CHAPTER 5

Vector Calculus

41. Line Integrals of a Vector Field

41.1. Vector Fields. Consider an air flow in the atmosphere. The air velocity varies from point to point. In order to describe the motion of the air, the air velocity must be defined as a function of position, which means that a velocity *vector* has to be assigned to every point in space. In other words, in contrast to ordinary functions, the air velocity is a *vector-valued* function on a point set in space.

Definition 41.1. (Vector Field).
Let E be a subset of a Euclidean space. A vector field $\mathbf{F}$ on E is a rule that assigns to each point P of E a unique vector $\mathbf{F}(P) = \langle F_1(P), F_2(P), F_3(P)\rangle$. The functions F_1, F_2, and F_3 are called the components of the vector field $\mathbf{F}$. The set E is called the domain of the vector field $\mathbf{F}$.

Let $\mathbf{r}$ be a position vector of a point P in a set E of an $n-$dimensional Euclidean space relative to some rectangular coordinate system. Then the components of a vector field $\mathbf{F}$ on E are functions of n variables:

$$\mathbf{F}(\mathbf{r}) = \langle F_1(\mathbf{r}),\ F_2(\mathbf{r}),\ F_3(\mathbf{r})\rangle\,.$$

In particular, if the domain of a vector field lies in space, then $\mathbf{r} = \langle x, y, z\rangle$ and the components of a vector field are functions of three variables. For example, a stationary flow of fluid or air can be described by a velocity vector field of three variables. If the flow is not stationary; that is, it can also change with time, then such a flow is described by a vector field of four variables (x, y, z, t) where t is time. In general, one can think of a vector field as *a rule that assigns a unique element of a Euclidean space to each element of a set in another Euclidean space*. The dimension of the Euclidean space where the vector field takes its values determines the number of components of the vector field, and the dimension of the domain determines the number of variables on which the vector field depends. Here only two- or three-dimensional vector fields of two or three variables will be studied. A vector field is said to be *continuous* if its components are continuous. A vector field is said to be *differentiable* if its components are differentiable. For example, a vector field $\mathbf{F}(x, y, z)$ is differentiable if its components are differentiable functions of the variables (x, y, z).

A simple example of a vector field is the gradient of a function, $\mathbf{F}(\mathbf{r}) = \boldsymbol{\nabla} f(\mathbf{r})$. The components of this vector field are partial derivatives of f:

$$\mathbf{F}(\mathbf{r}) = \boldsymbol{\nabla} f(\mathbf{r}) \quad \Leftrightarrow \quad F_1(\mathbf{r}) = f'_x(\mathbf{r}), \quad F_2(\mathbf{r}) = f'_y(\mathbf{r}), \quad F_3(\mathbf{r}) = f'_z(\mathbf{r}).$$

Many physical quantities are described by vector fields. Electric and magnetic fields are vector fields whose components are functions of position in space and time. All modern communication devices (radio, TV, cell phones, etc.) use electromagnetic waves. Visible light is also electromagnetic waves. The propagation of electromagnetic waves in space is described by differential equations that relate electromagnetic fields at each point in space and each moment of time to a distribution of electric charges and currents (e.g., antennas). The gravitational force looks constant near the surface of the Earth, but on the scale of the solar system this is not so. If one thinks about a planet as a homogeneous ball of mass M, then the gravitational force exerted by it on a point mass m depends on the position of the point mass relative to the planet's center according to Newton's law of gravity:

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^3}\,\mathbf{r} = \left\langle -GMm\frac{x}{r^3},\ -GMm\frac{y}{r^3},\ -GMm\frac{z}{r^3}\right\rangle,$$

where G is Newton's gravitational constant, $\mathbf{r}$ is the position vector relative to the planet's center, and $r = \|\mathbf{r}\|$ is its length. The force is proportional to the position vector and hence parallel to it at each point. The minus sign indicates that $\mathbf{F}$ is directed opposite to $\mathbf{r}$, that is, the force is *attractive*; the gravitational force pulls toward its source (the planet). *The magnitude* of the force

$$\|\mathbf{F}\| = \frac{GMm}{r^2}$$

decreases with increasing distance r. So the gravitational vector field can be visualized by plotting vectors of length $\|\mathbf{F}\|$ at each point in space pointing toward the origin. The magnitudes of these vectors become smaller for points farther away from the origin. At each point P in space, the vector $\mathbf{F}$ is directed along the line through the origin and the point P. This observation leads to the concept of *flow lines* of a vector field.

41.2. Flow Lines of a Vector Field.

Definition **41.2**. (Flow Lines of a Vector Field).
The flow line of a vector field $\mathbf{F}$ *in space is a spatial curve* C *such that, at any point* $\mathbf{r}$ *of* C*, the vector field* $\mathbf{F}(\mathbf{r})$ *is tangent to* C*.*

The direction of $\mathbf{F}$ defines the *orientation* of flow lines. The direction of a tangent vector $\mathbf{F}$ is shown by arrows on the flow lines as depicted in the left panel of Fig. **41.1**. For example, the flow lines of the planet's gravitational field are straight lines oriented toward the center of the planet. Flow lines of a gradient vector field $\mathbf{F} = \boldsymbol{\nabla} f \neq \mathbf{0}$ are normal to level surfaces of the function f and oriented in the direction in which f increases most rapidly (Theorem **24.2**). They are the curves of steepest ascent of the function f. Flow lines of the velocity vector field of the air are often shown in weather forecasts to indicate the wind direction over large areas. For example, flow lines of the air velocity in a hurricane would look like closed loops around the eye of the hurricane.

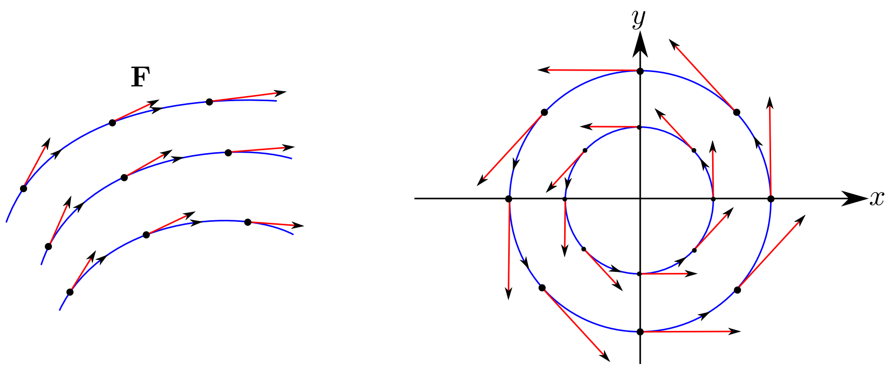


FIGURE **41.1**. **Left**: Flow lines of a vector field $\mathbf{F}$ are curves to which the vector field is tangential. The flow lines are oriented by the direction of the vector field. **Right**: Flow lines of the vector field $\mathbf{F} = (-y, x, 0)$ in Example **41.1** are concentric circles oriented counterclockwise. The magnitude $\|\mathbf{F}\| = \sqrt{x^2 + y^2}$ is constant along the flow lines and linearly increases with the increasing distance from the origin.

The qualitative behavior of flow lines may be understood by plotting vectors $\mathbf{F}$ at several points $\mathbf{r}_i$ and sketching curves through them so that the vectors $\mathbf{F}_i = \mathbf{F}(\mathbf{r}_i)$ are tangent to the curves. Finding the exact shape of the flow lines requires solving differential equations. If $\mathbf{r} = \mathbf{r}(t)$ is a parametric equation of a flow line, then $\mathbf{r}'(t)$ is parallel to $\mathbf{F}(\mathbf{r}(t))$. So the derivative $\mathbf{r}'(t)$ must be proportional to $\mathbf{F}(\mathbf{r}(t))$, which defines *a system of differential equations* for the components of the vector function $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = q(t)\mathbf{F}(\mathbf{r}(t))\,,$$

for some *positive* continuous function $q(t)$. The shape of flow lines is independent of the choice of $q(t)$ because one can always *reparameterize* the flow line by choosing the parameter $s = s(t)$ such that

$$ds = s'(t)dt = q(t)dt \quad \Rightarrow \quad \frac{d\mathbf{r}(s)}{ds} = \mathbf{F}(\mathbf{r}(s))\,.$$

By the inverse function theorem $s(t)$ is one-to-one because $s'(t) = q(t) > 0$ and in the new parameterization $d\mathbf{r}/dt = (d\mathbf{r}/ds)(ds/dt) = (d\mathbf{r}/ds)q$ so that q is cancelled in the above equation for the flow line. To find a flow line through a particular point $\mathbf{r}_0$, the differential equations must be supplemented by *initial* conditions, e.g., $\mathbf{r}(t_0) = \mathbf{r}_0$. If the equations have a unique solution, then the flow line through $\mathbf{r}_0$ exists and is given by the solution. Methods of finding solutions of a system of differential equations is the subject of courses on differential equations.

EXAMPLE **41.1**. *Analyze flow lines of the planar vector field* $\mathbf{F} = \langle -y, x, 0\rangle$.

SOLUTION: By noting that $\mathbf{F}\cdot\mathbf{r}=0$, it is concluded that at any point $\mathbf{F}$ is perpendicular to the position vector $\mathbf{r}=\langle x,y,0\rangle$ in the plane. So flow lines are curves whose tangent vector is perpendicular to the position vector. If $\mathbf{r}=\mathbf{r}(t)$ is a parametric equation of such a curve, then

$$\mathbf{r}(t)\cdot\mathbf{r}'(t)=0 \quad\Rightarrow\quad \frac{d}{dt}\Big(\mathbf{r}(t)\cdot\mathbf{r}(t)\Big)=0 \quad\Rightarrow\quad \|\mathbf{r}(t)\|^2=\text{const}\,;$$

the latter equation implies that $\mathbf{r}(t)$ traverses a circle centered at the origin (or a part of it). So flow lines are concentric circles. At the point $(1,0,0)$, the vector field is directed along the y axis: $\mathbf{F}(1,0,0)=\langle 0,1,0\rangle=\hat{\mathbf{e}}_2$. Therefore, the flow lines are oriented counterclockwise. The magnitude $\|\mathbf{F}\|=\sqrt{x^2+y^2}$ remains constant on each circle and increases with increasing circle radius. The flow lines are shown in the right panel of Fig. **41.1**. □

41.3. Line Integral of a Vector Field. The work done by a constant force $\mathbf{F}$ in moving an object along a straight line is given by

$$W=\mathbf{F}\cdot\mathbf{d},$$

where $\mathbf{d}$ is the displacement vector (Section **3.6**). Suppose that the force varies in space and the displacement trajectory is no longer a straight line. What is the work done by the force? This question is evidently of great practical significance. To answer it, the concept of the line integral of a vector field was developed.

Let C be a smooth curve that goes from a point $\mathbf{r}_a$ to a point $\mathbf{r}_b$ and has a length L. Consider a partition of C by segments C_i, $i=1,2,...,N$, of length $\Delta s=L/N$. Let $\mathbf{d}_i$ be a vector from the initial point of C_i to its final point. Since the curve is smooth, each partition segment C_i can be approximated by a straight line segment of length Δs oriented along the unit tangent vector $\hat{\mathbf{T}}(\mathbf{r}_i^*)$ at a sample point $\mathbf{r}_i^*\in C_i$ so that

$$\mathbf{d}_i=\hat{\mathbf{T}}(\mathbf{r}_i^*)\Delta s\,. \tag{41.1}$$

(see the left panel of Fig. **41.3**). Recall that if $\mathbf{r}(s)$ is the natural parameterization of the curve, then $\mathbf{r}'(s)=\hat{\mathbf{T}}(s)\equiv\hat{\mathbf{T}}(\mathbf{r}(s))$ (the last notation is to explicitly indicate that the unit tangent vector is taken at the point $\mathbf{r}(s)$). Suppose that $\mathbf{r}_i=\mathbf{r}(s_i)$ and $\mathbf{r}_{i+1}=\mathbf{r}(s_{i+1})$, where $s_i=i\Delta s$, are the position vectors of the endpoints of C_i. Then, for any $s_i^*\in[s_i,s_{i+1}]$ one infers by using the linearization of $\mathbf{r}(s)$ at s_i^* that

$$\begin{aligned}\mathbf{d}_i &= \mathbf{r}_{i+1}-\mathbf{r}_i=\mathbf{r}(s_{i+1})-\mathbf{r}(s_i)=\mathbf{r}(s_{i+1})-\mathbf{r}(s_i^*)+\mathbf{r}(s_i^*)-\mathbf{r}(s_i)\\ &= \mathbf{r}'(s_i^*)(s_{i+1}-s_i^*)+\mathbf{r}'(s_i^*)(s_i^*-s_i)=\mathbf{r}'(s_i^*)(s_{i+1}-s_i)\\ &= \hat{\mathbf{T}}(\mathbf{r}_i^*)\Delta s\,,\end{aligned}$$

where terms decreasing to zero faster than Δs have been neglected. Thus, variations of a sample point within C_i result only in changing terms that decreases to zero faster than Δs so that the approximation (**41.1**) becomes

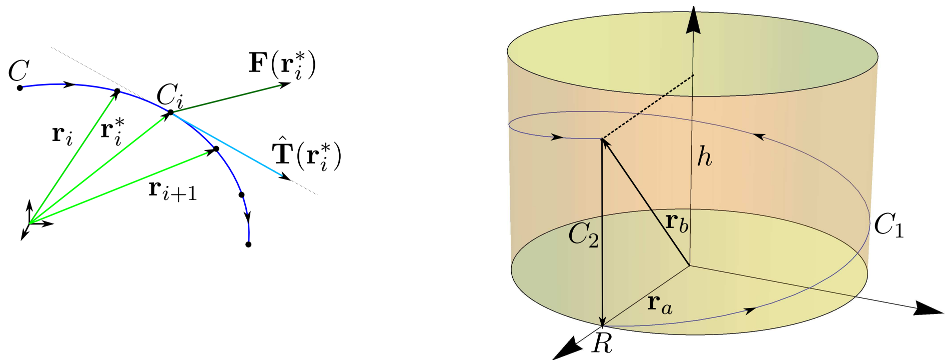


FIGURE **41.2**. **Left**: To calculate the work done by a continuous force $\mathbf{F}(\mathbf{r})$ in moving a point object along a smooth curve C, the latter is partitioned into segments C_i of arclength Δs. The work done by the force along a partition segment is $\mathbf{F}(\mathbf{r}_i^*)\cdot\mathbf{d}_i$ where the displacement vector is approximated by the oriented segment of length Δs that is tangent to the curve at a sample point $\mathbf{r}_i^*$, i.e., $\mathbf{d}_i = \hat{\mathbf{T}}(\mathbf{r}_i^*)\Delta s$ where $\hat{\mathbf{T}}$ is the unit tangent vector along the curve. **Right**: An illustration to Example **41.2**. The closed contour of integration in the line integral consists of two smooth pieces, one turn of the helix C_1 and the straight line segment C_2. The line integral is the sum of line integrals along C_1 and C_2.

more accurate as $N \to \infty$. The work along the segment C_i can therefore be approximated by

$$\Delta W_i = \mathbf{F}(\mathbf{r}_i^*) \cdot \hat{\mathbf{T}}(\mathbf{r}_i^*)\, \Delta s \quad \Rightarrow \quad W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_N\,.$$

The actual work should not depend on the choice of sample points. This problem is resolved by the usual trick of integral calculus by refining a partition, finding the low and upper sums, and taking their limits. If these limits exist and coincide, the limiting value should not depend on the choice of sample points and is the sought-after work. Note if one sums N terms of order $\Delta s \sim 1/N$, the result is of order $N \cdot (1/N) = 1$ (a number) in the limit $N \to \infty$, while the sum of N terms each of which is decreasing to zero faster than $\Delta s \sim 1/N$ is expected to vanish in this limit. Put

$$\Delta W_i = F_T(\mathbf{r}_i^*)\, \Delta s\,, \qquad F_T(\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}}(\mathbf{r})\,,$$

where $\hat{\mathbf{T}}(\mathbf{r})$ denotes the unit tangent vector at a point $\mathbf{r}$ in C. The scalar function F_T is called the *tangential component* of $\mathbf{F}$ to the curve C. The approximate total work looks like a Riemann sum of F_T along C. Its convergence is guaranteed for any choice of sample points if the corresponding upper and lower sums converge to the same value. If $M_i = \sup_{C_i} F_T(\mathbf{r})$ and

$m_i = \inf_{C_i} F_T(\mathbf{r})$, then

$$m_i \Delta s \le W_i \le M_i \Delta s \quad \Rightarrow \quad \sum_{i=1}^{N} m_i \Delta s \le \sum_{i=1}^{N} W_i \le \sum_{n=1}^{N} M_i \Delta s\,.$$

Therefore, *if the function F_T is integrable on the curve C, then the upper and lower sums converge to the same limit and the work is the line integral of the tangential component $\mathbf{F} \cdot \hat{\mathbf{T}}$ of the force.*

Definition **41.3**. (Line Integral of a Vector Field).
The line integral of a vector field $\mathbf{F}$ along a smooth curve C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \int_C F_T(\mathbf{r})\, ds\,,$$

where $\hat{\mathbf{T}}$ is the unit tangent vector to C, provided the tangential component $\mathbf{F} \cdot \hat{\mathbf{T}}$ of the vector field is integrable on C.

The integrability of $\mathbf{F} \cdot \hat{\mathbf{T}}$ is defined in the sense of line integrals for ordinary functions (see Definition **38.1**). In particular, the line integral of a *continuous* vector field over a smooth curve of a finite length always exists.

41.4. Evaluation of Line Integrals of Vector Fields. The line integral of a vector field is evaluated in much the same way as the line integral of a function (the line integral of the tangential component F_T).

Theorem **41.1**. (Evaluation of Line Integrals).
Let $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ be a continuous vector field on a spatial region E and let C be a smooth curve in E that originates from a point $\mathbf{r}_a$ and terminates at a point $\mathbf{r}_b$. Suppose that $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$, is a smooth parameterization of C oriented so that $\mathbf{r}(a) = \mathbf{r}_a$ and $\mathbf{r}(b) = \mathbf{r}_b$. Then

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b \Big(F_1(\mathbf{r}(t))x'(t) + F_2(\mathbf{r}(t))y'(t) + F_3(\mathbf{r}(t))z'(t) \Big)\, dt. \qquad (\mathbf{41.2})$$

Proof. The unit tangent vector reads $\hat{\mathbf{T}} = \mathbf{r}'/\|\mathbf{r}'\|$ and $ds = \|\mathbf{r}'\|\, dt$ so that $\hat{\mathbf{T}}\, ds = \mathbf{r}'(t)\, dt$. As the curve is smooth, $\hat{\mathbf{T}}(t)$ is continuous on $[a, b]$ and, by continuity of the vector field, the tangential component F_T is also continuous on the curve, $F_T(\mathbf{r}(t)) = \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{T}}(t)$, as the dot product of two continuous functions. Then by Theorem **38.4**

$$\int_C F_T\, ds = \int_a^b F_T(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

which is the conclusion of the theorem. □

Equation (**41.2**) also holds if C is piecewise smooth and $\mathbf{F}$ is bounded and not continuous at a finite number of points of C, much like in the case of the line integral of ordinary functions. Owing to the representation (**41.2**) and the relations $dx = x'dt$, $dy = y'dt$, and $dz = z'dt$, the line integral is often written in the form:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz \tag{41.3}$$

For a smooth parametric curve $\mathbf{r}(t)$, the differential $d\mathbf{r} = \langle dx, dy, dz\rangle$ is tangent to the curve.

In contrast to the line integral of ordinary functions, the line integral of a vector field depends on the orientation of C. The orientation of C is fixed by the conditions $\mathbf{r}(a) = \mathbf{r}_a$ and $\mathbf{r}(b) = \mathbf{r}_b$ for a vector function $\mathbf{r}(t)$, where $a \le t \le b$, provided the vector function traces out the curve only once. If $\mathbf{r}(t)$ traces out C from $\mathbf{r}_b$ to $\mathbf{r}_a$, then the orientation is reversed, and such a curve is denoted by $-C$. The line integral changes its sign when the orientation of the curve is reversed:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r} \tag{41.4}$$

because the direction of the derivative $\mathbf{r}'(t)$ is reversed for all t. If C is piecewise smooth (e.g., the union of smooth curves C_1 and C_2), then the additivity of the integral should be used to evaluate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Line integral along a parametric curve. A parametric curve is defined by a vector function $\mathbf{r}(t)$ on $[a, b]$ (recall Definition **10.4**). The vector function $\mathbf{r}(t)$ may trace its range (as a point set in space) or some parts of it several times as t changes from a to b. Furthermore two *different* vector functions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ on $[a, b]$ may have the same range. For example, $\mathbf{r}_1 = (\cos t, \sin t, 0)$ and $\mathbf{r}_1(t) = (\cos(2t), \sin(2t), 0)$ have the same range on $[0, 2\pi]$, which is the circle of unit radius, but $\mathbf{r}_2(t)$ traces out the circle twice. The line integral over a parametric curve is *defined* by the relation (**41.2**). A parametric curve is much like the trajectory of a particle that can pass through the same points multiple times. So, the relation (**41.2**) defines the work done by a non-constant force $\mathbf{F}$ along a *particle's trajectory* or *parametric curve* $\mathbf{r} = \mathbf{r}(t)$.

The evaluation of a line integral includes the following basic steps:

Step 1. If the curve C is defined as a point set in space by some geometrical means, then find its parametric equations $\mathbf{r} = \mathbf{r}(t)$ that agree with the orientation of C. Here it is useful to remember that, if $\mathbf{r}(t)$ corresponds to the orientation opposite to the required one, then it can still be used according to (**41.4**);

Step 2. Restrict the range of t to an interval $[a, b]$ so that C is traced out only once by $\mathbf{r}(t)$;

Step 3. Substitute $\mathbf{r} = \mathbf{r}(t)$ into the arguments of $\mathbf{F}$ to obtain the values of $\mathbf{F}$ on C and calculate the derivative $\mathbf{r}'(t)$ and the dot product $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$;

Step 4. Evaluate the (ordinary) integral (**41.2**).

EXAMPLE **41.2**. *Evaluate the line integral of $\mathbf{F} = \langle -y, x, z^2\rangle$ along a closed curve C that consists of two parts. The first part is one turn of a helix of radius R, which winds about the z axis counterclockwise as viewed from the top of the z axis, starting from the point $\mathbf{r}_a = \langle R, 0, 0\rangle$ and ending at the point $\mathbf{r}_b = \langle R, 0, 2\pi h\rangle$. The second part is a straight line segment from $\mathbf{r}_b$ to $\mathbf{r}_a$.*

SOLUTION: Let C_1 be one turn of the helix and let C_2 be the straight line segment. Two line integrals have to be evaluated. The parametric equations of the helix are

$$\mathbf{r}(t) = \langle R\cos t, R\sin t, ht\rangle \quad \Rightarrow \quad \mathbf{r}(0) = \mathbf{r}_a\,,\ \mathbf{r}(2\pi) = \mathbf{r}_b \quad \Rightarrow \quad 0 \le t \le 2\pi$$

as required by the orientation of C_1. Note the positive signs at $\cos t$ and $\sin t$ in the parametric equations that are necessary to make the helix winding about the z axis counterclockwise (see Study Problem **10.1**). Therefore,

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -R\sin t, R\cos t, h\rangle\,, \\
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle -R\sin t, R\cos t, h^2t^2\rangle \cdot \langle -R\sin t, R\cos t, h\rangle \\
&= R^2 + h^3t^2, \\
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} (R^2 + h^3t^2)\, dt \\
&= 2\pi R^2 + \frac{(2\pi h)^3}{3}.
\end{aligned}$$

The parametric equations of the line through two points $\mathbf{r}_a$ and $\mathbf{r}_b$ are $\mathbf{r}(t) = \mathbf{r}_a + \mathbf{v}t$, where $\mathbf{v} = \mathbf{r}_b - \mathbf{r}_a$ is the vector parallel to the line, or in the components

$$\mathbf{r}(t) = \langle R, 0, 0\rangle + t\langle 0, 0, 2\pi h\rangle = \langle R, 0, 2\pi ht\rangle\,, \quad 0 \le t \le 1$$

but $\mathbf{r}(0) = \mathbf{r}_a$ and $\mathbf{r}(1) = \mathbf{r}_b$ whereas $\mathbf{r}_b$ must be the initial point of C_2. So the found parametric equations describe the curve $-C_2$ (it has the opposite orientation). One has $\mathbf{r}'(t) = \langle 0, 0, 2\pi h\rangle$ and hence

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle 0, R, (2\pi h)^2t^2\rangle \cdot \langle 0, 0, 2\pi h\rangle = (2\pi h)^3t^2, \\
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= -\int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = -(2\pi h)^3 \int_0^1 t^2\, dt = -\frac{(2\pi h)^3}{3}.
\end{aligned}$$

The line integral along C is the sum of these integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi R^2\,.$$

$\square$

EXAMPLE **41.3**. *Evaluate the work done by the force*

$$\mathbf{F} = \langle 3x^2 + yz,\, 2y + zx,\, 3z^2xy\rangle$$

along the curve C that consists of three straight line segments connecting the points $(0,0,0) \to (1,0,0) \to (1,2,0) \to (1,2,-1)$; C is oriented from $(0,0,0)$ to $(1,2,-1)$.

