Placement Exam with Solutions, MAC 3474 Honors Calculus III, Fall 2023

1 (5 pts). Let $f(x)$ and $g(x)$ be functions of a real variable $-\infty < x < \infty$ and

$$\lim_{x\to a} f(x) = A\,, \qquad \lim_{x\to a} g(x) = B\,.$$

Assess each of the following statements as TRUE or FALSE. No reasoning or supporting arguments are required.
(i) If $A > 0$, then there exists an interval (c, d) containing a such that $f(x) > 0$ for each x in (c, d), with possible exception of a.
(ii) If $A > 0$, then there exists an interval (c, d) containing a such that $f(x) > 0$ for each x in (c, d).
(iii) If $A < 0$, then it is possible that $f(x) > 0$ for some x in any interval (c, d) containing a.
(iv) If $A \neq B$, then it is possible that $f(x) - g(x)$ can have both positive and negative values in any interval containing a, with possible exception of a.
(v) If $f(x) < g(x)$ for all x in (c, d) containing a, with possible exception of a, then $A < B$.

SOLUTION: (i) TRUE. By the definition of the limit, for any $\varepsilon > 0$ one can find an interval $0 < |x - a| < \delta$ in which $A - \varepsilon < f(x) < A + \varepsilon$. In particular, take $0 < \varepsilon < A$ so that $f(x) > 0$.
(ii) FALSE. It is possible that $f(a) \neq A$ and $f(a) < 0$.
(iii) TRUE. This can only happen if $f(a) > 0$.
(iv) FALSE. By the limit laws, $h(x) = f(x) - g(x) \to A - B$ and, hence, $h(x)$ must have the same sign as $A - B$ for all x near a as in Part (i).
(v) FALSE. Take $f(x) = A - 2|x| < g(x) = A - |x|$ and let $x \to 0$ so that $A = B$.

2 (1 pt). Let

$$f(x) = \frac{\sin(ax)}{|x|} \quad \text{if} \quad x \neq 0\,, \quad f(0) = a\,.$$

Find all values of real parameter a for which f is continuous everywhere.

SOLUTION: Let $a \neq 0$. Then the left and right limits at $x = 0$ are not equal:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(ax)}{x} = a\,, \quad \lim_{x\to 0^-} f(x) = -\lim_{x\to 0^-} \frac{\sin(ax)}{x} = -a$$

So, f is not continuous at $x = 0$ if $a \neq 0$. If $a = 0$, then $f(x) = 0$ which is a continuous function everywhere. Thus, $a = 0$ is the only value at which f is continuous everywhere.

3 (1 pt). The function f is defined by the equation

$$f(x) = \begin{cases} x & , \quad x \text{ a rational number} \\ 0 & , \quad x \text{ an irrational number} \end{cases}$$

Find the largest set on which the function f is continuous.

SOLUTION. Continuity requires $f(x) \to f(a)$ as $x \to a$. This means there exists an interval (b, c) containing a in which $|f(x) - f(a)|$ is less that any preassigned positive number.

Let $a \neq 0$ be rational. Then $f(a) = a \neq 0$ and $|f(x) - f(a)|$ is equal to either $|a|$ or $|x - a|$ because any interval containing a has irrational numbers where $f(x) = 0$. Therefore it would not be possible to find an interval containing a in which $|f(x) - f(a)|$ is less than any preassigned positive number, say, $|a|/2$. So, f is not continuous at any rational non-zero number. If $a \neq 0$ is irrational, then $f(a) = 0$ and $|f(x) - f(a)|$ is equal to either $|x|$ or 0 because any interval containing a has rational numbers where $f(x) = x$. But $|x|$ cannot be arbitrary small everywhere near $a \neq 0$. Thus, f is not continuous at any irrational non-zero number. Let $a = 0$. In this case, $|f(x) - f(a)| = |f(x)| = |x|$ which can be made arbitrary small for all x sufficiently close to 0. Thus, $f(x) \to f(0) = 0$ as $x \to 0$ and, hence, f is continuous at $x = 0$.

