **1.1**. **Left**: Any point P in space can be viewed as the intersection of three coordinate planes $x = x_0$, $y = y_0$, and $z = z_0$; hence, P can be given an algebraic description as an ordered triple of numbers $P = (x_0, y_0, z_0)$.
Right: Translation of the coordinate system. The origin is moved to a point (x_0, y_0, z_0) relative to the old coordinate system while the coordinate axes remain parallel to the axes of the old system. This is achieved by translating the origin first along the x axis by the distance x_0 (as shown in the figure), then along the y axis by the distance y_0, and finally along the z axis by the distance z_0. As a result, a point P that had coordinates (x, y, z) in the old system will have the coordinates $x' = x - x_0$, $y' = y - y_0$, and $z' = z - z_0$ in the new coordinate system.

1.6. Rigid Transformations of the Coordinate System. Since the origin and directions of the axes of a coordinate system can be chosen arbitrarily, the coordinates of a point depend on this choice. Suppose a point P has coordinates (x, y, z). Consider a new coordinate system whose axes are parallel to the corresponding axes of the old coordinate system, but whose origin is shifted to the point O' with coordinates $(x_0, 0, 0)$. It is straightforward to see that the point P would have the coordinates $(x - x_0, y, z)$ relative to the new coordinate system (Figure **1.1**, right panel). Similarly, if the origin is shifted to a point O' with coordinates (x_0, y_0, z_0), while the axes remain parallel to the corresponding axes of the old coordinate system, then the coordinates

(x', y', z') of P in the new coordinate system are

$$x' = x - x_0, \quad y' = y - y_0, \quad z' = z - z_0. \tag{1.1}$$

This transformation of the coordinate system is obviously a *parallel transport* (all points are moved parallel the segment OO').

Furthermore, consider a *rotation* in space under which the origin O does not move. Since the distances between points on different coordinate axes are preserved under rotations, the coordinate axes remains perpendicular and the numerical grid on them is preserved as well, while the orientation of the axes changes. Indeed, if A and B are points on two distinct coordinate axes and A' and B' their respective images under the rotation, then $|OA| = |OA'|$, $|OB| = |OB'|$, $|AB| = |A'B'|$ and therefore

$$|OA|^2 + |OB|^2 = |AB|^2 \qquad \Rightarrow \qquad |OA'|^2 + |OB'|^2 = |A'B'|^2$$

that is, the new axes are perpendicular. Thus, a new coordinate system can also be obtained by a rotation about the origin. The coordinates of the same point in space are different in the original and rotated rectangular coordinate systems. Algebraic relations between old and new coordinates can be established (Section **3.4**). They are somewhat more complicated than the relations between old and new coordinates in the case of a parallel translation of the coordinate system discussed above. A simple case, when a coordinate system is rotated about one of its axes (e.g., the z axis does not change under a rotation), is investigated in Study Problem **1.2**.

It is important to emphasize that no physical or geometrical quantity should depend on the choice of a coordinate system. For example, the length of a straight line segment must be the same in any coordinate system, while the coordinates of its endpoints depend on the choice of the coordinate system. When studying a practical problem, a coordinate system can be chosen in any way convenient to describe objects in space and facilitate the use of algebraic rules for real numbers (coordinates) to compute physical and geometrical characteristics of the objects. The numerical values of these characteristics should not depend on the choice of the coordinate system.

Remark. A coordinate system can always be chosen so that the xy coordinate plane coincides with a particular plane in space. If rotations are such that it preserve the z axis (a rotation about the z axis), they can be uniquely specified by the angle between the old and new x axis counted counterclockwise from the former to the latter (see Study

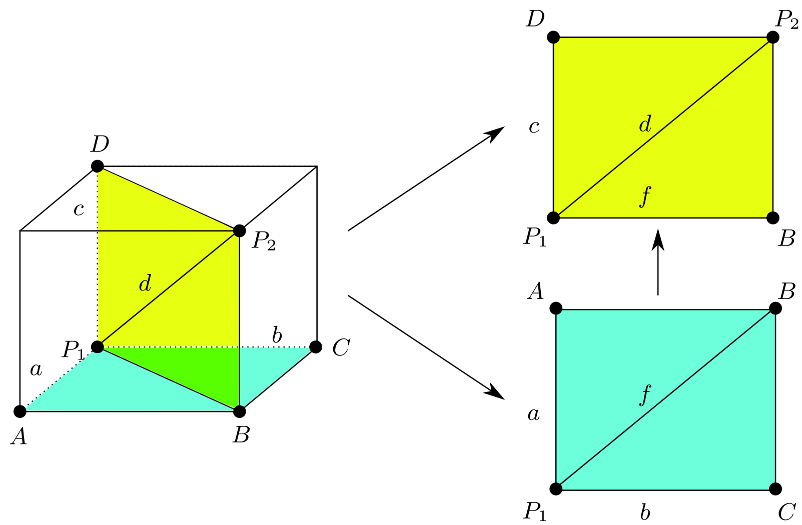


FIGURE **1.2**. Distance between two points with coordinates $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. The line segment P_1P_2 is viewed as the largest diagonal of the rectangular box whose faces are the coordinate planes corresponding to the coordinates of the points. Therefore, the distances between the opposite faces are $a = |x_1 - x_2|$, $b = |y_1 - y_2|$, and $c = |z_1 - z_2|$. The length of the diagonal d is obtained by the double use of the Pythagorean theorem in each of the indicated rectangles: $d^2 = c^2 + f^2$ (top right) and $f^2 = a^2 + b^2$ (bottom right).

Problem **1.2**). An algebraic description of a general rotation in space is discussed in Study Problem **5.1**.

1.7. Distance Between Two Points. Consider two points in space, P_1 and P_2. Let their coordinates relative to some rectangular coordinate system be (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. How can one calculate the distance between these points, or the length of a straight line segment with endpoints P_1 and P_2? The point P_1 is the intersection point of three coordinate planes $x = x_1$, $y = y_1$, and $z = z_1$. The point P_2 is the intersection point of three coordinate planes $x = x_2$, $y = y_2$, and $z = z_2$. These six planes contain faces of the rectangular box whose largest diagonal is the straight line segment between the points P_1 and P_2. The question therefore is how to find the length of this diagonal.

Consider three sides of this rectangular box that are adjacent, say, at the vertex P_1. The side parallel to the x axis lies between the coordinate planes $x = x_1$ and $x = x_2$ and is perpendicular to them. So the length of this side is $|x_2 - x_1|$. The absolute value is necessary as the difference $x_2 - x_1$ may be negative, depending on the values of x_1 and x_2, whereas the distance must be nonnegative. Similar arguments lead to the conclusion that the lengths of the other two adjacent sides are $|y_2 - y_1|$ and $|z_2 - z_1|$. If a rectangular box has adjacent sides of length a, b, and c, then the length d of its largest diagonal satisfies the equation

$$d^2 = a^2 + b^2 + c^2 .$$

Its proof is based on the Pythagorean theorem (see Figure **1.2**). Consider the rectangular face that contains the sides a and b. The length f of its diagonal is determined by the Pythagorean theorem $f^2 = a^2 + b^2$. Consider the cross section of the rectangular box by the plane that contains the face diagonal f and the side c. This cross section is a rectangle with two adjacent sides c and f and the diagonal d. They are related as $d^2 = f^2 + c^2$ by the Pythagorean theorem, and the desired conclusion follows.

Put $a = |x_2 - x_1|$, $b = |y_2 - y_1|$, and $c = |z_2 - z_1|$. Then $d = |P_1P_2|$ is the distance between P_1 and P_2. The distance formula is immediately found:

$$(\mathbf{1.2}) \qquad |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .$$

Note that the numbers (coordinates) (x_1, y_1, z_1) and (x_2, y_2, z_2) depend on the choice of the coordinate system, whereas the number $|P_1P_2|$ *remains the same* in any coordinate system! For example, if the origin of the coordinate system is translated to a point (x_0, y_0, z_0) while the orientation of the coordinate axes remains unchanged, then, according to rule (**1.1**), the coordinates of P_1 and P_2 relative to the new coordinate become $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and $(x_2 - x_0, y_2 - y_0, z_2 - z_0)$, respectively. The numerical value of the distance does not change because the coordinate differences, $(x_2 - x_0) - (x_1 - x_0) = x_2 - x_1$ (similarly for the y and z coordinates), do not change. For example, the distance between the points $P_1 = (-1, 0, 2)$ and $P_2 = (1, 2, 3)$, where the coordinates are measured in meters, is

$$|P_1P_2| = \sqrt{(1 - (-1))^2 + (0 - 2)^2 + (3 - 2)^2} = \sqrt{4 + 4 + 1} = 3\,m$$

EXAMPLE **1.1**. *A point moves* 3 *units of length parallel to a line, then it moves* 6 *units parallel to the second line that is perpendicular to the first line, and then it moves* 6 *units parallel to the third line that is perpendicular to the first and second lines. Find the distance between the initial and final positions.*

SOLUTION: The distance between points does not depend on the choice of the coordinate system. Let the origin be positioned at the initial point of the motion and the coordinate axes be directed parallel to the corresponding three mutually perpendicular lines parallel to which the point has moved. In this coordinate system, the trajectory of the motion consists of three straight line segments $(0,0,0) \to (3,0,0) \to (3,6,0) \to (3,6,6)$. The distance between the final point $(3,6,6)$ and the origin $(0,0,0)$ is

$$D = \sqrt{3^2 + 6^2 + 6^2} = \sqrt{9(1+4+4)} = 9\,.$$

□

Verification of the distance axioms. Does the distance defined by (**1**.**2**) satisfy the distance axioms? The first two axioms is not difficult to verify. The distance $|AB|$ is obviously non-negative for any A and B. If $A = B$, then their coordinates coincide and, hence, $|AB| = 0$. Conversely, if $|AB| = 0$, the sum of squares of the differences of the coordinates of A and B must be equal to zero, which is possible only if each difference is equal to zero. So, the points have the same coordinates and therefore $A = B$. Thus, the first axiom is fulfilled.

Clearly, $|AB| = |BA|$ because $(x_1 - x_2)^2 = (x_2 - x_1)^2$ in (**1**.**2**) and similarly for the other coordinates.

To verify the triangle inequality, consider the triangle ABC. Construct the line through C that is perpendicular to the line through A and B and intersects it at a point D. Then

$$|AC|^2 = |AD|^2 + |CD|^2 \geq |AD|^2 \quad \Rightarrow \quad |AC| \geq |AD|$$

The equality is possible only if C lies in the line segment AB. Similarly,

$$|BC| \geq |BD|\,.$$

Since D lies in the line through A and B

$$|AD| + |BD| \geq |AB|$$

and the equality is possible only if D lies in the line segment AB. Therefore, the triangle inequality holds:

$$|AC| + |BC| \geq |AD| + |BD| \geq |AB|\,.$$

Rigid Transformations in Space as Coordinate Transformations. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ relative to some rectangular coordinate system. If $P_1' = (x_1', y_1', z_1')$ and $P_2' = (x_2', y_2', z_2')$ are images of P_1 and P_2, respectively, under a rigid transformation, then

$$|P_1P_2| = |P_1'P_2'|$$

by squaring this equality and using the distance formula (**1.2**)

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=(x_2'-x_1')^2+(y_2'-y_1')^2+(z_2'-z_1')^2$$

Conversely, any coordinate transformation $(x,y,z)\to(x',y',z')$ for which the above relation holds is a rigid transformation. Evidently, the translation (**1.1**) satisfies this condition because $x_2'-x_1'=(x_2-x_0)-(x_1-x_0)=x_2-x_1$ and similarly $y_2'-y_1'=y_2-y_1$ and $z_2'-z_1=z_2-z_1$.

The origin can always be translated to P_1 so that in the new coordinate system $P_1=(0,0,0)$ and $P_2=(x_2-x_1,y_2-y_1,z_2-z_1)$. Then any rigid transformation under which at least one point remains fixed (the origin) is described by a coordinate transformation such that

$$x^2+y^2+z^2=(x')^2+(y')^2+(z')^2 \tag{1.3}$$

For example, a reflection about the origin

$$(x,y,z)\ \to\ (-x,-y,-z)=(x',y',z')$$

or a reflection in the xy plane

$$(x,y,z)\ \to\ (-x,y,z)=(x',y',z')$$

satisfy (**1.3**), and so do rotations about the origin.

In Study Problem **1.2** it is shown that under a counterclockwise rotation of the coordinate system about the z axis through an angle ϕ

$$x'=x\cos\phi+y\sin\phi\,,\quad y'=y\cos\phi-x\sin\phi\,,\quad z'=z \tag{1.4}$$

In particular, under such a rotation through the angle $\phi=\pi/4$:

$$x'=\frac{1}{\sqrt{2}}(x+y)\,,\quad y'=\frac{1}{\sqrt{2}}(y-x)\,,\quad z'=z\,. \tag{1.5}$$

It is not difficult to verify (**1.3**) (see Study Problem **1.2**). A general rotation in space is described in Study Problem **5.1**.

It should be noted that the coordinate system obtained by the reflection of the coordinate axes about the origin cannot be obtained by any rotation about the origin. Imagine that the thumb, index and middle fingers of your right are extended in the directions of the axes of a rectangular coordinate system, then it would not be possible to match it with a similar coordinate system obtained by the same fingers of your left hand by moving and rotating your wrists. It would only be possible to match the directions of a pair of fingers. For example, if the index and middle fingers of the right hand are moved so that they point in the same directions as the index and middle fingers of the left hand, then the thumbs would point in the opposite directions, that is, a reflection is needed to match these *left- and right-handed* coordinate system. Two coordinate systems (with the same origin) are said to have the same *handedness* if they can be obtained from another by a rotations. All coordinate systems are split into two classes: left-handed and right-handed systems. A reflection changes the handedness, while rotations and translations do not. A more detailed discussion of left- and right-handed coordinate system is postponed until Section **5.2**.

1.8. Spheres in Space. In this course, relations between two equivalent descriptions of objects in space – the geometrical and the algebraic – will always be emphasized. One of the course objectives is to learn how to interpret an algebraic equation by geometrical means and how to describe geometrical objects in space algebraically. One of the simplest examples of this kind is a sphere.

Geometrical Description of a Sphere. *A sphere is a set of points in space that are equidistant from a fixed point.* The fixed point is called the *center* of the sphere. The distance from the sphere center to any point of the sphere is called the *radius* of the sphere.

Algebraic Description of a Sphere. An algebraic description of a sphere implies finding an algebraic condition on coordinates (x, y, z) of points in space that belong to the sphere. So let the center of the sphere be a point P_0 with coordinates (x_0, y_0, z_0) (defined relative to some rectangular coordinate system). If a point P with coordinates (x, y, z) belongs to the sphere, then the numbers (x, y, z) must be such that the distance $|PP_0|$ is the same for any such P and equal to the radius of the sphere, denoted R, that is, $|PP_0| = R$ or

$$|PP_0|^2 = R^2$$

(see Figure **1.3**, left panel). Using the distance formula, this condition can be written as

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \,. \qquad (1.6)$$

For example, the set of points with coordinates (x, y, z) that satisfy the condition $x^2 + y^2 + z^2 = 4$ is a sphere of radius $R = 2$ centered at the origin $x_0 = y_0 = z_0 = 0$.

EXAMPLE **1.2**. *Find the center and the radius of the sphere* $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.

SOLUTION: In order to find the coordinates of the center and the radius of the sphere, the equation must be transformed to the standard form

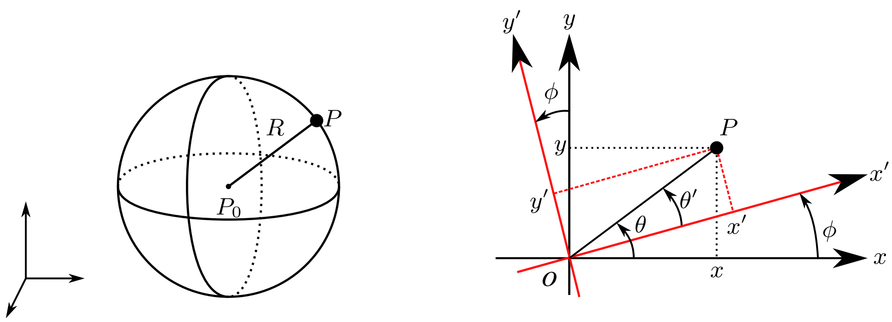


FIGURE **1.3**. **Left**: A sphere is defined as a point set in space. Each point P of the set has a fixed distance R from a fixed point P_0. The point P_0 is the center of the sphere, and R is the radius of the sphere. **Right**: An illustration to Study Problem **1.2**. Transformation of coordinates under a rotation of the coordinate system in a plane.

(**1.6**) by completing the squares:

$$x^2 - 2x = (x-1)^2 - 1\,,$$
$$y^2 + 4y = (y+2)^2 - 4\,,$$
$$z^2 - 6z = (z-3)^2 - 9\,.$$

Then the equation of the sphere becomes

$$(x-1)^2 - 1 + (y+2)^2 - 4 + (z-3)^2 - 9 + 5 = 0\,,$$
$$(x-1)^2 + (y+2)^2 + (z-3)^2 = 9\,.$$

The comparison with (**1.6**) shows that the center is at $(x_0, y_0, z_0) = (1, -2, 3)$ and the radius is $R = \sqrt{9} = 3$. □

1.9. Algebraic Description of Point Sets in Space. The idea of an algebraic description of a sphere can be extended to other sets in space. It is convenient to introduce a brief notation for an algebraic description of sets relative to some coordinate system. For example, for a set $\mathcal{S}$ of points in space with coordinates (x, y, z) such that they satisfy the algebraic condition (**1.6**), one writes

$$\mathcal{S} = \left\{ (x, y, z) \;\middle|\; (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \right\}.$$

This relation means that the set $\mathcal{S}$ is a collection of all points (x, y, z) such that (the vertical bar) their rectangular coordinates satisfy (**1.6**). The set

$$\mathcal{B} = \left\{ (x, y, z) \;\middle|\; (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \le R^2 \right\}$$

consists of all points whose distance from the point (x_0, y_0, z_0) does not exceed R. Therefore, it is a *(solid) ball* of radius R centered at (x_0, y_0, z_0). The boundary sphere also belongs to $\mathcal{B}$. If the inequality $\leq$ is replaced by the strict inequality $<$ in the above description of $\mathcal{B}$, then the boundary sphere of radius R must be excluded from $\mathcal{B}$. In the latter case, the ball is called *open*.

Similarly, the xy plane can be viewed as a set of points whose z coordinates vanish:

$$\mathcal{P} = \Big\{(x, y, z) \;\Big|\; z = 0\Big\}.$$

The solid region in space that consists of points whose coordinates are non-negative is called the *first octant*:

$$\mathcal{O}_1 = \Big\{(x, y, z) \;\Big|\; x \geq 0,\ y \geq 0,\ z \geq 0\Big\}.$$

The spatial region

$$\mathcal{B} = \Big\{(x, y, z) \;\Big|\; x > 0,\ y > 0,\ z > 0,\ x^2 + y^2 + z^2 < 4\Big\}$$

is the collection of all points in the portion of a ball of radius 2 that lies in the first octant. The strict inequalities imply that the boundary of this portion of the ball does not belong to the set $\mathcal{B}$.

1.10. Study Problems.

Problem 1.1. *Show that the coordinates of the midpoint of a straight line segment are*

$$\left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2},\ \frac{z_1 + z_2}{2}\right)$$

if the coordinates of its endpoints are (x_1, y_1, z_1) *and* (x_2, y_2, z_2)*.*

SOLUTION: Let P_1 and P_2 be the endpoints and let M be the point with coordinates equal half-sums of the corresponding coordinates of P_1 and P_2. One has to prove that

$$|MP_1| = |MP_2| = \frac{1}{2}|P_1P_2|.$$

These two conditions define M as the midpoint. Consider a rectangular box B_1 whose largest diagonal is P_1M. The length of its side parallel to the x axis is

$$\Big|\tfrac{1}{2}(x_1 + x_2) - x_1\Big| = \tfrac{1}{2}|x_2 - x_1|.$$

Similarly, its sides parallel to the y and z axes have the lengths $|y_2 - y_1|/2$ and $|z_2 - z_1|/2$, respectively. Consider a rectangular box B_2

whose largest diagonal is the segment MP_2. Then its side parallel to the x axis has the length

$$\left|x_2 - \tfrac{1}{2}(x_1 + x_2)\right| = \tfrac{1}{2}|x_2 - x_1|\,.$$

Similarly, the sides parallel to the y and z axes have lengths $|y_2 - y_1|/2$ and $|z_2 - z_1|/2$, respectively. Thus, the sides of B_1 and B_2 parallel to each coordinate axis have the same length. By the distance formula (**1.2**) the diagonals of B_1 and B_2 must have the same length $|P_2M| = |MP_1|$. The lengths of the sides of a rectangular box whose largest diagonal is P_1P_2 are $a = |x_2 - x_1|$, $b = |y_2 - y_1|$, and $c = |z_2 - z_1|$. They are twice as long as the corresponding sides of B_1 and B_2. If the length of each side of a rectangular box is scaled by a positive factor q, then the length d of its diagonal is also scaled by q:

$$\sqrt{(qa)^2 + (qb)^2 + (qc)^2} = \sqrt{q^2(a^2 + b^2 + c^2)} = q\sqrt{a^2 + b^2 + c^2} = qd\,.$$

In particular, this implies that $|MP_2| = \frac{1}{2}|P_1P_2|$. □

Problem **1.2**. *Let (x, y, z) be coordinates of a point P. Consider a new coordinate system that is obtained by rotating the x and y axes about the z axis counterclockwise as viewed from the top of the z axis through an angle ϕ. Let (x', y', z') be coordinates of P in the new coordinate system. Find the relations between the old and new coordinates. Verify Eq. (*__1.3__*).*

Solution: The height of P relative to the xy plane does not change upon rotation. So $z' = z$. It is therefore sufficient to consider rotations in the xy plane, that is, for points P with coordinates $(x, y, 0)$. Let $r = |OP|$ (the distance between the origin and P) and let θ be the angle counted from the positive x axis toward the ray OP counterclockwise (see Figure **1.3**, right panel). Then $x = r\cos\theta$ and $y = r\sin\theta$ (the polar coordinates of P). In the new coordinate system, the angle between the positive x' axis and the ray OP becomes $\theta' = \theta - \phi$. Therefore,

$$\begin{aligned} x' = r\cos\theta' = r\cos(\theta - \phi) &= r\cos\theta\cos\phi + r\sin\theta\sin\phi \\ &= x\cos\phi + y\sin\phi\,, \\ y' = r\sin\theta' = r\sin(\theta - \phi) &= r\sin\theta\cos\phi - r\cos\theta\sin\phi \\ &= y\cos\phi - x\sin\phi\,. \end{aligned}$$

These equations define the transformation $(x, y) \to (x', y')$ of the old coordinates to the new ones. The inverse transformation $(x', y') \to (x, y)$ can be found by solving the equations for (x, y). A simpler way is to note that if (x', y') are viewed as "old" coordinates and (x, y) as "new" coordinates, then the transformation that relates them is the rotation through the angle $-\phi$ (a clockwise rotation). Therefore the inverse relations can be obtained by swapping the "old" and "new" coordinates and replacing ϕ by $-\phi$ in the direct relations. This yields

$$x = x' \cos\phi - y' \sin\phi\,, \quad y = y' \cos\phi + x' \sin\phi\,.$$

Since $z' = z$ under rotations about the z axis, relation (**1.3**) has to be verified only for the x and y coordinates:

$$\begin{aligned}(x')^2 + (y')^2 &= (x\cos\phi + y\sin\phi)^2 + (y\cos\phi - x\sin\phi)^2 \\ &= x^2\cos^2\phi + 2xy\cos\phi\,\sin\phi + y^2\sin^2\phi \\ &\quad + x^2\sin^2\phi - 2xy\cos\phi\,\sin\phi + y^2\cos^2\phi \\ &= x^2 + y^2\end{aligned}$$

where the fundamental trigonometric identity $\cos^2\phi + \sin^2\phi = 1$ was used. □

Problem **1.3**. *Give a geometrical description of the set*

$$\mathcal{S} = \left\{(x, y, z)\ \middle|\ x^2 + y^2 + z^2 - 4z = 0\right\}.$$

Solution: The condition on the coordinates of points that belong to the set contains the sum of squares of the coordinates just like the equation of a sphere. The difference is that (**1.6**) contains the sum of perfect squares. So the squares must be completed in the above equation and the resulting expression has to be compared with (**1.6**). One has

$$z^2 - 4z = (z - 2)^2 - 4$$

so that the condition becomes

$$x^2 + y^2 + (z - 2)^2 = 4\,.$$

It describes a sphere of radius $R = 2$ that is centered at the point $(x_0, y_0, z_0) = (0, 0, 2)$; that is, the center of the sphere is on the z axis at a distance of 2 units above the xy plane. □

Problem **1.4**. *Give a geometrical description of the set*

$$\mathcal{C} = \left\{(x, y, z)\ \middle|\ x^2 + y^2 - 2x - 4y \le 4\right\}.$$

Solution: As in the previous problem, the condition can be written via the sum of perfect squares

$$(x - 1)^2 + (y - 2)^2 \le 9$$

by means the of relations $x^2 - 2x = (x-1)^2 - 1$ and $y^2 - 4y = (y-2)^2 - 4$. In the xy plane, the inequality describes the set of points whose distance from the point $(1, 2, 0)$ does not exceed $\sqrt{9} = 3$, which is the disk of radius 3 centered at the point $(1, 2, 0)$. As the algebraic condition imposes no restriction on the z coordinate of points in the set, in any plane $z = z_0$ parallel to the xy plane, the x and y coordinates satisfy the same inequality, and hence the corresponding points also form a disk of radius 3 with the center $(1, 2, z_0)$. Thus, the set is the union of all such disks, which is a solid cylinder of radius 3 whose axis is parallel to the z axis and passes through the point $(1, 2, 0)$. □

Problem **1.5**. *Give a geometrical description of the set*

$$\mathcal{P} = \left\{(x, y, z)\ \Big|\ z(y - x) = 0\right\} .$$

SOLUTION: The condition is satisfied if either $z = 0$ or $y = x$. The former equation describes the xy plane. The latter represents a line in the xy plane. Since it does not impose any restriction on the z coordinate, each point of the line can be moved up and down parallel to the z axis without leaving the set. The resulting set is a plane that contains the line $y = x$ in the xy plane and the z axis. Thus, the set $\mathcal{P}$ is the union of this plane and the xy plane. □

Problem **1.6**. *Find an algebraic description of a plane $\mathcal{P}$ that contains the point $A = (1, 2, 3)$ and is perpendicular to the line through A and $B = (2, 1, -1)$.*

SOLUTION: Let $P = (x, y, z)$ be a point in the plane $\mathcal{P}$. Then the straight line segments AP, BP, and AB must satisfy the Pythagorean theorem (see Section **1.2**)

$$P \in \mathcal{P} \quad \Leftrightarrow \quad AP \perp AB \quad \Leftrightarrow \quad |AP|^2 + |AB|^2 = |BP|^2 .$$

By the distance formula (**1.2**)

$$\begin{aligned}
|AB|^2 &= (2-1)^2 + (1-2)^2 + (-1-3)^2 = 18 ,\\
|AP|^2 &= (x-1)^2 + (y-2)^2 + (z-3)^2\\
&= x^2 + y^2 + z^2 - 2x - 4y - 6z + 14 ,\\
|BP|^2 &= (x-2)^2 + (y-1)^2 + (z+1)^2\\
&= x^2 + y^2 + z^2 - 4x - 2y + 2z + 6 .
\end{aligned}$$

Substituting these distances into the Pythagorean theorem and cancelling the quadratic terms, one finds an algebraic description of the plane as a point set in space:

$$\mathcal{P} = \left\{(x, y, z)\ \Big|\ x - y - 4z = -13\right\} .$$

□

1.11. Exercises.

1–2. Find the distance between the specified points.

1. $(1, 2, -3)$ and $(-1, 0, -2)$
2. $(-1, 3, -2)$ and $(-1, 2, -1)$

3–4. Determine whether the given points lie in a straight line.

3. $(8, 3, -3)$, $(-1, 6, 3)$, and $(2, 5, 1)$
4. $(-1, 4, -2)$, $(1, 2, 2)$, and $(-1, 2, -1)$

5. Determine whether the points $(1, 2, 3)$ and $(3, 2, 1)$ lie in the plane through $O = (1, 1, 1)$ and perpendicular to the line through O and $B = (2, 2, 2)$. If not, find a point in this plane.
6. Find the distance from the point $(1, 2, -3)$ to each of the coordinate planes and to each of the coordinate axes.
7. Find the length of the medians of the triangle with vertices $A = (1, 2, 3)$, $B = (-3, 2, -1)$, and $C = (-1, -4, 1)$.
8. Let the set $\mathcal{S}$ consist of points $(t, 2t, 3t)$ where $-\infty < t < \infty$. Find the point of $\mathcal{S}$ that is the closest to the point $(3, 2, 1)$. Sketch the set $\mathcal{S}$.
9–18. Give a geometrical description of the given set $\mathcal{S}$ of points defined algebraically and sketch the set:

9. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 + z^2 - 2x + 4y - 6z = 0\}$
10. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 + z^2 \geq 4\}$
11. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 + z^2 \leq 4,\ z > 0\}$
12. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 - 4y < 0,\ z > 0\}$
13. $\mathcal{S} = \{(x, y, z)\,|\,4 \leq x^2 + y^2 + z^2 \leq 9\}$
14. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 \geq 1,\ x^2 + y^2 + z^2 \leq 4\}$
15. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 + z^2 - 2z < 0,\ z > 1\}$
16. $\mathcal{S} = \{(x, y, z)\,|\,x^2 + y^2 + z^2 - 2z = 0,\ z = 1\}$
17. $\mathcal{S} = \{(x, y, z)\,|\,(x - a)(y - b)(z - c) = 0\}$
18. $\mathcal{S} = \{(x, y, z)\,|\,|x| \leq 1,\ |y| \leq 2,\ |z| \leq 3\}$

19–24. Sketch the given set of points and give its algebraic description.

19. A sphere whose diameter is the straight line segment AB, where $A = (1, 2, 3)$ and $B = (3, 2, 1)$.
20. Three spheres centered at $(1, 2, 3)$ that just barely touch the xy, yz, and xz planes, respectively.
21. Three spheres centered at $(1, -2, 3)$ that just barely touch the x, y, and z coordinate axes, respectively.
22. The largest solid cube that is contained in a ball of radius R centered at the origin. Solve the same problem if the ball is not centered at the origin. Compare the cases when the

boundaries of the solid are included into the set or excluded from it.

23. The solid region that is a ball of radius R that has a cylindrical hole of radius $R/2$ whose axis is at a distance of $R/2$ from the center of the ball. Choose a convenient coordinate system. Compare the cases when the boundaries of the solid are included into the set or excluded from it.

24. The part of a ball of radius R that lies between two parallel planes each of which is at a distance of $a < R$ from the center of the ball. Choose a convenient coordinate system. Compare the cases when the boundaries of the solid are included into the set or excluded from it.

25. Consider the points P such that the distance from P to the point $(-3, 6, 9)$ is twice the distance from P to the origin. Show that the set of all such points is sphere, and find its center and radius.

26. Find the volume of the solid whose boundaries are the spheres $x^2 + y^2 + z^2 - 6z = 0$ and $x^2 + y^2 - 2y + z^2 - 6z = -9$.

27. Find the volume of the solid that is described by the inequalities $|x - 1| \leq 2$, $|y - 2| \leq 1$, and $|z + 1| \leq 2$. Sketch the solid.

28. The solid region is described by the inequalities $|x - a| \leq a$, $|y - b| \leq b$, $|z - c| \leq c$, and $(y - b)^2 + (z - c)^2 \geq R^2$. If $R \leq \min(b, c)$, sketch the solid and find its volume.

29. Sketch the set of all points in the xy plane that are equidistant from two given points A and B. Let A and B be $(1, 2)$ and $(-2, -1)$, respectively. Give an algebraic description of the set.

30. Sketch the set of all points in space that are equidistant from two given points A and B. Let A and B be $(1, 2, 3)$ and $(-3, -2, -1)$, respectively. Give an algebraic description of the set.

31. A point $P = (x, y)$ belongs to the set $\mathcal{S}$ in the xy plane if $|PA| + |PB| = c$ where $A = (a, 0)$, $B = (-a, 0)$, and $c > 2a$. Show that $\mathcal{S}$ is an ellipse.

32. Determine whether the points $A = (1, 0, -1)$, $B = (3, 1, 1)$, and $C = (2, 2, -3)$ are vertices of a right-angled triangle.

2. Vectors in Space

2.1. Oriented Segments and Vectors. Suppose there is a point like object moving in space along a straight line with a constant rate, say, 5 m/s. If the object was initially at a point P_1, and in 1 second it arrives at a point P_2, then the distance traveled is $|P_1P_2| = 5$ m. The rate (or speed) 5 m/s does not provide a complete description of the motion of the object in space because it only answers the question "How fast does the object move?" but it does not say anything about "Where to does the object move?" Since the initial and final positions of the object are known, both questions can be answered, if one associates an *oriented segment* $\overrightarrow{P_1P_2}$ with the moving object (think of bow arrows). The arrow specifies the direction, "from P_1 to P_2," and the length $|P_1P_2|$ defines the rate (speed) at which the object moves. So, for every moving object, one can assign an oriented segment whose length equals its speed and whose direction coincides with the direction of motion. This oriented segment is called a *velocity*.

At two distinct moments of time a point like object moving with a constant speed along a straight line has different positions on the line. The two oriented segments corresponding to the velocity of the object at two different points have the same length and the same direction, but they are still different because their initial points do not coincide. Similarly, the oriented segments corresponding to the velocities of two objects moving along parallel lines with the same speed would also have the same length and the same direction, but their initial points do not coincide. On the other hand, the velocity should describe a particular physical property of the motion itself ("how fast and where to"), and therefore the spatial position where the motion occurs should not matter. This observation leads to the concept of a *vector* as an abstract mathematical object that *represents all oriented segments that are parallel and have the same length.*

Vectors will be denoted by boldface letters. *Two oriented segments* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *represent the same vector* **a** *if they lie either in the same line or in parallel lines, have the same direction and equal lengths* $|AB| = |CD|$. In this case, one also says that they are obtained by *parallel transport* from one another.

A representation of an abstract vector by a particular oriented segment is denoted by the equality $\mathbf{a} = \overrightarrow{AB}$ or $\mathbf{a} = \overrightarrow{CD}$. The fact that the oriented segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector is denoted by the equality $\overrightarrow{AB} = \overrightarrow{CD}$. Note that this geometrical definition of a vector is not related to any particular coordinate system.

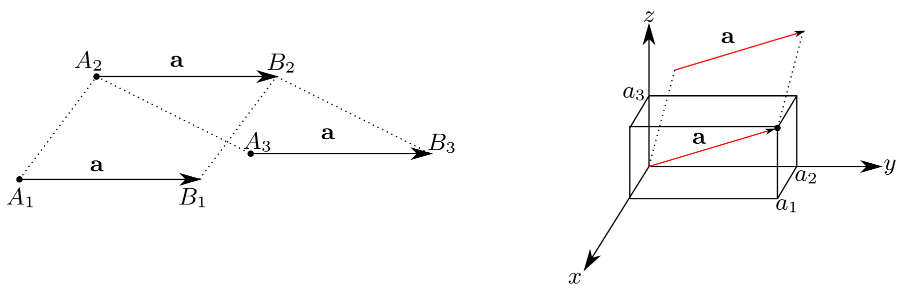


FIGURE **2.1**. **Left**: Oriented segments obtained from one another by parallel transport. They all represent the same vector. **Right**: A vector as an ordered triple of numbers. An oriented segment is transported parallel so that its initial point coincides with the origin of a rectangular coordinate system. The coordinates of the terminal point of the transported segment, (a_1, a_2, a_3), are components of the corresponding vector. So a vector can always be written as an ordered triple of numbers: $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. By construction, the components of a vector depend on the choice of the coordinate system.

2.2. Vector as an Ordered Triple of Numbers. Consider an oriented segment $\overrightarrow{AB}$ that represents a vector $\mathbf{a}$ (i.e., $\mathbf{a} = \overrightarrow{AB}$). An oriented segment $\overrightarrow{A'B'}$ represents the same vector if it is obtained by parallel transport of $\overrightarrow{AB}$. In particular, let us take $A' = O$, where O is the origin of some rectangular coordinate system. Then $\mathbf{a} = \overrightarrow{AB} = \overrightarrow{OB'}$. The direction and length of the oriented segment $\overrightarrow{OB'}$ is uniquely determined by the coordinates of the point B'. Continuing the analogy between vectors and velocities, one can say that a velocity of a point-like object is uniquely determined by the coordinates of the point at which the object arrives in one unit of time starting from the origin of some rectangular coordinate system. In turn, all points in space are in one-to-one correspondence with ordered triple of numbers being the coordinates of points in a rectangular coordinate system.

Thus, the following *algebraic* definition of a vector can be adopted.

DEFINITION **2.1**. (Vectors).
A vector in space is an ordered triple of real numbers:

$$\mathbf{a} = \langle a_1 \, , a_2 \, , a_3 \rangle \, .$$

The numbers a_1, a_2, and a_3 are called components *of the vector* $\mathbf{a}$.

Position vector of a point. Consider a point A with coordinates (a_1, a_2, a_3) in some rectangular coordinate system. The vector $\mathbf{a} = \overrightarrow{OA} = \langle a_1, a_2, a_3 \rangle$ is called the *position vector* of A relative to the given coordinate system. Similarly, the vector $\overrightarrow{PA}$ is called the *position vector of A relative to the point P*. There is a one-to-one correspondence between vectors and points in space: for every point there is a unique (position) vector and every vector defines a unique point as the position vector of the point relative to a fixed point (the zero vector corresponds to the fixed point). This implies, in particular, that, if the coordinate system is changed by a rigid transformation, the components of a vector $\mathbf{a}$ are transformed in the same way as the coordinates of a point whose position vector is $\mathbf{a}$.

Definition **2.2**. (Equality of Two Vectors).
Two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal or coincide if their corresponding components are equal:

$$\mathbf{a} = \mathbf{b} \quad \Longleftrightarrow \quad a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3\,.$$

This definition agrees with the geometrical definition of a vector as a class of all oriented segments that are parallel and have the same length. Indeed, if two oriented segments represent the same vector, then, after parallel transport such that their initial points coincide with the origin, their final points coincide too and hence have the same coordinates. By virtue of the correspondence between vectors and points in space, this definition reflects the fact that two same points should have the same position vectors.

Example **2.1**. *Find the components of a vector $\overrightarrow{P_1P_2}$ if the coordinates of P_1 and P_2 are (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively.*

Solution: Consider a rectangular box whose largest diagonal coincides with the segment P_1P_2 and whose sides are parallel to the coordinate axes. After parallel transport of the segment so that P_1 moves to the origin (see the right panel of Fig. **2.1** where $\mathbf{a} = \overrightarrow{P_1P_2}$), the coordinates of the other endpoint are the components of $\overrightarrow{P_1P_2}$. Alternatively, the coordinate system can be moved parallel so that the origin of the new coordinate system is the point P_1. Therefore,

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle,$$

according to the coordinate transformation law (**1.1**), where $P_0 = P_1$. $\square$

The above example shows that, in order to find the components of the vector $\overrightarrow{P_1P_2}$, one has to subtract the coordinates of the initial point P_1 from the corresponding coordinates of the final point P_2. For instance, if $P_1 = (1, -1, 2)$ and $P_2 = (0, 1, 4)$, then

$$\overrightarrow{P_1P_2} = \langle 0-1,\ 1-(-1),\ 4-2\rangle = \langle -1, 2, 2\rangle$$

Definition 2.3. (Norm of a Vector). *The number*

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

is called the norm *of a vector* **a**.

By Example **2.1** and the distance formula (**1.2**), the norm of a vector is the length of any oriented segment representing the vector. The norm of a vector is also called the *magnitude* or *length* of a vector.

Definition 2.4. (Zero Vector).
A vector with vanishing components, $\mathbf{0} = \langle 0, 0, 0\rangle$*, is called a* zero vector.

A vector **a** is a zero vector if and only if its norm vanishes:

$$\|\mathbf{a}\| = 0 \quad \Leftrightarrow \quad a_1 = a_2 = a_3 = 0\,.$$

Indeed, if $\mathbf{a} = \mathbf{0}$, then $a_1 = a_2 = a_3 = 0$ and hence $\|\mathbf{a}\| = 0$. For the converse, it follows from the condition $\|\mathbf{a}\| = 0$ that $a_1^2 + a_2^2 + a_3^2 = 0$, which is only possible if $a_1 = a_2 = a_3 = 0$, or $\mathbf{a} = \mathbf{0}$. Recall that an "if and only if" statement implies two statements. First, if $\mathbf{a} = \mathbf{0}$, then $\|\mathbf{a}\| = 0$ (the direct statement). Second, if $\|\mathbf{a}\| = 0$, then $\mathbf{a} = \mathbf{0}$ (the converse statement).

2.3. Vector Algebra. Continuing the analogy between the vectors and velocities of a moving object, consider two objects moving parallel but with different rates (speeds). Their velocities as vectors are parallel, but they have different magnitudes. What is the relation between the components of such vectors? Suppose that the objects are moving along lines parallel to the x axis and the direction of the motion is the same as that of the x axis. In this case the velocity vectors are $\mathbf{v} = \langle v, 0, 0\rangle$ and $\mathbf{u} = \langle u, 0, 0\rangle$ where v and u are the speeds. Evidently, the components of the vector **v** can be obtained by multiplying the components of **u** by the number $s = v/u$. This rules holds in general. Indeed, take a vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$. It can be viewed as the largest diagonal of a rectangular box with one vertex at the origin and the opposite vertex at the point (a_1, a_2, a_3). The adjacent sides of the rectangular box have lengths given by the corresponding components of **a** (modulo the signs if the components happen to be negative). When the lengths of the sides are scaled by a factor $s > 0$, a new rectangular box is obtained whose largest diagonal is parallel to **a**. The components of the vector representing this diagonal are obtained by multiplying the components of **a** by s. This geometrical observation leads to the following algebraic rule.

DEFINITION 2.5. (Multiplication of a Vector by a Number). *A vector* **a** *multiplied by a number* s *is a vector whose components are multiplied by* s*:*

$$s\mathbf{a} = \langle sa_1,\ sa_2,\ sa_3\rangle.$$

If $s > 0$, then the vector $s\mathbf{a}$ has the same direction as **a**. If $s < 0$, then the vector $s\mathbf{a}$ has the direction opposite to **a**. For example, the vector $-\mathbf{a}$ has the same magnitude as **a** but points in the direction opposite to **a**. Two vectors having opposite directions are sometimes called *anti-parallel*. Note that the non-zero opposite vectors **a** and $-\mathbf{a}$ cannot be obtained from one another by parallel transport. The magnitude of $s\mathbf{a}$ is:

$$\|s\mathbf{a}\| = \sqrt{(sa_1)^2 + (sa_2)^2 + (sa_3)^2} = \sqrt{s^2}\,\sqrt{a_1^2 + a_2^2 + a_3^2} = |s|\,\|\mathbf{a}\|\,;$$

that is, when a vector is multiplied by a number, its magnitude changes by the factor $|s|$.

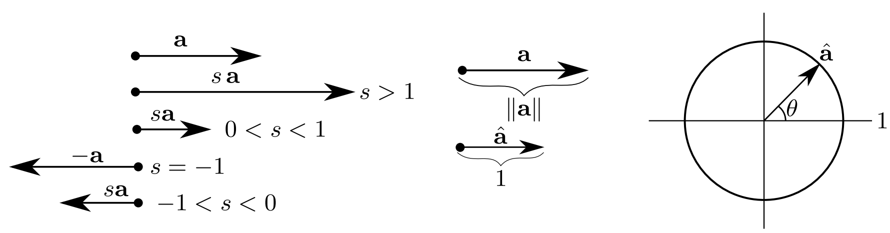


FIGURE 2.2. **Left**: Multiplication of a vector **a** by a number s. If $s > 0$, the result of the multiplication is a vector parallel to **a** whose length is scaled by the factor s. If $s < 0$, then $s\mathbf{a}$ is a vector whose direction is the opposite to that of **a** and whose length is scaled by $|s|$. **Middle**: Construction of a unit vector parallel to **a**. The unit vector $\hat{\mathbf{a}}$ is a vector parallel to **a** whose length is 1. Therefore, it is obtained from **a** by dividing the latter by its length $\|\mathbf{a}\|$, i.e., $\hat{\mathbf{a}} = s\mathbf{a}$, where $s = 1/\|\mathbf{a}\|$. **Right**: A unit vector in a plane can always be viewed as an oriented segment whose initial point is at the origin of a coordinate system and whose terminal point lies on the circle of unit radius centered at the origin. If θ is the polar angle in the plane, then $\hat{\mathbf{a}} = \langle\cos\theta, \sin\theta, 0\rangle$.

Changing units and multiplication by a number. In practical applications, components of a vector are often measured in certain units. If units are changed, then numerical values of the components change by a factor that converts the old unit to the new one. For example, let the position vector of a point in space be $\mathbf{a} = \langle 1, 2, -2\rangle$ whose components are given in centimeters. If now the unit of length is changed to millimeters, the components of **a** are scaled by the conversion factor $s = 10$ (1 cm = 10 mm): $\langle 10, 20, -20\rangle = 10\langle 1, 2, -2\rangle$. Naturally, the direction of a vector representing a physical quantity (e.g., velocity) cannot depend on the choice of units in which the components are measured, while a numerical value of the magnitude depends on it (in the above example $\|\mathbf{a}\| = 3\,\text{cm} = 30\,\text{mm}$).

Parallel vectors. The geometrical analysis of the multiplication of a vector by a number leads to the following simple algebraic criterion for two vectors being parallel. *Two nonzero vectors are parallel if and only if they are proportional:*

$$\mathbf{a} \parallel \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} = s\mathbf{b}$$

for some real s. If all the components of the vectors in question do not vanish, then this criterion may also be written as

$$\mathbf{a} \parallel \mathbf{b} \quad \Leftrightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\,, \tag{2.1}$$

which is easy to verify. If, say, $b_1 = 0$, then **b** is parallel to **a** when $a_1 = b_1 = 0$ and $a_2/b_2 = a_3/b_3$. Owing to the geometrical interpretation of $s\mathbf{b}$, all points in space whose position vectors are parallel to a given nonzero vector **b** form a line (through the origin) that is parallel to **b**.

It is terminologically, notationally, and algebraically convenient to regard the zero vector as being parallel to every vector, since $\mathbf{0} = 0\mathbf{b}$ for all vectors **b**. (Note, however, that for *nonzero* parallel vectors, *each* vector is a multiple of the other: if $\mathbf{a} = s\mathbf{b}$ and $\mathbf{a} \neq \mathbf{0}$, then $s \neq 0$ and $\mathbf{b} = \frac{1}{s}\mathbf{a}$. But if $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then **a** is a multiple of **b** (because $\mathbf{a} = s\mathbf{b}$ with $s = 0$), but **b** is not a multiple of **a**.)

DEFINITION 2.6. (Unit Vector).
A vector $\hat{\mathbf{a}}$ *is called a* unit vector *if its norm equals 1,* $\|\hat{\mathbf{a}}\| = 1$.

Any nonzero vector **a** can be turned into a unit vector $\hat{\mathbf{a}}$ that is parallel to **a**. The norm (length) of the vector $s\mathbf{a}$ reads $\|s\mathbf{a}\| = |s|\|\mathbf{a}\| = s\|\mathbf{a}\|$ if $s > 0$. So, by choosing $s = 1/\|\mathbf{a}\|$, the unit vector in the direction of **a** is obtained:

$$\hat{\mathbf{a}} = \frac{1}{\|\mathbf{a}\|}\,\mathbf{a} = \left\langle \frac{a_1}{\|\mathbf{a}\|}, \frac{a_2}{\|\mathbf{a}\|}, \frac{a_3}{\|\mathbf{a}\|} \right\rangle .$$

For example, owing to the trigonometric identity, $\cos^2\theta + \sin^2\theta = 1$, any unit vector in the xy plane can always be written in the form $\hat{\mathbf{a}} = \langle\cos\theta, \sin\theta, 0\rangle$, where θ is the angle counted from the positive x axis toward the vector $\mathbf{a}$ counterclockwise (see the right panel of Fig. **2.2**). In many practical applications, the components of a vector often have dimensions. For instance, the components of a position vector are measured in units of length (meters, inches, etc.), the components of a velocity vector are measured in, for example, meters per second, and so on. The magnitude of a vector $\mathbf{a}$ has the same dimension as its components. Therefore, the corresponding unit vector $\hat{\mathbf{a}}$ is dimensionless. It specifies only the direction of a vector $\mathbf{a}$.

