Placement Exam with solutions, MAC 3474 Honors Calculus III, Fall 2024

1 (8 pts). Assess each of the following statements as TRUE or FALSE. If the answer is FALSE, then support your answer by a counter example to the statement.
(i) Any bounded monotonic sequence of real numbers has a limit.
(ii) The limit of a convergent sequence of rational numbers is a rational number.
(iii) A continuous function on a bounded open interval (boundary points are not in the interval) always attains its maximal and minimal values in the interval.
(iv) Suppose that the derivative $f'(x)$ is continuous for $x < 0$ and $x > 0$. Suppose further that $f'(x) \to 2$ as $x \to 0^+$ (the right limit) and $f'(x) \to -1$ as $x \to 0^-$ (the left limit). Then the function f cannot be continuous at $x = 0$.
(v) Let a_n and b_n be numerical sequences such that the sum $a_n + b_n$ converges to some number c Then a_n and b_n converge to numbers a and b, respectively, and $c = a + b$.
(vi) If the function f is not continuous on an interval $[a, b]$, then the integral of f over this interval does not exist.
(vii) The integral $\int_a^b f(x)dx$ exists for every function f that is continuous on an open interval (a, b).
(viii) If the derivative of a function vanishes throughout the domain of the function, then the function is constant on its domain.

SOLUTION: (i) TRUE.
(ii) FALSE. Take a sequence of a decimal expansion of $\sqrt{2}$: 1, 1.4, 1.41, 1.414 and so on. This a sequence of rational numbers that converges to an irrational number.
(iii) FALSE. Take $f(x) = \tan(x)$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. It is continuous and monotonically increasing taking all real values. So, it has no maximal or minimal value.
(iv) FALSE. Let $f(x) = 2x$ if $x > 0$ and $f(x) = -x$ if $x \le 0$. This function is continuous at $x = 0$ and $f'(x)$ has the above properties.
(v) FALSE. Let $a_n = n$ and $b_n = -n$. These sequences have no limit. But $a_n + b_n = 0$ and, hence, the limit is zero.
(vi) FALSE. Let $[a, b] = [0, 2]$ and $f(x) = 1$ if $x \in [0, 1)$ and $f(x) = 2$ if $x \in [1, 2]$. The integral of f over $[0, 2]$ is the area under the graph of f and is equal to $1 + 2 = 3$.
(vii) FALSE. Let $(a, b) = (0, 1)$ and $f(x) = \frac{1}{x}$. Then the area under the graph is infinite (the improper integral diverges). (viii) FALSE. Let the domain be the union of two disjoint interval, e.g., $D = (0, 1) \cup (2, 3)$, and $f(x) = 1$ if $x \in (0, 1)$ and $f(x) = 2$ if $x \in (2, 3)$. Then $f'(x) = 0$ in D but $f(\frac{1}{2}) \ne f(\frac{3}{2})$ so f is not constant in D.

2 (1 pt). Let $f(x) = ax + b$. Find all values of a and b such that the composition $f(f(x)) = 4x - 9$.

SOLUTION: The following relation must hold for all x:

$$f(f(x)) = a(ax+b)+b = 4x-9 \quad \Rightarrow \quad a^2 = 4\,,\ b(a+1) = -9 \quad \Rightarrow \quad (a, b) = (2, -3) \text{ or } (-2, 3)\,.$$

3 (1 pt). Over which points of the x axis are the tangent lines to the graphs of $f(x) = x^3 + 8$ and $g(x) = 3x^2 - 3x + 9$ parallel?

SOLUTION: Tangent lines are parallel if they have the same slopes. The slope of a tangent line to the graph $y = f(x)$ at $x = a$ is the derivative $f'(a)$. Therefore

$$f'(x) = g'(x) \quad \Rightarrow \quad 3x^2 = 6x - 3 \quad \Rightarrow \quad (x - 1)^2 = 0 \quad \Rightarrow \quad x = 1\,.$$

4 (1 pt). For which values of a, b, and c, is it true that the graphs of $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx$ have the same tangent line at the point $(x, y) = (2, 2)$.

