CHAPTER 2

Vector Functions

10. Curves in Space and Vector Functions

To describe the motion of a pointlike object in space, its position vector must be specified at every moment of time. A vector is defined by three components in a coordinate system. Therefore, the motion of the object can be described by an ordered triple of real-valued functions of time. This observation leads to the concept of vector-valued functions of a real variable.

DEFINITION **10.1**. (Vector Function).
Let $\mathcal{D}$ be a set of real numbers. A vector function $\mathbf{r}(t)$ of a real variable t is a rule that assigns a unique vector to every value of t from $\mathcal{D}$. The set $\mathcal{D}$ is called the domain *of the vector function.*

Most commonly used rules to define a vector function are algebraic rules that specify components of a vector function in a coordinate system as functions of a real variable: $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. For example,

$$\mathbf{r}(t) = \langle \sqrt{1-t}\,,\ \ln(t)\,,\ t^2\rangle \quad \text{or}$$
$$x(t) = \sqrt{1-t}\,, \quad y(t) = \ln(t)\,, \quad z(t) = t^2\,.$$

Unless specified otherwise, the domain of the vector function is the set $\mathcal{D}$ of all values of t at which the algebraic rule makes sense; that is, all three components can be computed for any t from $\mathcal{D}$. In the above example,

$$\begin{aligned} x(t) = \sqrt{1-t} \quad &\Rightarrow \quad -\infty < t \le 1\,,\\ y(t) = \ln(t) \quad &\Rightarrow \quad 0 < t < \infty\,,\\ z(t) = t^2 \quad &\Rightarrow \quad -\infty < t < \infty\,. \end{aligned}$$

The domain of the vector function is *the intersection of the domains of its components*:

$$\mathcal{D} = (-\infty, 1] \cap (0, \infty) \cap (-\infty, \infty) = (0, 1]\,.$$

The *range* of a vector function is the collection of all vectors $\mathbf{r}(t)$. It can be visualized by viewing each $\mathbf{r}(t)$ as a position vector, that is, all vectors from the range are depicted so that they have a common initial point (the origin). The collection of all terminal points of position vectors defined by a vector function $\mathbf{r}(t)$ is called a *graph* of the vector function.

Suppose that the components of a vector function $\mathbf{r}(t)$ are continuous functions on an interval $\mathcal{D} = I = [a, b]$. Consider all vectors $\mathbf{r}(t)$, as t ranges over I, positioned so that their initial points are at a fixed point (e.g., the origin of a coordinate system). Then the terminal points of the vectors $\mathbf{r}(t)$

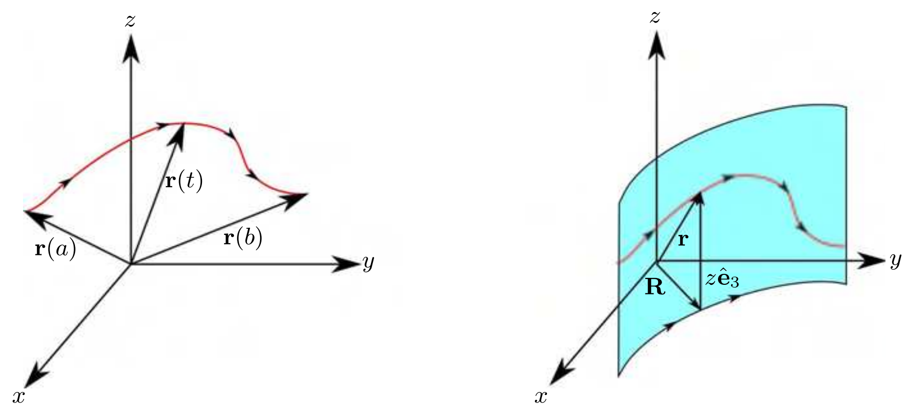

FIGURE **10.1**. **Left**: The terminal point of a vector $\mathbf{r}(t)$ whose components are continuous functions of t traces out a curve in space. **Right**: Graphing a space curve. Draw a curve in the xy plane defined by the parametric equations $x = x(t)$, $y = y(t)$. It is a "shadow" of the original curve in the xy plane and traced out by the vector $\mathbf{R}(t) = \langle x(t), y(t), 0\rangle$. This planar curve defines a cylindrical surface in space. The space curve in question lies in this surface. The space curve is obtained by raising or lowering the points of the planar curve along the surface by the amount $z(t)$, that is, $\mathbf{r}(t) = \mathbf{R}(t) + \hat{\mathbf{e}}_3 z(t)$. In other words, the graph $z = z(t)$ is wrapped around the cylindrical surface.

form a *curve* in space as depicted in Figure **10.1** (left panel). The simplest example is a straight line, which is described by a linear vector function $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. Thus, the graph of a vector function whose components are continuous functions is a curve in space.

10.1. Graphing Space Curves. To visualize the shape of a curve C traced out by a vector function, it is convenient to think about $\mathbf{r}(t)$ as a trajectory of motion. The position of a particle in space may be determined by its position in a plane and its height relative to that plane. For example, this plane can be chosen to be the xy plane. Then

$$\begin{aligned}\mathbf{r}(t) &= \langle x(t),\ y(t),\ z(t)\rangle \\ &= \langle x(t),\ y(t),\ 0\rangle + \langle 0,\ 0,\ z(t)\rangle \\ &= \mathbf{R}(t) + z(t)\hat{\mathbf{e}}_3.\end{aligned}$$

Consider the curve defined by the parametric equations $x = x(t)$, $y = y(t)$ in the xy plane. It is a "shadow" made by the original curve in the xy plane (if the light comes parallel to the z axis). One can mark a few points along the shadow curve corresponding to particular values of t, say, P_n with

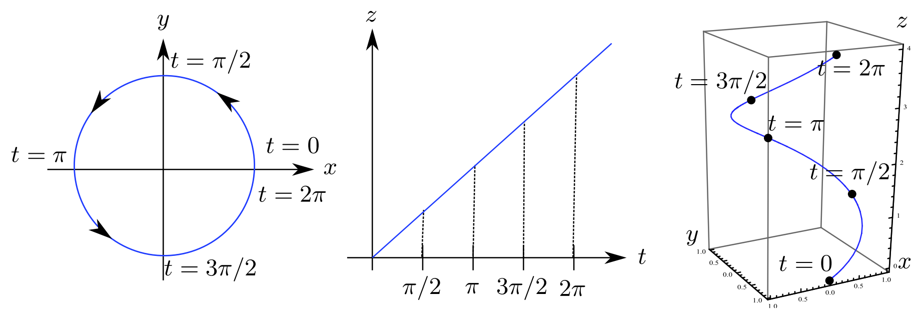

FIGURE **10.2**. Graphing a helix. **Left**: The shadow curve $\mathbf{R}(t) = \langle \cos t, \sin t, 0\rangle$ is a circle of unit radius, traced out counterclockwise. So the helix lies on the cylinder of unit radius whose symmetry axis is the z axis. **Middle**: The graph $z = z(t) = t$ is a straight line that defines the height of helix points relative to the circle traced out by $\mathbf{R}(t)$. **Right**: The graph of the helix $\mathbf{r}(t) = \mathbf{R}(t)+z(t)\hat{\mathbf{e}}_3$. As $\mathbf{R}(t)$ traverses the circle, the height $z(t) = t$ rises linearly. So the helix can be viewed as a straight line wrapped around the cylinder.

coordinates $(x(t_n), y(t_n))$, $n = 1, 2, ..., N$. Then the corresponding points of the curve C are obtained from them by moving the points P_n along the direction normal to the plane (i.e., along the z axis in this case), by the amount $z(t_n)$; that is, P_n goes up if $z(t_n) > 0$ or down if $z(t_n) < 0$. In other words, as a particle moves along the curve $x = x(t)$, $y = y(t)$, it ascends or descends according to the corresponding value of $z(t)$.

The curve can also be visualized by using a piece of paper. Consider a general cylinder with the horizontal trace being the curve $x = x(t)$, $y = y(t)$, like a wall of the shape defined by this curve. Then make a graph of the function $z(t)$ on a piece of paper (wallpaper) and glue it to the wall so that the t axis of the graph is glued to the curve $x = x(t)$, $y = y(t)$ while each point t on the t axis coincides with the corresponding point $(x(t), y(t))$ of the curve. After such a procedure, the graph of $z(t)$ along the wall would coincide with the curve C traced out by $\mathbf{r}(t)$. The procedure is illustrated in Figure **10.1** (right panel).

EXAMPLE **10.1**. *Graph the vector function* $\mathbf{r} = \langle \cos t, \sin t, t\rangle$, *where* t *ranges over the real line.*

SOLUTION: It is convenient to represent $\mathbf{r}(t)$ as the sum of a vector in the xy plane and a vector parallel to the z axis. In the xy plane, the curve

$$x = \cos t\,, \quad y = \sin t \quad \Leftrightarrow \quad x^2 + y^2 = 1$$

is the circle of unit radius traced out counterclockwise so that the point $(1, 0, 0)$ corresponds to $t = 0$. The circular motion is periodic with period 2π.

The height $z(t) = t$ rises linearly as the point moves along the circle. Starting from $(1, 0, 0)$, the curve makes one turn on the surface of the cylinder of unit radius climbing up by 2π. Think of a piece of paper with a straight line depicted on it that is wrapped around the cylinder. Thus, the curve traced by $\mathbf{r}(t)$ lies on the surface of a cylinder of unit radius and periodically winds about it climbing by 2π per turn. Such a curve is called a *helix*. The procedure is shown in Figure **10.2**. □

10.2. Limits and Continuity of Vector Functions.

Definition **10.2**. (Limit of a Vector Function).
A vector $\mathbf{r}_0$ *is called the* limit *of a vector function* $\mathbf{r}(t)$ *as* $t \to t_0$ *if*

$$\lim_{t\to t_0} \|\mathbf{r}(t) - \mathbf{r}_0\| = 0\,;$$

the limit is denoted as $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}_0$.

The left and right limits, $\lim_{t\to t_0^-} \mathbf{r}(t)$ and $\lim_{t\to t_0^+} \mathbf{r}(t)$, are defined similarly. This definition says that the length or norm of the vector $\mathbf{r}(t) - \mathbf{r}_0$ approaches 0 as t tends to t_0. The norm of a vector vanishes if and only if the vector is the zero vector. Therefore, the following theorem holds.

Theorem **10.1**. (Limit of a Vector Function).
Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ *and let* $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$. *Then the limit of a vector function exists if and only if the limits of its components exist:*

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}_0 \quad \Longleftrightarrow \quad \lim_{t\to t_0} x(t) = x_0\,, \quad \lim_{t\to t_0} y(t) = y_0\,, \quad \lim_{t\to t_0} z(t) = z_0.$$

This theorem reduces the problem of finding the limit of a vector function to the problem of finding limits of three ordinary functions.

Example **10.2**. *Let* $\mathbf{r}(t) = \langle \sin(t)/t\,,\ t\ln t\,, (e^t - 1 - t)/t^2\rangle$. *Find the limit of* $\mathbf{r}(t)$ *as* $t \to 0^+$ *or show that it does not exist.*

Solution: Recall from Calculus I that by the definition of the derivative of $f(t) = \sin(t)$

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t} = \lim_{t\to 0} \frac{\sin t}{t} = \cos 0 = 1$$

because $(\sin t)' = \cos t$. Since the limit exists, the left and right limits exist and are equal to the limit. Thus, the limit of the first component is 1. The limits of the other components may be investigated by l'Hospital's rule. Let us verify the hypotheses of l'Hospital's rule. One has $t\ln(t) = \ln(t)/t^{-1}$ so that the second component is a *ratio* of two *differentiable* functions $\ln(t)$ and t^{-1} that approach *infinity* as $t \to 0^+$ (the undetermined form $\frac{\infty}{\infty}$). The third component is also a *ratio* of two *differentiable* functions that approach *zero* as $t \to 0^+$ (the undetermined form $\frac{0}{0}$). Then *by l'Hospital's rule for either of these undetermined forms, if the limit of the ratio of the derivatives exists, then the original limit exists and is equal to the limit of the ratio of* *the derivatives*. The existence of the limit of the ratio of the derivatives is verified by the direct calculation:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-1}} = \lim_{t\to 0^+} \frac{(\ln t)'}{(t^{-1})'} = \lim_{t\to 0^+} \frac{t^{-1}}{-t^{-2}} = -\lim_{t\to 0^+} t = 0,$$

$$\lim_{t\to 0^+} \frac{e^t-1-t}{t^2} = \lim_{t\to 0^+} \frac{e^t-1}{2t} = \lim_{t\to 0^+} \frac{e^t}{2} = \frac{1}{2}\,.$$

where l'Hospital's rule has been used twice to calculate the last limit. Thus, $\lim_{t\to 0^+} \mathbf{r}(t) = \langle 1, 0, \frac{1}{2}\rangle$. □

Remark. Although l'Hospital's rule is a powerful tool to resolve undetermined forms, the use of basic properties of elementary functions and their approximations by Taylor polynomials is a better (and often technically simpler) method to analyze limits. Recall from Calculus II

$$\begin{aligned}
e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = 1 + t + \frac{t^2}{2} + O(t^3)\,,\\
\sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = t - \frac{t^3}{6} + O(t^5)\,,\\
\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = 1 - \frac{t^2}{2} + O(t^4)\,,\\
(1+t)^p &= 1 + pt + \frac{p(p-1)}{2!}\,t^2 + \frac{p(p-1)(p-2)}{3!}\,t^3 + \cdots\\
&= 1 + pt + \frac{p(p-1)}{2}\,t^2 + O(t^3)\,,\\
\ln(1+t) &= t - \frac{1!}{2!}\,t^2 + \frac{2!}{3!}t^3 - \frac{3!}{4!}\,t^4 + \cdots = t - \frac{t^2}{2} + O(t^3)\,,
\end{aligned}$$

where the symbol $O(t^n)$ denotes terms of order at least t^n, that is, $O(t^n) = Mt^n +$ possibly terms of higher order, for some constant $M \neq 0$. In other words, the characteristic property of $O(t^n)$ (the property that defines the symbol $O(t^n)$) is

$$\lim_{t\to 0} \frac{O(t^n)}{t^n} = M \neq 0\,.$$

Taylor polynomial approximations of degree n are obtained from the corresponding Taylor series by omitting terms $O(t^{n+1})$ (which is justified when the argument t is small) and used to analyze limits and local behavior of functions near $t = 0$. For example, the limit of the third component can be found as

$$\begin{aligned}
\frac{e^t-1-t}{t^2} &= \frac{(1+t+t^2/2+O(t^3))-1-t}{t^2}\\
&= \frac{t^2/2+O(t^3)}{t^2}\\
&= \frac{1}{2} + O(t) \to \frac{1}{2}
\end{aligned}$$

as $t \to 0$ because $O(t) \to 0$. This method becomes superior to l'Hospital's rule when functions in questions cannot be expressed in elementary functions (e.g. *special* functions) or such expressions are too complicated, while power expansions are known for most functions encountered in applications. The reader is encouraged to practice this method.

DEFINITION **10.3**. (Continuity of a Vector Function).
A vector function $\mathbf{r}(t)$, $t \in [a, b]$, *is said to be continuous at* $t = t_0 \in [a, b]$ *if*

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)\,.$$

A vector function $\mathbf{r}(t)$ *is continuous in the interval* $[a, b]$ *if it is continuous at every point of* $[a, b]$.

By Theorem **10.1**, *a vector function is continuous if and only if all its components are continuous functions.*

EXAMPLE **10.3**. *Let* $\mathbf{r}(t) = \langle \sin\,(2t)/t\,, t^2, e^t\rangle$ *for all* $t \neq 0$ *and* $\mathbf{r}(0) = \langle 1, 0, 1\rangle$. *Determine whether this vector function is continuous.*

SOLUTION: The components $y(t) = t^2$ and $z(t) = e^t$ are continuous for all real t and $y(0) = 0$ and $z(0) = 1$. The component $x(t) = \sin(2t)/t$ is continuous for all $t \neq 0$ because the ratio of two continuous functions is continuous. Using the Taylor approximation

$$x(t) = \frac{\sin(2t)}{t} = \frac{2t + O(t^3)}{t} = 2 + O(t^2) \quad \Rightarrow \quad \lim_{t \to 0} x(t) = 2 \neq x(0) = 1;$$

that is, $x(t)$ is not continuous at $t = 0$. Thus, $\mathbf{r}(t)$ is continuous everywhere, but $t = 0$. □

10.3. Space Curves and Continuous Vector Functions. A curve connecting two points in space as a point set can be obtained as a continuous transformation (or a deformation without breaking) of a straight line segment in space. Conversely, every such space curve can be continuously deformed to a straight line segment. So *a curve connecting two points in space is a continuous deformation of a straight line segment, and this deformation has a continuous inverse.*

A straight line segment can be viewed as an interval $a \leq t \leq b$ (a set of real numbers between a and b). Its continuous deformation can be described by a continuous vector functions $\mathbf{r}(t)$ on $[a, b]$. So *the range of a continuous vector function defines a curve in space.* Conversely, given a curve C as a point set in space, one might ask the question: What is a vector function that traces out a given curve in space? The answer to this question is not unique. For example, a line $\mathcal{L}$ as a point set in space is uniquely defined by its particular point and a vector $\mathbf{v}$ parallel to it. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are position vectors of two particular points of $\mathcal{L}$, then both vector functions $\mathbf{r}_1(t) = \mathbf{r}_1 + t\mathbf{v}$ and $\mathbf{r}_2(t) = \mathbf{r}_2 - 2t\mathbf{v}$ trace out the same line $\mathcal{L}$ because the vectors $-2\mathbf{v}$ and $\mathbf{v}$ are parallel.

The following, more sophisticated example is also of interest. Suppose one wants to find a vector function that traces out a semicircle of radius R. Let the semicircle be positioned in the upper part of the xy plane: $x^2 + y^2 = R^2$ and $y \geq 0$. The following three vector functions trace out the semicircle:

$$\begin{aligned} \mathbf{r}_1(t) &= \langle t,\ \sqrt{R^2 - t^2},\ 0\rangle\,, \quad -R \leq t \leq R\,, \\ \mathbf{r}_2(t) &= \langle R\cos t,\ R\sin t,\ 0\rangle\,, \quad 0 \leq t \leq \pi\,, \\ \mathbf{r}_3(t) &= \langle -R\cos t,\ R\sin t,\ 0\rangle\,, \quad 0 \leq t \leq \pi\,. \end{aligned}$$

This is easy to see by noting that the y components are non-negative in the specified intervals and the norm of these vector functions is constant for any value of t:

$$\|\mathbf{r}_i(t)\|^2 = x_i^2(t) + y_i^2(t) = R^2\,, \quad i = 1, 2, 3\,;$$

here $\mathbf{r}_i = \langle x_i, y_i, z_i\rangle$. The latter means that the endpoints of the vectors $\mathbf{r}_i(t)$ always remain on the circle of radius R. It can therefore be concluded that *there are many vector functions whose ranges define the same curve in space.*

Another observation is that there are vector functions that trace out the same curve in opposite directions as t increases from its smallest value a to its largest value b. In the above example, the vector function $\mathbf{r}_2(t)$ traces out the semicircle counterclockwise, while the functions $\mathbf{r}_1(t)$ and $\mathbf{r}_3(t)$ do so clockwise. So a vector function defines the *orientation* of a curve. However, this notion of the orientation of a curve should be regarded with caution because a vector function may traverse its range (or a part of it) several times. For example, the vector function

$$\mathbf{r}(t) = \langle R\cos t, R|\sin t|, 0\rangle$$

traces out the semicircle twice, back and forth, when t ranges from 0 to 2π. The vector function

$$\mathbf{r}(t) = (t^2, t^2, t^2)$$

is continuous on the interval $[-1, 1]$ and traces out the straight line segment, $x = y = z$, between the points $(0, 0, 0)$ and $(1, 1, 1)$ twice.

To emphasize the noted differences between space curves as point sets and continuous vector functions, the notion of a *parametric* curve is introduced.

DEFINITION **10.4**. (Parametric Curve)
A continuous vector function on an interval is called a parametric curve in space.

If the range of a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$ coincides with a given curve C as a point set in space, then the vector function is also called a *parameterization* of the curve C, the equations

$$C: \quad x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)\,, \quad a \leq t \leq b\,,$$

are called *parametric equations* of C, and t is called a *parameter*. As noted, a parameterization of a given space curve is not unique and there are different parametric equations that describe the very same space curve. A parametric curve $\mathbf{r}(t)$ is *closed* if

$$\mathbf{r}(a) = \mathbf{r}(b) \quad \Leftrightarrow \quad C \text{ is closed}$$

(the initial and terminal points of a curve traversed by $\mathbf{r}(t)$ coincide). For example, spatial curves that resemble the letter O or the infinity sign ∞ are closed. A curve in space is said to be *simple* if, loosely speaking, it does not intersect itself. To make this notion precise, it is rephrased in terms of parametric curves. A non-closed parametric curve $\mathbf{r}(t)$ is called *simple* if

$$\mathbf{r}(t_1) \neq \mathbf{r}(t_2) \quad \text{for any} \quad t_1 \neq t_2 \quad \Leftrightarrow \quad C \text{ is simple}.$$

For example, curves resembling the letters M and N are simple, while a curve that resembles the numeral 6 or the Greek letter α is not simple. A *closed* parametric curve $\mathbf{r}(t)$ is *simple* if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for any $t_1 \neq t_2$, except the case when $t_1 = a$ and $t_2 = b$. A point set C is a *simple curve* if there is a simple parametric curve whose range is C. A simple parametric curve whose range is a space curve C is also called a *simple parameterization* of the curve C. For example, a curve resembling the letter O is a simple closed curve, whereas a closed curve resembling the infinity sign ∞ is not simple. A simple parametric curve is always oriented. A vector function can also be defined on an infinite interval, e.g., $[a, \infty)$, or $(-\infty, b]$, or $(-\infty, \infty)$. In this case, a parametric curve is said to be simple if it is simple on any subinterval of the infinite interval.

Example **10.4**. *Find linear vector functions that orient the straight line segment between* $\mathbf{r}_1 = \langle 1, 2, 3\rangle$ *and* $\mathbf{r}_2 = \langle 2, 0, 1\rangle$ *from* $\mathbf{r}_1$ *to* $\mathbf{r}_2$ *and from* $\mathbf{r}_2$ *to* $\mathbf{r}_1$.

Solution: The vector $\mathbf{r}_2 - \mathbf{r}_1 = \langle 1, -2, -2\rangle$ is parallel to the line segment. So the vector equation $\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1)$ describes the line that contains the segment in question. The vector $\mathbf{r}_2 - \mathbf{r}_1$ is directed from $\mathbf{r}_1$ to $\mathbf{r}_2$. Therefore when t increases from $t = 0$, the terminal point of $\mathbf{r}(t)$ goes along the line from $\mathbf{r}_1$ toward $\mathbf{r}_2$, reaching the latter at $t = 1$. Thus, the segment is traversed from $\mathbf{r}_1$ to $\mathbf{r}_2$ by the vector function

$$\mathbf{r}(t) = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1) = \langle 1 + t,\ 2 - 2t,\ 3 - 2t\rangle, \quad 0 \le t \le 1.$$

Swapping the points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the above argument, it is concluded that the vector function

$$\mathbf{r}(t) = \mathbf{r}_2 + t(\mathbf{r}_1 - \mathbf{r}_2) = \langle 2 - t,\ 2t,\ 1 + 2t\rangle, \quad 0 \le t \le 1,$$

traverses the segment from $\mathbf{r}_2$ to $\mathbf{r}_1$. □

Example **10.5**. *Determine whether the parametric curve*

$$\mathbf{r}(t) = \langle \cos t\,,\ \sin(2t)\,,\ \sin^2(2t)\rangle\,, \quad 0 \le t \le 2\pi\,,$$

is closed or not and simple or not. Describe its shape and indicate the orientation, if any.

SOLUTION: Let us first investigate the vertical projection of the curve:

$$x = \cos t\,, \quad y = \sin(2t)\,.$$

From the graphs of $\cos t$ and $\sin(2t)$, it is clear that this planar curve originates from the point $(1, 0)$, goes into the first quadrant for $0 \le t \le \pi/2$, then crosses the origin at $t = \pi/2$. For $\pi/2 \le t \le \pi$, the curve lies in the third quadrant and crosses the x axis at $t = \pi$. As t increases from π to $3\pi/2$, the curve is in the second quadrant and passes through the origin again at $t = 3\pi/2$. Finally, for $3\pi/2 \le t \le 2\pi$, the curve is in the fourth quadrant and returns to the initial point at $t = 2\pi$. It resembles the infinity sign ∞. Since $\mathbf{r}(0) = \mathbf{r}(2\pi) = \langle 1, 0, 0\rangle$, the curve is closed. The curve is not simple because it intersects itself, $\mathbf{r}(\pi/2) = \mathbf{r}(3\pi/2) = \mathbf{0}$.

Now imagine a cylinder along the z axis with the cross section of the shape of the infinity sign ∞. To visualize the curve, wrap the graph $z = z(t) = \sin^2(2t)$ around this cylinder so that the zeros of the height $z(t)$ at $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$ match respectively the points $(1, 0)$, $(0, 0)$, $(-1, 0)$, $(0, 0)$, and $(1, 0)$ on the cross section of the cylinder by the xy plane. Clearly, the curve is oriented (the loop with $x \ge 0$ is traversed counterclockwise as viewed from the top of the z axis, while the loop with $x \le 0$ is traversed clockwise). □

10.4. Study Problems.

Problem **10.1**. *Find a vector function that traces out a helix of radius R that climbs up along its axis by h per one turn. Is such a helix unique?*

SOLUTION: Let the helix axis be the z axis. By making the mechanical analogy with the motion of a particle along the helix in question, the motion in the xy plane must be circular with radius R. Suitable parametric equations of the circle are

$$x^2 + y^2 = R^2 \quad \Rightarrow \quad x(t) = R\cos t\,, \quad y(t) = R\sin t\,.$$

With this parameterization of the circle, the motion has a period of 2π. On the other hand, $z(t)$ must rise linearly by h as t changes over the period. Therefore, $z(t) = ht/(2\pi)$. The vector function may be chosen in the form

$$\mathbf{r}(t) = \langle R\cos t,\ R\sin t,\ ht/(2\pi)\rangle\,.$$

Alternatively, one can take parametric equations of the circle in the form

$$x^2 + y^2 = R^2 \quad \Rightarrow \quad x(t) = R\cos t\,, \quad y(t) = -R\sin t\,.$$

In the latter parameterization, the circle is traced out clockwise, whereas it is traced out counterclockwise in the former parameterization. Consequently, the vector function

$$\mathbf{r}(t) = \langle R\cos t,\ -R\sin t,\ ht/(2\pi)\rangle$$

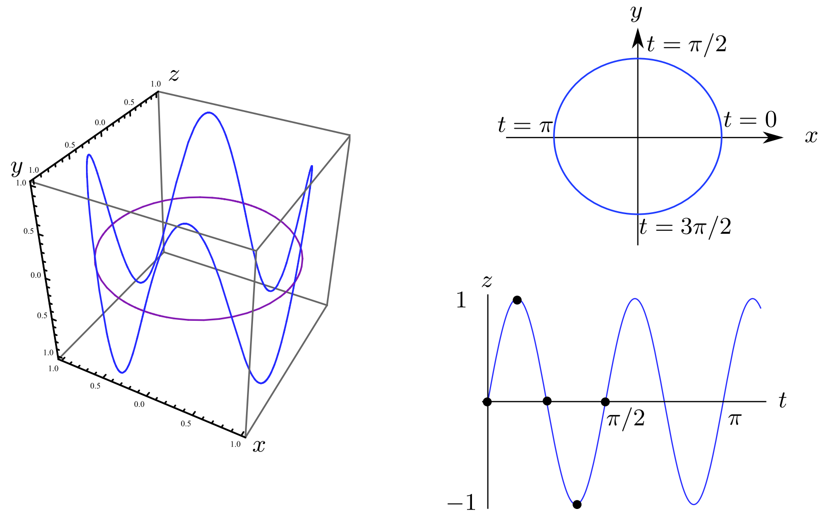

FIGURE **10.3**. Illustration to Study Problem **10.2**. **Left**: The curve lies on the cylinder of unit radius. It may be viewed as the graph of $z = \sin(4t)$ on the interval $0 \leq t \leq 2\pi$ wrapped around the cylinder. **Top right**: The circle (shadow) traced out by $\mathbf{R}(t) = \langle \cos t, \sin t, 0\rangle$. It defines the cylindrical surface on which the curve lies. **Bottom right**: The graph $z = z(t) = \sin(4t)$, which defines the height of points of the curve relative to the circle in the xy plane. The dots on the t axis indicate the points of intersection of the curve with the xy plane in the first quadrant.

also traces out a helix with required properties. The two helices are different despite that they share the same initial and terminal points. One helix winds about the z axis clockwise while the other counterclockwise. □

Problem **10.2**. *Sketch and/or describe the curve traced out by the vector function* $\mathbf{r}(t) = \langle \cos t, \sin t, \sin(4t)\rangle$ *if* t *ranges in the interval* $[0, 2\pi]$.

SOLUTION: The vector function $\mathbf{R}(t) = \langle \cos t, \sin t, 0\rangle$ traverses the circle of unit radius in the xy plane, counterclockwise, starting from the point $(1, 0, 0)$. As t ranges over the specified interval, the circle is traversed only once. The height $z(t) = \sin(4t)$ has a period of $2\pi/4 = \pi/2$. Therefore, the graph of $\sin(4t)$ makes four ups and four downs if $0 \leq t \leq 2\pi$. The curve $\mathbf{r}(t) = \mathbf{R}(t) + \hat{\mathbf{e}}_3 z(t)$ looks like the graph of $\sin(4t)$ wrapped around the cylinder of unit radius. It makes one up and one down in each quarter of the cylinder. The procedure is shown in Figure **10.3**. □

Problem **10.3**. *Sketch and/or describe the curve traced out by the vector function* $\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$.

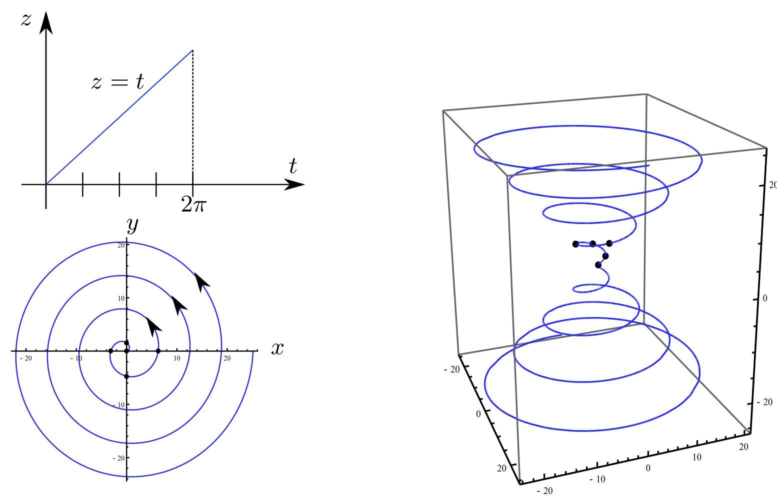

FIGURE **10.4**. Illustration to Study Problem **10.3**. **Left**: The height of the graph relative to the xy plane (top). The shadow curve $\mathbf{R}(t) = \langle t\cos t, t\sin t, 0\rangle$. For $t \geq 0$, it looks like an unwinding spiral (bottom). **Right**: For $t > 0$, the curve is traversed by the point moving along the spiral while rising linearly above the xy plane with the distance traveled along the spiral. It can be viewed as a straight line wrapped around the cone $x^2 + y^2 = z^2$.

SOLUTION: The components of $\mathbf{r}(t)$ satisfy the equation

$$x^2(t) + y^2(t) = t^2 \quad \Rightarrow \quad x^2(t) + y^2(t) = z^2(t)$$

for all values of t. Therefore, the curve lies on the double cone $x^2 + y^2 = z^2$. The shape of the planar parametric curve $x = x(t)$, $y = y(t)$ is easy to understand in polar coordinates:

$$x = r\cos\varphi\,,\ y = r\sin\varphi \quad \Rightarrow \quad r = t\,,\ \varphi = t$$

The curve represents a rotational motion (the angle φ increases linearly with time t) about the origin such that the distance from the origin r increases linearly with the angle of rotation. This is a *spiral* in the xy plane.

To obtain the spatial curve in question, each point of the spiral must be moved up or down in accord with the corresponding value of the height $z(t) = t$. If t increases from $t = 0$, the curve in question is traced by a point that rises linearly with the distance from the origin as it travels along the spiral. If t decreases from $t = 0$, instead of rising, the point would descend ($z(t) = t < 0$). So the curve winds about the axis of the double cone while remaining on its surface. The procedure is shown in Figure **10.4**. □

Problem **10.4**. *Find the portion of the elliptic helix* $\mathbf{r}(t) = \langle 2\cos(\pi t), t, \sin(\pi t)\rangle$ *that lies inside the ellipsoid* $x^2 + y^2 + 4z^2 = 13$.