EXAMPLE **2.2**. *Let $A = (1, 2, 3)$ and $B = (3, 1, 1)$. Find $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BA}$, the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, and the vector $\mathbf{c} = -2\overrightarrow{AB}$ and its norm.*

SOLUTION: By Example **2.1**,

$$\mathbf{a} = \langle 3-1, 2-1, 1-3\rangle = \langle 2, -1, -2\rangle\,.$$

The norm of $\mathbf{a}$ is

$$\|\mathbf{a}\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3\,.$$

The unit vector in the direction of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \tfrac{1}{3}\mathbf{a} = \langle\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}\rangle\,.$$

Using the rule of multiplication of vector by a number,

$$\mathbf{c} = -2\mathbf{a} = -2\langle 2, -1, -2\rangle = \langle -4, 2, 4\rangle\,,$$
$$\|\mathbf{c}\| = \|(-2)\mathbf{a}\| = |-2|\|\mathbf{a}\| = 2\|\mathbf{a}\| = 6\,.$$

The direction of $\overrightarrow{BA}$ is the opposite to $\overrightarrow{AB}$ and both the vectors have the same length. Therefore

$$\overrightarrow{BA} = -\overrightarrow{AB} \quad \Rightarrow \quad \mathbf{b} = -\mathbf{a} = \langle -2, 1, 2\rangle\,,$$

$\|\mathbf{b}\| = \|\mathbf{a}\| = 3$, and $\hat{\mathbf{b}} = -\hat{\mathbf{a}} = \langle -2/3,\ 1/3,\ 2/3\rangle$. □

The Parallelogram Rule. Suppose a person is walking on the deck of a ship with speed v m/s. In 1 second, the person goes a distance v from a point A to a point B of the deck. The velocity vector relative to the deck is $\mathbf{v} = \overrightarrow{AB}$ and $\|\mathbf{v}\| = |AB| = v$ (the speed). The ship moves relative to the water with a constant velocity so that in 1 second a point of the deck moves to a point D from a point C on the surface of the water. The ship's velocity vector relative to the water is then

$\mathbf{u} = \overrightarrow{CD}$ with magnitude $u = \|\mathbf{u}\| = |CD|$. What is the velocity vector of the person relative to the water?

Suppose the point A on the deck coincides with the point C on the surface of the water. Then the velocity vector is the displacement vector of the person relative to the water in 1 second. As the person walks on the deck along the segment AB, this segment travels the distance u parallel to itself along the vector $\mathbf{u}$ relative to the water. In 1 second, the point B of the deck is moved to a point B' on the surface of the water so that the displacement vector of the person relative to the water will be $\overrightarrow{AB'}$. Apparently, the displacement vector $\overrightarrow{BB'}$ coincides with the ship's velocity $\mathbf{u}$ because B travels the distance u parallel to $\mathbf{u}$. This suggests a simple geometrical rule for finding $\overrightarrow{AB'}$ as shown in Figure **2.3**. Take the vector $\overrightarrow{AB} = \mathbf{v}$, place the vector $\mathbf{u}$ (by transporting it parallel) so that its initial point coincides with B, and make the oriented segment with the initial point of $\mathbf{v}$ and the final point of $\mathbf{u}$ in this diagram. The resulting vector is the displacement vector of the person relative to the surface of the water in 1 second and hence defines the velocity of the person relative to the water. This geometrical procedure is called *addition of vectors*.

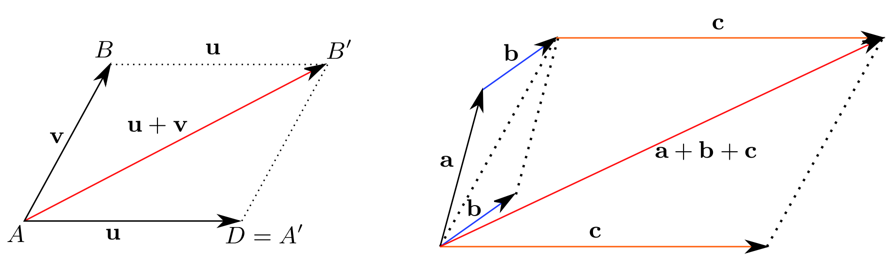


FIGURE **2.3**. **Left**: Parallelogram rule for adding two vectors. If two vectors form adjacent sides of a parallelogram at a vertex A, then the sum of the vectors is a vector that coincides with the diagonal of the parallelogram and originates at the vertex A. **Right**: Adding several vectors by using the parallelogram rule. Given the first vector in the sum, all other vectors are transported parallel so that the initial point of the next vector in the sum coincides with the terminal point of the previous one. The sum is the vector that originates from the initial point of the first vector and terminates at the terminal point of the last vector. It does not depend on the order of vectors in the sum.

Consider a parallelogram whose adjacent sides, the vectors $\mathbf{a}$ and $\mathbf{b}$, extend from the vertex of the parallelogram. The sum of the vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector, denoted $\mathbf{a}+\mathbf{b}$, that is the diagonal of the parallelogram extended from the same vertex. Note that the parallel sides of the parallelogram represent the same vector (they are parallel and have the same length). This geometrical rule for adding vectors is called the *parallelogram rule*. It follows from the parallelogram rule that the addition of vectors is *commutative*:

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a};$$

that is, the order in which the vectors are added does not matter. To add several vectors (e.g., $\mathbf{a}+\mathbf{b}+\mathbf{c}$), one can first find $\mathbf{a}+\mathbf{b}$ by the parallelogram rule and then add $\mathbf{c}$ to the vector $\mathbf{a}+\mathbf{b}$. Alternatively, the vectors $\mathbf{b}$ and $\mathbf{c}$ can be added first, and then the vector $\mathbf{a}$ can be added to $\mathbf{b}+\mathbf{c}$. According to the parallelogram rule, the resulting vector is the same:

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})\,.$$

This means that the addition of vectors is *associative*. So several vectors can be added in any order. Take the first vector, then move the second vector parallel to itself so that its initial point coincides with the terminal point of the first vector. The third vector is moved parallel so that its initial point coincides with the terminal point of the second vector, and so on. Finally, make a vector whose initial point coincides with the initial point of the first vector and whose terminal point coincides with the terminal point of the last vector in the sum. To visualize this process, imagine a man walking along the first vector, then going parallel to the second vector, then parallel to the third vector, and so on. The endpoint of his walk is independent of the order in which he chooses the vectors.

Algebraic Addition of Vectors.

DEFINITION 2.7. *The sum of two vectors* $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ *and* $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ *is a vector whose components are the sums of the corresponding components of* $\mathbf{a}$ *and* $\mathbf{b}$*:*

$$\mathbf{a}+\mathbf{b}=\langle a_1+b_1,\ a_2+b_2,\ a_3+b_3\rangle.$$

This definition is equivalent to the geometrical definition of adding vectors, that is, the parallelogram rule that has been motivated by studying the velocity of a combined motion. Indeed, put $\mathbf{a}=\overrightarrow{OA}$, where the endpoint A has the coordinates (a_1,a_2,a_3). A vector $\mathbf{b}$ represents all parallel segments of the same length $\|\mathbf{b}\|$. In particular, $\mathbf{b}$ is one such oriented segment whose initial point coincides with A. Suppose that $\mathbf{a}+\mathbf{b} = \overrightarrow{OC} = \langle c_1, c_2, c_3\rangle$, where C has coordinates (c_1, c_2, c_3). By the parallelogram rule, $\mathbf{b} = \overrightarrow{AC}$. Using the relation between the components of a vector and the coordinates of its endpoints (see Example **2.1**):

$$\mathbf{b} = \overrightarrow{AC} = \langle c_1 - a_1, c_2 - a_2, c_3 - a_3\rangle\,,$$

The equality of two vectors means the equality of the corresponding components, that is, $b_1 = c_1 - a_1$, $b_2 = c_2 - a_2$, and $b_3 = c_3 - a_3$, which implies that

$$c_1 = a_1 + b_1\,,\ c_2 = a_2 + b_2\,,\ c_3 = a_3 + b_3\,,$$

as required by Definition **2.7**.

Rules of Vector Algebra. Combining addition of vectors with multiplication by real numbers, the following simple rule can be established by either geometrical or algebraic means:

$$s(\mathbf{a}+\mathbf{b}) = s\mathbf{a} + s\mathbf{b}\,, \qquad (s+t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}\,.$$

The difference of two vectors can be defined as

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}\,.$$

In the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$, the sum of vectors $\mathbf{a}$ and $(-1)\mathbf{b}$ represents the vector that originates from the endpoint of $\mathbf{b}$ and ends at the endpoint of $\mathbf{a}$ because $\mathbf{b} + [\mathbf{a} + (-1)\mathbf{b}] = \mathbf{a}$ in accordance with the geometrical rule for adding vectors; that is $\mathbf{a} \pm \mathbf{b}$ are two diagonals of the parallelogram. The procedure is illustrated in Figure **2.4** (left panel).

EXAMPLE **2.3**. *An object travels* 3 *seconds with the velocity* $\mathbf{v} = \langle 1, 2, 4\rangle$*, where the components are given in meters per second, and then* 2 *seconds with the velocity* $\mathbf{u} = \langle 2, 4, 1\rangle$*. Find the distance between the initial and terminal points of the motion.*

SOLUTION: Let the initial and terminal points be A and B, respectively. Let C be the point at which the velocity was changed. Then $\overrightarrow{AC} = 3\mathbf{v}$ and $\overrightarrow{CB} = 2\mathbf{u}$. Therefore

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB} = 3\mathbf{v} + 2\mathbf{u} = 3\langle 1, 2, 4\rangle + 2\langle 2, 4, 1\rangle \\ &= \langle 3, 6, 12\rangle + \langle 4, 8, 2\rangle = \langle 7, 14, 14\rangle = 7\langle 1, 2, 2\rangle\end{aligned}$$

The distance $|AB|$ is the length (or the norm) of the vector $\overrightarrow{AB}$. So

$$|AB| = \|7\langle 1, 2, 2\rangle\| = 7\|\langle 1, 2, 2\rangle\| = 7\sqrt{1+4+4} = 21$$

meters. □

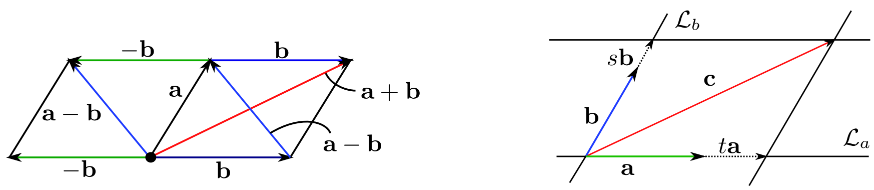


FIGURE 2.4. **Left**: Subtraction of two vectors. The difference $\mathbf{a} - \mathbf{b}$ is viewed as the sum of $\mathbf{a}$ and $-\mathbf{b}$, the vector that has the direction opposite to $\mathbf{b}$ and the same length as $\mathbf{b}$. The parallelogram rule for adding $\mathbf{a}$ and $-\mathbf{b}$ shows that the difference $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ is the vector that originates from the terminal point of $\mathbf{b}$ and ends at the terminal of $\mathbf{a}$ if $\mathbf{a}$ and $\mathbf{b}$ are adjacent sides of a parallelogram; that is, the sum $\mathbf{a} + \mathbf{b}$ and the difference $\mathbf{a} - \mathbf{b}$ are the two diagonals of the parallelogram. **Right**: Illustration to Study Problem **2.1**. Any vector in a plane can always be represented as a linear combination of two nonparallel vectors.

2.4. Study Problems.

Problem **2.1**. *Consider two nonparallel vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *in a plane. Show that any vector* $\mathbf{c}$ *in this plane can be written as a unique linear combination* $\mathbf{c} = t\mathbf{a} + s\mathbf{b}$ *for some real* t *and* s.

SOLUTION: By parallel transport, the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ can be moved so that their initial points coincide. The vectors $t\mathbf{a}$ and $s\mathbf{b}$ are parallel to $\mathbf{a}$ and $\mathbf{b}$, respectively, for all values of s and t. Consider the lines $\mathcal{L}_a$ and $\mathcal{L}_b$ that contain the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. Construct two lines through the terminal point of $\mathbf{c}$; one is parallel to $\mathcal{L}_a$ and the other to $\mathcal{L}_b$ as shown in Figure **2.4** (right panel). The points of intersection of these lines with $\mathcal{L}_a$ and $\mathcal{L}_b$ and the initial and terminal points of $\mathbf{c}$ form the vertices of the parallelogram whose diagonal is $\mathbf{c}$ and whose adjacent sides are parallel to $\mathbf{a}$ and $\mathbf{b}$. Therefore, $\mathbf{a}$ and $\mathbf{b}$ can always be scaled so that $t\mathbf{a}$ and $s\mathbf{b}$ become the adjacent sides of the constructed parallelogram. For a given $\mathbf{c}$, the reals t and s are uniquely defined by the proposed geometrical construction. By the parallelogram rule, $\mathbf{c} = t\mathbf{a} + s\mathbf{b}$. □

Problem **2.2**. *Find the coordinates of a point* B *that is at a distance of* 6 *units of length from the point* $A = (1, -1, 2)$ *in the direction of the vector* $\mathbf{v} = \langle 2, 1, -2 \rangle$.

SOLUTION: Coordinates of a point are the corresponding components of the position vector. The position vector of the point A is $\mathbf{a} = \overrightarrow{OA} =$

$\langle 1, -1, 2\rangle$. By the parallelogram rule the position vector of the point B is

$$\mathbf{b} = \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}\,,$$

where $\overrightarrow{AB}$ is parallel to the vector $\mathbf{v}$. Therefore

$$\overrightarrow{AB} = s\mathbf{v}\,,$$

where s is a positive number to be chosen so that the length $|AB| = s\|\mathbf{v}\|$ equals 6. Since $\|\mathbf{v}\| = 3$, one finds $s = 2$. Therefore,

$$\mathbf{b} = \mathbf{a} + s\mathbf{v} = \langle 1, -1, 2\rangle + 2\langle 2, 1, -2\rangle = \langle 5, 1, -2\rangle\,.$$

□

Problem **2.3**. *Consider a straight line segment with the endpoints $A = (1, 2, 3)$ and $B = (-2, -1, 0)$. Find the coordinates of the point C on the segment such that it is twice as far from A as it is from B.*

SOLUTION: The coordinates of C are the corresponding components of its position vector. Let $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle -2, -1, 0\rangle$, and $\mathbf{c}$ be position vectors of A, B, and C, respectively. The question is to express $\mathbf{c}$ via $\mathbf{a}$ and $\mathbf{b}$. One has

$$\mathbf{c} = \mathbf{a} + \overrightarrow{AC}\,.$$

Since C lies in the straight line segment connecting A and B, the vector $\overrightarrow{AC}$ is parallel to $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \langle -3, -3, -3\rangle$ and hence

$$\overrightarrow{AC} = s\overrightarrow{AB} = s(\mathbf{b} - \mathbf{a})\,.$$

To find s, note that $|AC| = 2|CB|$ and

$$|AB| = |AC| + |CB| = |AC| + \frac{1}{2}|AC| = \frac{3}{2}|AC| \quad \Rightarrow \quad |AC| = \frac{2}{3}|AB|$$

and therefore $s = \frac{2}{3}$. Thus,

$$\mathbf{c} = \mathbf{a} + \frac{2}{3}\overrightarrow{AB} = \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a}) = \langle 1, 2, 3\rangle - \langle 2, 2, 2\rangle = \langle -1, 0, 1\rangle\,.$$

□

Problem **2.4**. *In Study Problem **2.1**, let $\|\mathbf{a}\| = 1$, $\|\mathbf{b}\| = 2$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ be $2\pi/3$ (the smallest angle between oriented segments with a common initial point that represent the vectors $\mathbf{a}$ and $\mathbf{b}$). Find the coefficients s and t if the vector $\mathbf{c}$ has a norm of 6 and bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.*

Solution: It follows from the solution of Study Problem **2.1** that the numbers s and t do not depend on the coordinate system relative to which the components of all the vectors are defined. So choose the coordinate system so that $\mathbf{a}$ is parallel to the x axis and $\mathbf{b}$ lies in the xy plane. With this choice, $\mathbf{a} = \langle 1, 0, 0\rangle$ and

$$\mathbf{b} = \|\mathbf{b}\|\,\hat{\mathbf{b}} = \|\mathbf{b}\|\langle\cos(2\pi/3), \sin(2\pi/3), 0\rangle = \langle -1, \sqrt{3}, 0\rangle$$

where the unit vector $\hat{\mathbf{b}}$ has been found by the procedure given in the right panel of Fig. **2.2**. Similarly, $\mathbf{c}$ is the vector of length $\|\mathbf{c}\| = 6$ that makes the angle $\pi/3$ with the x axis, and therefore $\mathbf{c} = 6\hat{\mathbf{c}} = 6\langle\cos(\pi/3), \sin(\pi/3), 0\rangle = \langle 3, 3\sqrt{3}, 0\rangle$. Equating the corresponding components in the relation

$$\mathbf{c} = t\mathbf{a} + s\mathbf{b} \quad\Rightarrow\quad \langle 3, 3\sqrt{3}, 0\rangle = t\langle 1, 0, 0\rangle + s\langle -1, \sqrt{3}, 0\rangle\,,$$

one finds

$$\begin{aligned} 3 &= t - s \\ 3\sqrt{3} &= s\sqrt{3} \end{aligned} \quad\Rightarrow\quad \begin{aligned} t &= 3 + s = 6 \\ s &= 3 \end{aligned}$$

Hence, $\mathbf{c} = 6\mathbf{a} + 3\mathbf{b}$. □

Problem **2.5**. *According to the law of geometrical optics a ray of light is reflected by a flat mirror so that the incident and reflected rays lie in the plane that contains the line through the point of reflection and perpendicular to the mirror (called the normal line), and the incident and reflected rays make the same angle with the normal line. Suppose the three coordinate planes are all mirrored. A light ray strikes the mirrors. Determine the direction in which the reflected ray will go.*

Solution: Consider a plane that contains the incident and reflected rays. The normal line and the point of reflection O also lie in this plane. Construct a circle in this plane that is of unit radius and centered at the point O. Let A, B, and C be the points of intersection of the circle with the incident and reflected rays and the normal line, respectively. If $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors in the directions of the incident and reflected rays, respectively, then $\hat{\mathbf{v}} = \overrightarrow{OB}$ and $\hat{\mathbf{u}} = \overrightarrow{AO} = -\overrightarrow{OA}$. Since the angles COA and COB are equal, the components of $\overrightarrow{OB}$ and $\overrightarrow{OA}$ along the normal line are equal, too. Then it follows from $\hat{\mathbf{u}} = -\overrightarrow{OA}$ that the components of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ along the normal line are opposite. Since the segment OC bisects the angle AOB, the components of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ along the mirror coincide (see the right panel of Fig. **2.2** and set the normal line along the vertical axis). Thus, under a reflection from a plane mirror, only the component of $\hat{\mathbf{u}}$ along the normal line changes its sign. Therefore, after three consecutive reflections from each coordinate plane, all three components of $\hat{\mathbf{u}}$ change their signs, and the reflected ray will go parallel to the incident ray but in the exact opposite direction. For example, suppose the ray is reflected first by the xz plane, then by the yz plane, and finally by the xy plane. In this case, $\hat{\mathbf{u}} = \langle u_1, u_2, u_3\rangle \to \langle u_1, -u_2, u_3\rangle \to \langle -u_1, -u_2, u_3\rangle \to \langle -u_1, -u_2, -u_3\rangle = -\hat{\mathbf{u}}$. □

Cat's-eyes reflectors. The above reflection principle is used to design reflectors like the cat's-eyes on bicycles and those that mark lane borders on a road. No matter from which direction such a reflector is illuminated (e.g., by headlights of a car), the reflected light goes back to the light source so that the reflectors can always be seen by a car driver as bright spots.

2.5. Exercises.

1–5. Find the components and norms of each of the following vectors:

1. $\overrightarrow{AB}$ where $A = (1, 2, 3)$ and $B = (-1, 5, 1)$.
2. $\overrightarrow{BA}$ where $A = (1, 2, 3)$ and $B = (-1, 5, 1)$.
3. $\overrightarrow{AC}$ where C is the midpoint of the line segment AB with $A = (1, 2, 3)$ and $B = (-1, 5, 1)$.
4. The position vector of a point P obtained from the point $A = (-1, 2, -1)$ by moving the latter along a straight line by a distance of 3 units in the direction of the vector $\mathbf{u} = \langle 2, 2, 1\rangle$ then by a distance of 10 units in the direction of the vector $\mathbf{w} = \langle -3, 0, -4\rangle$.
5. The position vector of the vertex C of a triangle ABC in the first quadrant of the xy plane if A is at the origin, $B = (a, 0, 0)$, the angle at the vertex B is $2\pi/3$, and $|BC| = 2a$.

6. Are the points $A = (-3, 1, 2)$, $B = (1, 5, -2)$, $C = (0, 3, -1)$, and $D = (-2, 3, 1)$ vertices of a parallelogram?

7. If $A = (2, 0, 3)$, $B = (-1, 2, 0)$, and $C = (0, 3, 1)$, determine the point D such that A, B, C, and D are vertices of a parallelogram with sides AB, BC, CD, and DA.

8. A parallelogram has a vertex at $A = (1, 2, 3)$ and two sides $\mathbf{a} = \langle 1, 0, -2\rangle$ and $\mathbf{b} = \langle 3, -2, 6\rangle$ adjacent at A. Find the coordinates of the point of intersection of the diagonals of the parallelogram.

9. Draw two vectors $\mathbf{a}$ and $\mathbf{b}$ that are neither parallel and nor perpendicular. Sketch each of the following vectors: $\mathbf{a} + 2\mathbf{b}$, $\mathbf{b} - 2\mathbf{a}$, $\mathbf{a} - \frac{1}{2}\mathbf{b}$, and $2\mathbf{a} + 3\mathbf{b}$.

10. Draw three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in a plane, with none of them parallel to either of the others. Sketch each of the following vectors:

$\mathbf{a} + (\mathbf{b} - \mathbf{c})$, $(\mathbf{a} + \mathbf{b}) - \mathbf{c}$, $2\mathbf{a} - 3(\mathbf{b} + \mathbf{c})$, and $(2\mathbf{a} - 3\mathbf{b}) - 3\mathbf{c}$.

11. Let $\mathbf{a} = \langle 2, -1, -2\rangle$ and $\mathbf{b} = \langle -3, 0, 4\rangle$. Find unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. Express $6\hat{\mathbf{a}} - 15\hat{\mathbf{b}}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

12. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in the xy plane such that their sum $\mathbf{c} = \mathbf{a}+\mathbf{b}$ makes the angle $\pi/3$ with $\mathbf{a}$ and has the length twice the length of $\mathbf{a}$. Find $\mathbf{b}$ if $\mathbf{a}$ is based at the origin, has its terminal point in the first quadrant, makes an angle $\pi/3$ with the positive x-axis, and has length a. There are two vectors $\mathbf{b}$ with these properties. Find both of them.

13. Consider a triangle ABC. Let $\mathbf{a}$ be a vector from the vertex A to the midpoint of the side BC, let $\mathbf{b}$ be a vector from B to the midpoint of AC, and let $\mathbf{c}$ be a vector from C to the midpoint of AB. Use vector algebra to find $\mathbf{a} + \mathbf{b} + \mathbf{c}$, that is, do not resort to writing vectors in component-form; just use properties of vector addition, subtraction, and multiplication by scalars.

14. Let $\hat{\mathbf{u}}_k$, $k = 1, 2, ..., n$, be unit vectors in the plane such that the smallest angle between the two vectors $\hat{\mathbf{u}}_k$ and $\hat{\mathbf{u}}_{k+1}$ is $2\pi/n$. What is the sum $\mathbf{v}_n = \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 + \cdots + \hat{\mathbf{u}}_n$ for an even n? Sketch the sum for $n = 1$, $n = 3$, and $n = 5$. Compare the norms $\|\mathbf{v}_n\|$ for $n = 1, 3, 5$. Investigate the limit of $\mathbf{v}_n$ as $n \to \infty$ by studying the limit of $\|\mathbf{v}_n\|$ as $n \to \infty$.

15. Let $\hat{\mathbf{u}}_k$, $k = 1, 2, ..., n$, be unit vectors as defined in Exercise 14. Let $\mathbf{w}_k = \hat{\mathbf{u}}_{k+1} - \hat{\mathbf{u}}_k$ for $k = 1, 2, ..., n-1$ and $\mathbf{w}_n = \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_n$. Find the limit of $\|\mathbf{w}_1\| + \|\mathbf{w}_2\| + \cdots + \|\mathbf{w}_n\|$ as $n \to \infty$. Hint: Use a geometrical interpretation of the sum.

16. Suppose a wind is blowing at a speed of u mi/h in the direction that is $0 < \alpha < 90°$ degrees west of the northerly direction. A pilot is steering a plane in the direction that is $0 < \beta < 90°$ degrees east of the northerly direction at an airspeed (speed in still air) of $v > u$ mi/h. The true course of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The ground speed is the magnitude of the resultant. Find the true course and the ground speed of the plane. If α, u, and v are fixed, what is the direction in which the pilot should steer the plane to make the true course north?

17. Use vector algebra (do not resort to writing vectors in component-form) to show that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

18– 21. Describe geometrically the set of points whose position vectors $\mathbf{r}$ satisfy the given conditions.

18. $\|\mathbf{r} - \mathbf{a}\| = k$ and $\mathbf{r}$ lies in the xy plane, where $\mathbf{a}$ is a vector in the xy plane and $k > 0$.

19. $\|\mathbf{r}-\mathbf{a}\|+\|\mathbf{r}-\mathbf{b}\|=k$ and $\mathbf{r}$ lies in the xy plane, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in the xy plane and $k>\|\mathbf{a}-\mathbf{b}\|$.
20. $\|\mathbf{r}-\mathbf{a}\|=k$, where $\mathbf{a}$ is a vector in space and $k>0$.
21. $\|\mathbf{r}-\mathbf{a}\|+\|\mathbf{r}-\mathbf{b}\|=k$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in space and $k>\|\mathbf{a}-\mathbf{b}\|$.

22. Let point-like massive objects be positioned at P_i, $i=1,2,...,n$, and let m_i be the mass at P_i. The point P_0 is called the *center of mass* if

$$m_1\overrightarrow{P_0P_1}+m_2\overrightarrow{P_0P_2}+\cdots+m_n\overrightarrow{P_0P_n}=\mathbf{0}$$

Express the position vector $\mathbf{r}_0$ of P_0 in terms of the position vectors $\mathbf{r}_i$ of P_i. In particular, find the center of mass of three point masses, $m_1=m_2=m_3=m$, located at the vertices of a triangle ABC for $A=(1,2,3)$, $B=(-1,0,1)$, and $C=(1,1,-1)$.

23. Consider the graph $y=f(x)$ of a differentiable function and the line tangent to it at a point $x=a$. Express components of a vector parallel to the line in terms of the derivative $f'(a)$ and find a vector perpendicular to the line. In particular, find such vectors for the graph $y=x^2$ at the point $x=1$.

24. Let the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ have fixed lengths a, b, and c, respectively, while their direction may be changed. Is it always possible to achieve $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$? If not, formulate the most general condition under which it is possible.

25. Let the vectors $\mathbf{a}$ and $\mathbf{b}$ have fixed lengths, while their directions may be changed. Put $c_\pm=\|\mathbf{a}\pm\mathbf{b}\|$. It is always possible to achieve that $c_->c_+$, or $c_-=c_+$, or $c_-<c_+$? If so, give examples of the corresponding relative directions of $\mathbf{a}$ and $\mathbf{b}$.

26. A point object travels in the xy plane, starting from an initial point P_0. Its trajectory consists of straight line-segments, where the n^{th} segment starts at point P_{n-1} and ends at point P_n (for $n\geq 1$). When the object reaches P_n, it makes a 90–degree counterclockwise turn to proceed towards P_{n+1}. (Thus each segment, other than the first, is perpendicular to the preceding segment.) The length of the first segment is a. For $n\geq 2$, the n^{th} segment is s times as long as the $(n-1)^{\text{st}}$ segment, where s is a fixed number between 0 and 1 (strictly). If the object keeps moving forever,
(i) what is the farthest distance it ever gets from P_0, and
(ii) what is the distance between P_0 and the limiting position P_∞ that the object approaches?
Hint: Investigate the components of the position vector of the object in an appropriate coordinate system.

3. The Dot Product

DEFINITION **3.1**. (Dot Product).
The dot product $\mathbf{a}\cdot\mathbf{b}$ *of two vectors* $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ *and* $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ *is a number:*

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

By using this definition it is straightforward to verify that the dot product has the following properties:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}\,,$$
$$(s\mathbf{a})\cdot\mathbf{b} = s(\mathbf{a}\cdot\mathbf{b})\,,$$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c},$$

which hold for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and a number s. The first property states that the order in which two vectors are multiplied in the dot product does not matter; that is, the dot product is *commutative*. It is a trivial consequence of commutativity of multiplication of real numbers:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{b}\cdot\mathbf{a}\,.$$

The second property means that the result of the dot product does not depend on whether the vector $\mathbf{a}$ is scaled first and then multiplied by $\mathbf{b}$ or the dot product $\mathbf{a}\cdot\mathbf{b}$ is computed first and the result multiplied by s. The third relation shows that the dot product is *distributive*. Both the properties also follow from the algebraic rules for real numbers just as the first one.

EXAMPLE **3.1**. *Let* $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 2, -1, 1\rangle$ *and* $\mathbf{c} = \langle 1, 1, -1\rangle$. *Find* $\mathbf{a}\cdot(2\mathbf{b} - 5\mathbf{c})$.

SOLUTION: One has

$$\mathbf{a}\cdot\mathbf{b} = 1\cdot 2 + 2\cdot(-1) + 3\cdot 1 = 2 - 2 + 3 = 3$$

and, similarly,

$$\mathbf{a}\cdot\mathbf{c} = 1 + 2 - 3 = 0\,.$$

By the properties of the dot product:

$$\mathbf{a}\cdot(2\mathbf{b} - 5\mathbf{c}) = 2\mathbf{a}\cdot\mathbf{b} - 5\mathbf{a}\cdot\mathbf{c} = 2\cdot 3 - 5\cdot 0 = 6\,.$$

□

3.1. Geometrical Significance of the Dot Product. As it stands, the dot product is an algebraic rule for calculating a number out of six given numbers that are components of the two vectors involved. The components of a vector depend on the choice of the coordinate system. Recall that a rectangular coordinate system can be changed by a rigid transformation (a composition of rotations, translations, and reflections). Naturally, one should ask whether the numerical value of the dot product depends on the coordinate system relative to which the components of the vectors are determined. It turns out that it does not. Therefore,

> *the numerical value of the dot product represents an intrinsic geometrical quantity associated with two oriented segments in the product.*

To elucidate the geometrical significance of the dot product, note first the relation between the dot product and the norm (length) of a vector:

$$\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = \|\mathbf{a}\|^2 \qquad \text{or} \qquad \|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}\,.$$

Thus, if $\mathbf{a} = \mathbf{b}$ in the dot product, then the latter does not depend on the coordinate system with respect to which the components of $\mathbf{a}$ are defined. Next, consider the triangle whose adjacent sides are the vectors $\mathbf{a}$ and $\mathbf{b}$ as depicted in Figure **3.1** (left panel). Then the other side of the triangle can be represented by the difference $\mathbf{c} = \mathbf{b} - \mathbf{a}$. The squared length of this latter side is

$$\mathbf{c}\cdot\mathbf{c} = (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b}, \tag{3.1}$$

where the algebraic properties of the dot product have been used. Therefore, the dot product can be expressed via the geometrical invariants, namely, the lengths of the sides of the triangle:

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{c}\|^2\right). \tag{3.2}$$

This means that *the numerical value of the dot product is independent of the choice of a coordinate system.*

In particular, let us take the coordinate system in which the vector $\mathbf{a}$ is parallel to the x axis and the vector $\mathbf{b}$ lies in the xy plane as shown in Figure **3.1** (right panel). Let the angle between $\mathbf{a}$ and $\mathbf{b}$ be θ. By definition (given in Section **1.2**), this angle lies in the interval $[0, \pi]$ (it is the smallest angle between two adjoint straight line segments). When $\theta = 0$, the vectors $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. When $\theta = \pi/2$, they are said to be *orthogonal* or *perpendicular*, and they point in the opposite directions if $\theta = \pi$. In the chosen coordinate

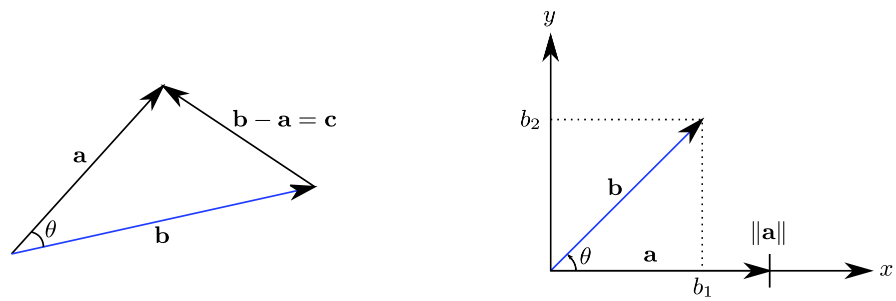


FIGURE **3.1**. **Left**: Independence of the dot product from the choice of a coordinate system. The dot product of two vectors that are adjacent sides of a triangle can be expressed via the lengths of the triangle sides as shown in (**3.2**). **Right**: Geometrical significance of the dot product. It determines the angle between two vectors as stated in (**3.3**). Two nonzero vectors are perpendicular if and only if their dot product vanishes. This follows from (**3.2**) and the Pythagorean theorem: $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{c}\|^2$ for a right-angled triangle.

system, $\mathbf{a} = \langle \|\mathbf{a}\|, 0, 0\rangle$ and $\mathbf{b} = \langle \|\mathbf{b}\|\cos\theta, \|\mathbf{b}\|\sin\theta, 0\rangle$. Hence,

$$(\mathbf{3.3}) \qquad \mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \qquad \text{or} \qquad \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\,.$$

Equation (**3.3**) reveals the geometrical significance of the dot product. It determines the angle between two oriented segments in space. It provides a simple algebraic method to establish a mutual orientation of two straight line segments in space.

THEOREM **3.1**. (Geometrical Significance of the Dot Product). *If θ is the angle between nonzero vectors* **a** *and* **b**, *then*

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta\,.$$

In particular, two nonzero vectors are orthogonal if and only if their dot product vanishes:

$$\mathbf{a}\perp\mathbf{b} \quad\Longleftrightarrow\quad \mathbf{a}\cdot\mathbf{b} = 0.$$

It is terminologically, notationally, and algebraically convenient to regard the zero vector as being perpendicular to every vector, since $\mathbf{0}\cdot\mathbf{b} = 0$ for all vectors **b**. Thus the zero vector is simultaneously parallel and perpendicular to every vector. It is easy to show that **0** is the *only* vector that is simultaneously parallel and perpendicular to *any* nonzero vector (Exercise **6**).

For a triangle with sides a, b, and c and an angle θ between sides a and b, the cosine law for triangles follows from the relation (**3**.1)

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

For a right-angled triangle, $c^2 = a^2 + b^2$ (the Pythagorean theorem).

EXAMPLE **3**.2. *Consider a triangle whose vertices are* $A = (1, 1, 1)$, $B = (-1, 2, 3)$*, and* $C = (1, 4, -3)$*. Find all the angles of the triangle.*

SOLUTION: Let the angles at the vertices A, B, and C be α, β, and γ, respectively. Then $\alpha + \beta + \gamma = 180°$. So it is sufficient to find any two angles. To find the angle α, define the vectors $\mathbf{a} = \overrightarrow{AB} = \langle -2, 1, 2\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle 0, 3, -4\rangle$. The initial point of these vectors is A, and hence the angle between the vectors coincides with α. Since $\|\mathbf{a}\| = 3$ and $\|\mathbf{b}\| = 5$, by the geometrical property of the dot product,

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{0+3-8}{15} = -\frac{1}{3} \quad \Rightarrow$$
$$\alpha = \cos^{-1}(-1/3) \approx 109.5° .$$

To find the angle β, define the vectors $\mathbf{a} = \overrightarrow{BA} = \langle 2, -1, -2\rangle$ and $\mathbf{b} = \overrightarrow{BC} = \langle 2, 2, -6\rangle$ with the initial point at the vertex B. Then the angle between these vectors coincides with β. Since $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 2\sqrt{11}$, and $\mathbf{a}\cdot\mathbf{b} = 4 - 2 + 12 = 14$, one finds $\cos\beta = 14/(6\sqrt{11})$ and $\beta = \cos^{-1}(7/(3\sqrt{11})) \approx 45.3°$. Therefore, $\gamma \approx 180° - 109.5° - 45.3° = 25.2°$. Note that the range of the function $\cos^{-1}$ must be taken from $0°$ to $180°$ in accordance with the definition of the angle between two vectors. □

3.2. Further geometrical properties of the dot product.

COROLLARY **3**.1. (Orthogonal decomposition of a vector)
Given a nonzero vector $\mathbf{a}$*, any vector* $\mathbf{b}$ *can be uniquely decomposed into the sum of two orthogonal vectors one of which is parallel to* $\mathbf{a}$*:*

$$\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\parallel, \quad \mathbf{b}_\perp = \mathbf{b} - s\mathbf{a}, \quad s = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}, \quad \mathbf{b}_\parallel = s\mathbf{a}$$

where $\mathbf{b}_\perp$ *is orthogonal to* $\mathbf{b}_\parallel$ *and* $\mathbf{a}$*, while* $\mathbf{b}_\parallel$ *is parallel to* $\mathbf{a}$*.*

Indeed, given $\mathbf{a}$ and $\mathbf{b}$, put $\mathbf{b}_\perp = \mathbf{b} - s\mathbf{a}$ and demand that $\mathbf{b}_\perp$ is orthogonal to $\mathbf{a}$, that is,

$$\mathbf{a}\cdot\mathbf{b}_\perp = 0 .$$

This condition uniquely determines the coefficient s:

$$\mathbf{a}\cdot\mathbf{b} - s\mathbf{a}\cdot\mathbf{a} = 0 \quad \Rightarrow \quad s = \mathbf{b}\cdot\mathbf{a}/\|\mathbf{a}\|^2\,.$$

The vectors $\mathbf{b}_\perp$ and $\mathbf{b}_\parallel$ are called the *orthogonal* and *parallel* components of $\mathbf{b}$ relative to the vector $\mathbf{a}$. Let $\hat{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\|$ be the unit vector along $\mathbf{a}$. Then

$$\mathbf{b}_\parallel = b_\parallel \hat{\mathbf{a}}\,, \qquad b_\parallel = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \|\mathbf{b}\|\cos\theta\,.$$

The vector $\mathbf{b}_\parallel$ is also called a *vector projection* of $\mathbf{b}$ onto $\mathbf{a}$, and the number $b_\parallel$ is called a *scalar projection* of $\mathbf{b}$ onto $\mathbf{a}$. The orthogonal decomposition $\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\parallel$ is shown in Fig. **3**.**3** (right panel). It is also easy to see from the figure that $\|\mathbf{b}_\perp\| = \|\mathbf{b}\|\sin\theta$.

EXAMPLE **3**.**3**. *Let* $\mathbf{a} = \langle 1,-2,1\rangle$ *and* $\mathbf{b} = \langle 5,1,9\rangle$. *Find the orthogonal decomposition*

$$\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\parallel$$

relative to the vector $\mathbf{a}$.

SOLUTION: One has $\mathbf{a}\cdot\mathbf{b} = 5-2+9 = 12$ and $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} = 1+(-2)^2+1 = 6$. Therefore $s = 12/6 = 2$,

$$\mathbf{b}_\parallel = s\mathbf{a} = 2\langle 1,-2,1\rangle = \langle 2,-4,2\rangle\,,$$

and

$$\mathbf{b}_\perp = \mathbf{b} - \mathbf{b}_\parallel = \langle 5,1,9\rangle - \langle 2,-4,2\rangle = \langle 3,5,7\rangle\,.$$

The result can also be verified: $\mathbf{a}\cdot\mathbf{b}_\perp = 3-10+7 = 0$, i.e., $\mathbf{a}$ is orthogonal to $\mathbf{b}_\perp$ as required. □

THEOREM **3**.**2**. (Cauchy-Schwarz Inequality).
For any two vectors $\mathbf{a}$ *and* $\mathbf{b}$,

$$|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|,$$

where the equality is reached only if the vectors are parallel.

This inequality is a direct consequence of the first relation in (**3**.**3**) and the inequality $|\cos\theta| \le 1$. The equality is reached only when $\theta = 0$ or $\theta = \pi$, that is, when $\mathbf{a}$ and $\mathbf{b}$ are parallel or anti-parallel.

THEOREM **3**.**3**. (Triangle Inequality).
For any two vectors $\mathbf{a}$ *and* $\mathbf{b}$,

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|.$$

PROOF. Put $\|\mathbf{a}\| = a$ and $\|\mathbf{b}\| = b$ so that $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2 = a^2$ and similarly $\mathbf{b}\cdot\mathbf{b} = b^2$. Using the algebraic rules for the dot product,

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}\\ &\le a^2 + b^2 + 2ab = (a+b)^2\,,\end{aligned}$$

where the Cauchy-Schwarz inequality has been used. By taking the square root of both sides, the triangle inequality is obtained. □

The triangle inequality has a simple geometrical meaning. Consider a triangle with sides **a**, **b**, and **c**. The directions of the vectors are chosen so that $\mathbf{c} = \mathbf{a} + \mathbf{b}$. The triangle inequality states that the length $\|\mathbf{c}\|$ cannot exceed the total length of the other two sides. It is also clear that the maximal length $\|\mathbf{c}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$ is attained only if **a** and **b** are parallel and point in the same direction. If they are antiparallel (they point in the opposite directions), then the length $\|\mathbf{c}\|$ becomes minimal and coincides with $|\|\mathbf{a}\| - \|\mathbf{b}\||$. The absolute value is necessary as the length of **a** may be less than the length of **b**. This observation can be stated in the following algebraic form:

$$\Big|\|\mathbf{a}\| - \|\mathbf{b}\|\Big| \le \|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|. \tag{3.4}$$

3.3. Direction Angles. Consider three unit vectors

$$\hat{\mathbf{e}}_1 = \langle 1,0,0\rangle\,,\quad \hat{\mathbf{e}}_2 = \langle 0,1,0\rangle\,,\quad \hat{\mathbf{e}}_3 = \langle 0,0,1\rangle$$

that are parallel to the coordinate axes x, y, and z, respectively. By the rules of vector algebra, any vector can be written as the sum of three mutually perpendicular vectors:

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3\,.$$

The vectors $a_1\hat{\mathbf{e}}_1$, $a_2\hat{\mathbf{e}}_2$, and $a_3\hat{\mathbf{e}}_3$ are adjacent sides of the rectangular box whose largest diagonal coincides with the vector **a** as shown in Figure **3.2** (right panel). Define the angle α as the smallest angle between the positive x semiaxis and the vector **a**. Evidently, $0 \le \alpha \le \pi$ for any nonzero vector **a**. In other words, the angle α is the angle between $\hat{\mathbf{e}}_1$ and **a**. Similarly, the angles β and γ are, by definition, the

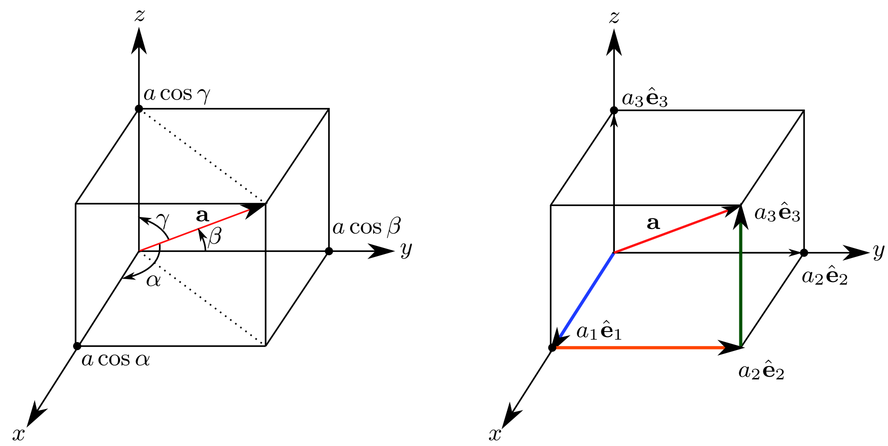


FIGURE **3.2.** **Left**: Direction angles of a vector are defined as the angles between the vector and three coordinates axes. Each angle ranges between 0 and π. They are the smallest angles between the oriented segment representing the vector and the *positive* coordinate semiaxes. The cosines of the direction angles of a vector are components of the unit vector parallel to that vector. **Right**: The decomposition of a vector **a** into the sum of three mutually perpendicular vectors that are parallel to the coordinate axes of a rectangular coordinate system. The vector is the diagonal of the rectangular box whose edges are formed by the vectors in the sum.

angles between **a** and the unit vectors $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$, respectively. Then

$$\cos\alpha = \frac{\hat{\mathbf{e}}_1 \cdot \mathbf{a}}{\|\hat{\mathbf{e}}_1\|\|\mathbf{a}\|} = \frac{a_1}{\|\mathbf{a}\|}\,,$$
$$\cos\beta = \frac{\hat{\mathbf{e}}_2 \cdot \mathbf{a}}{\|\hat{\mathbf{e}}_2\|\|\mathbf{a}\|} = \frac{a_2}{\|\mathbf{a}\|}\,,$$
$$\cos\gamma = \frac{\hat{\mathbf{e}}_3 \cdot \mathbf{a}}{\|\hat{\mathbf{e}}_3\|\|\mathbf{a}\|} = \frac{a_3}{\|\mathbf{a}\|}\,.$$

These cosines are nothing but the components of the unit vector parallel to **a**:

$$\hat{\mathbf{a}} = \frac{1}{\|\mathbf{a}\|}\,\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle\,.$$

Thus, the angles α, β, and γ uniquely determine the direction of a vector. For this reason, they are called *direction angles*. Note that they cannot be set independently because they always satisfy the condition $\|\hat{\mathbf{a}}\| = 1$ or

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,.$$

In practice (physics, mechanics, etc.), vectors are often specified by their magnitude $\|\mathbf{a}\| = a$ and direction angles. The components are then found by $a_1 = a\cos\alpha$, $a_2 = a\cos\beta$, and $a_3 = a\cos\gamma$.

3.4. Basis vectors and coordinate systems. Any vector $\mathbf{a}$, as a particular element $\langle a_1, a_2, a_3\rangle$ of the set of all ordered triples of numbers, is uniquely represented as a linear combination of three particular vectors $\hat{\mathbf{e}}_1 = \langle 1, 0, 0\rangle$, $\hat{\mathbf{e}}_2 = \langle 0, 1, 0\rangle$, and $\hat{\mathbf{e}}_3 = \langle 0, 0, 1\rangle$. They are called the *standard basis*. There are other triples of vectors with the characteristic property that any vector is a unique linear combination of them.

Given three three mutually orthogonal unit vectors $\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, any vector in space can be *uniquely* expanded into the sum

$$\mathbf{a} = s_1\hat{\mathbf{u}}_1 + s_2\hat{\mathbf{u}}_2 + s_3\hat{\mathbf{u}}_3$$

where the numbers s_i are the scalar projections of $\mathbf{a}$ onto $\hat{\mathbf{u}}_i$. Indeed, by Corollary **3.1** a vector $\mathbf{a}$ has the unique orthogonal decomposition relative to $\hat{\mathbf{u}}_1$:

$$\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp = (\mathbf{a}\cdot\hat{\mathbf{u}}_1)\hat{\mathbf{u}}_1 + \mathbf{a}_\perp = s_1\hat{\mathbf{u}}_1 + \mathbf{a}_\perp$$

where $s_1 = \mathbf{a}\cdot\hat{\mathbf{u}}_1$ (recall $\|\hat{\mathbf{u}}_1\| = 1$) and the vector $\mathbf{a}_\perp$ is perpendicular to $\hat{\mathbf{u}}_1$. Similarly, the vector $\mathbf{a}_\perp = \mathbf{a} - s_1\hat{\mathbf{u}}_1$ has the unique orthogonal decomposition relative to $\hat{\mathbf{u}}_2$. The parallel component of $\mathbf{a}_\perp$ relative to $\hat{\mathbf{u}}_2$ is $(\mathbf{a}_\perp\cdot\hat{\mathbf{u}}_2)\hat{\mathbf{u}}_2 = (\mathbf{a}\cdot\hat{\mathbf{u}}_2)\hat{\mathbf{u}}_2 = s_2\hat{\mathbf{u}}_2$, since $\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2 = 0$. The vector $\mathbf{b} = \mathbf{a} - s_1\hat{\mathbf{u}}_1 - s_2\hat{\mathbf{u}}_2$ is perpendicular to both $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ and therefore parallel to $\hat{\mathbf{u}}_3$. Hence, $\mathbf{b} = (\mathbf{b}\cdot\hat{\mathbf{u}}_3)\hat{\mathbf{u}}_3 = (\mathbf{a}\cdot\hat{\mathbf{u}}_3)\hat{\mathbf{u}}_3 = s_3\hat{\mathbf{u}}_3$ owing to the mutual orthogonality of $\hat{\mathbf{u}}_i$. By construction of the coefficients s_i, the vector $\mathbf{c} = \mathbf{a} - s_1\hat{\mathbf{u}}_1 - s_2\hat{\mathbf{u}}_2 - s_3\hat{\mathbf{u}}_3$ is perpendicular to three mutually perpendicular vectors $\hat{\mathbf{u}}_i$. Only the zero vector satisfies this condition in space, $\mathbf{c} = \mathbf{0}$, and therefore

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3 = s_1\hat{\mathbf{u}}_1 + s_2\hat{\mathbf{u}}_2 + s_3\hat{\mathbf{u}}_3$$

A triple of mutually orthogonal unit vectors is called an *orthonormal basis* in space. So with any orthonormal basis one can associate a rectangular coordinate system in which the coordinates of a point are given by the scalar projections of its position vector onto the basis vectors. Consider two coordinate systems with common origin associated with the basis $\hat{\mathbf{e}}_i$ and with the basis $\hat{\mathbf{u}}_i$. If $\mathbf{a}$ is a position vector of a point A, then components of $\mathbf{a}$ in these two bases are coordinates of A in the corresponding coordinate systems. The "new" coordinates s_i can be expressed in terms of the "old" coordinates a_i by means of the above equation:

$$s_i = \mathbf{a}\cdot\hat{\mathbf{u}}_i = a_1(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{u}}_i) + a_2(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{u}}_i) + a_3(\hat{\mathbf{e}}_3\cdot\hat{\mathbf{u}}_i)\,, \qquad i = 1, 2, 3\,.$$

Similarly, the inverse relations are

$$a_j = \mathbf{a}\cdot\hat{\mathbf{e}}_j = s_1(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{e}}_j) + s_2(\hat{\mathbf{u}}_2\cdot\hat{\mathbf{e}}_j) + s_3(\hat{\mathbf{u}}_3\cdot\hat{\mathbf{e}}_j)\,,\qquad j = 1,2,3\,.$$

These relations solve the problem of changing the coordinate system in space (when the origin is fixed) introduced in Section **1.6**. The numbers $\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_i$, $i, j = 1,2,3$, determine the transformation of coordinates under changing the coordinate system. For a fixed j, they are cosines of the direction angles of $\hat{\mathbf{e}}_j$ relative to the new coordinate system associated with the basis $\hat{\mathbf{u}}_i$. For a fixed i, they are cosines of the direction angles of $\hat{\mathbf{u}}_i$ relative to the old coordinate system. A further discussion is given in Study Problem **5.1**.