SOLUTION: The line through $(2,2)$ is $y = 2 + m(x-2)$ where m is the slope. The slopes of the tangent lines must be same. Therefore

$$f'(2) = g'(2) \quad \Rightarrow \quad 4 + a = 4 + c \quad \Rightarrow \quad a = c$$

The graphs must also contain the point $(2,2)$. Therefore

$$f(2) = 2\,,\ g(2) = 2 \quad \Rightarrow \quad 4 + 2a + b = 2\,,\ 4 + 2c = 2 \quad \Rightarrow \quad a = c = -1\,, b = 0\,.$$

5 (3 pts). Let $f(x) = x^2$ for x being a rational number, and $f(x) = 1$ otherwise. Find each of the following limits or show that the limit does not exist:

$$\text{(i)} \quad \lim_{x\to 1} f(x)\,, \qquad \text{(ii)} \quad \lim_{x\to 0} f(x)\,, \qquad \text{(iii)} \quad \lim_{x\to\infty} \frac{(\ln(f(x))^{2024}}{x}$$

SOLUTION: Recall that $f(x) \to A$ when $x \to a$ means that values of f can get arbitrary close to A and stay arbitrary close to A for all x close enough to a. Let us verify these two conditions for the limits in question.
(i) There are rational numbers arbitrary close to 1. Therefore values of f can get arbitrary close to $1^2 = 1$ by continuity of x^2. The values of f stay arbitrary close to 1 because if x is not rational, then $f(x) = 1$. So $\lim_{x\to 1} f(x) = 1$.
(ii) There are rational numbers arbitrary close to 0. Therefore the values of f can get arbitrary to $0^2 = 0$ by continuity of x^2. However the values of f do not stay arbitrary close to 0 because there are irrational numbers arbitrary close to 0 where $f(x) = 1$. Thus, the limit does not exist.
(iii) There are arbitrary large rational numbers. Therefore $\ln(f(x)) = \ln(x^2) = 2\ln(x)$ and

$$\lim_{x\to\infty} \frac{2^{2024}(\ln x)^{2024}}{x} = 0$$

because any power of the log function is increasing slower than any power function. By l'Hospital's rule

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{\frac{n}{x}(\ln x)^{n-1}}{1} = n \lim_{x\to\infty} \frac{(\ln x)^{n-1}}{x} = \cdots = n! \lim_{x\to\infty} \frac{1}{x} = 0$$

for any positive integer n. So, the values of f can get arbitrary close to 0 as $x \to \infty$. They also stay arbitrary close to 0 for all large enough x because if x is not rational, then $\ln(f(x)) = \ln(1) = 0$. Thus, the limit exists and is equal to 0.

6 (2 pts). Let $f(x) = |x|^q \sin(\frac{1}{x^2})$ if $x \neq 0$ and $f(0) = c$, where $1 < q < 2$.
(i) Find c for which f is continuous everywhere or show that no such c exists;
(ii) Is f is differentiable at $x = 0$ for some value of c? If so, find $f'(0)$.

SOLUTION: (i) The functions $|x|^q$ is continuous everywhere. The function $\sin(\frac{1}{x^2})$ is continuous for all $x \neq 0$ as the composition of two continuous functions. So, f is continuous at any $x \neq 0$. The continuity at $x = 0$ requires that

$$f(0) = \lim_{x\to 0} f(x) \quad \Rightarrow \quad c = \lim_{x\to 0} |x|^q \sin\left(\frac{1}{x^2}\right) = 0$$

because $|x|^q| \sin(u(x))| \le |x|^q \to 0$ as $x \to 0$ for any $u(x)$ (the sine function takes values in $[-1, 1]$).
(ii) The function $f(x)$ is differentiable for $x \neq 0$ because power functions and the sine function are

differentiable for $x \neq 0$. Continuity at $x = 0$ is necessary for differentiability (every differentiable function is continuous). So, $c = 0$ is necessary. By definition of the derivative

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} |x|^{q-1} \Big(\frac{1}{x^2} \Big) = 0$$

by the same argument because $q - 1 > 0$.

7 (1 pt). A tangent line to the curve $x^{2/3} + y^{2/3} = 1$ meets the x axis at A and the y axis at B. Show that the distance between A and B does not depend on the point of the curve at which the tangent line is constructed.

SOLUTION: Let a point (a, b) be on the curve. The tangent line is $y = b + m(x - a)$ where the slope m is obtaned by the implicit differentiation:

$$\frac{2}{3} x^{-\frac{1}{3}} dx + \frac{2}{3} y^{-\frac{1}{3}} dy = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\Big(\frac{y}{x} \Big)^{\frac{1}{3}} \quad \Rightarrow \quad m = -\Big(\frac{b}{a} \Big)^{\frac{1}{3}}$$

The line meets the y axis at $B = (0, b - ma)$ and the x axis at $A = (a - b/m, 0)$. The distance squared between A and B reads

$$\begin{aligned} |AB|^2 &= (b - ma)^2 + (a - b/m)^2 = \Big(b + b^{\frac{1}{3}} a^{\frac{2}{3}} \Big)^2 + \Big(a + a^{\frac{1}{3}} b^{\frac{2}{3}} \Big)^2 \\ &= b^{\frac{2}{3}} \Big(b^{\frac{2}{3}} + a^{\frac{2}{3}} \Big)^2 + a^{\frac{2}{3}} \Big(a^{\frac{2}{3}} + b^{\frac{2}{3}} \Big)^2 = b^{\frac{2}{3}} + a^{\frac{2}{3}} = 1 \,. \end{aligned}$$

So, the distance $|AB| = 1$ for any choice of (a, b).