Solution: The helix here is called *elliptic* because it lies on the surface of an elliptic cylinder. Indeed, in the xz plane, the parametric curve $x = 2\cos(\pi t)$, $z = \sin(\pi t)$ traverses the ellipse $x^2/4 + z^2 = 1$ because the latter equation is satisfied for all real t:

$$\frac{x^2(t)}{4} + z^2(t) = \cos^2(\pi t) + \sin^2(\pi t) = 1\,.$$

Therefore the curve remains on the surface of the elliptic cylinder parallel to the y axis. One turn around the ellipse occurs as t changes from 0 to 2 because the functions $\cos(\pi t)$ and $\sin(\pi t)$ have the period $2\pi/\pi = 2$. The helix rises by 2 along the y axis per turn because $y(t) = t$. Now, to solve the problem, one has to find the values of t at which the helix intersects the ellipsoid. The intersection happens when the components of $\mathbf{r}(t)$ satisfy the equation of the ellipsoid, that is, when

$$x^2(t) + y^2(t) + 4z^2(t) = 1 \quad \Rightarrow \quad 4 + t^2 = 13 \quad \Rightarrow \quad t = \pm 3\,.$$

The position vectors of the points of intersection are $\mathbf{r}(\pm 3) = \langle -2, \pm 3, 0\rangle$. The portion of the helix that lies inside the ellipsoid corresponds to the range $-3 \le t \le 3$. □

Problem **10.5**. *Consider two curves C_1 and C_2 traced out by the vector functions $\mathbf{r}_1(t) = \langle t^2, t, t^2 + 2t - 8\rangle$ and $\mathbf{r}_2(s) = \langle 8 - 4s, 2s, s^2 + s - 2\rangle$, respectively. Do the curves intersect? If so, find the points of intersection. Suppose two particles have the trajectories $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, where t is time. Do the particles collide?*

Solution: The curves intersect if there are values of the pair (t, s) such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. This vector equation is equivalent the system of three equations

$$\mathbf{r}_1(t) = \mathbf{r}_2(s) \;\Leftrightarrow\; \begin{cases} x_1(t) = x_2(s) \\ y_1(t) = y_2(s) \\ z_1(t) = z_2(s) \end{cases} \quad\Rightarrow\quad \begin{cases} t^2 = 8 - 4s \\ t = 2s \\ t^2 + 2t - 8 = s^2 + s - 2 \end{cases}$$

Substituting the second equation $t = 2s$ into the first equation, one finds that $(2s)^2 = 8 - 4s$ whose solutions are $s = -2$ and $s = 1$. One has yet to verify that the third equation holds for the pairs $(t, s) = (-4, -2)$ and $(t, s) = (2, 1)$ (otherwise, the z components do not match). A simple calculation shows that indeed both pairs satisfy the equation. So the position vectors of the points of intersection are

$$\mathbf{r}_1(-4) = \mathbf{r}_2(-2) = \langle 16, -4, 0\rangle \quad \text{and} \quad \mathbf{r}_1(2) = \mathbf{r}_2(1) = \langle 4, 2, 0\rangle\,.$$

Although the curves along which the particles travel intersect, this does not mean that the particles would necessarily collide because they may not arrive at a point of intersection at the same moment of time, just like two cars traveling along intersecting streets may or may not collide at the street intersection. The collision condition is more restrictive:

$$\mathbf{r}_1(t) = \mathbf{r}_2(t)\,;$$

the time of collision t must satisfy three conditions (the particles happen to be at the same time and the same point). For the problem at hand, these conditions cannot be fulfilled for any t because, among all the solutions of $\mathbf{r}_1(t) = \mathbf{r}_2(s)$, there is no solution for which $t = s$. Thus, the particles do not collide. □

Problem **10.6**. *Find a vector function that traces out the curve of intersection of the paraboloid* $z = x^2 + y^2$ *and the plane* $2x + 2y + z = 2$ *counterclockwise as viewed from the top of the* z *axis.*

SOLUTION: One has to find the components $x(t)$, $y(t)$, and $z(t)$ such that they satisfy the equations of the paraboloid and plane simultaneously for all values of t. This ensures that the endpoint of the vector $\mathbf{r}(t)$ remains on both surfaces, that is, traces out their curve of intersection (see Figure **10.5**). Consider first the motion in the xy plane described by the vector function $\langle x(t), y(t), 0\rangle$. Solving the plane equation for z, $z = 2 - 2x - 2y$, and substituting the solution into the paraboloid equation, one finds

$$2 - 2x - 2y = x^2 + y^2 \quad \Leftrightarrow \quad 4 = (x+1)^2 + (y+1)^2$$

by completing the squares. This equation describes a circle of radius 2 centered at $(-1, -1)$. Its parametric equations may be chosen as

$$x = x(t) = -1 + 2\cos t\,, \quad y = y(t) = -1 + 2\sin t\,.$$

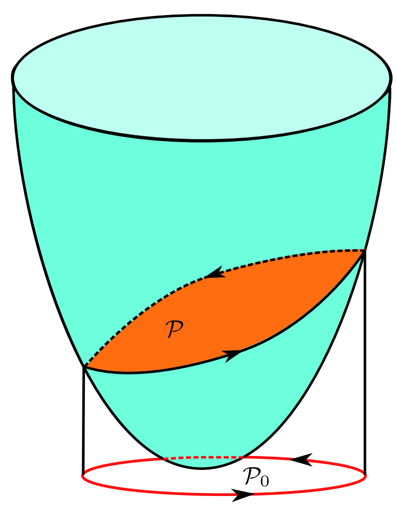

FIGURE **10.5**. Illustration to Study Problem **10.6**. The curve is an intersection of the paraboloid and the plane $\mathcal{P}$. It is traversed by the point simultaneously moving counterclockwise about the circle (shadow) in the xy plane (indicated by $\mathcal{P}_0$) and rising so that it remains on the paraboloid.

By construction, this circle is the vertical projection, $\langle x(t), y(t), z(t)\rangle \to \langle x(t), y(t), 0\rangle$, of the curve of intersection onto the xy plane (the plane $\mathcal{P}_0$ in Figure **10.5**). As t increases from 0 to 2π, the circle is traced out counterclockwise as required (the clockwise orientation can be obtained, e.g., by reversing the sign of $\sin t$). The height along the curve of intersection relative to the xy plane is $z(t) = 2 - 2x(t) - 2y(t)$. Thus,

$$\mathbf{r}(t) = \langle -1 + 2\cos t,\ -1 + 2\sin t,\ 6 - 4\cos t - 4\sin t\rangle\,,$$

where $t \in [0, 2\pi]$. □

Problem **10.7**. *Let* $\mathbf{v}(t) \to \mathbf{v}_0$ *and* $\mathbf{u}(t) \to \mathbf{u}_0$ *as* $t \to t_0$. *Prove the limit law for vector functions:* $\lim_{t\to t_0}(\mathbf{v}(t)\cdot\mathbf{u}(t)) = \mathbf{v}_0\cdot\mathbf{u}_0$ *using only Definition* **10.2**. *Then prove this law using Theorem* **10.1** *and basic limit laws for ordinary functions.*

SOLUTION: The idea is similar to the proof of the basic limit laws for ordinary functions given in Calculus I. One has to find an upper bound for $|\mathbf{v}\cdot\mathbf{u} - \mathbf{v}_0\cdot\mathbf{u}_0|$ in terms of $\|\mathbf{v} - \mathbf{v}_0\|$ and $\|\mathbf{u} - \mathbf{u}_0\|$. By Definition **10.2**, the latter quantities converge to zero as $t \to t_0$. The conclusion should follow from the squeeze principle. Consider the identities:

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{u} - \mathbf{v}_0\cdot\mathbf{u}_0 &= (\mathbf{v} - \mathbf{v}_0)\cdot\mathbf{u} + \mathbf{v}_0\cdot\mathbf{u} - \mathbf{v}_0\cdot\mathbf{u}_0\\
&= (\mathbf{v} - \mathbf{v}_0)\cdot\mathbf{u} + \mathbf{v}_0\cdot(\mathbf{u} - \mathbf{u}_0)\\
&= (\mathbf{v} - \mathbf{v}_0)\cdot(\mathbf{u} - \mathbf{u}_0) + (\mathbf{v} - \mathbf{v}_0)\cdot\mathbf{u}_0 + \mathbf{v}_0\cdot(\mathbf{u} - \mathbf{u}_0)
\end{aligned}$$

It follows from the inequality $0 \le |a+b| \le |a| + |b|$ and the Cauchy-Schwarz inequality (Theorem **3.2**) $|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|$ that

$$\begin{aligned}
0 &\le |\mathbf{v}\cdot\mathbf{u} - \mathbf{v}_0\cdot\mathbf{u}_0|\\
&\le |(\mathbf{v} - \mathbf{v}_0)\cdot(\mathbf{u} - \mathbf{u}_0)| + |(\mathbf{v} - \mathbf{v}_0)\cdot\mathbf{u}_0| + |\mathbf{v}_0\cdot(\mathbf{u} - \mathbf{u}_0)|\\
&\le \|\mathbf{v} - \mathbf{v}_0\|\|\mathbf{u} - \mathbf{u}_0\| + \|\mathbf{v} - \mathbf{v}_0\|\|\mathbf{u}_0\| + \|\mathbf{v}_0\|\|\mathbf{u} - \mathbf{u}_0\|
\end{aligned}$$

By Definition **10.2**, $\|\mathbf{v} - \mathbf{v}_0\| \to 0$ and $\|\mathbf{u} - \mathbf{u}_0\| \to 0$ as $t \to t_0$. So the right side of the above inequality converges to zero. By the squeeze principle, it is then concluded that $|\mathbf{v}\cdot\mathbf{u} - \mathbf{v}_0\cdot\mathbf{u}_0| \to 0$ as $t \to t_0$, which proves the assertion. A proof based on Theorem **10.1** is simpler. If $v_i(t)$ and $u_i(t)$, $i = 1, 2, 3$, are components of $\mathbf{v}(t)$ and $\mathbf{u}(t)$, respectively, then by Theorem **10.1** $v_i(t) \to v_{0i}$ and $u_i(t) \to u_{0i}$ as $t \to t_0$. Hence,

$$\begin{aligned}
\lim_{t\to t_0}\mathbf{v}(t)\cdot\mathbf{u}(t) &= \lim_{t\to t_0}\Big(v_1(t)u_1(t) + v_2(t)u_2(t) + v_3(t)u_3(t)\Big)\\
&= \lim_{t\to t_0} v_1(t)u_1(t) + \lim_{t\to t_0} v_2(t)u_2(t) + \lim_{t\to t_0} v_3(t)u_3(t)\\
&= v_{01}u_{01} + v_{02}u_{02} + v_{03}u_{03}\\
&= \mathbf{v}_0\cdot\mathbf{u}_0
\end{aligned}$$

where the basic limit laws for ordinary functions have been used. □

10.5. Exercises.

1–5. Find the domain of each of the following vector functions:

1. $\mathbf{r}(t) = \langle\ t,\ t^2,\ e^t\ \rangle$;
2. $\mathbf{r}(t) = \langle\ \sqrt{t},\ t^2,\ e^t\ \rangle$;
3. $\mathbf{r}(t) = \langle\ \sqrt{9-t^2},\ \ln t,\ \cos t\ \rangle$;
4. $\mathbf{r}(t) = \langle\ \ln(9-t^2),\ \ln|t|,\ (1+t)/(2+t)\ \rangle$;
5. $\mathbf{r}(t) = \langle\ \sqrt{t-1},\ \ln t,\ \sqrt{1-t}\ \rangle$.

6–15. Find each of the following limits or show that it does not exist:

6. $\lim_{t\to 1}\langle\ \sqrt{t},\ 2-t-t^2,\ 1/(t^2-2)\ \rangle$;
7. $\lim_{t\to 1}\langle\ \sqrt{t},\ 2-t-t^2,\ 1/(t^2-1)\ \rangle$;
8. $\lim_{t\to 0}\langle\ e^t,\ \sin t,\ t/(1-t)\ \rangle$;
9. $\lim_{t\to\infty}\langle\ e^{-t},\ 1/t^2,\ 4\ \rangle$;
10. $\lim_{t\to\infty}\langle\ e^{-t},\ (1-t^2)/t^2,\ \sqrt[3]{t}/(\sqrt{t}+t)\ \rangle$;
11. $\lim_{t\to-\infty}\langle\ 2,\ t^2,\ 1/\sqrt[3]{t}\ \rangle$;
12. $\lim_{t\to 0^+}\langle\ (e^{2t}-1)/t,\ (\sqrt{1+t}-1)/t,\ t\ln t\ \rangle$;
13. $\lim_{t\to 0}\langle\ \sin^2(2t)/t^2,\ t^2+2,\ (\cos t-1)/t^2\ \rangle$;
14. $\lim_{t\to 0}\langle\ (e^{2t}-1)/t,\ t\cot t,\ \sqrt{1+t}\ \rangle$;
15. $\lim_{t\to\infty}\langle\ e^{2t}/\cosh^2 t,\ t^{2012}e^{-t},\ e^{-2t}\sinh^2 t\ \rangle$.

16–22. Sketch each of the following curves and identify the direction in which the curve is traced out as the parameter t increases:

16. $\mathbf{r}(t) = \langle t, \cos(3t), \sin(3t)\rangle$;
17. $\mathbf{r}(t) = \langle 2\sin(5t), 4, 3\cos(5t)\rangle$;
18. $\mathbf{r}(t) = \langle 2t\sin t, 3t\cos t, t\rangle$;
19. $\mathbf{r}(t) = \langle \sin t, \cos t, \ln t\rangle$;
20. $\mathbf{r}(t) = \langle t,\ 1-t,\ (t-1)^2\rangle$;
21. $\mathbf{r}(t) = \langle t^2, t, \sin^2(\pi t)\rangle$;
22. $\mathbf{r}(t) = \langle \sin t,\ \sin t,\ \sqrt{2}\cos t\rangle$.

23. Two objects are said to collide if they are at the same position *at the same time*. Two trajectories are said to intersect if they have common points. Let t be the physical time. Let two objects travel along the space curves $\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{r}_2(t) = \langle 1+2t, 1+6t, 1+14t\rangle$. Do the objects collide? Do their trajectories intersect? If so, find the collision and intersection points.

24. Find a simple parameterization of the curve of intersection of the surfaces $x^2+y^2/4+z^2/9=1$, $y\ge 0$, and $z=x^2$. Sketch the curve.

25–32. Find two vector functions that traverse a given curve C in the opposite directions if C is the curve of intersection of two surfaces:

25. $y=x^2$ and $z=1$;
26. $x=\sin y$ and $z=x$;
27. $x^2+y^2=9$ and $z=xy$;
28. $x^2+y^2=z^2$ and $x+y+z=1$;
29. $z=x^2+y^2$ and $y=x^2$;
30. $x^2/4+y^2/9=1$ and $x+y+z=1$;

31. $x^2/2 + y^2/2 + z^2/9 = 1$ and $x - y = 0$;
32. $x^2 + y^2 - 2x = 0$ and $z = x^2 + y^2$.

33. Specify the parts of the curve $\mathbf{r}(t) = \langle \sin t,\ \cos t,\ 4\sin^2 t\rangle$ that lie above the plane $z = 1$ by restricting the range of the parameter t.

34. Find the values of the parameters a and b at which the curve $\mathbf{r}(t) = \langle 1 + at^2, b - t, t^3\rangle$ passes through the point $(1, 2, 8)$.

35–39. Find the values of a, b, and c, if any, for which each of the following vector functions is continuous: $\mathbf{r}(0) = \langle a, b, c\rangle$ and, for $t \neq 0$,

35. $\mathbf{r}(t) = \langle t,\ \cos^2 t,\ 1 + t + t^2\rangle$;
36. $\mathbf{r}(t) = \langle t,\ \cos^2 t,\ \sqrt{1 + t^2}\rangle$;
37. $\mathbf{r}(t) = \langle t,\ \cos^2 t,\ \ln|t|\rangle$;
38. $\mathbf{r}(t) = \langle \sin(2t)/t,\ \sinh(3t)/t,\ t\ln|t|\rangle$;
39. $\mathbf{r}(t) = \langle t\cot(2t),\ t^{1/3}\ln|t|,\ t^2 + 2\rangle$.

40. Suppose that the limits $\lim_{t\to a}\mathbf{v}(t)$ and $\lim_{t\to a}\mathbf{u}(t)$ exist. Prove the basic laws of limits for the following vector functions:

$$\begin{aligned}
\lim_{t\to a}(\mathbf{v}(t) + \mathbf{u}(t)) &= \lim_{t\to a}\mathbf{v}(t) + \lim_{t\to a}\mathbf{u}(t),\\
\lim_{t\to a}(s\mathbf{v}(t)) &= s\lim_{t\to a}\mathbf{v}(t),\\
\lim_{t\to a}(\mathbf{v}(t)\cdot\mathbf{u}(t)) &= \lim_{t\to a}\mathbf{v}(t)\cdot\lim_{t\to a}\mathbf{u}(t),\\
\lim_{t\to a}(\mathbf{v}(t)\times\mathbf{u}(t)) &= \lim_{t\to a}\mathbf{v}(t)\times\lim_{t\to a}\mathbf{u}(t).
\end{aligned}$$

41. Prove the last limit law in Exercise **40** directly from Definition **10.2**, i.e., without using Theorem **10.1**. Hint: see Study Problem **10.7**.

42–47. Let

$$\begin{aligned}
\mathbf{v}(t) &= \langle (e^{2t} - 1)/t,\ (\sqrt{1 + t} - 1)/t,\ t\ln|t|\rangle,\\
\mathbf{u}(t) &= \langle \sin^2(2t)/t^2,\ t^2 + 2,\ (\cos t - 1)/t^2\rangle,\\
\mathbf{w}(t) &= \langle t^{2/3},\ 2/(1 - t),\ 1 + t - t^2 + t^3\rangle.
\end{aligned}$$

Use the basic laws of limits established in Exercise **40** to find:

42. $\lim_{t\to 0}(2\mathbf{v}(t) - \mathbf{u}(t) + \mathbf{w}(t))$;
43. $\lim_{t\to 0}(\mathbf{v}(t)\cdot\mathbf{u}(t))$;
44. $\lim_{t\to 0}(\mathbf{v}(t)\times\mathbf{u}(t))$;
45. $\lim_{t\to 0}[\mathbf{w}(t)\cdot(\mathbf{v}(t)\times\mathbf{u}(t))]$;
46. $\lim_{t\to 0}[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t))]$;
47. $\lim_{t\to 0}[\mathbf{w}(t)\times(\mathbf{v}(t)\times\mathbf{u}(t)) + \mathbf{v}(t)\times(\mathbf{u}(t)\times\mathbf{w}(t)) + \mathbf{u}(t)\times(\mathbf{w}(t)\times\mathbf{v}(t))]$.

48. Suppose that the vector function $\mathbf{v}(t)\times\mathbf{u}(t)$ is continuous where $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are non-vanishing. Does this imply that both vector functions $\mathbf{v}(t)$ and $\mathbf{u}(t)$ are continuous? Support your arguments by examples.

49. Suppose that the vector functions $\mathbf{v}(t)\times\mathbf{u}(t)$ and $\mathbf{v}(t)\neq\mathbf{0}$ are continuous. Does this imply that the vector function $\mathbf{u}(t)$ is continuous? Support your arguments by examples.

11. Differentiation of Vector Functions

DEFINITION **11.1**. (Derivative of a Vector Function).
Suppose a vector function $\mathbf{r}(t)$ *is defined on an interval* $[a,b]$ *and* $t_0 \in [a,b]$. *If the limit*

$$\lim_{h\to 0} \frac{\mathbf{r}(t_0+h)-\mathbf{r}(t_0)}{h} = \mathbf{r}'(t_0) = \frac{d\mathbf{r}}{dt}(t_0)$$

exists, then it is called the derivative of a vector function $\mathbf{r}(t)$ *at* $t=t_0$, *and* $\mathbf{r}(t)$ *is said to be differentiable at* t_0. *For* $t_0=a$ *or* $t_0=b$, *the limit is understood as the right* ($h>0$) *or left* ($h<0$) *limit, respectively. If the derivative exists for all points in* $[a,b]$, *then the vector function* $\mathbf{r}(t)$ *is said to be differentiable on* $[a,b]$.

It follows from Theorem **10.1** that a vector function is differentiable if and only if all its components are differentiable:

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{h\to 0}\Big\langle \frac{x(t+h)-x(t)}{h},\ \frac{y(t+h)-y(t)}{h},\ \frac{z(t+h)-z(t)}{h}\Big\rangle \\ &= \langle x'(t)\,,\ y'(t)\,,\ z'(t)\rangle\,. \end{aligned} \tag{11.1}$$

For example,

$$\mathbf{r}(t) = \langle \sin(2t),\ t^2-t,\ e^{-3t}\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 2\cos(2t),\ 2t-1,\ -3e^{-3t}\rangle.$$

DEFINITION **11.2**. (Continuously Differentiable Vector Function).
If the derivative $\mathbf{r}'(t)$ *is a continuous vector function on an interval* $[a,b]$, *then the vector function* $\mathbf{r}(t)$ *is said to be continuously differentiable on* $[a,b]$.

Higher-order derivatives are defined similarly: the second derivative is the derivative of $\mathbf{r}'(t)$:

$$\mathbf{r}''(t) = (\mathbf{r}'(t))' = \langle x''(t)\,,\ y''(t)\,,\ z''(t)\rangle\,,$$

the third derivative is the derivative of $\mathbf{r}''(t)$:

$$\mathbf{r}'''(t) = (\mathbf{r}''(t))' = \langle x'''(t)\,,\ y'''(t)\,,\ z'''(t)\rangle\,,$$

and so on

$$\mathbf{r}^{(n)}(t) = (\mathbf{r}^{(n-1)}(t))' = \langle x^{(n)}(t)\,,\ y^{(n)}(t)\,,\ z^{(n)}(t)\rangle\,,$$

provided all components are differentiable sufficiently many times.

11.1. Differentiation Rules. The following rules of differentiation of vector functions can be deduced from the rule (**11.1**).

THEOREM **11.1**. (Differentiation Rules).
Suppose $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *are differentiable vector functions and* $f(t)$ *is a real-valued differentiable function. Then*

$$\begin{aligned}
\frac{d}{dt}\Big[\mathbf{v}(t)+\mathbf{u}(t)\Big] &= \mathbf{v}'(t)+\mathbf{u}'(t),\\
\frac{d}{dt}\Big[f(t)\mathbf{v}(t)\Big] &= f'(t)\mathbf{v}(t)+f(t)\mathbf{v}'(t),\\
\frac{d}{dt}\Big[\mathbf{v}(t)\cdot\mathbf{u}(t)\Big] &= \mathbf{v}'(t)\cdot\mathbf{u}(t)+\mathbf{v}(t)\cdot\mathbf{u}'(t),\\
\frac{d}{dt}\Big[\mathbf{v}(t)\times\mathbf{u}(t)\Big] &= \mathbf{v}'(t)\times\mathbf{u}(t)+\mathbf{v}(t)\times\mathbf{u}'(t).\\
\frac{d}{dt}\Big[\mathbf{v}(f(t))\Big] &= f'(t)\mathbf{v}'(f(t)).
\end{aligned}$$

The proof is based on a straightforward use of the rule given in Eq. (**11.1**) and basic rules of differentiation for ordinary functions and left as an exercise to the reader.

EXAMPLE **11.1**. *Find the first and second derivatives of the vector function* $\mathbf{r}(t) = (\mathbf{a}+t^2\mathbf{b})\times(\mathbf{c}-t\mathbf{d})$ *where* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{d}$ *are constant vectors.*

SOLUTION: By the product rule,

$$\begin{aligned}
\mathbf{r}'(t) &= (\mathbf{a}+t^2\mathbf{b})'\times(\mathbf{c}-t\mathbf{d})+(\mathbf{a}+t^2\mathbf{b})\times(\mathbf{c}-t\mathbf{d})'\\
&= 2t\mathbf{b}\times(\mathbf{c}-t\mathbf{d})-(\mathbf{a}+t^2\mathbf{b})\times\mathbf{d}\\
\mathbf{r}''(t) &= (2t\mathbf{b})'\times(\mathbf{c}-t\mathbf{d})+2t\mathbf{b}\times(\mathbf{c}-t\mathbf{d})'-(\mathbf{a}+t^2\mathbf{b})'\times\mathbf{d}\\
&= 2\mathbf{b}\times(\mathbf{c}-t\mathbf{d})-2t\mathbf{b}\times\mathbf{d}-2t\mathbf{b}\times\mathbf{d}\\
&= 2\mathbf{b}\times\mathbf{c}-6t\mathbf{b}\times\mathbf{d}
\end{aligned}$$

Alternatively, the cross product can be calculated first and then differentiated:

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{a}\times\mathbf{c}-t\mathbf{a}\times\mathbf{d}+t^2\mathbf{b}\times\mathbf{c}-t^3\mathbf{b}\times\mathbf{d}\\
\mathbf{r}'(t) &= -\mathbf{a}\times\mathbf{d}+2t\mathbf{b}\times\mathbf{c}-3t^2\mathbf{b}\times\mathbf{d}\\
\mathbf{r}''(t) &= 2\mathbf{b}\times\mathbf{c}-6t\mathbf{b}\times\mathbf{d}
\end{aligned}$$

□

11.2. Differential of a Vector Function. If $\mathbf{r}(t)$ is differentiable, then

(**11.2**) $$\mathbf{r}(t+h)-\mathbf{r}(t) = h\mathbf{r}'(t)+h\mathbf{u}(h)\,,$$

where $\mathbf{u}(h)$ approaches the zero vector, $\mathbf{u}(h)\to\mathbf{0}$, as $h\to 0$. Indeed, by the definition of the derivative,

$$\lim_{h\to 0}\mathbf{u}(h) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}-\mathbf{r}'(t) = \mathbf{0}\,.$$

Therefore, the components of the difference $\mathbf{r}(t+h)-\mathbf{r}(t)-h\mathbf{r}'(t)$ converge to 0 *faster* than h. Suppose that $\mathbf{r}'(t_0)$ does not vanish. Consider a linear vector function $\mathbf{L}(t)$ with the property $\mathbf{L}(t_0)=\mathbf{r}(t_0)$. Its general form is

$$\mathbf{L}(t)=\mathbf{r}(t_0)+(t-t_0)\mathbf{v}\,,$$

where $\mathbf{v}$ is a constant vector. For t close to t_0, $\mathbf{L}(t)$ is a *linear approximation* to $\mathbf{r}(t)$ in the sense that the approximation error $\|\mathbf{r}(t)-\mathbf{L}(t)\|$ becomes smaller with decreasing $|t-t_0|$. It follows from Eq.(**11.2**) that

$$\mathbf{r}(t)-\mathbf{L}(t)=(\mathbf{r}'(t_0)-\mathbf{v})h+h\mathbf{u}(h)\,,\quad h=t-t_0\,.$$

By the triangle inequality

$$\Big|\|\mathbf{a}\|-\|\mathbf{b}\|\Big|\leq\|\mathbf{a}+\mathbf{b}\|\leq\|\mathbf{a}\|+\|\mathbf{b}\|$$

the error of the linear approximation is bounded as

$$\Big|\|\mathbf{r}'(t_0)-\mathbf{v}\|-\|\mathbf{u}(h)\|\Big|\leq\frac{\|\mathbf{r}(t)-\mathbf{L}(t)\|}{|h|}\leq\|\mathbf{r}'(t_0)-\mathbf{v}\|+\|\mathbf{u}(h)\|\,.$$

Since $\|\mathbf{u}(h)\|$ tends to zero as $h\to 0$, the lower and upper bounds (the left and right sides of the inequality) approach the same value $\|\mathbf{r}'(t_0)-\mathbf{v}\|$. This allows us to conclude that the approximation error decreases *linearly* with decreasing h:

$$\|\mathbf{r}(t)-\mathbf{L}(t)\|=\|\mathbf{r}'(t_0)-\mathbf{v}\|\,|h|\,,\quad h\to 0\,,$$

provided $\mathbf{v}\neq\mathbf{r}'(t_0)$. *Only when* $\mathbf{v}=\mathbf{r}'(t_0)$*, the approximation error decreases faster than* h:

$$\|\mathbf{r}(t)-\mathbf{L}(t)\|=\|\mathbf{u}(h)\|\,|h|\,,\quad h\to 0\,.$$

Thus, the linear vector function

$$\mathbf{L}(t)=\mathbf{r}(t_0)+\mathbf{r}'(t_0)(t-t_0) \tag{11.3}$$

is the *best linear approximation* of $\mathbf{r}(t)$ near $t=t_0$. Provided the derivative does not vanish, $\mathbf{r}'(t_0)\neq\mathbf{0}$, the linear vector function $\mathbf{L}(t)$ defines a line passing through the point $\mathbf{r}(t_0)$. This line is called the *tangent line* to the curve traced out by $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$.

The analogy can be made with the tangent line to the graph $y=f(x)$ at a point (x_0,y_0) where $y_0=f(x_0)$. The equation of the tangent line is

$$y=y_0+f'(x_0)(x-x_0)$$

(recall Calculus I). The graph is a curve in the xy plane whose parametric equations are

$$C:\quad y=f(x)\quad\Leftrightarrow\quad x=t\,,\ y=f(t)\quad\Leftrightarrow\quad\mathbf{r}(t)=\langle t,f(t)\rangle\,.$$

Put $\mathbf{r}(t_0)=\langle x_0,y_0\rangle$. Then the tangent line is traversed by the linear vector function (**11.3**) where

$$\mathbf{r}'(t_0)=\langle 1,f'(t_0)\rangle\,.$$

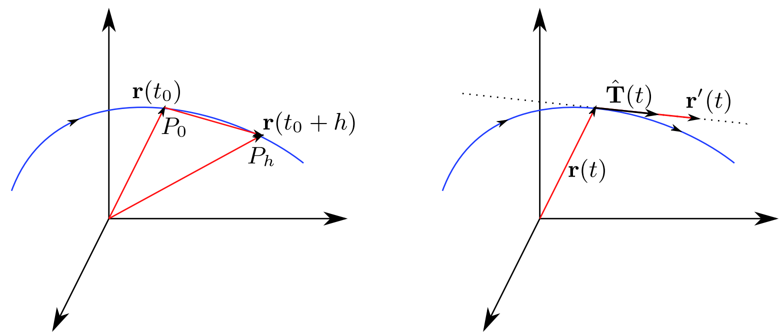

FIGURE **11.1**. **Left**: A secant line through two points of the curve, P_0 and P_h. As h gets smaller, the direction of the vector $\vec{P_0P_h} = \mathbf{r}(t_0+h) - \mathbf{r}(t_0)$ becomes closer to the tangent to the curve at P_0. **Right**: The derivative $\mathbf{r}'(t)$ defines a tangent vector to the curve at the point with the position vector $\mathbf{r}(t)$. It also specifies the direction in which $\mathbf{r}(t)$ traverses the curve with increasing t. $\hat{\mathbf{T}}(t)$ is the unit tangent vector.

The parametric equations of this line define the tangent line to the graph:

$$\mathcal{L}: \quad x = x_0 + (t - t_0)\,, \quad y = y_0 + f'(t_0)(t - t_0) \quad \Rightarrow \quad y = y_0 + f'(x_0)(x - x_0)$$

because $x - x_0 = t - t_0$.

DEFINITION **11.3**. (Differential of a Vector Function).
Let $\mathbf{r}(t)$ be a differentiable vector function. Then the vector

$$d\mathbf{r}(t) = \mathbf{r}'(t)\,dt$$

is called the differential *of $\mathbf{r}(t)$.*

In particular, the derivative is the ratio of the differentials, $\mathbf{r}'(t) = d\mathbf{r}/dt$. Recall that the differential dt is an independent variable that describes infinitesimal variations of t such that higher powers of dt can be neglected. In this sense the definition of the differential is the linearization of Eq. (**11.2**) in $dt = h$ (when terms decreasing to zero faster than h are neglected). At any particular $t = t_0$, the differential $d\mathbf{r}(t_0) = \mathbf{r}'(t_0)dt \neq \mathbf{0}$ defines the tangent line

$$\mathbf{L}(t) = \mathbf{r}(t_0) + d\mathbf{r}(t_0) = \mathbf{r}(t_0) + \mathbf{r}'(t_0)dt, \quad t = t_0 + dt\,.$$

Thus, *the differential $d\mathbf{r}(t)$ at a point of the curve $\mathbf{r}(t)$ is the increment of the position vector along the line tangent to the curve at that point.*

11.3. Geometrical Significance of the Derivative. Among all the lines through a particular point $\mathbf{r}(t_0)$ of the curve traversed by a vector function $\mathbf{r}(t)$, the line parallel to the derivative $\mathbf{r}'(t_0)$ has been shown to approximate best the curve near $\mathbf{r}(t_0)$. Let P_0 and P_h have position vectors $\mathbf{r}(t_0)$ and $\mathbf{r}(t_0+h)$. Then

$$\overrightarrow{P_0P_h} = \mathbf{r}(t_0+h) - \mathbf{r}(t_0)$$

is a secant vector. As $h \to 0$, the direction of the vector $\overrightarrow{P_0P_h}$ becomes tangential to the curve as depicted in Figure **11.1**. On the other hand, it follows from (**11.2**) that, for small enough $h = dt$,

$$\overrightarrow{P_0P_h} = d\mathbf{r}(t_0) = \mathbf{r}'(t_0)h\,, \quad h \to 0\,.$$

This qualitative geometrical consideration justifies that the line through P_0 and parallel to the derivative $\mathbf{r}'(t_0) \neq \mathbf{0}$ was called *tangent*. So, the geometrical significance of the derivative of a vector function is that *the vector* $\mathbf{r}'(t) \neq \mathbf{0}$ *is tangent to the curve traversed by the vector function* $\mathbf{r}(t)$. The direction of the tangent vector also defines the orientation of the curve, i.e., the direction in which the curve is traced out by $\mathbf{r}(t)$.

Example **11.2**. *Find the line tangent to the curve* $\mathbf{r}(t) = \langle 2t, t^2-1, t^3+2t\rangle$ *at the point* $P_0(2,0,3)$.

Solution: By the geometrical property of the derivative, a vector parallel to the line is $\mathbf{v} = \mathbf{r}'(t_0)$, where t_0 is the value of the parameter t at which $\mathbf{r}(t_0) = \langle 2,0,3\rangle$ is the position vector of P_0. Therefore, $t_0 = 1$. Then

$$\mathbf{v} = \mathbf{r}'(1) = \langle 2, 2t, 3t^2+2\rangle\Big|_{t=1} = \langle 2,2,5\rangle\,.$$

Parametric equations of the line through $P_0 = (2,0,3)$ and parallel to $\mathbf{v}$ are

$$x = 2+2t\,, \quad y = 2t\,, \quad z = 3+5t\,.$$

□

11.4. Smooth curves. Let $\mathbf{r}(t)$ be a parameterization of a curve C. Can a tangent line be defined at a point of C where the derivative $\mathbf{r}'(t)$ vanishes? It turns out that the answer depends on intrinsic geometrical properties of the curve C and not so much on a particular parameterization of C that happens to have the zero derivative at a particular point.

Suppose first that the derivative $\mathbf{r}'(t)$ exists and does not vanish. Then, at any point of the curve traced out by $\mathbf{r}(t)$, a *unit tangent vector* can be defined by

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\,.$$

In Section **10.3**, spatial curves were identified with continuous vector functions. Intuitively, a smooth curve as a point set in space should have a unit tangent vector that is continuous along the curve. Recall also that, for any curve as a point set in space, there are many vector functions whose range coincides with the curve.

Definition **11.4**. (Smooth Curve).
A point set C in space is called a smooth curve *if C is a curve for which there exists a simple, continuously differentiable parameterization $\mathbf{r}(t)$ whose derivative does not vanish and, if the curve is closed $\mathbf{r}(a) = \mathbf{r}(b)$, then in addition $\mathbf{r}'(a) = s\mathbf{r}'(b)$ for some $s > 0$.*

Clearly, a smooth curve always has a tangent line at any point. It is its *characteristic property*. The tangent line is parallel to the unit tangent vector at that point. A smooth parametric curve $\mathbf{r}(t)$ is *oriented* by the direction of the unit tangent vector $\hat{\mathbf{T}}(t)$. If a smooth curve is closed $\mathbf{r}(a) = \mathbf{r}(b)$, then the unit tangent vectors $\hat{\mathbf{T}}(a)$ and $\hat{\mathbf{T}}(b)$ must coincide, which is guaranteed by the condition $\mathbf{r}'(a) = s\mathbf{r}'(b)$ for some $s > 0$. Indeed,

$$\hat{\mathbf{T}}(a) = \frac{1}{\|\mathbf{r}'(a)\|}\,\mathbf{r}'(a) = \frac{s}{\|s\mathbf{r}'(b)\|}\,\mathbf{r}'(b) = \frac{s}{|s|}\,\hat{\mathbf{T}}(b) = \hat{\mathbf{T}}(b) \quad \text{if} \quad s > 0\,.$$

If $s < 0$, then $\hat{\mathbf{T}}(a) = -\hat{\mathbf{T}}(b)$ and the unit tangent vector is discontinuous.