EXAMPLE **3.4**. *Verify that the vectors Let* $\mathbf{u}_1 = \langle 1,-1,0\rangle$, $\mathbf{u}_2 = \langle 1,1,1\rangle$, *and* $\mathbf{u}_3 = \langle -1,-1,2\rangle$ *are mutually orthogonal. Find the corresponding unit vectors* $\hat{\mathbf{u}}_i$, $i = 1,2,3$. *They form an orthonormal basis. Consider a new rectangular coordinate system with the same origin and whose axes are oriented along the basis vectors* $\hat{\mathbf{u}}_i$. *If* $P = (1,2,3)$, *find the coordinates of the point* P *in the new coordinate system.*

SOLUTION: One has $\mathbf{u}_1\cdot\mathbf{u}_2 = 1-1 = 0$. So the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. Similarly, $\mathbf{u}_1\cdot\mathbf{u}_3 = 0$ and $\mathbf{u}_2\cdot\mathbf{u}_3 = 0$. Then

$$\hat{\mathbf{u}}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{u}_1 = \frac{1}{\sqrt{2}}\langle 1,-1,0\rangle,$$
$$\hat{\mathbf{u}}_2 = \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2 = \frac{1}{\sqrt{3}}\mathbf{u}_2 = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle,$$
$$\hat{\mathbf{u}}_3 = \frac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3 = \frac{1}{\sqrt{6}}\mathbf{u}_3 = \frac{1}{\sqrt{6}}\langle -1,-1,2\rangle$$

Let $\mathbf{r} = \overrightarrow{OP} = \langle 1,2,3\rangle$ be the position vector of the point P. Then

$$\mathbf{r} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3 = s_1\hat{\mathbf{u}}_1 + s_2\hat{\mathbf{u}}_2 + s_3\hat{\mathbf{u}}_3$$

where s_i, $i = 1,2,3$, are the coordinates of P in the new coordinate system. It follows from this equation that

$$s_1 = \mathbf{r}\cdot\hat{\mathbf{u}}_1 = \tfrac{1}{\sqrt{2}}(1-2+0) = -\tfrac{1}{\sqrt{2}},$$
$$s_2 = \mathbf{r}\cdot\hat{\mathbf{u}}_2 = \tfrac{1}{\sqrt{3}}(1+2+3) = 2\sqrt{3},$$
$$s_3 = \mathbf{r}\cdot\hat{\mathbf{u}}_3 = \tfrac{1}{\sqrt{6}}(1+2-6) = -\tfrac{\sqrt{3}}{\sqrt{2}}\,.$$

□

DEFINITION **3.2**. (Basis in Space)
A triple of vectors $\mathbf{u}_1$, $\mathbf{u}_2$, *and* $\mathbf{u}_3$ *is called a* basis *in space if any vector* $\mathbf{a}$ *can be uniquely represented as a linear combination of them:* $\mathbf{a} =$

$s_1\mathbf{u}_1+s_2\mathbf{u}_2+s_3\mathbf{u}_3$. *The coefficients* s_1, s_2, *and* s_3 *are called components of* $\mathbf{a}$ *in the basis. If the basis vectors are mutually orthogonal, the basis is called* orthogonal *and if, in addition, they are unit vectors, then the basis is called* orthonormal.

A basis may contain vectors that are not necessarily orthogonal or unit. For example, a vector in a plane is a *unique* linear combination of two given non-parallel vectors in the plane (Study Problem **2.1**). In this sense, any two non-parallel vectors in a plane define a (non-orthogonal) basis in a plane.

DEFINITION **3.3**. (Coplanar vectors)
Three vectors are called coplanar *if one of them is a linear combination of the others.*

Coplanar vectors lie in a plane. Evidently, three mutually orthogonal unit vectors are not coplanar. There are other triple of vectors that are not in a plane and, hence, none of them is a linear combination of the other two. Such vectors are called *linearly independent.* Thus, *three vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are linearly independent if and only if the vector equation* $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=\mathbf{0}$ *has only a trivial solution* $x=y=z=0$ because otherwise one of the vectors is a linear combination of the others. For example if $x\neq 0$, then $\mathbf{a}=-(y/x)\mathbf{b}-(z/x)\mathbf{c}$. It can be proved that *any three linearly independent vectors form a basis in space* (Study Problems **3.1** and **3.2**). So any vector in space is a linear combination of *three* non-coplanar vectors just like any vector in a plane is a linear combination of *two* non-parallel vectors in the plane and any vector in a line is a multiple of *one* non-zero vector in the line. For this reason, a line, a plane, and space are said to have *dimensions* one, two, and three, respectively.

3.5. Non-rectangular coordinate systems. Let nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be linearly independent in space. Consider two points P and Q whose respective position vectors are $\mathbf{p}$ and $\mathbf{q}$ relative to some rectangular coordinate system. Then there are unique decompositions

$$\mathbf{p}=p_1\mathbf{a}+p_2\mathbf{b}+p_3\mathbf{c},\quad \mathbf{q}=q_1\mathbf{a}+q_2\mathbf{b}+q_3\mathbf{c}$$

Two vectors coincide, $\mathbf{p}=\mathbf{q}$, coincide if and only if their corresponding decomposition coefficients are equal, $p_1=q_1$, $p_2=q_2$, and $p_3=q_3$. Indeed, the condition $\mathbf{p}=\mathbf{q}$ is equivalent to $\mathbf{p}-\mathbf{q}=\mathbf{0}$ and by rules of vector algebra

$$\mathbf{p}-\mathbf{q}=(p_1-q_1)\mathbf{a}+(p_2-q_2)\mathbf{b}+(p_3-q_3)\mathbf{c}=\mathbf{0}$$

By linear independence of the vectors **a**, **b**, and **c**, this equation has only trivial solution, $p_1 - q_1 = 0$, $p_2 - q_2 = 0$, and $p_3 - q_3 = 0$.

Consider three lines through the origin along the vectors **a**, **b**, and **c**. Let us turn the lines into coordinate axes oriented parallel to the vectors **a**, **b**, and **c** by defining the coordinate grid on them so that the distance between points in each line is measured respectively in units of $\|\mathbf{a}\|$, in units of $\|\mathbf{b}\|$, and in units of $\|\mathbf{c}\|$. Then any point P in space is uniquely represented by the ordered triple of coordinates (p_1, p_2, p_3) which are the decomposition coefficients of the position vector of P over the vectors **a**, **b**, and **c**. Note that the length of $p_1\mathbf{a}$ (a vector in the first coordinate line) is $|p_1|$ in units of $\|\mathbf{a}\|$ because $\|p_1\mathbf{a}\| = |p_1|\|\mathbf{a}\|$ (and similarly for the other axes).

The constructed coordinate system is *non-rectangular* because its axes are not perpendicular but it is just as good as any rectangular coordinate system to describe sets of points in space as collections of ordered triple of numbers which have a clear geometrical interpretation in terms of rules of vector algebra. Any relation between vectors can be written as relations between the corresponding components in this non-rectangular coordinate system. Geometrical quantities do not depend on the choice of a coordinate system. For example, the distance between two points P and Q with position vectors **p** and **q**, respectively, is always given by

$$|QP| = \|\mathbf{p} - \mathbf{q}\| = \sqrt{(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{q})}$$

in *any* coordinate system (rectangular or non-rectangular) and can be computed using the properties of the dot product when the decompositions of **p** and **q** are given in the basis (orthogonal or non-orthogonal) associated with a chosen coordinate system. Similarly, the angle between two straight line segments can be computed in any convenient coordinate system by means of Eq. (**3.2**). So *the rules of vector algebra allows us to analyze relations between geometrical quantities without any reference to a particular coordinate system.*

3.6. Applications of the dot product.

Static Problems. A force applied to an object is a vector because it has the direction in which it acts and a strength or magnitude. If mass, distance, and time are measured in kilograms, meters, and seconds, respectively, then the magnitude of a force (or components of the force vector) are measured in newtons, $1\ N = 1\ kg \cdot m/s^2$. According to Newton's mechanics, a pointlike object that was at rest remains at rest if the vector sum of all forces applied to it vanishes. This is the fundamental law of statics:

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}.$$

This vector equation implies three scalar equations that require vanishing each of the three components of the total force. A system of objects is at rest if all its elements are at rest. Thus,

for any element of a system at rest, the scalar projection of the total force onto any vector vanishes.

In particular, the components of the total force should vanish in *any orthonormal basis* or, as a point of fact, *they vanish in any basis in space* (see Study Problem **3.1**). This principle is used to determine either the magnitudes of some forces or the values of some geometrical parameters at which the system in question is at rest.

EXAMPLE **3.5**. *Let a ball of mass m be attached to the ceiling by two ropes so that the smallest angle between the first rope and the ceiling is θ_1 and the angle θ_2 is defined similarly for the second rope. Find the magnitudes of the tension forces in the ropes.*

SOLUTION: The system in question is shown in Fig. **3.3** (left panel). The equilibrium condition is

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{G} = \mathbf{0}\,.$$

This vector equation can be analyzed in any convenient basis. Let $\hat{\mathbf{e}}_1$ be a unit vector that is horizontal and directed from left to right and $\hat{\mathbf{e}}_2$ be a unit vector directed upward. They form an orthonormal basis in the plane in which the system lies. Using the scalar projections, the forces can be expanded in this basis as

$$\begin{aligned}
\mathbf{T}_1 &= -T_1 \cos\theta_1 \hat{\mathbf{e}}_1 + T_1 \sin\theta_1 \hat{\mathbf{e}}_2,\\
\mathbf{T}_2 &= T_2 \cos\theta_2 \hat{\mathbf{e}}_1 + T_2 \sin\theta_2 \hat{\mathbf{e}}_2,\\
\mathbf{G} &= -mg\hat{\mathbf{e}}_2
\end{aligned}$$

where T_1 and T_2 are the magnitudes of the tension forces. The scalar projections of the total force onto the horizontal and vertical directions defined by $\hat{\mathbf{e}}_1$ and $\mathbf{e}_2$ should vanish:

$$-T_1 \cos\theta_1 + T_2 \cos\theta_2 = 0\,, \quad T_1 \sin\theta_1 + T_2 \sin\theta_2 - mg = 0,$$

This system is then solved for T_1 and T_2. By multiplying the first equation by $\sin\theta_1$ and the second by $\cos\theta_1$ and then adding them, one gets

$$T_2(\cos\theta_2 \sin\theta_1 + \cos\theta_1 \sin\theta_2) = mg\cos\theta_1 \quad \Rightarrow \quad T_2 = \frac{mg\cos\theta_1}{\sin(\theta_1+\theta_2)} \,.$$

Substituting T_2 into the first equation, the tension in the first rope is obtained:

$$T_1 = \frac{T_2\cos\theta_2}{\cos\theta_1} = \frac{mg\cos\theta_2}{\sin(\theta_1+\theta_2)} \,.$$

□

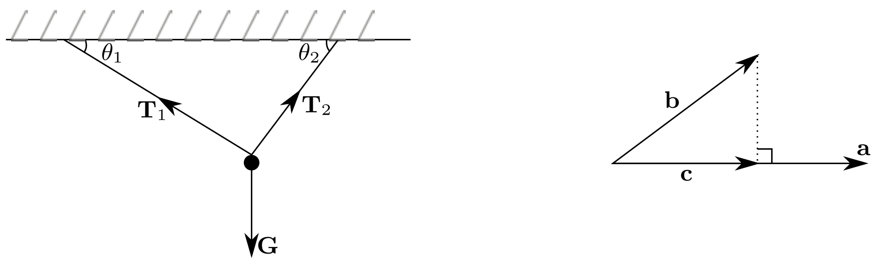


FIGURE **3.3**. **Left**: Illustration to Example **3.5**. At equilibrium, the vector sum of all forces acting on the ball vanishes. The components of the forces are easy to find in the coordinate system in which the x axis is horizontal and the y axis is vertical. **Right**: The vector $\mathbf{c} = \mathbf{b}_{\|}$ is the vector projection of a vector $\mathbf{b}$ onto $\mathbf{a}$. The line through the terminal points of $\mathbf{b}$ and $\mathbf{c}$ is perpendicular to $\mathbf{a}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\|\mathbf{b}\|\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. It is positive if $\theta < \pi/2$, or vanishes if $\theta = \pi/2$, or is negative if $\theta > \pi/2$.

Work Done by a Force. Suppose that an object of mass m moves with speed v. The quantity $K = mv^2/2$ is called the *kinetic energy* of the object. Suppose that the object has moved along a straight line segment from a point P_1 to a point P_2 under the action of a constant force $\mathbf{F}$. A law of physics states that a change in an object's kinetic energy is equal to the work W done by this force:

$$K_2 - K_1 = \mathbf{F}\cdot\overrightarrow{P_1P_2} = W \,,$$

where K_1 and K_2 are the kinetic energies at the initial and final points of the motion, respectively. Energy and work are measured in joules, $1\,J = 1\,N\cdot m = 1\ kg\cdot m^2/s^2$.

EXAMPLE **3.6**. *Let an object slide on an inclined plane without friction under the gravitational force. The magnitude of the gravitational force is equal to mg where m is the mass of the object and g is a universal constant for all objects near the surface of the Earth, $g \approx 9.8\ m/s^2$. Find the final speed v of the object if the relative height of the initial and final points is h and the object was initially at rest.*

SOLUTION: Choose the coordinate system so that the displacement vector $\overrightarrow{P_1P_2}$ and the gravitational force are in the xy plane. Let the y axis be vertical so that the gravitational force is $\mathbf{F} = \langle 0, -mg, 0\rangle$, where m is the mass and g is the acceleration of the free fall. The initial point is chosen to have the coordinates $(0, h, 0)$ while the final point is $(L, 0, 0)$, where L is the distance the object travels in the horizontal direction while sliding. The displacement vector is $\overrightarrow{P_1P_2} = \langle L, -h, 0\rangle$. Since $K_1 = 0$, one has

$$\begin{aligned} \frac{mv^2}{2} &= W = \mathbf{F} \cdot \overrightarrow{P_1P_2} \\ &= \langle 0, -mg, 0\rangle \cdot \langle L, -h, 0\rangle = mgh \quad \Rightarrow \quad v = \sqrt{2gh}\,. \end{aligned}$$

Note that the speed is independent of the mass of the object and the inclination angle of the plane (its tangent is h/L); it is fully determined by the relative height only. □

3.7. Study Problems.

Problem **3.1**. (General basis in space).
Let $\mathbf{u}_i$, $i = 1, 2, 3$, be three linearly independent (non-coplanar) vectors. Show that they form a basis in space, that is, any vector $\mathbf{a}$ can be uniquely expanded into the sum $\mathbf{a} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_3$.

SOLUTION: A solution employs the same approach as in the solution of Study Problem **2.1**. Let P_1 be the parallelogram with adjacent sides $\mathbf{u}_2$ and $\mathbf{u}_3$, P_2 be the parallelogram with sides $\mathbf{u}_1$ and $\mathbf{u}_3$, P_3 be the parallelogram with sides $\mathbf{u}_1$ and $\mathbf{u}_2$. Consider a box whose faces are the parallelograms P_1, P_2, and P_3. This box is called a *parallelepiped*. Let the vectors $\mathbf{u}_i$ and a vector $\mathbf{a}$ have common initial point. Consider three planes through the initial point of $\mathbf{a}$ that contain the parallelograms P_1, P_2, P_3 and three planes through the terminal point of $\mathbf{a}$ such that the first plane is parallel to the plane containing the parallelogram P_1, the second one is parallel to P_2, and the third one is parallel to P_3. These six planes enclose a parallelepiped whose diagonal is the vector $\mathbf{a}$ and whose adjacent sides are *parallel* to the vectors $\mathbf{u}_i$ and therefore are proportional to them, that is, the adjacent edges are the vectors $s_1\mathbf{u}_1$, $s_2\mathbf{u}_2$ and $s_3\mathbf{u}_3$ where the numbers s_1, s_2, and s_3 are uniquely determined by the proposed construction of the parallelepiped. Hence, by the parallelogram rule of adding vectors $\mathbf{a} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_3$. Note that the same geometrical construction has been used to expand a vector in an *orthonormal* basis $\hat{\mathbf{e}}_i$ as shown in Fig. **3.2** where the angles between adjacent sides of the box are not necessarily right angles, while the opposite faces are still parallel. □

Problem **3.2**. *Let* $\mathbf{u}_1 = \langle 1, 1, 0\rangle$, $\mathbf{u}_2 = \langle 1, 0, 1\rangle$, *and* $\mathbf{u}_3 = \langle 2, 2, 1\rangle$. *Show that these vectors are linearly independent and, hence, form a basis in space. Find the components of* $\mathbf{a} = \langle 1, 2, 3\rangle$ *in this basis.*

SOLUTION: If the vectors $\mathbf{u}_i$ are not linearly independent, then there should exist numbers c_1, c_2, and c_3 which do not simultaneously vanish such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

Indeed, this algebraic condition means that one of the vectors is a linear combination of the other two whenever c_i do not vanish simultaneously. This vector equation can be written in the components:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \langle c_1 + c_2 + 2c_3,\ c_1 + 2c_3,\ c_2 + c_3\rangle = \langle 0, 0, 0, \rangle$$

and therefore

$$\begin{cases} c_1 + c_2 + 2c_3 = 0 \\ c_1 + 2c_3 = 0 \\ c_2 + c_3 = 0 \end{cases} \iff \begin{cases} c_1 + c_2 + 2c_3 = 0 \\ c_1 = -2c_3 \\ c_2 = -c_3 \end{cases}$$

The substitution of the last two equations into the first one yields $-c_3 - 2c_3 + 2c_3 = 0$ or $c_3 = 0$ and, hence, $c_1 = c_2 = 0$. Thus the vectors $\mathbf{u}_i$ are linearly independent and form a basis in space. For any vector, $\mathbf{a} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_3$ where the numbers s_i, $i = 1, 2, 3$, are components of $\mathbf{a}$ in the basis $\mathbf{u}_i$. By writing this vector equation in components for $\mathbf{a} = \langle 1, 2, 3\rangle$, the system of equations is obtained:

$$\begin{cases} s_1 + s_2 + 2s_3 = 1 \\ s_1 + 2s_3 = 2 \\ s_2 + s_3 = 3 \end{cases} \iff \begin{cases} s_1 + s_2 + 2s_3 = 1 \\ s_1 = 2 - 2s_3 \\ s_2 = 3 - s_3 \end{cases}$$

The substitution of the last two equations into the first one yields $s_3 = 4$ and hence $s_1 = -6$ and $s_2 = -1$ so that $\mathbf{a} = -6\mathbf{u}_1 - \mathbf{u}_2 + 4\mathbf{u}_3$. □

Problem **3.3**. *Describe the set of all points in space whose position vectors* $\mathbf{r}$ *satisfy the condition* $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$. *Hint:* Note that the position vector satisfying the condition $\|\mathbf{r} - \mathbf{c}\| = R$ describes a sphere of radius R whose center has the position vector $\mathbf{c}$.

SOLUTION: The equation of a sphere can also be written in the form

$$\|\mathbf{r}-\mathbf{c}\|^2 = (\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c}) = R^2\,.$$

The equation $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0$ can be transformed into the sphere equation by completing the squares. Put

$$\mathbf{c} = \frac{1}{2}(\mathbf{a}+\mathbf{b})\,, \qquad \mathbf{R} = \frac{1}{2}(\mathbf{a}-\mathbf{b})\,,$$

and $R = \|\mathbf{R}\|$. Then it is easy to verify that

$$\begin{aligned}\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b} &= \frac{1}{4}\Big(\mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}\Big)-\mathbf{a}\cdot\mathbf{b}\\ &= \frac{1}{4}\Big(\mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}\Big) = \mathbf{R}\cdot\mathbf{R} = R^2 > 0\,.\end{aligned}$$

Using the algebraic properties of the dot product and the vector $\mathbf{c}$,

$$\begin{aligned}(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) &= \mathbf{r}\cdot\mathbf{r}-\mathbf{r}\cdot(\mathbf{a}+\mathbf{b})+\mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{r}\cdot\mathbf{r}-2\mathbf{r}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{r}\cdot\mathbf{r}-2\mathbf{r}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{b}\\ &= (\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c})-R^2\end{aligned}$$

Hence, the set is a sphere of radius R, and its center is positioned at $\mathbf{c}$. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of points A and B, then, by the parallelogram rule, the center of the sphere is the midpoint of the straight line segment AB and the segment AB is a diameter of the sphere, $|AB| = \|\mathbf{b}-\mathbf{a}\| = 2R$. □

3.8. Exercises.

1–5. Find the dot product $\mathbf{a}\cdot\mathbf{b}$ for the given vectors $\mathbf{a}$ and $\mathbf{b}$.

- **1**. $\mathbf{a} = \langle 1,2,3\rangle$ and $\mathbf{b} = \langle -1,2,0\rangle$
- **2**. $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$ where $A = (1,-2,1)$, $B = (2,-1,3)$, and $C = (1,1,1)$.
- **3**. $\mathbf{a} = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$ and $\mathbf{b} = 3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$
- **4**. $\mathbf{a} = \mathbf{u}_1 + 3\mathbf{u}_2 - \mathbf{u}_3$ and $\mathbf{b} = 3\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3$ where $\mathbf{u}_n$, $n = 1,2,3$, are orthogonal vectors and $\|\mathbf{u}_n\| = n$.
- **5**. $\mathbf{a} = 2\mathbf{c} - 3\mathbf{d}$ and $\mathbf{b} = \mathbf{c} + 2\mathbf{d}$ if $\mathbf{c}$ is a unit vector that makes the angle $\pi/3$ with the vector $\mathbf{d}$ and $\|\mathbf{d}\| = 2$

6. Let $\mathbf{a}$ be a nonzero vector. Show that $\mathbf{0}$ is the only vector that is both parallel and perpendicular to $\mathbf{a}$.

7–9. Are the given vectors $\mathbf{a}$ and $\mathbf{b}$ orthogonal, parallel, or neither?

- **7**. $\mathbf{a} = \langle 5,2\rangle$ and $\mathbf{b} = \langle -4,-10\rangle$
- **8**. $\mathbf{a} = \langle 1,-2,1\rangle$ and $\mathbf{b} = \langle 0,1,2\rangle$

9. For what values of b are the vectors $\langle -6\,, b\,, 2\rangle$ and $\langle b\,, b^2\,, b\rangle$ orthogonal?
10. Use the dot product to find all unit vectors that are perpendicular to the vectors $\langle 1, 1, -2\rangle$ and $\langle 1, -2, 4\rangle$.
11. Find the angle at the vertex A of a triangle ABC for $A = (1, 0, 1)$, $B = (1, 2, 3)$, and $= C(0, 1, 1)$. Express the answer in radians.
12. Find the cosines of the angles of a triangle ABC for $A = (0, 1, 1)$, $B = (-2, 4, 3)$, and $C = (1, 2, -1)$.
13. Consider a triangle whose one side is a diameter of a circle and the vertex opposite to this side is on the circle. Use vector algebra to prove that any such triangle is right-angled. *Hint*: Consider position vectors of the vertices relative to the center of the circle.
14. Let $\mathbf{a} = s\hat{\mathbf{u}} + \hat{\mathbf{v}}$ and $\mathbf{b} = \hat{\mathbf{u}} + s\hat{\mathbf{v}}$ where the angle between unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is $\pi/3$. Find the values of s for which the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal, minimal, or zero if such values exist.
15. Consider a cube whose edges have length a. Find the angle between its largest diagonal and any edge adjacent to the diagonal.
16. Consider a parallelepiped with adjacent sides $\mathbf{a} = \langle 1, -2, 2\rangle$, $\mathbf{b} = \langle -2, -2, 1\rangle$, and $\mathbf{c} = \langle -1, -1, -1\rangle$ (see the definition of a parallelepiped in Study Problem **3.1**). It has four vertex-to-opposite-vertex diagonals. Express them in terms of **a**, **b**, and **c** and find the largest one. Find the angle between the largest diagonal and the adjacent sides of the parallelepiped.
17. Let $\mathbf{a} = \langle 1, 2, 2\rangle$. For the vector $\mathbf{b} = \langle -2, 3, 1\rangle$, find the scalar and vector projections of **b** onto **a** and construct the orthogonal decomposition $\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\|$ relative to **a**.
18. Find the scalar and vector projections of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$ if $A = (0, 0, 4)$, $B = (0, 3, -2)$, and $C = (3, 6, 2)$.
19. Find all vectors that have a given length a and make an angle $\pi/3$ with the positive x axis and the angle $\pi/4$ with the positive z axis.
20. Find the components of all unit vectors $\hat{\mathbf{u}}$ that make an angle ϕ with the positive z axis. *Hint:* Put $\hat{\mathbf{u}} = a\hat{\mathbf{v}} + b\hat{\mathbf{e}}_3$, where $\hat{\mathbf{v}}$ is a unit vector in the xy plane. Find a, b, and all $\hat{\mathbf{v}}$ using the polar angle in the xy plane.
21. If $\mathbf{c} = \|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a}$, where **a** and **b** are non zero vectors, show that **c** bisects the angle between **a** and **b**. *Hint*: compare the angle between **c** and **a** to the angle between **c** and **b**.
22. A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus meet at right angles.
23. Consider a parallelogram with adjacent sides of length a and b. If d_1 and d_2 are the lengths of the diagonals, prove the parallelogram

law: $d_1^2 + d_2^2 = 2(a^2 + b^2)$. *Hint:* Consider the vectors $\mathbf{a}$ and $\mathbf{b}$ that are adjacent sides of the parallelogram and express the diagonals via $\mathbf{a}$ and $\mathbf{b}$. Use the dot product to evaluate $d_1^2 + d_2^2$.

24. Consider a right-angled triangle whose adjacent sides at the right angle have lengths a and b. Let P be a point in space at a distance c from all three vertices of the triangle ($c \geq a/2$ and $c \geq b/2$). Find the angles between the line segments connecting P with the vertices of the triangle. *Hint*: Consider vectors with the initial point P and terminal points at the vertices of the triangle.

25. Show that the vectors $\mathbf{u}_1 = \langle 1, 1, 2\rangle$, $\mathbf{u}_2 = \langle 1, -1, 0\rangle$, and $\mathbf{u}_3 = \langle 2, 2, -2\rangle$ are mutually orthogonal. For a vector $\mathbf{a} = \langle 4, 3, 4\rangle$ find the scalar orthogonal projections of $\mathbf{a}$ onto $\mathbf{u}_i$, $i = 1, 2, 3$, and the numbers s_i such that $\mathbf{a} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_3$.

26. For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ find all vectors coplanar with $\mathbf{a}$ and $\mathbf{b}$ that have the same vector projection onto $\mathbf{a}$ as the vector $\mathbf{b}$. Express these vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

27. A point object traveled 3 meters from a point A in a particular direction, then it changed the direction by 60° and traveled 4 meters, and then it changed the direction again so that it was traveling at 60° with each of the previous two directions. If the last stretch was 2 meters long, how far from A is the object?

28. Two balls of the same mass m are connected by a piece of rope of length h. Then the balls are attached to different points on a horizontal ceiling by a piece of rope with the same length h so that the distance L between the points is greater than h but less than $3h$. Find the equilibrium positions of the balls and the magnitude of tension forces in the ropes.

29. A cart is pulled up a 20° slope a distance of 10 meters by a horizontal force of 30 newtons. Determine the work.

30. Two tug boats are pulling a barge against the river stream. One tug is pulling with the force of magnitude 20 (in some units) and at the angle 45° to the stream and the second with the force of magnitude 15 at the angle 30° so that the angle between the pulling ropes is $45° + 30° = 75°$. If the barge does not move in the direction of the stream, what is the drag force exerted by the stream on the barge? Does the barge move in the direction perpendicular to the stream?

31. A ball of mass m is attached by three ropes of the same length a to a horizontal ceiling so that the attachment points on the ceiling form a triangle with sides of length a. Find the magnitude of the tension force in the ropes.

32. Four dogs are at the vertices of a square. Each dog starts running toward its neighbor on the right. The dogs run with the same speed v. At every moment of time each dog keeps running in the direction of its right neighbor (its velocity vector always points to the neighbor). Eventually, the dogs meet in the center of the square. When will this happen if the sides of the square have length a? What is the distance traveled by each dog? *Hint*: Is there a particular direction from the center of the square relative to which the velocity vector of a dog has the same component at each moment of time?

4. The Cross Product

4.1. Determinant of a Square Matrix.

DEFINITION **4.1**. *The determinant of a* 2×2 *matrix is the number computed by the following rule:*

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

that is, the product of the diagonal elements minus the product of the off-diagonal elements.

DEFINITION **4.2**. *The determinant of a* 3×3 *matrix* A *is the number obtained by the following rule:*

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= \sum_{k=1}^{3} (-1)^{k+1} a_{1k}\det A_{1k},$$

$$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \quad A_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

where the matrices A_{1k}, $k = 1, 2, 3$, *are obtained from the original matrix* A *by removing the row and column containing the element* a_{1k}.

It is straightforward to verify that the determinant can be expanded over any row or column:

$$\det A = \sum_{k=1}^{3} (-1)^{k+m} a_{mk}\det A_{mk} \quad \text{for any } m = 1, 2, 3,$$

$$\det A = \sum_{m=1}^{3} (-1)^{k+m} a_{mk}\det A_{mk} \quad \text{for any } k = 1, 2, 3,$$

where the matrix A_{mk} is obtained from A by removing the row and column containing a_{mk}. This definition of the determinant is extended recursively to $N \times N$ square matrices by letting k and m range over $1, 2, ..., N$.

In particular, the determinant of a triangular matrix (i.e., the matrix all of whose elements either above or below the diagonal vanish) is the product of its diagonal elements:

$$\det\begin{pmatrix} a_1 & b & c \\ 0 & a_2 & d \\ 0 & 0 & a_3 \end{pmatrix} = \det\begin{pmatrix} a_1 & 0 & 0 \\ b & a_2 & 0 \\ c & d & a_3 \end{pmatrix} = a_1a_2a_3$$

for any numbers b, c, and d. For a lower triangular matrix (the one on the right), the result follows from the expansion of the determinant over the first row, while the expansion over the first column proves the claim for an upper triangular matrix (the one on the left). Also, it follows from the expansion of the determinant over any column or row that, *if any two rows or any two columns are swapped in the matrix, its determinant changes sign*. For 2×2 matrices, this is easy to see directly from Definition **4.1**. In general, if the matrix B is obtained from A by swapping the first and second rows, that is $b_{1k} = a_{2k}$ and $b_{2k} = a_{1k}$, then the matrices B_{2k} and A_{1k} coincide and so do their determinants. By expanding $\det B$ over its *second* row $b_{2k} = a_{1k}$, one infers that

$$\begin{aligned}\det B &= \sum_{k=1}^{3}(-1)^{2+k} b_{2k} \det B_{2k} = \sum_{k=1}^{3}(-1)^{2+k} a_{1k} \det A_{1k} \\ &= -\sum_{k=1}^{3}(-1)^{1+k} a_{1k} \det A_{1k} = -\det A\end{aligned}$$

This argument can be applied to prove that the determinant changes its sign under swapping any two rows or columns in a square matrix of any dimension.

EXAMPLE **4.1**. *Calculate* $\det A$, *where*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}.$$

SOLUTION: Expanding the determinant over the first row yields

$$\begin{aligned}\det A &= 1\cdot\det\begin{pmatrix} 1 & 3 \\ 2 & 1\end{pmatrix} - 2\cdot\det\begin{pmatrix} 0 & 3 \\ -1 & 1\end{pmatrix} + 3\cdot\det\begin{pmatrix} 0 & 1 \\ -1 & 2\end{pmatrix} \\ &= 1(1-6) - 2(0+3) + 3(0+1) = -8\,.\end{aligned}$$

Alternatively, expanding the determinant over the second row yields the same result:

$$\begin{aligned}\det A &= -0\cdot\det\begin{pmatrix} 2 & 3 \\ 2 & 1\end{pmatrix} + 1\cdot\det\begin{pmatrix} 1 & 3 \\ -1 & 1\end{pmatrix} - 3\cdot\det\begin{pmatrix} 1 & 2 \\ -1 & 2\end{pmatrix} \\ &= 0 + 1(1+3) - 3(2+2) = -8\,.\end{aligned}$$

One can check that the same result can be obtained by expanding the determinant over any row or column. □

4.2. The Cross Product of Two Vectors.

DEFINITION **4.3**. (Cross Product).
Let $\hat{\mathbf{e}}_1 = \langle 1, 0, 0\rangle$, $\hat{\mathbf{e}}_2 = \langle 0, 1, 0\rangle$ *and* $\hat{\mathbf{e}}_3 = \langle 0, 0, 1\rangle$. *The cross product of two vectors* $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ *and* $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ *is a vector that is the determinant of the formal matrix expanded over the first row:*

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\\ &= \det\begin{pmatrix}a_2 & a_3\\ b_2 & b_3\end{pmatrix}\hat{\mathbf{e}}_1 - \det\begin{pmatrix}a_1 & a_3\\ b_1 & b_3\end{pmatrix}\hat{\mathbf{e}}_2 + \det\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}\hat{\mathbf{e}}_3\\ &= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle. \qquad (\mathbf{4.1})\end{aligned}$$

Note that the first row of the matrix consists of the unit vectors parallel to the coordinate axes rather than numbers. For this reason, it is referred as to a *formal* matrix. The use of the determinant is merely a compact way to write the algebraic rule to compute the components of the cross product.

EXAMPLE **4.2**. *Evaluate the cross product* $\mathbf{a}\times\mathbf{b}$ *if* $\mathbf{a} = \langle 1, 2, 3\rangle$ *and* $\mathbf{b} = \langle 2, 0, 1\rangle$.

SOLUTION: By the definition

$$\begin{aligned}\langle 1, 2, 3\rangle \times \langle 2, 0, 1\rangle &= \det\begin{pmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & 2 & 3\\ 2 & 0 & 1\end{pmatrix}\\ &= \det\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}\hat{\mathbf{e}}_1 - \det\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}\hat{\mathbf{e}}_2 + \det\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix}\hat{\mathbf{e}}_3\\ &= (2-0)\hat{\mathbf{e}}_1 - (1-6)\hat{\mathbf{e}}_2 + (0-4)\hat{\mathbf{e}}_3\\ &= 2\hat{\mathbf{e}}_1 + 5\hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3 = \langle 2, 5, -4\rangle\,.\end{aligned}$$

□

Properties of the cross product. The cross product has the following properties that follow from its definition:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= -\mathbf{b}\times\mathbf{a},\\ (\mathbf{a}+\mathbf{c})\times\mathbf{b} &= \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{b},\\ (s\mathbf{a})\times\mathbf{b} &= s(\mathbf{a}\times\mathbf{b})\,.\end{aligned}$$

The first property is obtained by swapping the components of $\mathbf{b}$ and $\mathbf{a}$ in (**4.1**). Alternatively, recall that the determinant of a matrix changes its sign if two rows are swapped in the matrix (the rows $\mathbf{a}$ and $\mathbf{b}$

in Definition **4.3**). So the cross product is skew-symmetric, i.e., it is *not commutative* and the order in which the vectors are multiplied is essential; changing the order leads to the opposite vector. In particular, if $\mathbf{b} = \mathbf{a}$, then $\mathbf{a} \times \mathbf{a} = -\mathbf{a} \times \mathbf{a}$ or $2(\mathbf{a} \times \mathbf{a}) = \mathbf{0}$ or

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}\,.$$

The cross product is *distributive* according to the second property. To prove it, change a_i to $a_i + c_i$, $i = 1, 2, 3$, in (**4.1**). If a vector $\mathbf{a}$ is scaled by a number s and the resulting vector is multiplied by $\mathbf{b}$, the result is the same as the cross product $\mathbf{a} \times \mathbf{b}$ computed first and then scaled by s (change a_i to sa_i in (**4.1**) and then factor out s).

The double cross product satisfies the so called “bac-cab” rule

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{4.2}$$

and the Jacobi identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}\,. \tag{4.3}$$

The “bac-cab” rule expresses the double cross product as a linear combination of the vectors $\mathbf{b}$ and $\mathbf{c}$. Note that the second and third terms in the left side of Eq. (**4.3**) are obtained from the first by *cyclic permutations* of the vectors. The proofs the “bac-cab” rule and the Jacobi identity are given in Study Problems **4.3** and **4.4**. The Jacobi identity implies that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

This means that the multiplication of vectors defined by the cross product is *not associative* in contrast to multiplication of numbers. This observation is further discussed in Study Problem **4.5**.

EXAMPLE **4.3**. *Calculate* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *if* $\mathbf{a} = \langle 1, 2, -1\rangle$, $\mathbf{b} = \langle 2, -1, 2\rangle$, *and* $\mathbf{c} = \langle 3, 1, 2\rangle$.

SOLUTION: Using the “bac-cab” rule:

$$\mathbf{a} \cdot \mathbf{c} = 3 + 2 - 2 = 3\,, \quad \mathbf{a} \cdot \mathbf{b} = 2 - 2 - 2 = -2$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 3\mathbf{b} - (-2)\mathbf{c} = 3\langle 2, -1, 2\rangle + 2\langle 3, 1, 2\rangle = \langle 12, -1, 10\rangle\,.$$

□

4.3. Geometrical Significance of the Cross Product. The above algebraic definition of the cross product uses a particular coordinate system relative to which the components of the vectors are defined. Does the cross product change under rigid transformations of the rectangular coordinate system with respect to which the components of the vectors are determined to compute the cross product? In contrast to a similar question about the dot product, here one should investigate both the *direction* and the *magnitude* of the cross product.

Let us first investigate the mutual orientation of the oriented segments $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}\times\mathbf{b}$. A simple algebraic calculation leads to the following result:

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0.$$

Swapping the vectors $\mathbf{a}$ and $\mathbf{b}$ in this relation and using the skew symmetry of the cross product, it is also concluded that

$$0 = \mathbf{b}\cdot(\mathbf{b}\times\mathbf{a}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})\,.$$

By the geometrical property of the dot product, the cross product must be perpendicular to both vectors $\mathbf{a}$ and $\mathbf{b}$:

$$(4.4)\qquad \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0 \quad\Longleftrightarrow\quad \mathbf{a}\times\mathbf{b}\perp\mathbf{a}\ \text{ and }\ \mathbf{a}\times\mathbf{b}\perp\mathbf{b}.$$

Let us calculate the length of the cross product. By Definition **4.1**,

$$\begin{aligned}
\|\mathbf{a}\times\mathbf{b}\|^2 &= (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})\\
&= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2\\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2\\
&= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2
\end{aligned}$$

where the third equality is obtained by computing the squares of the components of the cross product and regrouping terms in the obtained expression. The last equality uses the definitions of the norm and the dot product. Next, recall the geometrical property of the dot product (**3.3**). If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\begin{aligned}
\|\mathbf{a}\times\mathbf{b}\|^2 &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2\cos^2\theta\\
&= \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta
\end{aligned}$$

Since $0\le\theta\le\pi$, $\sin\theta\ge 0$ and the square root of the both sides of this equation can be taken with the result that

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

This relation shows that length of the cross product defined by (**4.1**) does not depend on the choice of the coordinate system as it is expressed via the geometrical invariants, the lengths of $\mathbf{a}$ and $\mathbf{b}$ and the angle between them.

Now consider the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$. If $\|\mathbf{a}\|$ is the length of its base, then $h = \|\mathbf{b}\|\sin\theta$ is its height. Therefore the norm of the cross product, $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|h = A$, is the area of the parallelogram.

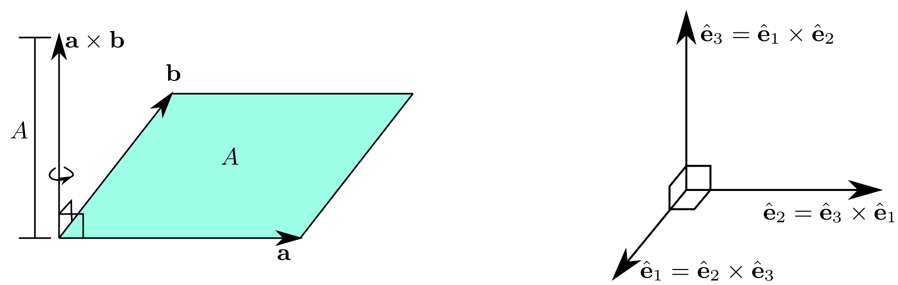


FIGURE **4.1**. **Left**: Geometrical interpretation of the cross product of two vectors. The cross product is a vector that is perpendicular to both vectors in the product. Its length equals the area of the parallelogram whose adjacent sides are the vectors in the product. If the fingers of the right hand curl in the direction of a rotation from the first to second vector through the smallest angle between them, then the thumb points in the direction of the cross product of the vectors.
Right: Illustration to Study Problem **4.2**.

Owing to that the angles between the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}\times\mathbf{b} \neq \mathbf{0}$ found in (**4.4**) as well as that their lengths are preserved under rotations of the coordinate system, the coordinate system can be oriented so that $\mathbf{a}$ is along the x axis, $\mathbf{b}$ is in the xy plane, while $\mathbf{a}\times\mathbf{b}$ is parallel to the z axis. In this coordinate system, $\mathbf{a} = \langle a_1, 0, 0\rangle$, where $a_1 = \|\mathbf{a}\|$, and $\mathbf{b} = \langle b_1, b_2, 0\rangle$ where $b_1 = \|\mathbf{b}\|\cos\theta$ and $b_2 = \|\mathbf{b}\|\sin\theta > 0$ if $\mathbf{b}$ lies either in the first or second quadrant of the xy plane and $b_2 = -\|\mathbf{b}\|\sin\theta < 0$ if $\mathbf{b}$ lies either in the third or fourth quadrant. By Definition **4.1**,

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & 0 & 0 \\ b_1 & b_2 & 0 \end{pmatrix} = a_1 b_2 \hat{\mathbf{e}}_3 = \langle 0, 0, a_1 b_2\rangle\,.$$

In the former case ($b_2 > 0$), the cross product is directed along the z axis: $\mathbf{a}\times\mathbf{b} = \langle 0, 0, A\rangle$, where $A > 0$ is the area of the parallelogram. In the latter case ($b_2 < 0$), $\mathbf{a}\times\mathbf{b} = \langle 0, 0, -A\rangle$ and the cross product has the opposite direction. It turns out that the direction of the cross product in both the cases can be described by a simple rule known as the *right-hand rule*: *If the fingers of the right hand curl in the direction of a rotation from* $\mathbf{a}$ *toward* $\mathbf{b}$ *through the smallest angle between them, then the thumb points in the direction of* $\mathbf{a}\times\mathbf{b}$.

In particular, by Definition **4.3**, $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle = \langle 0, 0, 1\rangle = \hat{\mathbf{e}}_3$. If $\mathbf{a}$ is orthogonal to $\mathbf{b}$, then *the relative orientation* *of the triple of vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{a}\times\mathbf{b}$ *is the same as that of the standard basis vectors* $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, *and* $\hat{\mathbf{e}}_3$. Furthermore, given two non-parallel vectors, one can construct an orthogonal basis associated with them using the cross product (see Exercise **10**).

The stated geometrical properties (*independent of the choice of a coordinate system*) are depicted in the left panel of Fig. **4.1** and summarized in the following theorem.

THEOREM **4.1**. (Geometrical Significance of the Cross Product). *The cross product* $\mathbf{a}\times\mathbf{b}$ *of vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is the vector that is perpendicular to both vectors,* $\mathbf{a}\times\mathbf{b}\perp\mathbf{a}$ *and* $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$, *has a magnitude equal to the area of the parallelogram with adjacent sides* $\mathbf{a}$ *and* $\mathbf{b}$, *and is directed according to the right-hand rule.*

Two useful consequences can be deduced from this theorem.

COROLLARY **4.1**. *Two nonzero vectors are parallel if and only if their cross product vanishes:*

$$\mathbf{a}\times\mathbf{b}=\mathbf{0} \quad \Longleftrightarrow \quad \mathbf{a}\parallel\mathbf{b}\,.$$

If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, then the area of the corresponding parallelogram vanishes, $\|\mathbf{a}\times\mathbf{b}\|=0$, which is only possible if the adjacent sides of the parallelogram are parallel. Conversely, for two parallel vectors, there is a number s such that $\mathbf{a}=s\mathbf{b}$. Hence, $\mathbf{a}\times\mathbf{b}=(s\mathbf{b})\times\mathbf{b}=s(\mathbf{b}\times\mathbf{b})=\mathbf{0}$.

If in the cross product $\mathbf{a}\times\mathbf{b}$ the vector $\mathbf{b}$ is changed by adding to it any vector parallel to $\mathbf{a}$, the cross product does not change:

$$\mathbf{a}\times(\mathbf{b}+s\mathbf{a})=\mathbf{a}\times\mathbf{b}+s(\mathbf{a}\times\mathbf{a})=\mathbf{a}\times\mathbf{b}$$

Let $\mathbf{b}=\mathbf{b}_\perp+\mathbf{b}_\parallel$ be the orthogonal decomposition of $\mathbf{b}$ relative to a non-zero vector $\mathbf{a}$. By Corollary **4.1**, $\mathbf{a}\times\mathbf{b}_\parallel=\mathbf{0}$ because $\mathbf{b}_\parallel$ is parallel to $\mathbf{a}$. It is then concluded that *the cross product depends only on the component* $\mathbf{b}_\perp$ *of* $\mathbf{b}$ *that is orthogonal to* $\mathbf{a}$. Thus, $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}_\perp$ and $\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}_\perp\|$.

Area of a triangle. One of the most important applications of the cross product is in calculations of the areas of planar figures in space.

COROLLARY **4.2**. (Area of a Triangle). *Let vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *be two sides of a triangle and have the same initial point at a vertex of a triangle. Then the area of the triangle is*

$$\text{Area}\,\triangle=\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|.$$

Indeed, by the geometrical construction, the area of the triangle is half of the area of a parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 4.4. *Let $A = (1, 1, 1)$, $B = (2, -1, 3)$, and $C = (-1, 3, 1)$. Find the area of the triangle ABC and a vector orthogonal to the plane that contains the triangle.*

SOLUTION: Take two vectors with the initial point at any of the vertices of the triangle which form the adjacent sides of the triangle at that vertex. For example, $\mathbf{a} = \overrightarrow{AB} = \langle 1, -2, 2\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle -2, 2, 0\rangle$. Then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \langle (-2) \cdot 0 - 2 \cdot 2,\ -(1 \cdot 0 - (-2) \cdot 2),\ 1 \cdot 2 - (-2) \cdot (-2)\rangle \\ &= \langle -4, -4, -2\rangle = -2\langle 2, 2, 1\rangle \,.\end{aligned}$$

Therefore

$$\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\| = \frac{1}{2} \cdot 2\|\langle 2, 2, 1\rangle\| = 3$$

is the area of the triangle ABC by Corollary **4.2**. The units here are squared units of length used to measure the coordinates of the triangle vertices (e.g., m^2 if the coordinates are measured in meters).

