1. Consider the Dirichlet problem
\[-\Delta u(x, y) = 5 \sin(2\pi x) \sin(\pi y), \quad u\big|_{\partial\Omega} = 0, \quad \Omega = (-1, 1) \times (-2, 2) .\]

Which of the following functions
\[u(x, y) = C \sin \left(\frac{\pi k}{2} x\right) \sin \left(\frac{\pi m}{4} y\right), \quad k, m = 1, 2, \ldots\]

where \(C\) is a constant, is the solution to it, if any?

Solution: The function \(u(x, y)\) satisfies the boundary condition for any \(k, m\) and \(C\). A direct evaluation of \(\Delta u\) shows that
\[\left(\frac{\pi^2 k^2}{4} + \frac{\pi^2 m^2}{16}\right) C \sin \left(\frac{\pi k}{2} x\right) \sin \left(\frac{\pi m}{4} y\right) = 5 \sin(2\pi x) \sin(\pi y)\]

This relation must hold for all \((x, y)\). Owing to the linear independence of the functions \(\sin(\pi x k/2)\) and \(\sin(\pi y m/4)\), this is possible if and only if \(k = 4\) and \(m = 4\), and, in this case, \(C = 1/\pi^2\).

2. Solve the boundary value problem
\[-\Delta u(x, y) = 5 \sin(2\pi x) \sin(\pi y), \quad (x, y) \in \Omega = (-1, 1) \times (-2, 2) ,
\]
\[u\big|_{x=\pm 1} = 0, \quad u\big|_{y=-2} = 3 \sin(\pi x), \quad u\big|_{y=2} = 4 \sin(2\pi x)\]

Hint: Use the result of Problem 1.

Solution: The solution to a boundary value problem can always be written as the sum
\[u(x, y) = u_f(x, y) + U(x, y)\]

where \(u_f(x, y)\) is the solution of the associated problem in which the boundary data are set to zero, while \(U\) is the solution to the associated problem in which the inhomogeneity is set to zero. The function found in Problem 1
\[u_f(x, y) = \frac{1}{\pi^2} \sin(2\pi x) \sin(\pi y)\]

satisfies the equation and vanishes at the boundary of the rectangle and, hence, is the solution to the first associated problem. The boundary data are zero on the vertical edges of the rectangle. Therefore the solution to the second associated problem can be expanded over the basis in the horizontal interval \([-1, 1]\):
\[X_k(x) = \sin[\nu_k(x + 1)], \quad \sin(2\nu_k) = 0, \quad \nu_k = \frac{\pi k}{2}, \quad k = 1, 2, \ldots\]

The boundary data contain only \(X_2(x) = -\sin(\pi x)\) and \(X_4(x) = \sin(2\pi x)\):
\[U\big|_{y=-2} = -3X_2(x), \quad U\big|_{y=2} = 4X_4(x)\]
Therefore the solution must be a linear combination of them

\[ U(x, y) = Y_2(y) \sin(\pi x) + Y_4(y) \sin(2\pi x) \]

where the expansion coefficients are solutions to the boundary value problems

\[
\begin{align*}
Y_2''(y) - \pi^2 Y_2(y) &= 0, \\
Y_2(-2) &= -3, \quad Y_2(2) = 0 \quad \Rightarrow \quad Y_2(y) = -3 \frac{\sinh[\pi(2 - y)]}{\sinh(4\pi)} \\
Y_4''(y) - 4\pi^2 Y_4(y) &= 0, \\
Y_4(-2) &= 0, \quad Y_4(2) = 4 \quad \Rightarrow \quad Y_4(y) = 4 \frac{\sinh[2\pi(2 + y)]}{\sinh(8\pi)}
\end{align*}
\]

3. Consider the Neumann problem

\[-\Delta u(x, y) = f(x, y), \quad \frac{\partial u}{\partial n}\bigg|_{\partial \Omega} = v(x, y)\bigg|_{\partial \Omega}, \quad \Omega = (-1, 1) \times (0, 2),\]

If \( f(x, y) = 6a^2y \) and \( v(x, y) = ax^2y^2 \), find all values of the parameter \( a \) at which the Neumann problem has a solution. How should the inhomogeneity \( f \) and the boundary data \( v \) be modified in order for a solution to the Neumann problem to be the sum of solutions to two Neumann problems in one of which the inhomogeneity is set to zero, while in the other the boundary data are set zero? Do NOT solve the problems, just formulate them!