SOLUTION: Let C_1, C_2, and C_3 be three line segments of C. Since the line segments are parallel to the coordinate axes,

$$\begin{aligned}
C_1: &\quad (x,y,z) = (x,0,0)\,, \quad d\mathbf{r} = \langle dx,0,0\rangle\,, \quad 0 \le x \le 1\,,\\
C_2: &\quad (x,y,z) = (1,y,0)\,, \quad d\mathbf{r} = \langle 0,dy,0\rangle\,, \quad 0 \le y \le 2\,,\\
C_3: &\quad (x,y,z) = (1,2,z)\,, \quad d\mathbf{r} = \langle 0,0,-dz\rangle\,, \quad -1 \le z \le 0\,,
\end{aligned}$$

the sign of dz has been reversed because the line $(1,2,z)$ starts at $(1,2,-1)$ and ends at $(1,2,0)$ as z increases from -1 to 0, whereas C_3 should have the opposite orientation. Note that the dot product $\mathbf{F}\cdot d\mathbf{r}$ depends only on the first component F_1 on C_1, the second component F_2 on C_2, and the third component F_3 on C_3. Therefore

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 F_1(x,0,0)dx + \int_0^2 F_2(1,y,0)dy - \int_{-1}^0 F_3(1,2,z)dz\\
&= \int_0^1 3x^2dx + \int_0^2 2ydy - \int_{-1}^0 6z^2dz\\
&= 1 + 4 - 2\cdot(0-(-1)^3) = 3\,.
\end{aligned}$$

□

41.5. Study Problems.

Problem **41.1**. *Find the line integral of the vector field $\mathbf{F} = g(r)(\mathbf{a}\times\mathbf{r})$, where $\mathbf{a}$ is a constant vector, g is continuous function, and $r = \|\mathbf{r}\|$, along a straight line segment parallel to $\mathbf{a}$.*

SOLUTION: For a straight line segment parallel to $\mathbf{a}$, the tangent vector $d\mathbf{r}$ is parallel to $\mathbf{a}$. Therefore

$$\mathbf{F}\cdot d\mathbf{r} = g(r)(\mathbf{a}\times\mathbf{r})\cdot d\mathbf{r} = 0$$

as the triple product of coplanar vectors vanishes. Thus, the line integral of $\mathbf{F}$ vanishes for any such straight line segment. □

Problem **41.2**. *Find the work done by the force $\mathbf{F} = \langle 2x, 3y^2, 4z^3\rangle$ along any smooth curve originating from the point $(0,0,0)$ and ending at the pont $(1,1,1)$.*

SOLUTION: If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ are parametric equations of a smooth curve, $a \le t \le b$, such that $\mathbf{r}(a) = \langle 0,0,0\rangle$ and $\mathbf{r}(b) = \langle 1,1,1\rangle$, then by the chain rule

$$\mathbf{F}\cdot d\mathbf{r} = 2xdx + 3y^2dy + 4z^3dz = d\Big(x^2 + y^3 + z^4\Big)\,.$$

Note that here the vector field is taken on the curve $\mathbf{F} = \mathbf{F}(\mathbf{r}(t))$ so that $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the above equation and the chain rule applies. By the fundamental theorem of calculus $\int_a^b df(t) = f(b) - f(a)$ and therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b d\Big(x^2(t) + y^3(t) + z^4(t)\Big) = (x^2(t) + y^3(t) + z^4(t))\Big|_a^b = 3\,.$$

□

Problem **41.3**. *Find the work done by the engines of a space craft of mass m against the gravitational pull of a planet of mass M if the space craft moved from the position $\mathbf{r}_a$ to $\mathbf{r}_b$ relative to the center of the planet.*

SOLUTION: Let $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, be a smooth parameterization of the trajectory of the space craft. Then

$$\mathbf{F} \cdot d\mathbf{r} = -GMm\,\frac{\mathbf{r} \cdot d\mathbf{r}}{\|\mathbf{r}\|^3} = -\frac{1}{2}GMm\,\frac{d(\mathbf{r} \cdot \mathbf{r})}{\|\mathbf{r}\|^3} = -\frac{1}{2}GMm\,\frac{du}{u^{3/2}}\,,$$

where $u = \mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2$. Put $u_a = \|\mathbf{r}_a\|^2$ and $u_b = \|\mathbf{r}_b\|^2$. The force exerted by the engines should compensate the gravitational force and, hence, its tangential component must be opposite to the tangential component of the gravitational force of the planed exerted on the space craft. For any trajectory going from $\mathbf{r}_a$ to $\mathbf{r}_b$, the work done by the engines is

$$\begin{aligned} W &= -\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}(t) = \frac{1}{2}GMm \int_{u_a}^{u_b} \frac{du}{u^{3/2}} \\ &= -GMm\; u^{-1/2}\Big|_{u_a}^{u_b} = \frac{GMm}{\|\mathbf{r}_a\|} - \frac{GMm}{\|\mathbf{r}_b\|}\,. \end{aligned}$$

□

Problem **41.4**. *A magnetic field $\mathbf{B}(\mathbf{r})$ exerts the Lorentz force $\mathbf{F} = (e/c)\mathbf{v} \times \mathbf{B}$ on a charged particle (see Study Problem* **12.3***), where $\mathbf{v}$ is the velocity of the particle. Prove that the work done by the Lorentz force is always zero for any trajectory of the particle.*

SOLUTION: Let $\mathbf{r}(t)$ be a trajectory of the particle. Then $d\mathbf{r}(t) = \mathbf{r}'(t)dt = \mathbf{v}(t)dt$ and, hence, along the trajectory

$$\mathbf{F} \cdot d\mathbf{r} = (e/c)(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt = 0$$

as the triple product of coplanar vectors vanishes. Therefore, the work done by the Lorentz force is zero for any trajectory. □

41.6. Exercises.

1–6. Sketch flow lines of the given planar vector field.

1. $\mathbf{F} = \langle ax, by\rangle$, where a and b are positive constants;
2. $\mathbf{F} = \langle ay, bx\rangle$, where a and b are positive constants;
3. $\mathbf{F} = \langle ay, bx\rangle$, where the constants a and b have different signs;

4. $\mathbf{F} = \nabla u$, $u = \tan^{-1}(y/x)$;
5. $\mathbf{F} = \nabla u$, $u = \ln[(x^2 + y^2)^{-1/2}]$;
6. $\mathbf{F} = \nabla u$, $u = \ln[(x-a)^2 + (y-b)^2]$.

7–15. Sketch flow lines of the given vector field in space.

7. $\mathbf{F} = \langle ax, by, cz\rangle$ where a, b, and c are positive constants;
8. $\mathbf{F} = \langle ax, by, cz\rangle$ where a and b are positive constants, while c is a negative constant;
9. $\mathbf{F} = \langle y, -x, a\rangle$ where a is a constant;
10. $\mathbf{F} = \nabla \|\mathbf{r}\|$, $\mathbf{r} = \langle x, y, z\rangle$;
11. $\mathbf{F} = \nabla \|\mathbf{r}\|^{-1}$, $\mathbf{r} = \langle x, y, z\rangle$;
12. $\mathbf{F} = \nabla u$, $u = (x/a)^2 + (y/b)^2 + (z/c)^2$;
13. $\mathbf{F} = \nabla u$, $u = \sqrt{x^2 + y^2 + (z+c)^2} + \sqrt{x^2 + y^2 + (z-c)^2}$ where c is a positive constant;
14. $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a constant vector and $\mathbf{r} = \langle x, y, z\rangle$;
15. $\mathbf{F} = \nabla u$, $u = z/\sqrt{x^2 + y^2 + z^2}$.

16. A ball rotates at a constant rate ω about its diameter parallel to a unit vector $\mathbf{n}$. If the origin of the coordinate system is set at the center of the ball, find the velocity vector field as a function of the position vector $\mathbf{r}$ of a point of the ball.

17–30. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the given vector field $\mathbf{F}$ and the specified curve C.

17. $\mathbf{F} = \langle y, xy, 0\rangle$ and C is the parametric curve $\mathbf{r}(t) = \langle t^2, t^3, 0\rangle$, $0 \le t \le 1$;
18. $\mathbf{F} = \langle z, yx, zy\rangle$ and C is the ellipse $x^2/a^2 + y^2/b^2 = 1$ oriented clockwise;
19. $\mathbf{F} = \langle z, yx, zy\rangle$ and C is the parametric curve $\mathbf{r}(t) = \langle 2t, t + t^2, 1 + t^3\rangle$ from the point $(-2, 0, 0)$ to the point $(2, 2, 2)$;
20. $\mathbf{F} = \langle -y, x, z\rangle$ and C is the boundary of the part of the paraboloid $z = a^2 - x^2 - y^2$ that lies in the first octant; C is oriented counterclockwise as viewed from the top of the z axis;
21. $\mathbf{F} = \langle -z, 0, x\rangle$ and C is the boundary of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant; C is oriented clockwise as viewed from the top of the z axis;
22. $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a}$ is a constant vector, $\mathbf{r} = \langle x, y, z\rangle$, and C is straight line segment from $\mathbf{r}_1$ to $\mathbf{r}_2$;
23. $\mathbf{F} = \langle y \sin z,\ z \sin x,\ x \sin y\rangle$ and C is the parametric curve $\mathbf{r} = \langle \cos t,\ \sin t,\ \sin(5t)\rangle$, $0 \le t \le 2\pi$;
24. $\mathbf{F} = \langle y,\ -xz,\ y(x^2 + z^2)\rangle$ and C is the intersection of the cylinder $x^2 + z^2 = 1$ with the plane $x + y + z = 1$ that is oriented counterclockwise as viewed from the top of the y axis;
25. $\mathbf{F} = \langle -y \sin(\pi z^2), x \cos(\pi z^2), e^{xyz}\rangle$ and C is the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$; C is oriented counterclockwise as viewed from the top of the z axis;

26. $\mathbf{F} = \langle e^{\sqrt{y}}, e^x, 0\rangle$ and C is the parabola concave up in the xy plane from the origin to the point $(1,1)$;
27. $\mathbf{F} = \langle x, y, z\rangle$ and C is an elliptic helix $\mathbf{r}(t) = \langle a\cos t, b\sin t, ct\rangle$, $0 \le t \le 2\pi$;
28. $\mathbf{F} = \langle y^{-1}, z^{-1}, x^{-1}\rangle$ and C is the straight line segment from the point $(1,1,1)$ to the point $(2,4,8)$;
29. $\mathbf{F} = \langle e^{y-z}, e^{z-x}, e^{x-y}\rangle$ and C is the straight line segment from the origin to the point $(1,3,5)$;
30. $\mathbf{F} = \langle y+z, -x, 3y-3x\rangle$ and C the shortest arc on the sphere $x^2+y^2+z^2 = 25$ from the point $(3,4,0)$ to the point $(0,0,5)$.

31. Find the work done by a constant force $\mathbf{F}$ in moving a point object along a smooth path from a point $\mathbf{r}_a$ to a point $\mathbf{r}_b$.
32. Find the work done by the force $\mathbf{F} = f'(r)\mathbf{r}/r$ in moving a point object along a smooth path from a point $\mathbf{r}_a$ to a point $\mathbf{r}_b$ where the derivative f' of f is a continuous function of $r = \|\mathbf{r}\|$.
33–34. Find the work done by the force $\mathbf{F} = \langle -y, x, c\rangle$, where c is a constant, in moving a point object along each of the following curves.

33. the circle $x^2 + y^2 = 1$, $z = 0$;
34. the circle $(x-2)^2 + y^2 = 1$, $z = 0$.

35. The force acting on a charged particle that moves in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ is $\mathbf{F} = e\mathbf{E} + (e/c)\mathbf{v}\times\mathbf{B}$ where $\mathbf{v}$ is the velocity of the particle, e is its electric charge, and c is the speed of light in the vacuum. Find the work done by the force along a trajectory originating from a point $\mathbf{r}_a$ and ending at the point $\mathbf{r}_b$ if the electric and magnetic fields are constant.

42. Fundamental Theorem for Line Integrals

Recall the fundamental theorem of calculus, which asserts that, if the derivative $f'(x)$ is continuous on an interval $[a,b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

It appears that there is an analog of this theorem for line integrals.

42.1. Conservative Vector Fields.

DEFINITION **42.1**. (Conservative Vector Field and Its Potential). *A vector field* $\mathbf{F}$ *in a region* E *is said to be conservative if there is a differentiable function* f*, called a* potential *of* $\mathbf{F}$*, such that* $\mathbf{F} = \boldsymbol{\nabla} f$ *in* E*.*

Conservative vector fields play a significant role in many practical applications. It has been proved earlier (see Study Problem **24.3**) that if a particle moves along a trajectory $\mathbf{r} = \mathbf{r}(t)$ under the force $\mathbf{F} = -\boldsymbol{\nabla} U$, then its energy $E = mv^2/2 + U(\mathbf{r})$, where $v = \|\mathbf{v}\|$ and $\mathbf{v} = \mathbf{r}'$ is the velocity, is conserved along the trajectory, $dE/dt = 0$. In particular, Newton's gravitational force is conservative,

$$\mathbf{F} = -\boldsymbol{\nabla} U\,, \quad U(\mathbf{r}) = -\frac{GMm}{\|\mathbf{r}\|}\,. \tag{42.1}$$

The result of Study Problem **24.3** shows that the work done by the gravitational force in moving a point object of mass m does not depend on the trajectory of the object and is determined by the values of its potential U at the endpoints of the trajectory

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = U(\mathbf{r}_b) - U(\mathbf{r}_a)\,.$$

It turns out that this is a common feature of all conservative vector fields.

THEOREM **42.1**. (Fundamental Theorem for Line Integrals). *Let* C *be a smooth curve in a region* E *with initial and terminal points* $\mathbf{r}_a$ *and* $\mathbf{r}_b$*, respectively. Let* f *be a function on* E *whose gradient* $\boldsymbol{\nabla} f$ *is continuous on* C*. Then*

$$\int_C \boldsymbol{\nabla} f\cdot d\mathbf{r} = f(\mathbf{r}_b) - f(\mathbf{r}_a). \tag{42.2}$$

PROOF. Let $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, be a smooth parameterization of C such that $\mathbf{r}(a) = \mathbf{r}_a$ and $\mathbf{r}(b) = \mathbf{r}_b$. Then, by (**41.2**) and the chain rule,

$$\int_C \boldsymbol{\nabla} f\cdot d\mathbf{r} = \int_a^b (f'_x x' + f'_y y' + f'_z z')\,dt = \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt = f(\mathbf{r}_b) - f(\mathbf{r}_a).$$

The latter equality holds by the fundamental theorem of calculus and the continuity of the partial derivatives of f and $\mathbf{r}'(t)$ for a smooth curve. □

42.2. Path Independence of Line Integrals.

Definition **42.2**. (Path Independence of Line Integrals).
A continuous vector field $\mathbf{F}$ *has path-independent line integrals if*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two simple, piecewise-smooth curves in the domain of $\mathbf{F}$ *with the same endpoints.*

Recall that a curve is simple if it does not intersect itself (see Section **10.3**). An important consequence of the fundamental theorem for line integrals is that the work done by a continuous conservative force, $\mathbf{F} = \boldsymbol{\nabla} f$, is *path-independent*. So a criterion for a vector field to be conservative would be advantageous for evaluating line integrals because for a conservative vector field a curve may be deformed at convenience without changing the value of the integral. Let us introduce a special notation of a line integral along a closed curve

$$\oint_C \mathbf{F} \cdot d\mathbf{r}\,.$$

A circle on the integral sign indicates that the line integral is evaluated along a closed curve C.

Theorem **42.2**. (Path-Independent Property).
Let $\mathbf{F}$ *be a continuous vector field on an open region* E*. Then* $\mathbf{F}$ *has path-independent line integrals if and only if its line integral vanishes along every piecewise-smooth, simple, closed curve* C *in* E*. In that case, there exists a function* f *such that* $\mathbf{F} = \boldsymbol{\nabla} f$*:*

$$\mathbf{F} = \boldsymbol{\nabla} f \quad \Longleftrightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Proof. Suppose first that there is a function f such that $\mathbf{F} = \boldsymbol{\nabla} f$ in E. Then by the fundamental theorem for line integrals, the line integral of $\mathbf{F}$ vanishes along any simple closed curve in E because the initial and terminal points of C coincide. Conversely, suppose that $\mathbf{F}$ has vanishing line integrals along any simple closed curve in E. Pick a point $\mathbf{r}_0$ in E and consider any smooth curve C from $\mathbf{r}_0$ to a point $\mathbf{r} = \langle x, y, z\rangle$ in E. The idea is to prove that the function

$$f(\mathbf{r}) = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{42.3}$$

is a potential of $\mathbf{F}$, that is, to prove that $\boldsymbol{\nabla} f = \mathbf{F}$ under the condition that the line integral of $\mathbf{F}$ vanishes for every closed curve in E. This "guess" for f is motivated by the fundamental theorem for line integrals (**42.2**), where $\mathbf{r}_b$ is replaced by a generic point $\mathbf{r}$ in E. The potential is defined up to an additive constant ($\boldsymbol{\nabla}(f + \text{const}) = \boldsymbol{\nabla} f$) so the choice of a fixed point $\mathbf{r}_0$ is irrelevant. The value of f is independent of the choice of C. Indeed, consider two such curves C_1 and C_2. Then the union of C_1 and $-C_2$ (the curve C_2

whose orientation is reversed) is a closed curve, and the line integral along it vanishes by the hypothesis. On the other hand, this line integral is the sum of line integrals along C_1 and $-C_2$. By the property (**41.4**), the line integrals along C_1 and C_2 coincide. To calculate the derivative

$$f'_x(\mathbf{r}) = \lim_{h\to 0} \frac{f(\mathbf{r} + h\hat{\mathbf{e}}_1) - f(\mathbf{r})}{h}, \quad \hat{\mathbf{e}}_1 = \langle 1, 0, 0\rangle\,,$$

let us express the difference $f(\mathbf{r} + h\hat{\mathbf{e}}_1) - f(\mathbf{r})$ via a line integral. Note that E is open, which means that a ball of sufficiently small radius centered at any point in E is contained in E (i.e., $\mathbf{r} + h\hat{\mathbf{e}}_1$ in E for a sufficiently small h). Since the value of f is path-independent, for the point $\mathbf{r}+h\hat{\mathbf{e}}_1$, the curve can be chosen so that it goes from $\mathbf{r}_0$ to $\mathbf{r}$ and then from $\mathbf{r}$ to $\mathbf{r} + h\hat{\mathbf{e}}_1$ along the straight line segment. Denote the latter by ΔC. Therefore,

$$f(\mathbf{r} + h\hat{\mathbf{e}}_1) - f(\mathbf{r}) = \int_{\Delta C} \mathbf{F}\cdot d\mathbf{r}$$

because the line integral of $\mathbf{F}$ from $\mathbf{r}_0$ to $\mathbf{r}$ is path-independent. A vector function that traces out ΔC is

$$\Delta C:\ \mathbf{r}(t) = \langle t, y, z\rangle\,, \quad x \le t \le x + h\,.$$

Therefore,

$$\mathbf{r}'(t) = \hat{\mathbf{e}}_1 \quad \Rightarrow \quad \mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r}(t) = \mathbf{F}(\mathbf{r}(t))\cdot \hat{\mathbf{e}}_1 dt = F_1(t, y, z)dt\,.$$

Thus,

$$\begin{aligned} f'_x(\mathbf{r}) &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} F_1(t,y,z)\,dt = \lim_{h\to 0}\frac{1}{h}\Big(\int_a^{x+h} - \int_a^x\Big)F_1(t,y,z)\,dt \\ &= \frac{\partial}{\partial x}\int_a^x F_1(t,y,z)\,dt = F_1(x,y,z) = F_1(\mathbf{r}) \end{aligned}$$

by the continuity of F_1. The equalities $f'_y = F_2$ and $f'_z = F_3$ are established similarly. The details are omitted. □

Although the path independence property does provide a necessary and sufficient condition for a vector field to be conservative, it is rather impractical to verify (one cannot evaluate line integrals along every closed curve!). A more feasible and practical criterion is needed, which is established next. It is worth noting that Eq. (**42.3**) gives a practical method of finding a potential if the vector field is found to be conservative (technical details are given in Study Problem **42.2**).

42.3. The Curl of a Vector Field. According to the rules of vector algebra, the product of a vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and a number s is defined by $s\mathbf{a} = \langle sa_1, sa_2, sa_3\rangle$. By analogy, the gradient $\boldsymbol{\nabla} f$ can be viewed as the formal product of the vector $\boldsymbol{\nabla} = \langle \partial/\partial x,\ \partial/\partial y,\ \partial/\partial z\rangle$ and a scalar f:

$$\boldsymbol{\nabla} f = \Big\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big\rangle f = \Big\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big\rangle\,.$$

The components of $\boldsymbol{\nabla}$ are not ordinary numbers, but rather they are *operators* (i.e., symbols standing for a specified operation that has to be carried out). For example, $(\partial/\partial x)f$ means that the operator $\partial/\partial x$ is applied to a function f and the result of its action on f is the partial derivative of f with respect to x. The directional derivative $D_{\mathbf{u}}f$ can be viewed as the result of the action of the operator

$$D_{\mathbf{u}} = \hat{\mathbf{u}} \cdot \boldsymbol{\nabla} = u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}$$

on a function f. In what follows, the formal vector $\boldsymbol{\nabla}$ is viewed as an operator whose action obeys the rules of vector algebra.

DEFINITION **42.3**. (Curl of a Vector Field).
The curl of a differentiable vector field $\mathbf{F}$ *is*

$$\operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F}.$$

The curl of a vector field is a vector field whose components can be computed according to the definition of the cross product:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} \\ &= \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{\mathbf{e}}_1 + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{\mathbf{e}}_2 + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{\mathbf{e}}_3. \end{aligned}$$

When calculating the components of the curl, the product of a component of $\boldsymbol{\nabla}$ and a component of $\mathbf{F}$ means that the component of $\boldsymbol{\nabla}$ operates on the component of $\mathbf{F}$, producing the corresponding partial derivative.

EXAMPLE **42.1**. *Find the curl of the vector field* $\mathbf{F} = \langle yz, xyz, x^2\rangle$.

SOLUTION:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xyz & x^2 \end{pmatrix} \\ &= \Big\langle \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(xyz),\ -\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial z}(yz),\ \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(yz)\Big\rangle \\ &= \langle -xy,\ y - 2x,\ yz - z\rangle. \end{aligned}$$

□

The geometrical significance of the curl of a vector field will be discussed later in the section devoted to Stokes' theorem. Here the curl is used to formulate sufficient conditions for a vector field to be conservative.

On the Use of the Operator $\boldsymbol{\nabla}$. The rules of vector algebra are useful to simplify algebraic operations involving the operator $\boldsymbol{\nabla}$. For example,

$$\operatorname{curl} \boldsymbol{\nabla} f = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = (\boldsymbol{\nabla} \times \boldsymbol{\nabla}) f = \mathbf{0}$$

because the cross product of a vector with itself vanishes. However, this formal algebraic manipulation should be adopted with precaution because it contains a tacit assumption that the action of the components of $\boldsymbol{\nabla} \times \boldsymbol{\nabla}$ on f vanishes. The latter imposes conditions on the class of functions for which such formal algebraic manipulations are justified. Indeed, according to the definition,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f'_x & f'_y & f'_z \end{pmatrix} = (f''_{zy} - f''_{yz},\ f''_{zx} - f''_{xz},\ f''_{xy} - f''_{yx}).$$

This vector vanishes, provided the order in which the partial derivatives are taken does not matter. In other words, Clairaut's theorem must hold for the class of functions for which formal algebraic manipulations with the operator $\boldsymbol{\nabla}$ are justified. Thus, *the rules of vector algebra can be used to simplify the action of an operator involving* $\boldsymbol{\nabla}$ *if the partial derivatives of a function on which this operator acts are continuous up to the order determined by that action.*

42.4. Test for a Vector Field to Be Conservative. A conservative vector field with continuous partial derivatives in a region E has been shown to have the vanishing curl:

$$\mathbf{F} = \boldsymbol{\nabla} f \quad \Longrightarrow \quad \operatorname{curl} \mathbf{F} = \mathbf{0}.$$

Unfortunately, the converse is *not* true in general. In other words, the vanishing of the curl of a vector field does *not* guarantee that the vector field is conservative. The converse is true only if the region in which the curl vanishes belongs to a special class. Recall that an open region E was defined as a *connected* set (Definition **28.1**); that is any two points of E can be connected by a curve that lies in E. In other words, E cannot be represented as the union of two or more non-intersecting (disjoint) regions.

DEFINITION **42.4**. (Simply Connected Region).
A region E is simply connected *if every closed curve in E can be continuously shrunk to a point in E while remaining in E throughout the deformation.*

Naturally, the entire Euclidean space is simply connected. A ball in space is also simply connected. If E is the region outside a ball, then it is also simply connected. However, if E is obtained by removing a line (or a cylinder) from the entire space, then E is not simply connected. Indeed, take a circle such that the line pierces through the disk bounded by the circle. There is no way this circle can be continuously contracted to a point of E without crossing the line. A solid torus is not simply connected. (Explain why!) A simply connected region D in a plane cannot have "holes" in it (see Figure **42.1**).

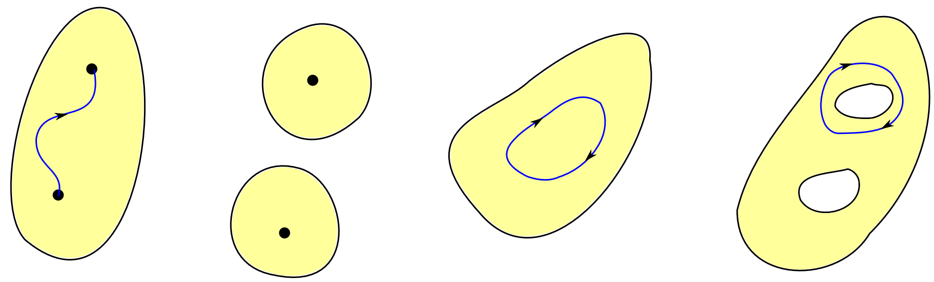


FIGURE **42.1**. From to left to right: A planar connected set (any two points in it can be connected by a continuous curve that lies in the set); a planar disconnected set (there are points in it which cannot be connected by a continuous curve that lies in the set); a planar simply connected set (every simple closed curve in it can be continuously shrunk to a point in it while remaining in the set throughout the deformation); a planar region that is not simply connected (it has holes).

THEOREM **42.3**. (Test for a Vector Field to Be Conservative). *Suppose* $\mathbf{F}$ *is a vector field whose components have continuous partial derivatives on a simply connected open region* E*. Then* $\mathbf{F}$ *is conservative in* E *if and only if its curl vanishes for all points of* E*:*

$$\operatorname{curl}\mathbf{F} = \mathbf{0} \text{ on simply connected } E \quad \Longleftrightarrow \quad \mathbf{F} = \boldsymbol{\nabla} f \text{ on } E.$$

This theorem follows from Stokes' theorem discussed later in this chapter and has two useful consequences. First, *the test for the path-independence of line integrals*:

$$\operatorname{curl}\mathbf{F} = \mathbf{0} \text{ on simply connected } E \quad \Longleftrightarrow \quad \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

for any two curves C_1 and C_2 in E originating from a point $\mathbf{r}_a$ in E and terminating at another point $\mathbf{r}_b$ in E. It follows from Theorem **42.2** for the curve C that is the union of C_1 and $-C_2$. Second, the test for vanishing line integrals along closed paths:

$$\operatorname{curl}\mathbf{F} = \mathbf{0} \text{ on simply connected } E \quad \Longleftrightarrow \quad \oint_C \mathbf{F}\cdot d\mathbf{r} = 0,$$

where C is a closed curve in E. The condition that E is simply connected is crucial here. Even if $\operatorname{curl}\mathbf{F} = \mathbf{0}$, but E is not simply connected, the line integral of $\mathbf{F}$ may still depend on the path and the line integral along a closed path may not vanish! An example is given in Study Problem **42.1**.

Equation (**42.1**) shows that Newton's gravitational force can be written as the gradient of the function $U(\mathbf{r})$ everywhere except the origin. Therefore, its curl vanishes in the region E that is the entire space with one point removed; it is simply connected. Hence, the work done by the gravitational force is *independent* of the path traveled by the object and determined by the difference of values of its potential U (called also a *potential energy*) at the initial and terminal points of the path. More generally, since the work done by a force equals the change of the kinetic energy (see Section **3.6**), the motion under a conservative force $\mathbf{F} = -\boldsymbol{\nabla} U$ has the fundamental property that *the sum of kinetic and potential energies,* $m\mathbf{v}^2/2 + U(\mathbf{r})$*, is conserved along a trajectory of the motion* (recall Study Problem **24.3**).