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors, any vector in the plane that contains the triangle is a linear combination $s\mathbf{a} + t\mathbf{b}$. Therefore the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to any such vector and, hence, to the plane because $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

$$(\mathbf{a} \times \mathbf{b}) \cdot (s\mathbf{a} + t\mathbf{b}) = s(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + t(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0 \,.$$

Thus, a vector $\mathbf{n}$ perpendicular to the plane containing a triangle ABC is a scalar multiple of $\overrightarrow{AB} \times \overrightarrow{AC}$. For example, one can take

$$\mathbf{n} = -\frac{1}{2}\mathbf{a} \times \mathbf{b} = \langle 2, 2, 1\rangle \,.$$

Note that all scalar multiples of the vector $\mathbf{a} \times \mathbf{b}$ lie in the same line (perpendicular to the plane in question). The choice $\mathbf{a} = \overrightarrow{CB}$ and $\mathbf{b} = \overrightarrow{CA}$ or $\mathbf{a} = \overrightarrow{BA}$ and $\mathbf{b} = \overrightarrow{BC}$ would give the same answer (modulo the sign change in the cross product). □

Area of a quadrilateral. Consider a quadrilateral in space. Its four vertices lie in a plane. Let the coplanar vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors originating from one of the vertices and terminating at the other three vertices of the quadrilateral so that the vectors $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of the quadrilateral, while $\mathbf{c}$ is its diagonal (the oriented segment $\mathbf{c}$ lies in the quadrilateral). The area of the quadrilateral is the sum of the area of the triangles with adjacent sides $\mathbf{a}$ and $\mathbf{c}$ and the area of the triangle with adjacent sides $\mathbf{b}$ and $\mathbf{c}$:

$$A = \frac{1}{2}\|\mathbf{c} \times \mathbf{a}\| + \frac{1}{2}\|\mathbf{c} \times \mathbf{b}\| \,.$$

Since the vectors **a**, **b** and **c** lie in a plane, the cross products $\mathbf{c}\times\mathbf{a}$ and $\mathbf{c}\times\mathbf{b}$ are perpendicular to the plane. Moreover, by the right-hand rule they point in the opposite directions (they are anti-parallel). Therefore the vectors $\mathbf{c}\times\mathbf{a}$ and $-(\mathbf{c}\times\mathbf{b}) = \mathbf{c}\times(-\mathbf{b})$ point in the same direction. The length of the sum of two parallel vectors is equal to the sum of their lengths. This validates the following equalities:

$$A = \frac{1}{2}\|\mathbf{c}\times\mathbf{a}\| + \frac{1}{2}\|\mathbf{c}\times(-\mathbf{b})\| = \frac{1}{2}\|\mathbf{c}\times\mathbf{a}+\mathbf{c}\times(-\mathbf{b})\|$$
$$= \frac{1}{2}\|\mathbf{c}\times(\mathbf{a}-\mathbf{b})\|\,.$$

The vector $\mathbf{a}-\mathbf{b}$ is another diagonal of the quadrilateral. Note that the oriented segment $\mathbf{a}-\mathbf{b}$ may not lie in the quadrilateral (as an example, sketch the quadrilaterals with $\mathbf{c} = -\mathbf{a}-\mathbf{b}$ or $\mathbf{c} = \frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$).

Corollary **4.3**. (Area of a quadrilateral)
Let **c** *and* **d** *be oriented segments connecting the opposite vertices of a quadrilateral. Then the area of the quadrilateral is*

$$A = \frac{1}{2}\|\mathbf{c}\times\mathbf{d}\|\,.$$

Alternatively, one can say that if c and d are the lengths of the diagonals of a quadrilateral and θ is the angle between them, then the area of the quadrilateral is $A = \frac{1}{2}cd\sin\theta$.

A parallelogram with adjacent sides **a** and **b** is a particular quadrilateral whose opposite sides are parallel. Let us verify Corollary **4.3** in this case. The diagonals of the parallelogram are $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{a}-\mathbf{b}$. Using the properties of the cross product

$$\begin{aligned}\mathbf{c}\times\mathbf{d} &= (\mathbf{a}+\mathbf{b})\times\mathbf{d} = \mathbf{a}\times\mathbf{d}+\mathbf{b}\times\mathbf{d}\\ &= \mathbf{a}\times(\mathbf{a}-\mathbf{b})+\mathbf{b}\times(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}\\ &= \mathbf{0}-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}+\mathbf{0} = -2(\mathbf{a}\times\mathbf{b})\end{aligned}$$

Therefore

$$\frac{1}{2}\|\mathbf{c}\times\mathbf{d}\| = \frac{1}{2}\,\|-2(\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\,|-2|\,\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\times\mathbf{b}\| = A$$

where the latter equality follows from the geometrical properties of the cross product. For a general quadrilateral, a diagonal **c** extended from the vertex at which the vectors **a** and **b** are adjacent sides is a linear combinations $\mathbf{c} = t\mathbf{a}+s\mathbf{b}$ where the numbers t and s are nonzero and uniquely determined (see Study Problem **2.1**). The numbers t and s must be both either positive or negative in order for **c** to be a diagonal of the quadrilateral. If $t > 0$ and $s < 0$ or $t < 0$ and $s > 0$, then $\mathbf{c}$ cannot be a diagonal and can only be viewed as a side of a quadrilateral adjacent to either $\mathbf{b}$ or $\mathbf{a}$, respectively, (or, either $\mathbf{a}$ or $\mathbf{b}$ becomes a diagonal), as follows from Study Problem **2.1**. So, the area of a quadrilateral can also be expressed in terms of t, s, and $\|\mathbf{a} \times \mathbf{b}\|$ as is illustrated in the following example.

EXAMPLE **4.5**. *Let* $\mathbf{a} = \langle 1, 2, -1\rangle$, $\mathbf{b} = \langle 2, 1, 1\rangle$ *and* $\mathbf{c} = t\mathbf{a} + s\mathbf{b}$, *where* t *and* s *are non-zero numbers, be position vectors of three vertices of a quadrilateral relative to one particular vertex so that* $\mathbf{c}$ *is a diagonal of the quadrilateral. Express the area* A *of the quadrilateral in terms of* t, s, *and* $\|\mathbf{a} \times \mathbf{b}\|$. *Put* $t = 3$ *and* $s = 1$ *and then calculate* A *for given* $\mathbf{a}$ *and* $\mathbf{b}$.

SOLUTION: The other diagonal of the quadrilateral is $\mathbf{d} = \mathbf{a} - \mathbf{b}$. Then by analogy with the case of a parallelogram:

$$\begin{aligned}
\mathbf{c} \times \mathbf{d} &= (t\mathbf{a} + s\mathbf{b}) \times \mathbf{d} = t(\mathbf{a} \times \mathbf{d}) + s(\mathbf{b} \times \mathbf{d}) \\
&= -t(\mathbf{a} \times \mathbf{b}) + s(\mathbf{b} \times \mathbf{a}) = -t(\mathbf{a} \times \mathbf{b}) - s(\mathbf{a} \times \mathbf{b}) \\
&= -(t + s)\mathbf{a} \times \mathbf{b} \\
A &= \frac{1}{2}\|\mathbf{c} \times \mathbf{d}\| = \frac{|t + s|}{2}\|\mathbf{a} \times \mathbf{b}\| \\
\mathbf{a} \times \mathbf{b} &= \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} = \langle 2 - (-1), -(1 - (-2)), 1 - 4\rangle \\
&= \langle 3, -3, -3\rangle = 3\langle 1, -1, -1\rangle \\
\|\mathbf{a} \times \mathbf{b}\| &= \|3\langle 1, -1, -1\rangle\| = 3\|\langle 1, -1, -1\rangle\| = 3\sqrt{3} \\
A &= \frac{|3 + 1|}{2}\, 3\sqrt{3} = 6\sqrt{3}\,.
\end{aligned}$$

□

4.4. Applications in physics. Torque. *Torque* or *moment of force* is the tendency of a force to rotate an object about an axis or a pivot. Just as a force is a push or a pull, a torque can be thought of as a twist. If $\mathbf{r}$ is the vector from a pivot point to the point where a force $\mathbf{F}$ is applied, then the torque is defined as the cross product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}\,.$$

The torque depends only on the component $\mathbf{F}_\perp$ of the force that is orthogonal to $\mathbf{r}$, i.e., $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_\perp$. If θ is the angle between $\mathbf{r}$ and $\mathbf{F}$, then the magnitude of the torque $\tau = \|\boldsymbol{\tau}\|$ is

$$\tau = \|\mathbf{r}\|\|\mathbf{F}_\perp\| = rF\sin\theta\,,$$

where $r = \|\mathbf{r}\|$ is the distance from the pivot point to the point where the force of magnitude F is applied. One can think of $\mathbf{r}$ as a lever attached to a pivot point and the force $\mathbf{F}$ is applied to the other end of the lever to rotate it about the pivot point. Naturally, the lever would not rotate if the force is parallel to it ($\theta = 0$ or $\theta = \pi$), whereas the maximal rotational effect is created when the force is applied in the direction perpendicular to the lever ($\theta = \pi/2$). The direction of $\boldsymbol{\tau}$ determines the axis about which the lever rotates. By the property of the cross product, this axis is perpendicular to the plane containing the force and position vectors. According to the right hand rule, the rotation occurs counterclockwise when viewed from the top of the torque vector. When driving a car, a torque is applied to the steering wheel to change the direction of the car. When a bolt is tightened by applying a force to a wrench, the produced turning effect is the torque.

An extended object is said to be *rigid* if the distance between any two its points remains constant in time regardless of external forces exerted on it. Let P be a fixed (pivot) point about which a rigid object can rotate. Suppose that the forces $\mathbf{F}_i$, $i = 1, 2, ..., n$, are applied to the object at the points whose position vectors relative to the point P are $\mathbf{r}_i$. The *Principle of Moments* states that a rigid object does not rotate about the point P if it was initially at rest and the total torque vanishes:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \cdots + \boldsymbol{\tau}_n = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \cdots + \mathbf{r}_n \times \mathbf{F}_n = \mathbf{0}$$

If, in addition, the total force vanishes $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}$, then a rigid object remains at rest and will not rotate about any other pivot point. Indeed, suppose that the torque about P vanishes and let $\mathbf{r}_0$ be a position vector of P relative to another point P_0. Then the position vectors of the points at which the forces are applied relative to the new pivot point P_0 are $\mathbf{r}_i + \mathbf{r}_0$. The total torque or the total moment of the forces about P_0 also vanishes:

$$\begin{aligned}\boldsymbol{\tau}_0 &= (\mathbf{r}_1 + \mathbf{r}_0) \times \mathbf{F}_1 + \cdots + (\mathbf{r}_n + \mathbf{r}_0) \times \mathbf{F}_n \\ &= \mathbf{r}_1 \times \mathbf{F}_1 + \cdots + \mathbf{r}_n \times \mathbf{F}_n + \mathbf{r}_0 \times (\mathbf{F}_1 + \cdots + \mathbf{F}_n) \\ &= \boldsymbol{\tau} + \mathbf{r}_0 \times \mathbf{F} = \mathbf{0}\end{aligned}$$

because by the hypothesis $\boldsymbol{\tau} = \mathbf{0}$ and $\mathbf{F} = \mathbf{0}$. *The conditions* $\boldsymbol{\tau} = \mathbf{0}$ *and* $\mathbf{F} = \mathbf{0}$ *comprise the fundamental law of statics for rigid objects.*

EXAMPLE 4.6. *The ends of rigid rods of length L_1 and L_2 are rigidly joined at the angle $\pi/2$. A ball of mass m_1 is attached to the free end of the rod of length L_1 and a ball of mass m_2 is attached to the free end of the rod of length L_2. The system is hanged by the joining point so that the system can rotate freely about it under the gravitational force. Find the equilibrium position of the system if the masses of the rods can be neglected as compared to the masses of the balls.*

SOLUTION: The gravitational forces have magnitudes $F_1 = m_1 g$ and $F_2 = m_2 g$ for the first and second ball, respectively (g is the free fall acceleration). They are directed downward and, therefore, lie in the plane that contains the position vectors of the balls relative to the pivot point. So the torques of the gravitational forces are orthogonal to this plane, and the equilibrium condition $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = \mathbf{0}$ is equivalent to $\tau_1 - \tau_2 = 0$ where $\tau_{1,2}$ are the magnitudes of the torques. The minus sign follows from the right hand rule by which the vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are parallel but have opposite directions. In other words, the gravitational forces applied to the balls generate opposite rotational moments. When the latter are equal in magnitude, the system is at rest.

In the plane that contains the system, let θ_1 and θ_2 be the smallest angles between the rods and a horizontal line. They determine the equilibrium position of the system. The objective is therefore to find these angles. Since the rods are perpendicular, $\theta_1+\theta_2 = \pi/2$. The angle between the position vector of the first ball and the gravitational force acting on it is $\phi_1 = \pi/2 - \theta_1$, and similarly $\phi_2 = \pi/2 - \theta_2$ is the angle between the position vector of the second ball and the gravitational force acting on it. Therefore $\tau_1 = L_1 F_1 \sin\phi_1$ and $\tau_2 = L_2 F_2 \sin\phi_2$. Owing to the identity $\sin(\pi/2 - \theta) = \cos\theta$, it follows that

$$\tau_1 = \tau_2 \quad \Leftrightarrow \quad m_1 L_1 \cos\theta_1 = m_2 L_2 \cos\theta_2 \quad \Leftrightarrow \quad \tan\theta_1 = \frac{m_1 L_1}{m_2 L_2}$$

where the relation $\theta_2 = \pi/2 - \theta_1$ has been used. □

4.5. Study Problems.

Problem 4.1. *Find the most general vector* $\mathbf{r}$ *that satisfies the equations* $\mathbf{a}\cdot\mathbf{r} = 0$ *and* $\mathbf{b}\cdot\mathbf{r} = 0$*, where* $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero, nonparallel vectors. Give a geometrical interpretation of all vectors satisfying these conditions.*

SOLUTION: The conditions imposed on $\mathbf{r}$ hold if and only if the vector $\mathbf{r}$ is orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$. Therefore, it must be parallel to their cross product. Thus,

$$\mathbf{r} = t(\mathbf{a}\times\mathbf{b})\,, \qquad -\infty < t < \infty\,.$$

If $\mathbf{r}$ is a position vector of a point relative to a fixed point O in space, then the vectors $\mathbf{r} = t(\mathbf{a}\times\mathbf{b})$, $-\infty < t < \infty$, are position vectors of all points in the line through O parallel to the vector $\mathbf{a}\times\mathbf{b}$. □

Problem **4.2**. *Use geometrical means to find the cross products of the unit vectors parallel to the coordinate axes.*

SOLUTION: Consider $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2$. Since $\hat{\mathbf{e}}_1\perp\hat{\mathbf{e}}_2$ and $\|\hat{\mathbf{e}}_1\| = \|\hat{\mathbf{e}}_2\| = 1$, their cross product must be a unit vector perpendicular to both $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. There are only two such vectors, $\pm\hat{\mathbf{e}}_3$. By the right-hand rule,

$$\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3\,.$$

Similarly, the other cross products are shown to be obtained by cyclic permutations of the indices 1, 2, and 3 in the above relation. A permutation of any two indices leads to a change in sign (e.g., $\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3$). Since a cyclic permutation of three indices $\{ijk\}\to\{kij\}$ (and so on) consists of two permutations of any two indices, the relation between the unit vectors can be cast in the form

$$\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k\,, \quad \{ijk\} = \{123\} \text{ and cyclic permutations.}$$

□

Problem **4.3**. *Prove the "bac − cab" rule* (**4.2**).

SOLUTION: If $\mathbf{c}$ and $\mathbf{b}$ are parallel, $\mathbf{b} = s\mathbf{c}$ for some real s, then the relation is true because both its sides vanish. If $\mathbf{c}$ and $\mathbf{b}$ are not parallel, then by the remark after Corollary **4.1** the double cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ depends only on the component of $\mathbf{a}$ that is orthogonal to $\mathbf{b}\times\mathbf{c}$. This component lies in the plane containing $\mathbf{b}$ and $\mathbf{c}$ and hence is a linear combination of them (see Study Problem **2.1**). So without loss of generality,

$$\mathbf{a} = t\mathbf{b} + p\mathbf{c}\,.$$

Also, using the orthogonal decomposition of $\mathbf{c}$ relative to $\mathbf{b}$:

$$\mathbf{b}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}_\perp\,, \quad \mathbf{c}_\perp = \mathbf{c} - s\mathbf{b}\,, \quad s = \frac{\mathbf{c}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\,,$$

where $\mathbf{c}_\perp$ is the component of $\mathbf{c}$ orthogonal to $\mathbf{b}$ (note $\mathbf{b}\cdot\mathbf{c}_\perp = 0$). The vectors $\mathbf{b}$, $\mathbf{c}_\perp$ and $\mathbf{b}\times\mathbf{c}_\perp$ are mutually orthogonal and oriented according to the right hand rule. In particular, $\|\mathbf{b}\times\mathbf{c}\| = \|\mathbf{b}\|\|\mathbf{c}_\perp\|$. By applying the right hand rule twice, it is concluded that $\mathbf{b}\times(\mathbf{b}\times\mathbf{c}_\perp)$

has the direction opposite to $\mathbf{c}_\perp$. Since $\mathbf{b}$ and $\mathbf{b} \times \mathbf{c}_\perp$ are orthogonal, $\|\mathbf{b} \times (\mathbf{b} \times \mathbf{c}_\perp)\| = \|\mathbf{b}\|\|\mathbf{b} \times \mathbf{c}_\perp\| = \|\mathbf{b}\|^2\|\mathbf{c}_\perp\|$. Therefore

$$\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{c}_\perp\|\mathbf{b}\|^2 = \mathbf{b}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{b} \cdot \mathbf{b})$$

By swapping the vector $\mathbf{b}$ and $\mathbf{c}$ in this equation, one also obtains

$$\mathbf{c} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}_\perp\|\mathbf{c}\|^2 = -\mathbf{c}(\mathbf{c} \cdot \mathbf{b}) + \mathbf{b}(\mathbf{c} \cdot \mathbf{c})$$

It follows from these relations that

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= t\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) + p\mathbf{c} \times (\mathbf{b} \times \mathbf{c}) \\ &= \mathbf{b}[(t\mathbf{b} + p\mathbf{c}) \cdot \mathbf{c}] - \mathbf{c}[(t\mathbf{b} + p\mathbf{c}) \cdot \mathbf{b}] \\ &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})\end{aligned}$$

□

Problem 4.4. *Prove the Jacobi identity* (**4.3**)

SOLUTION: By the "bac-cab" rule (**4.2**) applied to each term,

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) &= \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \\ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{c})\end{aligned}$$

By adding these equalities, it is easy to see that the coefficients at each of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in the right side are added up to make 0. □

Problem 4.5. *Consider all vectors in a plane. Any such vector* $\mathbf{a}$ *can be uniquely determined by specifying its length* $a = \|\mathbf{a}\|$ *and the angle* θ_a *that is counted from the positive* x *axis toward the vector* $\mathbf{a}$ *(i.e.,* $0 \leq \theta_a < 2\pi$*). The relation* $\langle a_1, a_2\rangle = \langle a\cos\theta_a, a\sin\theta_a\rangle$ *establishes a one-to-one correspondence between ordered pairs* (a_1, a_2) *and* (a, θ_a)*. Define the vector product of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *as the vector* $\mathbf{c}$ *for which* $c = ab$ *and* $\theta_c = \theta_a + \theta_b$*. Show that, in contrast to the cross product, this product is associative and commutative, that is, that* $\mathbf{c}$ *does not depend on the order of vectors in the product.*

SOLUTION: Let us denote the vector product by a small circle to distinguish it from the dot and cross products, $\mathbf{a} \circ \mathbf{b} = \mathbf{c}$. Since $\mathbf{c} = \langle ab\cos(\theta_a + \theta_b), ab\sin(\theta_a + \theta_b)\rangle$, the commutativity of the vector product $\mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$ follows from the commutativity of the product and addition of numbers: $ab = ba$ and $\theta_a + \theta_b = \theta_b + \theta_a$. Similarly, the associativity of the vector product $(\mathbf{a} \circ \mathbf{b}) \circ \mathbf{c} = \mathbf{a} \circ (\mathbf{b} \circ \mathbf{c})$ follows from the associativity of the product and addition of ordinary numbers: $(ab)c = a(bc)$ and $(\theta_a + \theta_b) + \theta_c = \theta_a + (\theta_b + \theta_c)$. □

Remark. The vector product introduced for vectors in a plane is known as the *product of complex numbers* which can be viewed as two-dimensional vectors. It is interesting to note that no commutative and associative vector product (i.e., "vector times vector = vector") can be defined in a Euclidean space of more than two dimensions.

Problem 4.6. *Let* $\mathbf{u}$ *be a vector rotating in the xy plane about the z axis. Given a vector* $\mathbf{v}$*, find the position of* $\mathbf{u}$ *such that the magnitude of the cross product* $\mathbf{v} \times \mathbf{u}$ *is maximal.*

SOLUTION: For any two vectors,

$$\|\mathbf{v} \times \mathbf{u}\| = \|\mathbf{v}\|\|\mathbf{u}\| \sin\theta\,,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$. The magnitude of $\mathbf{v}$ is fixed, while the magnitude of $\mathbf{u}$ does not change when rotating. Therefore, the absolute maximum of the cross-product magnitude is reached when

$$\sin\theta = 1 \quad \Rightarrow \quad \cos\theta = 0 \quad \Rightarrow \quad \mathbf{v} \perp \mathbf{u}\,,$$

i.e., when the vectors are orthogonal. The corresponding algebraic condition is

$$\mathbf{v} \cdot \mathbf{u} = 0\,.$$

Since $\mathbf{u}$ is rotating in the xy plane, its components are

$$\mathbf{u} = \langle \|\mathbf{u}\|\cos\phi, \|\mathbf{u}\|\sin\phi, 0\rangle\,, \quad 0 \le \phi < 2\pi\,,$$

where ϕ is the angle counted counterclockwise from the x axis toward the current position of $\mathbf{u}$. Put $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then the direction of $\mathbf{u}$ is determined by the equation

$$\mathbf{v} \cdot \mathbf{u} = \|\mathbf{u}\|(v_1\cos\phi + v_2\sin\phi) = 0 \quad \Rightarrow \quad \tan\phi = -\frac{v_1}{v_2}\,.$$

This equation has two solutions in the range $0 \le \phi < 2\pi$:

$$\phi = -\tan^{-1}(v_1/v_2) \quad \text{and} \quad \phi = -\tan^{-1}(v_1/v_2) + \pi\,.$$

Geometrically, these solutions correspond to the case when $\mathbf{u}$ is parallel to the line $v_2 y + v_1 x = 0$ in the xy plane. □

4.6. Exercises.

1–7. Find the cross product $\mathbf{a} \times \mathbf{b}$ for the given vectors $\mathbf{a}$ and $\mathbf{b}$.

1. $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 0, 1\rangle$
2. $\mathbf{a} = \langle 1, -1, 2\rangle$ and $\mathbf{b} = \langle 3, -2, 1\rangle$
3. $\mathbf{a} = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$ and $\mathbf{b} = 3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$
4. $\mathbf{a} = 2\mathbf{c} - \mathbf{d}$, $\mathbf{b} = 3\mathbf{c} + 4\mathbf{d}$ where $\mathbf{c} \times \mathbf{d} = \langle 1, 2, 3\rangle$.

5. $\mathbf{a} = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$ and $\mathbf{b} = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$ where the vectors $\mathbf{u}_i$, $i = 1, 2, 3$, are mutually orthogonal, have the same length 3, and $\mathbf{u}_1 \times \mathbf{u}_2 = 3\mathbf{u}_3$. Express the answer as a linear combination of $\mathbf{u}_i$ (not their cross products).
6. $\mathbf{a}$ has length 3 units, lies in the xy plane, and points from the origin to the first quadrant at the angle $\pi/3$ to the x axis and $\mathbf{b}$ has length 2 units and points from the origin in the direction of the z axis.
7. $\mathbf{a}$ and $\mathbf{b}$ point from the origin to the second and first quadrant of the xy plane, respectively, so that $\mathbf{a}$ makes the angle $15°$ with the y axis and $\mathbf{b}$ makes the angle $75°$ with the x axis, and $\|\mathbf{a}\| = 2$, $\|\mathbf{b}\| = 3$.

8. Let $\mathbf{a} = \langle 3, 2, 1\rangle$, $\mathbf{b} = \langle -2, 1, -1\rangle$, and $\mathbf{c} = \langle 1, 0, -1\rangle$. Find $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$, and $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$. Verify the Jacobi identity.

9. Let $\mathbf{a}$ be a unit vector orthogonal to $\mathbf{b}$ and $\mathbf{c}$. If $\mathbf{c} = \langle 1, 2, 2\rangle$, find the length of the vector $\mathbf{a} \times [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})]$

10. Given two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$, show that the vectors $\mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ are mutually orthogonal, and, hence, form an orthogonal basis in space.

11. Suppose $\mathbf{a}$ lies in the xy plane, its initial point is at the origin, and its terminal point is in first quadrant of the xy plane. Let $\mathbf{b}$ be parallel to $\hat{\mathbf{e}}_3$. Use the right-hand rule to determine whether the angle between $\mathbf{a} \times \mathbf{b}$ and the unit vectors parallel to the coordinate axes lies in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$ or equals $\pi/2$.

12. If vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ have the initial point at the origin and lie, respectively, in the positive quadrants of the xy, yz, and xz planes, determine the octants in which the pairwise cross products of these vectors lie by specifying the signs of the components of the cross products.

13. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar vectors, find $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})$.

14. Find a unit vector perpendicular to the vectors $\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3$.

15. Find the area of a triangle whose vertices lie on the different coordinate axes at distances a, b, and c from the origin.

16. Find the area of a triangle ABC for $A(1, 0, 1)$, $B(1, 2, 3)$, and $C(0, 1, 1)$ and a nonzero vector perpendicular to the plane containing the triangle.

17. Use the cross product to show that the area of the triangle whose vertices are midpoints of the sides of a triangle with area A is $A/4$. *Hint*: Define sides of the triangle of area A as vectors and express the sides of the other triangle in question in terms of these vectors.

18. Consider a triangle whose vertices are midpoints of any three sides of a parallelogram. If the area of the parallelogram is A, find the area of the triangle. *Hint*: Define adjacent sides of the parallelogram as vectors and express the sides of the triangle in terms of these vectors.

19. Let $A = (1, 2, 1)$ and $B = (-1, 0, 2)$ be vertices of a parallelogram. If the other two vertices are obtained by moving A and B along straight lines by a distance of 3 units in the direction of the vector $\mathbf{a} = \langle 2, 1, -2\rangle$, find the area of the parallelogram.

20. Consider four points in space. Suppose that the coordinates of the points are known. Describe a procedure based on the properties of the cross product to determine whether the points are in one plane. In particular, are the points $(1, 2, 3)$, $(-1, 0, 1)$, $(1, 3, -1)$, and $(0, 1, 2)$ in one plane?

21. Let the sides of a triangle have lengths a, b, and c and let the angles at the vertices opposite to the sides a, b, and c be, respectively, α, β, and γ. Prove that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Hint: Define the sides as vectors and express the area of the triangle in terms of the vectors adjacent at each vertex of the triangle.

22. A polygon $ABCD$ in a plane is a part of the plane bounded by four straight line segments AB, BC, CD, and DA. Suppose that the polygon is *convex*, that is, a straight line segment connecting any two points in the polygon lies in the polygon. If the coordinates of the vertices are specified, describe the procedure based on vector algebra to calculate the area of the polygon. In particular, let a convex polygon be in the xy plane and put $A = (0, 0)$, $B = (x_1, y_1)$, $C = (x_2, y_2)$, and $D = (x_3, y_3)$. Express the area in terms of x_i and y_i, $i = 1, 2, 3$.

23. Consider a parallelogram. Construct another parallelogram whose adjacent sides are diagonals of the first parallelogram. Find the relation between the areas of the parallelograms.

24. Given two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$, show that any vector $\mathbf{r}$ in space can be written as a linear combination $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{a}\times\mathbf{b}$ and that the numbers x, y, and z are unique for every $\mathbf{r}$. Express z in terms of $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$. In particular, put $\mathbf{a} = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 1, 1, 0\rangle$. Find the coefficients x, y, and z for $\mathbf{r} = \langle 1, 2, 3\rangle$. *Hint:* See Study Problems **4.1** and **2.1**.

25. A tetrahedron is a solid with four vertices and four triangular faces. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces and directions perpendicular to the faces and pointing outward.

Show that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. *Hint*: Set up vectors being the egdes of a tetrahedron. There are six edges. So all these vectors can be expressed as linear combinations of particular three non-coplanar vectors. Use the cross product to find the vectors $\mathbf{v}_j$, $j = 1, 2, 3, 4$, in terms of the three non-coplanar vectors.

26. If $\mathbf{a}$ is a non-zero vector, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

27. Given two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$, construct three mutually orthogonal unit vectors $\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, one of which is parallel to $\mathbf{a}$. Are such unit vectors unique? In particular, put $\mathbf{a} = \langle 1, 2, 2\rangle$ and $\mathbf{b} = \langle 1, 0, 2\rangle$ and find $\hat{\mathbf{u}}_i$.

28. Find the area of a quadrilateral $ACDB$ if $A = (1, 0, -1)$, $B = (2, 1, 2)$, $C = (0, 1, 2)$, and $\overrightarrow{AD} = 2\overrightarrow{AC} - \overrightarrow{AB}$.

29. Find the area of a quadrilateral $ABCD$ whose vertices are obtained as follows. The vertex B is the result of moving A by a distance of 6 units along the vector $\mathbf{u} = \langle 2, 1, -2\rangle$, C is obtained from B by moving the latter by a distance of 5 units along the vector $\mathbf{v} = \langle -3, 0, 4\rangle$, and $\overrightarrow{CD} = \mathbf{v} - \mathbf{u}$.

30. Let $\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, be an orthonormal basis in space with the property that $\hat{\mathbf{u}}_3 = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$. If a_1, a_2, and a_3 are the components of vector $\mathbf{a}$ relative to this basis and b_1, b_2, and b_3 are the components of $\mathbf{b}$, show that the components of the cross product $\mathbf{a} \times \mathbf{b}$ can also be computed by the determinant rule given in Definition **4.3** where $\hat{\mathbf{e}}_i$ are replaced by $\hat{\mathbf{u}}_i$. *Hint*: Use the "bac-cab" rule to find all pairwise cross products of the basis vectors $\hat{\mathbf{u}}_i$.

31. Let the angle between the rigid rods in Example **4.6** be $0 < \varphi < \pi$. Find the equilibrium position of the system.

31. Two rigid rods of the same length are rigidly attached to a ball of mass m so that the angle between the rods is $\pi/2$. A ball of mass $2m$ is attached to one of the free ends of the system. The remaining free end is used to hang the system. Find the angle between the rod connecting the pivot point and the ball of mass m and the vertical axis along which the gravitational force is acting. Assume that the masses of the rods can be neglected as compared to m.

33. Three rigid rods of the same length are rigidly joined by one end so that the rods lie in a plane and the other end of each rod is free. Let three balls of masses m_1, m_2, and m_3 are attached to the free ends of the rods. The system is hanged by the joining point and can rotate freely about it. Assume that the masses of the rods can be neglected as compared to the masses of the balls. Find the angles between the rods at which the system remains in a horizontal plane under gravitational forces acting vertically.

5. The Triple Product

DEFINITION 5.1. *The* triple product *of three vectors* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *is a number obtained by the rule:* $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$*.*

It follows from the algebraic definition of the cross product and the definition of the determinant of a 3×3 matrix that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = a_1 \det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3\end{pmatrix} - a_2 \det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3\end{pmatrix} + a_3 \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2\end{pmatrix}$$
$$= \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{pmatrix}.$$

This provides a convenient way to calculate the numerical value of the triple product. If two rows of a matrix are swapped, then its determinant changes sign. Therefore, *the sign of the triple product changes under swapping any two vectors in it*:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})\,.$$

This means, in particular, that the absolute value of the triple product is independent of the order of the vectors in the triple product. Also, the value of *the triple product is invariant under cyclic permutations of vectors in it*:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\,,$$

because a cyclic permutation is obtained by swapping vectors in two different pairs.

5.1. Geometrical Significance of the Triple Product. Suppose that $\mathbf{b}$ and $\mathbf{c}$ are not parallel (otherwise, $\mathbf{b}\times\mathbf{c}=\mathbf{0}$). Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$ as shown in Figure 5.1 (left panel). If $\mathbf{a}\perp\mathbf{b}\times\mathbf{c}$ (i.e., $\theta=\pi/2$), then the triple product vanishes. Let $\theta\neq\pi/2$. Consider parallelograms whose adjacent sides are pairs of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. They enclose a non-rectangular box whose edges are the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

A box with parallelogram faces is called a *parallelepiped* with adjacent sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The cross product $\mathbf{b}\times\mathbf{c}$ is orthogonal to the face containing the vectors $\mathbf{b}$ and $\mathbf{c}$, whereas $A=\|\mathbf{b}\times\mathbf{c}\|$ is the area of this face of the parallelepiped (the area of the parallelogram with adjacent sides $\mathbf{b}$ and $\mathbf{c}$). By the geometrical property of the dot product,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = A\|\mathbf{a}\|\cos\theta\,.$$

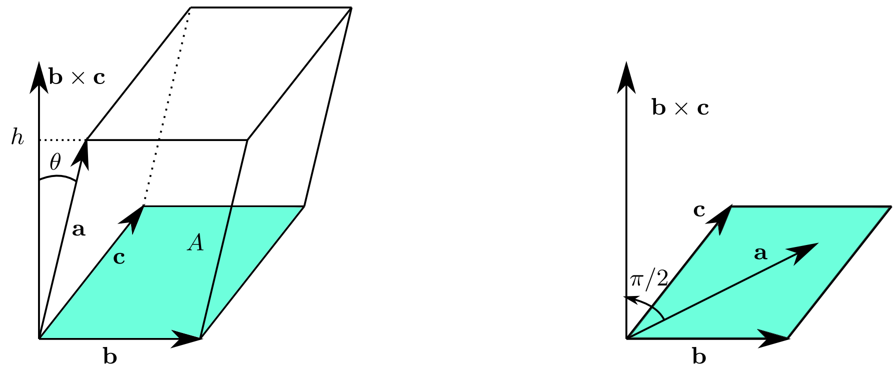


FIGURE 5.1. **Left**: Geometrical interpretation of the triple product as the volume of the parallelepiped whose adjacent sides are the vectors in the product: $h = \|\mathbf{a}\|\cos\theta$, $A = \|\mathbf{b}\times\mathbf{c}\|$, $V = hA = \|\mathbf{a}\|\,\|\mathbf{b}\times\mathbf{c}\|\cos\theta = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. Here $0 < \theta < \pi/2$ and, hence, $\cos\theta > 0$. If the terminal point of $\mathbf{a}$ lies below the plane containing $\mathbf{b}$ and $\mathbf{c}$ (not shown in the figure), then $\theta > \pi/2$ and $\cos\theta < 0$. In this case, $h = -\|\mathbf{a}\|\cos\theta$ and $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.
Right: Test for the coplanarity of three vectors. Three vectors are coplanar if and only if their triple product vanishes: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

On the other hand, the distance between the two faces parallel to both $\mathbf{b}$ and $\mathbf{c}$ (or the height of the parallelepiped) is $h = \|\mathbf{a}\|\cos\theta$ if $\theta < \pi/2$ and $h = -\|\mathbf{a}\|\cos\theta$ if $\theta > \pi/2$ or,

$$h = \|\mathbf{a}\||\cos\theta|\,.$$

The volume of the parallelepiped is $V = Ah$. This proves to the following theorem.

THEOREM 5.1. (Geometrical Significance of the Triple Product). *The volume V of a parallelepiped whose adjacent sides are the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the absolute value of their triple product:*

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|\,.$$

Thus, the triple product is a convenient algebraic tool for calculating volumes. Note also that the vectors can be taken in any order in the triple product to compute the volume because the triple product only changes its sign when two vectors are swapped in it.

EXAMPLE 5.1. *Find the volume of a parallelepiped with adjacent sides $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle -2, 0, 1\rangle$, and $\mathbf{c} = \langle 2, 1, 2\rangle$.*

SOLUTION: The expansion of the determinant over the first row yields

$$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = \det\begin{pmatrix} -2 & 0 & 1\\ 1 & 2 & 3\\ 2 & 1 & 2\end{pmatrix} = -2(4-3)+1(1-4) = -5$$

Taking the absolute value of the triple product, the volume is obtained, $V = |-5| = 5$. The components of the vectors must be given in the same units of length, e.g., meters. Then the volume is 5 cubic meters. □

In Section **3.4**, it is stated that three nonzero linearly independent vectors are not coplanar (see Definition **3.3**) and form a basis in space. Simple algebraic criteria for three vectors to be either coplanar or linearly independent can be deduced from Theorem **5.1**.

COROLLARY **5.1**. (Test for three vectors to be coplanar). *Three vectors are coplanar if and only if their triple product vanishes:*

$$\mathbf{a},\ \mathbf{b},\ \mathbf{c} \text{ are coplanar} \quad\Longleftrightarrow\quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0.$$

Indeed, if the vectors are coplanar (Figure **5.1**, right panel), then the cross product of any two vectors must be perpendicular to the plane where the vectors are and therefore the triple product vanishes. If, conversely, the triple product vanishes, then either $\mathbf{b}\times\mathbf{c} = \mathbf{0}$ or $\mathbf{a}\perp\mathbf{b}\times\mathbf{c}$. In the former case, $\mathbf{b}$ is parallel to $\mathbf{c}$, or $\mathbf{c} = t\mathbf{b}$, and hence $\mathbf{a}$ always lies in a plane with $\mathbf{b}$ and $\mathbf{c}$. In the latter case, all three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are perpendicular to $\mathbf{b}\times\mathbf{c}$ and therefore must be in one plane (perpendicular to $\mathbf{b}\times\mathbf{c}$).

COROLLARY **5.2**. (Test for three vectors to form a basis in space) *Three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent and, hence, form a basis in space if and only if their triple product does not vanish.*

Given three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, for every vector $\mathbf{a}$ one can find unique numbers s_1, s_2, and s_3 such that

$$\mathbf{a} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_3$$

if and only if $\mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3)\neq 0$.

EXAMPLE **5.2**. *Determine whether the points $A = (1,1,1)$, $B = (2,0,2)$, $C = (3,1,-1)$, and $D = (0,2,3)$ are in the same plane.*

SOLUTION: Consider the vectors $\mathbf{a} = \overrightarrow{AB} = \langle 1,-1,1\rangle$, $\mathbf{b} = \overrightarrow{AC} = \langle 2,0,-2\rangle$, and $\mathbf{c} = \overrightarrow{AD} = \langle -1,1,2\rangle$. The points in question are in the same plane if and only if the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar, or

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$ by Corollary **5.1**. The evaluation of the triple product yields

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{pmatrix} = 1(0+2)+1(4-2)+1(2-0) = 6 \neq 0\,.$$

Therefore, the points are not in the same plane. □

EXAMPLE **5.3**. *Can the vector* $\mathbf{a} = \langle 1,1,1\rangle$ *be represented as a linear combination of the vectors* $\mathbf{u}_1 = \langle 1,2,3\rangle$, $\mathbf{u}_2 = \langle 2,1,-6\rangle$, *and* $\mathbf{u}_3 = \langle 1,1,-1\rangle$*?*

SOLUTION: Any vector in space is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ if they form a basis. Let us verify first whether or not they form a basis. By Corollary **5.2**,

$$\mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3) = \det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -6 \\ 1 & 1 & -1 \end{pmatrix} = 1(-1+6)-2(-2+6)+3(2-1) = 0$$

these vectors do not form a basis. By Corollary **5.1**, they are coplanar. Note that $\mathbf{u}_3 = \frac{1}{3}(\mathbf{u}_1+\mathbf{u}_2)$. Therefore only if the vector $\mathbf{a}$ lies in the same plane as the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, then $\mathbf{a}$ is a linear combination of two non-parallel vectors, say, $\mathbf{u}_1$ and $\mathbf{u}_3$. Since the following triple product does not vanish,

$$\mathbf{a}\cdot(\mathbf{u}_1\times\mathbf{u}_3) = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = 1(-2-3)-1(-1-3)+1(1-2) = -2,$$

the vector $\mathbf{a}$ is not coplanar with the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Therefore $\mathbf{a}$ is not a linear combination of them. □

EXAMPLE **5.4**. *Can the points* $A = (1,0,1)$, $B = (4,3,4)$, $C = (1,3,1)$, *and* $D = (-2,3,-2)$ *be vertices of a quadrilateral? If so, find the areas of the quadrilaterals* $ABCD$ *and* $ABDC$*. The order of letters specifies the adjacent sides at each vertex. For example, at the vertex* B *the sides* BC *and* BD *are adjacent in the quadrilateral* $ABCD$*, while in* $ABDC$*, the adjacent sides at* B *are* BA *and* BD*.*

SOLUTION: With a four points in a plane one can construct a quadrilateral. So, the first objective is to determine whether the given points are in a plane. As in Example **5.2**, put $\mathbf{a} = \overrightarrow{AB} = \langle 3,3,3\rangle$, $\mathbf{b} = \overrightarrow{AC} =$

$\langle 0,3,0\rangle$ and $\mathbf{c} = \overrightarrow{AD} = \langle -3,3,-3\rangle$. Then

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \det\begin{pmatrix} 0 & 3 & 0 \\ -3 & 3 & -3 \\ 3 & 3 & 3 \end{pmatrix} = -3(-9+9) = 0\,.$$

By Corollary **5.1** the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar, and, hence, the points A, B, C, and D are in one plane and can be vertices of a quadrilateral.

The diagonals in the quadrilateral $ABCD$ are $\mathbf{u} = \overrightarrow{AC} = \langle 0,3,0\rangle$ and $\mathbf{v} = \overrightarrow{BD} = \langle -6,0,-6\rangle$ By Corollary **4.3**, the area is

$$A_{abcd} = \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\|\langle -18,0,18\rangle\| = \frac{18}{2}\|\langle -1,0,1\rangle\| = 9\sqrt{2}\,.$$

The diagonals in the quadrilateral $ABDC$ are $\mathbf{u} = \overrightarrow{AD} = \langle -3,3,-3\rangle$ and $\mathbf{v} = \overrightarrow{BC} = \langle -3,0,-3\rangle$. By Corollary **4.3**, the area is

$$A_{abdc} = \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\|\langle -9,0,9\rangle\| = \frac{9}{2}\|\langle -1,0,1\rangle\| = \frac{9\sqrt{2}}{2}\,.$$

□

5.2. Right- and left-handed coordinate systems. In Section **3.4** it was shown that *with any triple of mutually orthogonal unit vectors* $\hat{\mathbf{u}}_i$, $i = 1,2,3$, *one can associate a rectangular coordinate system.* Any vector in space is uniquely expanded into the sum

$$\mathbf{r} = x\hat{\mathbf{u}}_1 + y\hat{\mathbf{u}}_2 + z\hat{\mathbf{u}}_3\,.$$

where the ordered triple of numbers (x,y,z) is determined by scalar projections of $\mathbf{r}$ onto $\hat{\mathbf{u}}_i$. If $\mathbf{r}$ is a position vector of a point P relative to a particular point O, $\mathbf{r} = \overrightarrow{OP}$, then (x,y,z) are coordinates of P relative to the rectangular coordinate system with the origin at O and whose axes are oriented parallel to the vectors $\hat{\mathbf{u}}_i$.

The vector $\hat{\mathbf{u}}_1\times\hat{\mathbf{u}}_2$ must be parallel to $\hat{\mathbf{u}}_3$ because the latter is orthogonal to both $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$. Furthermore, owing to the orthogonality of $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$,

$$\|\hat{\mathbf{u}}_1\times\hat{\mathbf{u}}_2\| = \|\hat{\mathbf{u}}_1\|\|\hat{\mathbf{u}}_2\| = 1 \quad\Rightarrow\quad \hat{\mathbf{u}}_1\times\hat{\mathbf{u}}_2 = \pm\hat{\mathbf{u}}_3\,.$$

Consequently,

$$\hat{\mathbf{u}}_3\cdot(\hat{\mathbf{u}}_1\times\hat{\mathbf{u}}_2) = \hat{\mathbf{u}}_1\cdot(\hat{\mathbf{u}}_2\times\hat{\mathbf{u}}_3) = \pm 1$$

or, owing to the mutual orthogonality of the vectors, $\hat{\mathbf{u}}_2\times\hat{\mathbf{u}}_3 = \pm\hat{\mathbf{u}}_1$. So all rectangular coordinate systems are divided into two classes. A coordinate system is called *right-handed* if $\hat{\mathbf{u}}_1 \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) = 1$ and a coordinate system for which $\hat{\mathbf{u}}_1 \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) = -1$ is called *left-handed*:

$$\hat{\mathbf{u}}_1 \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) = \begin{cases} 1\,, & \text{right}-\text{handed systems} \\ -1\,, & \text{left}-\text{handed systems} \end{cases}$$

The coordinate system associated with the standard basis $\hat{\mathbf{u}}_1 = \hat{\mathbf{e}}_1 = \langle 1, 0, 0\rangle$, $\hat{\mathbf{u}}_2 = \hat{\mathbf{e}}_2 = \langle 0, 1, 0\rangle$, and $\hat{\mathbf{u}}_3 = \hat{\mathbf{e}}_3 = \langle 0, 0, 1\rangle$ is *right-handed* because $\hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = 1$.

A right-handed system can be visualized as follows. With the thumb, index, and middle fingers of the *right hand* at right angles to each other (with the index finger pointed straight), the middle finger points in the direction of $\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3$ when the thumb represents $\hat{\mathbf{u}}_2$ and the index finger represents $\hat{\mathbf{u}}_3$. A left-handed system is obtained by the reflection $\hat{\mathbf{u}}_1 \to -\hat{\mathbf{u}}_1$ and therefore is visualized by the fingers of the *left hand* in the same way. Since the dot product cannot be changed by rotations and translations in space, the handedness of the coordinate system does not change under simultaneous rotations and translations of the triple of vectors $\hat{\mathbf{u}}_i$ (three mutually orthogonal fingers of the left hand cannot be made pointing in the same direction as the corresponding fingers of the right hand by any rotation of the hand). The reflection of all three vectors $\hat{\mathbf{u}}_i \to -\hat{\mathbf{u}}_i$ or just one of them turns a right-handed system into a left-handed one and vice versa. A mirror reflection of a right-handed system is the left-handed one.

Remark. Under a reflection of a basis $\hat{\mathbf{u}}_i \to -\hat{\mathbf{u}}_i$, components of a vector change the sign: $\mathbf{a} \to -\mathbf{a}$. The cross product of two vectors does not change under the reflection: $(-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{a} \times \mathbf{b}$. For this reason, the cross product is often called a *pseudo* vector. In physics, a transformation under which left and right coordinate systems are swapped is called a *parity transformation*. Laws of physics are formulated in terms of vectors associated with physical quantities like forces, velocities, etc. It turns out, the fundamental laws of physics are not invariant under the parity transformation. In other words, the laws of nature are different in right- and left-handed coordinate systems. There are observable processes in subatomic systems (the so called *weak* interactions) which would be different in our universe and its mirror copy.

5.3. Distances Between Lines and Planes. In Sections **1.3** and **1.5**, a geometrical description of lines and planes was given as point sets in space. If the lines or planes in space are not intersecting, then how can one find the distance between them? This question can be answered using the geometrical properties of the triple and cross products (Theorems

4.1 and **5.1**). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sets of points in space. Let a point A_1 belong to $\mathcal{S}_1$, let a point A_2 belong to $\mathcal{S}_2$, and let $|A_1A_2|$ be the distance between them.

Definition **5.2**. (Distance Between Sets in Space).
The distance D between two sets of points in space, $\mathcal{S}_1$ and $\mathcal{S}_2$, is the largest number that is less than or equal to all the numbers $|A_1A_2|$ when the point A_1 ranges over $\mathcal{S}_1$ and the point A_2 ranges over $\mathcal{S}_2$.

Naturally, if the sets have at least one common point, the distance between them vanishes. If $\mathcal{S}_1$ is a closed interval $[0,1]$ on, say, the x axis and $\mathcal{S}_2$ is $[2,3]$ on the same axis, then $\mathcal{S}_1$ and $\mathcal{S}_2$ do not have common points. The distance between them is the smallest number $x_2 - x_1$ where $2 \le x_2 \le 3$ and $0 \le x_1 \le 1$ which is apparently 1 (the minimum is attained when $x_2 = 2$ and $x_1 = 1$).

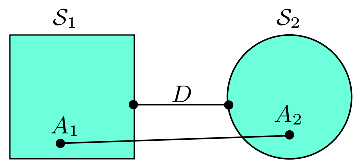
The distance between sets may vanish even if the sets have no common points. For example, let $\mathcal{S}_1$ be an open interval $(0,1)$ on the x axis, while $\mathcal{S}_2$ is the interval $(1,2)$ on the same axis. Apparently, the sets have no common points (the point $x = 1$ does not belongs to either of them). The distance is the largest number D such that $D \le |x_1 - x_2|$, where $0 < x_1 < 1$ and $1 < x_2 < 2$. The value of $|x_1 - x_2| > 0$ can be made smaller than any preassigned positive number by taking x_1 and x_2 close enough to 1. Since the distance $D \ge 0$, the only possible value is $D = 0$.

Intuitively, the sets are separated by a single point that is not an "extended" object, and hence the distance between them should vanish. In other words, there are situations in which the minimum of $|A_1A_2|$ is not attained for some $A_1 \in \mathcal{S}_1$, or some $A_2 \in \mathcal{S}_2$, or both. Nevertheless, the distance between the sets is still well (uniquely) defined as the largest number that is less than or equal to all numbers $|A_1A_2|$. Such a number is called the *infimum* of the set of numbers $|A_1A_2|$ and denoted $\inf |A_1A_2|$. Thus,

$$D = \inf |A_1A_2| \,, \quad A_1 \in \mathcal{S}_1 \,, \quad A_2 \in \mathcal{S}_2 \,.$$

The notation $A_1 \in \mathcal{S}_1$ stands for "a point A_1 belongs to the set $\mathcal{S}_1$," or simply "A_1 is an element of $\mathcal{S}_1$." The definition is illustrated in Figure **5.2** (left panel).

In Section **1.5** the distance between parallel planes is defined as the length of a straight line segment between two points of intersection of the planes with a line perpendicular to them. This definition is consistent with Definition **5.2** if $\mathcal{S}_1$ and $\mathcal{S}_2$ are parallel planes.



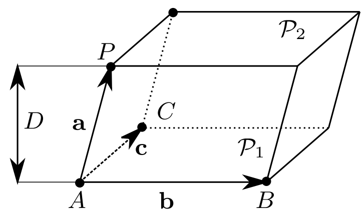


FIGURE 5.2. **Left**: Distance between two point sets $\mathcal{S}_1$ and $\mathcal{S}_2$ defined as the largest number that is less than or equal to all distances $|A_1A_2|$, where A_1 ranges over all points in $\mathcal{S}_1$ and A_2 ranges over all points in $\mathcal{S}_2$. **Right**: Distance between two parallel planes (Corollary **5.3**). Consider a parallelepiped whose opposite faces lie in the planes $\mathcal{P}_1$ and $\mathcal{P}_2$. Then the distance D between the planes is the height of the parallelepiped, which can be computed as the ratio $D = V/A$, where $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped and $A = \|\mathbf{b} \times \mathbf{c}\|$ is the area of the face.