How does one determine whether a given parametric curve is smooth or not? Note that if $\mathbf{r}'(t)$ is continuous and never vanishes, then $\hat{\mathbf{T}}(t)$ is continuous. In particular, with the definition above, a smooth curve does indeed have a continuous unit tangent vector. Therefore, *if a curve does not have a continuous unit tangent vector, it cannot be smooth.* This enables us to conclude that some curves are not smooth, based on properties deduced from a *single* parameterization. This is important because one cannot possibly test all parameterizations to see whether one of them meets the conditions in Definition **11.4**. The following example illustrates this concept.

Example **11.3**. *Investigate whether the planar curves traversed by the vector functions $\mathbf{r}_1(t) = \langle t^3, t^2\rangle$ and $\mathbf{r}_2(t) = \langle t^3, t^5\rangle$ are smooth.*

Solution: The first component of each vector function $x(t) = t^3$ is one-to-one for all t ($x(t_1) = x(t_2)$ implies $t_1 = t_2$). Therefore both the curves are simple. The vector functions are continuously differentiable everywhere,

$$\mathbf{r}_1'(t) = \langle 3t^2, 2t\rangle\,, \qquad \mathbf{r}_2'(t) = \langle 3t^2, 5t^2\rangle\,,$$

but both the derivatives vanish at the origin, $\mathbf{r}_1'(0) = \mathbf{r}_2'(0) = \mathbf{0}$. The unit tangent vectors $\hat{\mathbf{T}}_1(0)$ and $\hat{\mathbf{T}}_2(0)$ are not defined at $t = 0$ for the given parameterizations. Let us investigate the left and right limits of the unit tangent vectors at $t = 0$. One has

$$\|\mathbf{r}_1'(t)\| = (9t^4 + 4t^2)^{1/2} = 2|t|(1 + 9t^2/4)^{1/2}\,.$$

Therefore

$$\hat{\mathbf{T}}_1(t) = \frac{1}{\|\mathbf{r}_1'(t)\|}\,\mathbf{r}_1'(t) = \frac{1}{\sqrt{1 + 9t^2/4}}\left\langle \frac{3t^2}{2|t|}, \frac{t}{|t|}\right\rangle.$$

If $t > 0$, then $t/|t| = 1$ and, if $t < 0$, then $t/|t| = -1$. Hence,

$$\lim_{t\to 0^+}\hat{\mathbf{T}}_1(t) = \langle 0, 1\rangle\,, \qquad \lim_{t\to 0^-}\hat{\mathbf{T}}_1(t) = \langle 0, -1\rangle\,.$$

This shows that the unit tangent vector is discontinuous at the point $\mathbf{r}(0) = \mathbf{0}$ and the curve is not smooth. To visualize the curve near this point, let us solve the equation $x = t^3$ for t, $t = x^{1/3}$, and substitute the latter into $y = t^2$ to obtain $y = x^{2/3}$. So the curve traversed by $\mathbf{r}_1(t)$ is the graph $y = x^{2/3}$, which has a *cusp* at $x = 0$. (The graph lies in the positive half-plane $y \geq 0$ and approaches the y axis tangentially, forming a horn-like shape at the origin.) It should be stressed that the presence of a cusp was established from a singe (given) parameterization of the curve.

For the second curve, a similar analysis shows that

$$\begin{aligned}
&\|\mathbf{r}_2'(t)\| = (9t^4 + 25t^8)^{1/2} = 3t^2(1 + 25t^4/9)^{1/2}\,,\\
&\hat{\mathbf{T}}_2(t) = \frac{1}{\|\mathbf{r}_2'(t)\|}\,\mathbf{r}_2'(t) = \frac{1}{\sqrt{1+25t^4/9}}\left\langle 1\,, \frac{5t^2}{3}\right\rangle,\\
&\lim_{t\to 0^+}\hat{\mathbf{T}}_2(t) = \lim_{t\to 0^-}\hat{\mathbf{T}}_2(t) = \langle 1, 0\rangle\,.
\end{aligned}$$

The latter property allows us to *define* the unit tangent at the point where the derivative vanishes by $\hat{\mathbf{T}}_2(0) = \langle 1, 0\rangle$ so that $\hat{\mathbf{T}}_2(t)$ is *continuous* everywhere (in particular, $\lim_{t\to 0}\hat{\mathbf{T}}_2(t) = \hat{\mathbf{T}}_2(0)$) and, hence, the curve is smooth.

The fact that the derivative of a vector function that traverses a smooth curve vanishes at some point of the curve does not contradict Definition **11.4** because there exists another vector function $\mathbf{R}(s) = \langle s, s^{5/3}\rangle$ that has the same range since $\mathbf{R}(t^3) = \mathbf{r}_2(t)$ and whose derivative is continuous and never vanishes, $\mathbf{R}'(s) = \langle 1, 5s^{2/3}/3\rangle \neq \mathbf{0}$ for all s. It is easy to see that the second curve is the graph $y = x^{5/3}$ which has no cusp at $x = 0$ (it is tangent to the x axis and has an inflection point at $x = 0$). □

11.5. Study Problems.

Problem **11.1**. *Determine whether the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is smooth, where $a > 0$ is a parameter. If it is not smooth at particular points, investigate its behavior near those points by approximating the cycloid by the graph of a power function.*

SOLUTION: The existence of a continuous unit tangent vector has to be verified. Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Since $x'(t) = a(1 - \cos t) \geq 0$ for all t, and $x'(t) = 0$ only when t is a multiple of 2π, $x(t)$ is monotonically increasing. In particular, $x(t)$ is one-to-one, so C is simple. Since $y'(t) = a\sin t$, the derivatives $x'(t)$ and $y'(t)$ vanish simultaneously if and only if $t = 2\pi n$ for some integer n. Thus, $\mathbf{r}'(t) \neq 0$ unless $t = 2\pi n$, so C is smooth except possibly at the points $\mathbf{r}(2\pi n) = \langle 2\pi n a, 0\rangle$; that is, the portion of C between two consecutive such points is smooth, but it is not yet known whether C is smooth at those points. Since

$$\|\mathbf{r}'(t)\| = a\sqrt{2(1-\cos t)} = a\sqrt{4\sin^2(t/2)} = 2a|\sin(t/2)|\,,$$

the components of the unit tangent vector for $t \neq 2\pi n$ are

$$T_1(t) = \frac{x'(t)}{\|\mathbf{r}'(t)\|} = |\sin(t/2)|\,, \qquad T_2(t) = \frac{y'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sin t}{2|\sin(t/2)|}\,.$$

Owing to the periodicity of the sine and cosine functions, it is sufficient to investigate the point corresponding to $t = 0$. If there exists a continuous unit tangent vector, then the limit $\lim_{t\to 0}\hat{\mathbf{T}}(t)$ should exist and be the unit tangent vector at the point corresponding to $t = 0$. By Theorem **10.1**, the existence of the limits of the components $T_1(t)$ and $T_2(t)$ as $t \to 0$ has to be verified. Evidently, $T_1(t) \to 0$ as $t \to 0$, but the limit $\lim_{t\to 0} T_2(t)$ does not exist. Indeed, by the trigonometric identity $\sin t = 2\sin(t/2)\cos(t/2)$ the left and right limits are different:

$$\lim_{t\to 0^+} T_2(t) = \lim_{t\to 0^+} \frac{2\sin(t/2)\cos(t/2)}{2\sin(t/2)} = \lim_{t\to 0^+}\cos(t/2) = 1\,,$$
$$\lim_{t\to 0^-} T_2(t) = \lim_{t\to 0^-} \frac{2\sin(t/2)\cos(t/2)}{-2\sin(t/2)} = -\lim_{t\to 0^-}\cos(t/2) = -1\,.$$

Therefore the left and right limits of the unit tangent vector do not coincide at $t = 0$:

$$\lim_{t\to 0^+}\hat{\mathbf{T}}(t) = \langle 0, 1\rangle \neq \langle 0, -1\rangle = \lim_{t\to 0^-}\hat{\mathbf{T}}(t)\,.$$

This means that $\hat{\mathbf{T}}(t)$ cannot be continuously extended across the point $(0, 0)$, so C is not smooth there (as well as at $(2\pi n, 0)$).

A local behavior of the cycloid near $(0, 0)$ may be investigated by using the Taylor polynomial approximations near $t = 0$ of the trigonometric functions involved:

$$\sin t = t - t^3/6 + O(t^5)\,, \quad \cos t = 1 - t^2/2 + O(t^4)\,.$$

So neglecting terms t^4 and higher for small t, the cycloid is approximated by the curve

$$x = a(t - \sin(t)) \approx \frac{a}{6}\,t^3\,, \quad y = a(1 - \cos(t)) \approx \frac{a}{2}\,t^2\,.$$

Expressing t via x from the first equation and substituting it into the other equation,

$$t = \Big(\frac{6x}{a}\Big)^{1/3} \quad\Rightarrow\quad y = \frac{a}{2}\Big(\frac{6x}{a}\Big)^{2/3} = cx^{2/3}\,, \quad c = \Big(\frac{9a}{2}\Big)^{1/3}\,,$$

it is concluded that near the point $(0, 0)$ the cycloid behaves as the graph $y = cx^{2/3}$, which has a cusp at $(0, 0)$. □

Problem **11.2**. *Prove that, for any smooth curve on a sphere, a tangent vector at any point P is orthogonal to the vector from the sphere center to the point P.*

Solution: Let $\mathbf{r}_0$ be the position vector of the center of a sphere of radius R. The position vector $\mathbf{r}$ of any point of the sphere satisfies the equation

$$\|\mathbf{r} - \mathbf{r}_0\| = R \quad\Rightarrow\quad (\mathbf{r} - \mathbf{r}_0)\cdot(\mathbf{r} - \mathbf{r}_0) = R^2$$

because $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$ for any vector $\mathbf{a}$. Let $\mathbf{r}(t)$ be a vector function that traces out a curve on the sphere. In particular, the parameterization of the curve can be chosen so that $\mathbf{r}'(t) \neq \mathbf{0}$ (the curve is smooth). Then, for all values of t,

$$(\mathbf{r}(t)-\mathbf{r}_0)\cdot(\mathbf{r}(t)-\mathbf{r}_0) = R^2\,.$$

Differentiating both sides of the latter relation (put $\mathbf{u}(t)=\mathbf{v}(t)=\mathbf{r}(t)+\mathbf{r}_0$ in the third equation in Theorem **11.1**) and using the basic properties of the dot product, one infers

$$\begin{aligned}&\mathbf{r}'(t)\cdot(\mathbf{r}(t)-\mathbf{r}_0)+(\mathbf{r}(t)-\mathbf{r}_0)\cdot\mathbf{r}'(t)=0\,,\\ &2\,\mathbf{r}'(t)\cdot(\mathbf{r}(t)-\mathbf{r}_0)=0\,,\\ &\mathbf{r}'(t)\cdot(\mathbf{r}(t)-\mathbf{r}_0)=0 \quad\Longleftrightarrow\quad \mathbf{r}'(t)\perp\mathbf{r}(t)-\mathbf{r}_0\,.\end{aligned}$$

If $\mathbf{r}(t)$ is the position vector of P and O is the center of the sphere, then $\overrightarrow{OP}=\mathbf{r}(t)-\mathbf{r}_0$, and hence the tangent vector $\mathbf{r}'(t)$ at P is orthogonal to $\overrightarrow{OP}$ for any t or at any point P of the curve. □

Problem **11.3**. *Let $\mathbf{u}(t)$ be differentiable and $\|\mathbf{u}(t)\|=k$ where k is a constant. Show that $\mathbf{u}(t)$ and $\mathbf{u}'(t)$ are orthogonal. Use the result to find the unit tangent vector as a function of t to the exponential helix $\mathbf{r}(t)=\langle e^t\cos t,\, e^t\sin t,\, e^t\rangle$. Is the curve smooth?*

SOLUTION: Since $\|\mathbf{u}(t)\|^2=\mathbf{u}(t)\cdot\mathbf{u}(t)$, by differentiating the equation $\mathbf{u}(t)\cdot\mathbf{u}(t)=k^2$, one infers that the derivative $\mathbf{u}'(t)$ is orthogonal to $\mathbf{u}(t)$ (see the third equation in Theorem **11.1**):

$$\Big(\mathbf{u}(t)\cdot\mathbf{u}(t)\Big)'=0 \quad\Rightarrow\quad 2\mathbf{u}'(t)\cdot\mathbf{u}(t)=0 \quad\Leftrightarrow\quad \mathbf{u}'(t)\perp\mathbf{u}(t)\,.$$

The vector function $\mathbf{r}(t)$ can be written in the form

$$\mathbf{r}(t)=e^t\mathbf{u}(t)+e^t\hat{\mathbf{e}}_3\,,\quad \mathbf{u}(t)=\langle\cos t,\,\sin t,\,0\rangle\,,\quad \|\mathbf{u}(t)\|=1\,.$$

By the rules of differentiation (see the second equation in Theorem **11.1**):

$$\mathbf{r}'(t)=e^t\mathbf{u}(t)+e^t\mathbf{u}'(t)+e^t\hat{\mathbf{e}}_3=e^t\Big(\mathbf{u}(t)+\mathbf{u}'(t)+\hat{\mathbf{e}}_3\Big)\,.$$

The vectors $\mathbf{u}(t)$, $\mathbf{u}'(t)=\langle-\sin t,\cos t,0\rangle$, and $\hat{\mathbf{e}}_3$ are mutually orthogonal unit vectors so that $\|\mathbf{u}(t)+\mathbf{u}'(t)+\hat{\mathbf{e}}_3\|=\sqrt{3}$ (the length of the diagonal of a cube with edges of unit length). Therefore

$$\begin{aligned}\|\mathbf{r}'(t)\|=\sqrt{3}\,e^t \quad\Rightarrow\quad \hat{\mathbf{T}}(t)&=\frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)=\frac{1}{\sqrt{3}}\Big(\mathbf{u}(t)+\mathbf{u}'(t)+\hat{\mathbf{e}}_3\Big)\\ &=\frac{1}{\sqrt{3}}\langle\cos t-\sin t,\,\sin t+\cos t,\,1\rangle\,.\end{aligned}$$

The reader is advised to compare the technicalities involved if $\hat{\mathbf{T}}(t)$ is computed by differentiating $\mathbf{r}(t)$ component-wise. The function $\hat{\mathbf{T}}(t)$ is continuous because its components are continuous. So the curve is smooth. □

11.6. Exercises.

1–6. Find the derivatives and differentials of each of the following vector functions:

1. $\mathbf{r}(t) = \langle 1,\ 1+t,\ 1+t^3\rangle$;
2. $\mathbf{r}(t) = \langle \cos t,\ \sin^2(t),\ t^2\rangle$;
3. $\mathbf{r}(t) = \langle \ln(t),\ e^{2t},\ te^{-t}\rangle$;
4. $\mathbf{r}(t) = \langle \sqrt[3]{t-2},\ \sqrt{t^2-4},\ t\rangle$;
5. $\mathbf{r}(t) = \mathbf{a} + \mathbf{b}t^2 - \mathbf{c}e^t$;
6. $\mathbf{r}(t) = t\mathbf{a} \times (\mathbf{b} - \mathbf{c}e^t)$.

7. Sketch the curve traversed by the vector function $\mathbf{r}(t) = \langle 2,\ t-1,\ t^2+1\rangle$. Indicate the direction in which the curve is traversed by $\mathbf{r}(t)$ with increasing t. Sketch the position vectors $\mathbf{r}(0)$, $\mathbf{r}(1)$, $\mathbf{r}(2)$ and the vectors $\mathbf{r}'(0)$, $\mathbf{r}'(1)$, $\mathbf{r}'(2)$. Repeat the procedure for the vector function $\mathbf{R}(t) = \mathbf{r}(-t) = \langle 2,\ -t-1,\ t^2+1\rangle$ for $t = -2, -1, 0$.

8–12. Determine if the curve traced out by each of the following vector functions is smooth for a specified interval of the parameter. If the curve is not smooth at a particular point, graph it near that point.

8. $\mathbf{r}(t) = \langle t,\ t^2,\ t^3\rangle,\ \ 0 \le t \le 1$;
9. $\mathbf{r}(t) = \langle t^2,\ t^3,\ 2\rangle,\ \ -1 \le t \le 1$;
10. $\mathbf{r}(t) = \langle t^{1/3},\ t,\ t^3\rangle,\ \ -1 \le t \le 1$;
11. $\mathbf{r}(t) = \langle t^5,\ t^3,\ t^4\rangle, -1 \le t \le 1$;
12. $\mathbf{r}(t) = \langle \sin^3 t,\ 1,\ t^2\rangle,\ \ -\pi/2 \le t \le \pi/2$.

13. Determine whether a cardioid described by the polar graph $r = 1 - a\cos\theta$, $0 \le \theta \le 2\pi$, is smooth, where $|a| \le 1$. Sketch the cardioid for $a = 0$, $a = \pm 1/2$, and $a = \pm 1$. Hint: to find parametric equations of the cardioid, use the relations between the polar and rectangular coordinates and θ as a parameter.

14–15. Find the parametric equations of the tangent line to each of the following curves at a specified point:

14. $\mathbf{r}(t) = \langle t^2 - t, t^3/3, 2t\rangle$, $P_0 = (6, 9, 6)$;
15. $\mathbf{r}(t) = \langle \ln t,\ 2\sqrt{t},\ t^2\rangle$, $P_0 = (0, 2, 1)$.

16–17. Find the unit tangent vector to the curve traversed by the specified vector function at the given point P_0:

16. $\mathbf{r}(t) = \langle 2t+1,\ 2\tan^{-1} t,\ e^{-t}\rangle,\ \ \ P_0(1, 0, 1)$;
17. $\mathbf{r}(t) = \langle \cos(\omega t),\ \cos(3\omega t),\ \sin(\omega t)\rangle,\ \ \ P_0(1/2, -1, \sqrt{3}/2)$, where ω is a positive constant.

18. Find $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ if $\mathbf{r}(t) = \langle t,\ t^2-1,\ t^3+2\rangle$.

19. Is there a point on the curve $\mathbf{r}(t) = \langle t^2 - t,\ t^3/3,\ 2t\rangle$ at which the tangent line is parallel to the vector $\mathbf{v} = \langle -5/2, 2, 1\rangle$? If so, find the point.

20. Let $\mathbf{r}(t) = \langle e^t,\ 2\cos t,\ \sin(2t)\rangle$. Use the best linear approximation $\mathbf{L}(t)$ near $t = 0$ to estimate $\mathbf{r}(0.2)$. Use a calculator to assess the accuracy $\|\mathbf{r}(0.2) - \mathbf{L}(0.2)\|$ of the estimate. Repeat the procedure for $\mathbf{r}(0.7)$ and $\mathbf{r}(1.2)$. Compare the errors in all three cases.

21. Find the point of intersection of the plane $y + z = 3$ and the curve $\mathbf{r}(t) = \langle \ln t,\ t^2,\ 2t\rangle$. Find the angle between the normal of the plane and the tangent line to the curve at the point of intersection.
22. Does the curve $\mathbf{r}(t) = \langle 2t^2,\ 2t,\ 2-t^2\rangle$ intersect the plane $x+y+z = -3$? If not, find a point on the curve that is closest to the plane. What is the distance between the curve and the plane. Hint: Express the distance between a point on the curve and the plane as a function of t, then solve the extreme value problem.
23. Find the point of intersection of two curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2\rangle$ and $\mathbf{r}_2(s) = \langle 3-s, s-2, s^2\rangle$. If the angle at which two curves intersect is defined as the angle between their tangent lines at the point of intersection, find the angle at which the above two curves intersect.
24. State the condition under which the tangent lines to the curve $\mathbf{r}(t)$ at two distinct points $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$ are intersecting, or skew, or parallel. Let $\mathbf{r}(t) = \langle 2\sin(\pi t),\ \cos(\pi t),\ \sin(\pi t)\rangle$, $t_1 = 0$, and $t_2 = 1/2$. Determine whether the tangent lines at these points are intersecting and, if so, find the point of intersection.
25. Suppose a smooth curve $\mathbf{r}(t)$ does not intersect a plane through a point P_0 and orthogonal to a vector $\mathbf{n}$. Assume that, among the points on the curve, there is one that is closest to the plane. What is the angle between $\mathbf{n}$ and a tangent vector to the curve at the point that is the closest to the plane?
26. Suppose $\mathbf{r}(t)$ is twice differentiable. Show that $(\mathbf{r}(t) \times \mathbf{r}'(t))' = \mathbf{r}(t) \times \mathbf{r}''(t)$.
27. Suppose that $\mathbf{r}(t)$ is differentiable three times. Show that $[\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))]' = \mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}'''(t))$.
28. Let $\mathbf{r}(t)$ be a differentiable vector functions. Show that $(\|\mathbf{r}(t)\|)' = \mathbf{r}(t)\cdot\mathbf{r}'(t)/\|\mathbf{r}(t)\|$ at all t for which $\mathbf{r}(t) \neq \mathbf{0}$.
29. A space warship can fire a laser cannon forward along the tangent line to its trajectory. If the trajectory is traversed by the vector function $\mathbf{r}(t) = \langle t, t, t^2+4\rangle$ in the direction of increasing t and the target is the sphere $x^2 + y^2 + z^2 = 1$, find the part of the trajectory in which the laser cannon can hit the target. Hint: If a line $\mathcal{L}$ is tangent to the trajectory at $t = t_0$, then the target is hit when the distance between $\mathcal{L}$ and the origin is less or equal 1. State this geometrical condition as an algebraic condition on t_0. To solve this algebraic condition, show that the trajectory is a parabola in the plane $y = x$. So, find points on the parabola at which its tangent is at a distance less or equal 1 to the origin.
30–32. A plane *normal* to a curve at a point P_0 is the plane through P_0 whose normal is tangent to the curve at P_0. For each of the following curves find a suitable parameterization, the tangent line, and the normal plane at a specified point:

30. $y = x,\ z = x^2,\ P_0 = (1, 1, 1)$;
31. $x^2 + z^2 = 10,\ y^2 + z^2 = 10,\ P_0 = (1, 1, 3)$;
32. $x^2 + y^2 + z^2 = 6,\ x + y + z = 0,\ P_0 = (1, -2, 1)$;

33. Show that tangent lines to a circular helix have a constant angle with the axis of the helix.

34. Consider a line through the origin. Any such line sweeps a circular cone when rotated about the z axis and, for this reason, is called a *generating* line of a cone. Prove that the curve $\mathbf{r}(t) = (e^t \cos t, e^t \sin t, e^t)$ intersects all generating lines of the cone $x^2 + y^2 = z^2$ at the same angle. Hint: Show that parametric equations of a line in the cone are $x = s\cos\theta$, $y = s\sin\theta$, $z = s$. Define the points of intersection of the line and the curve and find the angle at which they intersect.

12. Integration of Vector Functions

DEFINITION **12.1**. (Definite Integral of a Vector Function).
Let $\mathbf{r}(t)$ *be defined on the interval* $[a, b]$. *The vector whose components are the definite integrals of the corresponding components of* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ *is called the* definite integral *of* $\mathbf{r}(t)$ *over the interval* $[a, b]$ *and denoted as*

$$\int_a^b \mathbf{r}(t)\,dt = \Big\langle \int_a^b x(t)\,dt\,,\ \int_a^b y(t)\,dt\,,\ \int_a^b z(t)\,dt \Big\rangle. \tag{12.1}$$

If the integral (**12.1**) *exists, then* $\mathbf{r}(t)$ *is said to be integrable on* $[a, b]$.

By this definition, a vector function is integrable if and only if all its components are integrable functions. Recall that a continuous real-valued function is integrable. Therefore, the following theorem holds.

THEOREM **12.1**. *If a vector function is continuous on the interval* $[a, b]$, *then it is integrable on* $[a, b]$.

EXAMPLE **12.1**. *Find the integral of* $\mathbf{r}(t) = \langle t/\pi, \sin t, \cos t\rangle$ *over the interval* $[0, \pi]$.

SOLUTION: The components of $\mathbf{r}(t)$ are continuous on $[0, \pi]$. Therefore, by the fundamental theorem of calculus,

$$\begin{aligned}\int_0^\pi \mathbf{r}(t)dt &= \Big\langle \int_0^\pi (t/\pi)dt,\ \int_0^\pi \sin t\,dt,\ \int_0^\pi \cos t\,dt \Big\rangle \\ &= \Big\langle \frac{t^2}{2\pi}\Big|_0^\pi,\ -\cos t\Big|_0^\pi,\ \sin t\Big|_0^\pi \Big\rangle = \langle \pi/2,\ 2, 0\rangle\,.\end{aligned}$$

□

DEFINITION **12.2**. (Indefinite Integral of a Vector Function).
A vector function $\mathbf{R}(t)$ *is called an* antiderivative *of* $\mathbf{r}(t)$ *if* $\mathbf{R}'(t) = \mathbf{r}(t)$. *The* indefinite integral $\int \mathbf{r}(t)dt$ *of a vector function* $\mathbf{r}(t)$ *is the collection of all antiderivatives* $\mathbf{R}(t)$ *of* $\mathbf{r}(t)$.

The indefinite integral of $\mathbf{r}(t)$ can also be viewed as the most general vector function whose derivative is $\mathbf{r}(t)$. Let $\mathbf{R}(t) = \langle X(t), Y(t), Z(t)\rangle$ be an antiderivative of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ on an interval. Then the functions $X(t)$, $Y(t)$, and $Z(t)$ are antiderivatives of $x(t)$, $y(t)$, and $z(t)$, respectively. Recall from Calculus I that two functions that have derivatives equal on an interval can differ at most by a constant function on this interval. Therefore

$$\int x(t)\,dt = X(t) + +c_1\,,\quad \int y(t)\,dt = Y(t) + c_2\,,\quad \int z(t)\,dt = Z(t) + c_3\,,$$

where c_1, c_2, and c_3 are constants. The latter relations can be combined into a single vector relation:

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{c}\,,$$

where $\mathbf{c}$ is an arbitrary constant vector.

Recall further from Calculus I that for a function $x(t)$ continuous on $[a,b]$, its particular antiderivative is given by

$$X(t) = \int_a^t x(u)du, \quad a \le t \le b\,,$$

and $X(t)$ is differentiable on (a,b) and satisfies the condition $X(a) = 0$. Therefore a particular antiderivative of a continuous vector function $\mathbf{r}(t)$ is

$$\mathbf{R}(t) = \int_a^t \mathbf{r}(u)du, \quad a \le t \le b\,.$$

The vector function $\mathbf{R}(t)$ is *differentiable* on (a,b) and satisfies the condition $\mathbf{R}(a) = 0$. The indefinite integral on $[a,b]$ is obtained by adding a constant vector, $\mathbf{R}(t) \to \mathbf{R}(t) + \mathbf{c}$. This observation allows us to extend the fundamental theorem of calculus to vector functions.

THEOREM **12.2**. (Fundamental Theorem of Calculus for Vector Functions). *If* $\mathbf{r}(t)$ *is continuous on* $[a,b]$*, then*

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}(t)$ *is any antiderivative of* $\mathbf{r}(t)$*, that is, a vector function such that* $\mathbf{R}'(t) = \mathbf{r}(t)$*.*

EXAMPLE **12.2**. *Find* $\mathbf{r}(t)$ *if* $\mathbf{r}'(t) = \langle 2t, 1, 6t^2\rangle$ *and* $\mathbf{r}(1) = \langle 2, 1, 0\rangle$*.*

SOLUTION: Taking the antiderivative of $\mathbf{r}'(t)$, one finds

$$\mathbf{r}(t) = \int \langle 2t,\ 1,\ 6t^2\rangle\,dt = \langle t^2,\ t,\ 3t^3\rangle + \mathbf{c}.$$

The constant vector $\mathbf{c}$ is determined by the condition $\mathbf{r}(1) = \langle 2, 1, 0\rangle$, which gives

$$\langle 1, 1, 3\rangle + \mathbf{c} = \langle 2, 1, 0\rangle \quad \Rightarrow \quad \mathbf{c} = \langle 2, 1, 0\rangle - \langle 1, 1, 3\rangle = \langle 1, 0, -3\rangle\,,$$

and, hence, $\mathbf{r}(t) = \langle t^2 + 1, t, 3t^3 - 3\rangle$. □

In general, the solution of the equation $\mathbf{r}'(t) = \mathbf{v}(t)$ satisfying the condition $\mathbf{r}(t_0) = \mathbf{r}_0$ can be written in the form

$$\mathbf{r}'(t) = \mathbf{v}(t) \ \text{ and } \ \mathbf{r}(t_0) = \mathbf{r}_0 \quad \Rightarrow \quad \mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^t \mathbf{v}(u)\,du$$

if $\mathbf{v}(t)$ is a continuous vector function on an interval (t_0 lies in the interval of continuity). As noted above, if the integrand is a continuous function on an interval, then the derivative of the integral with respect to its upper limit is the value of the integrand at that limit. Therefore, $\mathbf{r}'(t) = (d/dt)\int_{t_0}^t \mathbf{v}(u)\,du = \mathbf{v}(t)$, and hence $\mathbf{r}(t)$ is an antiderivative of $\mathbf{v}(t)$. When $t = t_0$, the integral vanishes and $\mathbf{r}(t_0) = \mathbf{r}_0$ as required.

12.1. Applications to Mechanics. Let $\mathbf{r}(t)$ be the position vector of a particle as a function of time t. The first derivative

$$\mathbf{r}'(t) = \mathbf{v}(t)$$

is called the *velocity* of the particle. The magnitude of the velocity vector $v(t) = \|\mathbf{v}(t)\|$ is called the *speed*. The speed of a car is a number shown on the speedometer. The velocity defines the direction in which the particle travels and the instantaneous rate at which it moves in that direction. The second derivative

$$\mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t)$$

is called the *acceleration*. If m is the mass of a particle and $\mathbf{F}$ is the force acting on the particle, then according to Newton's second law, the acceleration and force are related as

$$\mathbf{F} = m\mathbf{a}\,.$$

If the time is measured in seconds, the length in meters, and the mass in kilograms, then the force is given in newtons, $1\,N = 1\,kg \cdot m/s^2$.

If the force is known as a vector function of time, then Newton's second law determines a particle's trajectory. The problem of finding the trajectory amounts to reconstructing the vector function $\mathbf{r}(t)$ if its second derivative

$$\mathbf{r}''(t) = \frac{1}{m}\,\mathbf{F}(t)$$

is known; that is, $\mathbf{r}(t)$ is given by a *second antiderivative* of $(1/m)\mathbf{F}(t)$. Indeed, the velocity $\mathbf{v}(t)$ is an antiderivative of $(1/m)\mathbf{F}(t)$, and the position vector $\mathbf{r}(t)$ is an antiderivative of the velocity $\mathbf{v}(t)$. As shown in the previous section, an antiderivative is not unique, unless its value at a particular point is specified. So *the trajectory of motion is uniquely determined by Newton's equation, provided the position and velocity vectors are specified at a particular moment of time*, for example,

$$\mathbf{r}(t_0) = \mathbf{r}_0\,, \qquad \mathbf{v}(t_0) = \mathbf{v}_0\,.$$

The latter conditions are called *initial conditions*. Given the initial conditions, the trajectory of motion is uniquely defined by the relations:

$$\mathbf{v}(t) = \mathbf{v}_0 + \frac{1}{m}\int_{t_0}^{t} \mathbf{F}(u)\,du\,, \quad \mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(u)\,du \tag{12.2}$$

if the force is a continuous vector function of time. If the force is piecewise continuous, then the *initial value problem* has to be solved for each interval of continuity. The values of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ at the end point of a preceding interval of continuity serve as the initial values for the next adjacent interval of continuity.

Remark. If the force is a function of a particle's position, $\mathbf{F} = \mathbf{F}(\mathbf{r}(t))$, then the Newton's equation becomes a system of *ordinary differential equations* that is a set of some relations between components of the vector function $\mathbf{r}(t)$, its derivatives, and time. Solving the initial value problem for a system of differential equations requires a special technique to integrate such equations. Sometimes such problems can be solved by simpler means. Examples are given in Study Problems **12.3**, **15.4**, and **15.6**.

EXAMPLE **12.3**. (Motion Under a Constant Force).
Prove that the trajectory of motion under a constant force is a parabola if the initial velocity is not parallel to the force.