EXAMPLE **42.2**. *Evaluate the line integral of the vector field*

$$\mathbf{F} = \langle F_1, F_2, F_3 \rangle = \langle yz,\, xz + z + 2y,\, xy + y + 2z \rangle$$

along the path C *that consists of straight line segments* AB_1*,* B_1B_2*, and* B_2D*, where the initial point is* $A = (0,0,0)$*,* $B_1 = (2010, 2011, 2012)$*,* $B_2 = (102, 1102, 2102)$*, and the terminal point is* $D = (1,1,1)$*.*

SOLUTION: The path looks complicated enough to check whether $\mathbf{F}$ is conservative before evaluating the line integral using the parametric equations of C. First, note that the components of $\mathbf{F}$ are polynomials and hence have continuous partial derivatives in the entire space. Therefore, if its curl vanishes, then $\mathbf{F}$ is conservative in the entire space by Theorem **42.3** as the entire space is simply connected:

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz + z + 2y & xy + y + 2z \end{pmatrix} \\ &= \Big\langle (F_3)'_y - (F_2)'_z,\ -(F_3)'_x + (F_1)'_z,\ (F_2)'_x - (F_1)'_y \Big\rangle \\ &= \langle x + 1 - (x+1),\ -y + y,\ z - z \rangle = \mathbf{0}.\end{aligned}$$

Thus, $\mathbf{F}$ is conservative. Now there are two options to finish the problem.

Option 1. One can use the path-independence of the line integral, which means that one can pick any other curve C_1 connecting the initial point A and the terminal point D to evaluate the line integral in question. For example, a straight line segment connecting A and D is simple enough to evaluate the line integral. Its parametric equations are $\mathbf{r} = \mathbf{r}(t) = \langle t, t, t \rangle$, where $0 \le t \le 1$. Therefore,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle t^2,\ t^2 + 3t,\ t^2 + 3t \rangle \cdot \langle 1,\ 1,\ 1 \rangle = 3t^2 + 6t$$

and hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^2 + 6t)\, dt = 4.$$

Option 2. The procedure of Section **20.1** may be used to find a potential f of $\mathbf{F}$ (see also the study problems at the end of this section for an alternative procedure). The line integral is then found by the fundamental theorem for line integrals. Put $\boldsymbol{\nabla} f = \mathbf{F}$. Then the problem is reduced to finding f

from its first-order partial derivatives (the existence of f has already been established). Following the procedure of Section **20.1**,

$$f'_x = F_1 = yz \quad \Rightarrow \quad f(x,y,z) = xyz + g(y,z),$$

for some function $g(y,z)$. The substitution of f into the second equation $f'_y = F_2$ yields

$$xz + g'_y(y,z) = xz + z + 2y \quad \Rightarrow \quad g(y,z) = y^2 + zy + h(z),$$

for some function $h(z)$. The substitution of $f = xyz + y^2 + zy + h(z)$ into the third equation $f'_z = F_3$ yields

$$xy + y + h'(z) = xy + y + 2z \quad \Rightarrow \quad h(z) = z^2 + c,$$

where c is a constant. Thus, a potential of the vector field in question is $f(x,y,z) = xyz + yz + z^2 + y^2 + c$ and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 4$$

by the fundamental theorem for line integrals. $\square$

42.5. Study Problems.

Problem **42.1**. *Consider the vector field in space*

$$\mathbf{F} = \langle F_1,\ F_2,\ F_3 \rangle = \left\langle -\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ 2z \right\rangle.$$

(i). *Show that* $\operatorname{curl} \mathbf{F} = \mathbf{0}$ *in the domain of* $\mathbf{F}$.

(ii). *Let* $\theta = \theta(x,y)$ *be the polar angle as a function of rectangular coordinates* (x,y) *as defined in Section* **32.1**. *Show that*

$$\mathbf{F} = \boldsymbol{\nabla} f\,, \quad f(x,y,z) = \theta(x,y) + z^2\,,$$

at all points where f *is differentiable.*

(iii). *Evaluate the line integral of* $\mathbf{F}$ *along the circle* C*:* $x^2 + y^2 = R^2$ *in the plane* $z = a$. *The circle is oriented counterclockwise as viewed from the top of the* z *axis. Does the result contradict to the fundamental theorem for line integrals? Explain.*

(iv). *Is there a subregion of the domain of* $\mathbf{F}$ *where* $\mathbf{F}$ *is conservative?*

SOLUTION: (i). Since the first two components do not depend on z and the third component does not depend on x and y, it follows from Definition **42.3** that

$$\boldsymbol{\nabla} \times \mathbf{F} = \Big((F_2)'_x - (F_1)'_y\Big)\hat{\mathbf{e}}_3 = \Big(\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}\Big)\hat{\mathbf{e}}_3 = \mathbf{0}$$

for all (x,y,z) that are not on the z axis (that is, $(x,y) \neq (0,0)$) as the vector field is not defined on the z axis.

(ii). For definitiveness, let us use Eq. (**32.1**) that defines $\theta(x,y)$ in the range

$[0, 2\pi)$ for all $(x, y) \neq (0, 0)$. It follows from Eq. (**32.1**) that $\theta(x, y)$ has continuous partial derivatives for $x < 0$ and for $x > 0$, $y \neq 0$:

$$f'_x = \theta'_x = -\frac{x}{x^2 + y^2} = F_1\,, \quad f'_y = \theta'_y = \frac{y}{x^2 + y^2} = F_2\,, \quad f'_z = 2z = F_3$$

The function $\theta(x, y)$ is not continuous in the positive x axis ($x > 0$ and $y = 0$) and, hence, is not differentiable there. If $x = 0$, then $\theta(0, y) = \pi/2$ for $y > 0$ and $\theta(0, y) = 3\pi/2$ for $y < 0$. In either case, $\theta'_y(0, y) = 0 = F_2(0, y)$. The partial derivative $\theta'_x(0, y)$, $y \neq 0$, is calculated by using the definition of partial derivatives. Put $p = 1$ if $y > 0$ and $p = -1$ if $y < 0$. Then for $y \neq 0$

$$\lim_{x\to 0^+} \frac{\theta(x, y) - \theta(0, y)}{x} = \lim_{x\to 0^+} \frac{\tan^{-1}(\frac{y}{x}) - p\frac{\pi}{2}}{x} = -\lim_{x\to 0^+} \frac{y}{x^2 + y^2} = -\frac{1}{y}$$

$$\lim_{x\to 0^-} \frac{\theta(x, y) - \theta(0, y)}{x} = \lim_{x\to 0^-} \frac{\tan^{-1}(\frac{y}{x}) + p\frac{\pi}{2}}{x} = -\lim_{x\to 0^-} \frac{y}{x^2 + y^2} = -\frac{1}{y}$$

where l'Hospital's rule has been used to find the limits. Since the left and right limits exist and coincide, it is concluded that

$$\theta'_x(0, y) = \lim_{x\to 0} \frac{\theta(x, y) - \theta(0, y)}{x} = -\frac{1}{y} = F_1(0, y)$$

$$\Rightarrow \quad \mathbf{F} = \boldsymbol{\nabla} f\,,$$

for all (x, y, z) except *the points on the half of the xz plane with* $x \geq 0$. Note that θ'_x and θ'_y are continuous at all points of the y axis except the origin.

(iii). Parametric equations of the curve C can be chosen in the form

$$\mathbf{r}(t) = \langle R\cos t,\ R\sin t,\ a\rangle\,, \quad 0 \leq t \leq 2\pi\,.$$

Then $\mathbf{r}'(t) = \langle -R\sin t, R\cos t, 0\rangle$ and

$$\begin{aligned}
\mathbf{F}\cdot d\mathbf{r} &= \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt \\
&= \langle -R^{-1}\sin t, R^{-1}\cos t, 2a\rangle \cdot \langle -R\sin t, R\cos t, 0\rangle dt \\
&= (\sin^2 t + \cos^2 t)dt = dt \\
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} dt = 2\pi\,.
\end{aligned}$$

The result does not contradict the fundamental theorem for line integrals. First note that the domain of $\mathbf{F}$ is *not simply connected*. Indeed, the domain of $\mathbf{F}$ is the entire space with the z axis removed. Any closed curve encircling the z axis cannot be continuously shrunk to a point without crossing the z axis. Thus, the fact that the curl of $\mathbf{F}$ vanishes is not sufficient to conclude that $\mathbf{F}$ has the path-independence property (Theorem **42.3**). Second, note that $\mathbf{F} = \boldsymbol{\nabla} f$ does not hold in the entire domain of $\mathbf{F}$ because the function f is only differentiable in space with the half-plane $\theta(x, y) = 0$ removed. In turn, this implies that the chain rule used to prove Eq. (**42.2**) is *not applicable* for any curve that crosses the half-plane $\theta(x, y) = 0$ (see the left panel of Fig. **42.2**). The line integral of $\boldsymbol{\nabla} f$ along a curve encircling the z axis must be viewed as an improper integral where the initial and terminal

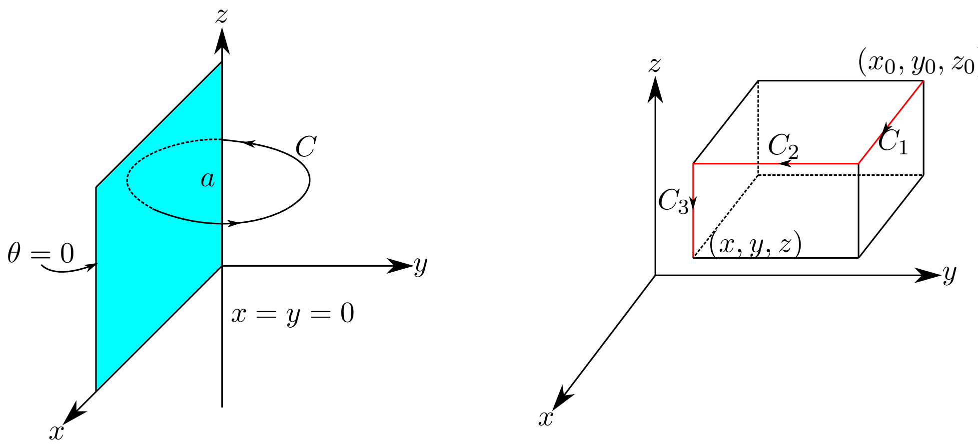


FIGURE **42.2**. **Left**: An illustration to Study Problem **42.1**. **Right**: An illustration to Study Problem **42.2**. To find a potential of a conservative vector field, one can evaluate its line integral from any point (x_0, y_0, z_0) to a generic point (x, y, z) along the rectangular contour C that is the union of the straight line segments C_1, C_2, and C_3 parallel to the coordinate axes.

points of the curve approach the same point on the half-plane where $\boldsymbol{\nabla} f$ does not exist. If the fundamental theorem for line integrals is applied to such a curve, then no contradiction arises because the values of f on the opposite sides of the half-plane differ exactly by 2π in full accordance with the conclusion of the theorem.
(iv). The vector field $\mathbf{F}$ is conservative in the region E that is the entire space with the half-plane $\theta(x, y) = 0$ removed and its potential is given by f (up to an additive constant). Indeed, the region E is *simply connected* as closed curves encircling the z axis are no longer contained in E (the point of intersection of the curve with the half-plane is not in E) and $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ in it. By Theorem **42.3**, $\mathbf{F}$ is conservative in E. The line integral of $\mathbf{F}$ over any closed curve in E vanishes (the total variation of θ along such a curve is zero). With any other definition, the function $\theta(x, y)$ must exhibit the $2\pi-$discontinuity on some ray extended from the origin as the polar angle has to change from 0 to 2π along any closed curve encircling the origin in the xy plane, and the above conclusions hold for any other definition of $\theta(x, y)$. $\square$

Remark. The example considered is not merely a mathematical exercise to illustrate subtleties of the path-independence property of vector fields, which is only of academic interest. In fact, non-conservative vector fields with zero curl in not simply connected regions of space do occur in nature. They describe vortices in fluid flows. They are also used in theoretical foundations for the existence of magnetic monopoles, fundamental particles that are believed to carry magnetic charges and whose properties may shed light on the early evolution of our Universe. A search for magnetic monopoles is still underway.

Problem **42.2**. *Prove that if* $\mathbf{F} = \langle F_1, F_2, F_3\rangle$ *is conservative, then its potential is*

$$f(x,y,z) = \int_{x_0}^{x} F_1(t,y_0,z_0)\,dt + \int_{y_0}^{y} F_2(x,t,z_0)\,dt + \int_{z_0}^{z} F_3(x,y,t)\,dt,$$

where (x_0,y_0,z_0) *is any point in the domain of* $\mathbf{F}$. *Use this equation to find a potential of* $\mathbf{F}$ *from Example* **42.2**.

Solution: In (**42.3**), take C that consists of three straight line segments, $(x_0,y_0,z_0) \to (x,y_0,z_0) \to (x,y,z_0) \to (x,y,z)$ as depicted in the right panel of Fig. **42.2**. The parametric equations of the first segment are

$$C_1 : \quad \mathbf{r}(t) = \langle t, y_0, z_0\rangle\,, \quad x_0 \le t \le x\,.$$

Therefore, $\mathbf{r}'(t) = \langle 1,0,0\rangle$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = F_1(t,y_0,z_0)$. So the line integral of $\mathbf{F}$ along C_1 gives the first term in the above expression for f. Similarly, the second term is the line integral of $\mathbf{F}$ along the second segment

$$C_2 : \quad \mathbf{r}(t) = \langle x, t, z_0\rangle\,, \quad y_0 \le t \le y\,,$$

so that $\mathbf{r}'(t) = \langle 0,1,0\rangle$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = F_2(x,t,z_0)$. The third term is the line integral of $\mathbf{F}$ along the third segment

$$C_3 : \quad \mathbf{r}(t) = \langle x, y, t\rangle\,, \quad z_0 \le t \le z\,,$$

so that $\mathbf{r}'(t) = \langle 0,0,1\rangle$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = F_3(x,y,t)$.

In Example **42.2**, it was established that $\mathbf{F} = \langle F_1,F_2,F_3\rangle = \langle yz,\, xz+z+2y,\, xy+y+2z\rangle$ is conservative. For simplicity, choose $(x_0,y_0,z_0) = (0,0,0)$. Then

$$\begin{aligned} f(x,y,z) &= \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt \\ &= 0 + y^2 + (xyz + yz + z^2) = xyz + yz + z^2 + y^2, \end{aligned}$$

which naturally coincides with f found by a different (longer) method. □

Problem **42.3**. (Operator $\boldsymbol{\nabla}$ in curvilinear coordinates)
Let the transformation $(u,v,w) \to (x,y,z)$ *be a change of variables. If* $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$, *and* $\hat{\mathbf{e}}_w$ *are unit vectors normal to the coordinate surfaces (see Eq.* (**36.7**) *in Section* **36.6**), *show that*

$$\boldsymbol{\nabla} = \|\boldsymbol{\nabla} u\|\hat{\mathbf{e}}_u \frac{\partial}{\partial u} + \|\boldsymbol{\nabla} v\|\hat{\mathbf{e}}_v \frac{\partial}{\partial v} + \|\boldsymbol{\nabla} w\|\hat{\mathbf{e}}_w \frac{\partial}{\partial w}$$

In particular, find the $\boldsymbol{\nabla}$ *operator in the cylindrical and spherical coordinates.*

SOLUTION: By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial}{\partial w}$$

and similarly for $\partial/\partial y$ and $\partial/\partial z$. Then

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\mathbf{e}}_1\frac{\partial}{\partial x} + \hat{\mathbf{e}}_2\frac{\partial}{\partial y} + \hat{\mathbf{e}}_3\frac{\partial}{\partial z} \\
&= \left(\frac{\partial u}{\partial x}\hat{\mathbf{e}}_1 + \frac{\partial u}{\partial y}\hat{\mathbf{e}}_2 + \frac{\partial u}{\partial z}\hat{\mathbf{e}}_3\right)\frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial x}\hat{\mathbf{e}}_1 + \frac{\partial v}{\partial y}\hat{\mathbf{e}}_2 + \frac{\partial v}{\partial z}\hat{\mathbf{e}}_3\right)\frac{\partial}{\partial v} \\
&\quad + \left(\frac{\partial w}{\partial x}\hat{\mathbf{e}}_1 + \frac{\partial w}{\partial y}\hat{\mathbf{e}}_2 + \frac{\partial w}{\partial z}\hat{\mathbf{e}}_3\right)\frac{\partial}{\partial w} \\
&= \boldsymbol{\nabla} u\frac{\partial}{\partial u} + \boldsymbol{\nabla} v\frac{\partial}{\partial v} + \boldsymbol{\nabla} w\frac{\partial}{\partial w} \\
&= \|\boldsymbol{\nabla} u\|\,\hat{\mathbf{e}}_u\frac{\partial}{\partial u} + \|\boldsymbol{\nabla} v\|\,\hat{\mathbf{e}}_v\frac{\partial}{\partial v} + \|\boldsymbol{\nabla} w\|\,\hat{\mathbf{e}}_w\frac{\partial}{\partial v}\,,
\end{aligned}$$

where the unit vectors are defined in (**36.7**). Making use of equations (**36.8**), (**36.9**), (**36.11**), and (**36.10**), the operator $\boldsymbol{\nabla}$ is obtained in the cylindrical and spherical coordinates:

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\,\hat{\mathbf{e}}_\theta\,\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_3\,\frac{\partial}{\partial z}\,, \\
\boldsymbol{\nabla} &= \hat{\mathbf{e}}_\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho}\,\hat{\mathbf{e}}_\phi\,\frac{\partial}{\partial \phi} + \frac{1}{\rho\sin\phi}\,\hat{\mathbf{e}}_\theta\,\frac{\partial}{\partial \theta}\,.
\end{aligned}$$

□

42.6. Exercises.

1–5. Calculate the curl $\boldsymbol{\nabla}\times\mathbf{F}$ of the given vector field $\mathbf{F}$ on its domain.

1. $\mathbf{F} = \langle xyz, -y^2x, 0\rangle$;
2. $\mathbf{F} = \langle \cos(xz),\ \sin(yz), 2\rangle$;
3. $\mathbf{F} = \langle h(x), g(y), f(z)\rangle$, where the functions h, g, and f are differentiable.
4. $\mathbf{F} = \langle \ln(xyz), \ln(yz), \ln z\rangle$;
5. $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a}$ is a constant vector and $\mathbf{r} = \langle x, y, z\rangle$.

6. Suppose that a vector field $\mathbf{F}(\mathbf{r})$ and a function $f(\mathbf{r})$ are differentiable. Use the vector algebra rules for the operator $\boldsymbol{\nabla}$ to show that $\boldsymbol{\nabla}\times(f\mathbf{F}) = f(\boldsymbol{\nabla}\times\mathbf{F}) + \boldsymbol{\nabla} f\times\mathbf{F}$.

7. Use the vector algebra rules for the operator $\boldsymbol{\nabla}$ to find $\boldsymbol{\nabla}\times(\mathbf{c}\times\mathbf{r}f(r))$ where $r = \|\mathbf{r}\|$, f is differentiable, and $\mathbf{c}$ is a constant vector.

8. A fluid, filling the entire space, rotates at a constant rate ω about an axis parallel to a unit vector $\hat{\mathbf{n}}$. Find the curl of the velocity vector field at a generic point $\mathbf{r}$. Assume that the position vector $\mathbf{r}$ originates from a point on the axis of rotation.

9–16. Determine whether the given vector field $\mathbf{F}$ is conservative in its domain and, if it is, find its potential.

9. $\mathbf{F} = \langle 2xy,\ x^2 + 2yz^3,\ 3z^2y^2 + 1\rangle$;
10. $\mathbf{F} = \langle yz,\ xz + 2y\cos z,\ xy - y^2 \sin z\rangle$;
11. $\mathbf{F} = \langle e^y,\ xe^y - z^2,\ -2yz\rangle$;
12. $\mathbf{F} = \langle 6xy + z^4y,\, 3x^2 + z^4x,\, 4z^3xy\rangle$;
13. $\mathbf{F} = \Big\langle yz(2x + y + z),\, xz(x + 2y + z),\, xy(x + y + 2z)\Big\rangle$;
14. $\mathbf{F} = \langle -y(x^2 + y^2)^{-1} + z,\, x(x^2 + y^2)^{-1},\, x\rangle$;
15. $\mathbf{F} = \langle y\cos(xy),\, x\cos(xy),\, z + y\rangle$;
16. $\mathbf{F} = \langle -yz/x^2,\, z/x,\, y/x\rangle$.

17–21. Determine first whether the given vector field $\mathbf{F}$ has the path-independence property (or it is conservative) in its domain and then evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ by making a convenient deformation of the curve C if applicable.

17. $\mathbf{F} = \langle y^2z^2 + 2x + 2y,\ 2xyz^2 + 2x,\ 2xy^2z + 1\rangle$ and C consists of there line segments: $(1, 1, 1) \to (a, b, c) \to (1, 2, 3)$;
18. $\mathbf{F} = \langle zx\,, yz\,, z^2\rangle$ and C is the part of the helix $\mathbf{r}(t) = \langle 2\sin t\,, -2\cos t\,, t\rangle$ that lies inside the ellipsoid $x^2 + y^2 + 2z^2 = 6$ and oriented in the direction of increasing t.
19. $\mathbf{F} = \langle y - z^2\,,\ x + \sin z\,, y\cos z - 2xz\rangle$ and C is one turn of a helix of radius a from $(a, 0, 0)$ to $(a, 0, b)$.
20. $\mathbf{F} = g(r^2)\mathbf{r}$ where $\mathbf{r} = \langle x, y, z\rangle$, $r = \|\mathbf{r}\|$, g is differentiable, and C is a smooth curve from a point on the sphere $x^2 + y^2 + z^2 = a^2$ to a point on the sphere $x^2 + y^2 + z^2 = b^2$. What is the work done by the force $\mathbf{F}$ if $g = -1/r^3$?
21. $\mathbf{F} = \Big\langle 2(y+z)^{1/2},\ -x(y+z)^{3/2},\ -x(y+z)^{-3/2}\Big\rangle$ and C is a smooth curve from the point $(1, 1, 3)$ and $(2, 4, 5)$.

22. Suppose that $\mathbf{F}$ and $\mathbf{G}$ are continuous on a simply connected open region E. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{G} \cdot d\mathbf{r}$ for any smooth closed curve C in E if there is a function f with continuous partial derivatives in E such that $\mathbf{F} - \mathbf{G} = \boldsymbol{\nabla} f$.
23. Use the properties of the gradient to show that the vectors $\hat{\mathbf{e}}_r = \langle\cos\theta, \sin\theta\rangle$ and and $\hat{\mathbf{e}}_\theta = \langle -\sin\theta, \cos\theta\rangle$ are unit vectors orthogonal to the coordinate curves $r(x, y) = \text{const}$ and $\theta(x, y) = \text{const}$ of polar coordinates. Given a planar vector field, put $\mathbf{F} = F_r\hat{\mathbf{e}}_r + F_\theta\hat{\mathbf{e}}_\theta$. Use the chain rule to express the curl of a planar vector field $\mathbf{F}(r, \theta)$ in polar coordinates (r, θ) in terms of F_r, F_θ, $\hat{\mathbf{e}}_r$, and $\hat{\mathbf{e}}_\theta$.
24. Evaluate the pairwise cross products of the unit vectors (**36.11**) and the pairwise cross products of the unit vectors (**36.10**). Use the obtained relations and the result of Study Problem **42.3** to express the curl of a vector

field in spherical and cylindrical coordinates:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \frac{1}{\rho\sin\phi}\left(\frac{\partial(\sin\phi F_\theta)}{\partial\phi}-\frac{\partial F_\phi}{\partial\theta}\right)\hat{\mathbf{e}}_\rho \\
&\quad+\frac{1}{\rho}\left(\frac{1}{\sin\phi}\frac{\partial F_\rho}{\partial\theta}-\frac{\partial(\rho F_\theta)}{\partial\rho}\right)\hat{\mathbf{e}}_\phi+\frac{1}{\rho}\left(\frac{\partial(\rho F_\phi)}{\partial\rho}-\frac{\partial F_\rho}{\partial\phi}\right)\hat{\mathbf{e}}_\theta\,, \\
\boldsymbol{\nabla}\times\mathbf{F} &= \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta}-\frac{\partial F_\theta}{\partial z}\right)\hat{\mathbf{e}}_r+\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)\hat{\mathbf{e}}_\theta \\
&\quad+\frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)\hat{\mathbf{e}}_z\,,
\end{aligned}$$

where the field $\mathbf{F}$ is decomposed over the bases (**36.11**) and (**36.10**): $\mathbf{F} = F_\rho\hat{\mathbf{e}}_\rho + F_\phi\hat{\mathbf{e}}_\phi + F_\theta\hat{\mathbf{e}}_\theta$ and $\mathbf{F} = F_r\hat{\mathbf{e}}_r + F_\theta\hat{\mathbf{e}}_\theta + F_z\hat{\mathbf{e}}_z$.
Hint: Show $\partial\,\hat{\mathbf{e}}_\rho/\partial\phi = \hat{\mathbf{e}}_\phi$, $\partial\,\hat{\mathbf{e}}_\rho/\partial\theta = \sin\theta\,\hat{\mathbf{e}}_\theta$, and similar relations for the partial derivatives of other unit vectors.

43. Green's Theorem

Green's theorem should be regarded as the counterpart of the fundamental theorem of calculus for the double integral.

DEFINITION **43.1**. (Orientation of Planar Closed Curves). *A simple closed curve C in a plane whose single traversal is counterclockwise (clockwise) is said to be positively (negatively) oriented.*

A simple closed curve divides the plane into two connected regions. If a planar region D is bounded by a simple closed curve, then the positively oriented boundary of D is denoted by the symbol ∂D (see the left panel of Fig. **43.1**).

Recall that a simple closed curve can be regarded as a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ on $[a, b]$ such that $\mathbf{r}(a) = \mathbf{r}(b)$ and, for any $t_1 \neq t_2$ in the open interval (a, b), $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$; that is, $\mathbf{r}(t)$ traces out C only once without self-intersection. A positive orientation means that $\mathbf{r}(t)$ traces out its range counterclockwise. For example, the vector functions $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ and $\mathbf{r}(t) = \langle \cos t, -\sin t\rangle$ on the interval $[0, 2\pi]$ define the positively and negatively oriented circles of unit radius, respectively.