COROLLARY **5.3**. (Distance Between Parallel Planes).
The distance between parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ is given by

$$D = \frac{|\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})|}{\|\overrightarrow{AB} \times \overrightarrow{AC}\|},$$

where A, B, and C are any three points in the plane $\mathcal{P}_1$ that are not on the same line, and P is any point in the plane $\mathcal{P}_2$.

PROOF. Since the points A, B, and C are not on the same line, the vectors $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AC}$ are not parallel and their cross product is a vector perpendicular to the planes (see Figure **5.2**, right panel). Consider the parallelepiped with adjacent sides $\mathbf{a} = \overrightarrow{AP}$, $\mathbf{b}$, and $\mathbf{c}$. Two of its faces, the parallelograms with adjacent sides $\mathbf{b}$ and $\mathbf{c}$, lie in the parallel planes, one in $\mathcal{P}_1$ and the other in $\mathcal{P}_2$. The distance between the planes is, by construction, the height of the parallelepiped which is equal to V/A_p, where A_p is the area of the face parallel to $\mathbf{b}$ and $\mathbf{c}$ and V is the volume of the parallelepiped. The conclusion follows from the geometrical properties of the triple and cross products: $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ and $A_p = \|\mathbf{b} \times \mathbf{c}\|$. □

Similarly, the distance between two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$ can be determined. Two lines are parallel if they are not intersecting and lie in the same plane. Let A and B be any two points on the line $\mathcal{L}_1$ and let C be any point on the line $\mathcal{L}_2$. Consider the parallelogram with adjacent sides $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$ as depicted in Figure **5.3** (left panel). The distance between the lines is the height of this parallelogram which is $D = A_p/\|\mathbf{a}\|$, where $A_p = \|\mathbf{a} \times \mathbf{b}\|$, is the area of the parallelogram and $\|\mathbf{a}\|$ is the length of its base.

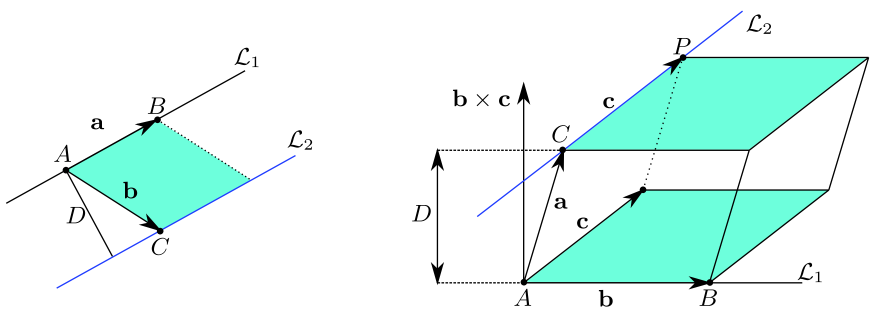


FIGURE 5.3. **Left**: Distance between two parallel lines. Consider a parallelogram whose two parallel sides lie in the lines. Then the distance between the lines is the height of the parallelogram (Corollary **5.4**): $D = A/\|\mathbf{a}\|$ where $A = \|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram. **Right**: Distance between skew lines. Consider a parallelepiped whose two non parallel edges AB and CP in the opposite faces lie in the skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then the distance between the lines is the height of the parallelepiped, which can be computed as the ratio $D = V/A$, where $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped and $A = \|\mathbf{b} \times \mathbf{c}\|$ is the area of the face (Corollary **5.5**).

COROLLARY **5.4**. (Distance Between Parallel Lines).
The distance between two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$D = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{AB}\|},$$

where A and B are any two distinct points on the line $\mathcal{L}_1$ and C is any point on the line $\mathcal{L}_2$.

DEFINITION **5.3**. (Skew Lines).
Two lines that are not intersecting and not parallel are called skew lines.

To determine the distance between skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$, consider any two points A and B on $\mathcal{L}_1$ and any two points C and P on $\mathcal{L}_2$. Define the vectors $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{CP}$ that are parallel to lines $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Since the lines are not parallel, the cross product

$\mathbf{b} \times \mathbf{c}$ does not vanish. The lines $\mathcal{L}_1$ and $\mathcal{L}_2$ lie in the parallel planes perpendicular to $\mathbf{b}\times\mathbf{c}$ (by the geometrical properties of the cross product, $\mathbf{b}\times\mathbf{c}$ is perpendicular to $\mathbf{b}$ and $\mathbf{c}$). The distance between the lines coincides with the distance between these parallel planes. Consider the parallelepiped with adjacent sides $\mathbf{a} = \overrightarrow{AC}$, $\mathbf{b}$, and $\mathbf{c}$ as shown in Figure **5.3** (right panel). The lines lie in the parallel planes that contain the faces of the parallelepiped parallel to the vectors $\mathbf{b}$ and $\mathbf{c}$. Thus, the distance between skew lines is the distance between the parallel planes containing them. By Corollary **5.3**, this distance is $D = V/A_p$ where V and $A_p = \|\mathbf{b}\times\mathbf{c}\|$ are, respectively, the volume of the parallelepiped and the area of its base.

COROLLARY **5.5**. (Distance between non-parallel lines).
The distance between two skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$D = \frac{|\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{CP})|}{\|\overrightarrow{AB}\times\overrightarrow{CP}\|},$$

where A and B are any two distinct points on $\mathcal{L}_1$, while C and P are any two distinct points on $\mathcal{L}_2$.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are intersecting, then they lie in a plane and the vectors $\overrightarrow{AC}$, $\overrightarrow{AB}$, and $\overrightarrow{CP}$ are coplanar. Their triple product vanishes and so does the distance between $\mathcal{L}_1$ and $\mathcal{L}_2$ as required.

As a consequence of the obtained distance formulas, the following criterion for mutual orientation of two lines in space holds.

COROLLARY **5.6**. (Positions of two lines in space)
Let $\mathcal{L}_1$ be a line through A and $B \neq A$, and $\mathcal{L}_2$ be a line through C and $P \neq C$. Then
(1) *$\mathcal{L}_1$ and $\mathcal{L}_2$ are skew if and only if*

$$\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{CP}) \neq 0\,;$$

(2) *$\mathcal{L}_1$ and $\mathcal{L}_2$ intersect if and only if*

$$\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{CP}) = 0\,;$$

(3) *$\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel if and only if*

$$\overrightarrow{AB}\times\overrightarrow{CP} = \mathbf{0} \quad \text{and} \quad \overrightarrow{AC}\times\overrightarrow{CP} \neq \mathbf{0}\,;$$

(4) *$\mathcal{L}_1$ and $\mathcal{L}_2$ coincide if and only if*

$$\overrightarrow{AB}\times\overrightarrow{CP} = \mathbf{0} \quad \text{and} \quad \overrightarrow{AC}\times\overrightarrow{CP} = \mathbf{0}\,.$$

PROOF. By Corollary **5.5**, for non-parallel lines $\overrightarrow{AB}\times\overrightarrow{CP}\neq\mathbf{0}$ (Corollary **4.1**) and the distance or the triple product $\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{CP})$ vanishes if and only if the lines are intersecting and does not vanish if and only if the lines are skew. If the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide, then the vectors $\overrightarrow{AB}$, $\overrightarrow{CP}$, and $\overrightarrow{AC}$ are parallel to each other and, hence, their cross products vanish. Conversely, the conditions $\overrightarrow{AB}\times\overrightarrow{CP}=\mathbf{0}$ and $\overrightarrow{AC}\times\overrightarrow{CP}=\mathbf{0}$ imply that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CP}$ and that $\overrightarrow{AC}$ is parallel to $\overrightarrow{CP}$ and, hence, $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$. The latter means that the point C also belongs to $\mathcal{L}_1$ and therefore $\mathcal{L}_1=\mathcal{L}_2$. If $\overrightarrow{AB}\times\overrightarrow{CP}=\mathbf{0}$, but $\overrightarrow{AC}\times\overrightarrow{CP}\neq\mathbf{0}$, then C cannot be a point of $\mathcal{L}_1$ because the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel, and the lines must be parallel. □

EXAMPLE **5.5**. *Find the distance between the line through the points $A=(1,1,2)$ and $B=(1,2,3)$ and the line through $C=(1,0,-1)$ and $P=(-1,1,2)$.*

SOLUTION: Let $\mathbf{a}=\overrightarrow{AB}=\langle 0,1,1\rangle$ and $\mathbf{b}=\overrightarrow{CP}=\langle -2,1,3\rangle$. Then

$$\mathbf{a}\times\mathbf{b}=\langle 3-1,-(0+2),0+2\rangle=\langle 2,-2,2\rangle\neq\mathbf{0}\,.$$

So the lines are not parallel by Part (3) in Corollary **5.6**. Put $\mathbf{c}=\overrightarrow{AC}=\langle 0,-1,-3\rangle$. Then

$$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\langle 0,-1,-3\rangle\cdot\langle 2,-2,2\rangle=0+2-6=-4\neq 0\,.$$

By Part (1) in Corollary **5.6**, the lines are skew. Next,

$$\|\mathbf{a}\times\mathbf{b}\|=\|\langle 2,-2,2\rangle\|=\|2\langle 1,-1,1\rangle\|=2\|\langle 1,-1,1\rangle\|=2\sqrt{3}\,.$$

By Corollary **5.5** the distance between the lines is

$$D=\frac{|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|}{\|\mathbf{a}\times\mathbf{b}\|}=\frac{|-4|}{2\sqrt{3}}=\frac{2}{\sqrt{3}}\,.$$

□

5.4. Study Problems.

Problem **5.1**. (Rotations in space)
Let $\overrightarrow{OP}$ be a position vector of a point P relative to a point O. If $\hat{\mathbf{e}}_j$, $j=1,2,3$, and $\hat{\mathbf{u}}_i$, $i=1,2,3$, are orthonormal bases associated with two rectangular coordinate systems with a common origin at O, then

$$\overrightarrow{OP}=a_1\hat{\mathbf{e}}_1+a_2\hat{\mathbf{e}}_2+a_3\hat{\mathbf{e}}_3=a_1'\hat{\mathbf{u}}_1+a_2'\hat{\mathbf{u}}_2+a_3'\hat{\mathbf{u}}_3$$

where (a_1,a_2,a_3) and (a_1',a_2',a_3') are "old" and "new" coordinates of P (Section **3.4**). *Let V be a 3×3 matrix elements $v_{ij}=\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_j$. Define vectors $\mathbf{v}_i=\langle v_{i1},v_{i2},v_{i3}\rangle$ and $\mathbf{w}_j=\langle v_{1j},v_{2j},v_{3j}\rangle$, $i,j=1,2,3$, whose*

components are the rows and columns of V, respectively.
(i). *Show that for a rotation of the coordinate system about the origin, the rows of V are mutually orthonormal, the columns of V are also mutually orthonormal, and V has unit determinant:*

$$(\mathbf{5.1})\qquad \mathbf{v}_i\cdot\mathbf{v}_j=\mathbf{w}_i\cdot\mathbf{w}_j=\begin{cases}1 & \text{if}\ \ i=j\\ 0 & \text{if}\ \ i\neq j\end{cases},\quad \text{and}\quad \det V=1$$

Consider the ordered triples $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ *and* $\mathbf{a}'=\langle a_1',a_2',a_3'\rangle$
(ii). *Show that the "new" and "old" coordinates are related as*

$$(\mathbf{5.2})\qquad \begin{aligned} a_i' &= v_{i1}a_1+v_{i2}a_2+v_{i3}a_3=\mathbf{v}_i\cdot\mathbf{a}, \quad i=1,2,3,\\ a_j &= v_{1j}a_1'+v_{2j}a_2'+v_{3j}a_3'=\mathbf{w}_j\cdot\mathbf{a}' \quad j=1,2,3\,, \end{aligned}$$

(iii). *How many independent parameters can the matrix V have for a generic rotation in space? In particular, find the rotation matrix V for a rotation about one of the coordinate axes* (Study Problem **1.2**) *and verify the properties* (**5.1**).
Hint: Use the orthogonal decompositions of vectors $\hat{\mathbf{u}}_i$ in the basis $\hat{\mathbf{e}}_j$ and the orthogonal decomposition of vector $\hat{\mathbf{e}}_j$ in the basis $\hat{\mathbf{u}}_i$. Establish relations between components of $\mathbf{v}_i$ and $\mathbf{w}_i$ and the expansion coefficients. Prove that $\hat{\mathbf{u}}_i\cdot\hat{\mathbf{u}}_j=\mathbf{v}_i\cdot\mathbf{v}_j$, $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j=\mathbf{w}_i\cdot\mathbf{w}_j$, and

$$(\mathbf{5.3})\qquad \hat{\mathbf{u}}_1\cdot(\hat{\mathbf{u}}_2\times\hat{\mathbf{u}}_3)=\det V\ \hat{\mathbf{e}}_1\cdot(\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3)$$

Solution: Put $v_{ij}=\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_j$, $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, and $\mathbf{a}'=\langle a_1',a_2',a_3'\rangle$. Then

$$\begin{aligned} a_i' &= \hat{\mathbf{u}}_i\cdot\overrightarrow{OP}=\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_1a_1+\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_2a_2+\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_3a_3\\ &= v_{i1}a_1+v_{i2}a_2+v_{i3}a_3=\mathbf{v}_i\cdot\mathbf{a}\\ a_j &= \hat{\mathbf{e}}_j\cdot\overrightarrow{OP}=\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_1a_1'+\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_2a_2'+\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_3a_3'\\ &= v_{1j}a_1'+v_{2j}a_2'+v_{3j}a_3'=\mathbf{w}_j\cdot\mathbf{a}' \end{aligned}$$

For a fixed j, the numbers v_{ij} are scalar projections of $\hat{\mathbf{e}}_j$ onto $\hat{\mathbf{u}}_i$, $i=1,2,3$, and hence are components of $\hat{\mathbf{e}}_j$ relative to the basis $\hat{\mathbf{u}}_i$. Thus, the jth column of V coincides with the components of $\hat{\mathbf{e}}_j$ in the basis $\hat{\mathbf{u}}_i$. Similarly, the ith row of V coincides with the components of $\hat{\mathbf{u}}_i$ in the basis $\hat{\mathbf{e}}_j$. So making use of the orthogonal expansions

$$\begin{aligned} \hat{\mathbf{e}}_j &= (\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_1)\hat{\mathbf{u}}_1+(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_2)\hat{\mathbf{u}}_2+(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{u}}_3)\hat{\mathbf{u}}_3=v_{1j}\hat{\mathbf{u}}_1+v_{2j}\hat{\mathbf{u}}_2+v_{3j}\hat{\mathbf{u}}_3\\ \hat{\mathbf{u}}_i &= (\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1+(\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2+(\hat{\mathbf{u}}_i\cdot\hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3=v_{i1}\hat{\mathbf{e}}_1+v_{i2}\hat{\mathbf{e}}_2+v_{i3}\hat{\mathbf{e}}_3 \end{aligned}$$

it follows from the orthonormality of $\hat{\mathbf{u}}_i$ and the orthonormality of $\hat{\mathbf{e}}_j$, respectively, that

$$\begin{aligned} \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j &= v_{1i}v_{1j}+v_{2i}v_{2j}+v_{3i}v_{3j}=\mathbf{w}_i\cdot\mathbf{w}_j\\ \hat{\mathbf{u}}_i\cdot\hat{\mathbf{u}}_j &= v_{i1}v_{j1}+v_{i2}v_{j2}+v_{i3}v_{j3}=\mathbf{v}_i\cdot\mathbf{v}_j \end{aligned}$$

Owing the orthonormality of the basis vectors, the first relation in (**5.1**) is proved. Next, consider the cross product

$$\begin{aligned}\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3 &= (v_{21}\hat{\mathbf{e}}_1 + v_{22}\hat{\mathbf{e}}_2 + v_{23}\hat{\mathbf{e}}_3) \times (v_{31}\hat{\mathbf{e}}_1 + v_{32}\hat{\mathbf{e}}_2 + v_{33}\hat{\mathbf{e}}_3)\\ &= \det V_{11}(\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) - \det V_{12}(\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1) + \det V_{13}(\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2)\\ V_{11} &= \begin{pmatrix} v_{22} & v_{23}\\ v_{32} & v_{33}\end{pmatrix}, \quad V_{12} = \begin{pmatrix} v_{21} & v_{23}\\ v_{31} & v_{33}\end{pmatrix}, \quad V_{13} = \begin{pmatrix} v_{21} & v_{22}\\ v_{31} & v_{32}\end{pmatrix}\end{aligned}$$

where the skew symmetry of the cross product $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = -\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_i$ and the definition of the determinant of a 2×2 matrix have been used; the matrices V_{1i}, $i = 1, 2, 3$, are obtained by removing from V the row and column that contain v_{1i}. Using the symmetry of the triple product under cyclic permutations of the vectors, one has

$$\hat{\mathbf{u}}_1 \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) = \Big(v_{11}\det V_{11} - v_{12}\det V_{12} + v_{13}\det V_{13}\Big)\hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3)$$

Equation (**5.3**) follows from this relation and Definition **4.2** of the determinant of a 3×3 matrix. Now recall that the handedness of a coordinate system is preserved under rotations (Section **5.2**), $\hat{\mathbf{u}}_1 \cdot (\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3) = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = \pm 1$, and therefore $\det V = 1$. It is concluded that any combination of rotations and reflections is described by a matrix V whose rows and columns form orthonormal bases, and $\det V = \pm 1$. The handedness of a coordinate system is changed if $\det V = -1$.

The vector $\hat{\mathbf{u}}_3$ is determined by its three direction angles in the original coordinate system. Only two of these angles are independent. A rotation about the axis containing the vector $\hat{\mathbf{u}}_3$ does not affect $\hat{\mathbf{u}}_3$ and can be specified by a rotation angle in a plane perpendicular to the coordinate axis parallel to $\hat{\mathbf{u}}_3$. This angle determines the vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ relative to the original basis. So *a general rotation matrix V has three independent parameters.*

In particular, the matrix V for counterclockwise rotations about the z axis through an angle ϕ (see Study Problem **1.2**) is

$$V = \begin{pmatrix} \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}$$

because $\hat{\mathbf{u}}_1\cdot\hat{\mathbf{e}}_1 = \cos\phi$, $\hat{\mathbf{u}}_1\cdot\hat{\mathbf{e}}_2 = \sin\phi$ $\hat{\mathbf{u}}_2\cdot\hat{\mathbf{e}}_1 = -\sin\phi$, and $\hat{\mathbf{u}}_2\cdot\hat{\mathbf{e}}_2 = \cos\phi$ by the definition of the angle ϕ (see Fig. **1.3**, right panel). Relations (**5.1**) for V and its rows and columns

$$\begin{aligned}\mathbf{v}_1 &= \langle\cos\phi, \sin\phi, 0\rangle & \mathbf{w}_1 &= \langle\cos\phi, -\sin\phi, 0\rangle\\ \mathbf{v}_2 &= \langle-\sin\phi, \cos\phi, 0\rangle & \mathbf{w}_2 &= \langle\sin\phi, \cos\phi, 0\rangle\\ \mathbf{v}_3 &= \langle 0, 0, 1\rangle & \mathbf{w}_3 &= \langle 0, 0, 1\rangle\end{aligned}$$

are easy to verify. The result of Study Problem **1.2** can be stated in the form (**5.2**) where $\mathbf{a} = \langle x, y, z\rangle$ and $\mathbf{a}' = \langle x', y', z'\rangle$. □

Problem **5.2**. *Find the most general vector* $\mathbf{r}$ *that satisfies the equation* $\mathbf{a}\cdot(\mathbf{r}\times\mathbf{b}) = 0$*, where* $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero, nonparallel vectors.*

Solution: By the algebraic property of the triple product, $\mathbf{a}\cdot(\mathbf{r}\times\mathbf{b}) = \mathbf{r}\cdot(\mathbf{b}\times\mathbf{a}) = 0$. Hence, $\mathbf{r}\perp\mathbf{a}\times\mathbf{b}$. The vector $\mathbf{r}$ lies in a plane parallel to both $\mathbf{a}$ and $\mathbf{b}$ because $\mathbf{a}\times\mathbf{b}$ is orthogonal to these vectors. Any vector in the plane is a linear combination of any two nonparallel vectors in it:

$$\mathbf{r} = t\mathbf{a} + s\mathbf{b}$$

for any real t and s (see Study Problem **2.1**). □

Problem **5.3**. (Volume of a Tetrahedron). *A tetrahedron is a solid with four vertices and four triangular faces. Its volume* $V = \frac{1}{3}Ah$*, where* h *is the distance from a vertex to the opposite face and* A *is the area of that face. Given coordinates of the vertices* B*,* C*,* D*, and* P*, express the volume of the tetrahedron through them.*

Solution: Put $\mathbf{b} = \overrightarrow{BC}$, $\mathbf{c} = \overrightarrow{BD}$, and $\mathbf{a} = \overrightarrow{BP}$. The area of the triangle BCD is $A = \frac{1}{2}\|\mathbf{b}\times\mathbf{c}\|$ (Corollary **4.2**). The distance from P to the plane $\mathcal{P}_1$ containing the face BCD is the distance between $\mathcal{P}_1$ and the parallel plane $\mathcal{P}_2$ through the vertex P. By Corollary **5.4**,

$$V = \frac{1}{3}A\,\frac{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{\|\mathbf{b}\times\mathbf{c}\|} = \frac{1}{6}\left|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\right|.$$

So the volume of a tetrahedron with adjacent sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is one-sixth the volume of the parallelepiped with the same adjacent sides. Note the result does not depend on the choice of a vertex. Any vertex could have been chosen instead of B in the above solution. □

Problem **5.4**. (Systems of linear equations)
Consider a system of linear equations for the variables x*,* y*, and* z*:*

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Define vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$*,* $\mathbf{b} = \langle b_1, b_2, b_3\rangle$*,* $\mathbf{c} = \langle c_1, c_2, c_3\rangle$*, and* $\mathbf{d} = \langle d_1, d_2, d_3\rangle$*. Show that the system has a unique solution for any* $\mathbf{d}$ *if*

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \neq 0\,.$$

If $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$*, formulate conditions on* $\mathbf{d}$ *under which the system has a solution.*

SOLUTION: The system of linear equations can be cast in the vector form

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$$

This equation states that a given vector $\mathbf{d}$ is a linear combination of three given vectors. In Study Problem **3.1** it was demonstrated that any vector in space can be uniquely represented as a linear combination of three non-coplanar vectors. So by Corollary **5.1**, the numbers x, y, and z exist and are unique if $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \neq 0$.

When $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$, the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ lie in one plane. If $\mathbf{d}$ is not this plane, the system has no solution because $\mathbf{d}$ cannot be represented as a linear combination of vectors in this plane.

Suppose that two of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are not parallel. Then their cross product is orthogonal to the plane and $\mathbf{d}$ must be orthogonal to the cross product in order to be in the plane. If, say, $\mathbf{a}\times\mathbf{b} \neq \mathbf{0}$, that is, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent in the plane, then, whenever $\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b}) = 0$, the vector $\mathbf{d} - z\mathbf{c}$ also lies in this plane (as a linear combinations of two vectors in the plane). The latter implies (Study Problem **1.6**) that for any real z there exist unique numbers x and y such that

$$x\mathbf{a} + y\mathbf{b} = \mathbf{d} - z\mathbf{c}\,.$$

In this case the system has infinitely many solutions labeled by a real number z (if $\mathbf{c} \neq \mathbf{0}$).

Finally, it is possible that all the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are parallel, that is, all pairwise cross products vanish. Then $\mathbf{d}$ must be parallel to them. If, say, $\mathbf{a} \neq \mathbf{0}$, then the system has a solution if $\mathbf{d}\times\mathbf{a} = \mathbf{0}$. in this case, the vector $\mathbf{d} - y\mathbf{b} - z\mathbf{c}$ is parallel to $\mathbf{a}$ for any choice of numbers y and z, and there is a unique number x such that

$$x\mathbf{a} = \mathbf{d} - y\mathbf{b} - z\mathbf{c}\,.$$

In this case, the system also has infinitely many solutions labeled by a pair or real numbers (y, z) (if $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{c} \neq \mathbf{0}$).

□

5.5. Exercises.

1–5. Find the triple products $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$, and $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ for given vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

1. $\mathbf{a} = \langle 1, -1, 2\rangle$, $\mathbf{b} = \langle 2, 1, 2\rangle$, and $\mathbf{c} = \langle 2, 1, 3\rangle$.

2. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are, respectively, position vectors of the points $A = (1, 2, 3)$, $B = (1, -1, 1)$, and $C = (2, 0, -1)$ relative to the point $O = (1, 1, 1)$.
3. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar so that $\mathbf{c} = 2\mathbf{a} - 3\mathbf{b}$.
4. $\mathbf{a} = \mathbf{u}_1 + 2\mathbf{u}_2$, $\mathbf{b} = \mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$, and $\mathbf{c} = \mathbf{u}_2 - 3\mathbf{u}_3$ if $\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) = 2$
5. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are pairwise perpendicular and $\|\mathbf{a}\| = 1$, $\|\mathbf{b}\| = 2$, and $\|\mathbf{c}\| = 3$. Is the answer unique under the specified conditions?

6. Verify whether the vectors $\mathbf{a} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$, $\mathbf{b} = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$, and $\mathbf{c} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3$ are coplanar.

7. Consider the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle -1, 0, 1\rangle$ and $\mathbf{c} = \langle s, 1, 2s\rangle$ where s is a number.
(i) Find all values of s, if any, for which these vectors are coplanar.
(ii) If such s exists, find the area of the quadrilateral whose three vertices have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ relative to the fourth vertex.
Hint: Determine which of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is a diagonal of the quadrilateral.

8. Determine whether the points $A = (1, 2, 3)$, $B = (1, 0, 1)$, $C = (-1, 1, 2)$, and $D = (-2, 1, 0)$ are in one plane and, if not, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

9. Find:
(i) all values of s at which the points $A = (s, 0, s)$, $B = (1, 0, 1)$, $C = (s, s, 1)$, and $D = (0, 1, 0)$ are in the same plane;
(ii) all values of s at which the volume of the parallelepiped with adjacent edges AB, AC, and AD is 9 units.

10. Let $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 2, 1, 0\rangle$, and $\mathbf{c} = \langle 3, 0, 1\rangle$. Find the volume of the parallelepiped with adjacent sides $s\mathbf{a} + \mathbf{b}$, $\mathbf{c} - t\mathbf{b}$, and $\mathbf{a} - p\mathbf{c}$ if the numbers s, t, and p satisfy the condition $stp = 1$.

11. Let the numbers u, v, and w be such that $uvw = 1$ and $u^3 + v^3 + w^3 = 1$. Are the vectors $\mathbf{a} = u\hat{\mathbf{e}}_1 + v\hat{\mathbf{e}}_2 + w\hat{\mathbf{e}}_3$, $\mathbf{b} = v\hat{\mathbf{e}}_1 + w\hat{\mathbf{e}}_2 + u\hat{\mathbf{e}}_3$, and $\mathbf{c} = w\hat{\mathbf{e}}_1 + u\hat{\mathbf{e}}_2 + v\hat{\mathbf{e}}_3$ coplanar? If not, what is the volume of the parallelepiped with adjacent edges $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$?

12. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \det \begin{pmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{pmatrix}.$$

Hint: Put $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. Use the invariance of the triple product under cyclic permutations of vectors in it and the "bac-cab" rule (**4.2**).

13. Let P be a parallelepiped of volume V. Find the volumes of all parallelepipeds whose adjacent edges are diagonals of the adjacent faces of P.

14. Let P be a parallelepiped of volume V. Find the volumes of all parallelepipeds whose two adjacent edges are diagonals of two non-parallel faces of P, while the third adjacent edge is a diagonal of P (the segment connecting two vertices of P that does not lie in a face of P).

15. Given two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$, find the most general vector $\mathbf{r}$ that satisfies the conditions $\mathbf{a}\cdot(\mathbf{r}\times\mathbf{b}) = 0$ and $\mathbf{b}\cdot\mathbf{r} = 0$.

16. Let a set $\mathcal{S}_1$ be the circle $x^2 + y^2 = 1$ and let a set $\mathcal{S}_2$ be the line through the points $(0, 2)$ and $(2, 0)$. Find the distance between the sets $\mathcal{S}_1$ and $\mathcal{S}_2$.

17. Consider a plane through three points $A = (1, 2, 3)$, $B = (2, 3, 1)$, and $C = (3, 1, 2)$. Find the distance between the plane and a point P obtained from A by moving the latter 3 units along a straight line segment parallel to the vector $\mathbf{a} = \langle -1, 2, 2\rangle$.

18. Consider two lines. The first line passes through the points $(1, 2, 3)$ and $(2, -1, 1)$, while the other passes through the points $(-1, 3, 1)$ and $(1, 1, 3)$. Find the distance between the lines.

19. Find the distance between the line through the points $(1, 2, 3)$ and $(2, 1, 4)$ and the plane through the points $(1, 1, 1)$, $(3, 1, 2)$, and $(1, 2, -1)$. *Hint:* If the line is not parallel to the plane, then they intersect and the distance is 0. So check first whether the line is parallel to the plane. How can this be done?

20. Consider the line through the points $(1, 2, 3)$ and $(2, 1, 2)$. If a second line passes through the points $(1, 1, s)$ and $(2, -1, 0)$, find all values of s, if any, at which the distance between the lines is $3/2$ units.

21. Consider two parallel straight line segments in space. Formulate an algorithm to compute the distance between them if the coordinates of their end points are given. In particular, find the distance between AB and CD if:

(i) $A = (1, 1, 1)$, $B = (4, 1, 5)$, $C = (2, 3, 3)$, $D = (5, 3, 7)$;
(ii) $A = (1, 1, 1)$, $B = (4, 1, 5)$, $C = (3, 5, 5)$, $D = (6, 5, 9)$

Note that this distance does not generally coincide with the distance between the parallel lines containing AB and CD. The segments may even be in the same line at a nonzero distance.

22-25. Consider the parallelepiped with adjacent edges AB, AC, and AD where $A = (3, 0, 1)$, $B = (-1, 2, 5)$, $C = (5, 1, -1)$, $D = (0, 4, 2)$. Find the specified distances.

22. The distances between the edge AB and all other edges parallel to it.

23. The distances between the edge AC and all other edges parallel to it;
24. The distances between the edge AD and all other edges parallel to it;
25. The distances between all parallel planes containing the faces of the parallelepiped.
26. The distances between all skew lines containing the edges of the parallelepiped.

6. Planes in Space

6.1. An algebraic description of a plane in space. In Section **1.2**, a plane $\mathcal{P}$ through a point P_0 was defined as *a set points in space that consists of all straight lines through* P_0 *that are perpendicular to a given line through* P_0. A point P belongs to the plane $\mathcal{P}$ if and only if the straight line segment P_0P is perpendicular to a given (fixed) line $\mathcal{L}$ through through P_0. Relative to a rectangular coordinate system any point has coordinates $P = (x, y, z)$. Here the objective is to find an algebraic condition on the coordinates (x, y, z) under which the point belongs to a plane $\mathcal{P}$, that is, to define a plane algebraically in the same sense as points sets in space were defined in Section **1.9**. An example of an algebraic description of a plane was given in Study Problem **1.6**. It can be extended to a general plane. However, it turns out that a simpler solution can be obtained by means of vector algebra.

Consider the plane through the origin $P_0 = (0, 0, 0)$ and perpendicular to the z axis. It is the set of points whose coordinates (x, y, z) satisfy the equation $z = 0$. This algebraic condition can be restated in term of vectors. The position vector $\overrightarrow{P_0P} = \langle x, y, z\rangle$ of any point $P = (x, y, z)$ in this plane is perpendicular to $\hat{\mathbf{e}}_3$. By the geometrical properties of the dot product,

$$\overrightarrow{P_0P} \perp \hat{\mathbf{e}}_3 \quad \Leftrightarrow \quad \overrightarrow{P_0P} \cdot \hat{\mathbf{e}}_3 = \langle x, y, z\rangle \cdot \langle 0, 0, 1\rangle = 0 \quad \Leftrightarrow \quad z = 0\,.$$

Since the coordinate system can be arbitrarily chosen by translating the origin and rotating the coordinate axes, any plane in space can be obtained from the xy plane in a particular coordinate system by suitable rotations and translations and, hence, admits the following equivalent description:

> *a plane in space is a set of points whose position vectors relative to a particular point in the set are orthogonal to a given nonzero vector* $\mathbf{n}$

The vector $\mathbf{n}$ is called a *normal* of the plane. Note that a normal is not unique. If $\mathbf{n}$ is a normal, then $s\mathbf{n}$ is also a normal for any $s \neq 0$ because any vector orthogonal to $\mathbf{n}$ is also orthogonal to its multiple $s\mathbf{n}$ (which is parallel to $\mathbf{n}$). Thus, the geometrical description of a plane $\mathcal{P}$ in space entails specifying a point P_0 that belongs to $\mathcal{P}$ and a normal $\mathbf{n}$ of $\mathcal{P}$. Then a point P belongs to the plane $\mathcal{P}$ if an only if the vector $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$:

$$P \in \mathcal{P} \quad \Leftrightarrow \quad \overrightarrow{P_0P} \perp \mathbf{n}\,, \quad P_0 \in \mathcal{P}$$

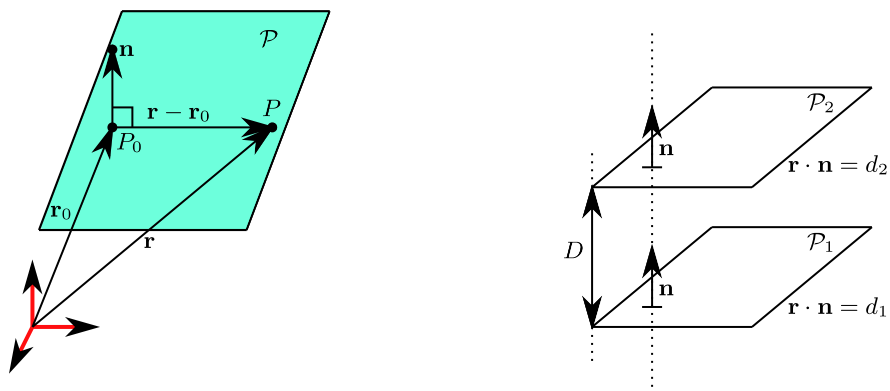


FIGURE **6.1**. **Left**: Algebraic description of a plane. If $\mathbf{r}_0$ is a position vector of a particular point in the plane and $\mathbf{r}$ is the position vector of a generic point in the plane, then the vector $\mathbf{r}-\mathbf{r}_0$ lies in the plane and is orthogonal to its normal, that is, $\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0$.
Right: Equations of parallel planes differ only by their constant terms. The difference of the constant terms determines the distance between the planes as stated in (**6.3**).

Let a plane $\mathcal{P}$ be defined by a point P_0 that belongs to it and a normal $\mathbf{n}$. In some coordinate system, the point P_0 has coordinates (x_0, y_0, z_0) and the vector $\mathbf{n}$ is specified by its components $\mathbf{n} = \langle n_1, n_2, n_3\rangle$. A generic point in space P has coordinates (x, y, z). An algebraic description of a plane amounts to specifying conditions on the variables (x, y, z) such that the point $P = (x, y, z)$ belongs to the plane $\mathcal{P}$. Let $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{r} = \langle x, y, z\rangle$ be the position vectors of a particular point P_0 in the plane and a generic point P in space, respectively. Then the position vector of P relative to P_0 is

$$\overrightarrow{P_0P} = \mathbf{r}-\mathbf{r}_0 = \langle x-x_0, y-y_0, z-z_0\rangle\,.$$

This vector lies in the plane $\mathcal{P}$ if it is orthogonal to the normal $\mathbf{n}$, according to the geometrical description of a plane (see Figure **6.1**, left panel). The algebraic condition equivalent to the geometrical one reads

$$\mathbf{n}\perp\overrightarrow{P_0P} \quad\Leftrightarrow\quad \mathbf{n}\cdot\overrightarrow{P_0P}=0 \quad\Leftrightarrow\quad \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0\,.$$

Thus, the following theorem has just been proved.

THEOREM **6.1**. (Equation of a plane).
A point with coordinates (x, y, z) belongs to a plane through a point $P_0 = (x_0, y_0, z_0)$ and normal to a vector $\mathbf{n} = \langle n_1, n_2, n_3\rangle$ if

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0 \qquad\text{or}\qquad \mathbf{n}\cdot\mathbf{r}=\mathbf{n}\cdot\mathbf{r}_0,$$

where $\mathbf{r} = \langle x, y, z\rangle$ *and* $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ *are position vectors of a generic point and a particular point* P_0 *in the plane.*

A normal to a given plane can always be obtained by taking the cross product of any two non-parallel vectors in the plane. Indeed, any vector in a plane is a linear combination of two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ (Study Problem **2.1**). The vector $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and, hence, to any linear combination of them.

EXAMPLE **6.1**. *Find an equation of the plane through three given points* $A = (1, 1, 1)$*,* $B = (2, 3, 0)$*, and* $C = (-1, 0, 3)$*. In particular, determine whether the points* $(5, 0, -3)$*,* $(-2, 1, 2)$*,* $(1, s, 1)$*, where* s *is real, lie in the plane.*

SOLUTION: A plane is specified by a particular point P_0 in it and by a vector $\mathbf{n}$ normal to it. Three points in the plane are given, so any of them can be taken as P_0, for example, $P_0 = A$ or $(x_0, y_0, z_0) = (1, 1, 1)$. A vector normal to a plane can be found as the cross product of any two nonparallel vectors in that plane (see Figure **6.2**, left panel). So put $\mathbf{a} = \overrightarrow{AB} = \langle 1, 2, -1\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle -2, -1, 2\rangle$. Then one can take

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle (4 - 1), -(2 - 2), -1 + 4\rangle = \langle 3, 0, 3\rangle\,.$$

An equation of the plane is

$$3(x - 1) + 0(y - 1) + 3(z - 1) = 0 \quad \Leftrightarrow \quad x + z = 2\,.$$

The coordinates of the point $(5, 0, -3)$ satisfy this equation and, hence, the point is in the plane, while the coordinates of the point $(-2, 1, 2)$ do not satisfy it and, hence, the point is not in this plane. The points $(1, s, 1)$ satisfy the equation for any s. They form a line through the point A ($s = 1$) that is parallel to the y axis. Note that if a particular component of $\mathbf{n}$ vanishes, then the equation of the plane does not contain the corresponding coordinate. This implies that the plane is parallel to the corresponding coordinate axis (or contains it). In the example considered, the y component of $\mathbf{n}$ vanishes and there is no y in the equation of the plane. The normal $\mathbf{n}$ is orthogonal to $\hat{\mathbf{e}}_2$ because $\mathbf{n} \cdot \hat{\mathbf{e}}_2 = 0$; that is, the y axis is orthogonal to $\mathbf{n}$ and hence parallel to the plane. □

6.2. Relative positions of planes in space. Given a nonzero vector $\mathbf{n}$ and a number d, consider a general linear equation

$$n_1 x + n_2 y + n_3 z = d \quad \text{or} \quad \mathbf{n} \cdot \mathbf{r} = d\,.$$

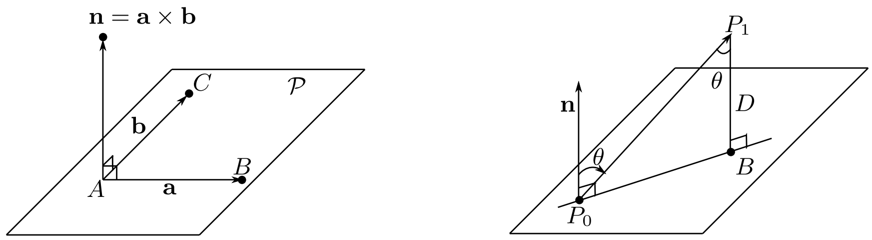


FIGURE **6.2**. **Left**: Illustration to Example **6.1**. The cross product of two non parallel vectors in a plane is a normal of the plane. **Right**: Distance between a point P_1 and a plane. An illustration to the derivation of the distance formula (**6.1**). The segment P_1B is parallel to the normal $\mathbf{n}$ so that the triangle P_0P_1B is right-angled. Therefore, $D = |P_1B| = |P_0P_1|\cos\theta$.

The set of solutions does not change if both sides are divided by a nonzero number, in particular, by the length of $\mathbf{n}$. So $\mathbf{n}$ can always be assumed to be a *unit* vector. It is always possible to find a particular solution of this equation, $\mathbf{r} = \mathbf{r}_0$. Indeed, since at least one component of $\mathbf{n}$ does not vanish, say, $n_1 \neq 0$, then $\mathbf{r}_0 = \langle d/n_1, 0, 0\rangle$. Therefore a general solution of the linear equation is $\mathbf{r} = \mathbf{r}_0 + \mathbf{p}$ where $\mathbf{p}$ is any vector orthogonal to $\mathbf{n}$. Any such $\mathbf{p}$ may be viewed as the position vector of a point in a plane relative to a particular point in the plane. So, *every plane can be described by a linear equation and every linear equation describes a plane.*

In notations of Section **1.9**, a plane in space is the point set

$$\mathcal{P} = \{\mathbf{r} \mid \hat{\mathbf{n}} \cdot \mathbf{r} = d\}$$

for a unit vector $\hat{\mathbf{n}}$ and a number d. Note that this description of a plane does not refer to any particular coordinate system, since the value of the dot product is independent of the choice of the coordinate system. The unit vector $\hat{\mathbf{n}}$ specifies the direction to which the plane is orthogonal, while the number d determines the position of the plane in space in the following way. Suppose that all points of the plane are transported parallel by a vector $\mathbf{a}$, i.e., $\mathbf{r} \to \mathbf{r} + \mathbf{a}$ for every position vector $\mathbf{r}$. The result is a new plane whose equation is $\hat{\mathbf{n}} \cdot (\mathbf{r} + \mathbf{a}) = d$ or

$$\hat{\mathbf{n}} \cdot \mathbf{r} = d - \hat{\mathbf{n}} \cdot \mathbf{a}\,.$$

The vector $\mathbf{a}$ has unique orthogonal decomposition relative to $\hat{\mathbf{n}}$ (Corollary **3.1**):

$$\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp = s\hat{\mathbf{n}} + \mathbf{a}_\perp\,, \quad s = \hat{\mathbf{n}} \cdot \mathbf{a}\,, \quad \mathbf{a}_\perp \cdot \hat{\mathbf{n}} = 0$$

where $\mathbf{a}_{\|}$ is parallel to $\hat{\mathbf{n}}$ and $\mathbf{a}_{\perp}$ is orthogonal to $\hat{\mathbf{n}}$. So the parallel transport of a plane by a vector $\mathbf{a}$ may be viewed the composition $\mathcal{T}$ of two parallel transports:

$$\mathcal{T}: \quad \mathbf{r} \to \mathbf{r} + \mathbf{a}_{\perp} \to (\mathbf{r} + \mathbf{a}_{\perp}) + \mathbf{a}_{\|} = \mathbf{r} + \mathbf{a}\,.$$

First, all points in the plane are transported *within* the plane by the vector $\mathbf{a}_{\perp}$. In this case, the plane as a point set does not change and neither does the number d because $\mathbf{n} \cdot \mathbf{a}_{\perp} = 0$. Then all points of the plane are transported parallel to the normal $\hat{\mathbf{n}}$ by the vector $\mathbf{a}_{\|}$. The result is a *parallel* plane. The number d changes by the amount $-\hat{\mathbf{n}} \cdot \mathbf{a}_{\|} = -s \neq 0$ for some real $s \neq 0$. So, *variations of d correspond to shifts of the plane parallel to itself along its normal* (see Figure **6.1**, right panel). All planes with parallel normals are either parallel or coincide.

Corollary **6.1**. (Relative positions of planes)
Two planes $\mathcal{P}_1 = \{\mathbf{r} \mid \mathbf{n}_1 \cdot \mathbf{r} = d_1\}$ and $\mathcal{P}_2 = \{\mathbf{r} \mid \mathbf{n}_2 \cdot \mathbf{r} = d_2\}$
(i) *are intersecting if their normals $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel;*
(ii) *are parallel $\mathcal{P}_1 \| \mathcal{P}_2$ if $\mathbf{n}_2 = s\mathbf{n}_1$ for some $s \neq 0$ and $d_2 \neq sd_1$;*
(iii) *coincide $\mathcal{P}_1 = \mathcal{P}_2$ if $\mathbf{n}_2 = s\mathbf{n}_1$ for some $s \neq 0$ and $d_2 = sd_1$.*

Note that in the last case the equations $\mathbf{n}_1 \cdot \mathbf{r} = d_1$ and $\mathbf{n}_2 \cdot \mathbf{r} = d_2$ are equivalent (they have the same set of solutions).

Example **6.2**. *Determine whether the planes $x - 2y - z = 5$ and $-2x + 4y + 2z = 1$ are parallel, or coincide, or neither.*

Solution: The normals of the planes are $\mathbf{n}_1 = \langle 1, -2, -1\rangle$ and $\mathbf{n}_2 = \langle -2, 4, 2\rangle$ (the components of a normal are the corresponding coefficients at the coordinates x, y, and z). By Eq. (**2.1**), it is easy to verify that the normals are proportional $\mathbf{n}_2 = -2\mathbf{n}_1$, but $d_2/d_1 = 1/5 \neq -2$, and, hence, the planes are parallel. □

Consider two non-parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$. It is clear that they intersect along a line $\mathcal{L}$. The plane $\mathcal{P}_2$ can always be obtained from $\mathcal{P}_1$ by a rigid rotation of the latter about the line $\mathcal{L}$ through an angle θ. The *smallest* rotation angle needed to obtained $\mathcal{P}_2$ from $\mathcal{P}_1$ (or $\mathcal{P}_1$ from $\mathcal{P}_2$) by rotation about their line of intersection is called the *angle between two intersecting planes.*

Let $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ be unit normals of two intersecting planes. Then $\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \cos\theta$ where $0 \leq \theta \leq \pi$ is the angle between the normals. The plane $\mathcal{P}_2$ is obtained from $\mathcal{P}_1$ by either a rotation through the angle θ or by a rotation through the angle $\pi - \theta$ in the opposite direction (like clockwise or counterclockwise). So, the angle between the planes is the smallest of these two angles. If $0 \leq \theta \leq \pi/2$, then the angle between the planes coincides with θ and, if $\pi/2 < \theta \leq \pi$, then it is $\pi - \theta$. A rotation of a plane through the angle π about a line in the plane does not change the plane as a point set in space, but its normal reverses the direction under such a rotation. So the second case is reduced to the first one by reversing the normal of one of the planes. Consequently, the angle θ between the planes is uniquely determined by the root of the equation $\cos\theta = |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2|$ in the interval $0 \leq \theta \leq \pi/2$. So the following algebraic definition of the angle between two planes may be adopted.

DEFINITION **6.1**. (Angle between two planes).
If $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *are normals of two planes, then the angle* $0 \leq \theta \leq \pi/2$ *satisfying the equation*

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2|$$

is called the angle between the planes*.*

The planes are perpendicular if the angle between them is $\pi/2$ (their normals are perpendicular). For example, the planes $x+y+z=1$ and $x+2y-3z=4$ are perpendicular because their normals $\mathbf{n}_1 = \langle 1,1,1\rangle$ and $\mathbf{n}_2 = \langle 1,2,-3\rangle$ are perpendicular: $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1+2-3 = 0$ (i.e., $\mathbf{n}_1 \perp \mathbf{n}_2$). If the angle between the planes is 0, then the planes are parallel (they may also coincide). The angle between two non-parallel planes is also also called the angle of intersection of the planes.

EXAMPLE **6.3**. *Find the angle at which the plane through the points* $A = (1,1,1)$*,* $B = (1,2,3)$*, and* $C = (2,0,1)$ *intersects the xy plane.*

SOLUTION: The vectors $\mathbf{a} = \overrightarrow{AB} = \langle 0,1,2\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle 1,-1,0\rangle$ are in the plane in question. Therefore its normal is

$$\mathbf{n}_1 = \mathbf{a} \times \mathbf{b} = \langle 0+2, -(0-2), 0-1\rangle = \langle 2,2,-1\rangle$$

and $\|\mathbf{n}_1\| = 3$. The vector $\mathbf{n}_2 = \hat{\mathbf{e}}_3 = \langle 0,0,1\rangle$ is a normal of the xy plane. Therefore the angle θ of intersection of the planes satisfies

$$\cos\theta = \frac{1}{3}|\mathbf{n}_1 \cdot \hat{\mathbf{e}}_3| = \frac{1}{3}|-1| = \frac{1}{3}$$

and, hence, $\theta = \cos^{-1}(1/3) \approx 1.23\,\text{rad} \approx 70.5^\circ$. □

6.3. The Distance Between a Point and a Plane. Consider the plane through a point P_0 with a normal $\mathbf{n}$. Let P_1 be a point in space. What is the distance between P_1 and the plane? Let the angle between $\mathbf{n}$ and the vector $\overrightarrow{P_0P_1}$ be θ (see Figure **6.2**, right panel). Then the distance in question is $D = \|\overrightarrow{P_0P_1}\|\cos\theta$ if $\theta \le \pi/2$ (the length of the straight line segment connecting P_1 and the plane along the normal $\mathbf{n}$). For $\theta > \pi/2$, $\cos\theta$ must be replaced by $-\cos\theta$ because $D \ge 0$. So

$$(\mathbf{6.1}) \qquad D = \|\overrightarrow{P_0P_1}\||\cos\theta| = \frac{\|\mathbf{n}\|\|\overrightarrow{P_0P_1}\||\cos\theta|}{\|\mathbf{n}\|} = \frac{|\mathbf{n}\cdot\overrightarrow{P_0P_1}|}{\|\mathbf{n}\|}\,.$$

Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be position vectors of P_0 and P_1, respectively. Then $\overrightarrow{P_0P_1} = \mathbf{r}_1 - \mathbf{r}_0$, and

$$(\mathbf{6.2}) \qquad D = \frac{|\mathbf{n}\cdot(\mathbf{r}_1 - \mathbf{r}_0)|}{\|\mathbf{n}\|} = \frac{|\mathbf{n}\cdot\mathbf{r}_1 - d|}{\|\mathbf{n}\|},$$

which is a bit more convenient than (**6.1**) if the plane is defined by an equation $\mathbf{n}\cdot\mathbf{r} = d$.