SOLUTION: Let $\mathbf{F}$ be a constant force. Without loss of generality, the initial conditions can be set at $t = 0$, $\mathbf{r}(0) = \mathbf{r}_0$, and $\mathbf{v}(0) = \mathbf{v}_0$. Then

$$\mathbf{v}(t) = \mathbf{v}_0 + \frac{1}{m}\int_0^t \mathbf{F}du = \mathbf{v}_0 + \frac{t}{m}\,\mathbf{F}\,,$$
$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(u)du = \mathbf{r}_0 + t\mathbf{v}_0 + \frac{t^2}{2m}\,\mathbf{F}\,.$$

If the vectors $\mathbf{v}_0$ and $\mathbf{F}$ are parallel, then they are proportional, $\mathbf{v}_0 = c\mathbf{F}$. In this particular case, the trajectory $\mathbf{r}(t) = \mathbf{r}_0 + (ct + t^2/(2m))\mathbf{F} = \mathbf{r}_0 + s\mathbf{F}$ lies in the straight line through $\mathbf{r}_0$ and parallel to $\mathbf{F}$. The parameter $s = ct + t^2/(2m)$ defines the position of the particle on the line as a function of time. Otherwise, the vector $\mathbf{r}(t) - \mathbf{r}_0$ is a linear combination of two non-parallel vectors $\mathbf{v}_0$ and $\mathbf{F}$ and hence must be orthogonal to $\mathbf{n} = \mathbf{F}\times\mathbf{v}_0$ by the geometrical property of the cross product. Therefore, the particle remains in the plane through $\mathbf{r}_0$ that is parallel to $\mathbf{F}$ and $\mathbf{v}_0$ or orthogonal to $\mathbf{n}$:

$$(\mathbf{r}(t) - \mathbf{r}_0)\cdot\mathbf{n} = 0\,, \qquad \mathbf{n} = \mathbf{F}\times\mathbf{v}_0$$

(see Figure **12.1**, left panel). The shape of a space curve does not depend on the choice of the coordinate system. Let us choose the coordinate system such that the origin is at the initial position $\mathbf{r}_0$ and the plane in which the trajectory lies coincides with the zy plane so that $\mathbf{F}$ is parallel to the z axis. In this coordinate system,

$$\mathbf{r}_0 = \langle 0,0,0\rangle\,, \quad \mathbf{F} = \langle 0,0,-F\rangle\,, \quad \mathbf{v}_0 = \langle 0, v_{0y}, v_{0z}\rangle\,.$$

The parametric equations of the trajectory of motion assume the form

$$x = 0\,, \quad y = v_{0y}t\,, \quad z = v_{0z}t - t^2F/(2m)\,.$$

The substitution of $t = y/v_{0y}$ from the second equation into the third equation yields

$$z = -\frac{Fv_{0y}^2}{2m}\,y^2 + \frac{v_{0z}}{v_{0y}}\,y\,,$$

which defines a parabola in the zy plane. Thus, the trajectory of motion under a constant force is a parabola through the point $\mathbf{r}_0$ that lies in the plane containing the force and initial velocity vectors $\mathbf{F}$ and $\mathbf{v}_0$. The parabola is concave in the direction of the force. In Figure **12.1**, the force vector points downward and the trajectory is concave down. □

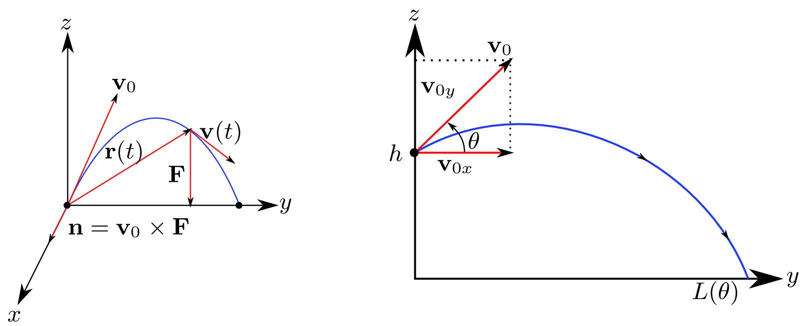

FIGURE **12.1**. **Left**: Motion under a constant force $\mathbf{F}$. The trajectory is a parabola that lies in the plane through the initial point of the motion $\mathbf{r}_0$ and orthogonal to the vector $\mathbf{n} = \mathbf{F} \times \mathbf{v}_0$, where the initial velocity $\mathbf{v}_0$ is assumed to be non-parallel to the force $\mathbf{F}$. **Right**: Motion of a projectile thrown at an angle θ and an initial height h. The trajectory is a parabola. The point of impact defines the range $L(\theta)$.

12.2. Motion Under a Constant Gravitational Force. The magnitude of the gravitational force that acts on an object of mass m near the surface of the Earth is mg, where $g \approx 9.8$ m/s^2 is a universal constant called the *acceleration of a free fall*. According to Example **12.3**, any projectile fired from some point follows a parabolic trajectory. This fact allows one to predict the exact positions of the projectile and, in particular, the point at which it impacts the ground. In practice, the initial speed v_0 of the projectile and angle of elevation θ at which the projectile is fired are known (see Figure **12.1**, right panel). Some practical questions are: At what elevation angle is the maximal range reached? At what elevation angle does the range attain a specified value (e.g., to hit a target)?

To answer these and related questions, choose the coordinate system such that the z axis is directed upward from the ground and the parabolic trajectory lies in the zy plane. The projectile is fired from the point $(0, 0, h)$, where h is the initial elevation of the projectile above the ground (firing from a hill). In the notation of Example **12.3**, $\mathbf{F} = \langle 0, 0, -mg\rangle$ (the z component of $\mathbf{F}$ is negative because the gravitational force is directed toward the ground, while the z axis points upward), $v_{0y} = v_0 \cos\theta$, and $v_{0z} = v_0 \sin\theta$. The trajectory is

$$y = tv_0 \cos\theta\,, \quad z = h + tv_0 \sin\theta - \frac{1}{2}gt^2\,, \quad t \geq 0\,.$$

It is interesting to note that the trajectory is independent of the mass of the projectile. Light and heavy projectiles would follow the same parabolic trajectory, provided they are fired from the same position, at the same speed, and at the same angle of elevation. The height of the projectile relative to the ground is given by $z(t)$. The horizontal displacement is $y(t)$. Let $t_L > 0$ be the moment of time when the projectile lands; that is, when $t = t_L$, the height vanishes, $z(t_L) = 0$. A positive solution of this equation is

$$t_L = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g}\,.$$

The distance L traveled by the projectile in the horizontal direction until it lands is the *range*:

$$L = y(t_L) = t_L v_0 \cos\theta\,.$$

Consider the case when the projectile is fired from the ground, $h = 0$. Then the travel time of the projectile and the range are:

$$h = 0 \quad \Rightarrow \quad t_L = \frac{2v_0 \sin\theta}{g}\,, \quad L = \frac{v_0^2 \sin(2\theta)}{g}\,.$$

The range attains its maximal value v_0^2/g when the projectile is fired at an angle of elevation $\theta = \pi/4$. The angle of elevation at which the projectile hits a target at a given range $L = L_0$ is

$$\theta = \frac{1}{2}\sin^{-1}\Big(\frac{L_0 g}{v_0^2}\Big)\,.$$

Note that this relation makes sense only if $L_0 g/v_0^2 \le 1$. It is impossible to hit the target at a range that exceeds the maximal range, $L_0 > v_0^2/g$.

If $h \neq 0$, the angle at which $L = L(\theta)$ attains its maximal values can be found by solving the equation $L'(\theta) = 0$, which defines critical points of the function $L(\theta)$. The angle of elevation at which the projectile hits a target at a given range is found by solving the equation $L(\theta) = L_0$. The technicalities are left to the reader.

Remark. In reality, the trajectory of a projectile deviates from a parabola because there is an additional force acting on a projectile moving in the atmosphere, the friction force. The friction force depends on the velocity of the projectile. A wind creates an additional force, a drag force (a projectile is dragged in the direction of the wind). So a more accurate analysis of the projectile motion in the atmosphere requires methods of differential equations.

12.3. Study Problems.

Problem **12.1**. *The acceleration of a particle is* $\mathbf{a} = \langle 2, 6t, 0\rangle$. *Find the position vector of the particle and its velocity in two units of time t if the particle was initially at the point* $(-1, -4, 1)$ *and had the velocity* $\langle 0, 2, 1\rangle$.

SOLUTION: The velocity vector is

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 2t, 3t^2, 0\rangle + \mathbf{c}\,.$$

The constant vector $\mathbf{c}$ is fixed by the initial condition $\mathbf{v}(0) = \langle 0, 2, 1\rangle$, which yields $\mathbf{c} = \langle 0, 2, 1\rangle$. Thus, $\mathbf{v}(t) = \langle 2t, 3t^2 + 2, 1\rangle$ and $\mathbf{v}(2) = \langle 4, 14, 1\rangle$. The position vector is

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \langle t^2, t^3 + 2t, t\rangle + \mathbf{c}\,.$$

Here the constant vector $\mathbf{c}$ is determined by the initial condition $\mathbf{r}(0) = \langle -1, -4, 1\rangle$, which yields $\mathbf{c} = \langle -1, -4, 1\rangle$. Thus, $\mathbf{r}(t) = \langle t^2-1, t^3+2t-4, t+1\rangle$ and $\mathbf{r}(2) = \langle 3, 8, 3\rangle$. Aternatively, the problem can also be solved using Eqs. (**12.2**):

$$\begin{aligned}\mathbf{v}(t) &= \langle 0, 2, 1\rangle + \int_0^t \mathbf{a}(u)du = \langle 0, 2, 1\rangle + \int_0^t \langle 2, 6u, 0\rangle du = \langle 2t, 3t^2 + 2, 1\rangle\,,\\ \mathbf{r}(t) &= \langle -1, -4, 1\rangle + \int_0^t \mathbf{v}(u)du = \langle t^2 - 1, t^3 + 2t - 4, t + 1\rangle\,.\end{aligned}$$

□

Problem **12.2**. *Show that if the velocity and position vectors of a particle remains orthogonal during the motion, then the trajectory lies on a sphere.*

SOLUTION: If $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal, then

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$$

for all t. On the other hand

$$(\mathbf{r}(t) \cdot \mathbf{r}(t))' = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0\,.$$

Therefore $\mathbf{r}(t) \cdot \mathbf{r}(t) = R^2 = \text{const}$ or $\|\mathbf{r}(t)\| = R$ for all t; that is, the particle remains at a fixed distance R from the origin all the time. □

Problem **12.3**. *A charged particle moving in a magnetic field $\mathbf{B}$ is subject to the Lorentz force $\mathbf{F} = (e/c)\mathbf{v} \times \mathbf{B}$, where e is the electric charge of the particle and c is the speed of light in vacuum. Assume that the magnetic field is a constant vector parallel to the z axis and the initial velocity is $\mathbf{v}(0) = \langle v_\perp, 0, v_\parallel\rangle$ (here $v_\perp$ and $v_\parallel$ are the components of the initial velocity in the direction perpendicular and parallel to the magnetic field, respectively). Show by verifying Newton's second law $m\mathbf{r}''(t) = \mathbf{F}(t)$ that the trajectory is a helix:*

$$\mathbf{r}(t) = \langle R\sin(\omega t),\, R\cos(\omega t),\, v_\parallel t\rangle, \quad \omega = \frac{eB}{mc}, \quad R = \frac{v_\perp}{\omega},$$

where $B = \|\mathbf{B}\|$ is the magnitude of the magnetic field and m is the particle mass.

SOLUTION: Newton's second law reads

$$m\mathbf{v}' = \frac{e}{c}\,\mathbf{v} \times \mathbf{B}.$$

Put $\mathbf{B} = \langle 0, 0, B\rangle$. Then

$$\begin{aligned} \mathbf{v} &= \mathbf{r}' = \langle \omega R\cos(\omega t), -\omega R\sin(\omega t), v_{\|}\rangle\,, \\ \mathbf{v}\times\mathbf{B} &= \langle -\omega RB\sin(\omega t), -\omega RB\cos(\omega t), 0\rangle, \\ \mathbf{v}' &= \langle -\omega^2 R\sin(\omega t), -\omega^2 R\cos(\omega t), 0\rangle = \frac{\omega}{B}\,\mathbf{v}\times\mathbf{B}\,. \end{aligned}$$

The substitution of these relations into Newton's second law yields

$$m\mathbf{v}' = \frac{e}{c}\,\mathbf{v}\times\mathbf{B} \quad\Rightarrow\quad \frac{m\omega}{B}\,\mathbf{v}\times\mathbf{B} = \frac{e}{c}\,\mathbf{v}\times\mathbf{B} \quad\Rightarrow\quad \omega = \frac{eB}{mc}\,.$$

Since $\mathbf{v}(0) = \langle \omega R, 0, v_{\|}\rangle = \langle v_\perp, 0, v_{\|}\rangle$, it follows that $R = v_\perp/\omega$. □

Remark. The rate at which the helix rises along the magnetic field is determined by the magnitude (speed) of the initial velocity component $v_{\|}$ parallel to the magnetic field, whereas the radius of the helix is determined by the magnitude of the initial velocity component $v_\perp$ perpendicular to the magnetic field. A particle makes one full turn about the magnetic field in time $T = 2\pi/\omega = 2\pi mc/(eB)$, that is, the larger the magnetic field, the faster the particle rotates about it. If $v_{\|} = 0$, then a charged particle is trapped by a magnetic field (it remains on a circular orbit). This effect is used in a device called a *cyclotron* to trap charged particles in a magnetic field. The magnetic field is made so that it has a constant magnitude but its direction changes slowly in space, remaining tangential to a circle. Even if $v_{\|} \neq 0$, the particle is winding about this circle and thus remains trapped is a spatial region that looks like a doughnut (think of a particle moving along a helix whose axis is a circle of radius that is larger than the radius of the helix). An accurate mathematical description of the motion of a charged particle in a cyclotron would require the methods of differential equations because the magnetic field is a function of position in space. This "trapping" effect of a magnetic field is responsible for a beautiful natural phenomenon known as *polar lights* (see Study Problem **15.4**).

12.4. Exercises.

1–7. Find the indefinite and definite integrals over specified intervals for each of the following functions:

1. $\mathbf{r}(t) = \langle 1,\ 2t,\ 3t^2\rangle,\ \ 0\le t\le 2\,;$
2. $\mathbf{r}(t) = \langle \sin t,\ t^3,\ \cos t\rangle,\ \ -\pi\le t\le \pi\,;$
3. $\mathbf{r}(t) = \langle t^2,\ t\sqrt{1-t},\ \sqrt{t}\rangle,\ \ 0\le t\le 1\,;$
4. $\mathbf{r}(t) = \langle t\ln t,\ t^2,\ e^{2t}\rangle,\ \ 0\le t\le 1\,;$
5. $\mathbf{r}(t) = \langle 2\sin t\cos t,\ 3\sin t\cos^2 t,\ 3\sin^2 t\cos t\rangle,\ \ 0\le t\le \pi/2\,;$
6. $\mathbf{r}(t) = \mathbf{a} + \cos(t)\mathbf{b},\ \ 0\le t\le\pi$, $\mathbf{a}$ and $\mathbf{b}$ are constant vectors;
7. $\mathbf{r}(t) = \mathbf{a}\times(\mathbf{u}'(t)+\mathbf{b}),\ \ 0\le t\le 1$, if $\mathbf{u}'(t)$ is continuous and $\mathbf{u}(0) = \mathbf{a}$ and $\mathbf{u}(1) = \mathbf{a}-\mathbf{b}$.

8–11. Find $\mathbf{r}(t)$ if the derivatives $\mathbf{r}'(t)$ and $\mathbf{r}(t_0)$ are given:

8. $\mathbf{r}'(t) = \langle 1,\ 2t,\ 3t^2\rangle,\quad \mathbf{r}(0) = \langle 1,2,3\rangle\,;$

9. $\mathbf{r}'(t) = \langle t-1,\ t^2,\ \sqrt{t}\rangle, \quad \mathbf{r}(1) = \langle 1,0,1\rangle$;
10. $\mathbf{r}'(t) = \langle \sin(2t)\ , 2\cos t,\ \sin^2 t\rangle, \quad \mathbf{r}(\pi) = \langle 1,2,3\rangle$;
11. $\mathbf{r}'(t) = \langle 2t,\ e^t,\ 4t^3\rangle\ , \quad \mathbf{r}(0) = \langle 1,\ 3,\ 0\rangle$.

12–14. Find the solution $\mathbf{r}(t)$ of each of the following initial value problems:

12. $\mathbf{r}''(t) = \langle 0,\ 2,\ 6t\rangle, \quad \mathbf{r}(0) = \langle 1,2,3\rangle, \quad \mathbf{r}'(0) = \langle 1,0,-1\rangle$;
13. $\mathbf{r}''(t) = \langle t^{1/3},\ t^{1/2},\ 6t\rangle, \quad \mathbf{r}(1) = \langle 1,0,-1\rangle, \quad \mathbf{r}'(0) = \langle 1,2,0\rangle$;
14. $\mathbf{r}''(t) = \langle -\sin t,\ \cos t,\ 1/t\rangle, \quad \mathbf{r}(\pi) = \langle 1,-1,0\rangle, \quad \mathbf{r}'(\pi) = \langle -1,0,2\rangle$.

15. Solve the equation $\mathbf{r}''(t) = \mathbf{a}$ where $\mathbf{a}$ is a constant vector if $\mathbf{r}(0) = \mathbf{b}$ and $\mathbf{r}(t_0) = \mathbf{c}$ for some $t = t_0 \neq 0$.

16. Find the most general vector function whose n^{th} derivative vanishes, $\mathbf{r}^{(n)}(t) = \mathbf{0}$, in an interval.

17. Show that a continuously differentiable vector function $\mathbf{r}(t)$ satisfying the equation $\mathbf{r}''(t) \times \mathbf{r}'(t) = \mathbf{0}$, where $\mathbf{r}'(t)$ is never zero, traverses a straight line (or a part of it).

18. If a particle was initially at point $(1,2,1)$ and had velocity $\mathbf{v} = \langle 0,1,-1\rangle$, find the position vector of the particle after it has been moving with acceleration $\mathbf{a}(t) = \langle 1,0,t\rangle$ for 2 units of time.

19. A particle of unit mass moves under a constant force $\mathbf{F}$. If a particle was initially at the point $\mathbf{r}_0$ and passed through the point $\mathbf{r}_1$ after 2 units of time, find the initial velocity of the particle. What was the velocity of the particle when it passed through $\mathbf{r}_1$?

20. A particle of mass of 1 kg was initially at rest. Then during 2 seconds a constant force of magnitude of 3 N was applied to the particle in the direction of $\langle 1,2,2\rangle$. How far is the particle from its initial position in 4 seconds?

21. The velocity of a particle is $\mathbf{v}(t) = \langle 2t, 5, 2t-16\rangle$. Find its position $\mathbf{r}(t)$ when the speed of the particle is minimal if $\mathbf{r}(0) = \mathbf{0}$.

22. A projectile is fired from the ground at an initial speed of 400 m/s and at an angle of elevation of 30°. Find the range of the projectile, the maximum height reached, and the speed at impact.

23. A ball of mass m is thrown southward into the air at an initial speed of v_0 at an angle of θ to the ground. An east wind applies a steady force of magnitude F to the ball in a westerly direction. Find the trajectory of the ball. Where does the ball land and at what speed? Find the deviation of the impact point from the impact point A when no wind is present. Is there any way to correct the direction and the initial speed in which the ball is thrown so that the ball still hits A? Is it possible to achieve the goal only by adjusting the direction, while keeping the initial speed fixed?

24. A rocket burns its on-board fuel while moving through space. Let $\mathbf{v}(t)$ and $m(t)$ be the velocity and mass of the rocket at time t. It can be shown that the force exerted by the rocket jet engines is $m'(t)\mathbf{v}_g$, where $\mathbf{v}_g$ is the velocity of the exhaust gases relative to the rocket. Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln(m(0)/m(t))\mathbf{v}_g$. The rocket is to accelerate in a straight line from rest to twice the speed of its own exhaust gases. What fraction of its initial mass would the rocket have to burn as fuel?

25. The acceleration of a projectile is $\mathbf{a}(t) = \langle 0, 2, 6t\rangle$. The projectile is shot from $(0, 0, 0)$ with an initial velocity $\mathbf{v}(0) = \langle 1, -2, -10\rangle$. It is supposed to destroy a target located at $(2, 0, -12)$. The target can be destroyed if the projectile's speed is at least 3.1 at impact. Will the target be destroyed?

13. Arc Length of a Curve

Let C be a simple curve in space. Then there is a simple parameterization $\mathbf{r}(t)$, $a \le t \le b$, of C. The vector function $\mathbf{r}(t)$ traverses the curve C and establishes a one-to-one correspondence between points of C and points of the interval $[a, b]$ (except possibly for the case when C is closed, $\mathbf{r}(a) = \mathbf{r}(b)$), i.e., the end points of the interval correspond to the same point of the curve if the curve is closed). Consider a partition of the interval $[a, b]$,

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b\,.$$

This partition induces a *partition of the curve* which is a collection of points of C, P_k, $k = 0, 1, ..., N$, whose position vectors are $\mathbf{r}(t_k)$. In particular, P_0 and P_N are the end points of the curve (see Fig. **13.1** (left panel)). If the curve is closed, then $P_0 = P_N$. Let $D_N = \max_k(t_k - t_{k-1})$ be the maximal length among all the partition intervals. A partition is said to be *refined* if $D_{N'} < D_N$ for $N' > N$. Clearly it is possible to refine a partition so that $D_N \to 0$ as $N \to \infty$. Note that for each partition there is at least one partition interval whose length is D_N. Therefore a refinement is obtained by adding a partition point in each partition interval whose length is D_N (e.g., by adding the midpoint of each interval of length D_N to the partition). So by taking N large enough D_N can be made smaller than any preassigned positive number. So it will always be assumed that *under a refinement of a partition, $D_N \to 0$ as $N \to \infty$.*

Definition **13.1**. (Arc Length of a Curve).
Let C be a simple curve in space. Let a collection of points P_k be a partition of C, $k = 0, 1, ..., N$, and $|P_{k-1}P_k|$ be the distance between two neighboring partition points ($P_0 = P_N$ if the curve is closed). The arc length of a curve C is the limit

$$L = \lim_{N\to\infty} \sum_{k=1}^{N} |P_{k-1}P_k|,$$

where the partition is refined as $N \to \infty$, provided it exists and is independent of the choice of partition. If $L < \infty$, the curve is called measurable *or* rectifiable.

The geometrical meaning of this definition is rather simple. Here the sum of $|P_{k-1}P_k|$ is the length of a polygonal path with vertices at P_0, P_1,..., P_N in this order. As the partition becomes finer and finer, this polygonal path approaches the curve more and more closely (see Figure **13.1**, left panel). In certain cases, the arc length is given by the Riemann integral.

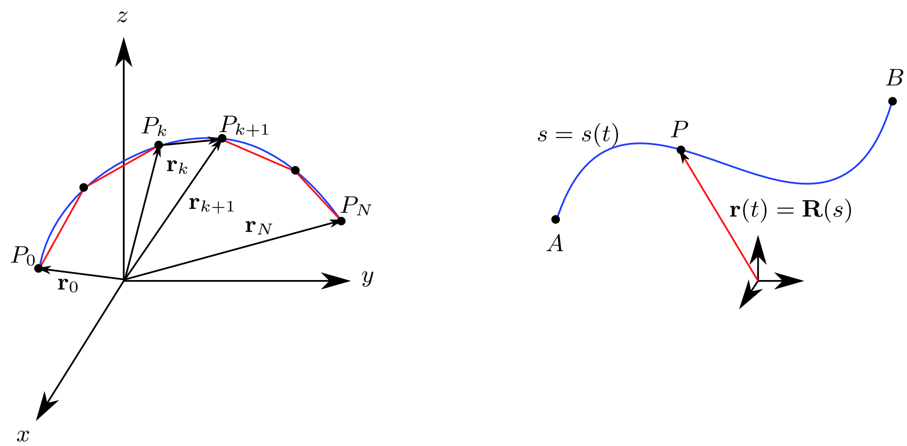

FIGURE **13.1**. **Left**: The arc length of a curve is defined as the limit of the sequence of lengths of polygonal paths through partition points of the curve. **Right**: Natural parameterization of a curve. Given a point A of the curve, the arc length s is counted from it to any point P of the curve. The position vector of P is a vector $\mathbf{R}(s)$. If the curve is traced out by another vector function $\mathbf{r}(t)$, then there is a relation $s = s(t)$ such that $\mathbf{r}(t) = \mathbf{R}(s(t))$.

THEOREM **13.1**. (Arc Length of a Curve).
Suppose that a curve C has a simple, continuously differentiable parameterization $\mathbf{r}(t)$, $a \le t \le b$. Then

$$L = \int_a^b \|\mathbf{r}'(t)\|\, dt\,.$$

A complete proof of the theorem goes beyond the scope of the course and is given in advanced calculus courses (it requires the concept of *uniform continuity*). Nevertheless the result may be understood from the following consideration. By the hypothesis, the curve C is simple. Hence, given a partition t_k of $[a, b]$ such that $t_0 = a < t_1 < \cdots < t_{N-1} < t_N = b$, there is a unique polygonal path with vertices P_k on C whose length is

$$L_N = \sum_{k=1}^{N} |P_{k-1}P_k| = \sum_{k=1}^{N} \|\mathbf{r}_k - \mathbf{r}_{k-1}\|\,.$$

where $\mathbf{r}_k = \mathbf{r}(t_k)$. Put $\Delta t_k = t_k - t_{k-1} > 0$, $k = 1, 2, ..., N$. Under a refinement of the partition $D_N = \max_k \Delta t_k \to 0$ as $N \to \infty$ and therefore $\Delta t_k \to 0$ for all k as $N \to \infty$. By the second hypothesis, $\mathbf{r}(t)$ traversing C is differentiable. Put $\mathbf{r}'_{k-1} = \mathbf{r}'(t_{k-1})$. The differentiability of $\mathbf{r}(t)$ implies that

$$\mathbf{r}_k - \mathbf{r}_{k-1} = \mathbf{r}'_{k-1}\,\Delta t_k + \mathbf{u}_k\,\Delta t_k$$

(see (**11.2**)). By the triangle inequality (**3.4**),

$$\|\mathbf{r}'_{k-1}\|\Delta t_k - \|\mathbf{u}_k\|\Delta t_k \le \|\mathbf{r}_k - \mathbf{r}_{k-1}\| \le \|\mathbf{r}'_{k-1}\|\Delta t_k + \|\mathbf{u}_k\|\Delta t_k.$$

The lower and upper bounds for the length of the polygonal path are obtained by taking the sum over k in this inequality:

$$\sum_{k=1}^{N} \|\mathbf{r}'_{k-1}\|\Delta t_k - E_N \le L_N \le \sum_{k=1}^{N} \|\mathbf{r}'_{k-1}\|\Delta t_k + E_N\,, \quad E_N = \sum_{k=1}^{N} \|\mathbf{u}_k\|\Delta t_k\,.$$

By the third hypothesis, the derivative $\mathbf{r}'(t)$ is continuous and so is its norm $\|\mathbf{r}'(t)\|$. A continuous function on $[a, b]$ is integrable. Therefore its Riemann sum converges:

$$\sum_{k=1}^{N} \|\mathbf{r}'_{k-1}\|\Delta t_k \to \int_a^b \|\mathbf{r}'(t)\|\, dt = L \quad \text{as} \quad N \to \infty\,.$$

under a refinement of a partition for any choice of sample points (in this case sample points are the left endpoints of partition intervals $\|\mathbf{r}'_{k-1}\| = \|\mathbf{r}'(t_{k-1})\|$). The conclusion of the theorem follows from the squeeze principle: $L_N \to L$ as $N \to \infty$ if

$$\lim_{N\to\infty} E_N = \lim_{N\to\infty} \sum_{k=1}^{N} \|\mathbf{u}_k\|\Delta t_k = 0\,.$$

Put $M_N = \max_k \|\mathbf{u}_k\|$ (the largest $\|\mathbf{u}_k\|$ for a given partition size N). Then

$$E_N = \sum_{k=1}^{N} \|\mathbf{u}_k\|\Delta t_k \le M_N \sum_{k=1}^{N} \Delta t_k = M_N(b-a)\,.$$

So, it is sufficient to show that $M_N \to 0$ as $N \to \infty$. In other words, all $\|\mathbf{u}_k\| \le M_N$ converge to zero *uniformly* under a refinement of the partition. By Eq. (**11.2**), $\|\mathbf{u}_k\|$ converges to 0 as $t_k \to t_{k-1}$ for a fixed k. So, intuitively M_N should converge to 0 because $t_k \to t_{k-1}$ (or $\Delta t_k \to 0$) for all k under a refinement of the partition. This conclusion can indeed be rigorously established: If $\mathbf{r}'(t)$ is continuous on a *closed* interval $[a, b]$, then $0 \le \|\mathbf{u}_k\| \le M_N \to 0$ as $N \to \infty$.

A few remarks on the use of Theorem **13.1** are in order. The length of a curve that has a simple, continuously differentiable parameterization $\mathbf{r}(t)$ on an *infinite* interval is defined as an improper integral. For example, if $a \le t < \infty$, then

$$L = \lim_{b\to\infty} \int_a^b \|\mathbf{r}'(t)\|\, dt\,.$$

The curve has a finite length if the integral converges ($L < \infty$).

The length is additive. So, if a curve C can be partitioned into finitely many simple pieces and each piece admits a simple, continuously differentiable parameterization, then the length of each piece can be found by

Theorem **13.1** and the length of C is the sum of lengths of each piece. This observation allows us to use Theorem **13.1** to compute the length of curves that are not simple.

Furthermore let $\mathbf{r}(t)$ be continuously differentiable on $[a, b]$ but does not necessarily define a one-to-one correspondence with its range C. Then the integral $\int_a^b \|\mathbf{r}'(t)\|\,dt$ exists but may not coincide with the length of the curve C as a point set in space because $\mathbf{r}(t)$ may traverse C or parts of C several times. Nevertheless, the value of the integral may be useful in applications. Suppose $\mathbf{r}(t)$ is a trajectory of a particle. Then its velocity is $\mathbf{v}(t) = \mathbf{r}'(t)$ and its speed is $v(t) = \|\mathbf{v}(t)\|$. The distance traveled by the particle in the time interval $[a, b]$ is given by

$$D = \int_a^b v(t)\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt.$$

If a particle travels along the same space curve (or some of its parts) several times, then the distance traveled does not coincide with the arc length L of the curve, $D \geq L$.

Example **13.1**. *Find the arc length of the curve* $\mathbf{r}(t) = \langle t^2, 2t, \ln t\rangle$, $1 \leq t \leq 2$.

Solution: The derivative $\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$ is continuous on $[1, 2]$. Its norm is

$$\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\Big(2t + \frac{1}{t}\Big)^2} = 2t + \frac{1}{t}\,.$$

Therefore, by Theorem **13.1**,

$$L = \int_1^2 \|\mathbf{r}'(t)\|dt = \int_1^2 \Big(2t + \frac{1}{t}\Big)dt = t^2\Big|_1^2 + \ln t\Big|_1^2 = 3 + \ln 2\,.$$

□

Example **13.2**. *Find the arc length of one turn of a helix of radius R that rises by h per each turn.*

Solution: Let the helix axis be the z axis (see Study Problem **10.1**). The helix is traced out by the vector function

$$\mathbf{r}(t) = \langle R\cos t, R\sin t, th/(2\pi)\rangle\,.$$

One turn is in a one-to-one correspondence with the interval $t \in [0, 2\pi]$ (because $z(t) = th/(2\pi)$ is one-to-one). So $\mathbf{r}(t)$ is a simple, continuously differentiable parameterization of one turn of the helix. Therefore,

$$\|\mathbf{r}'(t)\| = \|\langle -R\sin t,\ R\cos t, h/(2\pi)\rangle\| = \sqrt{R^2 + (h/(2\pi))^2}\,.$$

So the norm of the derivative turns out be constant. The arc length is

$$L = \int_0^{2\pi} \|\mathbf{r}'(t)\|\,dt = \sqrt{R^2 + (h/(2\pi))^2}\int_0^{2\pi} dt = \sqrt{(2\pi R)^2 + h^2}\,.$$

This result is rather easy to obtain without calculus. The helix lies on a cylinder of radius R. If the cylinder is cut parallel to its axis and unfolded into a strip, then one turn of the helix becomes the hypotenuse of the right-angled triangle with catheti $2\pi R$ and h. The result follows from the Pythagorean theorem. This consideration also shows that the length does not depend on whether the helix winds about its axis clockwise or counterclockwise. □

13.1. Reparameterization of a Curve. In Section **10.3**, it was shown that a space curve can be traversed by different vector functions. These vector functions are different parameterizations of the same curve. For example, a semicircle of radius R is traversed by the vector functions

$$\mathbf{r}(t) = \langle R\cos t,\ R\sin t,\ 0\rangle\,,\quad t\in[0,\pi]\,,$$
$$\mathbf{R}(u) = \langle u,\ \sqrt{R^2-u^2},\ 0\rangle\,,\quad u\in[-R,R]\,.$$

They are related to one another by the composition rule:

$$\mathbf{R}(u) = \mathbf{r}(t(u))\,,\quad t(u)=\cos^{-1}(u/R)\quad\text{or}$$
$$\mathbf{r}(t) = \mathbf{R}(u(t))\,,\quad u(t)=R\cos t\,.$$

This example illustrates the concept of a *reparameterization* of a curve. A reparameterization of a curve is a change of the parameter that labels points of the curve. It merely reflects a simple fact that there are many different vector functions which traverse the same space curve.

DEFINITION **13.2**. (Reparameterization of a Curve).
Let $\mathbf{r}(t)$ *traverse a curve* C *if* $t\in[a,b]$. *Let* $g(u)$ *be a continuous one-to-one function on an interval* $[a',b']$ *whose range is the interval* $[a,b]$, *i.e., for every* $a\le t\le b$ *there is just one* $a'\le u\le b'$ *such that* $t=g(u)$ *and vice versa. The vector function* $\mathbf{R}(u)=\mathbf{r}(g(u))$ *is called a* reparameterization *of* C.

The geometrical properties of the curve (e.g., its shape or length) do not depend on a parameterization of the curve because the vector functions $\mathbf{r}(t)$ and $\mathbf{R}(u)$ have the *same* range. A reparameterization of a curve is a technical tool to find parametric equations of the curve convenient for particular applications.

13.2. A Natural Parameterization of a Smooth Curve. Suppose one is traveling along a highway from town A to town B and comes upon an accident. How can the location of the accident be reported to the police? If one has a GPS navigator, one can report coordinates on the surface of the Earth. This implies that the police should use a specific (GPS) coordinate system to locate the accident. Is it possible to avoid any reference to a coordinate system? A simpler way to define the position of the accident is to report the distance traveled from A along the highway to the point where the accident happened (by using, e.g., mile markers). No coordinate system is needed to uniquely label all points of the highway by specifying the distance from a particular point A to the point of interest along the highway. This observation can be extended to all smooth curves (see Figure **13.1**, right panel).

DEFINITION **13.3**. (Natural or Arc Length Parameterization).
Let C be a smooth curve of length L. Let $\mathbf{r}(t)$, $t \in [a,b]$, be a simple, continuously differentiable parameterization of C such that $\mathbf{r}'(t) \neq \mathbf{0}$. Then the arc length $s = s(t)$ of the portion of the curve between $\mathbf{r}(a)$ and $\mathbf{r}(t)$ is a function of the parameter t:

$$s = s(t) = \int_a^t \|\mathbf{r}'(u)\| du\,, \quad 0 \le s \le L\,.$$

The vector function $\mathbf{R}(s) = \mathbf{r}(t(s))$ is called a natural or arc length parameterization *of C, where $t(s)$ is the inverse function of $s(t)$.*

For a smooth curve, the function $\mathbf{r}(t)$ is continuously differentiable and, hence, $\|\mathbf{r}'(t)\|$ is continuous on $[a,b]$. Therefore, the derivative $s'(t)$ exists and is obtained by differentiating the integral with respect to its upper limit:

$$s'(t) = \frac{d}{dt}\int_a^t \|\mathbf{r}'(u)\| du = \|\mathbf{r}'(t)\| > 0\,,$$

which is possible because the integrand $\|\mathbf{r}'(u)\|$ is continuous on $[a,b]$. The derivative $s'(t)$ is strictly positive because $\mathbf{r}'(t) \neq \mathbf{0}$ for a smooth curve. The existence of the inverse function of $s(t)$ is guaranteed by the inverse function theorem proved in Calculus I:

THEOREM **13.2**. (Inverse Function Theorem).
Let $s(t)$, $a \le t \le b$, have a continuous derivative such that $s'(t) > 0$ for $a < t < b$. Then there exists an inverse differentiable function $t = t(s)$, $c < s < d$, and $t'(s) = 1/s'(t)$, where $t = t(s)$ on the right side.