4 (1 pt). Find the limit or show that the limit does not exist

$$\lim_{x\to\infty}\left(\frac{3x^2-4x+1}{5x+4}\sin\left(\frac{2}{x}\right)\right)$$

Hint: Recall the behavior of $\sin(t)$ when $t \to 0$.

SOLUTION: First note that

$$\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right) = \lim_{t\to 0^+}\frac{\sin(2t)}{t} = 2$$

where $t = \frac{1}{x}$. Then by the limit laws, the limit in question can be reduced to the following expression by factoring out the leading power in the top and bottom of the ratio:

$$\lim_{x\to\infty}\left(\frac{3-\frac{4}{x}+\frac{1}{x^2}}{5+\frac{4}{x}}x\sin\left(\frac{2}{x}\right)\right) = \lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{1}{x^2}}{5+\frac{4}{x}}\lim_{x\to\infty}x\sin\left(\frac{2}{x}\right) = \frac{3}{5}\cdot 2 = \frac{6}{5}\,.$$

5 (1 pt). Show that the equation

$$4x^3 - 9x^2 - 6x + 2 = 0$$

has a root in each of the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$. Reasonings based on a graphing calculator are NOT acceptable.

SOLUTION: Recall that a continuous function f on $[a, b]$ takes all values between $f(a)$ and $f(b)$. This implies that the equation $f(x) = K$ has at least one root in (a, b) if K lies strictly between $f(a)$ and $f(b)$. Put $f(x) = 4x^3 - 9x^2 - 6x + 2$ and $K = 0$. Then for the intervals in question, one infers that

$$\begin{aligned}
&f(-1) = -4 - 9 + 6 + 2 = -5\,, \quad f(0) = 2 \quad \Rightarrow \quad f(-1) < K < f(0)\,,\\
&f(1) = 4 - 9 - 6 + 2 = -9\,, \quad f(0) = 2 \quad \Rightarrow \quad f(1) < K < f(0)\,,\\
&f(2) = 32 - 36 - 12 + 2 = -14\,, \quad f(3) = 9\cdot(12-9) - 18 + 2 = 11 \quad \Rightarrow \quad f(2) < K < f(3)
\end{aligned}$$

and, hence, the equation $f(x) = K = 0$ has at least one root in the indicated intervals.

6 (2 pts). The function f is defined by the equation

$$f(x) = ax^2 + \frac{b}{x^3}\,, \quad x \neq 0\,,$$

where a and b are numerical parameters.
(i) For which values of a and b will the graph $y = f(x)$ have a horizontal tangent line at $(x, y) = (1, 5)$?
(ii) Does the function f have a relative maximum or minimum at $x = 1$ for such a and b?

SOLUTION: (i) The graph has a horizontal tangent at $x = 1$ if $f'(1) = 0$, and the point $(1, 5)$ lies on the graph which implies that $f(1) = 5$. Since $f'(x) = 2ax - 3b/x^4$, the noted relations comprise two equations for two unknowns a and b:

$$\begin{cases} f'(1) &= 0 \\ f(1) &= 5 \end{cases} \Rightarrow \begin{cases} 2a - 3b &= 0 \\ a + b &= 5 \end{cases} \Rightarrow \begin{cases} a &= 3 \\ b &= 5 - a = 2 \end{cases}$$

(ii) The derivative $f'(x) = 6x - 6/x^4 = 6x(1 - 1/x^5)$ changes sign at $x = 1$ from negative to positive. By the first derivative test, f has a local minimum at $x = 1$.

7 (1 pt). Suppose that f is a continuous and bounded function on the whole real line, that is, there are numbers m and M such that

$$m \le f(x) \le M \quad \text{for all} \quad -\infty < x < \infty .$$

Is it true that f has a maximum and/or a minimum? Support your answer by appropriate arguments!