8 (1 pt). A man is in a boat 1 mile from the nearest point, A, of a straight shore. He wishes to arrive as soon as possible at a point B 3 miles along the shore from A. He can row 2 miles per hour and walk 4 miles per hour. Where should he land?

SOLUTION: Let x be the distance from A to the landing point. Then the distance from the initial position to the landing points is $\sqrt{1 + x^2}$ and $3 - x$ from B. The travel time as a function of x is

$$T(x) = \frac{1}{2} \sqrt{1 + x^2} + \frac{1}{4} (3 - x) \,, \quad 0 \leq x \leq 3 \,.$$

The extreme values of T are attained either at $x = 0$ or $x = 3$ or in $(0, 3)$ because T is continuous on a closed interval $[0, 3]$. This is a differentiable function and, hence, $T'(x) = 0$ at its minimum in $(0, 3)$ (if any). One has

$$T'(x) = \frac{x}{2\sqrt{1 + x^2}} - \frac{1}{4} = 0 \quad \Rightarrow \quad 3x^2 = 1 \quad \Rightarrow \quad x = \frac{\sqrt{3}}{3} \,.$$

It is not difficult to see that $T(\frac{\sqrt{3}}{3})$ is less than $T(0) = \frac{5}{4}$ or $T(3) = \frac{\sqrt{10}}{2}$. So, $x = \frac{\sqrt{3}}{3}$.

9 (1 pt). A function g is such that the second derivative g'' is continuous on the interval $[a, b]$. The equation $g(x) = 0$ has three different solutions in the open interval (a, b). Show that the equation $g''(x) = 0$ has at least one solution in (a, b).

SOLUTION: Let x_1, x_2, and x_3 be roots of $g(x)$. Since g'' is continuous on (a, b), the derivative g' is also continuous on (a, b) and, hence, g must be continuous on (a, b). Therefore, by

Rolle's theorem applied to the intervals $[x_1, x_2] \subset (a, b)$ and $[x_2, x_3] \subset (a, b)$ there exist two points $x'_1 \in (x_1, x_2)$ and $x'_2 \in (x_2, x_3)$ such that $g'(x'_1) = 0$ and $g'(x'_2) = 0$. By Rolle's theorem applied to g' in the interval $[x'_1, x'_2]$, there exists a point $x'' \in (x'_1, x'_2)$ such that $g''(x'') = 0$ as required.

10 (1 pt). Find the area of a planar region bounded by the graphs $y = x^4$ and $y = \sqrt{x}$.

SOLUTION: The graphs are intersecting at the points where $x^4 = \sqrt{x}$ or $x = 1$ and $x = 0$. Since $x^4 \le \sqrt{x}$ in $[0, 1]$, the area of a region bounded by the graphs reads

$$A = \int_0^1 (\sqrt{x} - x^4)\, dx = \Big(\frac{2x^{3/2}}{3} - \frac{x^5}{5}\Big)\Big|_0^1 = \frac{2}{3} - \frac{1}{5} = \frac{7}{15}\,.$$

11 (1 pt). Find the area of a planar region that is bounded by the curve $\sqrt{y} + \sqrt{x} = 1$ and the coordinate axes.

SOLUTION: The equation makes sense only if $y \ge 0$ and $x \ge 0$. So, the region is bounded from above by the graph $y = (1 - \sqrt{x})^2$, obtained by solving the equation for y, and by the x axis from below. Therefore

$$A = \int_0^1 (1 - \sqrt{x})^2 dx = \int_0^1 (1 - 2\sqrt{x} + x)\, dx = x - \frac{4x^{3/2}}{3} + \frac{x^2}{2}\Big|_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}\,.$$

12 (4 pts). Evaluate

(i) $\displaystyle\int_{-1}^{1} \frac{d}{dx} \arctan\Big(\frac{1}{x}\Big)\, dx$

(ii) $\displaystyle\int_0^1 x\sqrt{3 - 2x}\, dx$

(iii) $\displaystyle\int_0^{\pi} \cos^2(x)\sin^2(x)\, dx$

(iv) $\displaystyle\int_0^{\pi/2} \frac{\sin^2(\sqrt{x})}{\sin^2(\sqrt{\frac{\pi}{2} - x}) + \sin^2(\sqrt{x})}\, dx$

Hint: (iv) Change the integration variable: $y = \frac{\pi}{2} - x$.