Thus, the condition $s'(t) = \|\mathbf{r}'(t)\| > 0$ guarantees the existence of a one-to-one correspondence between the variables s and t and the existence of the differentiable inverse function $t = t(s)$. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be parametric equations of a smooth curve C. Then the parametric equations of C in the natural parameterization have the form

$$\mathbf{R}(s) = \langle x(t(s)),\, y(t(s)),\, z(t(s))\rangle.$$

EXAMPLE **13.3**. *Reparameterize the helix from Example **13.2** $\mathbf{r}(t) = \langle R\cos t, R\sin t, th/(2\pi)\rangle$ with respect to arc length measured from the point $(R,0,0)$ in the direction of increasing t.*

SOLUTION: The point $(R,0,0)$ corresponds to $t = 0$. Then

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \frac{L}{2\pi}\int_0^t du = \frac{Lt}{2\pi} \quad \Rightarrow \quad t(s) = \frac{2\pi s}{L}$$

where $L = \sqrt{(2\pi R)^2 + h^2}$ is the arc length of one turn of the helix (see Example **13.2**). Therefore

$$\mathbf{R}(s) = \mathbf{r}(t(s)) = \langle R\cos(2\pi s/L),\ R\sin(2\pi s/L),\ hs/L\rangle$$

In particular, $\mathbf{R}(0) = \langle R, 0, 0\rangle$ and $\mathbf{R}(L) = \langle R, 0, h\rangle$ are the position vector of the end points of one turn of the helix as required. □

Example **13.4**. *Find the coordinates of a point P that is $5\pi/3$ units of length away from the point $(4, 0, 0)$ along the helix $\mathbf{r}(t) = \langle 4\cos(\pi t), 4\sin(\pi t), 3\pi t\rangle$. Is there only one such point of the helix?*

Solution: If $\mathbf{R}(s)$ is the natural parameterization of the helix where s is counted from the point $(4, 0, 0)$, then the position vector of the point in question is given by $\mathbf{R}(5\pi/3)$. Thus, the first task is to find $\mathbf{R}(s)$. One has

$$\mathbf{r}'(u) = \langle -4\pi\sin(\pi u), 4\pi\cos(\pi u), 3\pi\rangle \quad \Rightarrow \quad \|\mathbf{r}'(u)\| = 5\pi.$$

The initial point of the helix corresponds to $t = 0$. So the arc length counted from $(4, 0, 0)$ as a function of t is

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t 5\pi\, du = 5\pi t \quad \Rightarrow \quad t(s) = \frac{s}{5\pi}\,.$$

The natural parameterization reads

$$\mathbf{R}(s) = \mathbf{r}(t(s)) = \langle 4\cos(s/5),\ 4\sin(s/5),\ 3s/5\rangle.$$

The position vector of P is $\mathbf{R}(5\pi/3) = \langle 2, 2\sqrt{3}, \pi\rangle$. There are two points of the helix at the specified distance from $(4, 0, 0)$ because the arclength can be counted in two directions from a given point. Note that $s(t)$ defined above is the arc length parameter counted from $(4, 0, 0)$ in the direction of *increasing* t (upward along the helix, $t > 0$). Accordingly, $s(t)$ can also be counted in the direction of *decreasing* t (downward along the helix, $t < 0$). In the latter case, $s(t) = -5\pi t > 0$. Hence, the position vector of the other point is $\mathbf{R}(-5\pi/3) = \langle 2, -2\sqrt{3}, -\pi\rangle$. □

It follows from Theorem **13.2** that *the derivative of a vector function that traverses a smooth curve C with respect to the natural parameter, the arclength, is a* unit *tangent vector to the curve.* Indeed, by the chain rule applied to the components of the vector function:

$$\begin{aligned}\frac{d\mathbf{r}(t)}{ds} &= \Big\langle \frac{dx(t)}{ds}, \frac{dy(t)}{ds}, \frac{dx(t)}{ds}\Big\rangle = \langle x'(t)t'(s),\ y'(t)t'(s),\ z'(t)t'(s)\rangle \\ &= t'(s)\langle x'(t),\ y'(t),\ z'(t)\rangle = \frac{1}{s'(t)}\,\mathbf{r}'(t) = \frac{1}{\|\mathbf{r}'(t)\|}\,\mathbf{r}'(t) = \hat{\mathbf{T}}(t)\end{aligned}$$

Thus, for a natural parameterization $\mathbf{r}(s)$ of a smooth curve C, the derivative $\mathbf{r}'(s)$ is a unit tangent vector to C, $\|\mathbf{r}'(s)\| = 1$. In Example **13.3**, a natural parameterization of a helix of radius R with one turn of a length L was obtained. The derivative of the corresponding vector function reads:

$$\begin{aligned}\mathbf{R}'(s) &= \frac{d}{dt}\langle R\cos(2\pi s/L),\ R\sin(2\pi s/L),\ hs/L\rangle \\ &= \langle -(2\pi R/L)\sin(2\pi s/L),\ (2\pi R/L)\cos(2\pi s/L),\ h/L\rangle \\ \|\mathbf{R}'(s)\| &= \sqrt{(2\pi R/L)^2 + (h/L)^2} = 1\,,\end{aligned}$$

where the fundamental trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ and the relation $L = \sqrt{(2\pi R)^2 + h^2}$ were used.

By definition, the arc length is independent of a parameterization of a space curve. For smooth curves, this can also be established through the change of variables in the integral that determines the arc length. Indeed, let $\mathbf{r}(t)$, $t \in [a, b]$, be a simple, continuously differentiable parameterization of a curve C of length L. Consider the change of the integration variable $t = t(s)$, $s \in [0, L]$. Then by the inverse function theorem $s'(t) = \|\mathbf{r}'(t)\|$ and $ds = s'(t)\,dt = \|\mathbf{r}'(t)\|\,dt$. Thus,

$$\int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^L ds = L$$

for any parameterization of the curve C satisfying the hypotheses of Theorem **13.1**.

13.3. Study problems.

Problem **13.1**. *Find the arclength of the part of the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the cylindrical surface $y = x\tan\sqrt{x^2 + y^2}$ from the origin to the point $(0, \pi/2, \pi/2)$. Hint: Use polar coordinates to obtain a parameterization of the curve.*

SOLUTION: Let $x = r\cos\theta$ and $y = r\sin\theta$. In the polar coordinates the equation of the cylindrical surface is simplified to $\tan\theta = \tan r$ or $r = \theta$. The latter is a polar graph that describes a spiral unwinding from the origin counterclockwise. The spiral is the cross section of the cylindrical surface in the xy plane. Using θ as a parameter so that $z = r = \theta$, the curve of intersection is traversed by the vector function

$$\mathbf{r}(\theta) = \langle \theta\cos\theta,\ \theta\sin\theta,\ \theta\rangle = \theta\langle\cos\theta,\ \sin\theta,\ 1\rangle = \theta\hat{\mathbf{u}}(\theta) + \theta\hat{\mathbf{e}}_3\,,$$

where $\hat{\mathbf{u}}(\theta) = \langle\cos\theta,\ \sin\theta,\ 0\rangle$ and $0 \le \theta \le \pi/2$ in order to select the part of the curve in question (note that $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}(\pi/2) = \langle 0, \pi/2, \pi/2\rangle$). This parameterization is simple (because $z(\theta) = \theta$ is one-to-one) and continuously differentiable. By the product rule

$$\mathbf{r}'(\theta) = \hat{\mathbf{u}}(\theta) + \theta\hat{\mathbf{u}}'(\theta) + \hat{\mathbf{e}}_3\,.$$

The vectors $\hat{\mathbf{u}}(\theta)$, $\hat{\mathbf{u}}'(\theta) = \langle -\sin\theta,\ \cos\theta,\ 0\rangle$, and $\hat{\mathbf{e}}_3$ are mutually orthogonal unit vectors. Therefore $\|\mathbf{r}'(\theta)\|^2 = 1 + \theta^2 + 1 = 2 + \theta^2$. The arclength is

$$L = \int_0^{\frac{\pi}{2}} \sqrt{2+\theta^2}\, d\theta = \ln\left(\frac{\theta}{\sqrt{2}} + \sqrt{1+\frac{\theta^2}{2}}\right) + \frac{\theta}{\sqrt{2}}\sqrt{1+\frac{\theta^2}{2}}\,\Bigg|_0^{\frac{\pi}{2}}$$
$$= \ln\left(\frac{\pi}{2\sqrt{2}} + \sqrt{1+\frac{\pi^2}{8}}\right) + \frac{\pi}{2\sqrt{2}}\sqrt{1+\frac{\pi^2}{8}}\,.$$

The integration can be carried out using the substitution

$$\theta = \sqrt{2}\sinh t\,, \quad d\theta = \sqrt{2}\cosh t dt\,, \quad \sqrt{2+\theta^2} = \sqrt{2}\cosh t\,.$$

Then

$$\int \sqrt{2+\theta^2}\, d\theta = 2\int \cosh^2(t)\, dt = \int (1+\cosh(2t))\, dt$$
$$= t + \frac{1}{2}\sinh(2t) = t + \sinh(t)\cosh(t)$$
$$= t + (\theta/\sqrt{2})\sqrt{1+\theta^2/2}$$

The variable t is expressed via θ by solving the quadratic equation in e^t:

$$\sqrt{2}\,\theta = 2\sinh t = e^t - e^{-t} \quad \Rightarrow \quad e^t = \theta/\sqrt{2} + \sqrt{1+\theta^2/2}$$

and taking the natural logarithm of both sides of the latter relation. □

13.4. Exercises.

1–6. Find the arc length of each of the following parametric curves:

1. $\mathbf{r}(t) = \langle 3\cos t,\ 2t,\ 3\sin t\rangle$, $\ -2 \le t \le 2$;
2. $\mathbf{r}(t) = \langle 2t,\ t^3/3,\ t^2\rangle$, $\ 0 \le t \le 1$;
3. $\mathbf{r}(t) = \langle 3t^2,\ 4t^{3/2},\ 3t\rangle$, $\ 0 \le t \le 2$;
4. $\mathbf{r}(t) = \langle e^t,\ \sqrt{2}\,t,\ e^{-t}\rangle$, $\ -1 \le t \le 1$;
5. $\mathbf{r}(t) = \langle \cosh t,\ \sinh t,\ t\rangle$, $\ 0 \le t \le 1$;
6. $\mathbf{r}(t) = \langle \cos t - t\sin t, \sin t + t\cos t, t^2\rangle$, $\ 0 \le t \le 2\pi$; Hint: Find the decomposition $\mathbf{r}(t) = \mathbf{v}(t) + t\mathbf{w}(t) + t^2\hat{\mathbf{e}}_3$ where $\mathbf{v}$, $\mathbf{w}$, and $\hat{\mathbf{e}}_3$ are mutually orthogonal, and $\mathbf{v}'(t) = \mathbf{w}(t)$, $\mathbf{w}'(t) = -\mathbf{v}(t)$. Use the Pythagorean theorem to calculate $\|\mathbf{r}'(t)\|$.

7. Find the arc length of the curve $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, $0 \le t < \infty$. Hint: Put $\mathbf{r}(t) = e^{-t}\mathbf{u}(t)$, differentiate, show that $\mathbf{u}(t)$ is orthogonal to $\mathbf{u}'(t)$, and use the Pythagorean theorem to calculate $\|\mathbf{r}'(t)\|$.

8. Find the arc length of the portion of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ that lies inside the sphere $x^2 + y^2 + z^2 = 2$.

9. Find the arc length of the portion of the curve $\mathbf{r}(t) = \langle 2t, 3t^2, 3t^3\rangle$ that lies between the planes $z = 3$ and $z = 24$.

10. Find the arc length of the portion of the curve $\mathbf{r}(t) = \langle \ln t, t^2, 2t\rangle$ that lies between the points of intersection of the curve with the plane $y - 2z + 3 = 0$.

11. Let C be the curve of intersection of the surfaces $z^2 = 2y$ and $3x = yz$.

Find the length of C from the origin to the point $(36, 18, 6)$.

12–15. For each of the following curves defined by given equations with a parameter $a > 0$, find suitable parametric equations and evaluate the arc length between a given point A and and a generic point $B = (x_0, y_0, z_0)$:

12. $y = a\sin^{-1}(x/a)$, $z = (a/4)\ln[(a-x)/(a+x)]$, $A = (0,0,0)$;

13. $(x-y)^2 = a(x+y)$, $x^2 - y^2 = 9z^2/8$, $A = (0,0,0)$; Hint: Use new variables $u = x+y$ and $v = x-y$ to find parametric equations;

14. $x^2 + y^2 = az$, $y = x\tan(z/a)$, $A = (0,0,0)$; Hint: Use polar coordinates in the $xy-$plane to find parametric equations;

15. $x^2+y^2+z^2 = a^2$, $\sqrt{x^2+y^2}\cosh(\tan^{-1}(y/x)) = a$, $A = (a,0,0)$; Hint: Represent the second equation as a polar graph.

16–20. Reparameterize each of the following curves with respect to the arc length measured from the point where $t = 0$ in the direction of increasing t:

16. $\mathbf{r}(t) = \langle t,\ 1-2t,\ 5+3t\rangle$;

17. $\mathbf{r}(t) = \frac{2t}{t^2+1}\hat{\mathbf{e}}_1 + (\frac{2}{t^2+1} - 1)\hat{\mathbf{e}}_3$;

18. $\mathbf{r}(t) = \langle \cosh t,\ \sinh t,\ t\rangle$;

19. $x = a(t - \sin t)$, $y = a(1-\cos t)$, $z = 0$, $a > 0$, and $0 \le t \le 2\pi$;

20. $\mathbf{r}(t) = e^t\langle \cos t,\ \sin t,\ 1\rangle$.

21. A particle travels from the point $(R, 0, 0)$ into the positive quadrant along a helix of radius R that rises h units of length per turn. If the z axis is the symmetry axis of the helix, find the position vector of the particle after it travels the distance $4\pi R$.

22. A particle travels along a curve traversed by the vector function $\mathbf{r}(u) = \langle u, \cosh u,\ \sinh u\rangle$ from the point $(0, 1, 0)$ with a constant speed $\sqrt{2}$ m/s so that its x coordinate increases. Find the position of the particle in one second.

23. Let C be a smooth closed curve whose arclength is L. Let $\mathbf{r}(t)$, $a \le t \le b$, be a simple, continuously differentiable parameterization of C. Prove that there is a number $a \le t^* \le b$ such that $\|\mathbf{r}'(t^*)\| = L/(b-a)$. Hint: Recall the integral mean value theorem.

24. A particle travels in space a distance D in time T. Show that there is a moment of time $0 \le t \le T$ at which the speed of the particle coincides with the average speed D/T.

14. Curvature of a Space Curve

Consider two curves passing through a point P. Both curves bend at P. Which one bends more than the other and how much more? The answer to this question requires a numerical characterization of bending, that is, a number computed at P for each curve with the property that it becomes larger as the curve bends more. Naturally, this number should not depend on a parameterization of a curve. Suppose that a curve is smooth so that a unit tangent vector can be attached to every point of the curve. A straight line does not bend (does not "curve") so it has the same unit tangent vector at all its points. If a curve bends, then its unit tangent vector must change along the curve.

The position on the curve can be specified in a coordinate- and parameterization-independent way by the arclength s counted from a particular point of the curve. If $\hat{\mathbf{T}}(s)$ is the unit tangent vector as a function of s, then its derivative $\hat{\mathbf{T}}'(s)$ vanishes for a straight line (see Figure **14.1**), while this is not the case for a general smooth curve. However, $\hat{\mathbf{T}}'(s)$ is not a number and cannot serve as the thought-after numerical measure of bending. Let $\Delta\theta$ be the angle between $\hat{\mathbf{T}}(s)$ and $\hat{\mathbf{T}}(s_0)$ for some s_0. Since $\hat{\mathbf{T}}(s)$ and $\hat{\mathbf{T}}(s_0)$ have the same (unit) length, a non-vanishing derivative $\hat{\mathbf{T}}'(s_0)$ implies that the rate of change $\Delta\theta/\Delta s$, where $\Delta s = s - s_0$, cannot vanish too. Yet, by geometrical reasonings, the rate $\Delta\theta/\Delta s$ is expected to increase as the curve bends more at a point corresponding to $s = s_0$. So, the rate $\Delta\theta/\Delta s$ in the limit $s \to s_0$ might be a good numerical measure of bending of a curve at a point.

By definition,

$$\hat{\mathbf{T}}'(s_0) = \lim_{s\to s_0} \frac{\hat{\mathbf{T}}(s) - \hat{\mathbf{T}}(s_0)}{s - s_0}\,.$$

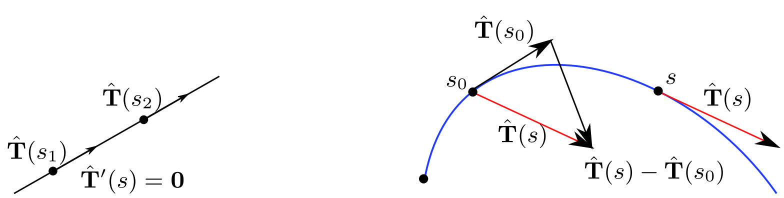

FIGURE **14.1**. **Left**: A straight line does not bend. The unit tangent vector has zero rate of change relative to the arc length parameter s. **Right**: Curvature of a smooth curve. The more a smooth curve bends, the larger the rate of change of the unit tangent vector relative to the arc length parameter becomes. So the magnitude of the derivative (curvature) $\|\hat{\mathbf{T}}'(s)\| = \kappa(s)$ can be taken as a geometrical measure of bending.

By continuity of the unit tangent vector

$$\Delta\theta \to 0 \quad \text{as} \quad \Delta s \to 0\,.$$

Since $\hat{\mathbf{T}}$ is a unit vector, $\|\hat{\mathbf{T}}\| = 1$ and $\hat{\mathbf{T}} \cdot \hat{\mathbf{T}} = 1$ and hence

$$(\hat{\mathbf{T}}(s) - \hat{\mathbf{T}}(s_0)) \cdot (\hat{\mathbf{T}}(s) - \hat{\mathbf{T}}(s_0)) = 2 - 2\cos\Delta\theta = 4\sin^2(\Delta\theta/2)$$

Recall that $\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ for any vector $\mathbf{a}$. Therefore the magnitude

$$\|\hat{\mathbf{T}}(s) - \hat{\mathbf{T}}(s_0)\| = 2\sin(\Delta\theta/2)$$

because $\Delta\theta \geq 0$. In the limit $\Delta s \to 0$, the sinus function can be approximated by its argument: $\sin x = x + O(x^3) = x(1 + O(x^2))$. Therefore

$$\|\hat{\mathbf{T}}'(s_0)\| = \lim_{\Delta s\to 0} \frac{2\sin(\Delta\theta/2)}{\Delta s} = \lim_{\Delta s\to 0} \frac{\Delta\theta}{\Delta s}\Big(1 + O(\Delta\theta^2)\Big) = \lim_{\Delta s\to 0} \frac{\Delta\theta}{\Delta s}\,,$$

and the number $\|\hat{\mathbf{T}}'(s_0)\|$ can be used as a numerical measure of bending (or *curvature*) of a curve at a point.

Definition **14.1**. (Curvature of a Smooth Curve).
Let C be a smooth curve and let its unit tangent vector $\hat{\mathbf{T}}(s)$ be a differentiable function of the arc length counted from a particular point of C. The number

$$\kappa(s) = \left\|\frac{d}{ds}\hat{\mathbf{T}}(s)\right\|$$

is called the curvature *of C at the point corresponding to the value s of the arc length.*

Let $\mathbf{r}(s)$ be the *natural parameterization* of a smooth curve (the parameter s is the arc length measured from a particular point on the curve). Then, as shown in the previous section, $\mathbf{r}'(s) = \hat{\mathbf{T}}(s)$, and therefore

$$\mathbf{r}'(s) = \mathbf{T}(s) \quad \Rightarrow \quad \kappa(s) = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|\,.$$

Example **14.1**. *Find the curvature of a helix of radius R that rises the distance h per turn.*

Solution: In Example **13.3**, the natural parameterization of the helix is obtained

$$\mathbf{r}(s) = \langle R\cos(2\pi s/L),\ R\sin(2\pi s/L),\ hs/L\rangle.$$

where $L = \sqrt{(2\pi R)^2 + h^2}$ is the arc length of one turn. Differentiating this vector function twice with respect to the arc length parameter s,

$$\begin{aligned}
\mathbf{r}''(s) &= \langle -(2\pi/L)^2 R\cos(2\pi s/L),\ -(2\pi/L)^2 R\sin(2\pi s/L),\ 0\rangle \\
&= -(2\pi/L)^2 R\,\langle\cos(2\pi s/L),\ \sin(2\pi s/L),\ 0\rangle\,, \\
\kappa(s) &= \|\mathbf{r}''(s)\| = (2\pi/L)^2 R = \frac{R}{R^2 + (h/2\pi)^2}\,,
\end{aligned}$$

where the relation $\|\langle\cos u,\ \sin u,\ 0\rangle\| = 1$ has been used. So the helix has a constant curvature. □

In practice, finding the natural parameterization of a smooth curve might be a tedious technical task. Therefore, a question of interest is to develop a method to calculate the curvature in any parameterization. Let $\mathbf{r}(t)$ be a vector function in $[a,b]$ that traces out a smooth curve C. The unit tangent vector as a function of the parameter t has the form

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\,.$$

So, to calculate the curvature as a function of t, the relation between the derivatives d/ds and d/dt has to be found. If $s = s(t)$ is the arc length as a function of t (see Definition **13.3**), then by the inverse function theorem (Theorem **13.2**), there exists an inverse differentiable function $t = t(s)$ that expresses the parameter t as a function of the arc length s and

$$\frac{dt(s)}{ds} = \frac{1}{(ds(t)/dt)} = \frac{1}{\|\mathbf{r}'(t)\|}\,.$$

By the chain rule:

$$\frac{d}{ds}\,\hat{\mathbf{T}} = \frac{dt}{ds}\,\frac{d}{dt}\,\hat{\mathbf{T}} = \frac{1}{\|\mathbf{r}'(t)\|}\,\frac{d}{dt}\,\hat{\mathbf{T}}$$

and therefore

$$\kappa(t) = \frac{\|\hat{\mathbf{T}}'(t)\|}{\|\mathbf{r}'(t)\|}\,. \tag{14.1}$$

Note that the existence of the curvature requires that $\mathbf{r}(t)$ be twice differentiable because $\hat{\mathbf{T}}(t)$ is proportional to $\mathbf{r}'(t)$. The ratio in Eq. (**14.1**) makes sense if $\mathbf{r}'(t) \neq \mathbf{0}$. Differentiation of the unit vector $\hat{\mathbf{T}}$ can sometimes be a rather technical task, too. The following theorem provides a more convenient way to calculate the curvature.

THEOREM **14.1**. (Curvature of a Curve).
Let a smooth curve be traced out by a twice-differentiable vector function $\mathbf{r}(t)$ *such that* $\mathbf{r}'(t) \neq \mathbf{0}$. *Then the curvature is*

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{14.2}$$

PROOF. Put $v(t) = \|\mathbf{r}'(t)\|$. With this notation,

$$\mathbf{r}'(t) = v(t)\hat{\mathbf{T}}(t)\,.$$

Differentiating both sides of this relation using the product rule, one infers

$$\mathbf{r}''(t) = v'(t)\hat{\mathbf{T}}(t) + v(t)\hat{\mathbf{T}}'(t) = \frac{v'(t)}{v(t)}\,\mathbf{r}'(t) + v(t)\hat{\mathbf{T}}'(t)\,. \tag{14.3}$$

Since the cross product of two parallel vectors vanishes, it follows from (**14.3**) that

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = v(t)\Big(\mathbf{r}'(t) \times \hat{\mathbf{T}}'(t)\Big)\,. \tag{14.4}$$

Now recall that $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ where θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. Therefore,

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = v(t)\|\mathbf{r}'(t)\times\hat{\mathbf{T}}'(t)\| = \|\mathbf{r}'(t)\|^2\|\hat{\mathbf{T}}'(t)\|\sin\theta, \tag{14.5}$$

where θ is the angle between $\hat{\mathbf{T}}'(t)$ and the tangent vector $\mathbf{r}'(t)$. Since $\hat{\mathbf{T}}(t)$ is a unit vector, one has $\|\hat{\mathbf{T}}(t)\|^2 = \hat{\mathbf{T}}(t)\cdot\hat{\mathbf{T}}(t) = 1$. By taking the derivative of both sides of the latter relation, it is concluded that the vectors $\hat{\mathbf{T}}'(t)$ and $\mathbf{r}'(t)$ are orthogonal:

$$\hat{\mathbf{T}}'(t)\cdot\hat{\mathbf{T}}(t) = 0 \;\Leftrightarrow\; \hat{\mathbf{T}}'(t)\perp\hat{\mathbf{T}}(t) \;\Leftrightarrow\; \hat{\mathbf{T}}'(t)\perp\hat{\mathbf{r}}'(t) \;\Leftrightarrow\; \theta = \frac{\pi}{2}$$

because $\mathbf{r}'(t)$ is parallel to $\hat{\mathbf{T}}(t)$. Hence, $\sin\theta = 1$. Then the claim (**14.2**) follows from (**14.1**) and (**14.5**):

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3\sin\theta} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}\,.$$

□

EXAMPLE **14.2**. *Find the curvature of the curve* $\mathbf{r}(t) = \langle \ln t, t^2, 2t\rangle$ *at the point* $P_0(0,1,2)$*.*

SOLUTION: The point P_0 corresponds to $t = 1$ because $\mathbf{r}(1) = \langle 0,1,2\rangle$ coincides with the position vector of P_0. Hence, one has to calculate $\kappa(1)$:

$$\begin{aligned}
\mathbf{r}'(1) &= \langle t^{-1},\ 2t,\ 2\rangle\Big|_{t=1} = \langle 1,2,2\rangle \quad\Rightarrow\quad \|\mathbf{r}'(1)\| = 3\,,\\
\mathbf{r}''(1) &= \langle -t^{-2},\ 2,\ 0\rangle\Big|_{t=1} = \langle -1,2,0\rangle\\
\mathbf{r}'(1)\times\mathbf{r}''(1) &= \langle -4,-2,4\rangle = 2\langle -2,-1,2\rangle\,,\\
\kappa\Big|_{P_0} &= \kappa(1) = \frac{\|\mathbf{r}'(1)\times\mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3} = \frac{2\,\|\langle 2,1,-2\rangle\|}{3^3} = \frac{6}{27} = \frac{2}{9}.
\end{aligned}$$

□

Equation (**14.2**) can be simplified in two particularly interesting cases. If a curve is planar (i.e., it lies in a plane), then, by choosing the coordinate system so that the xy plane coincides with the plane in which the curve lies, one has $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$. Since $\mathbf{r}'$ and $\mathbf{r}''$ are in the xy plane, their cross product is parallel to the z axis:

$$\mathbf{r} = \langle x(t), y(t), 0\rangle \quad\Rightarrow\quad \mathbf{r}'\times\mathbf{r}'' = \langle 0,0,x'y''-x''y'\rangle\,.$$

The substitution of this relation into Eq. (**14.2**) leads to the following result.

COROLLARY **14.1**. (Curvature of a Planar Curve).
The curvature of a planar smooth curve $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$*, where the derivatives* $x'(t)$ *and* $y'(t)$ *do not vanish simultaneously, is*

$$\kappa = \frac{|x'y''-x''y'|}{[(x')^2+(y')^2]^{3/2}}.$$

A further simplification occurs when the planar curve is a graph $y = f(x)$. The graph is traced out by the vector function $\mathbf{r}(t) = \langle t, f(t), 0\rangle$. Then, in Corollary **14.1**, $x'(t) = 1$, $x''(t) = 0$, and $y''(t) = f''(t) = f''(x)$, which leads to the following result.

Corollary **14.2**. (Curvature of a Graph).
The curvature of the graph $y = f(x)$ is

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

14.1. Geometrical Significance of the Curvature. Let us calculate the curvature of a circle of radius R. One has

$$\begin{aligned} x(t) &= R\cos t \\ y(t) &= R\sin t \end{aligned} \quad\Rightarrow\quad \begin{aligned} x'(t) &= -R\sin t \\ y'(t) &= R\cos t \end{aligned} \quad\Rightarrow\quad \begin{aligned} x''(t) &= -R\cos t \\ y''(t) &= -R\sin t \end{aligned}$$

By Corollary **14.1**

$$\begin{aligned} (x')^2 + (y')^2 &= R^2 \\ x'y'' - x''y' &= R^2 \end{aligned} \quad\Rightarrow\quad \kappa = \frac{R^2}{R^3} = \frac{1}{R}$$

Therefore, the curvature is constant along the circle and equals a reciprocal of its radius. The fact that the curvature is independent of its position on the circle can be anticipated from the rotational symmetry of the circle (it bends uniformly). Naturally, if two circles of different radii pass through the same point, then the circle of smaller radius bends more as it lies inside the bigger circle. Note also that the curvature has the dimension of the inverse length. This motivates the following definition.

Definition **14.2**. (Radius of Curvature).
The reciprocal of the curvature of a curve is called the radius of curvature $\rho = 1/\kappa$.

The radius of curvature is a function of a point on the curve. Let a planar curve have a curvature κ at a point P. Consider a circle of radius $\rho = 1/\kappa$ through the same point P that also has the same tangent line. Then the center of such a circle lies on the line through P that is perpendicular to the tangent line. There are only two circles with the described properties, only only one of them has the same concavity at P as the curve relative to the tangent line. In other words, the curve and the circle "bend" in the same way (or direction) at P, have the same tangent line and the curvature. This circle is called the *osculating circle* of the curve at a point P. One can say the curve looks like a circle of radius $1/\kappa$ (in units of length) near P.

Example **14.3**. *Find the osculating circle for the graph $y = \cos(2x)$ at the point $(0, 1)$.*

Solution: Since $y'(0) = -2\sin 0 = 0$, the tangent line to the graph is horizontal $y = 1$ and the y axis is the line normal to the tangent line at $(0, 1)$. Therefore the center of the osculating circle lies on the y axis down

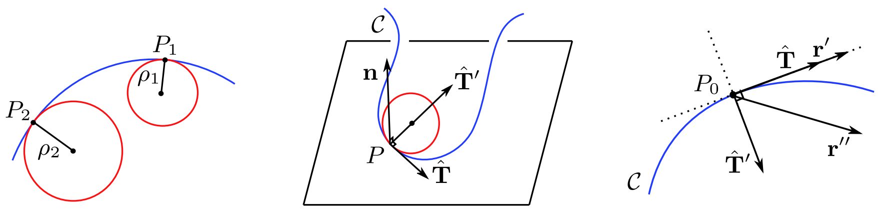

FIGURE **14.2**. **Left**: Radius of curvature. A smooth curve near a point P can be approximated by a portion of a circle of radius $\rho = 1/\kappa$. The curve bends in the same way as a circle of radius that is the reciprocal of the curvature. A large curvature at a point corresponds to a small radius of curvature. **Middle**: Osculating plane and osculating circle. The osculating plane at a point P contains the tangent vector $\hat{\mathbf{T}}$ and its derivative $\hat{\mathbf{T}}'$ at P and hence is orthogonal to $\mathbf{n} = \hat{\mathbf{T}} \times \hat{\mathbf{T}}'$. The osculating circle lies in the osculating plane, it has radius $\rho = 1/\kappa$, and its center is at a distance ρ from P in the direction of $\hat{\mathbf{T}}'$. One says that the curve "bends" in the osculating plane. **Right**: For a curve traced out by a vector function $\mathbf{r}(t)$, the derivatives $\mathbf{r}'$ and $\mathbf{r}''$ at any point P_0 lie in the osculating plane through P_0. So the normal to the osculating plane can also be computed as $\mathbf{n} = \mathbf{r}(t_0)' \times \mathbf{r}''(t_0)$, where $\mathbf{r}(t_0)$ is the position vector of P_0.

from $y = 1$ because the graph of $\cos(2x)$ is concave downward near $x = 0$. The curvature of the graph at $x = 0$ is found by Corollary **14.2**: $y'(x) = -2\sin(2x)$, $y''(x) = -4\cos(2x)$ and

$$\kappa(0) = \frac{|y''(0)|}{(1 + (y'(0))^2)^{3/2}} = \frac{|-4|}{1} = 4\,.$$

So the radius of curvature is $\rho_0 = \rho(0) = 1/\kappa(0) = 1/4$. The center (x_c, y_c) of the osculating circle lies on the y axis ρ_0 units of length down from $y = 1$, that is, at $(x_c, y_c) = (0, 3/4)$. The equation of the osculating circle is

$$(x - x_c)^2 + (y - y_c)^2 = \rho_0^2 \quad \text{or} \quad x^2 + (y - \tfrac{3}{4})^2 = \tfrac{1}{16}\,.$$

□

Let us now view a planar curve in space. Then there are infinitely many circles of radius $1/\kappa$ through the point P on the curve that share the same tangent line with the curve at P. By what principle can one select the osculating circle out of all such circles? Clearly, the same question arises for a general smooth curve in space. There are infinitely many planes containing the tangent line of a curve at a particular point. All such planes are obtained from another by rotations about the tangent line, and in any such plane one can take a circle of radius $1/\kappa$ through P. An example of the osculating circle to a planar curve already answered the question about the selection principle: the osculating circle should provide the *best approximation* to the curve near P. So, what is left is to quantify the notion of the "best approximation".

Let $\mathbf{r}(s)$ be a natural parameterization of a curve such that the position vector of P is $\mathbf{r}(0)$ (i.e., the arc length parameter is measured from P). Then

$$\mathbf{r}'(s)=\hat{\mathbf{T}}(s)\,,\quad \mathbf{r}''(s)=\hat{\mathbf{T}}'(s)\perp\hat{\mathbf{T}}(s)\,,\qquad \|\mathbf{r}''(s)\|=\kappa(s)\,.$$

The natural parameter s is equal to the arclength when counted in the direction of $\hat{\mathbf{T}}(0)$ from P and is equal to the negative of the arclength when counted in the opposite direction from P along the curve. Let $\mathbf{R}(s)$ be a natural parameterization of the circle of radius ρ that and passes through P (with the same agreement about negative and positive s as for the curve). Let us demand that the circle and the curve have the same tangent line and the same curvature at P. This implies that

$$\mathbf{r}(0)=\mathbf{R}(0)\,,\quad \mathbf{r}'(0)=\mathbf{R}'(0)=\hat{\mathbf{T}}(0)\,,\quad \|\mathbf{r}''(0)\|=\|\mathbf{R}''(0)\|=\kappa(0)\,.$$

Note that the unit tangent vectors to the curve and circle may have opposite directions in general. However this only refers to an agreement about the direction in which the curves are traced out by $\mathbf{r}(s)$ and $\mathbf{R}(s)$ and has nothing to do with the shape of the curves. The equality of the unite tangent vectors means that s increases in the same direction along the curve and circle (or they are oriented in the same way).