SOLUTION: The assertion is false. For example, take $f(x) = \arctan(x)$. This function is strictly increasing because $f'(x) = \frac{1}{1+x^2} > 0$ for all x and, hence, cannot have any local extremum. But it is bounded, $-\frac{\pi}{2} < f(x) < \frac{\pi}{2}$.

8 (2 pts). At a point P in the first (positive) quadrant on the curve $y = 7 - x^2$ a tangent line is drown, intersecting the coordinate axes at the points A and B. Find the position (coordinates) of P that makes the distance between A and B minimal.

SOLUTION: Let $0 < a < \sqrt{7}$. Then the equation of the tangent line to the graph $y = f(x) = 7 - x^2$ at $x = a$ reads

$$y = f(a) + f'(a)(x - a) = 7 - a^2 + 2a(x - a)$$

The line intersects the $y-$axis when $x = 0$ or $y = B = 7 + a^2$. The line intersects the $x-$axis when $y = 0$ or $x = A = \frac{7+a^2}{2a}$. The distance D between A and B reaches its minimum when D^2 is minimal

$$D^2(a) = A^2 + B^2 = (7 + a^2)^2\Big(1 + \frac{1}{4a^2}\Big)$$

Differentiating D^2 and setting the derivative to zero, an optimal a is found:

$$4a(7 + a^2)\Big(1 + \frac{1}{4a^2}\Big) - (7 + a^2)^2\,\frac{1}{2a^3} = 0 \;\Rightarrow\; 8a^4\Big(1 + \frac{1}{4a^2}\Big) = 7 + a^2 \;\Rightarrow\; 8a^4 + a^2 - 7 = 0$$

$$\Rightarrow\; a^2 = \frac{7}{8} \;\Rightarrow\; a = \frac{\sqrt{14}}{4}, \quad f(a) = \frac{49}{8}$$

9 (1 pt). A function f is such that its second derivative f'' is continuous on a closed interval $[a, b]$ and the equation $f(x) = 0$ has 3 roots in the open interval (a, b). Show that the equation $f''(x) = 0$ has at least one root in (a, b).

SOLUTION: By continuity of f'', the derivative f' is continuous too in (a, b). Let $a < x_1 < x_2 < x_3 < b$ be the roots of $f(x) = 0$. By Rolle's theorem f has local extrema between zeros, and there exists $x_1 < c_1 < x_2$ such that $f'(c_1) = 0$ and there exists $x_1 < c_2 < x_3$ such that $f'(c_2) = 0$. Repeating the same argument for the function $g(x) = f'(x)$ in the interval $[c_1, c_2]$, it is concluded that there exists $c_1 < c < c_2$ at which g has a local extremum and $g'(c) = f''(c) = 0$ as required.

10 (1 pt). Calculate the derivative $f'(x)$ if

$$f(x) = 2x + \sum_{n=1}^{\infty} \int_{-nx^2}^{nx^2} \Big(\sin(t)\Big)^{2023} dt$$

SOLUTION: First note that

$$\int_{-a}^{a} \Big(\sin(t)\Big)^{2023} dt = 0$$

for any a because the integrand is an odd function, $(\sin(-t))^{2023} = (-\sin(t))^{2023} = -(\sin(t))^{2023}$. Thus, $f(x) = 2x$ and $f'(x) = 2$.

11 (2 pts). Water is being withdrawn from a conical reservoir 8 ft. in diameter and 10 ft. deep at the constant rate of 5 cubic ft. per minute. How fast is the water level falling when the depth of water is 5 ft. if
(i) the vertex of the cone is down?
(ii) the vertex of the cone is up?

