Assignment 1 Solutions

1.3.14. Let S be a nonempty set and F a field. Let $\mathcal{C}(S,F)$ denote the set of all functions $f \in \mathcal{F}(S,F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S,F)$ is a subspace of $\mathcal{F}(S,F)$.

Solution. For $f \in \mathcal{F}(S,F)$, let $Q_f = \{s \in S : f(s) \neq 0\}$. Said another way, $\mathcal{C}(S,F)$ is the set of all functions $f \in \mathcal{F}(S,F)$ such that the set of all $s \in S$ for which $f(s) \neq 0$ is finite, i.e Q_f is a finite set.

Let $z : S \to F$ denote the zero function. That is, for all $s \in S$, $z(s) := 0$. Recall that this is the zero element of $\mathcal{F}(S,F)$, and of course $z \in \mathcal{C}(S,F)$ since it is zero everywhere so $Q_z = \emptyset$ (which is of course finite).

Let $c \in F$, and $f \in \mathcal{C}(S,F)$. If $c = 0$, then $cf = z \in \mathcal{C}(S,F)$, so suppose $c \neq 0$. We need for the set Q_{cf} to be finite, knowing that Q_f is finite. Suppose that $s \in Q_{cf}$, so that $(cf)(s) \neq 0$. Then $f(s) \neq 0$ (for if $f(s) = 0$, then $(cf)(s) = c(f(s)) = c \cdot 0 = 0$, despite that $(cf)(s) \neq 0$), so then $s \in Q_f$. As s was arbitrary, we can deduce that $Q_{cf} \subseteq Q_f$. As Q_f is finite, Q_{cf} must be finite! Hence $cf \in \mathcal{C}(S,F)$.

Now suppose $f, g \in \mathcal{C}(S,F)$ so that Q_f and Q_g are both finite. This implies that $Q_f \cup Q_g$ is finite, as the union of two finite sets is finite. Note now that $Q_{f+g} \subseteq Q_f \cup Q_g$. Indeed, suppose that $s \in Q_{f+g}$, so that $f(s) + g(s) \neq 0$. It must be the case that at least one of $f(s)$ or $g(s)$ is not zero, for it they were both zero, then $f(s) + g(s) = 0 + 0 = 0$, a contradiction. If it were the case that $f(s) \neq 0$, then $s \in Q_f \subseteq Q_f \cup Q_g$. If it were instead that $g(s) \neq 0$, then $s \in Q_g \subseteq Q_f \cup Q_g$. Either way, $s \in Q_f \cup Q_g$, and as s was arbitrary, we may conclude that $Q_{f+g} \subseteq Q_f \cup Q_g$. As $Q_f \cup Q_g$ is finite, Q_{f+g} is also finite and so $f + g \in \mathcal{C}(S,F)$.

Now, as $\mathcal{C}(S,F)$ has the zero element of $\mathcal{F}(S,F)$, is closed under scalar multiplication, and is closed under vector addition, it is a subspace. □

1.3.23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Solution. (a) As W_1 and W_2 are subspaces of V, we have $0 \in W_1$ and $0 \in W_2$ so $0 = 0 + 0 \in W_1 + W_2$.

Let $v \in W_1 + W_2$, and let $c \in F$. By definition, there must be $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$, and as these are subspaces, $cw_1 \in W_1$ and $cw_2 \in W_2$. Hence, $cv = c(w_1 + w_2) = cw_1 + cw_2 \in W_1 + W_2$.

Now suppose that $u, v \in W_1 + W_2$. Say that $u = w_1 + w_2$, and $v = t_1 + t_2$, with $w_1, t_1 \in W_1$ and $w_2, t_2 \in W_2$. Then $w_1 + t_1 \in W_1$ and $w_2 + t_2 \in W_2$ by respective closures under addition. If follows that $u + v = (w_1 + w_2) + (t_1 + t_2) = (w_1 + t_1) + (w_2 + t_2) \in W_1 + W_2$. Note that for any $a \in W_1$ and $b \in W_2$, $a = a + 0 \in W_1 + W_2$ and $b = 0 + b \in W_1 + W_2$, so $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

(b) Let $S \subseteq V$ be another subspace of V such that $W_1 \subseteq S$ and $W_2 \subseteq S$. We wish to show that $W_1 + W_2 \subseteq S$, so let $u \in W_1 + W_2$ with $u = w_1 + w_2$, $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_1 \in S$ and $w_2 \in S$ so by closure of S under addition, $u = w_1 + w_2 \in S$. □