Consider the quantity $\|\mathbf{r}(s)-\mathbf{R}(s)\|$. It has a simple meaning. It defines the distance between two points that are at a distance s from P along the curve and along the circle. So, for a circle that gives the "best approximation", this distance should be smallest for any given (small) value of s. Let us adopt this principle to figure out for which circle $\|\mathbf{r}(s)-\mathbf{R}(s)\|$ is minimal. To accomplish this task, recall from Calculus 1 that the Taylor polynomial of the second degree approximates best a twice differentiable function $x(s)$ near a point, say $s=0$ among all quadratic polynomials sharing the same value $x(0)$ and the same slope $x'(0)$ of the tangent line:

$$x(s)=x(0)+x'(0)s+\frac{1}{2}x''(0)s^2+s^2u(s)$$

The error of the approximation $s^2u(s)$ tends to zero faster than s^2, that is, $u(s)\to 0$ as $s\to 0$. In other words, put

$$p(s)=x(0)+x'(0)s+ax^2\,,\qquad u(s)=\frac{x(s)-p\,(s)}{s^2}\,.$$

Then the error of the approximation $s^2u(s)$ tends to zero faster than s^2, that is,

$$\lim_{s\to 0}u(s)=\lim_{s\to 0}\frac{x(s)-p(s)}{s^2}=0$$

if and only if $a=\frac{1}{2}x''(0)$. In this sense, the Taylor polynomial provides the best approximations near $s=0$ to a twice-differentiable function.

Using the Taylor approximation for each component of the vector functions $\mathbf{r}(s)$ and $\mathbf{R}(s)$, one finds

$$\mathbf{r}(s) = \mathbf{r}(0) + \mathbf{r}'(0)s + \tfrac{1}{2}\mathbf{r}''(0)s^2 + s^2\mathbf{u}(s)\,,$$
$$\mathbf{R}(s) = \mathbf{R}(0) + \mathbf{R}'(0)s + \tfrac{1}{2}\mathbf{R}''(0)s^2 + s^2\mathbf{v}(s)\,,$$

where $\mathbf{u}(s) \to \mathbf{0}$ and $\mathbf{v}(s) \to \mathbf{0}$ as $s \to 0$. Since the curve and the circle have a common point and the same unit tangent vector at $s = 0$, the approximation error reads

$$\|\mathbf{r}(s) - \mathbf{R}(s)\| = \frac{1}{2}\|\mathbf{r}''(0) - \mathbf{R}''(0) + \mathbf{w}(s)\|s^2\,,$$

where $\mathbf{w}(s) = 2(\mathbf{u}(s) - \mathbf{v}(s)) \to 0$ as $s \to 0$. Therefore

$$\lim_{s\to 0}\frac{\|\mathbf{r}(s) - \mathbf{R}(s)\|}{s^2} = \frac{1}{2}\lim_{s\to 0}\|\mathbf{r}''(0) - \mathbf{R}''(0) + \mathbf{w}(s)\|$$
$$= \frac{1}{2}\|\mathbf{r}''(0) - \mathbf{R}''(0)\| = 0$$

if and only if

$$\mathbf{r}''(0) = \mathbf{R}''(0)\,.$$

In other words, the error of the approximation decreases to zero in the fastest way (faster than s^2) if and only if the circle and and the curve have the *same derivative of the unit tangent vector* $\hat{\mathbf{T}}'(0)$ with respect to the natural parameter at the point P. This is a *stronger* condition than just the equality of the magnitudes $\kappa(0) = \|\mathbf{r}''(0)\| = \|\mathbf{R}''(0)\|$.

The vectors $\hat{\mathbf{T}}(0)$ and $\hat{\mathbf{T}}'(0)$ are orthogonal. Hence there exists a unique plane through P that contains these vectors. The circle that provides the best approximation to the curve near P lies in this plane as the plane and the circle have the same $\hat{\mathbf{T}}(0)$ and $\hat{\mathbf{T}}'(0)$. The radius of a circle is always perpendicular to the tangent line through the end point of the radius. The line through P and parallel to $\hat{\mathbf{T}}'(0)$ is perpendicular to the tangent line and, hence, passes through the center of the circle. By the geometrical interpretation of the derivative of a vector function the vector $\mathbf{T}'(s)$ points in the direction in which the curve bends as seen in Figs. **14.1** and **14.2** (the left panel); the curve is concave in the direction of $\hat{\mathbf{T}}'(s)$. This implies that the best approximating circle must lie in the plane through P that contains the vectors $\hat{\mathbf{T}}(0)$ and $\hat{\mathbf{T}}'(0)$, and the center of the circle must be $\rho(0) = 1/\kappa(0)$ units of length from the point P in the direction of $\hat{\mathbf{T}}'(0)$ (so that the curve and the circle have the same concavity at P).

DEFINITION **14.3**. (Osculating Plane and Circle).
The plane through a point P of a curve that is parallel to the unit tangent vector $\hat{\mathbf{T}}$ and its derivative $\hat{\mathbf{T}}' \neq \mathbf{0}$ at P is called the osculating plane *at P. The circle of radius $\rho = 1/\kappa$, where κ is the curvature at P, through P that lies in the osculating plane and whose center is in the direction of $\hat{\mathbf{T}}'$ from P is called the* osculating circle *at P.*

Unit normal vector. Suppose that the curvature does not vanish along a curve, $\kappa \neq 0$. Therefore the derivative $\hat{\mathbf{T}}'$ of the unit tangent vector does not vanish and one can define the unit vector

$$\hat{\mathbf{N}}(s) = \frac{1}{\|\hat{\mathbf{T}}'(s)\|}\,\hat{\mathbf{T}}'(s) = \frac{1}{\kappa(s)}\,\hat{\mathbf{T}}'(s)\,, \quad \kappa(s) \neq 0\,,$$

that is parallel to $\hat{\mathbf{T}}'$ at each point of the curve. This vector is called *the normal* of the curve. By construction, the unit normal and tangent vectors are perpendicular:

$$\hat{\mathbf{T}}(s) \perp \hat{\mathbf{N}}(s) \quad \Leftrightarrow \quad \hat{\mathbf{T}}(s)\cdot\hat{\mathbf{N}}(s) = 0$$

at any point on the curve.

Let $\mathbf{r}_0$ be the position vector of a particular point P_0 of the curve and $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$ be the unit tangent and normal vectors at P_0. Then any vector in the osculating plane through P_0 is a linear combination of $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$. In other words, $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$ form an orthonormal basis in the osculating plane. By Definition **14.3**, the *position vector of the center of of the osculating circle* at P_0 is

$$\mathbf{r}_c = \mathbf{r}_0 + \rho_0\hat{\mathbf{N}}_0\,, \quad \rho = \frac{1}{\kappa_0}\,.$$

Since any vector in the osculating plane is a linear combination of $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$, then parametric equations of the osculating circle may be taken in the form

$$\mathbf{R}(t) = \mathbf{r}_c + a(t)\hat{\mathbf{T}}_0 + b(t)\hat{\mathbf{N}}_0$$

where the functions $a(t)$ and $b(t)$ should satisfy the condition that the distance between a point on the circle to its center is ρ_0 for all t:

$$\|\mathbf{R}(t) - \mathbf{r}_c\| = \rho_0\,.$$

Owing to the orthogonality of $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$, that is, $\hat{\mathbf{T}}_0\cdot\hat{\mathbf{N}}_0 = 0$, this condition implies that

$$\begin{aligned}
\rho_0^2 &= (\mathbf{r}(t) - \mathbf{r}_c)\cdot(\mathbf{r}(t) - \mathbf{r}_c) \\
&= (a(t)\hat{\mathbf{T}}_0 + b(t)\hat{\mathbf{N}}_0)\cdot(a(t)\hat{\mathbf{T}}_0 + b(t)\hat{\mathbf{N}}_0) \\
&= a^2(t)\hat{\mathbf{T}}_0\cdot\hat{\mathbf{T}}_0 + b^2(t)\hat{\mathbf{N}}_0\cdot\hat{\mathbf{N}}_0 \\
&= a^2(t) + b^2(t)
\end{aligned}$$

Thus, the components of the vector $\mathbf{R}(t) - \mathbf{r}_c$ in the orthonormal basis $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$ satisfy an equation of the circle of radius ρ_0 and may chosen in the form $a(t) = \sin t$ and $b(t) = \pm\rho_0\cos t$ (the choice of the sign determines the direction in which the circle is traversed). If one takes $a(t) = -\rho_0\cos t$, then $\mathbf{R}(0) = \mathbf{r}_0$ (the point P_0 corresponds to $t = 0$ in the parametric equations of the osculating circle through P_0). This proves the following theorem.

THEOREM **14.2**. (Parametric equations of the osculating circle) *Suppose that a smooth curve C has a nonzero curvature κ_0 at a point P_0. Let $\hat{\mathbf{T}}_0$ and $\hat{\mathbf{N}}_0$ be the unit tangent and normal vectors at P_0. Then parametric equations of the osculating circle through P_0 are*

$$\mathbf{R}(t) = \mathbf{r}_0 + \rho_0(1 - \cos t)\hat{\mathbf{N}}_0 + \rho_0 \sin t\, \hat{\mathbf{T}}_0\,, \quad \rho_0 = \frac{1}{\kappa_0}\,, \quad 0 \le t \le 2\pi\,.$$

An example of constructing parametric equations of the osculating circle is given in Study Problem **14.4**. When the curve in question is planar, the osculating circle may be found by simpler means.

THEOREM **14.3**. (Equation of the Osculating Plane). *Let a curve C be traced out by a twice-differentiable vector function $\mathbf{r}(t)$. Let P_0 be a point of C such that its position vector is $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$ at which the vector $\mathbf{n} = \mathbf{r}'(t_0) \times \mathbf{r}''(t_0)$ does not vanish. An equation of the osculating plane through P_0 is*

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0\,, \quad \mathbf{n} = \langle n_1, n_2, n_3\rangle.$$

PROOF. It follows from (**14.3**) that the second derivative $\mathbf{r}''(t_0)$ lies in the osculating plane because it is a linear combination of $\hat{\mathbf{T}}(t_0)$ and $\hat{\mathbf{T}}'(t_0)$. Hence, the osculating plane contains the first and second derivatives $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$. Therefore, their cross product $\mathbf{n} = \mathbf{r}'(t_0)\times\mathbf{r}''(t_0)$ is perpendicular to the osculating plane, and the conclusion of the theorem follows. □

EXAMPLE **14.4**. *For the curve $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, find the osculating plane through the point* $(1, 1, 1)$.

SOLUTION: The point in question corresponds to $t = 1$. Then

$$\mathbf{r}'(1) = \langle 1, 2t, 3t^2\rangle\Big|_{t=1} = \langle 1, 2, 3\rangle$$
$$\mathbf{r}''(1) = \langle 0, 2, 6t\rangle\Big|_{t=1} = \langle 0, 2, 6\rangle$$

Therefore, the normal of the osculating plane is

$$\mathbf{n} = \mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 3\rangle \times \langle 0, 2, 6\rangle = \langle 6, -6, 2\rangle\,.$$

The osculating plane is $6(x-1) - 6(y-1) + 2(z-1) = 0$ or $3x - 3y + z = 1$. □

14.2. Study Problems.

Problem **14.1**. *Show that any smooth curve whose curvature vanishes is a portion of a straight line.*

SOLUTION: Let $\mathbf{r}(s)$ be a natural parameterization of a smooth curve. Then the derivative is a unit tangent vector to the curve, $\hat{\mathbf{T}}(s) = \hat{\mathbf{r}}'(s)$. By the definition of the curvature, $\kappa(s) = \|\hat{\mathbf{T}}'(s)\| = \|\mathbf{r}''(s)\|$. If $\kappa(s) = 0$, then $\mathbf{r}''(s) = \mathbf{0}$ for all s. Therefore the unit tangent vector $\mathbf{r}'(s) = \hat{\mathbf{T}}$ is a constant vector. The integration of this relation yields a vector equation of a straight line through some point $\mathbf{r}_0$ and parallel to $\hat{\mathbf{T}}$:

$$\kappa(s) = 0 \quad \Leftrightarrow \quad \mathbf{r}''(s) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{r}'(s) = \hat{\mathbf{T}} \quad \Leftrightarrow \quad \mathbf{r}(s) = \mathbf{r}_0 + \hat{\mathbf{T}}s\,.$$

□

Problem **14.2**. (Curvature of a Polar Graph)
Find the curvature of a polar graph $r = f(\theta)$ where $f(\theta)$ and $f'(\theta)$ do not vanish simultaneously, and f is twice differentiable. In particular, calculate the curvature of the spiral $r = \theta$.

SOLUTION: Using the relations $x = r\cos(\theta)$, $y = r\sin\theta$ between the rectangular and polar coordinates, parametric equations of the polar graph can be written in the form

$$\mathbf{r}(\theta) = \langle f(\theta)\cos\theta,\ f(\theta)\sin\theta\,,\ 0\rangle = f(\theta)\hat{\mathbf{u}}(\theta)$$

One has $\hat{\mathbf{u}}'(\theta) = \langle -\sin\theta, \cos\theta, 0\rangle$, which is a unit vector orthogonal to $\hat{\mathbf{u}}(\theta)$, and $\hat{\mathbf{u}}''(\theta) = -\hat{\mathbf{u}}(\theta)$. By the product rule for differentiation and by the properties of the cross product:

$$\begin{aligned}
\mathbf{r}'(\theta) &= f'(\theta)\hat{\mathbf{u}}(\theta) + f(\theta)\hat{\mathbf{u}}'(\theta)\,,\\
\mathbf{r}''(\theta) &= \Big(f''(\theta) - f(\theta)\Big)\hat{\mathbf{u}}(\theta) + 2f'(\theta)\hat{\mathbf{u}}'(\theta)\,,\\
\mathbf{r}'(\theta)\times\mathbf{r}''(\theta) &= 2(f'(\theta))^2\hat{\mathbf{u}}(\theta)\times\hat{\mathbf{u}}'(\theta) + f(\theta)\Big(f''(\theta) - f(\theta)\Big)\hat{\mathbf{u}}'(\theta)\times\hat{\mathbf{u}}(\theta)\\
&= \Big(2(f'(\theta))^2 - f(\theta)f''(\theta) + (f(\theta))^2\Big)\hat{\mathbf{u}}(\theta)\times\hat{\mathbf{u}}'(\theta)\,.
\end{aligned}$$

Since $\hat{\mathbf{u}}(\theta)$, $\hat{\mathbf{u}}'(\theta)$, and $\hat{\mathbf{u}}(\theta)\times\hat{\mathbf{u}}'(\theta)$ are mutually orthogonal unit vectors,

$$\begin{aligned}
\|\mathbf{r}'(\theta)\| &= \Big((f'(\theta))^2 + (f(\theta))^2\Big)^{1/2},\\
\|\mathbf{r}'(\theta)\times\mathbf{r}''(\theta)\| &= \Big|2(f'(\theta))^2 - f(\theta)f''(\theta) + (f(\theta))^2\Big|,\\
\kappa(\theta) &= \frac{\|\mathbf{r}'(\theta)\times\mathbf{r}''(\theta)\|}{\|\mathbf{r}'(\theta)\|^3}\\
&= \frac{\Big|2(f'(\theta))^2 - f(\theta)f''(\theta) + (f(\theta))^2\Big|}{\Big((f'(\theta))^2 + (f(\theta))^2\Big)^{3/2}}\,.
\end{aligned}$$

For the spiral $f(\theta) = \theta$, $f'(\theta) = 1$, and $f''(\theta) = 0$. Therefore

$$\kappa(\theta) = \frac{2+\theta^2}{(1+\theta^2)^{3/2}}$$

is the curvature of the spiral. □

Problem **14.3**. *Find the maximal curvature of the graph of the exponential, $y = e^x$, and the point(s) at which it occurs.*

SOLUTION: The curvature of the graph $y = e^x$ is calculated by Corollary **14.2**:

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}} .$$

Critical points are determined by $\kappa'(x) = 0$ or

$$\kappa'(x) = \frac{e^x(1 + e^{2x})^{1/2}(1 - 2e^{2x})}{(1 + e^{2x})^3} = 0 \quad \Rightarrow \quad 2e^{2x} = 1$$
$$\Rightarrow \quad x = -\frac{\ln 2}{2}.$$

From the shape of the graph of the exponential, it is clear that the found critical point corresponds to the (absolute) maximum of $\kappa(x)$ (maximal bending) and

$$\kappa_{\max} = \kappa\Big(-\frac{1}{2}\ln(2)\Big) = \frac{(1/\sqrt{2})}{[1 + (1/2)]^{3/2}} = \frac{2}{3\sqrt{3}} ,$$

where the relation $e^{2x} = 1/2$ at the critical point has been used. □

Problem **14.4**. *Let a curve C be the intersection of two quadric surfaces $4z = x^2$ and $6y = xz$. Find an equation of the osculating plane and parametric equations of the osculating circle through the point* $(2, 1/3, 1)$.

SOLUTION: One has to find a parameterization of the curve of intersection. If $x = x(t)$, $y = y(t)$, and $z = z(t)$ are parametric equations of the curve of intersection, then the functions $x(t)$, $y(t)$, and $z(t)$ must satisfy $4z = x^2$ and $6y = xz$ for all values of the parameter t. Put $x = 2t$. Then it follows from $4z = x^2$ and $6y = xz$ that $z = t^2$ and $y = t^3/3$. Thus, the curve is traversed by the vector function

$$\mathbf{r}(t) = \langle 2t, t^3/3, t^2\rangle$$

which is continuously differentiable. The point of interest corresponds to $t = 1$. The curve is smooth because $x'(t) = 2 > 0$ so that $x(t)$ is one-to-one by the inverse function theorem (the parameterization is simple) and $\mathbf{r}'(t) \neq \mathbf{0}$. Then using Theorem **14.3** the osculating plane $\mathcal{P}$ is found:

$$\mathbf{r}'(t) = \langle 2, t^2, 2t\rangle , \quad \mathbf{r}''(t) = \langle 0, 2t, 2\rangle ,$$
$$\mathbf{n} = \mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 2, 1, 2\rangle \times \langle 0, 2, 2\rangle = 2\langle -1, -2, 2\rangle$$
$$\mathcal{P} \ : \ -1(x - 1) - 2(y - 1) + 2(z - 1) = 0 \quad \text{or} \quad x + 2y - 2z = 1 .$$

Parametric equations of the osculating circle are found by using Theorem **14.2**. One has

$$\begin{aligned}
\|\mathbf{r}'(t)\| &= \sqrt{4+t^4+4t^2} = \sqrt{(2+t^2)^2} = 2+t^2\,,\\
\hat{\mathbf{T}}(t) &= \frac{1}{2+t^2}\langle 2, t^2, 2t\rangle \quad\Rightarrow\quad \hat{\mathbf{T}}_0 = \hat{\mathbf{T}}(1) = \frac{1}{3}\langle 2,1,2\rangle\,,\\
\hat{\mathbf{T}}'(t) &= -\frac{2t}{(2+t^2)^2}\langle 2, t^2, 2t\rangle + \frac{1}{2+t^2}\langle 0, 2t, 2\rangle\,,\\
\hat{\mathbf{T}}'(1) &= \frac{2}{9}\langle -2,2,1\rangle \quad\Rightarrow\quad \|\hat{\mathbf{T}}'(1)\| = \frac{2}{9}\|\langle -2,2,1\rangle\| = \frac{2}{3}\,,\\
\hat{\mathbf{N}}_0 &= \frac{1}{\|\hat{\mathbf{T}}'(1)\|}\hat{\mathbf{T}}'(1) = \frac{1}{3}\langle -2,2,1\rangle\,,\\
\kappa(1) &= \frac{\|\hat{\mathbf{T}}'(1)\|}{\|\mathbf{r}'(1)\|} = \frac{2}{9} \quad\Rightarrow\quad \rho_0 = \frac{1}{\kappa(1)} = \frac{9}{2}\,,
\end{aligned}$$

where Eq. (**14.1**) has been used to calculate the curvature. Let $\mathbf{r}_0 = \langle 1,1,1\rangle$. Then the vector function

$$\mathbf{R}(t) = \mathbf{r}_0 + \frac{9}{2}(1-\cos t)\hat{\mathbf{N}}_0 + \frac{9}{2}\sin t\,\hat{\mathbf{T}}_0\,, \quad 0 \le t \le 2\pi\,,$$

traverses the osculating circle through $\mathbf{r}_0$. Put $\mathbf{R}(t) = \langle X(t)\,, Y(t)\,, Z(t)\rangle$. Then parametric equations of the osculating circle are

$$\begin{aligned}
x &= X(t) = -1 + 3\cos t + 3\sin t\,,\\
y &= Y(t) = 4 - 3\cos t + \tfrac{3}{2}\sin t\,,\\
z &= Z(t) = \tfrac{5}{2} - \tfrac{3}{2}\cos t + 3\sin t\,.
\end{aligned}$$

It is not difficult to verify that the circle lies in the osculating plane by substituting these equations into the equation of the osculating plane obtained above. □

Problem **14.5**. *Consider a helix $\mathbf{r}(t) = \langle R\cos(\omega t), R\sin(\omega t), ht\rangle$, where R, ω, and h are numerical parameters. The arclength of one turn of the helix is a function of the parameter ω, $L = L(\omega)$, and the curvature at any fixed point of the helix is also a function of ω, $\kappa = \kappa(\omega)$. Use only geometrical arguments (no calculus) to find the limits of $L(\omega)$ and $\kappa(\omega)$ as $\omega \to \infty$.*

SOLUTION: The vector function $\mathbf{r}(t)$ traces out one turn of the helix when t ranges over the period of $\cos(\omega t)$ or $\sin(\omega t)$ (i.e., over the interval of length $2\pi/\omega$). Thus, the helix rises by $2\pi h/\omega = H(\omega)$ along the z axis per each turn. When $\omega \to \infty$, the height $H(\omega)$ tends to 0 so that each turn of the helix becomes closer and closer to a circle of radius R. Therefore,

$$\begin{aligned}
L(\omega) &\to 2\pi R \quad \text{(the circumference)}\\
\kappa(\omega) &\to \frac{1}{R} \quad \text{(the curvature of the circle)}
\end{aligned}$$

as $\omega \to \infty$.

A calculus approach requires a lot more work to establish this result:

$$\begin{aligned} L(\omega) &= \int_0^{2\pi/\omega} \|\mathbf{r}'(t)\|\,dt = \frac{2\pi}{\omega}\sqrt{(R\omega)^2+h^2} \\ &= 2\pi\sqrt{R^2+(h/\omega)^2} \to 2\pi R, \end{aligned}$$

$$\begin{aligned} \kappa(\omega) &= \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{R\omega^2[(R\omega)^2+h^2]^{1/2}}{[(R\omega)^2+h^2]^{3/2}} \\ &= \frac{R}{R^2+(h/\omega)^2} \to \frac{1}{R} \end{aligned}$$

as $\omega \to \infty$. □

14.3. Exercises.

1–10. Find the curvature of each of the following parameterized curves as a function of the parameter t, and find the radius of curvature at the indicated point P:

1. $\mathbf{r}(t) = \langle t,\ 1-t,\ t^2+1\rangle$, $P = (1,0,2)$;
2. $\mathbf{r}(t) = \langle t^2,\ t,\ 1\rangle$, $P = (4,2,1)$;
3. $y = \sin(x/2)$, $P = (\pi,1)$;
4. $\mathbf{r}(t) = \langle 4t^{3/2},\ -t^2,\ t\rangle$, $P = (4,-1,1)$;
5. $x = 1+t^2$, $y = 2+t^3$, $P = (2,1)$;
6. $x = e^t\cos t$, $y = 0$, $z = e^t\sin t$, $P = (1,0,0)$;
7. $\mathbf{r}(t) = \langle \ln t,\ \sqrt{t},\ t^2\rangle$, $P = (0,1,1)$;
8. $\mathbf{r}(t) = \langle \cosh t,\ \sinh t,\ 2+t\rangle$, $P = (1,0,2)$;
9. $\mathbf{r}(t) = \langle e^t,\ \sqrt{2}\,t,\ e^{-t}\rangle$, $P = (1,0,1)$;
10. $\mathbf{r}(t) = \langle \sin t - t\cos t,\ t,\ \cos t + t\sin t\rangle$, $P = (0,0,1)$.

11. Find the curvature of $\mathbf{r}(t) = \langle t, t^2/2, t^3/3\rangle$ at the point of its intersection with the surface $z = 2xy + 2/3$.

12. Find the maximal and minimal curvatures of the graph $y = \cos(ax)$ and the points at which they occur. Sketch the graph for $a = 1$ and mark the points of the maximal and minimal curvature, local maxima and minima of $\cos x$, and the inflection points.

13. Find the maximal and minimal curvatures of the graph $y = 1/x$.

14. Use a geometrical interpretation of the curvature to guess the point on the graphs $y = ax^2$ and $y = ax^4$ where the maximal curvature occurs. Then verify your guess by calculations.

15–17. Let $f(x)$ be twice continuously differentiable function and $\kappa(x)$ be the curvature of the graph $y = f(x)$.

15. Does κ attain a local maximum value at every local minimum and maximum of f? If not, state an additional condition on f under which the answer to this question is affirmative.
16. Prove that $\kappa = 0$ at inflection points of the graph.

17. Show by an example that the converse of the statement in Exercise **16** is not true, i.e., that the curvature vanishes at $x = x_0$ does not imply that the point $(x_0, f(x_0))$ is an inflection point.

18. Let f be twice differentiable at x_0. Let $T_2(x)$ be its Taylor polynomial of the second degree about $x = x_0$. Compare the curvatures of the graphs $y = f(x)$ and $y = T_2(x)$ at $x = x_0$.

19–21. Find the curvature of each of the following polar graphs:

19. $r = 1 + \cos\theta$;
20. $r = e^{\theta}$;
21. $r = |\sin(3\theta)|$.

22–26. Find the equation of the osculating circle for each of the following planar curves at a specified point P:

22. $y = x^2, \quad P = (0, 0)$;
23. $y = x - x^2/4, \quad P = (2, 1)$;
24. $y = 1/x, \quad P = (1, 1)$;
25. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (a cycloid), $\quad P = (a(\pi/2 - 1), a)$;
26. $x = \cos t$, $y = \sin(2t)$, $\quad P = (1, 0)$ and $P = (-1, 0)$.

27. Find the maximal and minimal curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > b$, and the points where they occur. Give the equations of the osculating circles at these points.

28. Let $\mathbf{r}(t) = \langle t^3, t^2, 0\rangle$. This curve is not smooth and has a cusp at $t = 0$. Find the curvature for $t \neq 0$ and investigate its limit as $t \to 0$.

29. Show that the cardioid $r = 1 + \cos\theta$ is not smooth at the origin. Investigate the curvature of the cardioid as the origin is approached along the cardioid.

30–31. Find an equation of the osculating plane for each of the following curves at a specified point:

30. $\mathbf{r}(t) = \langle 4t^{3/2},\ -t^2,\ t\rangle, \quad P = (4, -1, 1)$;
31. $\mathbf{r}(t) = \langle \ln t,\ \sqrt{t},\ t^2\rangle, \quad P = (0, 1, 1)$.

32. Find an equation for the osculating and normal planes for the curve $\mathbf{r}(t) = \langle \ln(t)\,,\, 2t\,,\, t^2\rangle$ at the point P_0 of its intersection with the plane $y - z = 1$. A plane is normal to a curve at a point if the tangent to the curve at that point is normal to the plane.

33. Is there a point on the curve $\mathbf{r}(t) = \langle t,\ t^2,\ t^3\rangle$ where the osculating plane is parallel to the plane $3x - 3y + z = 1$?

34. Prove that the trajectory of a particle has a constant curvature if the particle moves so that the magnitudes of its velocity and acceleration vectors are constant.

35. Consider a graph $y = f(x)$ such that $f''(x_0) \neq 0$. At a point (x_0, y_0) on the curve, where $y_0 = f(x_0)$, find the equation of the osculating circle in the form $(x - a)^2 + (y - b)^2 = R^2$. Hint: Show first that the vector $\langle 1, f'(x_0)\rangle$ is tangent to the graph and a vector orthogonal to it is $\langle -f'(x_0), 1\rangle$. Then consider two cases $f''(x_0) > 0$ and $f''(x_0) < 0$.

36. Let a smooth curve $\mathbf{r} = \mathbf{r}(t)$ be planar and lie in the xy plane. At a point (x_0, y_0) on the curve, find the equation of the osculating circle in the form $(x-a)^2+(y-b)^2 = R^2$. Hint: Use the result of Study Problem **14.4** to express the constants a, b, and R via x_0, y_0, and the curvature at (x_0, y_0).
37. Find parametric equations of the osculating circle to the curve $\mathbf{r}(t) = \langle 4t^{3/2},\ -t^2,\ t\rangle$ at the point $P = (4, -1, 1)$ by using the method of Study Problem **14.4**.

15. Applications to mechanics and geometry

15.1. Tangential and Normal Accelerations. Let $\mathbf{r}(t)$ be the trajectory of a particle (t is time). Then $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{v}'(t)$ are the velocity and acceleration of the particle. The magnitude of the velocity vector is the speed, $v(t) = \|\mathbf{v}(t)\|$. If $\hat{\mathbf{T}}(t)$ is the unit tangent vector to the trajectory, then $\hat{\mathbf{T}}'(t)$ is orthogonal to it. The unit vector $\hat{\mathbf{N}}(t) = \hat{\mathbf{T}}'(t)/\|\hat{\mathbf{T}}'(t)\|$ is called a unit *normal* to the trajectory. In particular, the osculating plane at any point of the trajectory contains $\hat{\mathbf{T}}(t)$ and $\hat{\mathbf{N}}(t)$. The differentiation of the relation $\mathbf{v}(t) = v(t)\hat{\mathbf{T}}(t)$ (see (**14.3**)) shows that that acceleration always lies in the osculating plane:

$$\mathbf{a} = v'\hat{\mathbf{T}} + v\hat{\mathbf{T}}' = v'\hat{\mathbf{T}} + v\|\mathbf{T}'\|\hat{\mathbf{N}}\,.$$

Furthermore, substituting the relations $\kappa = \|\hat{\mathbf{T}}'\|/v$ and $\rho = 1/\kappa$ into the latter equation, one finds (see Figure **15.1**, left panel) that

$$\begin{aligned} \mathbf{a} &= a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}}, \\ a_T &= v' = \hat{\mathbf{T}}\cdot\mathbf{a} = \frac{\mathbf{v}\cdot\mathbf{a}}{v}, \\ a_N &= \kappa v^2 = \frac{v^2}{\rho} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v}\,. \end{aligned}$$

DEFINITION **15.1**. (Tangential and Normal Accelerations).
Scalar projections a_T and a_N of the acceleration vector onto the unit tangent and normal vectors at any point of the trajectory of motion are called tangential and normal accelerations*, respectively.*

The tangential acceleration a_T determines the rate of change of a particle's speed, while the normal acceleration appears only when the particle makes a "turn". In particular, a circular motion with a constant speed, $v = v_0$, has no tangential acceleration, $a_T = 0$, and the normal acceleration is constant:

$$v = v_0 = \text{const} \quad\Rightarrow\quad a_T = v' = 0\,, \quad a_N = \kappa v_0^2 = \frac{v_0^2}{R}\,.$$

because the curvature of a circle is the reciprocal of its radius, $\kappa = 1/R$.

EXAMPLE **15.1**. *Let $\mathbf{r}(t) = \langle t, t^2/2, t^3/6\rangle$ be the position vector of a point particle as a function of time t. Find, the velocity, speed, acceleration, tangential and normal accelerations of the particle at the point $P = (2, 2, 4/3)$.*

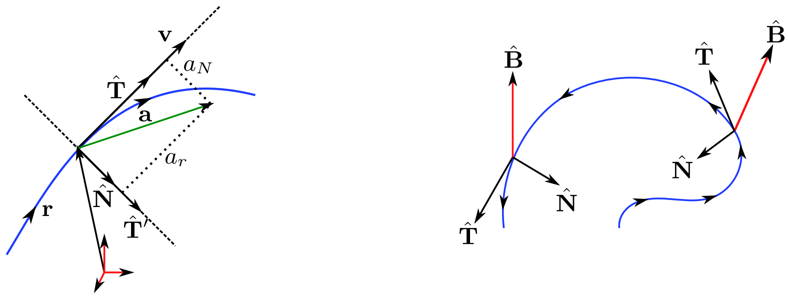

FIGURE **15.1**. **Left**: Decomposition of the acceleration $\mathbf{a}$ of a particle into normal and tangential components. The tangential component a_T is the scalar projection of $\mathbf{a}$ onto the unit tangent vector $\hat{\mathbf{T}}$. The normal component is the scalar projection of $\mathbf{a}$ onto the unit normal vector $\hat{\mathbf{N}}$. The vectors $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity vectors of the particle. **Right**: The tangent, normal, and binormal vectors associated with a smooth curve. These vectors are mutually orthogonal and have unit length. The binormal is defined by $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$. The shape of the curve is uniquely determined by the orientation of the triple of vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ as functions of the arc length parameter up to general rigid rotations and translations of the curve as the whole.

SOLUTION: The particle reaches the point P at $t = 2$ because $\mathbf{r}(2) = \langle 2, 2, 4/3\rangle$ is the position vector of P. Therefore

$$\begin{aligned}
\mathbf{v}\Big|_P &= \mathbf{v}(2) = \mathbf{r}'(2) = \langle 1, t, t^2/2\rangle\Big|_{t=2} = \langle 1, 2, 2\rangle\,,\\
v\Big|_P &= v(2) = \|\mathbf{v}(2)\| = \|\langle 1, 2, 2\rangle\| = 3\,,\\
\mathbf{a}\Big|_P &= \mathbf{a}(2) = \mathbf{r}''(2) = \langle 0, 1, t\rangle\Big|_{t=2} = \langle 0, 1, 2\rangle\,,\\
a_T\Big|_P &= a_T(2) = \frac{\mathbf{a}(2)\cdot\mathbf{v}(2)}{v(2)} = \frac{6}{3} = 2\,,\\
a_N\Big|_P &= a_N(2) = \frac{\|\mathbf{v}(2)\times\mathbf{a}(2)\|}{v(2)} = \frac{\|\langle 2, -2, 1\rangle\|}{3} = \frac{3}{3} = 1\,.
\end{aligned}$$

□

To gain an intuitive understanding of the tangential and normal accelerations, consider a car moving along a road. The speed of the car can be changed by pressing the gas or brake pedals. When one of these pedals is suddenly pressed, one can feel a force along the direction of motion of the car (the tangential direction). The car speedometer also shows that the speed changes, indicating that this force is due to the acceleration along the road (i.e., the tangential acceleration $a_T = v' \neq 0$). When the car moves along a straight road with a constant speed, its acceleration is 0. When the road takes a turn, the steering wheel must be turned in order to keep the car on the road, while the car maintains a constant speed. In this case, one can feel a force normal to the road. It is larger for sharper turns (a larger curvature or a smaller radius of curvature) and also grows when the same turn is passed with a greater speed. This force is due to the normal acceleration, $a_N = v^2/\rho$, and is called a *centrifugal force*. By Newton's law, its magnitude is

$$F = ma_N = m\kappa v^2 = \frac{mv^2}{\rho}\,,$$

where m is the mass of a moving object (e.g., a car).