THEOREM **43.1**. (Green's Theorem).
Let C be a positively oriented, piecewise-smooth, simple, closed curve in the plane and let D be the region bounded by $C = \partial D$. If the functions F_1 and F_2 have continuous partial derivatives in an open region that contains D, then

$$\iint_D \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA = \oint_{\partial D} F_1\, dx + F_2\, dy.$$

Just like the fundamental theorem of calculus, Green's theorem relates the derivatives of F_1 and F_2 in the integrand to the values of F_1 and F_2 on the boundary of the integration region. A proof of Green's theorem is rather involved. Here it is limited to the case when the region D is simple.

PROOF (FOR SIMPLE REGIONS). A simple region D admits two equivalent algebraic descriptions:

$$D = \{(x, y)\,|\, y_{\rm bot}(x) \le y \le y_{\rm top}(x)\,,\ a \le x \le b\}\,, \tag{43.1}$$

$$D = \{(x, y)\,|\, x_{\rm bot}(y) \le x \le x_{\rm top}(y)\,,\ c \le y \le d\}\,. \tag{43.2}$$

The idea of the proof is to establish the equalities

$$\oint_{\partial D} F_1\, dx = -\iint_D \frac{\partial F_1}{\partial y} dA\,, \qquad \oint_{\partial D} F_2\, dy = \iint_D \frac{\partial F_2}{\partial x} dA \tag{43.3}$$

using, respectively, (**43.1**) and (**43.2**). The conclusion of the theorem is then obtained by adding these equations. The technical details will be given to establish the first relation in (**43.3**) using the description (**43.1**) of D as a vertically simple region. The second relation in (**43.3**) is proved along the same line of reasoning by using the description (**43.2**) of D as a horizontally simple region.

The line integral is transformed into an ordinary integral first. The boundary ∂D contains four curves, denoted C_1, C_2, C_3, and C_4 (see the right panel of Fig. **43.1**). The curves C_1 and C_3 are the graphs $y = y_{\rm bot}(x)$ and $y = y_{\rm top}(x)$, respectively. Their parametric equations are

$$\begin{aligned} C_1: &\quad \mathbf{r} = \langle t,\ y_{\rm bot}(t)\rangle\,, \quad a \le t \le b\,,\\ C_3: &\quad \mathbf{r} = \langle t,\ y_{\rm top}(t)\rangle\,, \quad a \le t \le b\,. \end{aligned}$$

These vector functions traverse the graphs from left to right. The positive orientation of ∂D implies that the graph $y = y_{\rm bot}(x)$ must be oriented from left to right, whereas the graph $y = y_{\rm top}(x)$ from right to left. So the orientation of C_3 must be reversed to obtain the corresponding part of ∂D, which is achieved by changing the sign of the line integral along C_3 (the curve $-C_3$ is the part of ∂D). The boundary curves C_2 and C_4 (the sides of D) are segments of the vertical lines $x = b$ (oriented upward) and $x = a$ (oriented downward), which may collapse to a single point if the graphs $y = y_{\rm bot}(x)$ and $y = y_{\rm top}(x)$ intersect at $x = a$ or $x = b$ or both. The line integrals along C_2 and C_4 do not contribute to the line integral with respect to x along ∂D because $dx = 0$ along C_2 and C_4. By construction, $x = t$ and $dx = dt$ for the curves C_1 and C_3. Hence,

$$\begin{aligned} \oint_{\partial D} F_1\,dx &= \int_{C_1} F_1\,dx + \int_{-C_3} F_1\,dx\\ &= \int_a^b \Big(F(x, y_{\rm bot}(x)) - F(x, y_{\rm top}(x))\Big)\,dx, \end{aligned}$$

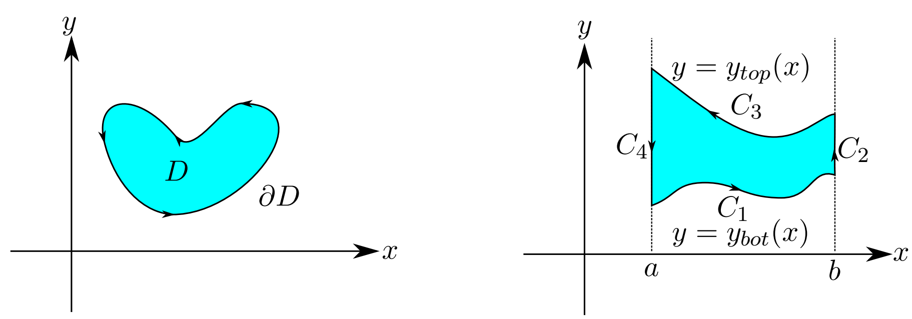


FIGURE **43.1**. **Left**: A simple closed planar curve encloses a (connected) region D in the plane. The positive orientation of the boundary of D means that the boundary curve ∂D is traversed counterclockwise. **Right**: A vertically simple region D is bounded by four smooth curves: two graphs C_1 and C_2 and two vertical lines C_2 $(x = b)$ and C_3 $(x = a)$. The boundary ∂D is the union of these curves oriented counterclockwise.

where the property (**41.4**) has been used. Next, the double integral is transformed into an ordinary integral by converting it to an iterated integral:

$$\begin{aligned}\iint_D \frac{\partial F_1}{\partial y} dA &= \int_a^b \int_{y_{\mathrm{bot}}(x)}^{y_{\mathrm{top}}(x)} \frac{\partial F_1}{\partial y}\, dy\, dx \\ &= \int_a^b \Big(F(x, y_{\mathrm{top}}(x)) - F(x, y_{\mathrm{bot}}(x))\Big) dx,\end{aligned}$$

where the latter equality follows from the fundamental theorem of calculus and the continuity of F_1 on an open interval that contains $[y_{\mathrm{bot}}(x), y_{\mathrm{top}}(x)]$ for any x in $[a, b]$ (the hypothesis of Green's theorem). Comparing the expression of the line and double integrals via ordinary integrals, the validity of the first relation in (**43.3**) is established. $\square$

Suppose that a smooth, oriented curve C divides a region D into two *simple* regions D_1 and D_2 (see the left panel of Fig. **43.2**). If the boundary ∂D_1 contains C (i.e., the orientation of C coincides with the positive orientation of ∂D_1), then ∂D_2 must contain the curve $-C$ and vice versa. Using the conventional notation $F_1\, dx + F_2\, dy = \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle F_1, F_2 \rangle$, one infers that

$$\begin{aligned}\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \iint_{D_1} \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA + \iint_{D_2} \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA \\ &= \iint_{D} \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA.\end{aligned}$$

The first equality holds because of the cancellation of the line integrals along C and $-C$ according to (**41.4**). The validity of the second equality follows from the proof of Green's theorem for simple regions. Finally, the equality is established by the additivity property of double integrals. By making use of similar arguments, the proof can be extended to a region D that can be represented as the union of finitely many simple regions.

Green's Theorem for Non-simply Connected Regions. Let regions D_1 and D_2 be bounded by simple, piecewise-smooth, closed curves and let D_2 lie in the interior of D_1 (see the right panel of Fig. **43.2**). Consider the region D that was obtained from D_1 by removing D_2 (the region D has a hole of the shape D_2). Making use of Green's theorem, one finds

$$\begin{aligned}\iint_{D} \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA &= \iint_{D_1} \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA - \iint_{D_2} \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA \\ &= \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{-\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}. \end{aligned} \tag{43.4}$$

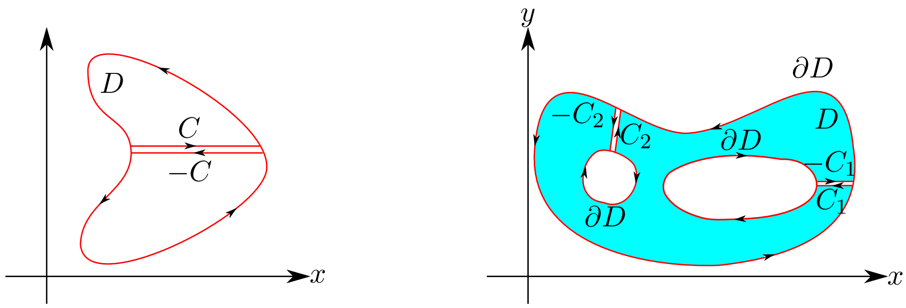
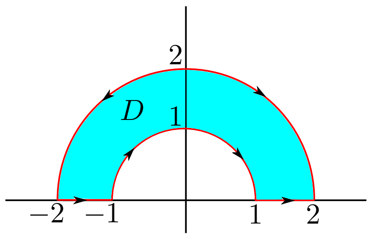


FIGURE **43.2**. **Left**: A region D is split into two regions by a curve C. If the boundary of the upper part of D has positive orientation, then the positively oriented boundary of the lower part of D has the curve $-C$. **Right**: Green's theorem holds for non-simply connected regions. The orientation of the boundaries of "holes" in D is obtained by making cuts along curves C_1 and C_2 so that D becomes simply connected. The positive orientation of the outer boundary of D induces the orientation of the boundaries of the "holes".

This establishes the validity of Green's theorem for not simply connected regions. The boundary ∂D consists of ∂D_1 and $-\partial D_2$; that is, the outer boundary has a positive orientation, while the inner boundary is negatively oriented. A similar line of reasoning leads to the conclusion that Green's theorem holds for any number of holes in D: all inner boundaries of D must be negatively oriented. Such orientation of the boundaries can also be understood as follows. Let a curve C connect a point of the outer boundary with a point of the inner boundary. Let us make a cut of the region D along C. Then the region D becomes simply connected and ∂D consists of a *continuous* curve (the inner and outer boundaries, and the curves C and $-C$). The boundary ∂D has to be positively oriented. The latter requires that the outer boundary be traced counterclockwise, while the inner boundary is traced clockwise (the orientation of C and $-C$ is chosen accordingly). By applying Green's theorem to ∂D, one can see that the line integrals over C and $-C$ are cancelled and (**43**.**4**) follows from the additivity of the double integral.

43.1. Evaluating Line Integrals via Double Integrals. Green's theorem provides a technically convenient tool to evaluate line integrals along planar closed curves. It is especially beneficial when the curve consists of several smooth pieces that are defined by different vector functions; that is, the line integral must be split into a sum of line integrals to be converted into ordinary integrals. Sometimes, the line integral turns out to be much more difficult to evaluate than the double integral.



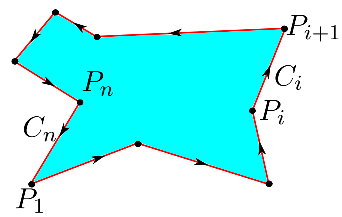


FIGURE **43.3**. **Left**: The integration curve in the line integral discussed in Example **43.1**. **Right**: A general polygon. Its area is evaluated in Example **43.3** by representing the area via a line integral.

EXAMPLE **43.1**. *Evaluate the line integral of*

$$\mathbf{F} = \langle y^2 + e^{\cos x},\ 3xy - \sin(y^4)\rangle$$

along the curve C that is the boundary of the half of the annulus: $1 \le x^2 + y^2 \le 4$ *and* $y \ge 0$*; C is oriented clockwise.*

The curve C consists of four smooth pieces, the half-circles of radii 1 and 2 and two straight line segments of the x axis, $[-2, -1]$ and $[1, 2]$ as shown in the left panel of Fig. **43.3**. Each curve can be easily parameterized and the line integral in question can be transformed into the sum of four ordinary integrals which are then evaluated. The reader is advised to pursue this avenue of actions to appreciate the following alternative way based on Green's theorem (this is not impossible to accomplish if one figures out how to handle the integration of the functions $e^{\cos x}$ and $\sin(y^4)$ whose antiderivatives are not expressible in elementary functions).
SOLUTION: The curve C is a simple, piecewise-smooth, closed curve and the components of $\mathbf{F}$ have continuous partial derivatives everywhere. Thus, Green's theorem applies if $\partial D = -C$ (because the orientation of C is negative) and D is the half-annulus. One has $\partial F_1/\partial y = 2y$ and $\partial F_2/\partial x = 3y$. By Green's theorem,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= -\oint_{\partial D}\mathbf{F}\cdot d\mathbf{r} = -\iint_D\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)dA = -\iint_D y\,dA \\ &= -\int_0^{\pi}\int_1^2 r\sin\theta\ r\,dr\,d\theta = -\int_0^{\pi}\sin\theta\,d\theta\int_1^2 r^2dr = -\frac{14}{3},\end{aligned}$$

where the double integral has been transformed to polar coordinates. The region D is the image of the rectangle $D' = [1, 2] \times [0, \pi]$ in the polar plane under the transformation $(r, \theta) \to (x, y)$. □

Changing the Curve of Integration in a Line Integral. If a planar vector field is not conservative, then its line integral along a curve C originating from a point A and terminating at a point B depends on C. If C' is another curve outgoing from A and terminating at B, what is the relation between the line integrals of $\mathbf{F}$ over C and C'? Green's theorem allows us to establish such a relation. Suppose that C and C' have no self-intersections and do not intersect each other. Then their union is a boundary of a simply connected region D. Let us reverse the orientation of one of the curves so that their union is the positively oriented boundary ∂D, say, ∂D is the union of C and $-C'$. Then

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} - \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

By Green's theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} + \iint_D \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA \tag{43.5}$$

which establishes the relation between lines integrals of a non-conservative planar vector field over two different curves that have common endpoints.

Example **43.2**. *Evaluate the line integral of the vector field*

$$\mathbf{F} = \langle 2y + \cos(x^2),\ x^2 + y^3 \rangle$$

along the curve C which consists of the line segments $(0,0) \to (1,1)$ and $(1,1) \to (0,2)$.

Solution: Let C' be the line segment $(0,0) \to (0,2)$. Then the union of C and $-C'$ is the boundary ∂D (positively oriented) of the triangular region D with vertices $(0,0)$, $(1,1)$, and $(0,2)$. The relation (**43.5**) can be applied to evaluate the line integral over C. The parametric equations of C' are $x = 0$, $y = t$, $0 \le t \le 2$. Hence, along C', $\mathbf{F} \cdot d\mathbf{r} = F_2(0,t)dt = t^3 dt$ and

$$\int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 t^3 dt = 4\,.$$

Then $\partial F_2/\partial x = 2x$ and $\partial F_1/\partial y = 2$. The region D admits an algebraic description as a vertically simple region: $x \le y \le 2 - x$, $0 \le x \le 1$. Hence,

$$\begin{aligned}\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA &= \iint_D (2x-2)dA = 2\int_0^1 (x-1)\int_x^{2-x} dydx \\ &= -4\int_0^1 (x-1)^2 dx = -\frac{4}{3}\,.\end{aligned}$$

Therefore, by Eq. (**43.5**)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4 - \frac{4}{3} = \frac{8}{3}\,.$$

□

43.2. Area of a Planar Region as a Line Integral. Consider the planar vector field $\mathbf{F} = \langle F_1, F_2\rangle$, where $F_2 = x$ and $F_1 = 0$. Then

$$\iint_D \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dA = \iint_D dA = A(D).$$

The area $A(D)$ can also be obtained if $\mathbf{F} = \langle -y, 0\rangle$ or $\mathbf{F} = \langle -y/2, x/2\rangle$. By Green's theorem, the area of D can be expressed by line integrals:

$$(\mathbf{43.6}) \qquad A(D) = \oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx = \frac{1}{2}\oint_{\partial D} x\, dy - y\, dx,$$

assuming, of course, that the boundary of D is a simple, piecewise-smooth, closed curve (or several such curves if D has holes). The reason the values of these line integrals coincide is simple. The difference of any two vector fields involved is the gradient of a function whose line integral along a closed curve vanishes owing to the fundamental theorem for line integrals. For example, for $\mathbf{F} = \langle 0, x\rangle$ and $\mathbf{G} = \langle -y, 0\rangle$, the difference is $\mathbf{F} - \mathbf{G} = \langle y, x\rangle = \boldsymbol{\nabla} f$, where $f(x, y) = xy$, so that

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} - \oint_{\partial D} \mathbf{G}\cdot d\mathbf{r} = \oint_{\partial D} (\mathbf{F} - \mathbf{G})\cdot d\mathbf{r} = \oint_{\partial D} \boldsymbol{\nabla} f\cdot d\mathbf{r} = 0.$$

The representation (**43.6**) of the area of a planar region as the line integral along its boundary is quite useful when the shape of D is too complicated to be computed using a double integral (e.g., when D is not simple and/or a representation of boundaries of D by graphs becomes technically difficult).

EXAMPLE **43.3**. (Area of a polygon)
Consider an arbitrary polygon whose vertices in counterclockwise order are (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). Find its area.

SOLUTION: Evidently, a generic polygon is not a simple region (e.g., it may have a star-like shape). So the double integral is not at all suitable for finding the area. In contrast, the line integral approach seems far more feasible as the boundary of the polygon consists of n straight line segments connecting neighboring vertices as shown in the right panel of Fig. **43.3**). If C_i is such a segment oriented from (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, ..., n-1$, then C_n goes from (x_n, y_n) to (x_1, y_1). A vector function that traces out a straight line segment from a point $\mathbf{r}_a$ to a point $\mathbf{r}_b$ is

$$\mathbf{r}(t) = \mathbf{r}_a + (\mathbf{r}_b - \mathbf{r}_a)t\,, \quad 0 \le t \le 1\,.$$

For the segment C_i, take $\mathbf{r}_a = (x_i, y_i)$ and $\mathbf{r}_b = (x_{i+1}, y_{i+1})$. Hence, parametric equations of C_i are

$$x(t) = x_i - (x_{i+1} - x_i)t = x_i + \Delta x_i\, t\,, \quad y(t) = y_i + (y_{i+1} - y_i)t = y_i + \Delta y_i\, t\,.$$

For the vector field $\mathbf{F} = \langle -y, x\rangle$ on C_i, one has

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t) = \langle -y(t), x(t)\rangle \cdot \langle \Delta x_i, \Delta y_i\rangle &= x_i\,\Delta y_i - y_i\,\Delta x_i \\ &= x_i y_{i+1} - y_i x_{i+1}; \end{aligned}$$

that is, the t dependence cancels out. Therefore, taking into account that C_n goes from (x_n, y_n) to (x_1, y_1), the area is

$$\begin{aligned}
A &= \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx = \frac{1}{2}\sum_{i=1}^{n}\int_{C_i} x\,dy - y\,dx \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\int_0^1 (x_i y_{i+1} - y_i x_{i+1})\,dt + \frac{1}{2}\int_0^1 (x_n y_1 - y_n x_1)\,dt \\
&= \frac{1}{2}\Big(\sum_{i=1}^{n-1}(x_i y_{i+1} - y_i x_{i+1}) + (x_n y_1 - y_n x_1)\Big).
\end{aligned}$$

$\square$

So Green's theorem offers an elegant way to find the area of a general polygon if the coordinates of its vertices are known. A simple, piecewise-smooth, closed curve C in a plane can always be approximated by a polygon. The area of the region enclosed by C can therefore be approximated by the area of a polygon with a large enough number of vertices, which is often used in many practical applications.

43.3. The Test for Planar Vector Fields to Be Conservative. Green's theorem can be used to prove Theorem **42.3** for planar vector fields. Consider a planar vector field $\mathbf{F} = \langle F_1(x,y), F_2(x,y), 0\rangle$. Its curl has only one component:

$$\boldsymbol{\nabla} \times \mathbf{F} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{pmatrix} = \hat{\mathbf{e}}_3\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big).$$

Suppose that the curl of $\mathbf{F}$ vanishes throughout a simply connected open region D, $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. By definition, any simple closed curve C in a simply connected region D can be shrunk to a point of D while remaining in D throughout the deformation (i.e., any such C bounds a subregion D_s of D). By Green's theorem, where $C = \partial D_s$,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{D_s}\Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)dA = \iint_{D_s} 0\;dA = 0$$

for any closed simple curve C in D. By the path-independence property (Theorem **42.2**), the vector field $\mathbf{F}$ is conservative in D.

43.4. Study Problems.

Problem **43.1**. *Evaluate the line integral of* $\mathbf{F} = \langle y + e^{x^2}, 3x - \sin(y^2)\rangle$ *along the counterclockwise-oriented boundary of* D *that is enclosed by the parabolas* $y = x^2$ *and* $x = y^2$.

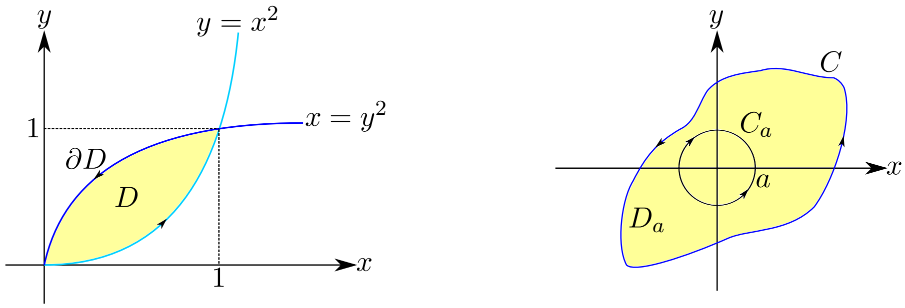


FIGURE **43.4**. **Left**: An illustration to Study Problem **43.1**. **Right**: An illustration to Study Problem **43.2**. The region D_a is bounded by a curve C and the circle C_a.

SOLUTION: One has $\partial F_1/\partial y = 1$ and $\partial F_2/\partial x = 3$. By Green's theorem,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D 2\, dA = 2 \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = 2 \int_0^1 (\sqrt{x} - x^2)\, dx = \frac{2}{3}.$$

The integration region D is shown in the left panel of Fig. **43.4**. □

Problem **43.2**. *Prove that the line integral of the planar vector field*

$$\mathbf{F} = \left\langle -\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2} \right\rangle$$

along any positively oriented, simple, smooth, closed curve C that encircles the origin is 2π and that it vanishes for any such curve that does not encircle the origin.

SOLUTION: It has been established (see Study Problem **42.1**) that the curl of this vector field vanishes in the domain that is the entire plane with the origin removed. If C does not encircle the origin, then $\partial F_2/\partial x - \partial F_1/\partial y = 0$ throughout the region encircled by C, and the line integral along C vanishes by Green's theorem. Given a closed curve C that encircles the origin, but does not go through it, one can always find a disk of a small enough radius a such that the curve C does not intersect it. Let D_a be the region bounded by the circle C_a of radius a and the curve C. Then $\partial F_2/\partial x - \partial F_1/\partial y = 0$ throughout D_a. Let C be oriented counterclockwise, while C_a is oriented clockwise. Then ∂D_a is the union of C and C_a. By Green's theorem,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = 0 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_a} \mathbf{F} \cdot d\mathbf{r} = \oint_{-C_a} \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

because $-C_a$ is the circle oriented counterclockwise and for such a circle the line integral has been found to be 2π (see Study Problem **42.1**). □

Problem **43.3**. (Volume of axially symmetric solids)
Let D be a region in the upper part of the xy plane ($y \geq 0$). Consider the

solid E obtained by rotation of D about the x axis. Show that the volume of the solid is given by

$$V(E) = -\pi \oint_{\partial D} y^2 dx$$

SOLUTION: Let dA be the area of a partition element of D that contains a point (x, y). If the partition element is rotated about the x axis, the point (x, y) traverses the circle of radius y (the distance from the point (x, y) to the x axis). The length of the circle is $2\pi y$. Consequently, the volume of the solid ring swept by the partition element is $dV = 2\pi y dA$. Taking the sum over the partition of D, the volume is expressed via the double integral over D:

$$V(E) = 2\pi \iint_D y dA$$

In Green's theorem, demand $\partial F_1/\partial y = 2y$ and $\partial F_2/\partial x = 0$ so that the above double integral is proportional to the left side of Green's equation. In particular, $F_1 = y^2$ and $F_2 = 0$ satisfy these conditions. By Green's theorem,

$$V(E) = \pi \iint_D \frac{\partial F_1}{\partial y} dA = -\pi \oint_{\partial D} F_1 dx = -\pi \oint_{\partial D} y^2 dx$$

as required. □

43.5. Exercises.

1–2. Evaluate the given line integral by two methods: (a) directly and (b) using Green's theorem.

1. $\oint_C xy^2 dx - y^2 x dy$, where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$; C is oriented counterclockwise;

2. $\oint_C 2yx dx + x^2 dy$, where C consists of the line segments from $(0, 1)$ to $(0, 0)$ and from $(0, 0)$ to $(1, 0)$ and the parabola $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$.

3–11. Evaluate the given line integral using Green's theorem.

3. $\oint_C x\sin(x^2)dx + (xy^2 - x^8)dy$, where C is the positively oriented boundary of the region between two circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$;

4. $\oint_C (y^3 dx - x^3 dy)$, where C is the positively oriented circle $x^2 + y^2 = a^2$;

5. $\oint_C (\sqrt{x} + y^3)dx + (x^2 + \sqrt{y})dy$, where C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$;

6. $\oint_C (y^4 - \ln(x^2 + y^2))dx + 2\tan^{-1}(y/x)dy$, where C is the positively oriented circle of radius $a > 0$ with the center (x_0, y_0) such that $x_0 > a$ and $y_0 > a$;

7. $\oint_C (x+y)^2 dx - (x^2+y^2)dy$, where C is a positively oriented triangle with the vertices $(1, 1)$, $(3, 2)$, and $(2, 5)$;

8. $\oint_C xy^2dx - x^2ydy$, where C is the negatively oriented circle $x^2+y^2 = a^2$;

9. $\oint_C (x+y)dx - (x-y)dy$, where C is the positively oriented ellipse $(x/a)^2 + (y/b)^2 = 1$;

10. $\oint_C e^x[(1-\cos y)dx - (y - \sin y)dy]$, where C is the positively oriented boundary of the region $0 \leq y \leq \sin x$, $0 \leq x \leq \pi$;

11. $\oint_C e^{-x^2+y^2}[\cos(2xy)dx - \sin(2xy)dy]$, where C is the positively oriented circle $x^2 + y^2 = a^2$.

12–14. Use the contour transformation law (**43.5**) to solve each of the following problems.

12. Let $\mathbf{F} = \langle (x+y)^2,\ -(x-y)^2 \rangle$. Find the difference between the line integrals of $\mathbf{F}$ over two curves C_1 and C_2 originating from the point $(1,1)$ and terminating at $(2,6)$ if C_1 in the straight line segment and C_2 is the parabola through $(1,1)$ and $(2,6)$ that also passes through $(0,0)$;

13. Find $\int_C (e^x \sin y - qx)dx + (e^x \cos y - q)dy$, where q is a constant and C is the upper part of the circle $x^2 + y^2 = ax$, $y \geq 0$, oriented from $(a,0)$ to $(0,0)$, $a > 0$;

14. Find $\int_C [g(y)e^x - qy]dx + [g'(y)e^x - q]dy$, where $g(y)$ and $g'(y)$ are continuous functions and C is a smooth curve from the point $P_1 = (x_1, y_1)$ to the point $P_2 = (x_2, y_2)$ such that it and the straight line segment P_1P_2 form the boundary of a region D of the area $A(D)$.