**Solution:** The solvability condition to be verified reads

\[ c_f + c_h + c_v = 0 \]

where

\[
\begin{align*}
c_f &= \int_{-1}^{1} \int_{0}^{2} f(x, y) dy dx = 6 \int_{-1}^{1} x^2 dx \int_{0}^{2} y dy = 6 \cdot \frac{2}{3} \cdot 2 = 8, \\
c_v &= \int_{0}^{2} [v(-1, y) + v(1, y)] dy = 2a \int_{0}^{2} y^2 dy = \frac{16a}{3}, \\
c_h &= \int_{-1}^{1} [v(x, 2) + v(x, 0)] dx = 4a \int_{-1}^{1} x^2 dx = \frac{8a}{3}
\end{align*}
\]

Therefore

\[ 8 + \frac{16a}{3} + \frac{8a}{3} = 8 + 8a = 0 \quad \Rightarrow \quad a = -1 \]

Since \( c_f \neq 0 \), the solution can be sought in the form \( u = U - c_f [(x + 1)^2 + y^2]/(4A) \), where \( A = 4 \) is the area of the rectangle, so that

\[-\Delta U = -\Delta u - \frac{c_f}{A} = f - 2 = 6x^2y^2 - 2 \]

The mean value of the new inhomogeneity \( 6x^2y^2 - 2 \) over the rectangle vanishes, and the solvability condition for \( U \) involves only the new boundary data:

\[
\frac{\partial U}{\partial n}\bigg|_{\partial \Omega} = \frac{\partial u}{\partial n}\bigg|_{\partial \Omega} + \frac{1}{2} \frac{\partial}{\partial n}[(x + 1)^2 + y^2]\bigg|_{\partial \Omega} \quad \Rightarrow \quad -U_x'(1, y) = -y^2, \quad U_x'(0, y) = 2 - y^2 \\
U_y'(x, 2) = 2 - 4x^2
\]

Note that \( (\partial/\partial n)|_{x=-1} = -(\partial/\partial x)|_{x=-1}, \ (\partial/\partial n)|_{x=1} = (\partial/\partial x)|_{x=1} \) and, similarly, \( (\partial/\partial n)|_{y=0} = -(\partial/\partial y)|_{y=0}, \ (\partial/\partial n)|_{y=2} = (\partial/\partial y)|_{y=2} \). A solution is \( U = U_1 + U_2 \), where \( U_1 \) is a solution to
the associated problem in which the boundary data are set to zero, while \( U_2 \) is a solution to
the second associated problem in which the inhomogeneity is set to zero.

4. Find a harmonic function in the annulus \( 1 < x^2 + y^2 < 9 \) that coincides with \( 3 + 2(x^2 - y^2) \)
on the inner boundary \( x^2 + y^2 = 1 \) and with \( 4xy \) on the outer boundary \( x^2 + y^2 = 9 \).

**Solution:** The boundary data have the following expansions over the trigonometric Fourier harmonics:

\[
3 + 2(x^2 - y^2) \bigg|_{r=1} = 3 + 2(\cos^2 \theta - \sin^2 \theta) = 3 + 2 \cos(2\theta)
\]

\[
4xy \bigg|_{r=3} = 36 \cos \theta \sin \theta = 18 \sin(2\theta)
\]

The harmonic function in question must be a linear combination of the harmonics in the boundary data:

\[
u(x, y) = R_0(r) + R_2^c(r) \cos(2\theta) + R_2^s(r) \sin(2\theta)
\]

where \( R_m(r) \), \( m = 0, 2 \), are solutions to the equation

\[
-\frac{1}{r} \left( r R'_m(r) \right)' + \frac{m^2}{r^2} R_m(r) = 0 \quad \Rightarrow \quad R_0 = C_1 + C_2 \ln r, \quad R_2^c = C_1 r^2 + \frac{C_2}{r^2}
\]