SOLUTION: (i) Let $H = 10$ ft. and $R = 8/2 = 4$ ft. be the height and radius of the reservoir, respectively, and h be the height of the water relative to the vertex of the cone. If r is the radius of the cone occupied by water, then the volume of water reads

$$V = \frac{1}{3}\pi r^2 h$$

By the similarity law

$$\frac{R}{H} = \frac{r}{h} \quad \Rightarrow \quad r = \frac{R}{H}h \quad \Rightarrow \quad V = \frac{\pi}{3}\Big(\frac{R}{H}\Big)^2 h^3$$

Differentiating this relation with respect to time a relation between rates of change of the volume and the height is obtained:

$$\frac{dV}{dt} = \pi\Big(\frac{R}{H}\Big)^2 h^2 \frac{dh}{dt} = 4\pi\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{4\pi}\frac{dV}{dt} = \frac{5}{4\pi} \text{ ft/min}$$

(ii) Let us keep the same notations, that is, h is the distance from the vertex of the cone to the water surface. Then the water volume reads

$$V = \frac{\pi}{3}R^2 H - \frac{\pi}{3}r^2 h = \frac{\pi}{3}R^2 H - \frac{\pi}{3}\Big(\frac{R}{H}\Big)^2 h^3$$

In this case, the water level is $H - h$. So, $\frac{d}{dt}(H-h) = -\frac{dh}{dt}$ and $H - h = 5$ implies that $h = 5$ because $H = 10$. Upon differentiation of V, the constant (the volume of the reservoir) disappears and the relation between the rates is the same as in Part (i). So, the answer is also the same.

12 (1 pts). The planar region bounded by the curves $y^2 = 4x$, $y = 0$, and $x = 1$ is rotated around the line $x = 1$. Find the volume of resulting solid.

SOLUTION: The solid is partitioned into cylindrical shells of radius r and height dy, where r is the distance from $(1, y)$ on the line $x = 1$ to (x, y) on the curve $y^2 = 4x$. The volume of each cylinder is $dV = \pi r^2 dy$ where $r = 1 - x = 1 - \frac{1}{4}y^2$. The coordinate y spans $[0, 2]$ in the solid. Therefore

$$V = \int_0^V dV = \int_0^2 \pi\Big(1 - \frac{1}{4}y^2\Big)^2 dy = \pi\Big(2 - \frac{1}{2}\cdot\frac{8}{3} + \frac{1}{16}\cdot\frac{32}{5}\Big) = \frac{16\pi}{15}$$

13 (1 pts). Find the indefinite integral (show your work!)

$$\int \frac{5x dx}{x^3 - x^2 + 4x - 4}$$

SOLUTION: The denominator has a root $x = 1$. Using the partial fraction decomposition

$$\frac{5x}{x^3 - x^2 + 4x - 4} = \frac{5x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4) + (Bx+C)(x-1)}{(x-1)(x^2+4)}$$

The numerators must be equal for all x. In particular, demanding the equality for $x = 1$, $x = 0$, and $x = -1$, the system of equations for A, B, and C is obtained and solved:

$$\begin{cases} 5A &= 5 \\ 4A - C &= 0 \\ 5A - 2(C - B) &= -5 \end{cases} \quad \Rightarrow \quad A = 1\,,\ C = 4\,,\ B = -1$$

Therefore the integral in question is equal to

$$\int \frac{dx}{x-1} - \int \frac{xdx}{x^2+4} + 4\int \frac{dx}{x^2+4} = \ln|x-1| - \frac{1}{2}\ln(x^2+4) + 2\arctan\left(\frac{x}{2}\right)$$

up to an additive constant, where $xdx = \frac{1}{2}d(x^2+1)$ was used to evaluate the second integral.

14 (1 pts). Evaluate (show your work!)

$$\int_0^1 x^2 \arctan(x)\, dx$$

SOLUTION: Using the integration by parts:

$$\begin{aligned} \int_0^1 x^2 \arctan(x)\, dx &= \frac{1}{3}\int_0^1 \arctan(x) dx^3 = \frac{1}{3}\arctan(x)\, x^3\Big|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{x^2+1}\, dx \\ &= \frac{1}{3}\arctan(1) - \frac{1}{3}\int_0^1 \left(x - \frac{x}{x^2+1}\right) dx \\ &= \frac{1}{3}\arctan(1) - \frac{1}{6} + \frac{1}{6}\ln(x^2+1)\Big|_0^1 = \frac{\pi - 2 + 2\ln(2)}{12} \end{aligned}$$

15 (1 pts). Does the improper integral

$$\int_0^\infty x^{2023} e^{-x}\, dx$$

converge? Support your answer by reasonings!