15. Use Green's theorem to find the work done by the force $\mathbf{F} = \langle 3xy^2 + y^3,\ y^4 \rangle$ in moving a particle along the circle $x^2 + y^2 = a^2$ from $(0,-a)$ to $(0,a)$ counterclockwise, $a > 0$.

16–23. Use a representation of the area of a planar region by the line integral to find the area of the specified region D.

16. D is bounded by an ellipse $x = a\cos t$, $y = b\sin t$, $0 \leq t \leq 2\pi$;

17. D is under one arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$;

18. D is the *astroid* enclosed by the curve $x = a\cos^3 t$, $y = a\sin^3 t$;

19. D is bounded by the curve $x(t) = a\cos^2 t$, $y(t) = b\sin(2t)$, where $0 \leq t \leq \pi$;

20. D is bounded by the parabola $(x+y)^2 = ax$ and by the x axis, $a > 0$;

21. D is bounded by one loop of the curve $x^3 + y^3 = 3axy$, $a > 0$. Hint: put $y = tx$;

22. D is bounded by the curve $(x^2+y^2)^2 = a^2(x^2 - y^2)$.
Hint: put $y = x\tan t$;

23. D is bounded by $(x/a)^n + (y/b)^n = 1$, $n > 0$.
Hint: $x = a\cos^{n/2} t$, $y = b\sin^{n/2} t$.

24. Let a curve C have fixed endpoints. Under what condition on the function $g(x,y)$ is the line integral $\int_C g(x,y)(ydx + xdy)$ independent of C?

25. Let D be a planar region bounded by a simple closed curve. If A is the area of D, show that the coordinates (x_c, y_c) of the centroid of D are

$$x_c = \frac{1}{2A}\oint_{\partial D} x^2 dy, \quad y_c = -\frac{1}{2A}\oint_{\partial D} y^2 dx$$

Hint: Use an approach similar to the derivation of (**43.6**).

26. Let a lamina with a constant surface mass density σ occupy a planar region D enclosed by a simple piecewise smooth curve. Show that its moments of inertia about the x and y axes are

$$I_x = -\frac{\sigma}{3}\oint_{\partial D} y^3 dx, \quad I_y = \frac{\sigma}{3}\oint_{\partial D} x^3 dy$$

Hint: Use an approach similar to the derivation of (**43.6**).

44. Flux of a Vector Field

The idea of a flux of a vector field stems from an engineering problem of mass transfer across a surface. Suppose there is a flow of a fluid or gas with a constant velocity $\mathbf{v}$ and a constant mass density σ (mass per unit volume). Let ΔA be a planar area element placed into the flow. At what rate is the fluid or gas carried by the flow across the area ΔA? In other words, what is the mass of fluid transferred across ΔA per unit time? This quantity is called a *flux* of the mass flow across the area ΔA.

Suppose first that the mass flow is normal to the area element. The flow may occur in two opposite directions. To distinguish these two cases, a unit normal vector $\hat{\mathbf{n}}$ is set on the area element. Suppose first that the flow occurs in the direction of $\hat{\mathbf{n}}$; that is, the velocity vector is a positive multiple of the normal: $\mathbf{v} = v\hat{\mathbf{n}}$, where $v = \|\mathbf{v}\|$ is the flow speed. Consider the cylinder with an axis parallel to $\mathbf{v}$ with cross section area ΔA and height $h = v\,\Delta t$, where Δt is a time interval. The volume of the cylinder is $\Delta V = h\,\Delta A = v\,\Delta t\,\Delta A$. In time Δt, all the mass stored in this cylinder is transferred by the flow across ΔA. This mass is $\Delta m = \sigma\,\Delta V = \sigma v\,\Delta t\,\Delta A$, and the flux is

$$\Delta\Phi = \frac{\Delta m}{\Delta t} = \sigma v\,\Delta A.$$

The flux becomes negative $\Delta\Phi = -\sigma v\,\Delta A$ if the velocity of the mass flow is a negative multiple of the normal $\mathbf{v} = -v\hat{\mathbf{n}}$. More generally, the flux depends on the angle between the normal $\hat{\mathbf{n}}$ and the velocity of the mass flow. If the velocity is parallel to the area element (or perpendicular to the normal), then no mass is transferred across it and the flux vanishes. A vector $\mathbf{v}$ can be uniquely decomposed into the sum of two orthogonal vectors one of which is parallel to $\hat{\mathbf{n}}$. If θ is the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$, then the component of $\mathbf{v}$ parallel to $\hat{\mathbf{n}}$ is $\mathbf{v}_n = v_n\hat{\mathbf{n}}$, where $v_n = v\cos\theta = \mathbf{v}\cdot\hat{\mathbf{n}}$ is the scalar projection of $\mathbf{v}$ onto $\hat{\mathbf{n}}$. Only the normal component $\mathbf{v}_n$ of the flow contributes to the flux (see the left panel of Fig. **44.1**):

$$\textbf{(44.1)}\qquad \Delta\Phi = \sigma v_n\,\Delta A = \sigma\mathbf{v}\cdot\hat{\mathbf{n}}\,\Delta A = \mathbf{F}\cdot\hat{\mathbf{n}}\,\Delta A = F_n\,\Delta A,$$

where the vector $\mathbf{F} = \sigma\mathbf{v}$ characterizes the mass flow ("how much" (σ) and "how fast" ($\mathbf{v}$)) and F_n is the scalar projection of $\mathbf{F}$ onto the normal $\hat{\mathbf{n}}$. Observe that the value of the flux is unambiguously defined by setting a normal vector to the surface element as the sign of the flux depends on the direction of the normal.

Suppose S is a smooth surface such that it has a continuous unit normal vector ($\hat{\mathbf{n}}$ is a continuous vector field on S). If the mass flow is not constant; that is, $\mathbf{F}$ becomes a vector field, then its flux across a surface S can be defined by partitioning S into small surface elements S_i, $i = 1, 2, ..., N$, whose surface areas are ΔS_i as shown in the right panel of Fig. **44.1**. Let $\mathbf{r}_i^*$ be a sample point in S_i and let $\hat{\mathbf{n}}_i = \hat{\mathbf{n}}(\mathbf{r}_i^*)$ be the unit normal to S_i at $\mathbf{r}_i^*$. If the size of S_i (the radius of the smallest ball containing S_i) is small, then, by neglecting variations of $\mathbf{F}$ and the normal $\hat{\mathbf{n}}$ within S_i, the flux across S_i

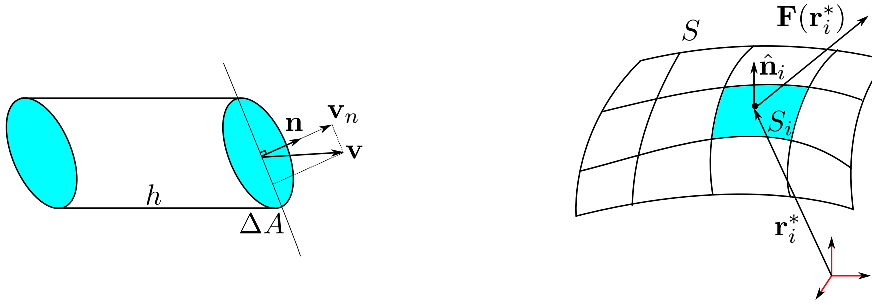


FIGURE **44.1**. **Left**: A mass transferred by a homogeneous mass flow with a constant velocity $\mathbf{v}$ across an area element ΔA in time Δt is $\Delta m = \sigma\Delta V$ where $\Delta V = h\Delta A$ is the volume of the cylinder with the cross section area ΔA and the hight $h = \Delta t v_n$; v_n is the scalar projection of $\mathbf{v}$ onto the normal $\mathbf{n}$. **Right**: A partition of a smooth surface S by elements S_i. If $\mathbf{r}_i^*$ is a sample point in S_i, $\hat{\mathbf{n}}_i$ is a unit normal to S at $\mathbf{r}_i^*$, and ΔS_i is the surface area of the partition element S_i, then the flux of a vector field $\mathbf{F}(\mathbf{r})$ across S_i is approximated by $\Delta\Phi_i = \mathbf{F}(\mathbf{r}_i^*)\cdot\hat{\mathbf{n}}_i\Delta S_i$.

can be approximated by (**44.1**), $\Delta\Phi_i \approx \mathbf{F}(\mathbf{r}_i^*)\cdot\hat{\mathbf{n}}_i\,\Delta S_i$. The approximation becomes better when $N\to\infty$ so that the sizes of S_i decrease to 0 uniformly and hence the total flux is

$$\Phi = \lim_{N\to\infty}\sum_{i=1}^{N}\Delta\Phi_i = \lim_{N\to\infty}\sum_{i=1}^{N}\mathbf{F}(\mathbf{r}_i^*)\cdot\hat{\mathbf{n}}_i\,\Delta S_i = \lim_{N\to\infty}\sum_{i=1}^{N} f_n(\mathbf{r}_i^*)\,\Delta S_i\,,$$

where $f_n = \mathbf{F}\cdot\mathbf{n}$ denotes the normal component of the vector field $\mathbf{F}$. A comparison with Eq. (**39.1**) shows that the sum in the above equation is nothing but the Riemann sum of the function $f_n(\mathbf{r})$ over a partition of the surface S. If the normal component $f_n(\mathbf{r})$ is integrable over S, then the above limit is the surface integral of $f_n(\mathbf{r})$ over S. For example, if $\mathbf{F}$ is continuous on S, then by continuity of $\hat{\mathbf{n}}$, f_n is continuous on S and, hence, integrable on S.

44.1. Orientable Surfaces. Consider a mass flow along a smooth surface. In this case the flow lines lie in the surface hence are perpendicular to the normal. In this case, the flux across any part of the surface is zero. Since the flux vanishes, no mass can get from side of the surface to the other, that is, *across* the surface. For example, a portion of a plane has two sides, say, the upper and lower (e.g., relative to a normal to the plane). A mass flow along the plane cannot transfer mass from one side to the other. A sphere also has two sides, the outer and inner sides. A mass flow along the sphere cannot transfer mass from one side to the other. Strangely enough,

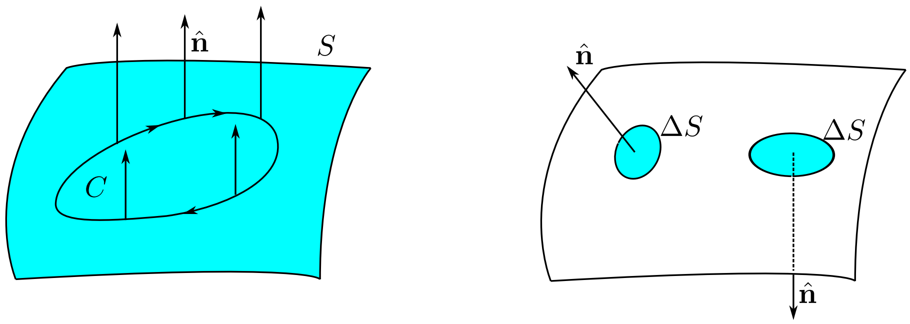


FIGURE **44.2**. **Left**: If there is a continuous unit normal vector $\hat{\mathbf{n}}$ on a surface S, then $\hat{\mathbf{n}}$ changes continuously along any closed curve C in S so that its initial direction should match the final direction. **Right**: A small patch ΔS of a surface S can be oriented in two different ways according to two possible choices of a unit normal vector, $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$. If there is a "one-sided" surface, a face-up patch can be transported along a closed curve in S to a face-down patch at the same position on S. If S has a boundary, then the closed curve is not allowed to cross the boundary.

there are smooth surfaces for which a particle (or a pont) can get from one side of the surface to the other just by sliding along the surface! In other words, there are smooth surfaces for which a tangential flow can transfer mass *across* the surface. So, the above definition of the flux is not suitable for such surfaces. The objective is therefore to describe the class of surfaces for which our definition of the flux is applicable.

Let S be a smooth surface. Since S is smooth, a continuous normal $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\mathbf{r})$ can always be defined in a neighborhood of any point on S as a continuous vector field. If two neighborhoods overlap, then the normal must be same at their common points. Suppose that, by doing this procedure of constructing $\hat{\mathbf{n}}$ in neighborhoods of some points of S and matching $\hat{\mathbf{n}}$ in their overlaps, a continuous normal $\hat{\mathbf{n}}$ can be obtained on the whole S. Then $\hat{\mathbf{n}}$ has to be continuous along every closed curve C in S. In other words, if one moves the obtained normal $\hat{\mathbf{n}}$ around a closed curve C in S, then the normal does not reverse its direction as illustrated in the left panel of Fig. **44.2**. If $\hat{\mathbf{n}}(\mathbf{r})$ is a unit normal at a point $\mathbf{r}$ of S, then $-\hat{\mathbf{n}}(\mathbf{r})$ is also a unit normal at $\mathbf{r}$. By continuity the direction of $\hat{\mathbf{n}}(\mathbf{r})$ defines one side of S, while the direction of $-\hat{\mathbf{n}}(\mathbf{r})$ defines the other side. For example, the outward normal of a sphere is continuous along any closed curve on the sphere (it remains outward along any closed curve) and hence defines the outer side of the sphere. If a normal on the sphere is chosen to be inward, then it is also continuous and defines the inner side of the sphere. Evidently, the flux is well defined for two-sided surfaces. (no tangential mass flow can transfer mass across a two-sided surface).

Are there one-sided surfaces? If such a surface exists, it should have quite remarkable properties. Take a point on it. In a sufficiently small neighborhood of this point, one can always think about two sides of a smooth surface. One side is defined by a normal $\hat{\mathbf{n}}$ (face-up patch), while the other has the same shape but its normal is $-\hat{\mathbf{n}}$ (face-down patch) as shown in the right panel of Fig. **44.2**. For a one-sided surface, face-up and face-down patches must be on the same side of the surface. This implies that there should exist a closed curve on the surface such that a face-up patch can be transported along it to a face-down patch at the same position on S and, if S has a boundary, this transport of a patch is not allowed to cross the boundary. This shows that a *continuous* normal cannot be defined on a one-sided surface.

Examples of One-Sided Surfaces. One-sided surfaces do exist. To construct an example, take a rectangular piece of paper. Put upward arrows on its vertical sides and glue these sides so that the arrows remain parallel. In doing so, a cylinder is obtained, which is a two-sided surface (there is no curve that traverses from one side to the other without crossing the boundary circles formed by the horizontal sides of the rectangle). The gluing can be done differently. Before gluing the vertical sides, twist the rectangle so that the arrows on its opposite sides have opposite directions and then glue the sides together. The procedure is shown in Fig. **44.3**. The resulting surface is the famous *Möbius strip* or *Möbius band* (named after the German mathematician August Möbius). It is one-sided. A face-up patch can be transported into a face-down patch at the same position along a closed curve in the band. The original rectangle is two-sided. Make a horizontal line that cuts the rectangle into two equal-area pieces. The line on one side cannot be continued to the line on the other side if the vertical edges of the rectangle are glued parallel to obtain a cylinder. Thanks to the twisting of the glued sides, the line on one side of the original rectangle becomes a continuation of the line on the other side of the rectangle, thus making a closed curve in the band along which a face-up patch can be transported into a face-down patch at the same position.

There are one-sided surfaces without boundaries (like a sphere). The most famous one is a *Klein bottle*. Take a bottle. Drill a hole in the side surface and in the bottom of the bottle. Suppose the neck of the bottle is flexible (a “rubber” bottle). Bend its neck and pull it through the hole on the bottle’s side surface (so that neck fits tightly into the hole). Finally, attach the edge of the bottle’s neck to the edge of the hole in the bottom of the bottle. The result is a surface without boundaries and it is one-sided (see the left panel of Fig. **44.4**). A small patch of this surface has two sides.

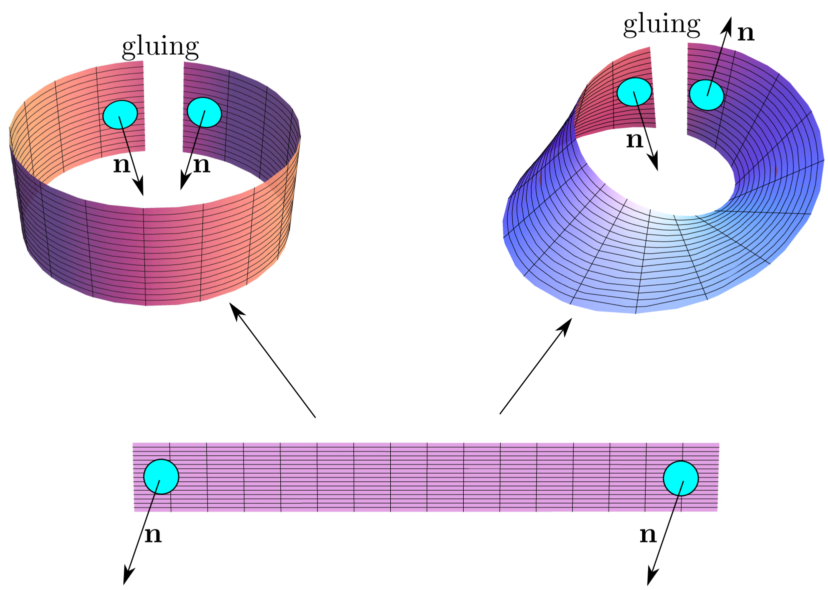


FIGURE **44.3**. A construction of two-sided surface (a portion of a cylinder) from a band by gluing its edges (left). A construction of one-sided surface (a Möbius band) from a band by gluing its edges after twisting the band (right).

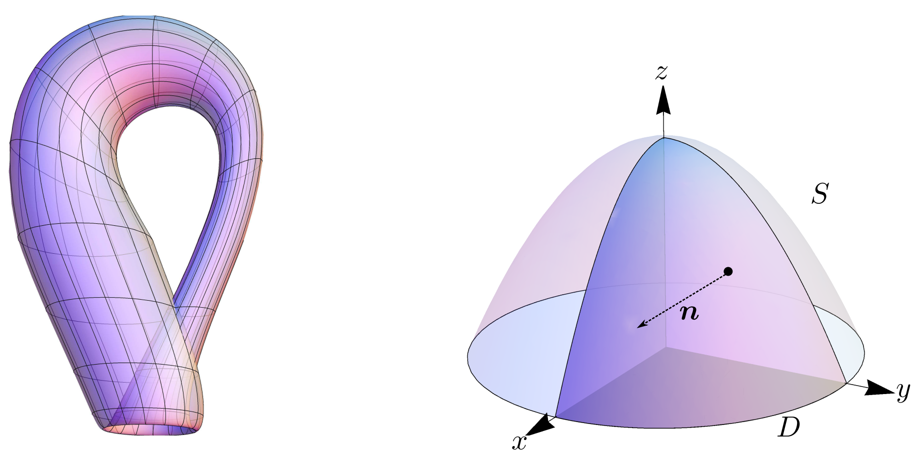


FIGURE **44.4**. **Left**: A Klein bottle is an example of one-sided closed surface (it has no boundaries). **Right**: An illustration to Example **44.1**

A bug sitting on one side of the patch can crawl *along* the surface and get to the other side of the patch.

Definition **44.1**. (Orientable Surface).
A smooth surface is called orientable *if there exists a continuous unit normal vector field on the surface.*

Definition **44.2**. (Orientation of an Orientable Smooth Surface)
An orientation *of an orientable smooth surface is a choice of continuous unit normal vector field.*

Definition **44.3**. (Oriented Smooth Surface)
An oriented *smooth surface is an orientable smooth surface together with an orientation.*

As already argued, every *connected* orientable surface has exactly two orientations. A flux of a vector field can only be defined across an orientable surface. Furthermore, there is a simple criterion whether a given smooth surface can be oriented.

Corollary **44.1**. (Test for a smooth surface to be orientable)
A smooth surface is orientable if and only if there is no closed curve in it such that a unit normal is reversed when moved around this curve.

For example, if a smooth surface is defined either as a graph of a function with continuous partial derivatives, or as a level set of a function whose gradient is continuous and non-vanishing, or by parametric equations, then a continuous unit normal vector field can be defined in a neighborhood of each point of a closed curve in the surface. It remains then to check if the directions of the normal match on overlaps of neighborhoods; that is, the defined normal is continuous on any such curve, e.g., by studying the values of the unit normal vector field $\hat{\mathbf{n}}(\mathbf{r})$ on a closed parametric curve $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, $\mathbf{r}(a) = \mathbf{r}(b)$.

44.2. Flux as a Surface Integral.

Definition **44.4**. (Flux of a Vector Field).
Let S be an oriented smooth surface, and let $\hat{\mathbf{n}}$ be the chosen unit normal vector field on S. The flux of a vector field $\mathbf{F}$ across S is the surface integral

$$\Phi = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS,$$

provided the normal component $\mathbf{F} \cdot \hat{\mathbf{n}}$ of the vector field is integrable on S.

The integrability of the normal component $F_n(\mathbf{r}) = \mathbf{F} \cdot \hat{\mathbf{n}}$ is defined in the sense of surface integrals of ordinary functions (see Definition **39.2**). In particular, the flux of a continuous vector field across a smooth oriented surface exists.

44.3. Evaluation of the Flux of a Vector Field. Suppose that a surface S is a graph $z = g(x, y)$ and g has continuous partial derivatives in a region D

bounded by piecewise smooth curves. There are two possible orientations of S. One of these is the continuous unit normal vector field

$$\hat{\mathbf{n}}(x,y) = \frac{1}{\|\mathbf{n}(x,y)\|}\,\mathbf{n}(x,y)\,, \quad \mathbf{n}(x,y) = \langle -g'_x(x,y),\ -g'_y(x,y),\ 1\rangle\,;$$

$\mathbf{n}$ is the normal vector used in Section **21.5** to find equations of tangent planes. Its z component is positive. For this reason, the graph is said to be *oriented upward*. Alternatively, one can take the normal vector in the opposite direction,

$$\mathbf{n}(x,y) = \langle g'_x(x,y),\ g'_y(x,y),\ -1\rangle\,.$$

the graph with this orientation is said to be *oriented downward*. Accordingly, the *upward* (*downward*) flux, denoted $\Phi_\uparrow$ ($\Phi_\downarrow$), of a vector field is associated with the upward (downward) orientation of the graph. When the orientation of a surface is reversed, the flux changes its sign:

$$\Phi_\uparrow = -\Phi_\downarrow.$$

Consider the upward-oriented graph $z = g(x,y)$. The unit normal vector reads

$$\hat{\mathbf{n}} = \frac{1}{\|\mathbf{n}\|}\,\mathbf{n} = \frac{1}{J}\,\langle -g'_x,\ -g'_y,\ 1\rangle\,, \quad J = \sqrt{1 + (g'_x)^2 + (g'_y)^2}.$$

In Section **39** it was established that the area of the portion of the graph above a planar region of area dA is $dS = J\,dA$. Therefore, in the flux across the surface area dS can be written in the form

$$\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \mathbf{F}\cdot\mathbf{n}\,\frac{1}{J}\,J\,dA = \mathbf{F}\cdot\mathbf{n}\,dA,$$

where the vector field must be evaluated on S, that is, $\mathbf{F} = \mathbf{F}(x,y,g(x,y))$ (in accord with Theorem **39.2**, the variable z is replaced by $g(x,y)$ because $z = g(x,y)$ for any point (x,y,z) in S). If the vector field F is continuous, then the dot product $\mathbf{F}\cdot\mathbf{n}$ is a continuous function on D so that the flux exists and is given by the double integral over D. The following theorem has been proved.

THEOREM **44.1**. (Evaluation of the Flux Across a Graph).
Suppose that S is a graph $z = g(x,y)$ of a function g on a region D bounded by piecewise smooth curves and partial derivatives of g are continuous and bounded on the interior of D. Let S be oriented upward by the normal vector $\mathbf{n} = \langle -g'_x, -g'_y, 1\rangle$, and let $\mathbf{F}$ be a continuous vector field on S. Then

$$\Phi_\uparrow = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_D F_n(x,y)\,dA,$$
$$F_n(x,y) = \mathbf{F}\cdot\mathbf{n}\Big|_{z=g(x,y)} = -g'_x F_1(x,y,g) - g'_y F_2(x,y,g) + F_3(x,y,g).$$

By this theorem, an evaluation of the flux involves the following basic steps:

Step 1. Represent S as a graph of a function g of two variables, e.g., $z = g(x, y)$ (i.e., find the function g using a geometrical description of S). If S cannot be represented as a graph of a single function, then it has to be split into pieces so that each piece can be described as a graph. By the additivity property, the surface integral over S is the sum of integrals over each piece.

Step 2. Find the region D that defines the part of the graph that coincides with S. One can think of D as the vertical projection of S onto the xy plane.

Step 3. Determine an orientation of S (upward or downward) from the problem description, $\mathbf{n} = p\langle -g'_x, -g'_y, 1\rangle$ where $p = 1$ and $p = -1$ for the upward and downward orientations, respectively. The sign of the flux is determined by the orientation. Calculate $F_n = \mathbf{F} \cdot \mathbf{n}$ as a function on D.

Step 4. Evaluate the double integral of F_n over D.