The constant \( C \) is found from the boundary conditions

\[
R_0(1) = 3, \quad R_0(3) = 0 \quad \Rightarrow \quad C_1 = 3, \quad C_2 = -3/\ln 3;
\]

\[
R_2^c(1) = 2, \quad R_2^c(3) = 0 \quad \Rightarrow \quad C_1 = -1/40, \quad C_2 = 81/40;
\]

\[
R_2^s(1) = 0, \quad R_2^s(3) = 18 \quad \Rightarrow \quad C_1 = 81/40, \quad C_2 = -81/40.
\]

5. Find a harmonic function in the disk \( x^2 + y^2 < 4 \) if the sum of the function and its outward normal derivative on the boundary circle \( x^2 + y^2 = 4 \) coincides with the function \( y^2x \).

**Hint:** \( 2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta) \).

**Solution:** The expansion of the boundary data over the trigonometric Fourier harmonics is

\[
y^2x \bigg|_{r=2} = 8 \sin^2(\theta) \cos(\theta) = 4 \sin(\theta) \sin(2\theta) = 2 \cos(\theta) - 2 \cos(3\theta)
\]

by the equation from the hint. Therefore the harmonic function in question must be a linear combination

\[
u(x, y) = R_1(r) \cos(\theta) + R_3(r) \cos(3\theta)
\]

where the expansion coefficients are regular solutions to the equation (as \( r = 0 \) is in the domain of the harmonic function)

\[
-\frac{1}{r} \left( r R'_m(r) \right)' + \frac{m^2}{r^2} R_m(r) = 0 \quad \Rightarrow \quad R_m(r) = Cr^m, \quad R'_m(r) = mCr^{m-1}, \quad m = 1, 3
\]

The constant \( C \) is found from the boundary condition:

\[
R_1(2) + R'_1(2) = 2, \quad \Rightarrow \quad 2C + C = 2, \quad \Rightarrow \quad C = 2/3;
\]

\[
R_3(2) + R'_3(2) = -2, \quad \Rightarrow \quad 8C + 12C = -2, \quad \Rightarrow \quad C = -1/10.
\]
6. Solve the boundary value problem in the annulus \(1 < r < 2\), where \(r\) is the distance from the origin in a plane spanned by \((x, y)\):

\[-\Delta u(x, y) = 48x(x^2 + y^2), \quad u\big|_{r=1} = x\big|_{r=1}, \quad u\big|_{r=2} = y\big|_{r=2}\]

**Solution:** The inhomogeneity and the boundary data have the following expansions over the trigonometric Fourier harmonics:

\[48x(x^2 + y^2) = 48r^3 \cos(\theta), \quad x\big|_{r=1} = \cos(\theta), \quad y\big|_{r=2} = 2\sin(\theta)\]

The solution is a linear combination of the harmonics involved in these expansions:

\[u(x, y) = R_c(r) \cos(\theta) + R_s(r) \sin(\theta)\]

The functions \(R_{c,s}(r)\) satisfy the equation (the radial part of the Poisson equation with \(m = 1\)):

\[-\frac{1}{r} \left(rR_c'(r)\right)' + \frac{1}{r^2} R_c = 48r^3, \quad -\frac{1}{r} \left(rR_s'(r)\right)' + \frac{1}{r^2} R_s = 0\]

A particular solution to the first equation can be found by the method of undetermined coefficients for equidimensional (Cauchy-Euler) equations (or by the Green’s function method). It must have the form \(R_p(r) = Cr^5\). The substitution to the equation yields

\[-25Ct^3 + Ct^3 = 48r^3 \quad \Rightarrow \quad C = -2\]

The general solutions are

\[R_c(r) = C_1r + \frac{C_2}{r} - 2r^5, \quad R_s(r) = C_1r + \frac{C_2}{r}\]