SOLUTION: Let $n > 0$. By integration by parts

$$\int_0^\infty x^n e^{-x} dx = -\int_0^\infty x^n de^{-x} = x^n e^{-x}\Big|_0^\infty + n\int_0^\infty x^{n-1}e^{-x} dx = n\int_0^\infty x^{n-1}e^{-x} dx$$

because $x^n e^{-x} \to 0$ as $x \to \infty$ for any n. If $n = 2023$, then repeating the integration by parts 2023 times, the integral is reduced to the integral of e^{-x} which is equal to 1 (the integral in question is therefore equal to 2023!).

16 (1 pts). Find

$$\int_0^a f(x)\, dx\,, \quad a > 0\,,$$

where f is the function defined in Problem 3, or show that the integral does not exist.
Hint: Use the definition of a Riemann integral.

SOLUTION: If f is integrable, then Riemann sums

$$R_N(f) = \sum_{j=1}^{N} f(x_j^*)\Delta x \,, \quad \Delta x = \frac{a}{N}$$

where $(j-1)\Delta x \le x_j^* \le j\Delta x$ are sample points, must converge to the *same* number as $N \to \infty$ for *any choice* of sample points. This is not possible for the function in question because any partition interval contains rational and irrational numbers. Indeed, if all x_j^* are irrational, then any Riemann sum is identically zero, whereas for rational sample points Riemann sums are strictly positive and do not vanish in the limit $N \to \infty$ because by the inequality $(j-1)\Delta x \le x_j^*$

$$R_N(f) \ge \frac{a^2}{N^2}\sum_{j=1}^{N}(j-1) = \frac{a^2}{N^2}(0+1+2\cdots+(N-1)) = a^2\frac{N(N-1)}{2N^2} \to \frac{1}{2}a^2 > 0\,.$$

Thus, the integral does not exist.

17 (2 pts). The function f is defined by the equation

$$f(x) = \begin{cases} \frac{1}{x+1} & , \quad x \ge 0 \\ \frac{1}{x-1} & , \quad x < 0 \end{cases}$$

(i) Give reasons that the following calculations are correct

$$\int_{-1}^{1} \frac{d}{dx} f(x)\, dx = f(x)\Big|_{-1}^{1} = f(1) - f(-1) = 1$$

or show that these calculations are false and find a correct answer if it exists.
(ii) Is it true that for any interval $[a,b]$ there exists a point c in it such that

$$\int_a^b f(x)\, dx = (b-a)\, f(c)\,, \quad a \le c \le b\,?$$

SOLUTION: (i) The function f has a jump discontinuity at $x = 0$ because $f(x) \to 1$ if $x \to 0^+$ and $f(x) \to -1$ if $x \to 0^-$. Therefore the derivative does not exist at $x = 0$, and the fundamental theorem of calculus does not apply. Thus, the reasoning is incorrect. The integral must be treated as an improper integral using intervals of continuity of $f'(x)$:

$$\begin{aligned} \int_{-1}^{1} \frac{d}{dx} f(x)\, dx &= \lim_{c\to 0^+} \int_{-1}^{-c} \frac{d}{dx} f(x) dx + \lim_{s\to 0^+} \int_{s}^{1} \frac{d}{dx} f(x) dx \\ &= \lim_{c\to 0^+} \big(f(-c) - f(-1)\big) + \lim_{s\to 0^+} \big(f(1) - f(s)\big) \\ &= -1 + \frac{1}{2} + \frac{1}{2} - 1 = -1 \end{aligned}$$

(ii) The assertion is false. Recall that the integral mean value theorem does not generally hold if the integrand is not continuous. In particular, the integral of f over $[-a,a]$ vanishes by symmetry, $f(-x) = -f(x)$ for any $x \ne 0$. However, $f(x)$ does not vanish anywhere in $[-a,a]$.