EXAMPLE **44.1**. *Evaluate the downward flux of the vector field* $\mathbf{F} = \langle xz, yz, z\rangle$ *across the part of the paraboloid* $z = 1 - x^2 - y^2$ *in the first octant.*

SOLUTION:
Step 1. The surface is the part of the graph $z = g(x, y) = 1 - x^2 - y^2$ in the first octant.
Step 2. The paraboloid intersects the xy plane ($z = 0$) along the circle $x^2 + y^2 = 1$. Therefore, the region D is the part of the disk bounded by this circle in the first quadrant.
Step 3. Since S is oriented downward,

$$\mathbf{n} = \langle g'_x,\ g'_y, -1\rangle = \langle -2x, -2y, -1\rangle$$

and the dot product of $\mathbf{F}$ and $\mathbf{n}$ on S is

$$F_n(x, y) = \langle xg,\ yg,\ g\rangle \cdot \langle -2x,\ -2y, -1\rangle = -(1 - x^2 - y^2)(1 + 2x^2 + 2y^2).$$

Step 4. The region D is the image of the rectangle $D' = [0, 1] \times [0, \pi/2]$ in the polar plane. Converting the double integral of F_n to polar coordinates and using Fubini's theorem,

$$\Phi_\downarrow = \iint_D F_n(x, y)\, dA = -\int_0^{\pi/2}\int_0^1 (1 - r^2)(1 + 2r^2)\, r\, dr\, d\theta = -\frac{5\pi}{24}.$$

The negative value of the downward flux means that the actual transfer of a quantity (like mass), whose flow is described by the vector field $\mathbf{F}$, occurs in the upward direction across S. □

EXAMPLE **44.2**. *Evaluate the flux of the vector field* $\mathbf{F} = \langle -x, y^2, z\rangle$ *across the part of the surface* $y = zx$ *that lies between the cylinders* $x^2+z^2 = 1$ *and* $x^2+z^2 = 4$*; the surface is oriented so that a normal to it has a positive* y *component.*

SOLUTION: It is convenient to view the surface as the graph $y = g(x, z) = xz$ over the region D in the xz plane:

$$D = \{(x, z) \,|\, 1 \le x^2 + z^2 \le 4\}$$

which is an annulus with the inner radius 1 and the outer radius 2. Therefore a normal to the surface can be chosen in the form

$$\mathbf{n} = \langle -g'_x,\ 1,\ -g'_z\rangle = \langle -z,\ 1,\ -x\rangle\,;$$

an upward normal relative to the direction of the y axis. Then

$$F_n(x, z) = \langle -x,\ (g(x, z))^2,\ z\rangle \cdot \langle -z,\ 1,\ -x\rangle = x^2 z^2\,.$$

The region D is the image of the rectangle $D' = [1, 2] \times [0, 2\pi]$ in the polar plane. Converting the double integral of F_n to polar coordinates in the xz plane,

$$\begin{aligned}
\Phi &= \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_D x^2 z^2 dA = \iint_{D'} (r\cos\theta)^2 (r\sin\theta)^2 r\, dA' \\
&= \int_0^{2\pi} \cos^2\theta \sin^2\theta\, d\theta \int_1^2 r^5 dr = \frac{1}{4}\int_0^{2\pi} \sin^2(2\theta) d\theta \cdot \frac{1}{6}\, r^6\Big|_1^2 \\
&= \frac{21}{8}\int_0^{2\pi} \frac{1}{2}\Big(1 - \cos(4\theta)\Big) d\theta = \frac{21\pi}{8}\,,
\end{aligned}$$

where the trigonometric double-angle formulas have been used to evaluate the integral. The integral of $\cos(4\theta)$ vanishes by periodicity. □

44.4. Parametric Surfaces. Suppose that an orientable smooth surface S is defined by parametric equations $\mathbf{r} = \mathbf{r}(u, v)$, where (u, v) span a region D. Then, by Theorem **39.3**, a normal vector to S can be chosen as

$$\mathbf{n} = \pm(\mathbf{r}'_u \times \mathbf{r}'_v) \ne \mathbf{0}$$

in the interior of D. The sign defines one of the two possible orientations of S. Since $\|\mathbf{n}\| = J$, where J determines the area transformation law

$$dS = J\, dA\,, \quad dA = du\, dv\,, \quad J = \|\mathbf{n}\|\,,$$

the flux of a vector field $\mathbf{F}$ across the surface area dS reads

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(u, v)) \cdot \hat{\mathbf{n}}\, dS &= \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{n}\, dA = \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}'_u \times \mathbf{r}'_v)\, dA \\
&= F_n(u, v)\, dA
\end{aligned}$$

and the flux of $\mathbf{F}$ across S is given by the double integral

$$\Phi = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}'_u \times \mathbf{r}'_v)\, dA = \iint_D F_n(u, v)\, dA.$$

Naturally, a graph $z = g(x, y)$ is described by the parametric equations $\mathbf{r}(u, v) = \langle u, v, g(u, v)\rangle$, which is a particular case of the above expression; it coincides with that given in Theorem **44.1** ($x = u$ and $y = v$). A description of surfaces by parametric equations is especially convenient for closed surfaces (i.e., when the surface cannot be represented as a graph of a single function).

EXAMPLE **44.3**. *Evaluate the outward flux of the vector field* $\mathbf{F} = \langle z^2x, z^2y, z^3\rangle$ *across the sphere of unit radius centered at the origin.*

SOLUTION: The parametric equations of the sphere of radius $R = 1$ are given in (**39.4**), and a normal vector is computed in Example **39.7**:

$$\mathbf{n} = \sin(\phi)\mathbf{r}(\phi,\theta)\,, \quad \mathbf{r}(\phi,\theta) = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi\rangle\,,$$

where the spherical angles (ϕ,θ) span their whole range $D = [0,\pi]\times[0,2\pi]$. This is an outward normal because $\sin\phi \geq 0$. It is convenient to represent $\mathbf{F} = z^2\mathbf{r}$ so that

$$\begin{aligned} F_n(\phi,\theta) &= \mathbf{F}(\mathbf{r}(\phi,\theta))\cdot\mathbf{n} = \cos^2\phi\sin\phi\ \mathbf{r}(\phi,\theta)\cdot\mathbf{r}(\phi,\theta) \\ &= \cos^2\phi\sin\phi\ \|\mathbf{r}(\phi,\theta)\|^2 = \cos^2\phi\sin\phi \end{aligned}$$

because $\|\mathbf{r}(\phi,\theta)\|^2 = R^2 = 1$. The outward flux reads

$$\begin{aligned} \Phi &= \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_D \cos^2\phi\sin\phi\,dA \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \cos^2\phi\sin\phi\,d\phi = \frac{4\pi}{3}. \end{aligned}$$

□

Non-orientable Parametric Surfaces. Nonorientable smooth surfaces can be described by the parametric equations $\mathbf{r} = \mathbf{r}(u,v)$ or by an algebraic equation $F(x,y,z) = 0$ (as a level surface of a function). For example, a Möbius band of width $2h$ with midcircle of radius R that lies in the xy plane is defined by parametric equations:

$$(\mathbf{44.2})\quad \mathbf{r}(u,v) = \Big\langle [R+u\cos(v/2)]\cos v,\ [R+u\cos(v/2)]\sin v,\ u\sin(v/2)\Big\rangle,$$

where the parameters (u,v) span the rectangle $D = [-h,h]\times[0,2\pi]$. This Möbius band can also be defined as a level set of a cubic polynomial (a *cubic* surface):

$$-R^2y + x^2y + y^3 - 2Rxz - 2x^2z - 2y^2z + yz^3 = 0.$$

This is verified by substituting the parametric equations into this algebraic equation and showing that the left side vanishes for all (u,v) in D.

Let us prove that the surface defined by the parametric equations (**44.2**) is not orientable. By Corollary **44.1**, one should analyze the behavior of a normal vector when the latter is moved around a closed curve in the surface. Consider the circle in the xy plane defined by the condition $u = 0$: $\mathbf{r}(0,v) = \langle R\cos v, R\sin v, 0\rangle$, $0 \leq v \leq 2\pi$. It is easy to show that

$$\begin{aligned} \mathbf{r}'_u(0,v) &= \langle \cos(v/2)\cos v,\ \cos(v/2)\sin v,\ \sin(v/2)\rangle\,, \\ \mathbf{r}'_v(0,v) &= \langle -R\sin v,\ R\cos v,\ 0\rangle\,. \end{aligned}$$

When $\mathbf{r}(0,v)$ returns to the initial point, that is, $\mathbf{r}(0,v+2\pi)=\mathbf{r}(0,v)$, the normal vector is reversed. Indeed,

$$\begin{aligned} \mathbf{r}'_u(0,v+2\pi) &= -\mathbf{r}'_u(0,v)\,, \quad \mathbf{r}'_v(0,v+2\pi)=\mathbf{r}'_v(0,v) \\ \Rightarrow \quad \mathbf{n}(0,v+2\pi) &= \mathbf{r}'_u(0,v+2\pi)\times\mathbf{r}'_v(0,v+2\pi) = -\mathbf{r}'_u(0,v)\times\mathbf{r}'_v(0,v) \\ &= -\mathbf{n}(0,v); \end{aligned}$$

that is, the surface defined by these parametric equations is *not* orientable because a normal vector is reversed when moved around a closed curve.

So, if a surface S is defined by parametric or algebraic equations, one still has to verify that it is orientable (i.e., it is two-sided!), when evaluating the flux across it; otherwise, the flux makes no sense.

44.5. Exercises.

1-7. Find the flux of a constant vector field $\mathbf{F}=\langle a,b,c\rangle$ across the specified surface S without evaluation of the flux integral using only geometrical means.

1. S is a rectangle of area A in each of the coordinate planes oriented along the coordinate axis orthogonal to the rectangle ($\hat{\mathbf{n}}=\hat{\mathbf{e}}_i$, $i=1,2,3$);

2. S is the part of the plane $(x/a)+(y/b)+(z/c)=1$ in the positive octant oriented outward from the origin and a, b, c are positive;

3. S is the boundary of the pyramid whose base is the square $[-q,q]\times[-q,q]$ in the xy plane and the vertex is $(0,0,h)$. S is oriented outward;

4. S is the cylinder $x^2+y^2=R^2$, $0\le z\le h$, oriented inward ($\hat{\mathbf{n}}$ is directed toward the axis of the cylinder);

5. S is the surface of a rectangular box oriented outward;

6. S is the sphere $x^2+y^2+z^2=R^2$ oriented outward;

7. S is a torus oriented inward.

8–20. Find the flux of the given vector field $\mathbf{F}$ across the specified oriented surface S.

8. $\mathbf{F}=\langle xy,zx,xy\rangle$ and S is the part of the paraboloid $z=1-x^2-y^2$ that lies above the square $[0,1]\times[0,1]$ and is oriented upward;

9. $\mathbf{F}=\langle y,\ -x,\ z^2\rangle$ and S is the part of the paraboloid $z=1-x^2-y^2$ that lies above the $xy-$plane and is oriented downward;

10. $\mathbf{F}=\langle xz\,,\ zy\,,\ z^2\rangle$ and where S is the part of the cone $z=\sqrt{x^2+y^2}$ beneath the plane $z=2$ in the first octant and is oriented upward;

11. $\mathbf{F}=\langle x,\ -z,\ y\rangle$ and S the part of the sphere $x^2+y^2+z^2=R^2$ in the first octant oriented toward the origin;

12. $\mathbf{F}=\mathbf{a}\times\mathbf{r}$ where $\mathbf{a}$ is a constant vector and S is the sphere of radius R oriented outward and centered at the origin;

13. $\mathbf{F}=\langle 2y+x,\ y+2z-x,\ z-y\rangle$ and S is the boundary of the cube with vertices $(\pm1,\pm1,\pm1)$ oriented outward.

14. $\mathbf{F} = c\mathbf{r}/\|\mathbf{r}\|^3$, where c is a constant, and S is the sphere of radius a that is centered at the origin and oriented inward;

15. $\mathbf{F} = \langle 2y, x, -z\rangle$ and S is the part of the paraboloid $y = 1 - x^2 - z^2$ in the first octant oriented so that $\hat{\mathbf{n}}$ has a positive y component;

16. $\mathbf{F} = \langle xy, zy, z\rangle$ and S is the part of the plane $2x - 2y - z = 3$ that lies inside the cylinder $x^2 + y^2 = 1$ and is oriented upward;

17. $\mathbf{F} = \langle x, y, z\rangle$ and S is the part of the paraboloid $x = z^2 + y^2$ that lies between the planes $x = 0$ and $x = 1$ and is oriented so that $\hat{\mathbf{n}}$ has a positive x component;

18. $\mathbf{F} = \langle x, y, z\rangle$ and S is the boundary of the solid region $0 < a^2 \le x^2 + y^2 + z^2 \le R^2$ oriented outward;

19. $\mathbf{F} = \langle f(x), g(y), h(z)\rangle$ where f, g, h are continuous functions and S is the boundary of the rectangular box $[0, a] \times [0, b] \times [0, c]$ oriented outward;

20. $\mathbf{F} = \langle y - z,\ z - x,\ x - y\rangle$ and S is the part of the cone $x^2 + y^2 = z^2$, $0 \le z \le h$, oriented away from the z axis.

21–26. Use parametric equations of the specified oriented surface S to evaluate the flux of the given vector field $\mathbf{F}$ across S.

21. $\mathbf{F} = \langle x,\ -y,\ z^2\rangle$ and S is the part of the double cone $z^2 = x^2 + y^2$ between the planes $z = -1$ and $z = 1$ oriented so that $\hat{\mathbf{n}}$ is directed away from the axis of the cone;

22. $\mathbf{F} = \langle z^2 + y^2,\ x^2 + z^2,\ x^2 + y^2\rangle$ and S is the boundary of the solid enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $y = 1$. S is oriented outward;

23. $\mathbf{F} = \langle y, x, z\rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies outside the double cone $z^2 = 3(x^2 + y^2)$ and is oriented toward the origin;

24. $\mathbf{F} = \langle -y,\ x,\ z\rangle$ and S is the torus with radii R and a oriented outward;

25. $\mathbf{F} = \langle x^{-1},\ y^{-1},\ z^{-1}\rangle$ and S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ oriented outward;

26. $\mathbf{F} = \langle x^2,\ y^2,\ z^2\rangle$ and S is the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$ oriented outward.

45. Stokes' Theorem

45.1. Vector Form of Green's Theorem. It was shown in Section **43.3** that the curl of a planar vector field $\mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y), 0\rangle$ is parallel to the z axis, $\boldsymbol{\nabla} \times F = (\partial F_2/\partial y - \partial F_1/\partial x)\hat{\mathbf{e}}_3$. This observation allows us to reformulate Green's theorem in the following vector form:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{e}}_3 \; dA.$$

Thus, *the line integral of a vector field along a closed simple curve is determined by the flux of the curl of the vector field across the surface bounded by this curve*. It turns out that this statement holds not only in a plane, but also in space. It is known as *Stokes' theorem*.

45.2. Positive (Induced) Orientation of a Closed Curve. Suppose S is an oriented smooth surface, with a chosen normal vector field $\hat{\mathbf{n}}$, and suppose that S is bounded by a simple closed curve C. Consider a tangent plane at a point $\mathbf{r}_0$ of S. Any circle in the tangent plane centered at $\mathbf{r}_0$ can always be oriented counterclockwise as viewed from the top of the normal vector $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0$ at $\mathbf{r}_0$. This circle is said to be *positively oriented relative to the orientation of S*. Since the surface is smooth, a circle of a sufficiently small radius can always be projected onto a closed simple curve C in S by moving each point of the circle parallel to $\hat{\mathbf{n}}_0$. This curve is also *positively oriented* relative to $\hat{\mathbf{n}}_0$. It can then be continuously (i.e., without breaking) deformed along S so that a part of it lies on the boundary of S after the deformation and the orientations of the boundary of S and C can be compared. The boundary of S is said to be *positively oriented* if it has the same orientation as C. The positively oriented boundary of S is denoted by ∂S. The procedure to define a positive orientation of the boundary of an oriented surface S is illustrated in Fig. **45.1** (left panel).

In other words, the positively oriented boundary ∂S of a surface S (or the *induced* orientation of the boundary of S) means that if one walks in the positive direction along the boundary with one's head pointing in the direction of $\hat{\mathbf{n}}$, then the surface S will always be on one's left. Let S be a graph $z = g(x,y)$ over D oriented upward. Then ∂S is obtained from ∂D (a positively oriented boundary of D) by lifting points of ∂D to S parallel to the z axis (see the right panel of Fig. **45.1**).

THEOREM **45.1**. (Stokes' Theorem).
Let S be an oriented, piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve that is positively oriented relative to the orientation of S. Let the components of a vector field $\mathbf{F}$ have continuous partial derivatives on an open spatial region that contains S. Then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS,$$

where $\hat{\mathbf{n}}$ is the unit normal vector on S.

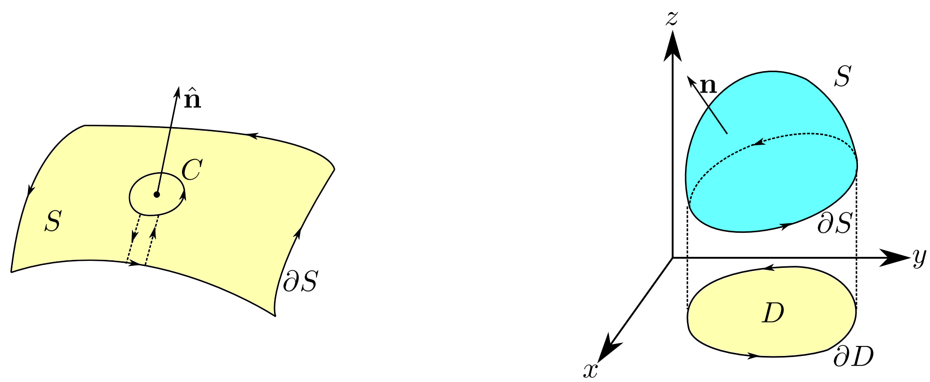


FIGURE **45.1**. **Left**: The positive (or induced) orientation of the boundary of an oriented surface S. The surface S is oriented by a normal vector $\hat{\mathbf{n}}$. Take a closed curve C in S that is oriented counterclockwise as viewed from the tip of the vector $\hat{\mathbf{n}}$. Deform this curve toward the boundary of S. The boundary of S has positive orientation if it coincides with the orientation of C. **Right**: The surface S is the graph of a function on D. If S is oriented upward, then the positively oriented boundary ∂S is obtained from the positively (counterclockwise) oriented boundary ∂D.

Stokes' theorem is difficult to prove in general. Here it is proved for a particular case when S is a graph of a function.

PROOF (FOR S BEING A GRAPH). Let S be an upward-oriented graph:

$$S: \quad z = g(x, y)\,, \quad (x, y) \in D\,,$$

where g has continuous second-order partial derivatives on D and D is a simple planar region whose boundary ∂D corresponds to the boundary ∂S. The upward orientation of the graph is defined by the normal vector

$$\mathbf{n} = \langle -g'_x, -g'_y, 1\rangle$$

and the upward flux of $\operatorname{curl}\mathbf{F}$ across S can be evaluated according to Theorem **44.1** in which $\mathbf{F}$ is replaced by $\boldsymbol{\nabla}\times\mathbf{F}$:

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_D (\operatorname{curl}\mathbf{F})_n\,dA,$$

$$(\operatorname{curl}\mathbf{F})_n = \operatorname{curl}\mathbf{F}\cdot\mathbf{n} = -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\frac{\partial z}{\partial x} - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial z}\Big)\frac{\partial z}{\partial y} + \\ + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big),$$

where the notations $\partial z/\partial x = g'_x$ and $\partial z/\partial y = g'_y$ are used to emphasize that z is not an independent variable on S but a function of x and y. Let $x = x(t)$ and $y = y(t)$, $a \le t \le b$, be parametric equations of ∂D so that $x(a) = x(b)$

and $y(a) = y(b)$ (∂D is a closed curve). Then the vector function

$$\mathbf{r}(t) = \langle x(t),\ y(t),\ g(x(t), y(t))\rangle\,, \quad a \le t \le b\,,$$

traces out the boundary ∂S, $\mathbf{r}(a) = \mathbf{r}(b)$. Making use of Theorem **41.1**, the line integral of $\mathbf{F}$ along ∂S can be evaluated. By the chain rule,

$$\mathbf{r}' = \langle x'\,, y'\,, g'_x x' + g'_y y'\rangle\,.$$

Therefore,

$$\mathbf{F}\cdot\mathbf{r}' = (F_1 + F_3 g'_x)x' + (F_2 + F_3 g'_y)y'\,, \quad F_i = F_i\Big(x(t), y(t), g(x(t), y(t))\Big)\,,$$

where $i = 1, 2, 3$, and hence

$$\begin{aligned}\oint_{\partial S}\mathbf{F}\cdot d\mathbf{r} &= \int_a^b [(F_1 + F_3 g'_x)x' + (F_2 + F_3 g'_y)y']\,dt \\ &= \oint_{\partial D}\Big(F_1 + F_3\frac{\partial z}{\partial x}\Big)dx + \Big(F_2 + F_3\frac{\partial z}{\partial y}\Big)dy\end{aligned}$$

because $x'\,dt = dx$ and $y'\,dt = dy$ along ∂D, where $z = g(x, y)$ in all components of $\mathbf{F}$. The latter line integral can be transformed into the double integral over D by Green's theorem (Theorem **43.1** where F_1 and F_2 are replaced by $F_1 + F_3 g'_x$ and $F_2 + F_3 g'_y$, respectively):

$$\begin{aligned}\oint_{\partial S}\mathbf{F}\cdot d\mathbf{r} &= \iint_D \Big[\frac{\partial}{\partial x}\Big(F_2 + F_3\frac{\partial z}{\partial y}\Big) - \frac{\partial}{\partial y}\Big(F_1 + F_3\frac{\partial z}{\partial x}\Big)\Big]dA \\ &= \iint_D (\operatorname{curl}\mathbf{F})_n\,dA = \iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS,\end{aligned} \tag{45.1}$$

where the middle equality is verified by the direct evaluation of the partial derivatives using the chain rule (which holds by the hypothesis that the vector field $\mathbf{F}$ has continuous partial derivatives and g has continuous second-order partial derivatives). For example,

$$\begin{aligned}\frac{\partial}{\partial x}F_2(x, y, g(x, y)) &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial g}{\partial x} \\ \frac{\partial}{\partial x}\Big(F_3\frac{\partial g}{\partial y}\Big) &= \Big(\frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial x}\Big)\frac{\partial g}{\partial y} + F_3\frac{\partial^2 g}{\partial x\partial y}\end{aligned}$$

The terms containing the mixed derivatives $g''_{xy} = g''_{yx}$ are cancelled out owing to Clairaut's theorem, while the other terms can be arranged to coincide with the expression for $(\operatorname{curl}\mathbf{F})_n$ found above. The last equality in (**45.1**) holds by Theorem **39.2** ($dS = J\,dA$ and $\mathbf{n} = J\hat{\mathbf{n}}$). □

45.3. Use of Stokes' Theorem. Stokes' theorem is very helpful for evaluating line integrals of vector fields along closed oriented curves of complicated shapes when a direct use of Theorem **41.1** is technically too involved. The procedure includes a few basic steps.

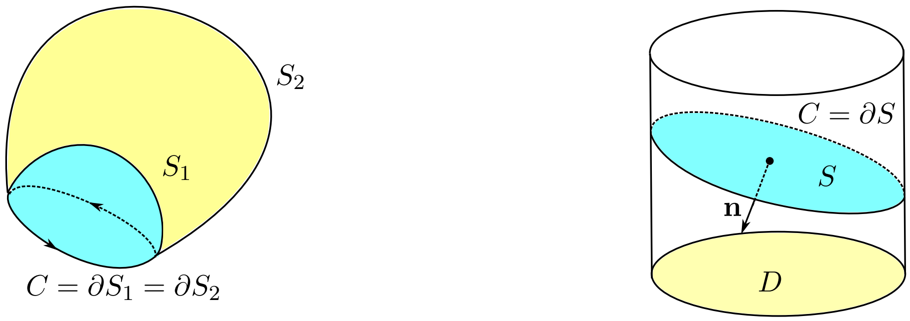


FIGURE **45.2**. **Left**: Given the curve $C = \partial S$, the surface S may have any desired shape in Stokes' theorem. The surfaces S_1 and S_2 have the same boundaries $\partial S_1 = \partial S_2 = C$. The line integral over C can be transformed to the flux integral either across S_1 or S_2. **Right**: An illustration to Example **45.1**. The integration contour C is the intersection of the cylinder and a plane. When applying Stokes' theorem, the simplest choice of a surface, whose boundary is C, is the part of the plane that lies inside the cylinder.

Step 1. Given an oriented simple closed curve C, choose *any* smooth orientable surface S whose boundary is C. Note that, according to Stokes' theorem, the value of the line integral is independent of the choice of S. This freedom should be used to make S as simple as possible (see the left panel of Fig. **45.2**).

Step 2. Determine the orientation of S such that the orientation of C is positive relative to the unit normal of S, that is, $C = \partial S$.

Step 3. Evaluate $\mathbf{B} = \operatorname{curl} \mathbf{F}$ and calculate the flux of $\mathbf{B}$ across S.

EXAMPLE **45.1**. *Evaluate the line integral of* $\mathbf{F} = \langle xy, yz, xz \rangle$ *along the curve of intersection of the cylinder* $x^2 + y^2 = 1$ *and the plane* $x + y + z = 1$. *The curve is oriented clockwise as viewed from above.*

SOLUTION:
Step 1. The curve C lies in the plane $x + y + z = 1$. Therefore, the simplest choice of S is the portion of this plane that lies within the cylinder. The surface S can be represented as the graph $z = g(x, y) = 1 - x - y$ over the region D that is $x^2 + y^2 \le 1$ (as shown in the right panel of Fig. **45.2**).
Step 2. Since C is oriented clockwise as viewed from above, the orientation of S must be downward to make the orientation positive relative to the normal on S, that is,

$$\mathbf{n} = \langle g'_x,\ g'_y,\ -1 \rangle = \langle -1,\ -1,\ -1 \rangle .$$

Step 3. The curl of $\mathbf{F}$ is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{F} = \det \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{pmatrix} = \langle -y, -z, -x \rangle \,.$$

Therefore, substituting $z = g(x, y)$ into the components of $\mathbf{B}$,

$$\begin{aligned} B_n(x,y) &= \mathbf{B} \cdot \mathbf{n} = \langle -y, -g, -x \rangle \cdot \langle -1, -1, -1 \rangle = g(x,y) + y + x = 1\,, \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \iint_D B_n(x,y)\, dA \\ &= \iint_D dA = A(D) = \pi\,. \end{aligned}$$

□

Changing the Curve of Integration in a Line Integral. In Section **43.1** it was shown how Green's theorem can be used to change the curve of integration in a line integral of a planar vector field. Stokes's theorem can also be used to change the curve of integration in a line integral of a general vector field. Let C_1 and C_2 be two simple smooth non-intersecting curves originating from a point A and terminating at a point B. Consider the line integrals of a vector field $\mathbf{F}$ along C_1 and C_2. The union of C_1 and C_2 is a closed simple curve C. Let C be oriented so that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\,.$$

Suppose S is a smooth orientable surface whose boundary is the closed curve C. The orientation of S is chosen so that $\partial S = C$. Assuming that the hypotheses of Stokes' theorem are fulfilled for the vector field $\mathbf{F}$, the line integral of $\mathbf{F}$ along C can be converted to the flux of $\operatorname{curl} \mathbf{F}$ across S, which yields the following relation between the line integrals of $\mathbf{F}$ along two curves with common endpoints.

COROLLARY **45.1**. (Changing the Curve of Integration in a Line Integral) *Let two simple curves C_1 and C_2 have common initial and terminal points A and B and be non-intersecting otherwise. Suppose that C_1 and C_2 are oriented from A to B so that the union of C_1 and $-C_2$ is the positively oriented boundary ∂S of an oriented surface S. Suppose that the hypotheses of Stokes' theorem hold for S and a vector field $\mathbf{F}$. Then*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \tag{45.2}$$

Note that any suitable surface may be used in relation (**45.2**).

EXAMPLE **45.2**. *Evaluate the line integral of*

$$\mathbf{F} = \langle xy^2 + ze^{xz}\,,\ -x^2y - z^2y\,,\ xe^{xz} \rangle$$

along the curve C that consists of two parabolas $z = 1-y^2$, $x = 0$, $0 \le y \le 1$ and $z = 1 - x^2$, $y = 0$, $0 \le x \le 1$. The curve C is oriented from the point $(0,1,0)$ to $(1,0,0)$.