The constants are found from the boundary conditions

\[R_c(1) = 1, \quad R_c(2) = 0 \quad \Rightarrow \quad C_1 = 125/3, \quad C_2 = -116/3\]
\[R_s(1) = 0, \quad R_s(2) = 2 \quad \Rightarrow \quad C_1 = 4/3, \quad C_2 = -4/3,\]

7. Find the constant \(a\) at which the Neumann problem is solvable in the disk \(r < 1\), where \(r\) is the distance from the origin in a plane spanned by \((x, y)\), and find all solutions to the problem:

\[-\Delta u(x, y) = x^2 + y^2, \quad \frac{\partial u}{\partial n}\big|_{r=1} = ax^2\big|_{r=1}\]

**Solution:** The solvability condition is

\[c_f + c_v = 0\]

where

\[c_f = \iint_{\Omega} (x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_0^1 r^2 \cdot r \, dr \, d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}\]
\[c_v = a \int_{\partial\Omega} x^2 \, ds = a \int_0^{2\pi} \cos^2(\theta) \, d\theta = \frac{a}{2} \int_0^{2\pi} \left(1 + \cos(2\theta)\right) \, d\theta = \pi a + 0\]
The Neumann problem has a solution if and only if
\[ \frac{\pi}{2} + \pi a = 0 \implies a = -\frac{1}{2} \]

In this case, the inhomogeneity and the boundary data have the following expansions over the trigonometric Fourier harmonics:
\[ x^2 + y^2 = r^2, \quad -\frac{1}{2} x^2 \bigg|_{r=1} = -\frac{1}{2} \cos^2(\theta) = -\frac{1}{4} - \frac{1}{4} \cos(2\theta) \]

Therefore the solution is a linear combination
\[ u(x, y) = R_0(r) + R_2(r) \cos(2\theta) \]

The functions \( R_m(r), m = 0, 2, \) are regular solutions to the radial part of the Poisson equation (as \( r = 0 \) is in the domain of the problem):
\[ -\frac{1}{r} \left(r R'_0(r)\right)' = r^2 \implies R_0(r) = C_1 - \frac{1}{16} r^4, \quad R_0(r) = -\frac{1}{4} r^3 \]
\[ -\frac{1}{r} \left(r R'_2(r)\right)' + \frac{4}{r^2} = 0 \implies R_2(r) = C_1 r^2, \quad R'_2(r) = 2C_1 r \]

where a particular solution to the first equation is found by the method of undetermined coefficients for the equidimensional equation. It must have the form \( R_p = Cr^4 \) so that the substitution into the equation yields \(-16r^2 = r^2 \) or \( C = -1/16 \). The constant \( C_1 \) is found from the boundary conditions:
\[ R'_0(1) = -\frac{1}{4} \implies -\frac{1}{4} = -\frac{1}{4}; \quad R'_2(1) = -\frac{1}{4} \implies C_2 = -\frac{1}{8} \]

The constant in \( R_0 \) remains arbitrary (the solution to the Neumann problem is unique up an additive constant). Note that the first boundary condition is fulfilled thanks to the solvability condition.

8 Extra credit. Someone says that there is a harmonic function \( u(x, y) \) in the region bounded by the upper arc of the circle \( x^2 + y^2 = 2, \ y \geq 1 \), and the horizontal line \( y = 1 \) such that
\[ \frac{\partial u}{\partial r} \bigg|_{r=2} = 12x^2y \bigg|_{r=2}, \quad \frac{\partial u}{\partial y} \bigg|_{y=1} = 3x^3, \]

where \( r \) is the distance from the origin. Should you believe this person? Save the bets and guessing for Las Vegas and a stock market!

Solution: The line integral of the boundary data over the boundary of the region must vanish in order for the Neumann problem to have a solution. This is obviously not so for the given data. Note that the integral of \( 3x^3 \) over the part of the line \( y = 1 \) in the boundary \((-a < x < a, \) where \( a \) defines the points of intersection of the line and the circle) vanishes by symmetry, while the boundary data are positive on the circular part of the boundary \( 12x^2y > 0 \) if \( y > 1 \). Hence, the line integral cannot be zero. No, such harmonic function exists.