When making a turn, the car does not slide off the road as long as the friction force between the tires and the road compensates for the centrifugal force. The maximal friction force depends on the road and tire conditions (e.g., a wet road and worn tires reduce substantially the maximal friction force). The centrifugal force is determined by the speed (the curvature of the road is fixed by the road shape). So, for a high enough speed, the centrifugal force can no longer be compensated for by the friction force, and the car would skid off the road. For this reason, suggested speed limit signs are often placed at highway exits. If one drives a car on a highway exit with a speed twice as high as the suggested speed, *the risk of skidding off the road is quadrupled, not doubled*, because the normal acceleration $a_N = v^2/\rho$ quadruples when the speed v is doubled.

EXAMPLE **15.2**. *A road takes a turn that has a parabolic shape,*

$$y = \frac{x^2}{2R}\,, \quad -R < x < R\,,$$

where (x, y) are coordinates of points of the road and R is a constant (all measured in meters). A safety assessment requires that the normal acceleration on the road should not exceed a threshold value a_m meters per second squared to avoid skidding off the road. If a car moves with a constant speed v meters per second along the road, find the portion of the road where the car might skid off the road.

SOLUTION: The normal acceleration of the car as a function of *position* (not time!) is

$$a_N(x) = \kappa(x)v^2\,.$$

By Corollary **14.2**, the curvature of the graph $y = x^2/(2R)$ is

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{R[1 + (x/R)^2]^{3/2}}\,.$$

The maximal curvature and hence the maximal normal acceleration are attained at $x = 0$. So, if the speed v is such that

$$a_N^{max} = a_N(0) = \frac{v^2}{R} < a_m \quad \Rightarrow \quad v < v_{max} = \sqrt{Ra_m}\,,$$

no accident can happen. Thus, a suggested speed limit sign $v < v_{max}$ can be placed for this turn.

If a car makes the turn exceeding the speed limit v_{max}, then the dangerous part of the road, where an accident can happen, is determined by the inequality

$$a_N(x) \geq a_m\,,$$
$$\frac{v^2}{R[1+(x/R)^2]^{3/2}} \geq a_m \quad \Rightarrow \quad |x| \leq R\sqrt{\nu-1}\,, \quad \nu = \left(\frac{v}{v_{max}}\right)^{4/3}.$$

The constant ν always exceeds 1 if $v > v_{max}$. Since the turn corresponds to the range $|x| < R$, the dangerous part of the turn is determined by the smallest number of R and $R\sqrt{\nu-1}$. Thus, the car can skid off the road when moving on the part of the road corresponding to the interval

$$-R\mu \leq x \leq R\mu\,, \quad \mu = \min\{1,\ \sqrt{\nu-1}\}\,.$$

Note that an accident can happen anywhere on the turning part of the road if the speed at which the car enters into this part of the road exceeds the speed limit just by the factor $2^{3/4} \approx 1.68$ (it corresponds to $\nu = 2$). □

15.2. Torsion and Frenet-Serret Formulas. The shape of a space curve as a point set is independent of a parameterization of the curve. A natural question arises: What parameters of the curve determine its shape? Suppose the curve is smooth enough so that the unit tangent vector $\hat{\mathbf{T}}(s)$ and its derivative $\hat{\mathbf{T}}'(s)$ can be defined as functions of the arc length s counted from an endpoint of the curve. Let $\hat{\mathbf{N}}(s)$ be the unit normal vector of the curve. If the curvature $\kappa(s)$ does not vanish, then $\hat{\mathbf{N}}(s)$ is uniquely defined by the relation $\hat{\mathbf{T}}'(s) = \kappa(s)\hat{\mathbf{N}}(s)$ because $\hat{\mathbf{T}}'(s)$ and $\hat{\mathbf{T}}(s)$ are orthogonal and, by definition, $\|\hat{\mathbf{T}}'(s)\| = \kappa(s)$.

DEFINITION **15.2**. (Binormal Vector).
Let $\hat{\mathbf{T}}$ *and* $\hat{\mathbf{N}}$ *be the unit tangent and normal vectors at a point of a curve. The unit vector* $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$ *is called the* binormal (unit) vector*.*

So, with every point of a smooth curve with a non-vanishing curvature, one can associate a *unique* triple of mutually orthogonal unit vectors so that one of them is tangent to the curve while the other two span the plane normal to the tangent vector (normal to the curve). If the curvature vanishes, one can still define two unit mutually orthogonal vectors in the plane normal to the curve. However, any rotation in this plane would produce another pair of unit vectors with the same properties so that the triple of unit vectors is not unique.

By a suitable rotation, the triple of vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ can be oriented parallel to the axes of any given coordinate system, that is, parallel to $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$, respectively. Indeed, $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ can always be made parallel to $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. Then, owing to the relation $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$, the binormal must be parallel to $\hat{\mathbf{e}}_3$. In other words, $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ define a local *right-handed* coordinate system at each point of the curve. The orientation of the unit tangent, normal, and binormal vectors relative to some coordinate system depends on the point of the curve. The triple of these vectors can only rotate as the point slides along the curve (the vectors are mutually orthogonal and unit at any point). Therefore, the rates with respect to the arc length at which these vectors change must be characteristic for the shape of the curve (see Figure **15.1**, right panel).

By the definition of the curvature, $\hat{\mathbf{T}}'(s) = \kappa(s)\hat{\mathbf{N}}(s)$. Suppose $\kappa(s) \neq 0$ so that $\hat{\mathbf{N}}(s)$ is uniquely defined. Next, consider the rate:

$$\hat{\mathbf{B}}' = (\hat{\mathbf{T}} \times \hat{\mathbf{N}})' = \hat{\mathbf{T}}' \times \hat{\mathbf{N}} + \hat{\mathbf{T}} \times \hat{\mathbf{N}}' = \hat{\mathbf{T}} \times \hat{\mathbf{N}}'$$

because $\hat{\mathbf{T}}'(s)$ is parallel to $\hat{\mathbf{N}}(s)$. It follows from this equation that $\hat{\mathbf{B}}'$ is perpendicular to $\hat{\mathbf{T}}$, and, since $\hat{\mathbf{B}}$ is a unit vector, its derivative must also be perpendicular to $\hat{\mathbf{B}}$. Thus, $\hat{\mathbf{B}}'$ must be parallel to $\hat{\mathbf{N}}$. This conclusion establishes the existence of another scalar quantity that characterizes the curve shape.

DEFINITION **15.3**. (Torsion of a Curve).
Let $\hat{\mathbf{N}}(s)$ and $\hat{\mathbf{B}}(s)$ be unit normal and binormal vectors of the curve as functions of the arc length s. Then

$$\frac{d\hat{\mathbf{B}}(s)}{ds} = -\tau(s)\hat{\mathbf{N}}(s)$$

and the number $\tau(s)$ is called the torsion of the curve.

By definition, the torsion is measured in units of a reciprocal length, just like the curvature, because the unit vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ are dimensionless.

EXAMPLE **15.3**. *Use the natural parameterization of a helix given in Example* **13.3** *to find the unit tangent, normal, and binormal vectors, $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$, as functions of the arclength parameter. Express the torsion and curvature of the helix in terms its radius R and the height per turn h.*

SOLUTION: To simplify notations, put $\omega = 2\pi/L$ and $a = h/(2\pi)$ so that $\omega = (R^2 + a^2)^{-1/2}$ and the natural parameterization from Example **13.3** reads

$$\mathbf{r}(s) = \langle R\cos(\omega s),\ R\sin(\omega s),\ a\omega s\rangle\,.$$

Then using the definitions of $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$, the curvature, and the torsion,

$$\begin{aligned}
\hat{\mathbf{T}}(s) &= \mathbf{r}'(s) = \langle -R\omega\sin(\omega s),\ R\omega\cos(\omega s),\ a\omega\rangle\,,\\
\hat{\mathbf{T}}'(s) &= R\omega^2\langle -\cos(\omega s),\ -\sin(\omega s),\ 0\rangle\,,\\
\kappa(s) &= \|\hat{\mathbf{T}}'(s)\| = R\omega^2 = \frac{R}{R^2 + a^2}\,, \quad a = \frac{h}{2\pi}\\
\hat{\mathbf{N}}(s) &= \frac{1}{\kappa(s)}\hat{\mathbf{T}}'(s) = \langle -\cos(\omega s),\ -\sin(\omega s),\ 0\rangle\,,
\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{B}}(s) &= \hat{\mathbf{T}}(s)\times\hat{\mathbf{N}}(s) = \langle a\omega\sin(\omega s),\ -a\omega\cos(\omega s),\ R\omega\rangle\\ \hat{\mathbf{B}}'(s) &= \langle a\omega^2\cos(\omega s),\ -a\omega^2\sin(\omega s),\ 0\rangle = -a\omega^2\hat{\mathbf{N}}(s)\,,\\ \tau(s) &= a\omega^2 = \frac{a}{R^2+a^2}\,,\quad a=\frac{h}{2\pi}\,.\end{aligned}$$

So, a helix has a constant torsion and a constant curvature. □

At any point of a curve, the binormal $\hat{\mathbf{B}}$ is perpendicular to the osculating plane. So, if the curve is planar, then $\hat{\mathbf{B}}$ does not change along the curve, $\hat{\mathbf{B}}'(s)=\mathbf{0}$, because the osculating plane at any point coincides with the plane in which the curve lies. *A planar curve has no torsion.* Thus, *the torsion is a local numerical characteristic that determines how fast the curve deviates from the osculating plane while bending in it with some curvature radius.*

It follows from the relation $\hat{\mathbf{N}}=\hat{\mathbf{B}}\times\hat{\mathbf{T}}$ (compare $\hat{\mathbf{e}}_2=\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1$) that

$$\hat{\mathbf{N}}' = (\hat{\mathbf{B}}\times\hat{\mathbf{T}})' = \hat{\mathbf{B}}'\times\hat{\mathbf{T}}+\hat{\mathbf{B}}\times\hat{\mathbf{T}}' = -\tau\hat{\mathbf{N}}\times\hat{\mathbf{T}}+\kappa\hat{\mathbf{B}}\times\hat{\mathbf{N}} = \tau\hat{\mathbf{B}}-\kappa\hat{\mathbf{T}}\,,$$

where the definitions of the torsion and curvature have been used. The obtained rates of change of the unit vectors are known as the *Frenet-Serret formulas or equations*:

$$\hat{\mathbf{T}}'(s) = \kappa(s)\hat{\mathbf{N}}(s)\,, \tag{15.1}$$

$$\hat{\mathbf{N}}'(s) = -\kappa(s)\hat{\mathbf{T}}(s)+\tau(s)\hat{\mathbf{B}}(s)\,, \tag{15.2}$$

$$\hat{\mathbf{B}}'(s) = -\tau(s)\hat{\mathbf{N}}(s)\,. \tag{15.3}$$

The Frenet-Serret equations form a system of *differential equations* for the components of $\hat{\mathbf{T}}(s)$, $\hat{\mathbf{N}}(s)$, and $\hat{\mathbf{B}}(s)$. If the curvature and torsion are continuous functions on an interval $0\le s\le L$, then the system can be proved to have a unique solution on this interval for every given set of the vectors $\hat{\mathbf{T}}(0)$, $\hat{\mathbf{N}}(0)$, and $\hat{\mathbf{B}}(0)$ at an initial point of the curve. A geometrical meaning of the Frenet-Serret equations is that they determine how the triple of unit mutually orthogonal vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ at a point P on the curve is rotated as the point P moves along the curve and that this rotation is *uniquely* determined by the shape of the curve.

In particular, for the helix discussed in Example **15.3**, the unit vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ rotates about the z axis, while remaining mutually orthogonal, at a constant rate relative to the distance s traveled along the helix that is determined by the curvature and torsion of the helix. Indeed, by Study Problem **1.2** variations of the angle $\theta=s\omega$ in $\hat{\mathbf{T}}(s)$, $\hat{\mathbf{N}}(s)$, and $\hat{\mathbf{B}}(s)$ correspond to rotations about the z axis. The angle of rotation changes at a constant rate $d\theta/ds=\omega=(\kappa^2+\tau^2)^{-1/2}$ determined by the constant curvature and torsion of the helix. On the other hand, the geometrical parameters of the helix (its radius R and the height h per turn) are uniquely determined by the curvature and torsion as Example **15.3** shows. So the shape of a helix is determined by its curvature and torsion. *The observation that the curvature and the torsion determine the shape of a smooth curve is of general nature.*

THEOREM **15.1**. (Shape of a Smooth Curve in Space).
Given the curvature and torsion as continuous functions of the arclength parameter of a smooth curve, the curve is uniquely determined by them up to rigid rotations and translations of the curve as a whole, provided the curvature is nowhere zero.

A proof of Theorem **15.1** requires a proof of the uniqueness of a solution to the Frenet-Serret equations, which goes beyond the scope of this course. It should also be noted that the Frenet-Serret equations have a *unique* solution even without the condition that the curvature does not vanish. However, *wherever* $\kappa(s) = 0$*, the objects* $\hat{\mathbf{N}}(s)$ *and* $\tau(s)$ *no longer have the meaning that they were defined to have* earlier in this section. Solutions of the Frenet-Serret equations with $\kappa = 0$ and various $\tau \neq 0$ may describe the same curve in space. For example, if $\kappa(s) = \tau(s) = 0$, then $\hat{\mathbf{T}}(s) = \hat{\mathbf{T}}(0)$, $\hat{\mathbf{N}}(s) = \hat{\mathbf{N}}(0)$, and $\hat{\mathbf{B}}(s) = \hat{\mathbf{B}}(0)$. If $\mathbf{r}(s)$ is a natural parameterization of a curve, then $\mathbf{r}'(s) = \hat{\mathbf{T}}(s) = \hat{\mathbf{T}}(0)$. The integration of this equation yields $\mathbf{r}(s) = \mathbf{r}_0 + \hat{\mathbf{T}}(0)s$, where $\mathbf{r}_0$ is a constant vector, which is a straight line (or a part of it if the range of s is restricted). On the other hand, if $\kappa(s) = 0$ and $\tau(s) = \tau_0 \neq 0$, then it follows from Eq. (**15.1**) that unit tangent vector is still a constant vector $\hat{\mathbf{T}}(s) = \hat{\mathbf{T}}(0)$ and, hence, the curve is a line. As has been noted earlier, the choice of $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ is not unique wherever $\kappa(s) = 0$ (they can be chosen up to any rotation in the normal plane). This freedom shows up in the solution of Eqs. (**15.2**) and (**15.3**) (which is easy to verify by differentiation)

$$\begin{aligned}\hat{\mathbf{N}}(s) &= \cos(\tau_0 s)\hat{\mathbf{N}}(0) + \sin(\tau_0 s)\hat{\mathbf{B}}(0)\,,\\ \hat{\mathbf{B}}(s) &= \cos(\tau_0 s)\hat{\mathbf{B}}(0) - \sin(\tau_0 s)\hat{\mathbf{N}}(0)\,.\end{aligned}$$

This solution describes a rotation through the angle $\theta = \tau_0 s$ of the initial unit vectors $\hat{\mathbf{N}}(0)$ and $\hat{\mathbf{B}}(0)$ in the plane orthogonal to the line as the point travels the distance s along the line. This rotation is not associated with the shape of the curve (a line parallel to $\hat{\mathbf{T}}(0)$ in the case considered) and has no geometrical significance whatsoever. This illustrates the necessity of the hypothesis of a non-vanishing curvature in Theorem **15.1**.

EXAMPLE **15.4**. *Use the Frenet-Serret equations to prove that a curve with a constant curvature* $\kappa(s) = \kappa_0 \neq 0$ *and zero torsion* $\tau(s) = 0$ *is a circle (or its portion) of radius* $R = 1/\kappa_0$.

SOLUTION: A vector function $\mathbf{r}(s)$ that satisfies the Frenet-Serret equations is sought in the basis of the initial tangent, normal, and binormal vectors: $\hat{\mathbf{e}}_1 = \hat{\mathbf{T}}(0)$, $\hat{\mathbf{e}}_2 = \hat{\mathbf{N}}(0)$, and $\hat{\mathbf{e}}_3 = \hat{\mathbf{B}}(0)$. Since the torsion is 0, the binormal does not change along the curve, $\hat{\mathbf{B}}(s) = \hat{\mathbf{e}}_3$. The curve is planar and lies in a plane orthogonal to $\hat{\mathbf{e}}_3$. Any unit vector $\hat{\mathbf{T}}$ orthogonal to $\hat{\mathbf{e}}_3$ can always be written as $\hat{\mathbf{T}} = \cos\varphi\hat{\mathbf{e}}_1 + \sin\varphi\hat{\mathbf{e}}_2$ where $\varphi = \varphi(s)$ such that $\varphi(0) = 0$. Owing to the relations $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_3$, a unit vector $\hat{\mathbf{N}}$ orthogonal to $\hat{\mathbf{T}}$

such that $\hat{\mathbf{T}} \times \hat{\mathbf{N}} = \hat{\mathbf{B}} = \hat{\mathbf{e}}_3$ must have the form $\hat{\mathbf{N}} = -\sin\varphi\hat{\mathbf{e}}_1 + \cos\varphi\hat{\mathbf{e}}_2$. Equation (**15.1**) gives

$$\hat{\mathbf{T}}' = -\varphi'\sin\varphi\hat{\mathbf{e}}_1 + \varphi'\cos\varphi\,\hat{\mathbf{e}}_2 = \varphi'\hat{\mathbf{N}} = \kappa_0\hat{\mathbf{N}} \quad \Rightarrow \quad \varphi'(s) = \kappa_0$$

and therefore $\varphi(s) = \kappa_0 s$ because $\varphi(0) = 0$. For a natural parameterization of the curve, $\mathbf{r}'(s) = \hat{\mathbf{T}}(s)$. Hence,

$$\begin{aligned} \mathbf{r}'(s) &= \cos(\kappa_0 s)\hat{\mathbf{e}}_1 + \sin(\kappa_0 s)\hat{\mathbf{e}}_2\,, \\ \mathbf{r}(s) &= \mathbf{r}_0 + \kappa_0^{-1}\sin(\kappa_0 s)\hat{\mathbf{e}}_1 - \kappa_0^{-1}\cos(\kappa_0 s)\hat{\mathbf{e}}_2 \end{aligned}$$

where $\mathbf{r}_0$ is a constant vector. Put $R = 1/\kappa_0$. By the Pythagorean theorem, the distance between any point of the curve and a fixed point $\mathbf{r}_0$ is constant:

$$\|\mathbf{r}(s) - \mathbf{r}_0\|^2 = \frac{1}{\kappa_0^2}\Big(\sin^2(\kappa_0 s) + \cos^2(\kappa_0 s)\Big) = R^2\,.$$

Since the curve is planar, it is a circle (or its portion) of radius R. □

A calculation of the torsion based on Definition **15.3** requires a natural parameterization of a smooth. The following theorem provides the method to compute torsion of a curve by using any suitable parameterization.

THEOREM **15.2**. (Torsion of a Curve).
Let $\mathbf{r}(t)$ be three times differentiable vector function that traverses a smooth curve whose curvature does not vanish. Then the torsion of the curve is

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}.$$

PROOF. Put $\|\mathbf{r}'(t)\| = v(t)$ (if $s = s(t)$ is the arc length as a function of t, then $s' = v$). By (**14.3**) and the definition of the curvature,

$$\mathbf{r}'' = v'\hat{\mathbf{T}} + \kappa v^2\hat{\mathbf{N}}\,, \tag{15.4}$$

and by (**14.4**) and the definition of the binormal,

$$\mathbf{r}' \times \mathbf{r}'' = v\hat{\mathbf{T}} \times \mathbf{r}'' = \kappa v^3\hat{\mathbf{B}}. \tag{15.5}$$

Differentiation of both sides of (**15.4**) gives

$$\mathbf{r}''' = v''\hat{\mathbf{T}} + v'\hat{\mathbf{T}}' + (\kappa' v^2 + 2\kappa v v')\hat{\mathbf{N}} + \kappa v^2\hat{\mathbf{N}}'.$$

The derivatives $\hat{\mathbf{T}}'(t)$ and $\hat{\mathbf{N}}'(t)$ are found by making use of the differentiation rule $d/ds = (1/s'(t))(d/dt) = (1/v)(d/dt)$ in the Frenet-Serret equations (**15.1**) and (**15.2**):

$$\hat{\mathbf{T}}' = \kappa v\hat{\mathbf{N}}\,, \quad \hat{\mathbf{N}}' = -\kappa v\hat{\mathbf{T}} + \tau v\hat{\mathbf{B}}\,.$$

Therefore,

$$\mathbf{r}''' = (v'' - \kappa^2 v^3)\hat{\mathbf{T}} + (3\kappa v v' + \kappa' v^2)\hat{\mathbf{N}} + \kappa\tau v^3\hat{\mathbf{B}}. \tag{15.6}$$

By (**15.5**), (**15.6**), and the mutual orthogonality of the tangent, normal, and binormal vectors

$$(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = \kappa v^3\hat{\mathbf{B}} \cdot \mathbf{r}''' = \kappa^2 v^6\tau\hat{\mathbf{B}} \cdot \hat{\mathbf{B}} = \kappa^2 v^6\tau\,.$$

Therefore,

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\kappa^2 v^6} = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2}, \qquad \kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{v^3},$$

and the conclusion of the theorem follows from Theorem **14.1**. □

Relation (**15.5**) shows that $\hat{\mathbf{B}}$ is the unit vector in the direction of $\mathbf{r}' \times \mathbf{r}''$. This observation offers a more convenient way for calculating the unit binormal vector than its definition because it uses any parametrization of a smooth curve. The unit tangent, normal, and binormal vectors at a particular point $\mathbf{r}(t_0)$ of the curve $\mathbf{r}(t)$ are

(**15.7**) $$\hat{\mathbf{T}}(t_0) = \frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|},$$

(**15.8**) $$\hat{\mathbf{B}}(t_0) = \frac{\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)}{\|\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)\|},$$

(**15.9**) $$\hat{\mathbf{N}}(t_0) = \hat{\mathbf{B}}(t_0) \times \hat{\mathbf{T}}(t_0).$$

EXAMPLE **15.5**. *Find the unit tangent, normal, and binormal vectors and the torsion of the curve* $\mathbf{r}(t) = \langle \ln t, t, t^2/2 \rangle$ *at the point* $(0, 1, 1/2)$.

SOLUTION: The point in question corresponds to $t = 1$. Therefore

$$\begin{aligned}
\mathbf{r}'(1) &= \langle t^{-1},\ 1,\ t\rangle\Big|_{t=1} = \langle 1,\ 1,\ 1\rangle \quad \Rightarrow \quad \|\mathbf{r}'(1)\| = \sqrt{3} \\
\mathbf{r}''(1) &= \langle -t^{-2},\ 0,\ 1\rangle\Big|_{t=1} = \langle -1,\ 0,\ 1\rangle \\
\mathbf{r}'''(1) &= \langle 2t^{-3},\ 0,\ 0\rangle\Big|_{t=1} = \langle 2,\ 0,\ 0\rangle \\
\mathbf{r}'(1) \times \mathbf{r}''(1) &= \langle 1,\ -2,\ 1\rangle \quad \Rightarrow \quad \|\mathbf{r}'(1) \times \mathbf{r}''(1)\| = \sqrt{6} \\
\hat{\mathbf{T}}(1) &= \tfrac{1}{\sqrt{3}}\langle 1,\ 1,\ 1\rangle \\
\hat{\mathbf{B}}(1) &= \tfrac{1}{\sqrt{6}}\langle 1,\ -2,\ 1\rangle \\
\hat{\mathbf{N}}(1) &= \tfrac{1}{\sqrt{6}\sqrt{3}}\langle 1,\ -2,\ 1\rangle \times \langle 1,\ 1,\ 1\rangle = \tfrac{1}{3\sqrt{2}}\langle -3,\ 0,\ 3\rangle \\
&= \tfrac{1}{\sqrt{2}}\langle -1,\ 0,\ 1\rangle \\
\tau(1) &= \frac{(\mathbf{r}'(1) \times \mathbf{r}''(1)) \cdot \mathbf{r}'''(1)}{\|\mathbf{r}'(1) \times \mathbf{r}''(1)\|^2} = \frac{2}{6} = \frac{1}{3}
\end{aligned}$$

□

Remark. A smooth curve C has a unit tangent vector at a point P. So a small part of the curve (a part of a small arclength s) containing P can be approximated by a piece of the tangent line of the same arclength s. If the curve C has a nonzero curvature at P, then a better approximation is given by a part of the osculating circle of the arclength s (see Study Problem **14.4**). If the curve C has a nonzero torsion at P, an even more accurate approximation is provided by a curve through P that has the same unit tangent vector at P, and a constant curvature and torsion equal to the curvature and torsion of the curve C at P. By Theorem **15.1** such a curve is unique. As shown in Example **15.3** (see also Study Problem **15.3**), it is a helix whose radius and length of each turn are uniquely determined by the curvature and torsion. These three successively more accurate approximations do not refer to any particular coordinate system or any particular parameterization of C as the approximation curves are fully determined as the point sets in space by the geometrical invariants of the curve C at P: the unit tangent vector, curvature, and torsion.

15.3. Study Problems.

Problem **15.1**. *Find the position vector* $\mathbf{r}(t)$ *of a particle as a function of time* t *if the particle moves clockwise along a circular path of radius* R *in the* xy *plane through* $\mathbf{r}(0) = \langle R, 0, 0\rangle$ *with a constant speed* v_0.

SOLUTION: For a circle of radius R in the xy plane through the point $(R, 0, 0)$,

$$\mathbf{r}(t) = \langle R\cos\varphi, R\sin\varphi, 0\rangle\,,$$

where $\varphi = \varphi(t)$ such that $\varphi(0) = 0$. Then the velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \varphi'\langle -R\sin\varphi, R\cos\varphi, 0\rangle\,.$$

Hence, the condition $\|\mathbf{v}(t)\| = v_0$ yields $R|\varphi'(t)| = v_0$ or $\varphi(t) = \pm(v_0/R)t$ and

$$\mathbf{r}(t) = \langle R\cos(\omega t), \pm R\sin(\omega t), 0\rangle\,,$$

where $\omega = v_0/R$ is the angular velocity. The second component must be taken with the minus sign because the particle revolves clockwise with increasing t (the second component should become negative immediately after $t = 0$). □

Problem **15.2**. *Let the particle position vector as a function of time* t *be* $\mathbf{r}(t) = \langle \ln(t), t^2, 2t\rangle$, $t > 0$. *Find the speed, tangential and normal accelerations, the unit tangent, normal, and binormal vectors, and the torsion of the trajectory at the point* $P_0(0, 1, 2)$.

SOLUTION: By Example **14.2**, the velocity and acceleration vectors at P_0 are $\mathbf{v} = \langle 1, 2, 2\rangle$ and $\mathbf{a} = \langle -1, 2, 0\rangle$. So the speed is $v = \|\mathbf{v}\| = 3$. The tangential acceleration is $a_T = \mathbf{v}\cdot\mathbf{a}/v = 1$. As $\mathbf{v}\times\mathbf{a} = 2\langle -2, -1, 2\rangle$, the normal acceleration is $a_N = \|\mathbf{v}\times\mathbf{a}\|/v = 6/3 = 2$. The unit tangent, binormal, and normal vectors are obtained by Eqs. (**15.7**)–(**15.9**), where $t_0 = 1$:

$$\begin{aligned}
\hat{\mathbf{T}} &= \frac{1}{v}\,\mathbf{v} = \frac{1}{3}\,\langle 1, 2, 2\rangle\,,\\
\hat{\mathbf{B}} &= \frac{1}{\|\mathbf{v}\times\mathbf{a}\|}\mathbf{v}\times\mathbf{a} = \frac{1}{3}\langle -2, -1, 2\rangle\,,\\
\hat{\mathbf{N}} &= \hat{\mathbf{T}}\times\hat{\mathbf{B}} = \frac{1}{3}\langle -2, 2, -1\rangle\,.
\end{aligned}$$

The torsion at P_0 is calculated by Theorem **15.2** with $t = 1$. One has $\mathbf{r}'''(1) = \langle 2/t^2, 0, 0\rangle|_{t=1} = \langle 2, 0, 0\rangle = \mathbf{b}$. Therefore,

$$\tau(1) = (\mathbf{v}\times\mathbf{a})\cdot\mathbf{b}/\|\mathbf{v}\times\mathbf{a}\|^2 = -\frac{8}{36} = -\frac{2}{9}\,.$$

□

Problem **15.3**. (Curves with Constant Curvature and Torsion). *Prove that all curves with a constant curvature $\kappa(s) = \kappa_0 \neq 0$ and a constant torsion $\tau(s) = \tau_0 \neq 0$ are helices by integrating the Frenet-Serret equations.*

SOLUTION: It follows from (**15.1**) and (**15.3**) that the vector

$$\mathbf{w} = \tau\hat{\mathbf{T}} + \kappa\hat{\mathbf{B}}$$

does not change along the curve, $\mathbf{w}'(s) = 0$. Indeed, because $\kappa'(s) = \tau'(s) = 0$, one has

$$\mathbf{w}' = \tau\hat{\mathbf{T}}' + \kappa\hat{\mathbf{B}}' = (\tau\kappa - \tau\kappa)\hat{\mathbf{N}} = \mathbf{0}\,.$$

By the Pythagorean theorem, $\|\mathbf{w}\| = (\kappa_0^2 + \tau_0^2)^{1/2}$. Consider two new unit vectors orthogonal to $\hat{\mathbf{N}}$:

$$\hat{\mathbf{w}} = \frac{1}{\|\mathbf{w}\|}\,\mathbf{w} = \sin\alpha\,\hat{\mathbf{T}} + \cos\alpha\,\hat{\mathbf{B}}, \qquad \hat{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{T}} - \sin\alpha\,\hat{\mathbf{B}}\,,$$

where $\cos\alpha = \kappa_0/\omega$, $\sin\alpha = \tau_0/\omega$, and $\omega = (\kappa_0^2 + \tau_0^2)^{1/2}$. By construction, the unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{w}}$, and $\hat{\mathbf{N}}$ are mutually orthogonal unit vectors, which is easy to verify by calculating the corresponding dot products, $\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} = \hat{\mathbf{w}}\cdot\hat{\mathbf{w}} = 1$ and $\hat{\mathbf{u}}\cdot\hat{\mathbf{w}} = 0$. Also,

$$\hat{\mathbf{w}}\times\hat{\mathbf{u}} = -\cos^2\alpha\,\hat{\mathbf{T}}\times\hat{\mathbf{B}} + \sin^2\alpha\,\hat{\mathbf{B}}\times\hat{\mathbf{T}} = (\cos^2\alpha + \sin^2\alpha)\hat{\mathbf{N}} = \hat{\mathbf{N}}\,.$$

By differentiating the vector $\hat{\mathbf{u}}$ and using the Frenet-Serret equations,

$$\hat{\mathbf{u}}' = \cos\alpha\,\hat{\mathbf{T}}' - \sin\alpha\,\hat{\mathbf{B}}' = (\kappa_0\cos\alpha + \tau_0\sin\alpha)\hat{\mathbf{N}} = \omega\hat{\mathbf{N}}\,.$$

Since $\hat{\mathbf{w}}(s) = \hat{\mathbf{w}}(0)$ is a constant unit vector, it is convenient to seek a solution in an orthonormal basis such that $\hat{\mathbf{e}}_3 = \hat{\mathbf{w}}(0)$ and $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$. In this basis $\hat{\mathbf{u}} = \cos\varphi\,\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_2$, where $\varphi = \varphi(s)$, is a unit vector in the plane orthogonal to $\hat{\mathbf{e}}_3$. The orientation of the basis vectors in the plane orthogonal to $\hat{\mathbf{e}}_3$ is defined up to a general rotation about $\hat{\mathbf{e}}_3$. This freedom is used to set $\hat{\mathbf{e}}_1 = \hat{\mathbf{u}}(0)$, which implies that the function $\varphi(s)$ satisfies the condition $\varphi(0) = 0$. Then the unit normal vector in this basis is

$$\hat{\mathbf{N}} = \hat{\mathbf{w}}\times\hat{\mathbf{u}} = \cos\varphi\,\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1 + \sin\varphi\,\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2 = \cos\varphi\,\hat{\mathbf{e}}_2 - \sin\varphi\,\hat{\mathbf{e}}_1$$

and

$$\hat{\mathbf{u}}' = -\varphi'\sin\varphi\,\hat{\mathbf{e}}_1 + \varphi'\cos\varphi\,\hat{\mathbf{e}}_2 = \varphi'\hat{\mathbf{N}}\,.$$

The comparison of this representation of $\hat{\mathbf{u}}'$ with that obtained before yields $\varphi'(s) = \omega$ or $\varphi(s) = \omega s$ owing to the condition $\varphi(0) = 0$. Expressing the vector $\hat{\mathbf{T}}$ via $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$,

$$\hat{\mathbf{T}} = \cos\alpha\,\hat{\mathbf{u}} + \sin\alpha\,\hat{\mathbf{w}}\,,$$

one infers (compare Example **15.4**)

$$\mathbf{r}'(s) = \hat{\mathbf{T}}(s) = \frac{\kappa_0}{\omega}\cos(\omega s)\,\hat{\mathbf{e}}_1 + \frac{\kappa_0}{\omega}\sin(\omega s)\,\hat{\mathbf{e}}_2 + \frac{\tau_0}{\omega}\,\hat{\mathbf{e}}_3$$

where $\mathbf{r}(s)$ is a natural parameterization of the curve. The integration of this equation gives

$$\mathbf{r}(s) = \mathbf{r}_0 + R\sin(\omega s)\,\hat{\mathbf{e}}_1 - R\cos(\omega s)\,\hat{\mathbf{e}}_2 + hs\,\hat{\mathbf{e}}_3, \quad R = \frac{\kappa_0}{\omega^2}\,, \quad h = \frac{\tau_0}{\omega},$$

This is a helix of radius R whose axis goes through the point $\mathbf{r}_0$ parallel to $\hat{\mathbf{e}}_3$; the helix climbs along its axis by $2\pi h/\omega$ per each turn.
Remark. The fact that a curve with constant nonzero curvature and torsion is a helix can also be established by Theorem **15.1**. By Example **15.3**, a helix has constant nonzero curvature and torsion. By Theorem **15.1**, the curve with such curvature and torsion is unique modulo rigid rotations and translations. Thus, the curve is a helix. □

Problem **15.4**. (Motion in a Constant Magnetic Field, Revisited).
The force acting on a charged particle moving in the magnetic field $\mathbf{B}$ *is given by* $\mathbf{F} = (e/c)\mathbf{v}\times\mathbf{B}$*, where* e *is the electric charge of the particle,* c *is the speed of light, and* $\mathbf{v}$ *is its velocity. Show that the trajectory of the particle in a constant magnetic field is a helix whose axis is parallel to the magnetic field by integrating Newton's equations of motion.*

Solution: In contrast to Study Problem **12.3**, here the shape of the trajectory is to be obtained directly from Newton's second law with arbitrary initial conditions. Choose the coordinate system so that the magnetic field is parallel to the z axis, $\mathbf{B} = B\hat{\mathbf{e}}_3$, where B is the magnitude of the magnetic field. Newton's law of motion, $m\mathbf{a} = \mathbf{F}$, where m is the mass of the particle, determines the acceleration, $\mathbf{a} = \mu\mathbf{v}\times\mathbf{B} = \mu B\mathbf{v}\times\hat{\mathbf{e}}_3$, where $\mu = e/(mc)$. First, note that $v_3' = a_3 = \hat{\mathbf{e}}_3\cdot\mathbf{a} = 0$. Hence, $v_3 = v_\parallel = \text{const}$ (the component of the velocity parallel to the magnetic field remains constant).