Distance Between Parallel Planes. If parallel planes are defined by the equations $\mathbf{n}\cdot\mathbf{r} = d_1$ and $\mathbf{n}\cdot\mathbf{r} = d_2$, then Eq. (**6.2**) allows us to obtain a simpler formula for the distance between them than that given in Corollary **5.3** (see Figure **6.1**, right panel):

$$(\mathbf{6.3}) \qquad D = \frac{|d_2 - d_1|}{\|\mathbf{n}\|}\,.$$

Indeed, the distance between two parallel planes is the distance between the first plane and a point $\mathbf{r}_2$ in the second plane. By (**6.2**), this distance is

$$D = \frac{|\mathbf{n}\cdot\mathbf{r}_2 - d_1|}{\|\mathbf{n}\|} = \frac{|d_2 - d_1|}{\|\mathbf{n}\|}$$

because $\mathbf{n}\cdot\mathbf{r}_2 = d_2$ for any point in the second plane.

EXAMPLE **6.4**. *Find an equation of a plane that is parallel to the plane $2x - y + 2z = 2$ and at a distance of 3 units from it.*

SOLUTION: Since the planes are parallel, their normals may be chosen to coincide with $\mathbf{n} = \langle 2, -1, 2\rangle$ (the normal of the given plane). Therefore, the problem is reduced to finding a particular point in each parallel plane. Let P_0 be a particular point in the given plane. Then a point in a parallel plane can be obtained from it by moving P_0 by a distance of 3 units along a straight line segment parallel to the normal $\mathbf{n}$. If $\mathbf{r}_0$ is the position vector of P_0, then a point on a parallel plane has a position vector $\mathbf{r}_0 + s\mathbf{n}$, where the displacement vector $s\mathbf{n}$ must have a length of 3, or $\|s\mathbf{n}\| = |s|\|\mathbf{n}\| = 3|s| = 3$ and therefore $s = \pm 1$. Naturally, there should be two planes parallel to the given one and at the same distance from it. To find a particular point on the given plane, one can set two coordinates to 0 and find the value of the third coordinate from the equation of the plane. Take, for instance, $P_0 = (1, 0, 0)$. Particular points on the parallel planes are

$$\mathbf{r}_0 \pm \mathbf{n} = \langle 1, 0, 0\rangle \pm \langle 2, -1, 2\rangle = \begin{cases} \langle 3, -1, 2\rangle \\ \langle -1, 1, -2\rangle \end{cases}.$$

Using these points in the standard equation of a plane, the equations of two parallel planes are obtained:

$$\mathbf{r}\cdot\mathbf{n} = (\mathbf{r}_0 \pm \mathbf{n})\cdot\mathbf{n} = 2 \pm 9 \quad \Rightarrow \quad \begin{aligned} 2x - y + 2z &= 11 \\ 2x - y + 2z &= -7 \end{aligned}.$$

An *alternative* algebraic solution is based on the distance formula (**6.3**) for parallel planes. An equation of a plane parallel to the given one should have the form $2x - y + 2z = d$. The number d is determined by solving Eq. (**6.3**) where $D = 3$:

$$\frac{|d-2|}{\|\mathbf{n}\|} = 3 \quad \Rightarrow \quad |d-2| = 9 \quad \Rightarrow \quad d = \pm 9 + 2\,.$$

□

6.4. Study Problems.

Problem **6.1**. *Find an equation of the plane that is normal to a straight line segment AB and bisects it if $A = (1, 1, 1)$ and $B = (-1, 3, 5)$.*

Solution: One has to find a particular point in the plane and its normal. Since AB is perpendicular to the plane, $\mathbf{n} = \overrightarrow{AB} = \langle -2, 2, 4\rangle$. The midpoint of the segment lies in the plane. Hence, $P_0 = (0, 2, 3)$ (the coordinates of the midpoints are the half sums of the corresponding coordinates of the endpoints by Study Problem **1.1**). The equation reads

$$-2x + 2(y-2) + 4(z-3) = 0 \quad \Rightarrow \quad -x + y + 2z = 8\,.$$

□

Problem **6.2**. *Find the plane through the point $P_0 = (1, 2, 3)$ that is perpendicular to the planes $x + y + z = 1$ and $x - y + 2z = 1$.*

Solution: One has to find a particular point in the plane and any vector orthogonal to it. The first part of the problem is easy to solve: P_0 is given. Let $\mathbf{n}$ be a normal of the plane in question. Then, from the geometrical description of a plane, it follows that

$$\mathbf{n} \perp \mathbf{n}_1 = \langle 1, 1, 1\rangle \quad \text{and} \quad \mathbf{n} \perp \mathbf{n}_2 = \langle 1, -1, 2\rangle\,.$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals of the given planes. So $\mathbf{n}$ is a vector orthogonal to two given vectors. By the geometrical property of the cross product, such a vector can be constructed as

$$\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2+1,\, -(2-1),\, -1-1\rangle = \langle 3, -1, -2\rangle\,.$$

Hence, the equation reads

$$3(x-1) - (y-2) - 2(z-3) = 0 \quad \text{or} \quad 3x - y - 2z = -5\,.$$

□

Problem 6.3. *Determine whether two planes $x + 2y - 2z = 1$ and $2x + 4y + 4z = 10$ are parallel and, if not, find the angle between them.*

Solution: The normals are $\mathbf{n}_1 = \langle 1, 2, -2\rangle$ and $\mathbf{n}_2 = \langle 2, 4, 4\rangle = 2\langle 1, 2, 2\rangle$. They are not proportional. Hence, the planes are not parallel. Since $\|\mathbf{n}_1\| = 3$, $\|\mathbf{n}_1\| = 6$, and $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2$, the angle is determined by $\cos\theta = 2/18 = 1/9$ or $\theta = \cos^{-1}(1/9)$. □

Problem 6.4. *Find a family of all planes that contains the straight line segment AB if $A = (1, 2, -1)$ and $B = (2, 4, 1)$.*

Solution: All the planes in question contain the point A. So it can be chosen as a particular point in every plane. Since the segment AB lies in every plane of the family, the question amounts to describing all vectors orthogonal to $\mathbf{a} = \overrightarrow{AB} = \langle 1, 2, 2\rangle$ which determine the normals of the planes in the family. It is easy to find a particular vector orthogonal to $\mathbf{a}$. For example, $\mathbf{b} = \langle 0, 1, -1\rangle$ is orthogonal to $\mathbf{a}$ because $\mathbf{a} \cdot \mathbf{b} = 0$. Next, the vector $\mathbf{a} \times \mathbf{b} = \langle -4, 1, 1\rangle$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Any vector orthogonal to $\mathbf{a}$ lies in a plane orthogonal to $\mathbf{a}$ and hence must be a linear combination of any two non-parallel vectors in this plane. So the sought-after normals are all linear combinations of $\mathbf{b}$ and $\mathbf{c} = \mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{n} = 0 \quad \Leftrightarrow \quad \mathbf{n} = s\mathbf{b} + t\mathbf{c}\,,$$

for any non-zero choice of numbers s and t, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually orthogonal, by construction. Since the length of each normal is irrelevant, the family of the planes is described by all unit vectors orthogonal to $\mathbf{a}$. Recall that any unit vector in a plane can be written in the form

$$\hat{\mathbf{n}}_\theta = \cos\theta\,\hat{\mathbf{u}}_1 + \sin\theta\,\hat{\mathbf{u}}_2\,,$$

where $\hat{\mathbf{u}}_{1,2}$ are two unit orthogonal vectors in the plane and $0 \le \theta < 2\pi$. Indeed,

$$\begin{aligned}\hat{\mathbf{n}} = s\hat{\mathbf{u}}_1 + t\hat{\mathbf{u}}_2 \quad \Rightarrow \quad 1 &= \|\hat{\mathbf{u}}\|^2 = (s\hat{\mathbf{u}}_1 + t\hat{\mathbf{u}}_2)\cdot(s\hat{\mathbf{u}}_1 + t\hat{\mathbf{u}}_2) \\ &= s^2\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_1 + t^2\hat{\mathbf{u}}_2\cdot\hat{\mathbf{u}}_2 + 2st\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2 \\ &= s^2 + t^2 \\ &\Rightarrow\ s = \cos\theta,\ t = \sin\theta\,.\end{aligned}$$

So put $\mathbf{n} = \mathbf{n}_\theta$ where

$$\hat{\mathbf{u}}_1 = \hat{\mathbf{b}} = \frac{1}{\|\mathbf{b}\|}\,\mathbf{b} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle\,, \quad \hat{\mathbf{u}}_2 = \hat{\mathbf{c}} = \frac{1}{\|\mathbf{c}\|}\,\mathbf{c} = \frac{1}{3\sqrt{2}}\langle -4, 1, 1\rangle\,.$$

The range for the parameter θ must be restricted to $0 \le \theta < \pi$ because $\mathbf{n}_{\theta+\pi} = -\mathbf{n}_\theta$. Therefore no new plane is obtained for $\pi \le \theta < 2\pi$. The family of the planes is described by equations

$$\hat{\mathbf{n}}_\theta \cdot (\mathbf{r} - \mathbf{r}_0) = 0\,.$$

where $0 \le \theta < \pi$ and $\mathbf{r}_0 = \langle 1, 2, -1\rangle$ (the position vector of A). After some simple algebraic transformations this equation assumes the form

$$(-4\sin\theta)x + (3\cos\theta + \sin\theta)y + (\sin\theta - 3\cos\theta)z = 9\cos\theta - 3\sin\theta\,.$$

□

6.5. Exercises.

1. Find an equation of the plane through the origin and parallel to the plane $2x - 2y + z = 4$. What is the distance between the two planes?

2. Do the planes $2x + y - z = 1$ and $4x + 2y - 2z = 10$ intersect?

3. Determine whether the planes $2x + y - z = 3$ and $x + y + z = 1$ are intersecting. If they are, find the angle between them.

4–**6**. Consider a parallelepiped with one vertex at the origin O at which the adjacent sides are the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 2, 1, 1\rangle$, and $\mathbf{c} = \langle -1, 0, 1\rangle$. Let OP be its diagonal extended from the vertex O. Find equations of the following planes.

- **4**. The planes that contain the faces of the parallelepiped.
- **5**. The planes that contain the diagonal OP and the diagonal of each of three its faces adjacent at P.
- **6**. The planes that contain parallel diagonals in the opposite faces of the parallelepiped.

7. Find an equation of the plane with x intercept $a \ne 0$, y intercept $b \ne 0$, and z intercept $c \ne 0$. What is the distance between the origin and the plane? Find the angles between the plane and the coordinate planes.

8. Show that the points $A = (1, 1, 1)$, $B = (1, 2, 3)$, $C = (2, 0, -1)$ and

$D = (3, 1, 0)$ are not in a plane and therefore vertices of a tetrahedron. Any two of the four faces of the tetrahedron are intersecting along one of its six edges. Find the angles of intersection of the face BCD with the other three faces.

9. Find equations of all planes that are perpendicular to the line through $(1, -1, 1)$ and $(3, 0, -1)$ and that are at the distance 2 from the point $(1, 2, 3)$.

10. Find an equation for the set of points that are equidistant from the points $(1, 2, 3)$ and $(-1, 2, 1)$. Give a geometrical description of the set.

11. Find an equation of the plane that is perpendicular to the plane $x + y + z = 1$ and contains the line through the points $(1, 2, 3)$ and $(-1, 1, 0)$.

12. To which of the planes $x + y + z = 1$ and $x + 2y - z = 2$ is the point $(1, 2, 3)$ the closest?

13–15. Give a geometrical description of each of the following families of planes where c is a numerical parameter.

13. $x + y + z = c$.
14. $x + y + cz = 1$.
15. $x \sin c + y \cos c + z = 1$.

16. Find values of c for which the plane $x+y+cz = 1$ is closest to the point $P = (1, 2, 1)$ and farthest from P.

17. Consider three planes with normals $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ such that each pair of the planes is intersecting. Under what condition on the normals are the three lines of intersection parallel or even coincide?

18. Find equations of all the planes that are perpendicular to the plane $x + y + z = 1$, have the angle $\pi/3$ with the plane $x + y = 1$, and pass through the point $(1, 1, 1)$.

19. Let $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 1, 0, -1\rangle$. Find an equation of the plane that contains the point $(1, 2, -1)$, the vector $\mathbf{a}$ and a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

20. Consider the plane $\mathcal{P}$ through three points $A = (1, 1, 1)$, $B = (2, 0, 1)$ and $C = (-1, 3, 2)$. Find all the planes that contain the segment AB and have the angle $\pi/3$ with the plane $\mathcal{P}$. Hint: see Study Problem **6.4**.

21. Find an equation of the plane that contains the line through $(1, 2, 3)$ and $(2, 1, 1)$ and cuts the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$ into two hemispheres.

22. Find all planes perpendicular to $\mathbf{n} = \langle 1, 1, 1\rangle$ whose intersection with the ball $x^2 + y^2 + z^2 \le R^2$ is a disk of area $\pi R^2/4$.

23. Find an equation of the plane that is tangent to the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 11 = 0$ at the point $(2, 1, 2)$. *Hint*:

What is the angle between a line tangent to a circle at a point P and the segment OP where O is the center of the circle? Extend this observation to a plane tangent to a sphere to determine a normal of the tangent plane.

24. Find the family of planes through the point $(0, 0, a)$, $a > R$, that are tangent to the sphere $x^2+y^2+z^2 = R^2$. *Hint*: Compare this family with the family of planes in Exercise **15**.

25. Consider a sphere of radius R centered at the origin and two points P_1 and P_2 whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$. Suppose that $\|\mathbf{r}_1\| > R$ and $\|\mathbf{r}_2\| > R$ (the points are outside the sphere). Find the equation $\mathbf{n}\cdot\mathbf{r} = d$ of the plane through P_1 and P_2 whose distance from the sphere is maximal. What is the distance? Hint: Show first that a normal of the plane can always be written in the form $\mathbf{n} = \mathbf{r}_1 + c(\mathbf{r}_2 - \mathbf{r}_1)$. Then find a condition to determine the constant c.

7. Lines in Space

Let a straight line pass through points A and B. By Section **1.3**, for any point P in the segment AB, the distances between the three points A, B, and P satisfy the condition $|AB| = |AP| + |PB|$. Here the objective is to give an algebraic description of a line in terms of conditions on coordinates of its points relative to some coordinate system just as planes in space were described in the previous Section.

Let us first reformulate the above geometrical description of a line (in terms of distances) using the vector algebra. Consider the line that coincides with a coordinate axis of a rectangular coordinate system, say, the x axis. Any point in it has the characteristic property that its position vector is proportional to the position vector of a particular point. For example, if $\mathbf{r}$ is a position vector of a point on the x axis, then $\mathbf{r} = x\hat{\mathbf{e}}_1 = \langle x, 0, 0\rangle$ for some x. By suitable rotations and translations of the coordinate system, the x axis can be transformed to any given line. Since translations and rotations preserve distances and angles, any line in space can also be given the following equivalent definition:

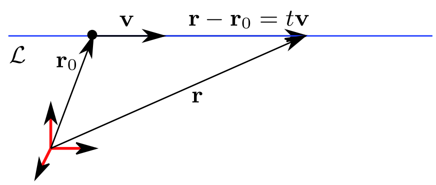
> *a line in space is a set of points whose position vectors relative to a particular point in the set are parallel to a given nonzero vector* $\mathbf{v}$.

The vector $\mathbf{v}$ is called a *tangent vector* of the line. It is not unique because any vector parallel to $\mathbf{v}$ is a also a tangent vector of the same line. Thus, the line is uniquely defined by a particular point in it and a non-zero vector parallel to the line. In order for a point P to be in a line $\mathcal{L}$ that passes through a point P_0 and is parallel to a non-zero vector $\mathbf{v}$, the vector $\overrightarrow{P_0P}$ must be parallel to $\mathbf{v}$:

$$P \in \mathcal{L} \quad \Leftrightarrow \quad \overrightarrow{P_0P} \parallel \mathbf{v}\,, \quad \mathbf{v} \parallel \mathcal{L}\,,\ P_0 \in \mathcal{L}$$

This geometrical condition is easy to turn into an algebraic condition in some coordinate system.

7.1. An Algebraic Description of a Line. In some coordinate system, a particular point P_0 of a line $\mathcal{L}$ has coordinates (x_0, y_0, z_0), and a vector parallel to $\mathcal{L}$ is defined by its components, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Let $\mathbf{r} = \langle x, y, z\rangle$ be a position vector of a generic point of $\mathcal{L}$ and let $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ be the position vector of P_0. Then the vector $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ is the position vector of P relative to P_0. By the geometrical description of the line, it must be parallel to $\mathbf{v}$. Since any two parallel vectors are proportional, a point (x, y, z) belongs to $\mathcal{L}$ if and only if $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$ for some real t (see the left panel of Fig. **7.1**).



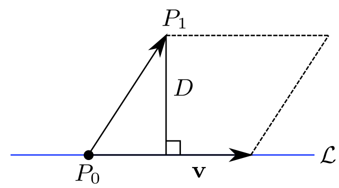


FIGURE **7.1**. **Left**: Algebraic description of a line $\mathcal{L}$ through $\mathbf{r}_0$ and parallel to a vector $\mathbf{v}$. If $\mathbf{r}_0$ and $\mathbf{r}$ are position vectors of particular and generic points of the line, then the vector $\mathbf{r}-\mathbf{r}_0$ is parallel to the line and hence must be proportional to a vector $\mathbf{v}$, that is, $\mathbf{r}-\mathbf{r}_0=t\mathbf{v}$ for some real number t. **Right**: Distance between a point P_1 and a line $\mathcal{L}$ through a point P_0 and parallel to a vector $\mathbf{v}$. It is the height of the parallelogram whose adjacent sides are the vectors $\vec{P_0P_1}$ and $\mathbf{v}$.

THEOREM **7.1**. (Vector and parametric equations of a line). *The coordinates of the points of the line $\mathcal{L}$ through a point $P_0=(x_0,y_0,z_0)$ and parallel to a vector $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ satisfy the vector equation*

$$\mathbf{r}=\mathbf{r}_0+t\mathbf{v}\,,\quad -\infty<t<\infty. \tag{7.1}$$

or the parametric equations

$$x=x_0+tv_1\,,\quad y=y_0+tv_2\,,\quad z=z_0+tv_3\,,\quad -\infty<t<\infty. \tag{7.2}$$

The parametric equations of the line can be solved for t. If none of the components of $\mathbf{v}$ vanishes, then $t=(x-x_0)/v_1$, $t=(y-y_0)/v_2$, and $t=(z-z_0)/v_3$. Equating the right sides of these equations, *symmetric* equations of a line are obtained that do not involve any parameter.

COROLLARY **7.1**. (Symmetric equations of a line) *Coordinates (x,y,z) of points of a line through a point (x_0,y_0,z_0) and parallel to a vector $\langle v_1,v_2,v_3\rangle$ with non-vanishing components satisfy the equations*

$$\frac{x-x_0}{v_1}=\frac{y-y_0}{v_2}=\frac{z-z_0}{v_3}, \tag{7.3}$$

Equations (**7.3**) make sense only if all the components of $\mathbf{v}$ do not vanish. If, say, $v_1=0$, then the first equation in (**7.2**) does not contain the parameter t at all. So the symmetric equations are written in the form

$$x=x_0\,,\quad \frac{y-y_0}{v_2}=\frac{z-z_0}{v_3}.$$

The first and second equations in (**7.3**) can also be written in the form

$$\begin{cases} v_2(x-x_0) - v_1(y-y_0) = 0 \\ v_3(y-y_0) - v_2(z-z_0) = 0 \end{cases}$$

Each of these equations describes a plane. So symmetric equations define a line in space is an intersection of two planes through the point (x_0, y_0, z_0) and whose normals are $\mathbf{n}_1 = \langle v_2, -v_1, 0\rangle$ and $\mathbf{n}_2 = \langle 0, v_3, -v_2\rangle$, respectively:

$$\mathcal{L} = \mathcal{P}_1 \cap \mathcal{P}_2\,, \quad \mathcal{P}_1 \perp \mathbf{n}_1 = \langle v_2, -v_1, 0\rangle\,,\ \mathcal{P}_2 \perp \mathbf{n}_2 = \langle 0, v_3, -v_2\rangle\,.$$

The reader is advised to verify that the vectors $\mathbf{v}$ and $\mathbf{n}_1 \times \mathbf{n}_2$ are parallel. Why? (See Example **7.2**.)

EXAMPLE **7.1**. *Find the vector, parametric, and symmetric equations of the line through the points $A = (1, 1, 1)$ and $B = (1, 2, 3)$. Give equations of two planes whose intersection is the line through AB.*

SOLUTION: Take $\mathbf{v} = \overrightarrow{AB} = \langle 0, 1, 2\rangle$ and $P_0 = A$. Then

$$\begin{aligned} &\mathbf{r} = \langle 1,1,1\rangle + t\langle 0,1,2\rangle, \\ &x = 1\,, \quad y = 1+t\,, \quad z = 1+2t\,, \\ &x = 1\,, \quad y - 1 = \frac{z-1}{2} \end{aligned}$$

are the vector, parametric, and symmetric equations of the line, respectively. According to the symmetric equations, the line is the intersection of two planes:

$$\begin{cases} x = 1 \\ 2y - z = 1 \end{cases}$$

□

Clearly, a line can always be described as the set of points of intersection of two non-parallel planes. Since the line of intersection lies in each plane, it must be orthogonal to the normals of these planes. Therefore a vector parallel to the line can always be chosen as the cross product of the normals.

EXAMPLE **7.2**. *Find the line that is the intersection of the planes $x + y + z = 1$ and $2x - y + z = 2$.*

SOLUTION: The normals of the planes are $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and $\mathbf{n}_2 = \langle 2, -1, 1\rangle$. So the vector

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 1, -3\rangle$$

is parallel to the line. To find a particular point of the line, note that its *three* coordinates (x_0, y_0, z_0) should satisfy *two* equations of the planes.

So one can choose one of the coordinates at will and find the other two from the equations of the planes. It follows from the parametric equations (**7.2**) that if, for example, $v_3 \neq 0$, then there is a value of t at which z vanishes, meaning that the line always contains a point with the vanishing z coordinate. Since $v_3 = -3 \neq 0$ for the line in question, put $z_0 = 0$. Then

$$\begin{cases} x_0 + y_0 = 1 \\ 2x_0 - y_0 = 2 \end{cases}$$

By adding these equation, $x_0 = 1$ and, hence, $y_0 = 0$. The parametric equations of the line of intersection are

$$x = 1 + 2t\,, \quad y = t\,, \quad z = -3t\,.$$

Alternative Solution: *Three* coordinates (x, y, z) of points of the line of intersection satisfy *two* equations of the planes. This system of equations can be solved for two variables, while the other variable is viewed as parameter:

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 1 - \tfrac{2}{3}z \\ y &= -\tfrac{1}{3}z \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 1 - \tfrac{2}{3}s \\ y &= -\tfrac{1}{3}s \\ z &= s \end{aligned}$$

where $-\infty < s < \infty$. The last three equations are parametric equations of the line of intersection. They appear to be different from those found before. Nevertheless, they describe the same set of points in space. Indeed, by setting $s = -3t$ the equivalence is established. This freedom in parametric equations of a line is easy to understand. The choice of a particular point of the line and a vector parallel to the line is not unique. In the above two sets of parametric equations, the vectors parallel to the line are different, $\mathbf{v} = \langle 2, 1, -3\rangle$ and $\mathbf{u} = \langle -\frac{2}{3}, -\frac{1}{3}, 1\rangle$ (note $\mathbf{v} = -3\mathbf{u}$), while a particular point is the same, $(1, 0, 0)$. □

Distance Between a Point and a Line. Let $\mathcal{L}$ be a line through P_0 and parallel to $\mathbf{v}$. What is the distance between a given point P_1 and the line $\mathcal{L}$? Consider a parallelogram with vertex P_0 and whose adjacent sides are the vectors $\mathbf{v}$ and $\overrightarrow{P_0P_1}$ as depicted in Figure **7.1** (right panel). The distance in question is the height of this parallelogram, which is therefore its area divided by the length of the base $\|\mathbf{v}\|$. If $\mathbf{r}_0$ and $\mathbf{r}_1$ are position vectors of P_0 and P_1, then $\overrightarrow{P_0P_1} = \mathbf{r}_1 - \mathbf{r}_0$ and hence

$$(\mathbf{7.4}) \qquad D = \frac{\|\mathbf{v} \times \overrightarrow{P_0P_1}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times (\mathbf{r}_1 - \mathbf{r}_0)\|}{\|\mathbf{v}\|}.$$

Equation (**7.4**) also follows from Corollary **5.4** because the distance between a line $\mathcal{L}$ and a point P_1 is the distance between $\mathcal{L}$ and a parallel line through P_1. So, in Corollary **5.4**, put $\mathbf{v} = \overrightarrow{AB}$, $A = P_0$, and $C = P_1$ to obtain Eq. (**7.4**).

7.2. Relative Positions of Lines in Space. Two lines in space can be intersecting, parallel, or skew. The criterion for relative positions of the lines in space is stated in Corollary **5.6**. Given an algebraic description of the lines established here, it can now be restated as follows.

COROLLARY **7.2**. (Relative positions of lines in space)
Let $\mathcal{L}_1$ be a line through P_1 and parallel to a vector $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathcal{L}_2$ be a line through P_2 and parallel to a vector $\mathbf{v}_2 \neq \mathbf{0}$. Put $\mathbf{r}_{12} = \overrightarrow{P_1P_2}$. Then

(1) *$\mathcal{L}_1$ and $\mathcal{L}_2$ are skew if and only if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}_{12}$ are not coplanar, or $\mathbf{r}_{12} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \neq 0$;*
(2) *$\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at a point if and only if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}_{12}$ are coplanar, but $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, or $\mathbf{r}_{12} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 0$ and $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$;*
(3) *$\mathcal{L}_1$ and $\mathcal{L}_2$ coincide if and only if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}_{12}$ are parallel, or $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ and $\mathbf{r}_{12} \times \mathbf{v}_1 = \mathbf{0}$;*
(4) *$\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, but $\mathbf{r}_{12}$ is not parallel to them, or $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ and $\mathbf{r}_{12} \times \mathbf{v}_1 \neq \mathbf{0}$.*

Indeed, the vector $\mathbf{v}_1$ can always be viewed as a vector with initial and terminal points on the line $\mathcal{L}_1$. The same observation is true for the vector $\mathbf{v}_2$ and the line $\mathcal{L}_2$. With this observation the equivalence of Corollary **5.6** to Corollary **7.2** is obvious.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be intersecting. How can one find the point of intersection? To solve this problem, consider the vector equations for the lines

$$\begin{aligned} \mathcal{L}_1 : &\quad \mathbf{r}_t = \mathbf{r}_1 + t\mathbf{v}_1 \\ \mathcal{L}_2 : &\quad \mathbf{r}_s = \mathbf{r}_2 + s\mathbf{v}_2 \end{aligned}$$

When changing the parameter t, the terminal point of $\mathbf{r}_t$ slides along the line $\mathcal{L}_1$, while the terminal point of $\mathbf{r}_s$ slides along the line $\mathcal{L}_2$ when changing the parameter s as depicted in Figure **7.2** (left panel). Note that the parameters of both lines are not related in any way according to the geometrical description of the lines. If two lines are intersecting, then there should exist a pair of numbers $(t, s) = (t_0, s_0)$ at which the terminal points of vectors $\mathbf{r}_t$ and $\mathbf{r}_s$ coincide:

$$\mathbf{r}_t = \mathbf{r}_s \quad \Rightarrow \quad (t, s) = (t_0, s_0)$$

If this equation has no solution, then the lines are not intersecting. The vector $\mathbf{r}_t$ at $t = t_0$ (or $\mathbf{r}_s$ at $s = s_0$) is the position vector of the point of

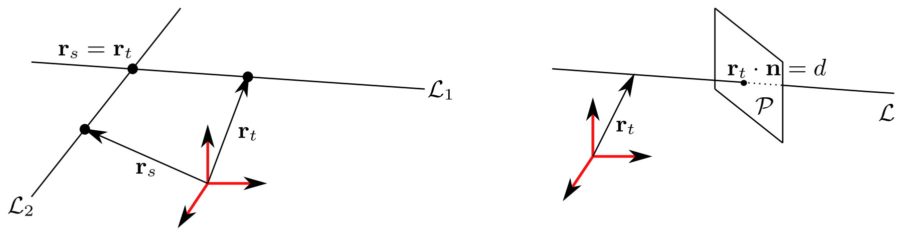


FIGURE **7.2**. **Left**: Intersection point of two lines $\mathcal{L}_1$ and $\mathcal{L}_2$. The terminal point of the vector $\mathbf{r}_t$ traverses $\mathcal{L}_1$ as t ranges over all real numbers, while the terminal point of the vector $\mathbf{r}_s$ traverses $\mathcal{L}_2$ as s ranges over all real numbers independently of t. If the lines are intersecting, then there should exist a pair of numbers $(t, s) = (t_0, s_0)$ such that the vectors $\mathbf{r}_t$ and $\mathbf{r}_s$ coincide, which means that their components must be the same. This defines three equations on two variables t and s. **Right**: The point of intersection of a line $\mathcal{L}$ and a plane $\mathcal{P}$. The terminal point of the vector $\mathbf{r}_t$ traverses $\mathcal{L}$ as t ranges over all real numbers. If the line intersects the plane defined by the equation $\mathbf{r} \cdot \mathbf{n} = d$, then there should exist a particular value of t at which the vector $\mathbf{r}_t$ satisfies the equation of the plane: $\mathbf{r}_t \cdot \mathbf{n} = d$.

intersection. Let $\mathbf{v}_i = \langle a_i, b_i, c_i \rangle$, $i = 1, 2$. Writing the vector equation $\mathbf{r}_t = \mathbf{r}_s$ in components, the following system of equations is obtained:

$$\begin{aligned} x_1 + ta_1 &= x_2 + sa_2, \\ y_1 + tb_1 &= y_2 + sb_2, \\ z_1 + tc_1 &= z_2 + sc_2. \end{aligned}$$

This system of equations is solved in a conventional manner, e.g., by expressing t via s from the first equation, substituting it into the second and third ones, and verifying that the resulting two equations have the *same* solution for s. Note that the system has three equations for only two variables. It is an *overdetermined* system, which may or may not have a solution. If the conditions of Parts (1) or (4) in Corollary **7.2** are satisfied, then the system has no solution (the lines are skew or parallel). If the conditions of Part (3) in Corollary **7.2** are fulfilled, then there is a unique solution. Naturally, if the lines coincide there will be infinitely many solutions. Let $(t, s) = (t_0, s_0)$ be a unique solution. Then the position vector of the point of intersection is $\mathbf{r}_1 + t_0\mathbf{v}_1$ or $\mathbf{r}_2 + s_0\mathbf{v}_2$:

$$\mathcal{L}_1 \cap \mathcal{L}_2 = P_0\,, \quad \overrightarrow{OP_0} = \mathbf{r}_1 + t_0\mathbf{v}_1 = \mathbf{r}_2 + s_0\mathbf{v}_2\,.$$

Two lines intersecting at a point P_0 lie in a plane. Each of the lines can be obtained from the other by rotation about P_0 in this plane. The *smallest* rotation angle is called the angle between the lines. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors parallel to the first and second line, respectively, then the angle between the lines coincides with the angle θ between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ if $0 \leq \theta \leq \pi/2$. If $\pi/2 < \theta \leq \pi$, then the smallest rotation angle is $\pi - \theta$. So the angle between two intersecting lines can be defined similarly to the angle between two intersecting planes (Definition **6.1**).

DEFINITION **7.1**. (Angle between two intersecting lines). *If* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are vectors parallel to two intersecting lines, then the angle* $0 \leq \theta \leq \pi/2$ *satisfying the equation*

$$\cos\theta = \frac{|\mathbf{v}_1 \cdot \mathbf{v}_2|}{\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|} = |\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2|$$

is called the angle between the lines.

EXAMPLE **7.3**. *Determine whether the lines*

$$\mathcal{L}_1: \quad x - 1 = y - 1 = -\frac{z+1}{2}$$
$$\mathcal{L}_2: \quad \frac{x-1}{2} = y = -\frac{z}{3}$$

are skew, intersecting, or parallel. If they are intersecting, find the point of intersection and the angle of intersection.

SOLUTION: The first line contains the point $P_1 = (1, 1, -1)$ and is parallel to the vector $\mathbf{v}_1 = \langle 1, 1, -2\rangle$. The second line contains the point $P_2 = (1, 0, 0)$ and is parallel to the vector $\mathbf{v}_2 = \langle 2, 1, -3\rangle$. So the lines are traversed by the position vectors

$$\mathcal{L}_1: \quad \mathbf{r}_t = \langle 1 + t, 1 + t, -1 - 2t\rangle$$
$$\mathcal{L}_2: \quad \mathbf{r}_s = \langle 1 + 2s, s, -3s\rangle$$

If the lines are intersecting, then the vector equation $\mathbf{r}_t = \mathbf{r}_s$ should have a solution for some (t, s). The vector equation implies that:

$$1 + t = 1 + 2s\,, \quad 1 + t = s\,, \quad -1 - 2t = -3s$$

This system has a unique solution $t = -2$ and $s = -1$. So the lines are intersecting at the point whose position vector is

$$\mathcal{L}_1 \cap \mathcal{L}_2: \quad \mathbf{r}_t\Big|_{t=-2} = \mathbf{r}_s\Big|_{s=-1} = \langle -1, -1, 3\rangle\,.$$

The angle between the lines is calculated by Definition **7.1**. One has $\mathbf{v}_1 \cdot \mathbf{v}_2 = 9$, $\|\mathbf{v}_1\| = \sqrt{6}$, and $\|\mathbf{v}_2\| = \sqrt{14}$. Therefore

$$\cos\theta = \frac{9}{\sqrt{6}\sqrt{14}} = \frac{3\sqrt{3}}{2\sqrt{7}} \quad \Rightarrow \quad \theta \approx 10.9^\circ$$

□

7.3. Relative Positions of Lines and Planes. Consider a line $\mathcal{L}$ and a plane $\mathcal{P}$:

$$\mathcal{L}: \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}\,; \qquad \mathcal{P}: \quad \mathbf{r}\cdot\mathbf{n} = d\,.$$

The question of interest is to determine whether they are intersecting or parallel. If the line does not intersect the plane (they have no common points), it must be in a plane parallel to $\mathcal{P}$. Therefore a vector $\mathbf{v}$ parallel to $\mathcal{L}$ must be orthogonal to a normal $\mathbf{n}$ of $\mathcal{P}$. A line perpendicular to $\mathbf{n}$ can also lie in the plane $\mathcal{P}$, which occurs if and only if a particular point $\mathbf{r}_0$ lies in the plane.

Corollary **7.3**. (Criterion for a line and a plane to be parallel). *A line $\mathcal{L}$, $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, is parallel to a plane $\mathcal{P}$, $\mathbf{r}\cdot\mathbf{n} = d$, if and only if a vector parallel to the line $\mathbf{v}$ and a normal $\mathbf{n}$ of the plane are orthogonal and a particular point of $\mathcal{L}$ is not in the plane $\mathcal{P}$:*

$$\mathcal{L} \parallel \mathcal{P} \quad \Leftrightarrow \quad \mathbf{v} \perp \mathbf{n} \text{ and } \mathbf{r}_0 \notin \mathcal{P} \quad \Leftrightarrow \quad \mathbf{v}\cdot\mathbf{n} = 0 \text{ and } \mathbf{r}_0\cdot\mathbf{n} \neq d.$$

If $\mathbf{v}$ and $\mathbf{n}$ are not orthogonal, $\mathbf{v}\cdot\mathbf{n} \neq 0$, then the line and the plane have one point of intersection and there should exist a unique value of the parameter t for which the position vector $\mathbf{r}_t = \mathbf{r}_0 + t\mathbf{v}$ of a point of $\mathcal{L}$ also satisfies an equation of the plane $\mathbf{r}\cdot\mathbf{n} = d$ (see Figure **7.2**, right panel). The value of the parameter t that corresponds to the point of intersection is determined by the equation

$$\mathbf{r}_t\cdot\mathbf{n} = d \quad \Rightarrow \quad \mathbf{r}_0\cdot\mathbf{n} + t\mathbf{v}\cdot\mathbf{n} = d \quad \Rightarrow \quad t = \frac{d - \mathbf{r}_0\cdot\mathbf{n}}{\mathbf{v}\cdot\mathbf{n}}.$$

The position vector of the point of intersection is found by substituting this value of t into the vector equation of the line $\mathbf{r}_t = \mathbf{r}_0 + t\mathbf{v}$. Note that if the equation $\mathbf{r}_t\cdot\mathbf{n} = d$ is satisfied for *all* values of t, which is only possible if $\mathbf{v}\cdot\mathbf{n} = 0$ and $\mathbf{r}_0\cdot\mathbf{n} = d$, then the line lies in the plane.

Example **7.4**. *A point object is traveling along the line*

$$\mathcal{L}: \quad x - 1 = \frac{y}{2} = \frac{z+1}{2}$$

with a constant speed $v = 6$ meters per second. If all coordinates are measured in meters and the initial position vector of the object is $\mathbf{r}_0 =$

$\langle 1, 0, -1\rangle$, *when does it reach the plane* $2x + y + z = 13$? *What is the distance traveled by the object?*

SOLUTION: Parametric equations of the line are

$$x = 1 + s\,, \quad y = 2s\,, \quad z = -1 + 2s\,.$$

The value of the parameter s at which the line intersects the plane is determined by the substitution of these equations into the equation of the plane:

$$2(1 + s) + 2s + (-1 + 2s) = 13 \quad \Leftrightarrow \quad 6s = 12 \quad \Leftrightarrow \quad s = 2$$

So the position vector of the point of intersection is $\mathbf{r} = \langle 3, 4, 3\rangle$. The distance between it and the initial point is

$$D = \|\mathbf{r} - \mathbf{r}_0\| = \|\langle 2, 4, 4\rangle\| = \|2\langle 1, 2, 2\rangle\| = 2\|\langle 1, 2, 2\rangle\| = 6$$

meters and the travel time is $T = D/v = 1$ sec. □

Remark. In this example the parameter s does not coincide with the physical time. If an object travels with a constant speed v along the line through $\mathbf{r}_0$ and parallel to a *unit* vector $\hat{\mathbf{v}}$, then its velocity vector is $\mathbf{v} = v\hat{\mathbf{v}}$ and its position vector is $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ where t is the physical time. Indeed, the vector $\mathbf{r} - \mathbf{r}_0$ is the displacement vector of the object along its trajectory and hence its length determines the distance traveled by the object: $\|\mathbf{r} - \mathbf{r}_0\| = \|\mathbf{v}t\| = vt$, which shows that the parameter $t > 0$ is the travel time.

EXAMPLE **7.5**. *Find an equation of the plane* $\mathcal{P}$ *that is perpendicular to the plane* $\mathcal{P}_1$, $x + y - z = 1$, *and contains the line* $x - 1 = y/2 = z + 1$.

SOLUTION: The plane $\mathcal{P}$ contains the given line and therefore its normal $\mathbf{n}$ must be orthogonal to the vector $\mathbf{v} = \langle 1, 2, 1\rangle$ that is parallel to the line (the components of $\mathbf{v}$ appear in the denominators in symmetric equations (**7.3**)). The plane $\mathcal{P}$ is perpendicular to the plane $\mathcal{P}_1$ and therefore $\mathbf{n}$ must be orthogonal to the normal $\mathbf{n}_1 = \langle 1, 1, -1\rangle$ of $\mathcal{P}_1$ (Definition **6.1**). So $\mathbf{n}$ is a nonzero vector orthogonal to both $\mathbf{n}_1$ and $\mathbf{v}$. Therefore, one can take

$$\begin{cases} \mathbf{n} \perp \mathbf{n}_1 = \langle 1, 1, -1\rangle \\ \mathbf{n} \perp \mathbf{v} \;\; = \langle 1, 2, 1\rangle \end{cases} \quad \Rightarrow \quad \mathbf{n} = \mathbf{n}_1 \times \mathbf{v} = \langle 3, -2, 1\rangle\,.$$

The line lies in $\mathcal{P}$ and therefore any of its points can be taken as a particular point of $\mathcal{P}$. For example, put $x = 1$ in the symmetric equations to get $0 = y/2 = z + 1$ and hence $P_0 = (1, 0, -1)$. An equation of $\mathcal{P}$ reads

$$3(x-1) - 2y + (z+1) = 0 \quad \text{or} \quad 3x - 2y + z = 2\,.$$

□

EXAMPLE 7.6. *Find the planes that are perpendicular to the line* $x = y/2 = -z/2$ *and have the distance* 3 *units from the point* $(-1, -2, 2)$ *on the line.*

SOLUTION: The line is parallel to the vector $\mathbf{v} = \langle 1, 2, -2\rangle$. So the planes have the same normal $\mathbf{n} = \mathbf{v}$. Particular points in the planes may be taken as the points of intersection of the line with the planes. These points are at the distance 3 from $\mathbf{r}_0 = \langle -1, -2, 2\rangle$ and their position vectors $\mathbf{r}$ should satisfy the condition $\|\mathbf{r} - \mathbf{r}_0\| = 3$. The substitution of the vector equation of the line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ into this condition yields the values of t at which the distance of 3 units from $\mathbf{r}_0$ along the line is reached:

$$3 = \|\mathbf{r} - \mathbf{r}_0\| = \|t\mathbf{v}\| = |t|\|\mathbf{v}\| = 3|t| \quad \Rightarrow \quad t = \pm 1\,.$$

So the position vectors of particular points in the planes are

$$\mathbf{r}\Big|_{t=\pm 1} = \mathbf{r}_0 \pm \mathbf{v} = \langle -1, -2, 2\rangle \pm \langle 1, 2, -2\rangle = \begin{cases} \langle 0, 0, 0\rangle \\ \langle -2, -4, 4\rangle \end{cases}$$

Equations of the planes are

$$\begin{aligned} &\mathcal{P}_1: \quad x + 2y - 2z = 0\,, \\ &\mathcal{P}_2: \quad (x+2) + 2(y+4) - 2(z-4) = 0 \quad \text{or} \quad x + 2y - 2z = -18\,. \end{aligned}$$

□

7.4. Parametric equations of a plane. Let P_0 be in a plane $\mathcal{P}$. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-parallel vectors in $\mathcal{P}$. Any vector in a plane is a linear combination of $\mathbf{a}$ and $\mathbf{b}$. Therefore for any point $P \in \mathcal{P}$ there exists a unique ordered pair of numbers (s, t) such that

$$\overrightarrow{P_0P} = s\mathbf{a} + t\mathbf{b}\,.$$

Let $\mathbf{r}$ and $\mathbf{r}_0$ be position vectors of the points P and P_0. Then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ so that

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{a} + t\mathbf{b}\,, \quad -\infty < t, s < \infty$$

This vector equation is equivalent to three scalar equations in some rectangular coordinate system. Put $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$. Then

$$x = x_0 + a_1 s + b_1 t\,, \quad y = y_0 + a_2 s + b_2 t\,, \quad z = z_0 + a_3 s + b_3 t\,,$$

where s and t are any real numbers (parameters). These equations are called *parametric equations* of a plane through a point (x_0, y_0, z_0) that contains two non-parallel vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$.

For a fixed value of s, parametric equation of the plane define a line $\mathcal{L}_s$ through the point $\mathbf{r}_s = \mathbf{r}_0 + s\mathbf{a}$ and parallel to the vector $\mathbf{b}$. So, all the lines $\mathcal{L}_s$, $-\infty < s < \infty$, are parallel, and the plane is the union of all such lines. Alternatively, if t is fixed, then the parametric equations of the plane define a line $\mathcal{L}_t$ through the point $\mathbf{r}_t = \mathbf{r}_0 + t\mathbf{b}$ and parallel to the vector $\mathbf{a}$. So all the lines $\mathcal{L}_t$, $-\infty < t < \infty$, are parallel, and the plane is their union:

$$\mathcal{P} = \bigcup_s \mathcal{L}_s = \bigcup_t \mathcal{L}_t$$

where the unions are taken over all real t and s.

7.5. Study Problems.

Problem **7.1**. *Let $\mathcal{L}_1$ be the line through $P_1 = (2, 3, 0)$ and parallel to $\mathbf{v}_1 = \langle 1, 2, -1\rangle$ and let $\mathcal{L}_2$ be the line through $P_2 = (0, -2, -2)$ and parallel to $\mathbf{v}_2 = \langle 2, 1, 0\rangle$. Determine whether the lines are parallel, intersecting, or skew and find the line $\mathcal{L}$ that is perpendicular to both $\mathcal{L}_1$ and $\mathcal{L}_2$ and intersects them.*

SOLUTION: The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not proportional, and hence the lines are either skew or intersecting by Corollary **7.2**. Put $\mathbf{r}_{12} = \overrightarrow{P_1P_2} = \langle -2, -5, -2\rangle$. Then

$$\mathbf{v}_1 \times \mathbf{v}_2 = \langle 1, -2, -3\rangle \quad \Rightarrow \quad \mathbf{r}_{12} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 14 \neq 0\,,$$

and the lines are skew by Corollary **7.2**. The vector $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$ is parallel to $\mathcal{L}$ because $\mathcal{L}$ is perpendicular to both the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. So the objective is to find a particular point of $\mathcal{L}$. Since $\mathcal{L}$ is intersecting $\mathcal{L}_1$ and $\mathcal{L}_2$, any of the points of intersection may be taken a particular point of $\mathcal{L}$. Let $\mathbf{r}_t = \mathbf{r}_1 + t\mathbf{v}_1$ be a position vector of a point of $\mathcal{L}_1$ and let $\mathbf{r}_s = \mathbf{r}_2 + s\mathbf{v}_2$ be a position vector of a point of $\mathcal{L}_2$ as shown in Figure **7.3** (left panel). The line $\mathcal{L}$ is orthogonal to both vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. As it intersects the lines $\mathcal{L}_1$ and $\mathcal{L}_2$, there should exist a pair of values (t, s) of the parameters at which the vector $\mathbf{r}_s - \mathbf{r}_t$ is parallel to $\mathcal{L}$; that is, the vector

$$\mathbf{r}_s - \mathbf{r}_t = \langle -2 + 2s - t,\ -5 + s - 2t,\ -2 + t\rangle$$

becomes orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$ when the values of t and s correspond to the points of intersection of $\mathcal{L}$ with $\mathcal{L}_1$ and $\mathcal{L}_2$. The corresponding

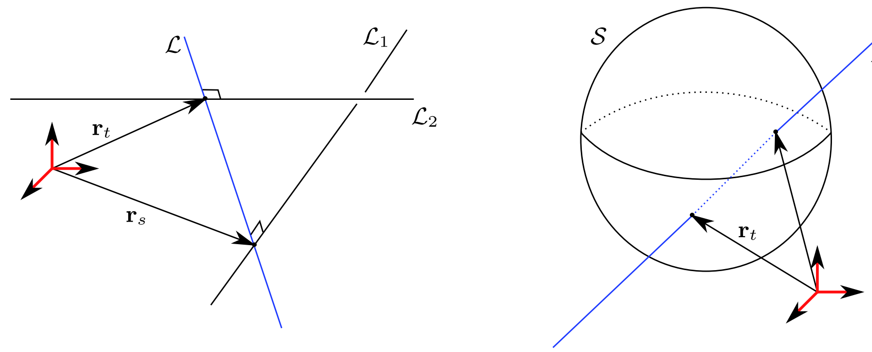


FIGURE **7.3**. **Left**: Illustration to Study Problem **7.1**. The vectors $\mathbf{r}_s$ and $\mathbf{r}_t$ trace out two given skewed lines $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. There are particular values of t and s at which the distance $\|\mathbf{r}_t - \mathbf{r}_s\|$ becomes minimal. Therefore, the line $\mathcal{L}$ through such points $\mathbf{r}_t$ and $\mathbf{r}_s$ is perpendicular to both $\mathcal{L}_1$ and $\mathcal{L}_2$. **Right**: Intersection of a line $\mathcal{L}$ and a sphere $\mathcal{S}$. An illustration to Study Problem **7.2**. The terminal point of the vector $\mathbf{r}_t$ traverses the line as t ranges over all real numbers. If the line intersects the sphere, then there should exist a particular value of t at which the components of the vector $\mathbf{r}_t$ satisfy the equation of the sphere. This equation is quadratic in t, and hence it can have two distinct real roots, or one multiple real root, or no real roots. These three cases correspond to two, one, or no points of intersection. One point of intersection occurs when the line is tangent to the sphere.

algebraic conditions are

$$\begin{aligned}
\mathbf{r}_s - \mathbf{r}_t \perp \mathbf{v}_1 \quad &\Longleftrightarrow \quad (\mathbf{r}_s - \mathbf{r}_t) \cdot \mathbf{v}_1 = -10 + 4s - 6t = 0,\\
\mathbf{r}_s - \mathbf{r}_t \perp \mathbf{v}_1 \quad &\Longleftrightarrow \quad (\mathbf{r}_s - \mathbf{r}_t) \cdot \mathbf{v}_2 = -9 + 5s - 4t = 0.
\end{aligned}$$

This system has the solution $t = -1$ and $s = 1$. Thus, the points with the position vectors $\mathbf{r}_{t=-1} = \mathbf{r}_1 - \mathbf{v}_1 = \langle 1, 1, 1\rangle$ and $\mathbf{r}_{s=1} = \mathbf{r}_2 + \mathbf{v}_2 = \langle 2, -1, -2\rangle$ belong to $\mathcal{L}$. In particular, the vector $\mathbf{v} = \mathbf{r}_{s=1} - \mathbf{r}_{t=0} = \langle 1, -2, -3\rangle$ is parallel to $\mathcal{L}$ (as expected, it is parallel to $\mathbf{v}_1 \times \mathbf{v}_2$). Taking a particular point of $\mathcal{L}$ with the position vector $\langle 1, 1, 1\rangle$, the parametric equations read $x = 1 + t$, $y = 1 - 2t$, $z = 1 - 3t$. □

Problem **7.2**. *Consider a line through the origin that is parallel to the vector* $\mathbf{v} = \langle 1, 1, 1\rangle$. *Find the part of this line that lies inside the sphere* $x^2 + y^2 + z^2 - x - 2y - 3z = 9$.