SOLUTION: The initial and final points of the curve C lie in the plane $z = 0$ in which the vector field $\mathbf{F}$ has much simpler form (set $z = 0$ in the components of $\mathbf{F}$). This suggests that the curve of integration should be deformed to a curve in the xy plane to simplify the evaluation of the integral. Consider the paraboloid $z = 1 - x^2 - y^2$. Its part S in the first octant is bounded by the curve C and by the part C' of the circle $x^2+y^2 = 1$ in the first quadrant of the xy plane. The curves C and C_1 have the same initial and final points. If S is oriented upward, then C is a part of ∂S according to the given orientation of C. By Eq. (**45.2**) the line integral in question can be transformed to the line integral of $\mathbf{F}$ along C' where C' should be oriented from the point $(0,1,0)$ to the point $(1,0,0)$. Let us take the standard parameterization of a circle

$$C' : \quad \mathbf{r}(t) = \langle \cos t,\ \sin t,\ 0\rangle, \quad 0 \le t \le \pi/2\,.$$

This vector function traverses C' from $(1,0,0)$ to $(0,1,0)$. Therefore this parameterization defines the orientation opposite to the required one and, hence, setting $C_1 = C$ and $C_2 = -C'$ in (**45.2**) one has

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{C'} \mathbf{F}\cdot d\mathbf{r} + \iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

The upward orientation of S, which is the graph $z = g(x,y) = 1 - x^2 - y^2$, is defined by the normal

$$\mathbf{n} = \langle -g'_x,\ -g'_y,\ 1\rangle = \langle 2x,\ 2y,\ 1\rangle\,,$$

where (x,y) span the part D of the disk $x^2 + y^2 \le 1$ in the first quadrant. Next,

$$\begin{aligned}
\mathbf{B} &= \operatorname{curl}\mathbf{F} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 + ze^{xz} & -x^2y - z^2y & xe^{xz}\end{pmatrix} \\
&= \langle 2zy,\ 0,\ -4xy\rangle\,, \\
B_n(x,y) &= \mathbf{B}\cdot\mathbf{n} = \langle 2gy,\ 0,\ -4xy\rangle\cdot\langle 2x,\ 2y,\ 1\rangle = 4gxy - 4xy \\
&= 4xy(g-1) = -4xy(x^2+y^2)\,, \\
\iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \iint_D B_n(x,y)\,dA = -4\int_0^{\pi/2}\sin\theta\cos\theta d\theta\int_0^1 r^5\,dr \\
&= -4\cdot\frac{1}{2}\cdot\frac{1}{6} = -\frac{1}{3}\,.
\end{aligned}$$

where the double integral has been converted to polar coordinates. Note that D is the image of the rectangle $D' = [0,1]\times[0,\pi/2]$ in the polar plane.

Finally, the line integral of $\mathbf{F}$ along C' has to be evaluated. One has

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) &= \langle \cos t\sin^2 t, -\sin t\cos^2 t, \cos t\rangle \cdot \langle -\sin t,\ \cos t,\ 0\rangle \\
&= -\cos t\sin^3 t - \sin t\cos^3 t = -\sin t\cos t(\sin^2 t + \cos^2 t) \\
&= -\sin t\cos t\,, \\
\int_{C'} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = -\int_0^{\pi/2} \sin t\cos t dt = -\frac{1}{2}\,, \\
\int_C \mathbf{F}\cdot d\mathbf{r} &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\,.
\end{aligned}$$

□

45.4. Geometrical Significance of the Curl. Stokes' theorem reveals the geometrical significance of the curl of a vector field. The line integral of a vector field along a closed curve C is often called the *circulation* of a vector field along C. Let $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{F}$ and let $\mathbf{B}_0 = \mathbf{B}(\mathbf{r}_0)$ at some point $\mathbf{r}_0$. Consider a plane through $\mathbf{r}_0$ normal to a unit vector $\hat{\mathbf{n}}$. Let S_a be a simple region in the plane such that $\mathbf{r}_0$ is an interior point of S_a. The unit vector $\hat{\mathbf{n}}$ defines an orientation of S_a and, as usual, ∂S_a is the positively oriented boundary curve of S_a. Let a be the radius of the smallest disk centered at $\mathbf{r}_0$ that contains S_a. If ΔS_a is the area of S_a, consider *the circulation of a vector field* $\mathbf{F}$ *per unit area* at a point $\mathbf{r}_0$ defined as the limit

$$\lim_{a\to 0^+} \frac{1}{\Delta S_a} \oint_{\partial S_a} \mathbf{F}\cdot d\mathbf{r}\,,$$

i.e., in the limit when the region S_a shrinks to the point $\mathbf{r}_0$. Then by virtue of Stokes' theorem and the integral mean value theorem,

$$\begin{aligned}
\lim_{a\to 0} \frac{1}{\Delta S_a} \oint_{\partial S_a} \mathbf{F}\cdot d\mathbf{r} &= \lim_{a\to 0} \frac{1}{\Delta S_a} \iint_{S_a} \mathbf{B}\cdot\hat{\mathbf{n}}\,dS = \mathbf{B}_0\cdot\hat{\mathbf{n}} = (\operatorname{curl}\mathbf{F})_0\cdot\hat{\mathbf{n}} \\
&= \|(\operatorname{curl}\mathbf{F})_0\|\cos\theta\,,
\end{aligned}$$

where θ is the angle between $\hat{\mathbf{n}}$ and $(\operatorname{curl}\mathbf{F})_0$, the curl of $\mathbf{F}$ at the point $\mathbf{r}_0$. Indeed, since the function $f(\mathbf{r}) = \mathbf{B}\cdot\hat{\mathbf{n}}$ is continuous on S_a, by the integral mean value theorem there is a point $\mathbf{r}_a$ in S_a such that the surface integral of f equals $\Delta S_a\, f(\mathbf{r}_a)$. As $a\to 0$, $\mathbf{r}_a\to\mathbf{r}_0$ and, by the continuity of f, $f(\mathbf{r}_a)\to f(\mathbf{r}_0)$. Thus, *the circulation of a vector field per unit area is maximal if the normal to the area element is in the same direction as the curl of the vector field (*$\theta = 0$*), and the maximal circulation equals the magnitude of the curl.*

This observation has the following mechanical interpretation illustrated in the left panel of Fig. **45.3**. Let $\mathbf{F}$ describe a fluid flow $\mathbf{F} = \mathbf{v}$, where $\mathbf{v}$ is the fluid velocity vector field. Imagine a tiny paddle wheel in the fluid at a point $\mathbf{r}_0$ whose axis of rotation is directed along $\hat{\mathbf{n}}$. The fluid exerts pressure on the paddles, causing the paddle wheel to rotate. The work done by the pressure force is determined by the line integral along the loop ∂S_a

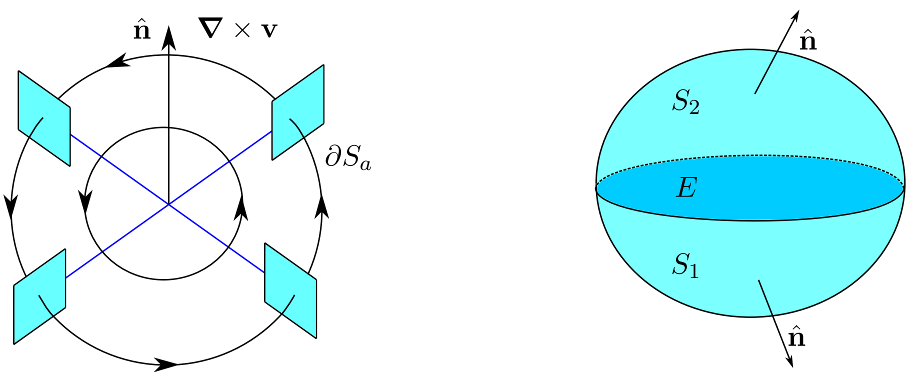


FIGURE **45.3**. **Left**: An illustration to the mechanical interpretation of the curl. A small paddle wheel whose axis of rotation is parallel to $\hat{\mathbf{n}}$ is placed into a fluid flow. The work done by the pressure force along the loop ∂S_a through the paddles causes rotation of the wheel. It is determined by the line integral of the velocity vector field $\mathbf{v}$. The work is maximal (the fastest rotation of the wheel) when $\hat{\mathbf{n}}$ is aligned parallel with $\boldsymbol{\nabla}\times\mathbf{v}$. **Right**: An illustration to Corollary **46.2**. The flux of a vector field across the surface S_1 can be related to the flux across S_2 by the divergence theorem if S_1 and S_2 have a common boundary and their union encloses a solid region E.

through the paddles. The more work done by the pressure force, the faster the wheel rotates. The wheel rotates fastest (maximal work) when its axis $\hat{\mathbf{n}}$ is parallel to $\operatorname{curl}\mathbf{v}$ because, in this case, the normal component of the curl, $(\boldsymbol{\nabla}\times\mathbf{v})\cdot\hat{\mathbf{n}} = \|\boldsymbol{\nabla}\times\mathbf{v}\|$, is maximal. For this reason, the curl is often called the *rotation* of a vector field and also denoted as $\operatorname{rot}\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F}$.

DEFINITION **45.1**. (Rotational Vector Field).
A vector field $\mathbf{F}$ *that can be represented as the curl of another vector field* $\mathbf{A}$*, that is,* $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{A}$*, is called a* rotational vector field.

The following theorem holds (the proof is omitted).

THEOREM **45.2**. (Helmholtz's Theorem).
Let $\mathbf{F}$ *be a vector field on a bounded open region* E *whose components have continuous second order partial derivatives. Then* $\mathbf{F}$ *can be decomposed into the sum of conservative and rotational vector fields; that is, there is a function* f *and a vector field* $\mathbf{A}$ *such that*

$$\mathbf{F} = \boldsymbol{\nabla} f + \boldsymbol{\nabla}\times\mathbf{A}.$$

The vector field $\mathbf{A}$ is called *a vector potential* of the field $\mathbf{F}$. The vector potential is not unique and determined up to adding the gradient of a function, $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla} g$, because

$$\boldsymbol{\nabla} \times (\mathbf{A} + \boldsymbol{\nabla} g) = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} g) = \boldsymbol{\nabla} \times \mathbf{A}$$

for any g that has continuous second order partial derivatives. Electromagnetic waves are rotational components of electromagnetic fields, while the Coulomb field created by static charges is conservative. The velocity vector field of an incompressible fluid (like water) is a rotational vector field.

45.5. Test for a Vector Field to Be Conservative. The test for a vector field to be conservative (Theorem **42.3**) follows from Stokes' theorem. Indeed, in a simply connected region E, any simple, closed curve can be shrunk to a point while remaining in E throughout the deformation. Therefore, for any such curve C, one can always find a surface S in E such that $\partial S = C$ (e.g., C can be shrunk to a point along such S). If $\operatorname{curl} \mathbf{F} = \mathbf{0}$ throughout E, then, by Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0$$

for any simple closed curve C in E. Alternatively, if $\operatorname{curl} \mathbf{F} = \mathbf{0}$ in a simply connected region E, then a curve C_1 can be continuously deformed along a surface S in E to a curve C_2 that has the same initial and terminal points. The path-independence property of $\mathbf{F}$ follows from Eq. (**45.2**) and, hence, $\mathbf{F}$ is conservative. The hypothesis that E is simply connected is crucial. For example, if E is the entire space with the z axis removed (see Study Problem **42.1**), then the z axis always pierces through any surface S bounded by a closed simple curve encircling the z axis, and one cannot claim that the curl vanishes everywhere on S.

45.6. Study problems.

Problem **45.1**. *Prove that the flux of a continuous rotational vector field* $\mathbf{F}$ *vanishes across any smooth, closed, and oriented surface. What can be said about a flux in a flow of an incompressible fluid?*

Solution: A continuous rotational vector field can be written as the curl of a vector field $\mathbf{A}$ whose components have continuous partial derivatives, $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$. Consider a smooth closed simple curve C in a smooth, oriented, and closed surface S. It cuts S into two pieces S_1 and S_2. Since S is a boundary of a solid connected region, it can be oriented either inward or outward. In either case, the induced orientations of the boundaries ∂S_1 and

∂S_2 are opposite: $\partial S_1 = -\partial S_2$. By virtue of Stokes theorem:

$$\begin{aligned}\iint_S (\boldsymbol{\nabla}\times\mathbf{A})\cdot\hat{\mathbf{n}}\,dS &= \iint_{S_1} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\hat{\mathbf{n}}\,dS + \iint_{S_2} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\hat{\mathbf{n}}\,dS \\ &= \oint_{\partial S_1} \mathbf{A}\cdot d\mathbf{r} + \oint_{\partial S_2} \mathbf{A}\cdot d\mathbf{r} \\ &= \oint_{\partial S_1} \mathbf{A}\cdot d\mathbf{r} + \oint_{-\partial S_1} \mathbf{A}\cdot d\mathbf{r} = 0\end{aligned}$$

Recall that the line integral changes its sign when the orientation of the curve is reversed. Since the flow of an incompressible fluid is described by a rotational vector field, the flux across a closed surface always vanishes in such a flow. □

45.7. Exercises.
1–3. Verify Stokes' theorem for the given vector field **F** and surface S by calculating the circulation of **F** along ∂S and the flux of $\boldsymbol{\nabla}\times\mathbf{F}$ across S. Choose an orientation of S.

1. $\mathbf{F} = \langle y,\ -x, z\rangle$ and S is the part of the sphere $x^2+y^2+z^2=2$ that lies above the plane $z=1$;
2. $\mathbf{F} = \langle x,\ y, xyz\rangle$ and S is the part of the plane $2x+y+z=4$ in the first octant;
3. $\mathbf{F} = \langle y, z, x\rangle$ and S is the part of the plane $x+y+z=0$ inside the sphere $x^2+y^2+z^2=a^2$.

4–14. Use Stokes' theorem to evaluate the line integral of the given vector field **F** along the specified closed contour C.

4. $\mathbf{F} = \langle x+y^2,\ y+z^2, z+x^2\rangle$ and C is the triangle traversed as $(1,0,0)\to(0,1,0)\to(0,0,1)\to(1,0,0)$;
5. $\mathbf{F} = \langle yz,\ 2xz, e^{xy}\rangle$ and C is the intersection of the cylinder $x^2+y^2=1$ and the plane $z=3$ oriented clockwise when viewed from above the plane;
6. $\mathbf{F} = \langle xy,\ 3z, 3y\rangle$ and C is the intersection of the plane $x+y=1$ and the cylinder $y^2+z^2=1$, C is oriented counterclockwise when viewed from the tip of the x axis;
7. $\mathbf{F} = \langle z,\ y^2,\ 2x\rangle$ and C is the intersection of the plane $x+y+z=5$ and the cylinder $x^2+y^2=1$, C is oriented counterclockwise when viewed from the tip of the z axis;
8. $\mathbf{F} = \langle -yz,\ xz,\ 0\rangle$ and C is the intersection of the hyperbolic paraboloid $z=y^2-x^2$ and the cylinder $x^2+y^2=1$, C is oriented clockwise when viewed from the tip of the z axis;
9. $\mathbf{F} = \langle z^2y/2,\ -z^2x/2,\ 0\rangle$ and C is the boundary of the part of the cone $z = 1-\sqrt{x^2+y^2}$ that lies in the first quadrant, C is oriented counterclockwise when viewed from the tip of the z axis;

10. $\mathbf{F} = \langle y - z,\ -x,\ x\rangle$ and C is the intersection of the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = x^2 + (y-1)^2$, C is oriented counterclockwise when viewed from the tip of the $z-$axis;

11. $\mathbf{F} = \langle y - z,\, z - x,\, x - y\rangle$ and C is the ellipse $x^2 + y^2 = a^2$, $(x/a) + (z/b) = 1$, $a > 0$, $b > 0$, oriented positively when viewed from the tip of the z axis;

12. $\mathbf{F} = \langle y + z,\, z + x,\, x + y\rangle$ and C is the ellipse $x = a\sin^2 t$, $y = 2a\sin t\cos t$, $z = a\cos^2 t$, $0 \le t \le \pi$, oriented in the direction of increasing t;

13. $\mathbf{F} = \langle y^2 - z^2,\, z^2 - x^2,\, x^2 - y^2\rangle$ and C is the intersection of the boundary surface of the cube $[0,a] \times [0,a] \times [0,a]$ by the plane $x + y + z = 3a/2$, oriented counterclockwise when viewed from the tip of the x axis;

14. $\mathbf{F} = \langle 3y^2 z,\, xyz,\, xy^2\rangle$ and C is the intersection of the parabolic cylinder $y = x^2$ with the circular cylinder $x^2 + z^2 = 1$, C is oriented clockwise when viewed from the tip of the y axis. Hint: Choose S to be a part of the parabolic cylinder.

15. Let C be a simple, closed curve in the plane $\mathbf{n} \cdot \mathbf{r} = d$ and let the area of a region in the plane bounded by C be A. If C is oriented counterclockwise when viewed from the tip of the vector $\mathbf{n}$, find

$$\oint_C (\mathbf{n} \times \mathbf{r}) \cdot d\mathbf{r}$$

16–18. Use Stoke's theorem to find the work done by the given force $\mathbf{F}$ in moving a particle along the specified closed curve C.

16. $\mathbf{F} = \langle -yz,\ zx,\ yx\rangle$ and C is the triangle: $(0,0,6) \to (2,0,0) \to (0,3,0) \to (0,0,6)$;

17. $\mathbf{F} = \langle -yz,\ xz,\ z^2\rangle$ and C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant that is traversed clockwise when viewed from the tip of the z-axis;

18. $\mathbf{F} = (y + \sin x,\ z^2 + \cos y,\ x^3)$ and C is traversed by $\mathbf{r}(t) = (\sin t,\ \cos t,\ \sin(2t))$ for $0 \le t \le 2\pi$, C is oriented in the direction of increasing t. Hint: Observe that C lies in the surface $z = 2xy$.

19. Find the line integral of $\mathbf{F} = (e^{x^2} - yz,\ e^{y^2} - xz,\ z^2 - xy)$ along C which is the helix $x = a\cos t$, $y = a\sin t$, $z = ht/(2\pi)$ from the point $(a,0,0)$ to the point $(a,0,h)$;
Hint: Supplement C by the straight line segment BA to make a closed curve and then use Stokes' theorem.

20. Suppose that a surface S satisfies the hypotheses of Stokes' theorem and the functions f and g have continuous partial derivatives. Show that:

$$\oint_{\partial S} (f\boldsymbol{\nabla} g) \cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) \cdot \hat{\mathbf{n}}\, dS,$$

Use the result to show that the circulation of the vector fields of the form $\mathbf{F} = f\boldsymbol{\nabla} f$ and $\mathbf{F} = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$ vanishes along ∂S.

21. Consider a rotationally symmetric solid. Let the solid be rotating about the symmetry axis at a constant rate ω (angular velocity measured in radians per unit time). Let $\mathbf{w}$ be the vector parallel to the symmetry axis such that $\|\mathbf{w}\| = \omega$ and the rotation is counterclockwise when viewed from the tip of $\mathbf{w}$. If the origin is on the symmetry axis, show that the linear velocity vector field in the solid is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ where $\mathbf{r}$ is the position vector of a point in the solid. Next, show that $\boldsymbol{\nabla} \times \mathbf{v} = 2\mathbf{w}$. This gives another relation between the curl of a vector field and rotations.

46. Gauss-Ostrogradsky (Divergence) Theorem

46.1. Divergence of a Vector Field.

DEFINITION **46.1**. (Divergence of a Vector Field).
Suppose that a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ *is differentiable. Then the scalar function*

$$\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

is called the divergence of a vector field.

EXAMPLE **46.1**. *Find the divergence of the vector field*

$$\mathbf{F} = \langle x^3 + \cos(yz),\ y + \sin(x^2 z),\ xyz \rangle .$$

SOLUTION: One has

$$\operatorname{div} \mathbf{F} = (x^3 + \cos(yz))'_x + (y + \sin(x^2 z))'_y + (xyz)'_z = 3x^2 + 1 + yx.$$

□

COROLLARY **46.1**. *A rotational vector field whose components have continuous partial derivatives is divergence free,*

$$\operatorname{div} \operatorname{curl} \mathbf{A} = 0 .$$

PROOF. By definition, a rotational vector field has the form $\mathbf{F} = \operatorname{curl} \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}$, where the components of $\mathbf{A}$ have continuous second partial derivatives because, by the hypothesis, the components of $\mathbf{F}$ have continuous partial derivatives. Therefore,

$$\operatorname{div} \mathbf{F} = \operatorname{div} \operatorname{curl} \mathbf{A} = \boldsymbol{\nabla} \cdot \operatorname{curl} \mathbf{A} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$$

by the rules of vector algebra (the triple product vanishes if any two vectors in it coincide). These rules are applicable because the components of $\mathbf{A}$ have continuous second partial derivatives (Clairaut's theorem holds for its components; see Section **42.3**). □

Laplace operator. Let $\mathbf{F} = \boldsymbol{\nabla} f$. Then $\operatorname{div}\mathbf{F} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = f''_{xx} + f''_{yy} + f''_{zz}$. The operator $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2$ is called the *Laplace operator*.

46.2. Another Vector Form of Green's Theorem. Green's theorem relates a line integral along a planar closed smooth simple curve C of the *tangential* component of a planar vector field to the flux of its curl across the region D bounded by C. Let us investigate the line integral of the *normal* component of a vector field. If a vector function $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$, $a \le t \le b$, traces out the boundary C of D in the positive (counterclockwise) direction, then the vectors

$$\hat{\mathbf{T}}(t) = \frac{1}{\|\mathbf{r}'(t)\|} \Big\langle x'(t),\ y'(t),\ 0 \Big\rangle , \qquad \hat{\mathbf{n}}(t) = \frac{1}{\|\mathbf{r}'(t)\|} \Big\langle y'(t),\ -x'(t),\ 0 \Big\rangle ,$$

$$\hat{\mathbf{T}} \cdot \hat{\mathbf{n}} = 0 ,$$

are the unit tangent vector and the outward unit normal vector to the curve C, respectively. That $\hat{\mathbf{n}}$ is directed outward the region D can easily be understood from the right-hand rule for the direction of the cross product. Indeed, the direct evaluation of the cross product shows that it is parallel to the z axis:

$$\hat{\mathbf{n}} \times \hat{\mathbf{T}} = \frac{1}{\|\mathbf{r}'(t)\|^2} \left\langle 0,\ 0,\ (x'(t))^2 + (y'(t))^2 \right\rangle = \langle 0, 0, 1\rangle = \hat{\mathbf{e}}_3 \,.$$

On the other hand, under the condition that $\hat{\mathbf{T}}$ defines the counterclockwise orientation of C, the cross product $\hat{\mathbf{n}} \times \hat{\mathbf{T}}$ points in the direction of the z axis if and only if $\hat{\mathbf{n}}$ is the outward normal. Similarly, if the parameterization of C is such that $\hat{\mathbf{T}}$ defines the *clockwise* orientation of C, then $\hat{\mathbf{n}}$ defined by the above equation is the *inward* normal.

Consider the line integral

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$$

of the normal component of a planar vector field $\mathbf{F}$ along C. One has $ds = \|\mathbf{r}'(t)\| dt$, and hence

$$\mathbf{F} \cdot \hat{\mathbf{n}}\, ds = F_1 y'\, dt - F_2 x'\, dt = F_1\, dy - F_2\, dx = \mathbf{G} \cdot d\mathbf{r},$$

where $\mathbf{G} = \langle -F_2, F_1\rangle$. By Green's theorem applied to the line integral of the vector field $\mathbf{G}$,

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{G} \cdot d\mathbf{r} = \iint_D \Big(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\Big)\, dA = \iint_D \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\Big)\, dA.$$

The integrand in the double integral is the divergence of $\mathbf{F}$. Thus, another vector form of Green's theorem has been obtained:

$$\oint_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA.$$

For a planar vector field (think of a mass flow on a plane), the line integral on the left side can be viewed as the outward flux of $\mathbf{F}$ across the boundary of a region D (e.g., the mass transfer by a planar flow across the boundary of D) . An extension of this form of Green's theorem to three-dimensional vector fields is known as the *divergence or Gauss-Ostrogradsky theorem.*

46.3. The Divergence Theorem. Let a solid region E be bounded by a closed smooth orientable surface S. If the surface is oriented outward (the normal vector points outside of E), then it is denoted $S = \partial E$ (the inward oriented boundary of a solid region is then $-\partial E$).

THEOREM **46.1**. (Gauss-Ostrogradsky (Divergence) Theorem).
Suppose E is a bounded, closed region in space that has a piecewise-smooth

boundary ∂E (oriented outward). If components of a vector field $\mathbf{F}$ have continuous partial derivatives in an open region that contains E, then

$$\iint_{\partial E} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV.$$

The divergence theorem states that the outward flux of a vector field across a closed surface S is given by the triple integral of the divergence of the vector field over the solid region bounded by S. It provides a convenient technical tool to evaluate the flux of a vector field across a closed surface.

Remark. It should be noted that the boundary ∂E may contain several disjoint pieces. For example, let E be a solid region with a cavity. Then ∂E consists of two pieces, the outer boundary and the cavity boundary. Both pieces are oriented outward in the divergence theorem.

Green's and divergence theorems are two- and three-dimensional extensions of the fundamental theorem of calculus:

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)\,.$$

It it relates the integral of the derivative to values of the function on the boundary of the region of integration, much like Green's and divergence theorems.

Example **46.2**. *Evaluate the outward flux of the vector field $\mathbf{F} = \langle xz\,, xy\,, yz\rangle$ across the boundary of the cube $E = [0,1]\times[0,1]\times[0,1]$.*

Solution: The components of the vector fields are polynomials and therefore have continuous partial derivatives everywhere. The boundary of the cube is a piecewise smooth surface. So the hypotheses of the divergence theorem are fulfilled. One has

$$\begin{aligned}\operatorname{div}\mathbf{F} &= (xz)'_x + (xy)'_y + (yz)'_z = z + x + y\,,\\ \iint_{\partial E}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dxdydz\\ &= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}\,,\end{aligned}$$

where Fubini's theorem has been used to evaluate the triple integral. □

Example **46.3**. *Evaluate the inward flux of the vector field*

$$\mathbf{F} = \langle 4xy^2z + e^z\,,\ 4yx^2z\,,\ z^4 + \sin(xy)\rangle$$

across the closed surface that is the boundary of the part of the ball $x^2+y^2+z^2 \le R^2$ in the first octant.

Solution: The divergence of the vector field is

$$\operatorname{div}\mathbf{F} = (4xy^2z + e^z)'_x + (4yx^2z)'_y + (z^4+\sin(xy))'_z = 4z(x^2+y^2+z^2).$$

By the hypotheses of the divergence theorem, the boundary of a solid region must be oriented outward. Since the reversal of an orientation of the surface changes the sign of the flux,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= -\iiint_E 4z(x^2+y^2+z^2)\,dV = -\iiint_{E'} 4\rho^3\sin\phi J dV' \\ &= -\int_0^{\pi/2} d\theta \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi \int_0^R 4\rho^5\,d\rho \\ &= -\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{4R^6}{6} = -\frac{\pi R^6}{6},\end{aligned}$$

where the triple integral has been converted to spherical coordinates, $dV = JdV'$, $J = \rho^2\sin\phi$, and the region E is the image of the rectangular box $E' = [0,R]\times[0,\pi/2]\times[0,\pi/2]$ spanned by spherical coordinates. The reader is advised to try to evaluate the flux *without* using the divergence theorem to appreciate the power of the latter! □

The divergence theorem can be used to change (simplify) the surface of integration in a flux integral.