Second, by the geometrical property of the cross product the acceleration and velocity remain orthogonal during the motion, and therefore the tangential acceleration vanishes, $a_T = \mathbf{v}\cdot\mathbf{a} = 0$. Hence, the speed of the particle is a constant of motion, $v = v_0$ (because $v' = a_T = 0$). Let us make the orthogonal decomposition of $\mathbf{v}$ relative to the magnetic field: $\mathbf{v} = \mathbf{v}_\perp + v_\parallel\hat{\mathbf{e}}_3$, where $\mathbf{v}_\perp$ is in the xy plane. Since $\|\mathbf{v}\| = v_0$, the magnitude of $\mathbf{v}_\perp$ is also constant, $\|\mathbf{v}_\perp\| = v_\perp = (v_0^2 - v_\parallel^2)^{1/2}$. The velocity vector can therefore be written in the form $\mathbf{v} = \langle v_\perp\cos\varphi, v_\perp\sin\varphi, v_\parallel\rangle$, where the function $\varphi = \varphi(t)$ is to be determined by the equations of motion:

$$\begin{aligned}\mathbf{a} &= \mu B\,\mathbf{v}\times\hat{\mathbf{e}}_3 = \mu B\,\langle v_\perp\sin\varphi,\ -v_\perp\cos\varphi,\ 0\rangle\,,\\ \mathbf{a} &= \mathbf{v}' = \varphi'\,\langle -v_\perp\sin\varphi,\ v_\perp\cos\varphi,\ 0\rangle\,.\end{aligned}$$

It follows from the comparison of these expressions that $\varphi'(t) = -\mu B$ or $\varphi(t) = -\mu Bt + \varphi_0 = \omega t + \varphi_0$ where $\omega = eB/(mc)$ is the so-called cyclotron frequency and the integration constant φ_0 is determined by the initial velocity: $\mathbf{v}(0) = \langle v_\perp \cos\varphi_0, v_\perp \sin\varphi_0, v_\| \rangle$, i.e., $\tan\varphi_0 = v_2(0)/v_1(0)$. Integration of the equation

$$\mathbf{r}'(t) = \mathbf{v}(t) = \langle v_\perp \cos(\omega t - \varphi_0),\ -v_\perp \sin(\omega t - \varphi_0),\ v_\| \rangle$$

yields the trajectory of motion:

$$\mathbf{r}(t) = \mathbf{r}_0 + \langle R\sin(\omega t - \varphi_0), R\cos(\omega t - \varphi_0), v_\| t \rangle,$$

where $R = v_\perp/\omega$. This equation describes a helix of radius R whose axis goes through $\mathbf{r}_0$ parallel to the z axis. So a charged particle moves along a helix that winds about force lines of the magnetic field. The particle revolves in the plane perpendicular to the magnetic field with frequency $\omega = eB/(mc)$. In each turn, the particle moves along the magnetic field a distance $h = 2\pi v_\|/\omega$. In particular, if the initial velocity is orthogonal to the magnetic field (i.e., $v_\| = 0$), then the trajectory is a circle of radius R.
The polar lights. The Sun produces a stream of charged particles (the solar wind). The magnetic field of the Earth plays the role of a shield from the solar wind as it traps the particles forcing them to travel along its force lines that are arcs connecting the magnetic poles of the Earth (which approximately coincide with the south and north poles). As a result, the solar wind particles can penetrate the lower atmosphere only near the magnetic poles of the Earth causing a spectacular phenomenon, the polar lights, by colliding with molecules of the oxygen and nitrogen in the atmosphere. □

Problem 15.5. *Suppose that the force acting on a particle of mass m is proportional to the position vector of the particle (such forces are called* central*). Prove that the angular momentum of the particle, $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$, is a constant of motion (i.e., $d\mathbf{L}/dt = 0$).*

SOLUTION: Since a central force $\mathbf{F}$ is parallel to the position vector $\mathbf{r}$, their cross product vanishes, $\mathbf{r}\times\mathbf{F} = \mathbf{0}$. By Newton's second law, $m\mathbf{a} = \mathbf{F}$ and hence $m\mathbf{r}\times\mathbf{a} = \mathbf{0}$. Therefore,

$$\frac{d\mathbf{L}}{dt} = m(\mathbf{r}\times\mathbf{v})' = m(\mathbf{r}'\times\mathbf{v} + \mathbf{r}\times\mathbf{v}') = m\mathbf{r}\times\mathbf{a} = \mathbf{0},$$

where $\mathbf{r}' = \mathbf{v}$, $\mathbf{v}' = \mathbf{a}$, and $\mathbf{v}\times\mathbf{v} = \mathbf{0}$ have been used. □

Problem 15.6. (Kepler's Laws of Planetary Motion). *Newton's law of gravity states that two masses m and M at a distance r are attracted by a force of magnitude GmM/r^2, where G is the universal constant (called* Newton's constant*). Prove Kepler's laws of planetary motion:*
1. A planet revolves around the Sun in an elliptical orbit with the Sun at one focus.
2. The line joining the Sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

SOLUTION: Let the Sun be at the origin of a coordinate system and let $\mathbf{r}$ be the position vector of a planet. The mass of the Sun is much larger than the mass of a planet and therefore a displacement of the Sun due to the gravitational pull from a planet can be neglected (e.g., the Sun is about 332946 times heavier than the Earth). Let $\hat{\mathbf{r}} = \mathbf{r}/r$ be the unit vector parallel to $\mathbf{r}$. Then the gravitational force is

$$\mathbf{F} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}} = -\frac{GMm}{r^3}\,\mathbf{r},$$

where M is the mass of the Sun and m is the mass of a planet. The minus sign is necessary because an attractive force must be opposite to the position vector. By Newton's second law, the trajectory of a planet satisfies the equation $m\mathbf{a} = \mathbf{F}$ and hence

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}.$$

The gravitional force is a central force, and, by Study Problem **15.5**, the vector $\mathbf{r}\times\mathbf{v} = \mathbf{l}$ is a constant of motion. One has $\mathbf{v} = \mathbf{r}' = (r\hat{\mathbf{r}})' = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'$. Using this identity, the constant of motion can also be written as

$$\mathbf{l} = \mathbf{r}\times\mathbf{v} = r\hat{\mathbf{r}}\times\mathbf{v} = r(r'\hat{\mathbf{r}}\times\hat{\mathbf{r}} + r\hat{\mathbf{r}}\times\hat{\mathbf{r}}') = r^2(\hat{\mathbf{r}}\times\hat{\mathbf{r}}').$$

Using the rule for the double cross product (see Study Problem **4.4**), one infers that

$$\mathbf{a}\times\mathbf{l} = -\frac{GM}{r^2}\hat{\mathbf{r}}\times\mathbf{l} = -GM\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{r}}') = GM\hat{\mathbf{r}}',$$

where $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$ has been used. On the other hand,

$$(\mathbf{v}\times\mathbf{l})' = \mathbf{v}'\times\mathbf{l} + \mathbf{v}\times\mathbf{l}' = \mathbf{a}\times\mathbf{l}$$

because $\mathbf{l}' = \mathbf{0}$. It follows from these two equations that

$$(\mathbf{15.10})\qquad (\mathbf{v}\times\mathbf{l})' = GM\hat{\mathbf{r}}' \quad\Longrightarrow\quad \mathbf{v}\times\mathbf{l} = GM\hat{\mathbf{r}} + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. The motion is characterized by two constant vectors $\mathbf{l}$ and $\mathbf{c}$. It occurs in the plane through the origin that is orthogonal to the constant vector $\mathbf{l}$ because $\mathbf{l} = \mathbf{r}\times\mathbf{v}$ must be orthogonal to $\mathbf{r}$. It also follows from (**15.10**) and $\mathbf{l}\cdot\hat{\mathbf{r}} = 0$ that the constant vectors $\mathbf{l}$ and $\mathbf{c}$ are orthogonal because $\mathbf{l}\cdot\mathbf{c} = 0$. It is therefore convenient to choose the coordinate system so that $\mathbf{l}$ is parallel to the z axis and $\mathbf{c}$ to the x axis as shown in Figure **15.3** (left panel).

The vector $\mathbf{r}$ lies in the xy plane. Let θ be the polar angle of $\mathbf{r}$ (i.e., $\mathbf{r}\cdot\mathbf{c} = rc\cos\theta$, where $c = \|\mathbf{c}\|$ is the length of $\mathbf{c}$). Then

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{l}) = \mathbf{r}\cdot(GM\hat{\mathbf{r}} + \mathbf{c}) = GMr + rc\cos\theta.$$

On the other hand, using a cyclic permutation in the triple product,

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{l}) = \mathbf{l}\cdot(\mathbf{r}\times\mathbf{v}) = \mathbf{l}\cdot\mathbf{l} = l^2,$$

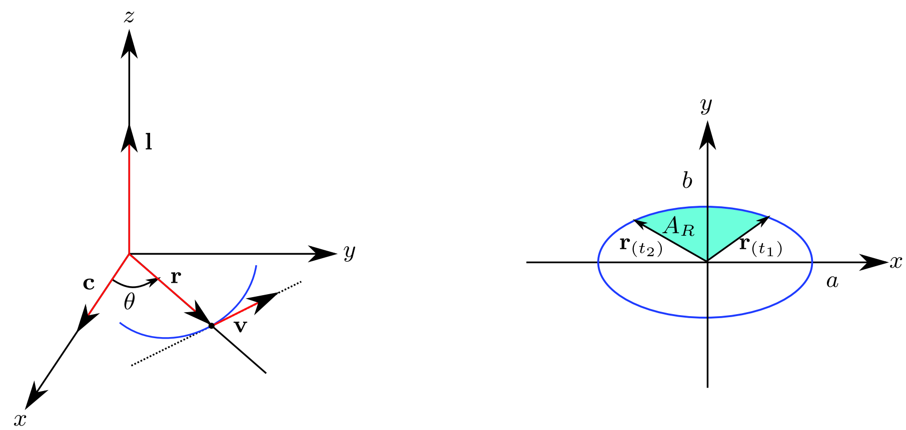

FIGURE **15.2**. **Left**: The setup of the coordinate system for the derivation of Kepler's first law. **Right**: An illustration to the derivation of Kepler's second law.

where $l = \|\mathbf{l}\|$ is the length of $\mathbf{l}$. The comparison of the last two equations yields the equation for the trajectory:

$$l^2 = r(GM + c\cos\theta) \quad \Longrightarrow \quad r = \frac{ed}{1 + e\cos\theta},$$

where $d = l^2/c$ and $e = c/(GM)$. This is the polar equation of a conic section with focus at the origin and eccentricity e (see Calculus II). Thus, *all possible trajectories of any massive body in a solar system are conic sections*! This is a quite remarkable result. Parabolas and hyperbolas do not correspond to a periodic motion. So a planet must follow an elliptic trajectory with the Sun at one focus. All objects coming to the solar system from outer space (i.e., those that are not confined by the gravitational pull of the Sun) should follow either parabolic or hyperbolic trajectories.

To prove Kepler's second law, put $\hat{\mathbf{r}} = \langle \cos\theta, \sin\theta, 0\rangle$ and hence $\hat{\mathbf{r}}' = \langle -\theta'\sin\theta, \theta'\cos\theta, 0\rangle$. Therefore,

$$\mathbf{l} = r^2(\hat{\mathbf{r}} \times \hat{\mathbf{r}}') = \langle 0, 0, r^2\theta'\rangle \quad \Longrightarrow \quad l = r^2\theta'\,.$$

The area of a sector with angle $d\theta$ swept by $\mathbf{r}$ is $dA = \frac{1}{2}r^2\,d\theta$ (see Calculus II; the area bounded by a polar graph $r = r(\theta)$). Hence,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{l}{2}.$$

For any moments of time t_1 and t_2, the area of the sector between $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$ is

$$A_{12} = \int_{t_1}^{t_2} \frac{dA}{dt}\,dt = \int_{t_1}^{t_2} \frac{l}{2}\,dt = \frac{l}{2}(t_2 - t_1).$$

Thus, the position vector $\mathbf{r}$ sweeps out equal areas in equal times (see Figure **15.3**, right panel).

Kepler's third law follows from the last equation. Indeed, the entire area of the ellipse A is swept when $t_2 - t_1 = T$ is the period of the motion. If the major and minor axes of the ellipse are $2a$ and $2b$, respectively, $a > b$, then $A = \pi ab = lT/2$ and $T = 2\pi ab/l$. Now recall that $ed = b^2/a$ for an elliptic conic section (see Calculus II) or $b^2 = eda = l^2a/(GM)$. Hence,

$$T^2 = \frac{4\pi^2a^2b^2}{l^2} = \frac{4\pi^2}{GM}\,a^3.$$

Note that the proportionality constant $4\pi^2/(GM)$ is independent of the mass of a planet; therefore, Kepler's laws are *universal* for all massive objects trapped by the Sun (planets, asteroids, and comets). □

15.4. Exercises.

1–7. For each of the following trajectories of a particle, find the velocity, speed, and the normal and tangential acceleration as functions of time, and their values at a specified point P:

1. $\mathbf{r}(t) = \langle t,\ 1-t,\ t^2+1\rangle\,, \quad P = (1,0,2)\,;$
2. $\mathbf{r}(t) = \langle t^2,\ t,\ 1\rangle\,, \quad P = (4,2,1)\,;$
3. $\mathbf{r}(t) = \langle 4t^{3/2},\ -t^2,\ t\rangle\,, \quad P = (4,-1,1)\,;$
4. $\mathbf{r}(t) = \langle \ln t,\ \sqrt{t},\ t^2\rangle\,, \quad P = (0,1,1)\,;$
5. $\mathbf{r}(t) = \langle \cosh t,\ \sinh t,\ 2+t\rangle\,, \quad P = (1,0,2)\,;$
6. $\mathbf{r}(t) = \langle e^t,\ \sqrt{2}\,t,\ e^{-t}\rangle\,, \quad P = (1,0,1)\,;$
7. $\mathbf{r}(t) = \langle \sin t - t\cos t,\ t^2,\ \cos t + t\sin t\rangle\,, \quad P = (0,0,1)\,.$

8. Find the normal and tangential accelerations of a particle with the position vector $\mathbf{r}(t) = \langle t^2+1, t, t^2-1\rangle$ when the particle is at the least distance from the origin.

9. Find the tangential and normal accelerations of a particle with the position vector $\mathbf{r}(t) = \langle R\sin(\omega t+\varphi_0), -R\cos(\omega t+\varphi_0), v_0t\rangle$, where R, ω, φ_0, and v_0 are constants (see Study Problem **15.4**).

10. The shape of a winding road can be approximated by the graph $y = L\cos(x/L)$, where the coordinates are in miles and $L = 0.1$ mile. The condition of the road is such that if the normal acceleration of a car on it exceeds $0.13g$, where g is the acceleration of the free fall, the car may skid off the road. Recommend a speed limit for this portion of the road.

11. A particle moves along the curve $y = x^2 + x^3$ in the direction of increasing x. If the acceleration of the particle at the point $(1,2)$ is $\mathbf{a} = \langle -3, -1\rangle$, find its normal and tangential accelerations.

12. Suppose that a particle moves so that its tangential acceleration a_T is constant, while the normal acceleration a_N remains 0. What is the trajectory of the particle?

13. Suppose that a particle moves in a plane so that its tangential acceleration a_T remains 0, while the normal acceleration a_N is constant. What is the trajectory of the particle? Hint: Investigate the curvature of the trajectory.

14. A race car moves with a constant speed v_0 along an elliptic track

$x^2/a^2 + y^2/b^2 = 1$, $a > b$. Find the maximal and minimal values of the magnitude of its acceleration and the points where they occur.

15. Does there exist a curve with zero curvature and a non-zero torsion? Explain the answer.

16–20. For each of the following curves, find the unit tangent, normal, and binormal vectors and the torsion at a specified point P:

16. $\mathbf{r}(t) = \langle t,\ 1-t,\ t^2+1\rangle\,,\quad P=(1,0,2)\,;$
17. $\mathbf{r}(t) = \langle t^3,\ t^2,\ 1\rangle\,,\quad P=(8,4,1)\,;$
18. $\mathbf{r}(t) = \langle 4t^{3/2},\ -t^2,\ t\rangle\,,\quad P=(4,-1,1)\,;$
19. $\mathbf{r}(t) = \langle \ln t,\ 2\sqrt{t},\ t^2\rangle\,,\quad P=(0,2,1)\,;$
20. $\mathbf{r}(t) = \langle \cosh t,\ \sinh t,\ 2+t\rangle\,,\quad P=(1,0,2)\,.$

21. Let $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$. Find the speed, the tangential and normal accelerations, the curvature and torsion, and the unit tangent vector, normal, and binormal as functions of time t.
Hint: To simplify calculations, find the decomposition $\mathbf{r}(t) = \mathbf{v}(t) - t\mathbf{w}(t) + t^2\hat{\mathbf{e}}_3$ where $\mathbf{v}$, $\mathbf{w}$, and $\hat{\mathbf{e}}_3$ are mutually orthogonal unit vectors such that $\mathbf{v}'(t) = \mathbf{w}(t)$, $\mathbf{w}'(t) = -\mathbf{v}(t)$. Use the properties of the cross products of mutually orthogonal unit vectors.

22. Let C be the curve of intersection of an ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2 = 1$ with the plane $2x - 2y + z = 0$. Find the torsion and the binormal $\hat{\mathbf{B}}$ along C.

Selected Answers and Hints to Exercises

Section 10. **1**. $-\infty < t < \infty$. **2**. $t \geq 0$. **3**. $0 < t \leq 3$. **4**. The union of three open intervals $(-3,-2)$, $(-2,0)$, and $(0,3)$. **5**. $t = 1$. **6**. $\langle 1,0,-1\rangle$. **7**. The limit does not exist. **8**. $\langle 1,0,0\rangle$. **9**. $\langle 0,0,4\rangle$. **10**. $\langle 0,-1,0\rangle$. **11**. The limit does not exist. **12**. $\langle 2,\frac{1}{2},0\rangle$. **13**. $\langle 4,2,-\frac{1}{2}\rangle$. **14**. $\langle 2,1,1\rangle$. **15**. $\langle 4,0,\frac{1}{4}\rangle$. **16**. A helix about the x axis, radius is 1; the helix rises by $2\pi/3$ units per turn. **17**. An ellipse that is the intersection of the elliptic cylinder $x^2/4 + z^2/9 = 1$ and the coordinate plane $y = 4$. **19**. The curve is obtained by wrapping the graph of $\ln t$ on the cylinder $x^2 + y^2 = 1$. The line parallel to the z axis through the point $(0,1,0)$ (it lies on the cylinder) is a vertical asymptote of the curve as $t \to 0^+$. **21**. The curve is obtained by wrapping the graph of $\sin^2(\pi t)$ to the parabolic cylinder $x = y^2$. **22**. The curve is the circle of intersection of the sphere $x^2 + y^2 + z^2 = 2$ and the plane $y = x$. **23**. No collision. Two points of intersection: $\mathbf{r}_1(1) = \mathbf{r}_2(0) = \langle 1,1,1\rangle$ and $\mathbf{r}_1(2) = \mathbf{r}_2(1/2) = \langle 2,4,8\rangle$. **24**. $x = t$, $z = t^2$, $y = 2(1 - t^2 - t^4/9)^{1/2}$. **25**. $\mathbf{r}(t) = \langle t,t^2,1\rangle$. **26**. $\mathbf{r}(t) = \langle \sin t,\, t,\, \sin t\rangle$. **27**. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, \frac{9}{2}\sin(2t)\rangle$. **29**. $\mathbf{r}(t) = \langle t,\, t^2,\, t^2{+}t^4\rangle$. **30**. $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1 - 2\cos t - 3\sin t\rangle$. **31**. $\mathbf{r}(t) = \langle \cos t, \cos t, 3\sin t\rangle$. **32**. $\mathbf{r}(t) = \langle 1 + \cos t, \sin t, 2 + 2\cos t\rangle$. **33**. $\frac{\pi}{6} \leq t \leq \frac{5\pi}{6}$ and $\frac{7\pi}{6} \leq t \leq \frac{11\pi}{6}$. **34**. $a = 0$ and $b = 4$. **35**. $\mathbf{r}(0) = \langle 0,1,1\rangle$. **36**. $\mathbf{r}(0) = \langle 0,1,1\rangle$. **37**. No such $\mathbf{r}(0)$ exists. **38**. $\mathbf{r}(0) = \langle 2,3,0\rangle$. **39**. $\mathbf{r}(0) = \langle \frac{1}{2},0,2\rangle$. **42**. $\langle 0,1,\frac{3}{2}\rangle$. **43**. 9. **44**. $\langle -\frac{1}{4},1,2\rangle$. **45**. 4. **46**. $\langle 3,-\frac{1}{4},\frac{1}{2}\rangle$. **47**. $\mathbf{0}$ (by the Jacobi identity).

Section 11. **1**. $\mathbf{r}'(t) = \langle 0,1,3t^2\rangle$. **2**. $\mathbf{r}'(t) = \langle -\sin t, \sin(2t), 2t\rangle$. **3**. $\mathbf{r}'(t) = \langle \frac{1}{t}, 2e^{2t}, e^{-t}(1-t)\rangle$. **4**. $\mathbf{r}'(t) = \langle \frac{1}{3}(t-2)^{-2/3}, t(t^2-4)^{-1/2}, 1\rangle$. **5**. $\mathbf{r}'(t) = 2t\mathbf{b} - e^t\mathbf{c}$. **6**. $\mathbf{r}'(t) = \mathbf{a}\times\mathbf{b} - e^t(t+1)\mathbf{a}\times\mathbf{c}$. **8**. smooth everywhere. **9**. smooth everywhere except the point $(0,0,2)$. **12**. smooth everywhere except the point $(0,1,0)$. **14**. $x = 6{+}5t$, $y = 9{+}9t$, $z = 6{+}2t$. **15**. $x = t$, $y = 2{+}t$, $z = 1 + 2t$. **16**. $\langle \frac{2}{3},\frac{2}{3},-\frac{1}{3}\rangle$. **18**.$\mathbf{r}'\cdot\mathbf{r}'' = 4t + 18t^3$, $\mathbf{r}'\times\mathbf{r}'' = \langle 6t^2,-6t,2\rangle$. **19**. Yes, at the point $\mathbf{r}(-2) = \langle 6,-\frac{8}{3},-4\rangle$. **21**. The point of intersection is $\mathbf{r}(1) = \langle 0,1,2\rangle$, the angle is $\cos^{-1}(\frac{4}{3\sqrt{2}})$. **22**. No intersection; the distance between the curve and the plane is $4/\sqrt{3}$. **23**. The point of intersection is $\mathbf{r}_1(1) = \mathbf{r}_2(2) = \langle 1,0,4\rangle$; the angle is $\cos^{-1}(\frac{1}{\sqrt{3}})$. **25**. $\pi/2$. **30**. $x = 1 + t$, $y = 1 + t$, $z = 1 + 2t$ (tangent line); $x + y + 2z = 4$ (normal plane). **31**. $x = 1 + 3t$, $y = 1 + 3t$, $z = 3 - t$ (tangent line); $3x + 3y - z = 3$ (normal plane). **32**. $x = 1 + t$, $y = -2$, $z = 1 - t$ (tangent line); $x - z = 0$ (normal plane).

Section 12. **1**. $\langle 2,4,8\rangle$ (definite integral). **2**. $\langle 0,0,0\rangle$ (definite integral). **3**. $\langle \frac{1}{3},\frac{4}{15},\frac{2}{3}\rangle$ (definite integral). **4**. $\langle -\frac{1}{4},\frac{1}{3},\frac{1}{2}(e^2-1)\rangle$ (definite integral). **5**. $\langle 1,1,1\rangle$ (definite integral). **6**. $\pi\mathbf{a}$ (definite integral). **7**. $\mathbf{0}$ (definite integral). **8**. $\langle 1{+}t, 2{+}t^2, 3{+}t^3\rangle$. **9**. $\langle \frac{1}{2}t^2{-}t{+}\frac{3}{2}, \frac{1}{3}t^3{-}\frac{1}{3}, \frac{2}{3}t^{3/2}{+}\frac{1}{3}\rangle$. **10**. $\langle \frac{3}{2}{-}\frac{1}{2}\cos(2t), 2{+}2\sin(t), 3{-}\frac{\pi}{2}{+}\frac{t}{2}{-}\frac{1}{4}\sin(2t)\rangle$. **12**. $\langle t{+}1, t^2{+}2, t^3{-}t{+}3\rangle$. **13**. $\langle t{+}\frac{9}{28}(t^{7/3}-$

$1), 2t + \frac{2}{15}(2t^{5/2} - 17), t^3 - 2\rangle$. **14**. $\langle 1 + \sin t,\ -2 - \cos t,\ t(1 + \ln(t/\pi)) - \pi\rangle$. **16**. $\mathbf{c}_n t^{n-1} + \mathbf{c}_{n-1} t^{n-2} + \cdots + \mathbf{c}_2 t + \mathbf{c}_1$ where $\mathbf{c}_k$, $k = 1, 2, ..., n$, are arbitrary constant vectors. Alternatively, $\mathbf{r}(t) = \langle P_{n-1}(t), Q_{n-1}(t), R_{n-1}(t)\rangle$ where P_{n-1}, Q_{n-1}, and R_{n-1} are polynomials of degree at most $n - 1$. **18**. $\mathbf{r}(2) = \langle 3, 4, \frac{1}{3}\rangle$. **20**. 18 meters. **22**. The maximal height is approximately 2 kilometers, the range is approximately 14 kilometers, and the speed at impact is 400 m/s. **23**. Let the x axis be from east to west, the y axis from north to south, the z axis is vertical, and the origin at the initial point. Then the point of impact is $x = 2v_0^2 F \sin^2\theta/(mg^2)$, $y = v_0^2 \sin(2\theta)/g$, $z = 0$. Change the initial velocity by adding a nonzero x component: $\mathbf{v}_0 = \langle u, v_0\cos\theta, v_0\sin\theta\rangle$ where $u = -Fv_0\sin\theta/(mg)$, then the impact point is at $x = z = 0$, $y = v_0^2\sin(2\theta)/g$ as if $F = 0$.

Section 13. **1**. $4\sqrt{13}$ **2**. $7/3$. **3**. 18. **4**. $2(e - e^{-1})$. **5**. $(e - e^{-1})/\sqrt{2}$. **7**. $\sqrt{3}$. **8**. $2\sqrt{2}$. **9**. 23. **10**. $8 + \ln 3$. **11**. 42. **14**. Parametric equations are $\mathbf{r}(t) = \langle a\sqrt{t}\cos t, a\sqrt{t}\sin t, at\rangle$; the arc length is $a\sqrt{t_0}(1 + 2t_0/3)$ where $t_0 = z_0/a$. **16**. $\mathbf{R}(s) = \langle \frac{s}{\sqrt{14}},\ 1 - \frac{2s}{\sqrt{14}},\ 5 + \frac{3s}{\sqrt{14}}\rangle$. **17**. $\mathbf{R}(s) = \sin(s)\hat{\mathbf{e}}_1 + \cos(s)\hat{\mathbf{e}}_3$ (the curve is a circle). **18**. $\mathbf{R}(s) = \langle \sqrt{1 + s^2/2},\ s/\sqrt{2},\ \sinh^{-1}(s/\sqrt{2}\rangle$; recall that $\sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1})$. **22**. The position vector is $\langle \ln(1 + \sqrt{2}),\ \sqrt{2},\ 1\rangle$.

Section 14. **1**. $\kappa(t) = (1 + 2t^2)^{-3/2}$, $\kappa(1) = 3^{-3/2}$. **2**. $\kappa(t) = 2(1 + 4t^2)^{3/2}$, $\kappa(2) = 2/17^{3/2}$. **3**. $\kappa(x) = \frac{1}{4}|\sin(x/2)|(1 + \frac{1}{4}\cos^2(x/2))^{-3/2}$, $\kappa(\pi) = \frac{1}{4}$. **9**. $\kappa(t) = \sqrt{2}(e^t + e^{-t})^{-2}$, $\kappa(0) = 2^{-3/2}$. **11**. $\sqrt{2}/3$. **12**. $x = \pi n/a$ where n is any integer. **17**. $y = x^4$, $x_0 = 0$. **18**. The curvatures are equal. **19**. $\kappa(\theta) = \frac{3}{2\sqrt{2}}(1 + \cos\theta)^{-1/2}$ (the curve is not smooth at the origin, $\theta = \pi$, where the curvature becomes infinite). **20**. $\kappa(\theta) = e^{-\theta}/\sqrt{2}$. **22**. $x^2 + (y - R)^2 = R^2$ where $R = 1/\kappa(0) = 1/2$. **24**. $(x - 2)^2 + (y - 2)^2 = 2$. **27**. $\kappa_{max} = a/b^2$ occurs at $(\pm a, 0)$, $\kappa_{min} = b/a^2$ occurs at $(0, \pm b)$; the osculating circle at $(a, 0)$ is $(x - a + b^2/a)^2 + y^2 = b^4/a^2$; the osculating circle at $(0, b)$ is $x^2 + (y - b + a^2/b)^2 = a^4/b^2$. **28**. $\kappa(t) \to \infty$ as $t \to 0$. **30**. $2x + 3y - 6z = -1$. **31**. $6x - 16y + z = -15$. **33**. Yes, at $(1, 1, 1)$.

Section 15. **1**. $\mathbf{v}(t) = \langle 1, -1, 2t\rangle$, $\mathbf{a}(t) = \langle 0, 0, 2\rangle$, $v(t) = \sqrt{2 + 4t^2}$, $a_T(t) = 4t/\sqrt{2 + 4t^2}$, $a_N(t) = 2/\sqrt{1 + 2t^2}$. The point P corresponds to $t = 1$. **2**. $\mathbf{v}(t) = \langle 2t, 1, 0\rangle$, $\mathbf{a}(t) = \langle 2, 0, 0\rangle$, $v(t) = \sqrt{1 + 4t^2}$, $a_T(t) = 4t/\sqrt{1 + 4t^2}$, $a_N(t) = 2/\sqrt{1 + 4t^2}$. The point P corresponds to $t = 2$. **3**. $\mathbf{v}(t) = \langle 6\sqrt{t}, -2t, 1\rangle$, $\mathbf{a}(t) = \langle 3/\sqrt{(t)}, -2, 0\rangle$, $v(t) = \sqrt{4t^2 + 36t + 1}$, $\|\mathbf{v} \times \mathbf{a}\| = \sqrt{36t + 4 + 9/t}$, $a_T(t) = (18 + 4t)/v(t)$, $a_N(t) = \|\mathbf{v} \times \mathbf{a}\|/v(t)$. The point P corresponds to $t = 1$. **4**. $\mathbf{v}(t) = \frac{1}{t}\langle 1, \sqrt{t}/2, 2t^2\rangle$, $\mathbf{a}(t) = \frac{1}{t^2}\langle -1, -\sqrt{t}/4, 2t^2\rangle$, $v(t) = \frac{1}{t}\sqrt{1 + 4t^4 + t/2}$, $\|\mathbf{v} \times \mathbf{a}\| = \frac{1}{t^3}\sqrt{\frac{9}{4}t^5 + 16t^4 + t/16}$, $\mathbf{v} \cdot \mathbf{a} = \frac{1}{t^3}(4t^2 - 1 - t/8)$, $a_T(t) = \mathbf{v} \cdot \mathbf{a}/v(t)$, $a_N(t) = \|\mathbf{v} \times \mathbf{a}\|/v(t)$. The point P corresponds to $t = 1$. **8**. $a_T = 0$ and $a_N = 2\sqrt{2}$. **10**. The maximal speed is 51 km/h

or 32 *mph*. A suitable speed limit is 30 *mph* or 50 *km/h*. **11**. A tangent vector is $\langle 1, 5\rangle$, and a normal vector $\langle -5, 1\rangle$. Hence the magnitudes of the tangential and normal accelerations are, respectively, $a_T = -8/\sqrt{26}$ and $a_N = 14/\sqrt{26}$. **12**. A straight line. **13**. A circle. **14**. Maximal acceleration $v_0^2 a/b^2$ occurs at $(\pm a, 0)$. **17**. $\hat{\mathbf{T}} = \frac{1}{\sqrt{10}}\langle 3, 1, 0\rangle$, $\hat{\mathbf{N}} = \frac{1}{\sqrt{10}}\langle 1, -3, 0\rangle$, $\hat{\mathbf{B}} = \langle 0, 0, -1\rangle$, $\tau = 0$. **18**. $\hat{\mathbf{T}} = \frac{1}{\sqrt{41}}\langle 6, -2, 1\rangle$, $\hat{\mathbf{N}} = \frac{1}{7\sqrt{41}}\langle -9, -38, -22\rangle$, $\hat{\mathbf{B}} = \frac{1}{7}\langle 2, 3, -6\rangle$, $\tau = -\frac{3}{49}$. **19**. $\hat{\mathbf{T}} = \frac{1}{\sqrt{6}}\langle 1, 1, 2\rangle$, $\hat{\mathbf{N}} = \frac{1}{\sqrt{606}}\langle -17, -11, 14\rangle$, $\hat{\mathbf{B}} = \frac{1}{\sqrt{101}}\langle 6, -8, 1\rangle$, $\tau = \frac{12}{101}$. **22**. $\hat{\mathbf{B}} = \frac{1}{3}\langle 2, -2, 1\rangle$, $\tau = 0$.