SOLUTION: (i) The function $\arctan(\frac{1}{x})$ has a jump discontinuity at $x = 0$ and, hence, not differentiable at $x = 0$ and continuously differentiable otherwise. Therefore the fundamental theorem of calculus cannot be applied on any interval that contains $x = 0$. So, the integral should be viewed as an improper integral:

$$\begin{aligned}\int_{-1}^{1} \frac{d}{dx} \arctan\Big(\frac{1}{x}\Big)\, dx &= \lim_{a\to 0^+} \int_{-1}^{-a} \frac{d}{dx} \arctan\Big(\frac{1}{x}\Big)\, dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{d}{dx} \arctan\Big(\frac{1}{x}\Big)\, dx \\ &= \lim_{a\to 0^+} \Big[\arctan\Big(-\frac{1}{a}\Big) - \arctan(-1)\Big] + \lim_{b\to 0^+} \Big[\arctan(1) - \arctan\Big(\frac{1}{b}\Big)\Big] \\ &= -\frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{2}\,.\end{aligned}$$

because the fundamental theorem of calculus applies on $[-1, -a]$ and $[b, 1]$.
(ii) Let $s = 3 - 2x$ so that $x = (3 - s)/2$ and $dx = -ds/2$. One has $s = 3$ when $x = 0$ and $s = 1$ when $x = 1$. Therefore

$$\begin{aligned}\int_0^1 x\sqrt{3 - 2x}\, dx &= -\frac{1}{4}\int_3^1 (3 - s)\sqrt{s}\, ds = \frac{1}{4}\int_1^3 (3s^{1/2} - s^{3/2})\, ds \\ &= \frac{s^{3/2}}{2} - \frac{s^{5/2}}{10}\Big|_1^3 = \frac{3\sqrt{3} - 2}{5}\end{aligned}$$

(iii) By the double angle formulas $\sin(2x) = 2\sin(x)\cos(x)$ and $\sin^2(a) = \frac{1}{2}(1-\cos(2a))$, the integral is reduced to

$$\int_0^\pi \cos^2(x)\sin^2(x)\,dx = \frac{1}{4}\int_0^\pi \sin^2(2x)\,dx = \frac{1}{8}\int_0^\pi (1-\cos(4x))\,dx = \frac{x}{8} - \frac{\sin(4x)}{32}\Big|_0^\pi = \frac{\pi}{8}$$

(iv) Let I denote the integral in question. Let $y = \frac{\pi}{2} - x$ so that $dy = -dx$ and $y = \frac{\pi}{2}$ when $x = 0$ and $y = 0$ when $x = \frac{\pi}{2}$. Therefore

$$\begin{aligned} I &= -\int_{\pi/2}^0 \frac{\sin^2(\sqrt{\frac{\pi}{2}-y})}{\sin^2(\sqrt{y}) + \sin^2(\sqrt{\frac{\pi}{2}-y})}\,dy = \int_0^{\pi/2} \frac{\sin^2(\sqrt{\frac{\pi}{2}-y}) + \sin^2(\sqrt{y}) - \sin^2(\sqrt{y})}{\sin^2(\sqrt{y}) + \sin^2(\sqrt{\frac{\pi}{2}-y})}\,dy \\ &= \int_0^{\pi/2} dy - I = \frac{\pi}{2} - I \quad \Rightarrow \quad 2I = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}. \end{aligned}$$

13 (2 pt). Find the limit or show that it does not exist

(i) $$\lim_{a\to 0} \frac{1}{a}\int_0^{2a} \left(x^{2024} + x^{2023} + \cdots + x + 1 + \sin(2024x)\right)^{2024} dx$$

(ii) $$\lim_{x\to 0} x^{-2024} e^{-\frac{1}{x^2}}$$

SOLUTION: (i) The integrand is a continuous function for all x, denoted by $f(x)$. By the integral mean value theorem, there exists a point x_a between 0 and $2a$ such that

$$\int_0^{2a} f(x)\,dx = 2af(x_a) \quad \Rightarrow \quad \lim_{a\to 0}\frac{1}{a}\int_0^{2a} f(x)\,dx = \lim_{a\to 0} 2f(x_a) = 2f(0) = 2$$

because $x_a \to 0$ as $x \to 0$ and by continuity of f, $f(x_a) \to f(0)$.

(ii) Put $s = \frac{1}{x^2}$. Then $s \to \infty$ for $x \to 0$. Therefore

$$\lim_{x\to 0} x^{-2024} e^{-\frac{1}{x^2}} = \lim_{s\to\infty} s^{1012} e^{-s} = \lim_{s\to\infty} \frac{s^{1012}}{e^s} = 1012 \lim_{s\to\infty} \frac{s^{1011}}{e^s} = \cdots = 1012! \lim_{s\to\infty} \frac{1}{e^s} = 0$$

by l'Hospital's rule.