Corollary **46.2**. (Changing the Surface in a Flux Integral) *Let the (outward oriented) boundary ∂E of a solid region E be the union of two surfaces S_1 and S_2 that have a common boundary curve and no other common points. Suppose that all the hypotheses of the divergence theorem hold. Then*

$$\iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV - \iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS. \tag{46.1}$$

Proof. The flux of a vector field $\mathbf{F}$ across ∂E is the sum of the flux integrals over S_1 and S_2 (see Fig. **45.3** (right panel)). On the other hand, the integral over ∂E can be expressed as a triple integral by the divergence theorem, which establishes the stated relation between the fluxes across S_1 and S_2. □

Note that the surfaces S_1 and S_2 in Eq. (**46.1**) are oriented by an *outward* normal. If, for example, S_1 is oriented by an *inward* normal (relative to the solid E the surfaces S_1 and S_2 enclose), then the sign of the flux integral over S_1 is reversed in Eq. (**46.1**).

Example **46.4**. *Evaluate the upward flux of the vector field*

$$\mathbf{F} = \langle z^2\tan^{-1}(y^2+1)\,,\ z^4\ln(x^2+1)\,,\ z\rangle$$

across the part of the paraboloid $z = 2 - x^2 - y^2$ that lies above the plane $z = 1$.

Solution: Consider the solid E bounded by the paraboloid and the plane $z = 1$ (see the left panel of Fig. **46.1**). Let S_2 be the part of the paraboloid that bounds E and let S_1 be the part of the plane $z = 1$ that bounds E. If S_2 is oriented upward and S_1 is oriented downward, then the boundary of E is oriented outward, and Corollary **46.2** applies. The surface S_1 is

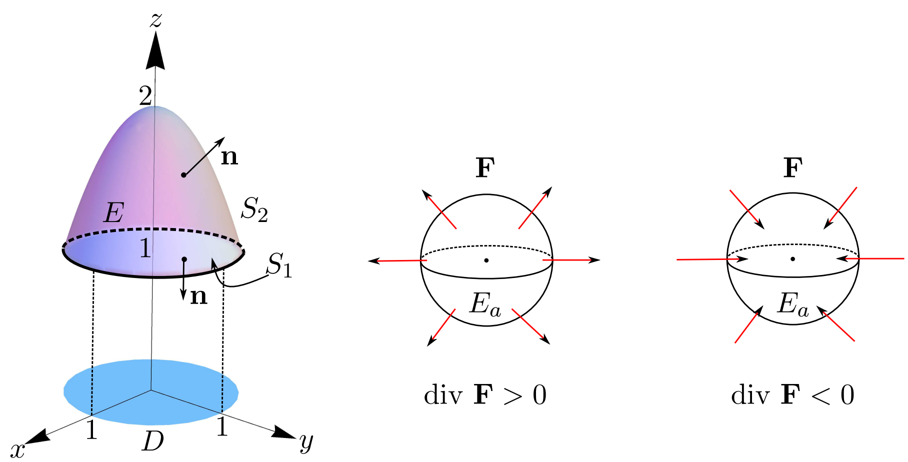


FIGURE **46.1**. **Left**: An illustration to Example **46.4**. The solid region E is enclosed by the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$. By Corollary **46.2**, the flux of a vector field across the part S_2 of the paraboloid oriented upward can be converted to the flux across the part S_1 of the plane $z = 1$ oriented downward. The union of S_1 and S_2 is a closed surface oriented outward. **Right**: The divergence of a vector field $\mathbf{F}$ determines the density of sources of $\mathbf{F}$. If div$\mathbf{F} > 0$ at a point, then the flux of $\mathbf{F}$ across a surface that encloses a small region E_a containing the point is positive (a "faucet"). If div$\mathbf{F} < 0$ at a point, then the flux of $\mathbf{F}$ across a surface that encloses a small region E_a containing the point is negative (a "sink").

the part of the plane $z = 1$ bounded by the curve of intersection of the paraboloid and the plane: $1 = 2 - x^2 - y^2$ or $x^2 + y^2 = 1$. So S_2 is the graph $z = g(x, y) = 1$ over D, which is the disk $x^2 + y^2 \leq 1$. The downward normal vector to S_1 is $\mathbf{n} = \langle g'_x, g'_y, -1\rangle = \langle 0, 0, -1\rangle$, and hence $F_n = \mathbf{F} \cdot \mathbf{n} = -F_3(x, y, g) = -F_3(x, y, 1) = -1$ on S_1 and

$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_D F_n(x, y)\, dA = -\iint_D dA = -A(D) = -\pi.$$

Next, the divergence of $\mathbf{F}$ is

$$\operatorname{div} \mathbf{F} = (z^2 \tan^{-1}(y^2 + 1))'_x + (z^4 \ln(x^2 + 1))'_y + (z)'_z = 0 + 0 + 1 = 1.$$

Hence,

$$\begin{aligned} \iiint_E \operatorname{div} \mathbf{F}\, dV &= \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_1^{2-r^2} r\, dz\, dr\, d\theta \\ &= 2\pi \int_0^1 (1 - r^2) r\, dr = \frac{\pi}{2}, \end{aligned}$$

where the triple integral has been converted to cylindrical coordinates for

$$E = \{(x, y, z) | z_{\text{bot}} = 1 \le z \le 2 - x^2 - y^2 = z_{\text{top}}\,,\ x^2 + y^2 \le 1\}\,,$$

which is the image of the region E' : $1 \le z \le 2 - r^2$ and (r, θ) span the rectangle $D' = [0, 1] \times [0, 2\pi]$. The upward flux of $\mathbf{F}$ across the paraboloid is now easy to find by Corollary **46.2**:

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV - \iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \frac{\pi}{2} + \pi = \frac{3\pi}{2}.$$

□

The reader is again advised to try to evaluate the flux directly via the surface integral to appreciate the power of the divergence theorem!

COROLLARY **46.3**. *The flux of a rotational vector field, whose components have continuous partial derivatives, across an oriented, closed, piecewise-smooth surface S vanishes:*

$$\iint_S \operatorname{curl} \mathbf{A} \cdot \hat{\mathbf{n}}\, dS = 0.$$

PROOF. The hypotheses of the divergence theorem are fulfilled. Therefore,

$$\iint_S \operatorname{curl} \mathbf{A} \cdot \hat{\mathbf{n}}\, dS = \iiint_E \operatorname{div} \operatorname{curl} \mathbf{A}\, dV = 0$$

by Corollary **46.1**. □

By Helmholtz's theorem, a vector field can always be decomposed into the sum of conservative and rotational vector fields. It follows then that only the conservative component of the vector field contributes to the flux across a closed surface:

$$\operatorname{div}(\boldsymbol{\nabla} f + \boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}^2 f + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}^2 f$$

So the divergence of a vector field is determined by the action of the Laplace operator on a scalar potential f of the vector field. This observation is further elucidated with the help of the concept of sources of a vector field.

46.4. Sources of a Vector Field. Consider a region E_a of volume ΔV_a whose (outward oriented) boundary ∂E_a is a connected surface (E_a has no cavities). Let $\mathbf{r}_0$ be an interior point of E_a. Let a be the radius of the smallest ball that contains E_a and is centered at $\mathbf{r}_0$. Let us calculate the outward flux *per unit volume* of a vector field $\mathbf{F}$ across the boundary ∂E_a, which is defined as the limit of the ratio

$$\lim_{a \to 0^+} \frac{1}{\Delta V_a} \iint_{\partial E_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS\,.$$

Suppose that components of $\mathbf{F}$ have continuous partial derivatives. By virtue of the divergence theorem and the integral mean value theorem,

$$\lim_{a \to 0} \frac{1}{\Delta V_a} \iint_{\partial E_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \lim_{a \to 0} \frac{1}{\Delta V_a} \iiint_{E_a} \operatorname{div} \mathbf{F}\, dV = \operatorname{div} \mathbf{F}(\mathbf{r}_0).$$

Indeed, by the continuity of $\operatorname{div}\mathbf{F}$ and the integral mean value theorem, there is a point $\mathbf{r}_a$ in E_a such that the triple integral equals $\Delta V_a \operatorname{div}\mathbf{F}(\mathbf{r}_a)$. In the limit $a \to 0^+$, $\mathbf{r}_a \to \mathbf{r}_0$ and $\operatorname{div}\mathbf{F}(\mathbf{r}_a) \to \operatorname{div}\mathbf{F}(\mathbf{r}_0)$ by continuity of partial derivatives of $\mathbf{F}$. Thus, if the divergence is positive $\operatorname{div}\mathbf{F}(\mathbf{r}_0) > 0$, the *outward* flux of the vector field across any small surface around $\mathbf{r}_0$ is positive. This, in turn, means that the flow lines of $\mathbf{F}$ are outgoing from $\mathbf{r}_0$ as if there is a *source* creating a flow at $\mathbf{r}_0$. Following the analogy with water flow, such a source is called a *faucet* (see the right panel of Fig. **46.1**). If $\operatorname{div}\mathbf{F}(\mathbf{r}_0) < 0$, the outward flux across any small surface around $\mathbf{r}_0$ is negative, which means that the *inward* flux is *positive*. Therefore the flow lines should go toward $\mathbf{r}_0$. Such a source is called a *sink* again by analogy with water flow. Thus,

- *the divergence of a vector field determines the density of sources of a vector field.*

For example, flow lines of a static electric field originate from positive electric charges and end on negative electric charges. The divergence of the electric field determines the electric charge density in space.

The divergence theorem states that the outward flux of a vector field across a closed surface is determined by the total source of the vector field in the region bounded by the surface. In particular, the outward flux of the electric field $\mathbf{E}$ across a closed surface S is determined by the total electric charge in the region enclosed by S. In contrast, the magnetic field $\mathbf{B}$ is a rotational vector field and, hence, is divergence free. So there are no *magnetic charges* also known as *magnetic monopoles*. These two laws of physics are stated in the form:

$$\operatorname{div}\mathbf{E} = 4\pi\sigma, \qquad \operatorname{div}\mathbf{B} = 0$$

where σ is the density of electric charges. Flow lines of a static magnetic field are closed, while flow lines of a static electric field end at points where electric charges are located (just like indicated by arrows in the right panel of Fig. **46.1**).

Remark. Magnetic monopoles are forbidden by the laws of classical electromagnetism. However the laws of quantum physics, which are more general than the laws of classical physics, allow for magnetic monopoles to exist if their magnetic charge is determined by the electric charge of an electron in a certain way. Such magnetic monopoles are known as Dirac monopoles. As of now, no Dirac monopole or a similar particle predicted by modern fundamental laws of physics have been found despite an extensive search.

46.5. Study Problems.

Problem **46.1**. (Volume of a solid as the surface integral)
Let E be bounded by a piecewise smooth surface $S = \partial E$ oriented by an

outward unit normal vector $\hat{\mathbf{n}} = \langle n_1, n_2, n_3\rangle$. *Prove that the volume of* E *is*

$$V(E) = \frac{1}{3}\iint_{\partial E} \hat{\mathbf{n}}\cdot\mathbf{r}\,dS = \iint_{\partial E} n_1 x\,dS = \iint_{\partial E} n_2 y\,dS = \iint_{\partial E} n_3 z\,dS$$

SOLUTION: Consider the vector field $\mathbf{F} = \langle x, y, z\rangle$. Then

$$\operatorname{div}\mathbf{F} = 1 + 1 + 1 = 3\,.$$

Then by virtue of the divergence theorem

$$\iint_{\partial E} \hat{\mathbf{n}}\cdot\mathbf{r}\,dS = \iint_{\partial E} \hat{\mathbf{n}}\cdot\mathbf{F}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = 3\iiint_E dV = 3V(E)$$

and the first equality in the asserted result follows. The other equalities are established similarly by considering the vector field $\mathbf{F} = \langle x, 0, 0\rangle$, or $\mathbf{F} = \langle 0, y, 0\rangle$, or $\mathbf{F} = \langle 0, 0, z\rangle$. □

46.6. Exercises.

1–8. Find the divergence of the specified vector field $\mathbf{F}$ in its domain. If $\mathbf{F}$ contains a general function of a vector field, express $\operatorname{div}\mathbf{F}$ in terms of the operator $\boldsymbol{\nabla}$ acting on that function or vector field and simplify the action of $\boldsymbol{\nabla}$ as much as possible.

1. $\mathbf{F} = \boldsymbol{\nabla} f$, where $f = \sqrt{x^2 + y^2 + z^2}\,$;
2. $\mathbf{F} = \mathbf{r}/r$, where $r = \|\mathbf{r}\| \neq 0\,$;
3. $\mathbf{F} = \mathbf{a}f(r)$, where $r = \|\mathbf{r}\|$, f is differentiable, and $\mathbf{a}$ is a constant vector;
4. $\mathbf{F} = \mathbf{r}f(r)$, where $r = \|\mathbf{r}\|$ and f is differentiable. Find all f for which the divergence vanishes everywhere except possibly at $r = 0\,$;
5. $\mathbf{F} = \mathbf{a}g$ where $\mathbf{a}$ is a constant vector and $\boldsymbol{\nabla} g$ is continuous and vanishes nowhere. If the divergence of $\mathbf{F}$ vanishes everywhere, what can be said about level sets of g?
6. $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a}$ is a constant vector;
7. $\mathbf{F} = \mathbf{a}\times\boldsymbol{\nabla} g$, where $\mathbf{a}$ is a constant vector;
8. $\mathbf{F} = \mathbf{a}\times\mathbf{G}$ where $\mathbf{a}$ is a constant vector.

9–12. Prove each of the following identities assuming that the appropriate partial derivatives of vector fields and functions exist and are continuous.

9. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\boldsymbol{\nabla} f\,$;
10. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}\,$;
11. $\operatorname{div}(\boldsymbol{\nabla} f\times\boldsymbol{\nabla} g)\,$;
12. $\operatorname{curl}\operatorname{curl}\mathbf{F} = \boldsymbol{\nabla}(\operatorname{div}\mathbf{F}) - \boldsymbol{\nabla}^2\mathbf{F}\,$.

13. Let $\mathbf{a}$ be a fixed vector and $\hat{\mathbf{n}}$ be the unit normal to a planar closed curve C directed outward from the region bounded by C. Show that

$$\oint_C \mathbf{a}\cdot\hat{\mathbf{n}}\,ds = 0\,.$$

14. Let C be a simple closed smooth curve in the xy plane and $\hat{\mathbf{n}}$ be the unit normal to C directed outward from the region D bounded by C. If

$A(D)$ is the area of D, find $\oint_C \mathbf{r}\cdot\hat{\mathbf{n}}ds$.

15–16. Verify the divergence theorem for the given vector field $\mathbf{F}$ on the region E by calculating both sides of the equation stated in the divergence theorem.

15. $\mathbf{F} = \langle 3x,\ yz,\ 3xz\rangle$ and E is the rectangular box $[0,a]\times[0,b]\times[0,c]$;

16. $\mathbf{F} = \langle 3x,\ 2y,\ z\rangle$ and E is the solid bounded by the paraboloid $z = a^2 - x^2 - y^2$ and the plane $z = 0$.

17. Let $\mathbf{a}$ be a constant vector and S be a closed smooth surface oriented outward by the unit normal vector $\hat{\mathbf{n}}$. Prove that

$$\iint_S \mathbf{a}\cdot\hat{\mathbf{n}}\,dS = 0\,.$$

18–30. Evaluate the flux of the given vector field $\mathbf{F}$ across the specified closed oriented surface S. In each case, determine the kind of source of $\mathbf{F}$ in the region enclosed by S (sink or faucet).

18. $\mathbf{F} = \langle x^2, y^2, z^2\rangle$ and S is the boundary of the rectangular box $[0,a]\times[0,b]\times[0,c]$ oriented outward;

19. $\mathbf{F} = \langle x^3, y^3, z^3\rangle$ and S is the sphere $x^2+y^2+z^2 = R^2$ oriented inward;

20. $\mathbf{F} = \langle xz,\ yz+\sin(xz),\ \cos(yx)\rangle$ and S is the boundary of the solid enclosed by the parabolic cylinder $z = 1-x^2$ and the planes $z = 0$, $y = 0$, $y+z = 1$. S is oriented outward;

21. $\mathbf{F} = \langle -xy^2,\ -yz^2,\ zx^2\rangle$ and S is the sphere $x^2+y^2+z^2 = 1$ oriented inward;

22. $\mathbf{F} = \langle xy,\ z^2y,\ zx\rangle$ and S is the boundary of the solid region inside the cylinder $x^2+y^2 = 4$ and between the planes $z = \pm 2$. S is oriented outward;

23. $\mathbf{F} = \langle xz^2,\ y^3/3,\ zy^2+xy\rangle$ and S is the boundary of the part of the ball $x^2+y^2+z^2 \le 1$ in the first octant oriented inward;

24. $\mathbf{F} = \langle yz,\ z^2x+y,\ z-xy\rangle$ and S is the boundary of the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$ oriented outward;

25. $\mathbf{F} = \langle x+\tan(yz),\ \cos(xz)-y,\ \sin(xy)+z\rangle$ and S is the boundary of the solid region between the sphere $x^2+y^2+z^2 = 2z$ and the cone $z = \sqrt{x^2+y^2}$ oriented outward;

26. $\mathbf{F} = \langle \tan(yz),\ \ln(1+z^2x^2),\ z^2+e^{yx}\rangle$ and S is the boundary of the smaller part of the ball $x^2+y^2+z^2 \le a^2$ between two half-planes $y = x/\sqrt{3}$ and $y = \sqrt{3}x$, $x \ge 0$, oriented inward;

27. $\mathbf{F} = \langle xy^2, xz, zx^2\rangle$ and S is the boundary of the solid bounded by two paraboloids $z = x^2+y^2$ and $z = 1+x^2+y^2$ and the cylinder $x^2+y^2 = 4$, oriented outward;

28. $\mathbf{F} = \langle x, y, z\rangle$ and S is the boundary of the solid obtained from the box $[0,2a]\times[0,2b]\times[0,2c]$ by removing the smaller box $[0,a]\times[0,b]\times[0,c]$. S is oriented inward;

29. $\mathbf{F} = \langle x - y + z,\, y - z + x,\, z - x + y\rangle$ and S is the surface $|x - y + z| + |y - z + x| + |z - x + y| = 1$ oriented outward. Hint: Use a suitable change of variables in the triple integral to simplify the equation of the boundary;

30. $\mathbf{F} = \langle x^3,\, y^3,\, z^3\rangle$ and S is the sphere $x^2 + y^2 + z^2 = x$ oriented outward.

31. Let S_1 and S_2 be two smooth oriented surfaces that have the same boundary and non-intersecting otherwise. Suppose that $\mathbf{F} = \operatorname{curl}\mathbf{A}$ and components of $\mathbf{F}$ have continuous partial derivatives. Compare the fluxes of $\mathbf{F}$ across S_1 and S_2.

32–35. Use the divergence theorem to find the flux of the given vector field $\mathbf{F}$ across the specified surface S by an appropriate deformation of S.

32. $\mathbf{F} = \langle xy^2,\ yz^2,\ zx^2 + x^2\rangle$ and S is the top half of the sphere $x^2 + y^2 + z^2 = 4$ oriented toward the origin;

33. $\mathbf{F} = \langle z\cos(y^2),\ z^2\ln(1+x^2),\ z\rangle$ and S is the part of the paraboloid $z = 2 - x^2 - y^2$ above the plane $z = 1$, S is oriented upward;

34. $\mathbf{F} = \langle yz,\, xz,\, xy\rangle$ and S is the cylinder $x^2 + y^2 = a^2$, $0 \le z \le b$, oriented outward from its axis of symmetry;

35. $\mathbf{F} = \langle y\tan z + x^3,\, x^2z^3,\, xy\rangle$ and S is the part of the cone $z = 1 - \sqrt{x^2 + y^2}$ that lies above the xy plane and is oriented upward.

36. The electric field $\mathbf{E}$ and the charge density σ are related by the Gauss law $\operatorname{div}\mathbf{E} = 4\pi\sigma$. Suppose the charge density is constant, $\sigma = k > 0$, inside the sphere $x^2 + y^2 + z^2 = R^2$ and zero otherwise. Find the outward flux of the electric field across the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ in the two following cases: first, when R is greater than any of a, b, c; second, when R is less than any of a, b, c.

37. Let $\mathbf{F}$ be a vector field such that $\operatorname{div}\mathbf{F} = \sigma_0 = \mathit{const}$ in a solid bounded region E and $\operatorname{div}\mathbf{F} = 0$ otherwise. Let S be a closed smooth surface oriented outward. Consider all possible relative positions of E and S in space (the solid region bounded by S has or does not have an overlap with E). If V is the volume of E, what are all possible values of the flux of $\mathbf{E}$ across S?

38. Use the vector form of Green's theorem to prove the Green's first and second identities:

$$\iint_D f\boldsymbol{\nabla}^2 g\,dA = \oint_{\partial D}(f\boldsymbol{\nabla}g)\cdot\hat{\mathbf{n}}\,ds - \iint_D g\boldsymbol{\nabla}^2 f\,dA$$

$$\iint_D (f\boldsymbol{\nabla}^2 g - f\boldsymbol{\nabla}^2 g)\,dA = \oint_{\partial D}(f\boldsymbol{\nabla}g - g\boldsymbol{\nabla}f)\cdot\hat{\mathbf{n}}\,ds$$

where D satisfy the hypotheses of Green's theorem and the appropriate partial derivatives of f and g exist and are continuous.

39–41. Use the result of Study Problem **46.1** to find the volume of a solid bounded by the specified surfaces.

39. The planes $z = \pm c$ and the parametric surface $x = a\cos u\cos v + b\sin u\sin v$, $y = a\cos u\sin v - b\sin u\cos v$, $z = c\sin u$;

40. The planes $x = 0$ and $z = 0$ and the parametric surface $x = u\cos v$, $y = u\sin v$, $z = -u + a\cos v$ where $u \geq 0$ and $a > 0$;

41. The torus $x = (a+R\cos u)\cos v$, $y = (a+R\cos u)\sin v$, $z = R\sin u$.

42. Use the results of Study Problem **42.3** to express the divergence of a vector field in the cylindrical and spherical coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho\sin\phi}\left(\frac{\partial(\sin\phi F_\phi)}{\partial \phi} + \frac{\partial F_\theta}{\partial \theta}\right)$$

Hint: Show $\partial\,\hat{\mathbf{e}}_\rho/\partial\phi = \hat{\mathbf{e}}_\phi$, $\partial\,\hat{\mathbf{e}}_\rho/\partial\theta = \sin\theta\,\hat{\mathbf{e}}_\theta$, and similar relations for partial derivatives of other unit vectors.

43. Use the results of Study Problem **42.3** to express the Laplace operator in the cylindrical and spherical coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial^2 \theta} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

Hint: Show $\partial\,\hat{\mathbf{e}}_\rho/\partial\phi = \hat{\mathbf{e}}_\phi$, $\partial\,\hat{\mathbf{e}}_\rho/\partial\theta = \sin\theta\,\hat{\mathbf{e}}_\theta$, and similar relations for partial derivatives of other unit vectors.

Selected Answers and Hints to Exercises

Section 41.6. **17.** $\frac{31}{40}$. **18.** 0. **19.** $\frac{944}{105}$. **20.** $\frac{\pi}{2}a^2$. **21.** $\frac{\pi}{2}a^2$. **22.** $\mathbf{r}_2\cdot(\mathbf{a}\times\mathbf{r}_1)$. **24.** 0. **25.** $-\pi$.

Section 42.6. **1.** $\langle 0, xy, -y^2 - xy\rangle$. **2.** $\langle -y\cos(yz), -x\sin(xz), 0\rangle$. **3.** $\mathbf{0}$. **4.** $\langle -\frac{1}{z}, \frac{1}{z}, -\frac{1}{y}\rangle$. **5.** $2\mathbf{a}$. **9.** $x^2y + y^2z^3 + z + c$. **10.** $xyz + y^2\cos z + c$. **11.** $xe^y - yz^2 + c$. **17.** 39. **18.** $\frac{2}{3}$. **19.** $-ab^2$.

Section 43.5. **1.** $-\frac{5}{3}$. **2.** 0. **3.** $\frac{15}{4}\pi$. **4.** $-\frac{3}{2}\pi a^4$. **12.** -2. **13.** $\frac{1}{2}qa^2$. **16.** πab. **17.** $3\pi a^2$.

Section 44.5. **1.** aA, bA, or cA if the surface is in the yz, xz, or xy planes, respectively. **2.** $\frac{3}{2}abc$. **3.** 0. **4.** 0. **5.** 0. **8.** $\frac{7}{12}$. **9.** $-\frac{\pi}{3}$. **10.** 0. **11.** $-\frac{\pi}{6}R^3$. **13.** 24. **15.** $\frac{13}{15} - \frac{\pi}{8}$. **18.** $4\pi(R^3 - a^3)$.

Section 45.7. **4.** -1. **5.** -3π. **6.** 0. **7.** $-\pi$. **9.** 0. **10.** -6π. **14.** 0. **15.** $2\|\mathbf{n}\|A$. **17.** 0. **18.** $-\pi$. **19.** $\frac{1}{3}h^3$.

Section 46.6. **1.** $\frac{2}{r}$, $r = \|\mathbf{r}\| \neq 0$. **2.** $\frac{2}{r}$, $r = \|\mathbf{r}\| \neq 0$. **6.** 0. **7.** 0. **13.** Hint: Use the form of Green's theorem as the divergence theorem for planar vector fields. **14.** $2A$. **15.** $3abc + \frac{1}{2}abc^2 + \frac{3}{2}a^2bc$. **16.** $3\pi a^4$. **18.** $a^2bc + ab^2c + abc^2$, faucet. **19.** $-\frac{12}{5}\pi R^5$, faucet. **20.** $\frac{16}{35}$, faucet. **21.** $\frac{4}{15}\pi$, sink. **22.** $\frac{64}{3}\pi$, faucet. **23.** $-\frac{\pi}{10}$, faucet. **24.** $\frac{4\pi}{3}(1 - \frac{1}{\sqrt{2}})$, faucet. **25.** π, faucet. **26.** 0, no source. **32.** $-\frac{84}{5}\pi$. **33.** $\frac{3\pi}{2}$. **36.** $16\pi^2kabc/3$ (first) and $16\pi^2kR^3/3$ (second). **37.** The flux changes from 0 to $\sigma_0 V$ (when S encloses E).