SOLUTION: The parametric equations of the line are

$$x = t\,,\quad y = t\,,\quad z = t\,.$$

If the line intersects the sphere, then there should exist particular values of t at which the coordinates of a point of the line also satisfy the equation for the sphere (see Figure **7.3**, right panel). In general, parametric equations of a line are linear in t, while an equation of a sphere is quadratic in the coordinates. Therefore, the equation that determines the values of t corresponding to the points of intersection is quadratic. A quadratic equation has two, one, or no real solutions. Accordingly, these cases correspond to two, one, and no points of intersection, respectively. In our case,

$$3t^2 - 6t = 9 \quad\Rightarrow\quad t^2 - 2t = 3 \quad\Rightarrow\quad t = -1 \quad\text{and}\quad t = 3\,.$$

The points of intersection are $A = (-1,-1,-1)$ and $B = (3,3,3)$. The line segment connecting them can be described by the parametric equations

$$AB:\quad x = t\,,\quad y = t\,,\quad z = t\,,\quad -1 \le t \le 3\,.$$

$\square$

Problem **7.3**. *Let $\mathcal{P}$ be a plane through a point P_0 and orthogonal to a vector $\mathbf{n} \neq \mathbf{0}$. Represent $\mathcal{P}$ as the union of lines through P_0.*

SOLUTION: Let $\mathbf{r}_0$ be the position vector of P_0. Given $\mathbf{n} = \langle n_1, n_2, n_3\rangle$ is not difficult to find a non-zero vector orthogonal to $\mathbf{n}$. For example,

$$\mathbf{m} = \langle n_2, -n_1, 0\rangle \quad\Rightarrow\quad \mathbf{m}\cdot\mathbf{n} = n_2n_1 - n_1n_2 = 0 \quad\Rightarrow\quad \mathbf{m} \perp \mathbf{n}$$

Then the vectors $\mathbf{m}$, $\mathbf{m}\times\mathbf{n}$, and $\mathbf{n}$ are mutually orthogonal. Therefore the unit vectors

$$\hat{\mathbf{u}}_1 = \frac{1}{\|\mathbf{m}\|}\,\mathbf{m}\,,\qquad \hat{\mathbf{u}}_2 = \frac{1}{\|\mathbf{m}\times\mathbf{n}\|}\,\mathbf{m}\times\mathbf{n}$$

are orthogonal to each other and parallel to the plane $\mathcal{P}$. Any unit vector in the plane $\mathcal{P}$ can therefore be written in the form (see Study Problem **6.4**)

$$\mathbf{v}_\theta = \cos\theta\,\hat{\mathbf{u}}_1 + \sin\theta\,\hat{\mathbf{u}}_2$$

for some $0 \le \theta < 2\pi$. Consequently, any line through P_0 that is also in the plane $\mathcal{P}$ can be described by the vector equation

$$\mathcal{L}_\theta:\quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_\theta\,,\quad 0 \le \theta < \pi\,.$$

The lines corresponding to the values of θ that differ by π are identical because $\mathbf{v}_{\theta+\pi} = -\mathbf{v}_\theta$. For this reason, the range of the parameter θ is restricted to the interval $[0, \pi)$. Then the plane $\mathcal{P}$ is the union:

$$\mathcal{P} = \bigcup_\theta \mathcal{L}_\theta ,$$

where the union is taken over $0 \leq \theta < \pi$. Recall that a plane was first defined as the point set of all the lines through a given point and perpendicular to a given line through that point. □

7.6. Exercises.

1–**7**. Find vector, parametric, and symmetric equations of the specified line.

1. The line containing the segment AB where $A = (1, 2, 3)$ and $B = (-1, 2, 4)$

2. The lines containing the diagonals of the parallelogram whose adjacent sides at the vertex $(1, 0, -1)$ are $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 2, 1\rangle$.

3. The line through the vertex A of a triangle ABC and perpendicular to the sides AB and AC if $A = (1, 0, -1)$, $B = (-1, 1, 2)$, and $C = (2, -1, -2)$

4. The line through the vertex C of a triangle ABC and parallel to the edge AB if $A = (1, 0, -1)$, $B = (-1, 1, 2)$, and $C = (2, -1, -2)$

5. A line through the origin that makes an angle of 60° with the x and y axes. Is such a line unique? Explain.

6. The line through the vertex $A = (1, 2, 3)$ of a parallelogram with adjacent sides at A being $\mathbf{a} = \langle 1, 2, 2\rangle$ and $\mathbf{b} = \langle -2, 1, -2\rangle$ that bisects the angle of the parallelogram at A. *Hint*: See Exercise **21** in Section **3.8**.

7. The line parallel to the vector $\langle 1, -2, 0\rangle$ that contains a diameter of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$.

8. Show that the line through $P_1 = (1, 2, -1)$ and parallel to $\mathbf{v}_1 = \langle 1, -1, 3\rangle$ coincides with the line through $P_2 = (0, 3, -4)$ and parallel to $\mathbf{v}_2 = \langle -2, 2, -6\rangle$ as points sets in space.

9. Do the lines $x - 1 = 2y = 3z$ and $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 7, 3, 2\rangle$ and $\mathbf{v} = \langle 6, 3, 2\rangle$, coincide as point sets in space?

10. Find parametric equations of the line through the point $(1, 2, 3)$ and perpendicular to the plane $x + y + 2z = 1$. Find the point of intersection of the line and the plane.

11. Find parametric and symmetric equations of the line of intersection of the planes $x + y + z = 1$ and $2x - 2y + z = 1$.

12–13. Determine whether the given lines are are parallel, skew, or intersecting. If they intersect, find the point of intersection and the angle between the lines.

12. The line through the points $(1, 2, 3)$ and $(2, -1, 1)$ and the line through the points $(0, 1, 3)$ and $(1, 0, 2)$

13. The lines $x = 1 + 2t$, $y = 3t$, $z = 2 - t$ and $x + 1 = y - 4 = (z - 1)/3$.

14. Let $\overrightarrow{AB} = \langle 1, 2, 2\rangle$, $\overrightarrow{AC} = \langle 2, -1, -2\rangle$, and $\overrightarrow{AD} = \langle 0, 3, 4\rangle$ be the adjacent sides of a parallelepiped. Show that the diagonal of the parallelepiped extended from the vertex A intersects the diagonal extended from the vertex D and find the angle between the diagonals.

15. Are the four lines containing the diagonals of a parallelepiped intersecting at a point? Prove your answer. If they are intersecting, find the position vector of the point of intersection relative to a vertex at which the adjacent sides of the parallelepiped are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

16. Find vector and parametric equations of the straight line segment from the point $(1, 2, 3)$ to the point $(-1, 1, 2)$.

17. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be position vectors of two points in space. Find a vector equation of the straight line segment from $\mathbf{r}_1$ to $\mathbf{r}_2$.

18. Find the distance from the point $(1, 2, 3)$ to the line $2x = y + 1$, $z = 3$.

19. Consider the plane $x + y - z = 0$ and a point $P = (1, 1, 2)$ in it. Find parametric equations of the lines through the origin that lie in the plane and are at a distance of 1 unit from P. *Hint*: A vector parallel to these lines can be taken in the form $\mathbf{v} = \langle 1, c, 1 + c\rangle$ where c is to be determined. Explain why!

20. Find the parallel lines intersecting the line $x = 2 + t$, $y = 1 + t$, $z = 2 + 2t$ at a right angle and parallel to the plane $x + 2y - 2z = 1$ that are at a distance of 1 unit from the plane. *Hint*: Find values of t at which the distance from a point in the given line to the plane is 1. This determines the points of intersection of the lines in question with the given line.

21. Find parametric equations of the line that is parallel to $\mathbf{v} = \langle 2, -1, 2\rangle$ and goes through the center of the sphere $x^2 + y^2 + z^2 = 2x + 6z - 6$. Restrict the range of the parameter to describe the part of the line that is inside the sphere.

22. Let the line $\mathcal{L}_1$ pass through the point $A(1, 1, 0)$ parallel to the vector $\mathbf{v} = \langle 1, -1, 2\rangle$ and let the line $\mathcal{L}_2$ pass through the point $B(2, 0, 2)$ parallel to the vector $\mathbf{w} = \langle -1, 1, 2\rangle$. Show that the lines are intersecting. Find the point C of intersection and parametric equations of the line $\mathcal{L}_3$ through C that is perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$.

23. Find parametric equations of the line through $(1, 2, 5)$ that is perpendicular to the line $x - 1 = 1 - y = z$ and intersects this line.

24. Find the distance between the lines $x = y = z$ and $x + 1 = y/2 = z/3$.

25. A small meteor moves with the speed v along a straight line parallel to a unit vector $\hat{\mathbf{u}}$. If the meteor passed the point $\mathbf{r}_0$, find the condition on $\hat{\mathbf{u}}$ so that the meteor hits an asteroid of the shape of a sphere of radius R centered at the point $\mathbf{r}_1$. Determine the position vector of the impact point.

26. A projectile is fired in the direction $\mathbf{v} = \langle 1, 2, 3 \rangle$ from the point $(1, 1, 1)$. Let the target be a disk of radius R centered at $(2, 3, 6)$ in the plane $2x - 3y + 4z = 19$. If the trajectory of the projectile is a straight line, determine whether it hits a target in two cases $R = 2$ and $R = 3$.

27. Consider a triangle ABC where $A = (1, 1, 1)$, $B = (3, 1, -1)$, $C = (1, 3, 1)$. Find the area of a polygon $DPQB$ where the vertices D and Q are the midpoints of CB and AB, respectively, and the vertex P is the intersection of the segments CQ and AD.

8. Euclidean Spaces.

The concept of a Euclidean space is studied in detail in courses of Linear Algebra. Here only a few facts are given which are necessary to define functions of several variables discussed in Chapter 3. The reader may skip this section and review it later before reading Chapter 3.

With every ordered pair of numbers (x, y), one can associate a point in a plane and its position vector relative to a fixed point $(0, 0)$ (the origin), $\mathbf{r} = \langle x, y\rangle$. With every ordered triple of numbers (x, y, z), one can associate a point in space and its position vector (again relative to the origin $(0, 0, 0)$), $\mathbf{r} = \langle x, y, z\rangle$. So the plane can be viewed as the set of all two-component vectors; similarly, space is the set of all three-component vectors. From this point of view, the plane and space have common characteristic features. First, their elements are vectors. Second, they are closed relative to addition of vectors and multiplication of vectors by a real number; that is, if $\mathbf{a}$ and $\mathbf{b}$ are elements of space or a plane and c is a real number, then $\mathbf{a}+\mathbf{b}$ and $c\mathbf{a}$ are also elements of a space (ordered triples of numbers) or a plane (ordered pairs of numbers). Third, the norm or length of a vector $\|\mathbf{r}\|$ vanishes if and only if the vector has zero components. Consequently, two elements of space or a plane coincide if and only if the norm of their difference vanishes, that is, $\mathbf{a} = \mathbf{b} \Leftrightarrow \|\mathbf{a} - \mathbf{b}\| = 0$. Finally, the dot product $\mathbf{a} \cdot \mathbf{b}$ of two elements is defined in the same way for two- or three-component vectors (plane or space) so that $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$. Since points and vectors are described by the same mathematical object, an ordered triple (or pair) of numbers, there is no necessity to make a distinction between them. So the space may be viewed as the set of all vectors (or ordered triple of numbers) in which the addition, multiplication by a number, and the dot product are defined. These observations can be extended to ordered n-tuples for any positive n and lead to the notion of a *Euclidean space*.

8.1. Higher-dimensional Euclidean spaces.

Definition 8.1. (Euclidean Space).
For each positive integer n, consider the set of all ordered n-tuples of real numbers. For any two elements $\mathbf{a} = \langle a_1, a_2, ..., a_n\rangle$ and $\mathbf{b} = \langle b_1, b_2, ..., b_n\rangle$ and a number c, put

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1 + b_1, a_2 + b_2, ..., a_n + b_n\rangle, \\ c\,\mathbf{a} &= \langle ca_1, ca_2, ..., ca_n\rangle, \\ \mathbf{a} \cdot \mathbf{b} &= a_1b_1 + a_2b_2 + \cdots + a_nb_n, \\ \|\mathbf{a}\| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\ . \end{aligned}$$

The set of all ordered $n-$*tuples in which the addition, the multiplication by a number, the dot product, and the norm are defined by these rules is called an* n-dimensional Euclidean space *and denoted* $\mathbb{R}^n$*. Elements of* $\mathbb{R}^n$ *are also called* vectors.

Two elements of a Euclidean space are said to coincide, $\mathbf{a} = \mathbf{b}$, if the corresponding components are equal, that is,

$$\mathbf{a} = \mathbf{b} \quad \Leftrightarrow \quad a_i = b_i\,,\ i = 1, 2, ..., n\,.$$

It follows that $\mathbf{a} = \mathbf{b}$ if and only if $\|\mathbf{a}-\mathbf{b}\| = 0$. Indeed, by the definition of the norm, $\|\mathbf{c}\| = 0$ if and only if $\mathbf{c} = (0, 0, ..., 0)$. Put $\mathbf{c} = \mathbf{a} - \mathbf{b}$. Then $\|\mathbf{a} - \mathbf{b}\| = 0$ if and only if $\mathbf{a} = \mathbf{b}$. The number $\|\mathbf{a} - \mathbf{b}\|$ is called the *distance* between two elements $\mathbf{a}$ and $\mathbf{b}$ of a Euclidean space.

The dot product in a Euclidean space has the same geometrical properties as in two and three dimensions. The Cauchy-Schwarz inequality can be extended to any Euclidean space (cf. Theorem **3.2**).

Theorem **8.1**. (Cauchy-Schwarz Inequality).

$$|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|$$

for any elements $\mathbf{a}$ *and* $\mathbf{b}$ *of* $\mathbb{R}^n$*, and the equality is reached if and only if* $\mathbf{a} = t\mathbf{b}$ *for some number* t.

Proof. Put $a = \|\mathbf{a}\|$ and $b = \|\mathbf{b}\|$, that is, $a^2 = \mathbf{a}\cdot\mathbf{a}$ and similarly $b^2 = \mathbf{b}\cdot\mathbf{b}$. If $b = 0$, then $\mathbf{b} = \mathbf{0}$, and the conclusion of the theorem holds. For $b \ne 0$ and any real t,

$$\|\mathbf{a} - t\mathbf{b}\|^2 = (\mathbf{a} - t\mathbf{b})\cdot(\mathbf{a} - t\mathbf{b}) \ge 0\,.$$

Therefore,

$$a^2 - 2tc + t^2b^2 \ge 0\,,$$

where $c = \mathbf{a}\cdot\mathbf{b}$. Completing the squares in the left side of this inequality,

$$\Big(bt - \frac{c}{b}\Big)^2 - \frac{c^2}{b^2} + a^2 \ \ge\ 0\,,$$

shows that the left side attains its absolute minimum when the expression in the parenthesis vanishes, i.e., at $t = c/b^2$. Since the inequality is valid for any t, it is satisfied for $t = c/b^2$, that is,

$$\begin{aligned} a^2 - c^2/b^2 \ge 0 \quad &\Rightarrow \quad c^2 \le a^2b^2 \quad \Rightarrow \quad |c| \le ab \\ &\Rightarrow \quad |\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|\,. \end{aligned}$$

The inequality becomes an equality if and only if $\|\mathbf{a} - t\mathbf{b}\|^2 = 0$ and hence if and only if $\mathbf{a} = t\mathbf{b}$. □

It follows from the Cauchy-Schwarz inequality that $\mathbf{a}\cdot\mathbf{b} = s\|\mathbf{a}\|\|\mathbf{b}\|$, where s is a number such that $|s| \le 1$. So one can always put $s = \cos\theta$, where $\theta \in [0, \pi]$. If $\theta = 0$, then $\mathbf{a} = t\mathbf{b}$ for some positive $t > 0$ (such elements of $\mathbb{R}^n$ are called *parallel*), and $\mathbf{a} = t\mathbf{b}$, $t < 0$, when $\theta = \pi$ (such elements of $\mathbb{R}^n$ are called *antiparallel*). The dot product vanishes when $\theta = \pi/2$. This allows one to *define* θ as the angle between two elements of a Euclidean space:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

much like in two and three dimensions. Two elements of $\mathbb{R}^n$ are called *orthogonal* if their dot product vanishes. Consequently, the triangle inequality (**3.4**) holds in a Euclidean space of any dimension.

8.2. A basis in a Euclidean space. Consider n elements $\hat{\mathbf{e}}_j$, $j = 1, 2, ..., n$, of $\mathbb{R}^n$ such that all components of $\hat{\mathbf{e}}_j$ are 0 except the j^{th} component which is 1, that is, $\hat{\mathbf{e}}_1 = \langle 1, 0, ..., 0\rangle$, $\hat{\mathbf{e}}_2 = \langle 0, 1, 0, ..., 0\rangle$, ..., $\hat{\mathbf{e}}_n = \langle 0, 0, ..., 0, 1\rangle$. Evidently, these elements are mutually orthogonal and have unit norm: $\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m = 0$ if $j \neq m$ and $\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_j = 1$ or $\|\hat{\mathbf{e}}_j\| = 1$. By the definition of a Euclidean space, every element $\mathbf{a} = \langle a_1, a_2, ..., a_n\rangle$ is a linear combination of $\hat{\mathbf{e}}_j$:

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + \cdots + a_n\hat{\mathbf{e}}_n\,.$$

DEFINITION **8.2**. (Standard basis in a Euclidean space)
The set of elements $\hat{\mathbf{e}}_j$, $j = 1, 2, ..., n$, is called the standard basis of an $n-$dimensional Euclidean space.

Note that any non-zero vector $\mathbf{a} = s\hat{e}_1 = \langle s, 0, ..., 0\rangle \neq \mathbf{0}$, which is a multiple of $\hat{\mathbf{e}}_1$ cannot be expressed as a linear combination of the other basis vectors $\hat{\mathbf{e}}_j$, $j = 2, 3, ..., n$, because the first component of a general linear combination $a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3 + \cdots + a_n\hat{\mathbf{e}}_n = \langle 0, a_2, a_3, ..., a_n\rangle$ is always zero. It is then not difficult to see that none of the standard basis vectors is a linear combination of the others. The standard basis is said to be a set of *linearly independent* vectors.

DEFINITION **8.3**. (A set of linearly independent vectors)
A set of vectors $\mathbf{u}_k$, $k = 1, 2, ..., m \leq n$, in an $n-$dimensional Euclidean space is called linearly independent if none of these vectors is a linear combination of the others.

Clearly, any m vectors ($m \leq n$) from the standard basis form a linearly independent set of vectors. It is proved in Linear Algebra that any set of n linearly independent vectors in an $n-$dimensional Euclidean space form a basis in the sense that every element of $\mathbb{R}^n$

is expressed as a unique linear combination of these vectors. Similarly to Section **3.4**, one can show that any set of n mutually orthogonal unit vectors $\hat{\mathbf{u}}_j$, $j = 1, 2, ..., n$, form an orthonormal basis in an $n-$dimensional Euclidean space in the sense that, for every element $\mathbf{a} = \langle a_1, a_2, ..., a_n\rangle = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + \cdots + a_n\hat{\mathbf{e}}_n$, there exist unique numbers s_j such that $\mathbf{a} = s_1\hat{\mathbf{u}}_1 + s_2\hat{\mathbf{u}}_2 + \cdots + s_n\hat{\mathbf{u}}_n$.

THEOREM **8.2**. (A criterion for linear independence)
Non-zero vectors $\mathbf{u}_j$, $j = 1, 2, ..., m \le n$, *in an* $n-$*dimensional Euclidean space are linearly independent if and only if the vector equation*

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_m\mathbf{u}_m = \mathbf{0}$$

has a trivial solution $s_1 = s_2 = \cdots = s_m = 0$.

Note that if the vector equation has a nontrivial solution, then one of the numbers s_j is not zero, say, $s_1 \ne 0$. The latter implies that $\mathbf{u}_1$ is a linear combination of the others: $\mathbf{u}_1 = -(s_2/s_1)\mathbf{u}_2 - \cdots - (s_m/s_1)\mathbf{u}_m$. The numbers s_2, s_3,...,s_m cannot vanish simultaneously because $\mathbf{u}_1 \ne \mathbf{0}$. So, the vectors cannot be linearly independent. Using this observation is not difficult to prove the theorem.

It can also be proved that there are exactly $n - 1$ nonzero linearly independent vectors orthogonal to a given nonzero vector in $\mathbb{R}^n$.

EXAMPLE **8.1**. *Show that the vectors* $\mathbf{u}_1 = \langle 1, 0, ..., 0\rangle$, $\mathbf{u}_2 = \langle 1, 1, 0, ..., 0\rangle$, ..., $\mathbf{u}_{n-1} = \langle 1, 1, ..., 1, 0\rangle$, *and* $\mathbf{u}_n = \langle 1, 1, ..., 1\rangle$ *in an* $n-$*dimensional Euclidean space are linearly independent and, hence, form a basis. Find components of the vector* $\mathbf{a} = \langle 1, 2, ..., n-1, n\rangle$ *in this basis.*

SOLUTION: By Theorem **8.2** one has to show that the vector equation $s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_n\mathbf{u}_n = \mathbf{0}$ admits only a trivial solution. Writing this equation in components, one infers that the numbers s_j, $j = 1, 2, ..., n$, satisfy the system of n equations

$$\begin{aligned}
&s_1 = 0\\
&s_1 + s_2 = 0\\
&\cdots\\
&s_1 + s_2 + \cdots s_{n-1} = 0\\
&s_1 + s_2 + \cdots s_{n-1} + s_n = 0
\end{aligned}$$

Substituting the first equation into the second one, it is concluded that $s_1 = s_2 = 0$. Continuing this procedure recursively downward to the third equation, then to the fourth equation, and so on. It is easy to see that $s_3 = 0$, $s_4 = 0$, and so on. Thus, the system has a trivial solution and the vectors in question are linearly independent.

To prove that these vectors form a basis, one has to show that there exist unique numbers s_j, $j = 1, 2, ..., n$, such that $\mathbf{a} = \langle a_1, a_2, ..., a_n \rangle = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_n\mathbf{u}_n$ for every $\mathbf{a}$. The latter vector equation is equivalent to the system of equations:

$$\begin{aligned}
&s_1 = a_1 \\
&s_1 + s_2 = a_2 \\
&\cdots \\
&s_1 + s_2 + \cdots s_{n-1} = a_{n-1} \\
&s_1 + s_2 + \cdots s_{n-1} + s_n = a_n
\end{aligned}$$

which has a unique solution $s_1 = a_1$, $s_2 = a_2 - s_1 = a_2 - a_1$, $s_3 = a_3 - (s_1 + s_2) = a_3 - a_2$ and so on, $s_j = a_j - a_{j-1}$ for $j = 2, 3, ..., n$. For the vector with component $a_j = j$ in the standard basis, the components in the basis of the vectors $\mathbf{u}_j$ are $s_1 = 1$ and $s_j = j - (j - 1) = 1$ for $j = 2, 3, ..., n$, that is, $\langle 1, 2, ..., n \rangle = \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n$. □

8.3. Lines and planes in a Euclidean space. In Section **1.3**, a line in space was defined using the concept of the *shortest path* connecting two points. The latter requires a description of how the *length* of a path can be measured, which is based on properties of the space we live in and other laws of physics (e.g., the speed of light is a universal constant). These physical laws themselves have to be tested and verified. Yet, their validity has been established only with some certainty. The history of physics shows that more accurate measurements invalidate established laws and every physics law has a limited validity. In contrast, mathematics is based on a pure logic that establishes properties of an object from its definition (a theorem is proved using definitions, axioms, and other established theorems), and, hence, it cannot use approximate laws of Nature. One can think of a mathematician as a person developing abstract mathematical concepts with no reference to a real world whatsoever, following only a logical consistency of the concepts developed. Whether or not these abstract mathematical concepts may be applied to the real world is a question yet to be answered! The concept of a Euclidean space (Definition **8.1**) has been shown to be quite useful. However Definition **8.1** does include the notion of the *shortest path* between two points or, in fact, any path connecting two points. So the logical consistency requires a *definition of a line in a Euclidean space.*

A line in a Euclidean space. Consider all elements of a Euclidean space that are multiples of a particular nonzero element $\mathbf{v}$, $\mathbf{r} = t\mathbf{v}$ where $-\infty < t < \infty$. All such elements also form a Euclidean space because the set of these vectors is *closed* relative to addition of its elements and multiplication of its elements by a real number:

$$\mathbf{r}_1 + \mathbf{r}_2 = t\mathbf{v} + s\mathbf{v} = (t+s)\mathbf{v}\,,$$
$$s\mathbf{r} = s(t\mathbf{r}) = (st)\mathbf{r}$$

for any real s and t, that is, the sum of any two elements of the set and a multiple of any element of the set belong to the set. This subset of a Euclidean space is said to be a *one-dimensional* Euclidean subspace because all of its elements are obtained as multiples of a *single* nonzero element; it is denoted by $\mathbb{R}$. A *line* in a Euclidean space $\mathbb{R}^n$ is defined as *a one-dimensional subspace* of $\mathbb{R}^n$. Take three elements $\mathbf{a} = t_a\mathbf{v}$, $\mathbf{b} = t_b\mathbf{v}$, and $\mathbf{c} = t_c\mathbf{v}$ of a one-dimensional subspace of $\mathbb{R}^n$ and calculate the pairwise distances between them, $d_{ab} = \|\mathbf{a}-\mathbf{b}\| = |t_a - t_b|v$, $d_{bc} = \|\mathbf{b}-\mathbf{c}\| = |t_b - t_c|v$, and $d_{ac} = \|\mathbf{a}-\mathbf{c}\| = |t_a - t_c|v$, where $v = \|\mathbf{v}\| > 0$. Suppose d_{ab} is the largest of the three. Then

$$d_{ab} = d_{bc} + d_{ac} \quad \text{or} \quad |t_a - t_b| = |t_b - t_c| + |t_a - t_c|\,.$$

The latter is identical to the distance property of a real number system discussed in Section **1.3**.

Let us fix a particular element $\mathbf{r}_0$ of $\mathbb{R}^n$. The element $\mathbf{r} - \mathbf{r}_0$ is called a *position vector* of an element $\mathbf{r}$ relative to $\mathbf{r}_0$. *A line through* $\mathbf{r}_0$ *is a collection of elements of* $\mathbb{R}^n$ *whose position vectors relative to* $\mathbf{r}_0$ *form a one-dimensional subspace.* Evidently, all such elements elements can be written in the form

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}\,, \quad -\infty < t < \infty$$

for some nonzero element $\mathbf{v}$. Note well that for every element of a line there is a *unique* real number t and vice versa.

A plane in a Euclidean space. A *plane* in a Euclidean space $\mathbb{R}^n$, $n \geq 2$, is defined as a two-dimensional subspace $\mathbb{R}^2$ of $\mathbb{R}^n$ in the sense that every element of $\mathbb{R}^2$ is obtained as a linear combination of *two nonzero linearly independent* elements $\mathbf{v}$ and $\mathbf{u}$. This definition is to be compared with the result of Study Problem **2.1**. Similarly to the case of $\mathbb{R}$, it is not difficult to verify that this subset of elements of $\mathbb{R}^n$ is closed relative to addition and multiplication by a number. *A plane through a particular element* $\mathbf{r}_0$ *is a collection of all elements of* $\mathbb{R}^n$ *whose position vectors relative to* $\mathbf{r}_0$ *are linear combinations of two linearly independent elements* $\mathbf{v}$ *and* $\mathbf{u}$*:*

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} + s\mathbf{u}\,, \quad -\infty < t, s < \infty\,.$$

Every element of a plane is *uniquely* determined by an ordered pair of real numbers (t, s) (which are elements of $\mathbb{R}^2$). If $n = 3$, then, owing to the geometrical properties of the cross product, the above *vector equation* of a plane can be written in the standard form $\mathbf{n}\cdot\mathbf{r}=\mathbf{n}\cdot\mathbf{r}_0$ given in Theorem **6.1** where the normal is $\mathbf{n}=\mathbf{v}\times\mathbf{u}$ (recall $\mathbf{n}\cdot\mathbf{v}=\mathbf{n}\cdot\mathbf{u}=0$). Note that in $\mathbb{R}^n$, $n>3$, there are $n-2$ linearly independent vectors orthogonal to both nonzero linearly independent vectors $\mathbf{v}$ and $\mathbf{u}$. Consequently, there are non-parallel planes that intersect at a *single* point! For example, consider two planes in $\mathbb{R}^4$ whose vector equations are $\mathbf{r}=t_1\hat{\mathbf{e}}_1+t_2\hat{\mathbf{e}}_2=\langle t_1,t_2,0,0\rangle$ and $\mathbf{r}=s_1\hat{\mathbf{e}}_3+s_2\hat{\mathbf{e}}_4=\langle 0,0,s_1,s_2\rangle$. The set of common elements of these planes satisfies the vector equation

$$\langle t_1,t_2,0,0\rangle=\langle 0,0,s_1,s_2\rangle \quad\Leftrightarrow\quad t_1=t_2=s_1=s_2=0$$

and, hence, consists of a single element $\langle 0,0,0,0\rangle$.

A hyperplane in a Euclidean space. In a Euclidean space $\mathbb{R}^n$ one can define an $m-$dimensional subspace $\mathbb{R}^m$, $m\leq n$, as a collection of all linear combination of m nonzero linearly independent vectors. Let $\mathbf{u}_j$, $j=1,2,...,m\leq n$ be nonzero linearly independent vectors in $\mathbb{R}^n$. The set of vectors

$$\mathbf{r}=\mathbf{r}_0+t_1\mathbf{u}_1+t_2\mathbf{u}_2+\cdots+t_m\mathbf{u}_m\,,\quad -\infty<t_1,t_2,...,t_m<\infty\,,$$

is called an $m-$*dimensional hyperplane* through $\mathbf{r}_0$. Every element of a hyperplane is *uniquely* described by an ordered $m-$tuple of numbers $(t_1,t_2,...,t_m)$, that is, by an element of $\mathbb{R}^m$. Conversely, for every $m-$tuple (or an element of $\mathbb{R}^m$) there is a unique element of a hyperplane.

8.4. Coordinate systems and bases. Given a basis $\mathbf{u}_j$, $j=1,2,...,m$, in $\mathbb{R}^m$, every vector is uniquely represented by a linear combination $\mathbf{r}=x_1\mathbf{u}_1+x_2\mathbf{u}_2+\cdots+x_m\mathbf{u}_m$. Consider m straight lines through $\langle 0,0,...,0\rangle$ that are parallel to the basis vectors and oriented by arrows in the direction of of the corresponding basis vector. The set of these straight lines is called a *coordinate system* in $\mathbb{R}^m$. An element $\mathbf{r}=x_1\mathbf{u}_1+x_2\mathbf{u}_2+\cdots+x_m\mathbf{u}_m$ of $\mathbb{R}^m$ is called a *point* that has *coordinates* $(x_1,x_2,...,x_m)$ relative to the coordinate system associated with the basis $\mathbf{u}_j$. If the basis is orthonormal, the coordinate system is called a *rectangular coordinate system.* The length of the position vector of a point $(x_1,x_2,...,x_m)$ is $\|\mathbf{r}\|=(x_1^2+x_2^2+\cdots+x_m)^{1/2}$. The distance between two points $P_1=(x_1,x_2,...,x_m)$ and $P_2=(y_1,y_2,...,y_m)$ is given by the norm of the vector

$$\overrightarrow{P_1P_2}=(y_1-x_1)\mathbf{u}_1+(y_2-x_2)\mathbf{u}_2+\cdots+(y_m-x_m)\mathbf{u}_m$$

If the basis is orthonormal (the associated coordinate system is rectangular), then

$$|P_1P_2| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_m - x_m)^2}\,.$$

Every point set in $\mathbb{R}^m$ can now be described algebraically in terms of conditions imposed on the coordinates of points in some coordinate system.

This completes an abstract mathematical model of our space as a three-dimensional Euclidean space. Now the model has no reference to any particular law of physics and is solely based on the properties of real numbers. The postulated properties of a Euclidean space (Definition **8.1**) and their consequences are to be compared with observed properties of the space we live in to decide how accurate this model is (e.g., does the light travels along straight lines in a Euclidean space, or, is the distance between two points coincides with the distance in a Euclidean space?).

Multidimensional spheres and balls. Given a coordinate system in $\mathbb{R}^m$, points sets can be described algebraically by imposing conditions on coordinates of points in $\mathbb{R}^m$. An *$m-$dimensional sphere* (or simply $m-$sphere) with radius R and center at $\mathbf{a}$ is a collection of elements of $\mathbb{R}^{m+1}$ whose distance from $\mathbf{a}$ is R. Take the coordinate system associated with the standard basis. Then $\mathbf{a} = \langle a_1, a_2, ..., a_{m+1}\rangle$ and $\mathbf{r} = \langle x_1, x_2, ..., x_{m+1}\rangle$ are position vectors of the sphere center and a generic point in $\mathbb{R}^m$, respectively. The point $\mathbf{r}$ belongs to a $m-$sphere of radius R and centered at $\mathbf{a}$ if

$$\|\mathbf{r}-\mathbf{a}\|^2 = R^2 \quad \Leftrightarrow \quad (x_1-a_1)^2+(x_2-a_2)^2+\cdots+(x_{m+1}-a_{m+1})^2 = R^2$$

Evidently, a one-dimensional sphere is a circle in a plane and a two-dimensional sphere is a sphere in space.

Similarly, the set of elements in $\mathbb{R}^n$ satisfying the condition

$$\|\mathbf{r} - \mathbf{a}\| < R$$

is called an *open ball* of radius R and centered at $\mathbf{a}$. If the equality is also allowed, $\|\mathbf{r} - \mathbf{a}\| \le R$, the ball is called *closed*.

9. Quadric Surfaces

In Section **1.9**, an algebraic description of point sets in space has been introduced. Spheres, cylinders, planes, and lines can be described algebraically by imposing certain conditions on the coordinates of points in the set. Lines and planes can be described by linear equations, while spheres and cylinders by quadratic equations. In Calculus 2, rotational surfaces have been discussed, such as circular paraboloids, cones, and ellipsoids. It turns out that all these surfaces can also be described by quadratic equations. In general, one can pose a question about a classification of all surfaces described by quadratic equations in some rectangular coordinate system.

Definition **9.1**. (Quadric Surface).
A nonempty set of points whose coordinates in a rectangular coordinate system satisfy the equation

$$Ax^2 + By^2 + Cz^2 + Fxy + Gxz + Hyz + Px + Qy + Vz + D = 0,$$

where numbers A, B, C, F, G, and H do not vanish simultaneously, and P, Q, V, and D are real numbers, is called a quadric surface *if it does not admit an alternative description in terms of equations linear in x, y, and z.*

The equation that defines quadric surfaces is the most general equation *quadratic* in all the coordinates. This is why surfaces defined by it are called *quadric*. A sphere provides a simple example of a quadric surface:

$$x^2 + y^2 + z^2 - R^2 = 0\,,$$

that is, $A = B = C = 1$, $D = -R^2$ (here R is the radius of the sphere), and other constants vanish in the general quadratic equation. If $B = C = 1$, $P = -1$, while the other constants vanish, the quadratic equation $x = y^2 + z^2$ defines a circular paraboloid whose symmetry axis is the x axis. On the other hand, if $A = B = 1$, $V = -1$, while the other constants vanish, the equation

$$z = x^2 + y^2$$

also defines a paraboloid that can be obtained from the former one by a rotation about the y axis through the angle $\pi/2$ under which $(x, y, z) \to (z, y, -x)$ so that $x = y^2 + z^2 \ \to\ z = y^2 + x^2$. Thus, there are quadric surfaces of the same shape described by different equations.

Sometimes a quadratic equation can be reduced to a linear one. For example, the quadratic equation

$$(x + 2y + 3z)^2 = 1 \quad \Leftrightarrow \quad x + 2y + 3z = \pm 1$$

admits an alternative description in terms of two linear equations that describe two parallel planes with the normal $\mathbf{n} = \langle 1, 2, 3\rangle$. Similarly, the set of solutions to the quadratic equation

$$(x-y)^2 + z^2 = 0 \quad \Leftrightarrow \quad y = x \text{ and } z = 0$$

coincides with the set of common solutions to two linear equations. In this case, it is the intersection of two planes which is the line $y = x$ in the xy plane. The set described by the quadratic equation

$$x^2 + y^2 + z^2 = 0 \quad \Leftrightarrow \quad (x, y, z) = (0, 0, 0)$$

consists of a single point. It is not difficult to give an example of a quadratic equation which has no solution:

$$x^2 + 2y^2 + 3z^2 = -1$$

There is no point (x, y, z) whose coordinates satisfy this equation because for any point the left side is non-negative, while the right side is negative. So, this quadratic equation describes an empty set.

When quadric surfaces are discussed, it is always assumed that the corresponding quadratic equation cannot be reformulated in terms of linear equations according to Definition **9.1** so that the set of solutions is not empty and not a collection of planes, lines , or isolated points (the latter can always be described by linear equations).

As has been already noted, two different quadratic equations can describe the very same surface. The shape of a surface does not change under rigid transformations because the distance between any two points of the shape is preserved. On the other hand, the equation that describes the shape would change under rigid transformations of the coordinate system. The freedom in choosing the coordinate system can be used to simplify the equation for quadric surfaces and obtain a classification of different shapes described by it. In other words, *two points sets in space are said to have the same shape if they can be transformed into one another by a rigid transformation.*

9.1. Quadric Cylinders. Consider first a simpler problem in which the equation of a quadric surface does not contain one of the coordinates, say, z (i.e., $C = G = H = V = 0$). Then the set $\mathcal{S}$,

$$\mathcal{S} = \Big\{(x, y, z)\Big| Ax^2 + By^2 + Fxy + Px + Qy + D = 0\Big\},$$

is the same curve in every horizontal plane $z = \text{const}$. For example, if $A = B = 1$, $F = P = Q = 0$, and $D = -R^2$, the cross section of the surface $\mathcal{S}$ by any horizontal plane is a circle $x^2 + y^2 = R^2$:

$$\mathcal{S} = \{(x, y, z)\,|\, x^2 + y^2 = R^2\}$$

So the surface $\mathcal{S}$ is a *cylinder* of radius R that is swept by the circle when the latter is shifted up and down parallel to the z axis. Similarly, *a general cylindrical shape is obtained by rigid translations of a curve in the xy plane up and down parallel to the z axis.* For example, a cylinder obtained by rigid translations of the parabola

$$\mathcal{S} = \{(x, y, z) \,|\, y = x^2\}$$

is called a *parabolic cylinder*.

In what follows, the notations introduced in Study Problem **1.2** will be used. The coordinates of a point in the original coordinate system will be denoted as (x, y, z) and the ordered triple (x', y', z') is used to denote the coordinates of the same point in a new coordinate system obtained by translations and rotations of the original coordinate system.

Theorem **9.1**. (Classification of Quadric Cylinders).
A general equation for quadric cylinders

$$\mathcal{S} = \Big\{(x, y, z)\Big|Ax^2 + By^2 + Fxy + Px + Qy + D = 0\Big\}$$

can be brought to one of the standard forms

$$A'x'^2 + B'y'^2 + D' = 0 \quad \text{or} \quad A'x'^2 + Q'y' = 0$$

by a rigid transformation of the coordinate system, provided A, B, and F do not vanish simultaneously. In particular, these forms define the quadric cylinders:

$$y' - ax'^2 = 0 \qquad \text{(parabolic cylinder)}, \quad Q' \neq 0$$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \qquad \text{(elliptic cylinder)}, \quad \frac{A'}{D'} < 0,\ \frac{B'}{D'} < 0,\ D' \neq 0$$

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \qquad \text{(hyperbolic cylinder)}, \quad A'B' < 0,\ D' \neq 0$$

The shapes of quadric cylinders are shown in Figure **9.1**. Other than quadric cylinders, the standard equations may define planes or a line for some particular values of the constants A', B', D', and Q' (points sets that admit an alternative description in terms linear equations). For example, for $A' = -B' = 1$ and $D' = 0$ the equation $x'^2 = y'^2$ defines two planes $x' \pm y' = 0$. For $A' = B' = 1$ and $D' = 0$, the equation $x'^2 + y'^2 = 0$ defines the line $x' = y' = 0$ (the z axis).

Proof of Theorem 9.1. Let (x', y') be coordinates in the coordinate system obtained by a rotation through an angle ϕ. The equation of $\mathcal{S}$ in the new coordinate system is obtained by the transformation:

$$(x, y) = (x' \cos\phi - y' \sin\phi,\ y' \cos\phi + x' \sin\phi)$$

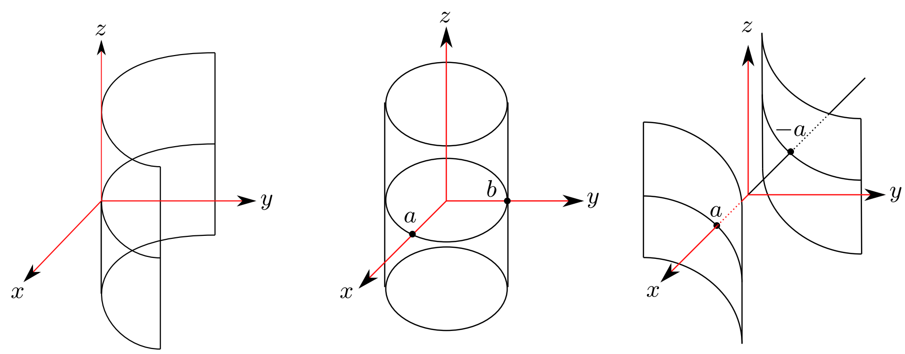


FIGURE **9.1**. **Left**: Parabolic cylinder. The cross section by any horizontal plane $z = \text{const}$ is a parabola $y = ax^2$. **Middle**: An elliptic cylinder. The cross section by any horizontal plane $z = \text{const}$ is an ellipse $x^2/a^2 + y^2/b^2 = 1$. **Right**: A hyperbolic cylinder. The cross section by any horizontal plane $z = \text{const}$ is a hyperbola $x^2/a^2 - y^2/b^2 = 1$.

according to Study Problem **1.2**. The angle ϕ can be chosen so that the equation for $\mathcal{S}$ does not contain the "mixed" term $x'y'$. Indeed, consider the transformation of quadratic terms in the equation for $\mathcal{S}$:

$$\begin{aligned}
x^2 &= \cos^2\phi\, x'^2 + \sin^2\phi\, y'^2 - 2\sin\phi\cos\phi\, x'y'\,,\\
&= \tfrac{1}{2}(1+\cos(2\phi))x'^2 + \tfrac{1}{2}(1-\cos(2\phi)y'^2 - \sin(2\phi)x'y'\,,\\
y^2 &= \sin^2\phi\, x'^2 + \cos^2\phi\, y'^2 + 2\sin\phi\cos\phi\, x'y'\\
&= \tfrac{1}{2}(1-\cos(2\phi))x'^2 + \tfrac{1}{2}(1+\cos(2\phi)y'^2 + \sin(2\phi)x'y'\,,\\
xy &= \sin\phi\cos\phi(x'^2 - y'^2) + (\cos^2\phi - \sin^2\phi)x'y'\\
&= \tfrac{1}{2}\sin(2\phi)(x'^2 - y'^2) + \cos(2\phi)x'y'\,.
\end{aligned}$$

Substituting these relations into the original quadratic equation, the coefficient F' at xy is obtained:

$$F' = (B - A)\sin(2\phi) + F\cos(2\phi)\,.$$

The angle ϕ is set so that $F' = 0$ or

$$(\mathbf{9.1})\qquad \tan(2\phi) = \frac{F}{A-B} \quad \text{if} \quad A \neq B \quad \text{and} \quad \phi = \frac{\pi}{4} \quad \text{if} \quad A = B.$$

Similarly, the coefficients A' and B' (the factors at x'^2 and y'^2) and P' and Q' (the factors at x' and y') are

$$A' = \tfrac{1}{2}[A + B + (A - B)\cos(2\phi) + F\sin(2\phi)],$$
$$B' = \tfrac{1}{2}[A + B - (A - B)\cos(2\phi) - F\sin(2\phi)],$$
$$P' = P\cos\phi + Q\sin\phi\,, \quad Q' = Q\cos\phi - P\sin\phi\,,$$

where ϕ satisfies (**9**.**1**). Depending on the values of A, B, and F, the following three cases can occur.

First, $A' = B' = 0$, which is only possible if $A = B = F = 0$. Indeed, under any rotation

$$Ax^2 + By^2 + Fxy = A'x'^2 + B'y'^2 + F'x'y'\,.$$

If $A' = B' = F' = 0$ for a particular ϕ (chosen to make $F' = 0$), then this combination should be identically zero in any other coordinated system obtained by rotation. In this case, $\mathcal{S}$ is defined by the linear equation $Px + Qy + D = 0$, which is a plane parallel to the z axis.

Second, only one of A' and B' vanishes. For establishing the shape it is irrelevant how the horizontal and vertical coordinates in the plane are called. Also, it is always possible to make an additional rotation through the angle $\pi/2$ under which

$$(x', y') \to (y', -x')$$

and hence $A'x'^2 + B'y'^2 + P'x' + Q'y' + D = 0$ becomes

$$A'y'^2 + B'x'^2 + P'y' - Q'x' + D = 0\,,$$

that is, the coefficients at x'^2 and y'^2 are swapped. So, without loss of generality, put $B' = 0$. In this case, the equation for $\mathcal{S}$ assumes the form

$$A'x'^2 + P'x' + Q'y' + D = 0\,.$$

This equation defines a quadric cylinder only if $Q' \neq 0$. By completing the squares, it becomes

$$A'\Big(x' - x_0\Big)^2 + Q'(y' - y_0) = 0\,, \quad x_0 = -\frac{P'}{2A'}\,,\ y_0 = \frac{A'x_0^2 - D}{Q'}\,.$$

After the *translation* of the coordinate system:

$$x' \to x' + x_0\,, \quad y' \to y' + y_0\,,$$

the equation is reduced to

$$A'x'^2 + Q'y' = 0$$

which defines a parabola $y' - ax'^2 = 0$ where $a = -A'/Q'$.

Third, both A' and B' do not vanish. Then, after the completion of squares, the equation $A'x'^2 + B'y'^2 + P'x' + Q'y' + D = 0$ has the form

$$A'(x' - x_0)^2 + B'(y' - y_0)^2 + D' = 0\,,$$
$$x_0 = -\frac{P'}{2A'}\,, \quad y_0 = -\frac{Q'}{2B'}\,, \quad D' = D - A'x_0^2 - B'y_0^2\,.$$

Finally, after the translation of the origin to the point (x_0, y_0), the equation becomes

$$A'x'^2 + B'y'^2 + D' = 0\,.$$

If $D' = 0$, then this equation defines two straight lines $y' = \pm mx'$, where $m = (-A'/B')^{-1/2}$, provided A' and B' have opposite signs. Otherwise, the equation has the solution $x' = y' = 0$ (a line). If $D' \neq 0$, then the equation can be written as

$$\left(-\frac{A'}{D'}\right) x'^2 + \left(-\frac{B'}{D'}\right) y'^2 = 1\,, \qquad D' \neq 0\,.$$

One can always assume that $A'/D' < 0$. Indeed, an additional rotation of the coordinate system through the angle $\pi/2$ swaps the axes, $(x', y') \to (y', -x')$, which can be used to reverse the sign of A'/D'. Now put $-A'/D' = 1/a^2$ and $B'/D' = \pm 1/b^2$ (depending on whether B'/D' is positive or negative) so that the equation becomes

$$\frac{x'^2}{a^2} \pm \frac{y'^2}{b^2} = 1\,.$$

When the plus is taken, this equation defines an ellipse. When the minus is taken, this equation defines a hyperbola. The proof is complete.

