Assignment 3 Solutions

2.3.4. For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2. Use Theorem 2.14 to compute the following vectors:

(a) $[T(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.

(b) $[T(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.

(c) $[T(A)]_\gamma$, where is $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(d) $[T(f(x))]_\gamma$, where $f(x) = 6 - x + 2x^2$.

Solution. From Exercise 5 of Section 2.2,

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$\beta = \{1, x, x^2\}$ and $\gamma = \{1\}$. Recall that if V is a vector space with basis $\beta = \{v_1, \dots, v_n\}$, and $x \in V$, then $[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ where $[x]_\beta = \sum_{i=1}^n a_i v_i$.

Also, Theorem 2.14 of the text says that if $\zeta = \{v_1, \dots, v_n\}$ and $\xi = \{w_1, \dots, w_m\}$ are bases for V and W, respectively, then if $T : V \longrightarrow W$ is a linear transformation and $u \in V$, we have $[T(u)]_\xi = [T]_\zeta^\xi [u]_\zeta$

(a) In Exercise 5, $T : M_{2\times2}(F) \longrightarrow M_{2\times2}(F)$ is $T(A) = A^t$. By Theorem 2.14, $[T(A)]_\alpha = [T]_\alpha^\alpha [A]_\alpha$. Note that

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so that $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Now,

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 6\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $[A]_\alpha = \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix}$. Now, $[T]_\alpha [A]_\alpha = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$

(b) In Exercise 5, $T : P_2(\mathbb{R}) \longrightarrow M_{2\times2}(\mathbb{R})$ is $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3)) \end{pmatrix}$. We now find $[T]_\beta^\alpha$. Note that

$$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Now, for $f(x) = 4 - 6x + 3x^2$, we have $[f(x)]_\beta = \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}$. Now, $[T(f(x))]_\beta = [T_\beta^\alpha[f(x)]_\beta = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$.

(c) In Exercise 5, $T : M_{2\times 2}(F) \to P_2(F)$ is $T(A) = \mathrm{tr}(A) = \sum_{i=1}^{2} A_{i,i} = A_{1,1} + A_{2,2}$.
We now compute $[T]_\alpha^\gamma$. Note,

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1,$$

$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 0,$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 0,$$

and

$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$$

so $[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$. It is clear that $[A]_\alpha = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$ so $[T(A)]_\gamma = [T]_\alpha^\gamma [A]_\alpha = \begin{pmatrix} 5 \end{pmatrix}$.

(d) In Exercise 5, $T : P_2(\mathbb{R}) \to \mathbb{R}$ is $T(f(x)) = f(2)$. Note that $T(1) = 1$, $T(x) = 2$, and $T(x^2) = 4$, so that $[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$. Also, for $f(x) = 6 - x + 2x^2$, $[f(x)]_\beta = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$, so that $[T(f(x))]_\gamma = [T]_\beta^\gamma [f(x)]_\beta = \begin{pmatrix} 12 \end{pmatrix}$. □

2.3.9. Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = O$, but $BA \neq O$.

Solution. We search for such transformations as follows. Note that for any linear transformation $S : F^2 \to F^2$, $T_0 \circ S = T_0$, and $S \circ T_0 = T_0$. Hence, in searching for such U and T, we cannot have either of U or T be the zero transformation, for then $T \circ U \neq T_0$ cannot be achieved. Furthermore, we do not want T to be surjective, for if so, then $U = T_0$. Indeed, if $(a, b) \in F^2 = T(F^2)$, then $(a, b) = T(c, d)$ for some $(c, d) \in F^2$. However, the condition that $U \circ T = T_0$ would imply that $U(a, b) = U(T(c, d)) = T_0(c, d) = (0, 0)$, so $U = T_0$.

This further implies that the range of T should be a one-dimensional subspace of F^2. If $T(F^2)$ were a 0-dimensional space, that would imply $T(F^2) = \{(0, 0)\}$ so $T = T_0$, which we said above cannot be the case to meet the conditions. If $T(F^2)$ were a 2-dimensional space, that would imply that $T(F^2)$ is all of F^2 so T is surjective, which cannot happen.

Continuing, we will use a fundamental technique to construct linear transformations as follows. Consider the standard basis $\beta = \{(0, 1), (1, 0)\}$, and the set $\{(0, 1)\}$, which generates the subspace $A = \{(0, a) : a \in F\} \subseteq F^2$. There exists (by Theorem 2.6 of the book), $T : F^2 \to F^2$ linear such that $T(0, 1) = (0, 1) = T(1, 0)$. Explicitly, for $(a, b) \in F^2$,

$$T(a, b) = T(a(1, 0) + b(0, 1)) = aT(1, 0) + bT(0, 1) = a(0, 1) + b(0, 1) = (0, a + b),$$

and note that $T(F^2) = A$.

Now, we want U to be zero on all of A, so as above we can also get linear $U : F^2 \to F^2$ such that $U(0, 1) = (0, 0)$ and $U(1, 0) = (1, 0)$. The first equality gets that for all $(a, b) \in F^2$, $U(T(a, b)) = U(0, a + b) = (a + b)U(0, 1) = (0, 0)$ so $U \circ T = T_0$. The second equality gets that $T(U(1, 0)) = T(1, 0) = (0, 1 + 0) = (0, 1) \neq T_0(1, 0)$. As $T \circ U$ disagrees with T_0 at $(1, 0)$, $T \circ U \neq T_0$.

Recall that O is the 2×2 zero matrix, and $[T_0]_\beta = O$. Consider $A = [U]_\beta$ and $B = [T]_\beta$. As $U \circ T = T_0$, $[U \circ T]_\beta = O$, and as $T \circ U \neq T_0$, $[T \circ U]_\beta \neq O$. By Theorem 2.11, we get that $AB = [U]_\beta [T]_\beta = [U \circ T]_\beta = O$, and $BA = [T]_\beta [U]_\beta = [T \circ U]_\beta \neq O$.

□

2.4.16. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Solution. As B is invertible, B^{-1} exists, and $BB^{-1} = I_n = B^{-1}B$, where I_n is the $n \times n$ identity matrix. Note that Φ is linear. If $H, K \in M_{n\times n}(F)$ and $c \in F$, then

$$\Phi(H+K) = B^{-1}(H+K)B = (B^{-1}H + B^{-1}K)B = B^{-1}HB + B^{-1}KB = \Phi(H) + \Phi(K)$$

and $\Phi(cH) = B^{-1}(cH)B = (cB^{-1})HB = c(B^{-1}HB) = c\Phi(H)$.

Now let $H \in M_{n\times n}(F)$. Then

$$H = I_n H I_n = (B^{-1}B)H(B^{-1}B) = B^{-1}(BHB^{-1})B = \Phi(BHB^{-1})$$

so that Φ is injective.

If $B^{-1}HB = \Phi(H) = \Phi(K) = B^{-1}KB$, then

$$H = B(B^{-1}HB)B^{-1} = B(B^{-1}KB)B^{-1} = K,$$

so Φ is linear. and hence an isomorphism. □