9.2. Classification of General Quadric Surfaces. The classification of general quadric surfaces can be carried out in the same way. The general quadratic equation can be written in the new coordinate system that is obtained by a translation (**1**.**1**) and rotation (**5**.**2**) described in Study Problem **5**.**1**. The rotational freedom can be used first to eliminate the "mixed" terms $x'y'$, $x'z'$, and $y'z'$, that is, three parameters that determine a rotation of the coordinate system (see Study Problem **5**.**1**) can be chosen so that $F' = 0$, $G' = 0$, and $H' = 0$ much like in the proof of Theorem **9**.**1**. After this rotation, the squares in the equation in the new coordinates are to be completed and the linear terms are then eliminated by a suitable translation, provided A', B', and C' do not vanish. If one of these coefficients vanishes, say, $C' = 0$, then the resulting equation is quadratic only in x' and y' and linear in z'. If two of them vanish, say, $B' = C' = 0$, then one can use an additional rotation about the x' axis to eliminate either y' or z' (the coordinate x' does not change under such rotations) and the subsequent completion of the squares for the remaining two variables. In this case, the final equation describes a quadric cylinder. The corresponding technicalities can be carried using the result of Study Problem **5.1**, but they are rather lengthy. Yet, they are done best by linear algebra methods. So the final result is given without a proof where x, y, and z denote coordinates in the rotated and translated coordinate system (to avoid too many primes in the equations).

THEOREM **9.2**. (Classification of Quadric Surfaces). *By rotation and translation of a coordinate system, a general equation for quadric surfaces can be brought to one of the standard forms:*

$$A'x^2 + B'y^2 + C'z^2 + D' = 0 \quad \text{or} \quad A'x^2 + B'y^2 + V'z = 0\,.$$

In particular, the standard forms describe quadric cylinders and one of the following six surfaces:

$$\begin{aligned}
&\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 && \text{(ellipsoid)}\,,\\
&\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} && \text{(elliptic double cone)},\\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 && \text{(hyperboloid of one sheet)}\,,\\
&-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 && \text{(hyperboloid of two sheets)}\,,\\
&\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} && \text{(elliptic paraboloid)},\\
&\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} && \text{(hyperbolic paraboloid)}\,.
\end{aligned}$$

The indicated six shapes are the counterparts in three dimensions of the conic sections in the plane discussed in Calculus 2.

9.3. Visualization of Quadric Surfaces. The shape of a quadric surface can be understood by studying intersections of the surface with the coordinate planes $x = x_0$, $y = y_0$, and $z = z_0$. These intersections are also called *traces*.

An Ellipsoid. If $a^2 = b^2 = c^2 = R^2$, then the ellipsoid becomes a sphere of radius R. So, intuitively, an ellipsoid is a sphere "stretched" along the coordinate axes (see Fig. **9.2** (left panel)). Traces of an ellipsoid in

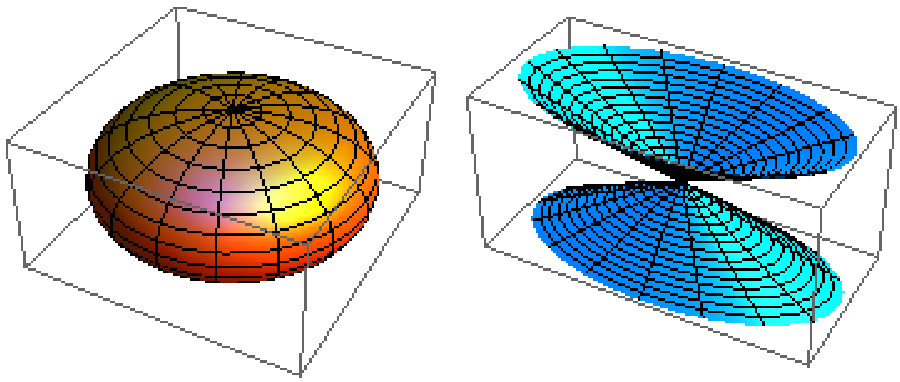


FIGURE **9.2**. **Left**: An ellipsoid. A cross section by any coordinate plane is an ellipse. **Right**: An elliptic double cone. A cross section by a horizontal plane z = const is an ellipse. A cross section by any vertical plane through the z axis is two lines through the origin.

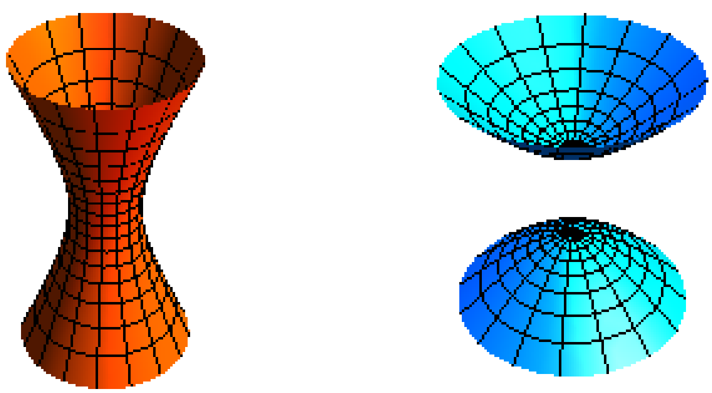


FIGURE **9.3**. **Left**: A hyperboloid of one sheet. A cross section by a horizontal plane z =const is an ellipse. A cross section by a vertical plane x =const or y =const is a hyperbola. **Right**: A hyperboloid of two sheets. A non-empty cross section by a horizontal plane is an ellipse. A cross section by a vertical plane is a hyperbola.

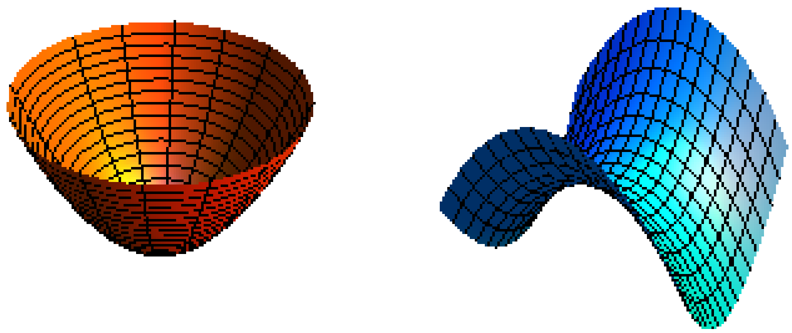


FIGURE **9.4**. **Left**: An elliptic paraboloid. A nonempty cross section by a horizontal plane is an ellipse. A cross section by a vertical plane is a parabola. **Right**: A hyperbolic paraboloid (a "saddle"). A cross section by a horizontal plane is a hyperbola. A cross section by a vertical plane is a parabola.

the planes $x = x_0$, $|x_0| < a$, are ellipses

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = k \quad \text{or} \quad \frac{y^2}{(b\sqrt{k})^2} + \frac{z^2}{(c\sqrt{k})^2} = 1, \quad k = 1 - \frac{x_0^2}{a^2} > 0.$$

As the plane $x = x_0$ gets closer to $x = a$ or $x = -a$, k becomes smaller and so does the ellipse because its major axes $b\sqrt{k}$ and $c\sqrt{k}$ decrease. Apparently, the traces in the planes $x = \pm a$ consist of a single point $(\pm a, 0, 0)$, and there is no trace in any plane $x = x_0$ if $|x_0| > a$. Traces in the planes $y = y_0$ and $z = z_0$ are also ellipses and exist only if $|y_0| \le b$ and $|z_0| \le c$. Thus, *the characteristic geometrical property of an ellipsoid is that its traces are ellipses.*

A Paraboloid. Suppose $c > 0$. Then the paraboloid lies above the xy plane because it has no trace in all horizontal planes below the xy plane, $z = z_0 < 0$. In the xy plane, its trace contains just the origin. Traces of the paraboloid in the planes $z = z_0 > 0$ are ellipses,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k \quad \text{or} \quad \frac{x^2}{(a\sqrt{k})^2} + \frac{y^2}{(b\sqrt{k})^2} = 1, \quad k = z_0/c > 0.$$

The ellipses become wider as z_0 gets larger because their major axes $a\sqrt{k}$ and $b\sqrt{k}$ grow with increasing k. Vertical traces (traces in the planes $x = x_0$ and $y = y_0$) are parabolas:

$$z - kc = \frac{c}{b^2}\,y^2, \quad k = \frac{x_0^2}{a^2} \quad \text{and} \quad z - kc = \frac{c}{a^2}\,x^2, \quad k = \frac{y_0^2}{b^2}.$$

Similarly, a paraboloid with $c < 0$ lies below the xy plane. So *the characteristic geometrical property of a paraboloid is that its horizontal traces are ellipses, while its vertical ones are parabolas* (see Fig. **9.4** (left panel)). If $a = b$, the paraboloid is also called a *circular paraboloid* because its horizontal traces are circles.

A Double Cone. Horizontal traces are ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k^2 \quad \text{or} \quad \frac{x^2}{(ak)^2} + \frac{y^2}{(bk)^2} = 1, \quad k = z_0/c.$$

So as $|z_0|$ grows, that is, as the horizontal plane moves away from the xy plane ($z = 0$), the ellipses become wider. In the xy plane, the cone has a trace that consists of a single point (the origin). The vertical traces in the planes $x = 0$ and $y = 0$ are a pair of lines

$$z = \pm(c/b)y \quad \text{and} \quad z = \pm(c/a)x\,.$$

Furthermore, the trace in any plane that contains the z axis is also a pair of straight lines (see Fig. **9.2**). Indeed, take parametric equations of a line in the xy plane through the origin,

$$x = v_1 t\,, \quad y = v_2 t\,.$$

Then the z coordinate of any point of the trace of the cone in the plane that contains the z axis and this line satisfies the equation

$$\frac{z^2}{c^2} = \Big[(v_1/a)^2 + (v_2/b)^2\Big]\,t^2 \quad \Rightarrow \quad z = \pm v_3 t\,,$$

where $v_3 = c\sqrt{(v_1/a)^2 + (v_2/b)^2}$. So the points of intersection form two straight lines through the origin:

$$x = v_1 t\,, \quad y = v_2 t\,, \quad z = \pm v_3 t\,, \quad -\infty < t < \infty\,.$$

Given an ellipse in a plane, consider a line through the center of the ellipse that is perpendicular to the plane. Fix a point P on this line that does not coincide with the point of intersection of the line and the plane. Then a double cone is the surface that contain all lines through P and points of the ellipse. The point P is called the *vertex* of the cone. So *the characteristic geometrical property of a cone is that horizontal traces are ellipses; its vertical traces in planes through the axis of the cone are straight lines* (see Fig. **9.2** (right panel)).

Vertical traces in the planes $x = x_0 \neq 0$ and $y = y_0 \neq 0$ are hyperbolas $y^2/b^2 - z^2/c^2 = k$, where $k = -x_0^2/a^2$, and $x^2/a^2 - z^2/c^2 = k$, where $k = -y_0^2/b^2$. Recall in this regard *conic sections* studied in Calculus II.

If $a = b$, the cone is called a *circular cone*. In this case, vertical traces in the planes containing the cone axis are a pair of lines with the same slope that is determined by the angle ϕ between the axis of the cone and any of these lines: $c/b = c/a = \cot\phi$. The equation of a circular double cone can be written as

$$z^2 = \cot^2(\phi)(x^2 + y^2)\,, \quad 0 < \phi < \pi/2\,.$$

The equation for an upper ($z \geq 0$) or lower ($z \leq 0$) cone of the double circular cone is

$$z = \pm\cot(\phi)\sqrt{x^2 + y^2}\,.$$

A Hyperbolic Paraboloid. Horizontal traces in the planes $z = z_0$ are hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = k, \quad k = \frac{z_0}{c}.$$

Suppose $c > 0$. If $z_0 > 0$ (horizontal planes above the xy plane), then $k > 0$. In this case the hyperbolas are symmetric about the x axis, and their branches lie either in $x > 0$ or in $x < 0$ (i.e., they do not intersect the y axis) because $x^2/a^2 = y^2/b^2 + k > 0$ (x cannot not vanish for any y). If $z_0 < 0$, then $k < 0$. In this case the hyperbolas are symmetric about the y axis, and their branches lie either in $y > 0$ or in $y < 0$ (i.e., they do not intersect the x axis) because $y^2/b^2 = x^2/a^2 - k > 0$ (y cannot vanish for any x). Vertical traces in the planes $x = x_0$ and $y = y_0$ are concave *down* and *up* parabolas, respectively:

$$z - z_0 = -\frac{c}{b^2}\,y^2, \quad z_0 = \frac{cx_0^2}{a^2} \quad \text{and} \quad z - z_0 = \frac{c}{a^2}\,x^2, \quad z_0 = -\frac{cy_0^2}{b^2}$$

Take the parabolic trace the zx plane $z = (c/a^2)x^2$ (i.e. in the plane $y = y_0 = 0$). The traces in the perpendicular planes $x = x_0$ are parabolas whose vertices are $(x_0, 0, z_0)$, where $z_0 = (c/a^2)x_0^2$, and hence lie on the parabola $z = (c/a^2)x^2$ in the zx plane. This observation suggests that the hyperbolic paraboloid is swept by the parabola in the zy plane, $z = -(c/b^2)y^2$, when the latter is moved parallel so that its vertex remains on the parabola $z = (c/a^2)x^2$ in the perpendicular plane. The obtained surface has the characteristic shape of a "saddle" (see Fig. **9.4** (right panel)).

A Hyperboloid of One Sheet. Traces in horizontal planes $z = z_0$ are ellipses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k^2 \quad \text{or} \quad \frac{x^2}{(ka)^2} + \frac{y^2}{(kb)^2} = 1, \quad k = \sqrt{1 + z_0^2/c^2} \geq 1$$

The ellipse is the smallest in the xy plane ($z_0 = 0$ and $k = 1$). The major axes of the ellipse, ka and kb, grow as the horizontal plane gets away from the xy plane because k increases. The surface looks like a tube with ever expanding elliptic cross section. The vertical cross section of the "tube" by the planes $x = 0$ and $y = 0$ are hyperbolas:

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$

So, *the characteristic geometrical property of a hyperboloid of one sheet is that its horizontal traces are ellipses and its vertical traces are hyperbolas* (see Fig. **9.3** (left panel)).

A Hyperboloid of Two Sheets. A distinctive feature of this surface is that it consists of two sheets (see Fig. **9.3** (right panel)). Indeed, the trace in the plane $z = z_0$ satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0^2}{c^2} - 1$$

which has no solution if $z_0^2/c^2 - 1 < 0$ or $-c < z_0 < c$. So one sheet lies above the plane $z = c$ and the other lies below the plane $z = -c$. Horizontal traces in the planes $z = z_0 > c$ or $z = z_0 < -c$ are ellipses whose major axes increase with increasing $|z_0|$. The upper sheet touches the plane $z = c$ at the point $(0, 0, c)$, while the lower sheet touches the plane $z = -c$ at the point $(0, 0, -c)$. These points are called *vertices* of a hyperboloid of two sheets. Vertical traces in the planes $x = 0$ and $y = 0$ are hyperbolas:

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$$

Thus, the characteristic geometrical properties of hyperboloids of one sheet and two sheets are similar, apart from the fact that the latter one consists of two sheets. Also, in the asymptotic region $|z| \gg c$ (reads "$|z|$ is much larger than c"), the hyperboloids approach the surface of the double cone. Indeed, in this case, $z^2/c^2 \gg 1$, and hence the equations for hyperboloids can be well approximated by the double cone equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm 1 + \frac{z^2}{c^2} \approx \frac{z^2}{c^2}\,, \qquad \frac{z^2}{c^2} \gg 1\,,$$

the numbers ± 1 can be neglected as compared to z^2/c^2 for large enough z^2. In the region $z > 0$, the hyperboloid of one sheet approaches the double cone from below, while the hyperboloid of two sheets approaches it from above. For $z < 0$, the converse holds. In other words, the hyperboloid of two sheets lies "inside" the cone, while the hyperboloid of one sheet lies "outside" it. Both the hyperboloids become closer to the double cone as z^2 gets larger (away from the xy plane).

9.4. Shifted quadric surfaces. If the origin of a coordinated system is shifted to a point (x_0, y_0, z_0) without any rotation of the coordinate axes, then the coordinates of a point in space are translated

$$(x, y, z) \to (x - x_0, y - y_0, z - z_0)\,.$$

Therefore any equation of the form $f(x, y, z) = 0$ becomes a "shifted" equation in the new coordinates:

$$f(x, y, z) = 0 \quad \to \quad f(x - x_0, y - y_0, z - z_0) = 0\,.$$

If the equation $f(x, y, z) = 0$ defines a surface in space, then the equation $f(x - x_0, y - y_0, z - z_0) = 0$ defines the very same surface that has been translated as the whole (each point of the surface is shifted by the same vector $\langle x_0, y_0, z_0\rangle$). For example, the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = \frac{(z - z_0)^2}{c^2}$$

describes a double elliptic cone whose axis is parallel to the z axis and whose vertex is at (x_0, y_0, z_0). Equations of shifted quadric surfaces can be reduced to the standard form by completing the squares.

Example **9.1**. *Classify the quadric surface*

$$9x^2 + 36y^2 + 4z^2 - 18x + 72y + 16z + 25 = 0$$

Solution: Let us complete the squares for each of the variables:

$$\begin{aligned} 9x^2 - 18x &= 9(x^2 - 2x) = 9[(x-1)^2 - 1] = 9(x-1)^2 - 9 \\ 36y^2 + 72y &= 36(y^2 + 2y) = 36[(y+1)^2 - 1] = 36(y+1)^2 - 36 \\ 4z^2 + 16z &= 4(z^2 + 4z) = 4[(z+2)^2 - 4] = 4(z+2)^2 - 16 \end{aligned}$$

The equation becomes $9(x-1)^2 + 36(y+1)^2 + 4(z+2)^2 = 36$ and by dividing it by 36 the standard form is obtained

$$\frac{(x-1)^2}{16} + (y+1)^2 + \frac{(z+2)^2}{9} = 1$$

This equation describes an ellipsoid with the center at $(1, -1, -2)$ and major axes $a = 4$, $b = 1$, and $c = 3$. □

EXAMPLE **9.2**. *Classify the surface* $x^2 + 2y^2 - 4y - 2z = 0$

SOLUTION: By completing the squares

$$2y^2 - 4y = 2(y^2 - 2y) = 2[(y-1)^2 - 1]\,,$$

the equation can written in the form

$$x^2 + 2(y-1)^2 - 2 - 2z = 0 \quad \text{or} \quad z + 1 = \frac{x^2}{2} + (y-1)^2$$

which is an elliptic paraboloid with the vertex at $(0, 1, -1)$ because it is obtained from the standard equation $z = x^2/2 + y^2$ by the shift of the coordinate system $(x, y, z) \to (x, y-1, z+1)$. □

EXAMPLE **9.3**. *Classify the surface* $x^2 - 4y^2 + z^2 - 2x - 8z + 1 = 0$.

SOLUTION: By completing the squares, the equation is transformed to

$$(x-1)^2 - 1 - 4y^2 + 4(z+1)^2 - 4 + 1 = 0\,,$$
$$\frac{(x-1)^2}{4} + (z+1)^2 - y^2 = 1$$

which is a hyperboloid of one sheet whose axis is the line through $(1, 0, -1)$ that is parallel to the y axis. □

EXAMPLE **9.4**. *Use an appropriate rotation in the* $xy-$*plane to reduce the equation* $z = 2xy$ *to the standard form and classify the surface.*

SOLUTION: Let (x', y') be coordinates in the rotated coordinate system through the angle ϕ. By Study Problem **1.2**, the old coordinates (x, y) are expressed via the new ones (x', y'):

$$x = x' \cos\phi - y' \sin\phi, \quad y = y' \cos\phi + x' \sin\phi$$

In the new coordinate system, the equation

$$z = 2xy = 2x'^2 \cos\phi \sin\phi - 2y'^2 \cos\phi \sin\phi + 2x'y'(\cos^2\phi - \sin^2\phi)$$

would have the standard form if the coefficient at $x'y'$ vanishes. So, put $\phi = \pi/4$. Then $2\sin\phi\cos\phi = \sin(2\phi) = 1$ and

$$z = x'^2 - y'^2$$

which is a hyperbolic paraboloid (a saddle). □

9.5. Study Problems.

Problem **9.1**. *Classify the quadric surface* $3x^2 + 3z^2 - 2xz = 4$.

SOLUTION: The equation does not contain one variable (the y coordinate). The surface is a cylinder parallel to the y axis. To determine the type of cylinder, consider a rotation of the coordinate system in the xz plane and choose the rotation angle so that the coefficient at the xz term vanishes in the transformed equation. The rotation angle is obtained from Eq. (**9.1**) for $A = B = 3$ and $F = -2$ so that

$$\phi = \frac{\pi}{4}\,.$$

Then

$$A' = \frac{1}{2}(A + B - F) = 4\,, \quad B' = \frac{1}{2}(A + B + F) = 2\,.$$

So, in the new coordinates, the equation becomes

$$4x'^2 + 2z'^2 = 4 \quad \Leftrightarrow \quad x'^2 + \frac{z'^2}{2} = 1\,,$$

which is an ellipse with semi-axes $a = 1$ and $b = \sqrt{2}$. The surface is an elliptic cylinder parallel to the y axis. □

Problem **9.2**. *Classify the quadric surface* $x^2 - 2x + y - z = 0$.

SOLUTION: The characteristic feature of this quadratic equation is that the "mixed" terms xy, xz, and yz are absent. By completing the squares, the equation can be transformed into the form

$$(x - 1)^2 + (y - 1) + z = 0\,.$$

After shifting the origin to the point $(1, 1, 0)$, the equation becomes

$$x^2 + y - z = 0\,.$$

Consider rotations of the coordinate system about the x axis (see remarks in the beginning of Section **9.2** about the case when $B' = C' = 0$):

$$(y, z) = (\cos\phi\, y' + \sin\phi\, z',\ \cos\phi\, z' - \sin\phi\, y')\,.$$

Under this rotation,

$$y - z = (\cos\phi + \sin\phi)y' + (\sin\phi - \cos\phi)z'\,.$$

Therefore, by setting $\phi = \pi/4$, the variable z' is eliminated and the equation assumes one of the standard forms

$$x^2 + \sqrt{2}\, y' = 0$$

which corresponds to a parabolic cylinder. □

Problem **9.3**. *Classify the quadric surface* $x^2+z^2-2x+2z-y=0$.

SOLUTION: By completing the squares, the equation can be transformed into the form

$$(x-1)^2+(z+1)^2-(y+2)=0\,.$$

The latter can be brought into one of the standard forms by shifting the origin to the point $(1,-2,-1)$:

$$x^2+z^2=y\,,$$

which is a circular paraboloid. Its symmetry axis is parallel to the y axis (the line of intersection of the planes $x=1$ and $z=-1$) and its vertex is $(1,-2,-1)$. □

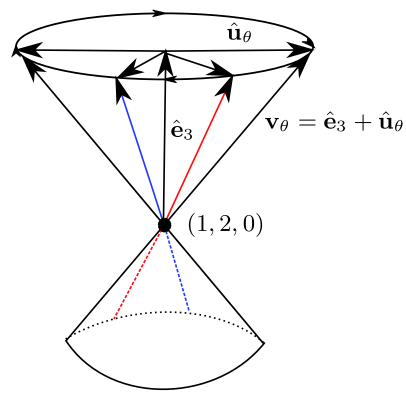


FIGURE **9.5**. An illustration to Study Problem **9.4**. The vector $\hat{\mathbf{u}}_\theta$ rotates about the vertical line so that the line through $(1,2,0)$ and parallel to $\mathbf{v}_\theta$ sweeps a double cone with the vertex at $(1,2,0)$.

Problem **9.4**. *Sketch and/or describe the set of points in space formed by a family of lines through the point* $(1,2,0)$ *and parallel to* $\mathbf{v}_\theta = \langle\cos\theta,\sin\theta,1\rangle$*, where* $\theta\in[0,2\pi]$ *labels lines in the family.*

SOLUTION: The parametric equations of each line are

$$x=1+t\cos\theta\,,\quad y=2+t\sin\theta\,,\quad z=t\,.$$

Therefore,

$$(x-1)^2+(y-2)^2=z^2$$

for all values of t and θ. Thus, the lines form a double cone whose axis is parallel to the z axis and whose vertex is $(1,2,0)$. Alternatively, one could notice that the vector $\mathbf{v}_\theta$ rotates about the z axis as θ changes.

Indeed, put $\mathbf{v}_\theta = \hat{\mathbf{u}} + \hat{\mathbf{e}}_z$, where $\hat{\mathbf{u}} = \langle \cos\theta, \sin\theta, 0\rangle$ is the unit vector in the xy plane as shown in Figure **9.5**. It rotates as θ changes, making a full turn as θ increases from 0 to 2π. So the set in question can be obtained by rotating a particular line, say, the one corresponding to $\theta = 0$, about the vertical line through $(1, 2, 0)$. The line sweeps the double cone. □

9.6. Exercises.

1–10. Use traces to sketch and identify each of the following surfaces:

1. $y^2 = x^2 + 9z^2$;
2. $y = x^2 - z^2$;
3. $4x^2 + 2y^2 + z^2 = 4$;
4. $x^2 - y^2 + z^2 = -1$;
5. $y^2 + 4z^2 = 16$;
6. $x^2 - y^2 + z^2 = 1$;
7. $x^2 + 4y^2 - 9z^2 + 1 = 0$;
8. $x^2 + z = 0$;
9. $x^2 + 9y^2 + z = 0$;
10. $y^2 - 4z^2 = 16$.

11–15. Reduce each of the following equations to one of the standard forms, classify the surface, and sketch it:

11. $x^2 + y^2 + 4z^2 - 2x + 4y = 0$;
12. $x^2 - y^2 + z^2 + 2x - 2y + 4z + 2 = 0$;
13. $x^2 + 4y^2 - 6x + z = 0$;
14. $y^2 - 4z^2 + 2y - 16z = 0$;
15. $x^2 - y^2 + z^2 - 2x + 2y = 0$.

16–20. Use rotations in the appropriate coordinate plane to reduce each of the following equations to one of the standard forms and classify the surface:

16. $6xy + x^2 + y^2 = 1$;
17. $3y^2 + 3z^2 - 2yz = 1$;
18. $x - yz = 0$;
19. $xy - z^2 = 0$;
20. $2xz + 2x^2 - y^2 = 0$.

21. Find an equation for the surface obtained by rotating the line $y = 2x$ about the y axis. Classify the surface.

22. Find an equation for the surface obtained by rotating the curve $y = 1 + z^2$ about the y axis. Classify the surface.

23. Find equations for the family of surfaces obtained by rotating the curves $x^2 - 4y^2 = k$ about the y axis where k is real. Classify the surfaces.

24. Find an equation for the surface consisting of all points that are equidistant from the point $(1, 1, 1)$ and the plane $z = 2$.

25. Sketch the solid region bounded by the surface $z = \sqrt{x^2 + y^2}$ from below and by $x^2 + y^2 + z^2 - 2z = 0$ from above.

26. Sketch the solid region bounded by the surfaces $y = 2 - x^2 - z^2$, $y = x^2 + z^2 - 2$, and $x^2 + z^2 = 1$.

27. Sketch the solid region bounded by the surfaces $x^2 + y^2 = R^2$ and $x^2 + z^2 = R^2$

28. Find an equation for the surface consisting of all points P for which the distance from P to the y axis is twice the distance from P to the zx plane. Identify the surface.

29. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines through (a, b, c) and parallel to $\mathbf{v} = \langle 1, 1, 2(b-a)\rangle$ and $\mathbf{u} = \langle 1, -1, -2(a+b)\rangle$ both lie entirely on this paraboloid. Deduce from this result that the hyperbolic paraboloid can be generated by the motion of a straight line. Show that hyperboloids of one sheet, cones, and cylinders can also be obtained by the motion of a straight line.

Remark. The fact that hyperboloids of one sheet are generated by the motion of a straight line is used to produce gear transmissions. The cogs of the gears are the generating lines of the hyperboloids.

30. Find an equation for the cylinder of radius R whose axis goes through the origin and is parallel to a vector $\mathbf{v}$.

31. Show that the curve of intersection of the surfaces $x^2 - 2y^2 + 3z^2 - 2x + y - z = 1$ and $2x^2 - 4y^2 + 6z^2 + x - y + 2z = 4$ lies in a plane.

32. The projection of a point set $\mathcal{S}$ onto the xy plane is obtained by setting the z coordinates of all points of $\mathcal{S}$ to zero. The projections of $\mathcal{S}$ onto the other two coordinate axes are defined similarly. What are the curves that bound the projections of the ellipsoid $x^2 + y^2 + z^2 - xy = 1$ onto the coordinate planes?

Selected Answers and Hints to Exercises

Section 1.11. **1**. 3. **2**. $\sqrt{2}$. **3**. The points lie in a line. **4**. The points do not lie in a line. **5**. If $P = (x, y, z)$ is a point in the plane in question, then $|OP|^2 + |OB|^2 = |PB|^2$ or $x + y + z = 3$. The given points are not in this plane. **6**. Let D_a be the distance to the a axis, and D_{ab} be the distance to the ab plane. Then $D_x = \sqrt{13}$, $D_y = \sqrt{10}$, $D_z = \sqrt{5}$, $D_{xy} = 3$, $D_{xz} = 2$, $D_{yz} = 1$. **7**. $3\sqrt{3}$, $3\sqrt{3}$, and 6. **8**. $\mathcal{S}$ is the straight line through the origin and the point $(1, 2, 3)$. The point of $\mathcal{S}$ that is closest to $(3, 2, 1)$ is $(5/7,\ 10/7,\ 15/7)$ (minimize the squared distance from $(3, 2, 1)$ to $(t, 2t, 3t)$ relative to the parameter t). **9**. The sphere centered at $(1, -2, 3)$ of radius $\sqrt{14}$. **10**. The whole space with the ball of radius 2 centered at the origin removed; the boundary sphere is included into the set. **11**. The upper half of the the ball of radius 2 centered at the origin; the spherical part of the boundary is included into the set, while the bottom (planar) boundary is excluded. **14**. The ball centered at the origin of radius 2 that has a cylindrical hole of radius 1 along the z axis. **15**. The part of the sphere of radius 1 centered at $(0, 0, 1)$ that lies above the coordinate plane $z = 1$. **16**. The circle $x^2 + y^2 = 1$ of radius 1 in the plane $z = 1$. The center of the circle is at $(0, 0, 1)$. **17**. The union of three coordinate planes $x = a$, $y = b$, and $z = c$. **18**. The rectangular box whose faces lie in the coordinate planes $x = \pm 1$, $y = \pm 2$, and $z = \pm 3$. **19**. $(x-2)^2 + (y-2)^2 + (z-2)^2 = 2$. **20**. $(x-1)^2 + (y-2)^2 + (z-3)^2 = R^2$ where $R = 3$, $R = 1$, and $R = 2$ for the spheres touching the xy, yz, and xz planes, respectively. **21**. $(x-1)^2 + (y+2)^2 + (z-3)^2 = R^2$ where $R = \sqrt{13}$, $R = \sqrt{10}$, and $R = \sqrt{5}$ for the spheres touching the x, y, and z axes, respectively. **23**. $x^2 + y^2 + z^2 \leq R^2$ and $x^2 + (y - R/2)^2 \geq R^2/4$. **26**. $104\pi/3$. **27**. 32. **28**. $8abc - 2\pi R^2 a$. **29**. The line $x + y = 0$. **30**. The plane perpendicular to the segment AB and passing through the midpoint $P_0 = (-1, 0, 1)$ of AB. If $P = (x, y, z)$ is a point of the set, then $|PA|^2 = |PB|^2$ or $x + y + z = 0$.

Section 2.5. **1**. $\langle -2, 3, -2\rangle$. **2**. $\langle 2, -3, 2\rangle$. **3**. $\langle -1, \frac{3}{2}, -1\rangle$. **4**. $\langle -5, 4, -8\rangle$. **5**. $\langle 2a, \sqrt{3}a, 0\rangle$. **6**. Yes. $\overrightarrow{AC} + \overrightarrow{AD} = \overrightarrow{AB}$. **7**. $D = (3, 1, 4)$. **8**. $(3, 1, , 5)$. **11**. $\hat{\mathbf{a}} = \langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\rangle$, $\hat{\mathbf{b}} = \langle -\frac{3}{5}, 0, \frac{4}{5}\rangle$, $2\mathbf{a} - 3\mathbf{b}$. **12**. $\langle -\frac{3}{2}a, \frac{\sqrt{3}}{2}a, 0\rangle$ and $\langle \frac{3}{2}a, -\frac{\sqrt{3}}{2}a, 0\rangle$, where a is the length of $\mathbf{a}$. **13**. $\mathbf{0}$. **15**. 2π (the length of a circle of radius 1). **16**. If the x axis is directed from west to east, and the y axis from south to north, then the velocity is $\langle v\sin\beta - u\sin\alpha, u\cos\alpha + v\cos\beta, 0\rangle$, the speed is $(u^2 + v^2 + 2uv\cos(\alpha + \beta))^{1/2}$;

the aircraft flies north if $\beta = \arcsin((u/v)\sin\alpha)$. **18**. A circle of radius k centered at the point whose position vector is **a**. **19**. An ellipse whose foci have position vectors **a** and **b**. **20**. A sphere of radius k centered at the point whose position vector is **a**. **21**. A surface obtained by rotating the ellipse in Exercise **19** about the line through its foci (called an ellipsoid). **22**. $\mathbf{r}_0 = \frac{1}{m}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n)$ where $m = m_1 + m_2 + \cdots + m_n$ is the total mass; for the triangle ABC, $\mathbf{r}_0 = \langle \frac{1}{3}, 1, 1\rangle$. **23**. A tangent vector is $\langle a, f'(a)\rangle$, a normal vector is $\langle -f'(a), a\rangle$; if $f(x) = x^2$ and $a = 1$, then the tangent and normal vectors are, respectively, $\langle 1, 2\rangle$ and $\langle -2, 1\rangle$. **24**. If **c** is the largest vector, then by the parallelogram rule $\|\mathbf{a}\| + \|\mathbf{b}\| > \|\mathbf{c}\|$.

Section 3.8. **1**. 3. **2**. -3. **3**. -4. **4**. -30. **5**. -21. **7**. Neither. **8**. Orthogonal. **9**. $b = \pm 2$ and $b = 0$. **10**. $\pm\frac{1}{\sqrt{5}}\langle 0, 2, 1\rangle$. **11**. $\pi/3$. **12**. If α, β, and γ are the angles at the vertices A, B, and C, respectively, then $\cos\alpha = -\frac{3}{\sqrt{6}\sqrt{17}}$, $\cos\beta = \frac{20}{\sqrt{17}\sqrt{29}}$, $\cos\gamma = \frac{9}{\sqrt{6}\sqrt{29}}$. **14**. Minimum at $s = -2$ and $\mathbf{a}\cdot\mathbf{b} = 0$ at $s = -2 \pm \sqrt{3}$. **15**. $\cos^{-1}(1/\sqrt{3})$. **16**. $\mathbf{d}_1 = \mathbf{a}+\mathbf{b}+\mathbf{c}$, $\mathbf{d}_2 = \mathbf{a}+\mathbf{b}-\mathbf{c}$, $\mathbf{d}_3 = \mathbf{a}-\mathbf{b}+\mathbf{c}$, $\mathbf{d}_4 = \mathbf{c}+\mathbf{b}-\mathbf{a}$. **17**. The scalar projection is $b_\parallel = 2$, the vector projection is $\mathbf{b}_\parallel = (2/3)\mathbf{a}$, and $\mathbf{b}_\perp = \mathbf{b} - \mathbf{b}_\parallel$. **18**. The scalar projection is $-15/\sqrt{34}$, the vector projection is $-\frac{15}{34}\langle 3, 3, 4\rangle$. **19**. $\langle a/2,\ \pm a/2,\ a/\sqrt{2}\rangle$. **24**. $2\sin^{-1}(2\sqrt{a^2+b^2}/c)$, $2\sin^{-1}(2a/c)$, and $2\sin^{-1}(2b/c)$. **25**. $\mathbf{a} = \frac{5}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3$. **26**. If $\mathbf{b}_\parallel$ is the vector projection of **b** onto **a**, then $\mathbf{b}_\perp = \mathbf{b} - \mathbf{b}_\parallel$ is orthogonal to **a**, and the vectors $\mathbf{b} + t\mathbf{b}_\perp$, where t is real, have the same projection onto **a** as **b**. **27**. $\sqrt{55}$. **28**. The equilibrium configuration is the symmetric trapezoid with the largest side of length L on the ceiling. If θ is the angle between the rope connecting the ball to the ceiling and the vertical line, then $\sin\theta = (L-h)/(2h)$, $0 < \theta < \pi/2$, the tension in the rope connecting the balls is $mg\tan\theta$, and the tension in the ropes connecting the balls to the ceiling is $mg/\cos\theta$. **29**. $300\cos(20^\circ)$ N. **30**. $10\sqrt{2} + \frac{15}{2}\sqrt{3}$ is the drag force; the barge moves in the direction perpendicular to the stream.

Section 4.6. **1**. $\langle 2, -4, 2\rangle$. **2**. $\langle 3, 5, 1\rangle$. **3**. $\langle 1, -4, -11\rangle$. **4**. $11(\mathbf{c}\times\mathbf{d}) = \langle 11, 22, 33\rangle$. **5**. $15\mathbf{u}_1 + 6\mathbf{u}_2 - 9\mathbf{u}_3$. **6**. $\langle 3\sqrt{3}, -3, 0\rangle$. **7**. The angle between the vectors is $\theta = \pi/6$. By the right hand rule, $\mathbf{a}\times\mathbf{b} = \langle 0, 0, c\rangle$, where $c = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = 3$. **8**. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \langle 1, 2, -7\rangle$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a}) = \langle -2, 2, 6\rangle$, $\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \langle 1, -4, 1\rangle$. **9**. The vector is $-\mathbf{c}$ and $\|-\mathbf{c}\| = 3$. **11**. Let α, β, and γ be the direction angles of $\mathbf{a}\times\mathbf{b}$. Then $0 < \alpha < \pi/2$, $\pi/2 < \beta < \pi$, and $\gamma = \pi/2$. **12**. $\mathbf{a}\times\mathbf{b}$ has a negative y component and positive x and z components; $\mathbf{a}\times\mathbf{c}$ has a positive x component and negative y and

z components; $\mathbf{b}\times\mathbf{c}$ has a negative z component and positive x and y components. **13**. **0**. **15**. $\frac{1}{2}(b^2c^2+a^2c^2+a^2b^2)^{1/2}$. **16**. $\langle -1,-1,1\rangle$ and the area is $\sqrt{3}$. **18**. $A/4$. **19**. $\sqrt{17}$. **20**. Points A, B, C, and D are in a plane if and only if $\overrightarrow{AB}\times\overrightarrow{AC}$ and $\overrightarrow{DB}\times\overrightarrow{DC}$ are parallel (or proportional). **23**. If A_1 is the area of the given parallelogram and A_2 is the area in question, then $A_2 = 2A_1$. **26**. Yes. **27**. Make unit vectors parallel or antiparallel, respectively, to $\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$, and $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})$ (mutually orthogonal vectors); $\hat{\mathbf{u}}_1 = \frac{1}{3}\mathbf{a} = \frac{1}{3}\langle 1,2,2\rangle$, $\hat{\mathbf{u}}_2 = \frac{1}{\sqrt{5}}\langle 2,0,-1\rangle$, $\hat{\mathbf{u}}_3 = \frac{1}{3\sqrt{5}}\langle -2,5,-4\rangle$. **28**. $\sqrt{46}/2$. **29**. $3\sqrt{29}/2$. **32**. $\tan^{-1}(2/3)$.

Section 5.5. **1**. 3, -3, 3, respectively. **2**. 4, -4, 4, respectively. **3**. 0. **4**. 14, -14, 14, respectively. **5**. $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\| = 6$ (the volume of a rectangular box); then $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \pm 6$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$. **6**. Not coplanar. **7**. $s = 2/3$. **8**. A, B, C, and D are not in a plane; $V = 8$. **9**. Part (i): $s = 1$; Part (ii): $s = 4$ and $s = -2$. **10**. 24. **11**. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 3uvw - (u^3+v^3+w^3) = 2 \neq 0$ (not coplanar); the volume is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = 2$. **13**. The diagonals are $\mathbf{d}_1 = \mathbf{a}+p\mathbf{b}$, $\mathbf{d}_2 = \mathbf{b}+s\mathbf{c}$, and $\mathbf{d}_3 = \mathbf{a}+t\mathbf{c}$, where the numbers p, s, and t take values 1 or -1; then $\mathbf{d}_1\cdot(\mathbf{d}_2\times\mathbf{d}_3) = (t+ps)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where $t+ps$ is 0 (the diagonals are coplanar, the volume is zero) or ± 2 and the volume is $2V$. **15**. $\mathbf{a}\cdot(\mathbf{r}\cdot\mathbf{b}) = \mathbf{r}\cdot(\mathbf{b}\times\mathbf{a}) = 0$; so $\mathbf{r}\perp\mathbf{a}\times\mathbf{b}$ or $\mathbf{r} = s\mathbf{a}+t\mathbf{b}$ where s and t are any real numbers; if, in addition, $\mathbf{r}\perp\mathbf{b}$, then $t = -s(\mathbf{a}\cdot\mathbf{b})/\|\mathbf{b}\|^2$ and $\mathbf{r}$ is any vector orthogonal to $\mathbf{b}$ that also lies in the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$. **16**. $\sqrt{2}-1$. **17**. $\sqrt{3}$. **18**. $3/\sqrt{38}$. **20**. $s = (-5\pm 3\sqrt{5})/2$.

Section 6.5. **1**. $2x-2y+z=0$, $D = \frac{4}{3}$. **2**. No. **3**. $0 < \cos^{-1}(\sqrt{2}/3) < \pi/2$. **4**. The faces through the origin: $x-5y+3z=0$, $x-3y+z=0$, and $x-2y+z=0$; the faces through the point P: $x-5y+3z=2$, $x-3y+z=-2$, and $x-2y+z=1$. **5**. $3x-7y+3z=0$, $x+y-z=0$, and $x-4y+2z=0$. **6**. The planes through the origin are the same as in **5**; the other three planes are $3x-9y+5z=2$, $y-z=-1$, and $3x-7y+3z=-2$. **7**. $x/a+y/b+z/c=1$, the distance is $D = abc/\sqrt{a^2b^2+a^2c^2+b^2c^2}$, and the angles are $\cos^{-1}(D/|a|)$ (for the yz plane), $\cos^{-1}(D/|b|)$ (for the xz plane), and $\cos^{-1}(D/|c|)$ (for the xy plane), where the inverse cosine value is taken in the interval $(0,\pi/2)$. **8**. The faces ABC, ACD, and ABD make the angles with the face BCD, $\cos^{-1}(\frac{13}{\sqrt{5}\sqrt{38}})$, $\cos^{-1}(\frac{23}{2\sqrt{7}\sqrt{19}})$, and $\cos^{-1}(\frac{7}{\sqrt{5}\sqrt{38}})$, respectively. **9**. $2x+y-2z=-8$ and $2x+y-2z=4$. **10**. $x+z=2$, the plane through the midpoint of the segment AB and perpendicular to it. **11**. $2x-y-z=-3$. **12**. P lies in the second plane (the distance is 0) and is at the distance $5/\sqrt{3}$ from the first plane. **13**. Parallel planes orthogonal to $\langle 1,1,1\rangle$. **14**. Planes intersecting the coordinate axes at $(1,0,0)$, $(0,1,0)$ and $(0,0,1/c)$; or all planes containing the line $x+y=1$ in the xy plane and intersecting the z axis. **15**. All planes through the point $(0,0,1)$ orthogonal to the vectors $\hat{\mathbf{e}}_3+\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector in the xy plane; or all planes obtained from the plane $y+z=1$ by rotations about the z axis. **16**. The squared distance between P and the plane is $(2+c)^2/(2+c^2)$; the critical points of this function, $c=-2$ and $c=1$, are zeros of its derivative; if $c=-2$, P lies in the plane (the distance is 0); for $c=1$, the distance takes the largest value of $\sqrt{3}$. **17**. $\mathbf{n}_1\cdot(\mathbf{n}_2\times\mathbf{n}_3)=0$ (the normals are coplanar). **19**. $4x+y-2z=3$. **21**. $2x+z=5$. **22**. $x+y+z=\pm 3R/2$. **23**. $x-y-z=-1$.

Section 7.6. **1**. $x=1-2t$, $y=2$, $z=3+t$; or $y=2$, $(1-x)/2=z-3$. **2**. The through the given vertex: $x=1$, $y=t$, $z=-1+t$; or $x=1$, $y=z+1$; the line containing the other diagonal: $x=t$, $y=2$, $z=t$; or $y=2$, $x=z$. **3**. $x=1+2t$, $y=t$, $z=-1+t$; or $x-1=2y=2(z+1)$. **4**. $x=2-2t$, $y=-1+t$, $z=-2+3t$; or $(x-2)/2=y+1=(z+2)/3$. **5**. Four such lines exist. Put $\mathbf{v}=\langle\cos\alpha,\cos\beta,\cos\gamma\rangle$ where α, β, and γ are the direction angles. If the line parallel to $\mathbf{v}$ makes the angle $\pi/3$ with the x axis, then $\cos\alpha=\pm1/2$ (similarly for $\cos\beta$). Since $-\mathbf{v}$ defines the same line, only 4 vectors out of 8 distinct sets of values of the triple $(\cos\alpha,\cos\beta,\cos\gamma)$ define distinct lines: $x=\pm t/2$, $y=t/2$, $z=\pm t/\sqrt{2}$, and $x=\pm t/2$, $y=\pm t/2$, $z=\pm t/\sqrt{2}$. **6**. $x=1-t$, $y=2+3t$, $z=3$; or $3(1-x)=y-2$, $z=3$. **7**. Take the center of the sphere as a particular point of the line; then $x=1+t$, $y=-2-2t$, $z=3$; or $2x=-y$, $z=3$. **8**. The lines have the same symmetric equations $x=3-y=(z+4)/3$ and, hence, coincide. **9**. Yes (they have the same symmetric equations). **10**. $x=1+t$, $y=2+t$, $z=3+2t$; the point of intersection is at $t=-4/3$. **11**. $x=3t$, $y=t$, $z=1-4t$; or $x/3=y=(1-z)/4$. **12**. The lines are intersecting at $(2,-1,1)$. **13**. The lines are skew. **14**. If P is the point of intersection, then $\overrightarrow{AP}=\langle\frac{3}{2},2,2\rangle$ and the angle of intersection is $\cos^{-1}(\frac{15}{\sqrt{41}\sqrt{29}})$. **15**. $\frac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. **17**. $\mathbf{r}=\mathbf{r}_1+(\mathbf{r}_2-\mathbf{r}_1)t$ where $0\leq t\leq 1$. **18**. $1/\sqrt{5}$. **21**. $x=1+2t$, $y=-t$, $z=3+2t$, where $-\frac{2}{3}\leq t\leq\frac{2}{3}$. **23**. $x=1+4t$, $y=2-7t$, $z=5-11t$. **24**. $1/\sqrt{6}$. **25**. $\|(\mathbf{r}_1-\mathbf{r}_0)\times\hat{\mathbf{u}}\|\leq R$. **26**. The projectile misses the target if $R=2$ and hits it if $R=3$.

Section 9.6. **1**. A double elliptic cone, the y axis is the axis of the cone. **2**. A "saddle" or hyperbolic paraboloid. **3**. An ellipsoid wit $a=1$, $b=\sqrt{2}$, $c=2$. **4**. A hyperboloid of two sheets, the y axis is the axis of the hyperboloid. **5**. An elliptic cylinder, the x axis is the axis of the cylinder. **6**. A hyperboloid of one sheet, the y axis is the axis of the hyperboloid. **7**. A hyperboloid of two sheets, the z axis is the axis of the hyperboloid. **8**. A parabolic cylinder parallel to the y axis. **9**. An elliptic paraboloid (concave down), the z axis is the axis of the paraboloid. **10**. A hyperbolic cylinder parallel to the x axis. **11**. An ellipsoid with the center at $(1,-2,0)$ and $a=b=\sqrt{5}$, $c=\sqrt{5}/2$. **12**. A hyperboloid of one sheet with the center at $(-1,-1,-2)$ and the axis parallel to the y axis. **13**. An elliptic paraboloid concave downward and with the vertex at $(3,0,9)$. **14**. A hyperbolic cylinder with the axis through

$(0, -1, -2)$ and parallel to the x axis. **15**. A double cone with the vertex at $(1, 1, 0)$ and the axis parallel to the y axis. **16**. A hyperbolic cylinder. **18**. A "Saddle" or hyperbolic paraboloid. **21**. The double cone $\frac{1}{4}y^2 = z^2 + x^2$. **20**. An elliptic double cone. Consider a rotation in the xz plane through an angle ϕ so that $\sin(2\phi) = -1/\sqrt{2}$ and $\cos(2\phi) = 1/\sqrt{2}$. **24**. The paraboloid $(x-1)^2 + (y-1)^2 = -2z + 3$ (concave down, the vertex at $(1, 1, \frac{3}{2})$). **25**. A part of the ball $x^2 + y^2 + (z-1)^2 \leq 1$ that lies inside the cone whose vertex $(0, 0, 0)$ is a point of the surface of the ball and whose axis (the z axis) contains a diameter of the ball. **28**. The double circular cone $x^2 + z^2 = 4y^2$ about the y axis. **30**. $\|\mathbf{r} \times \mathbf{v}\|^2 = R^2\|\mathbf{v}\|^2$ where $\mathbf{r} = \langle x, y, z\rangle$. **31**. Multiply the first equation by 2 and subtract the result from the second equation to show that all points of intersection satisfy $5x - 3y + 4z = 2$ and, hence, lie in the plane.