Assignment 4 Solutions

2.5.7. In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L along the line perpendicular to L. (See the definition of *projection* in the exercises of Section 2.1.)

Solution. **(a)** Let $m \neq 0$, we follow the argument shown in Example 3 in the special case of $m = 2$. Recall that linear transformations defined on finite-dimensional vector spaces are completely determined by where they map any particular basis elements for a fixed basis. Hence, we will find a basis β' for $\mathbb{R}^2$ that we will know how to map the elements of under this T.

Inspired by visual geometry, we would expect the reflection about L to satisfy $T(1, m) = (1, m)$ as points on the line are unaffected by reflection (any point on the line could have been chosen, but this was a simple pick). Hence, the point $(1, m)$ can be one element of β'. Any other point on the line y cannot be the second element of β' since they would be linearly dependent with $(1, m)$, so we choose a point not on the line. Now, $(-m, 1)$ is not on the line as $1 \neq m(-m)$ since $m(-m) < 0$. Further, is must be that $T(-m, 1) = (m, -1)$

So $\beta' = \{(1, m), (-m, 1)\}$ is an ordered basis for $\mathbb{R}^2$, and let $\beta = \{(1, 0), (0, 1)\}$ be the standard ordered basis. Let $(a, b) \in \mathbb{R}^2$ be arbitrary, and suppose that $T(a, b) = (c, d)$, for some $c, d \in \mathbb{R}$. We must find an expression for c and d in terms of a and b. Note that $[T(a, b)]_\beta = [(c, d)]_\beta = \begin{pmatrix} c \\ d \end{pmatrix}$ and also $[T(a, b)]_\beta = [T]_\beta[(a, b)]_\beta = [T]_\beta \begin{pmatrix} a \\ b \end{pmatrix}$ so that the equation $\begin{pmatrix} c \\ d \end{pmatrix} = [T]_\beta \begin{pmatrix} a \\ b \end{pmatrix}$ would give us an expression for c and d if we knew $[T]_\beta$, which is what we compute now. Let $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta}$ be the change of coordinate matrix from β'-coordinates to β-coordinates. By Theorem 2.23, $Q^{-1}[T]_\beta Q = [T]_{\beta'}$ so that $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

It is easy to see that

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

so that

$$Q^{-1} = \frac{1}{1 + m^2} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$

as seen through the standard inverse formula. It can be verified by computation that

$$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \frac{1}{1 + m^2} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix}$$

and so

$$\begin{pmatrix} c \\ d \end{pmatrix} = [T]_\beta \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{1 + m^2} \begin{pmatrix} (1 - m^2)a + 2mb \\ 2ma + (m^2 - 1)b \end{pmatrix}$$

which gives that for $(a, b) \in \mathbb{R}$,

$$T(a, b) = \frac{1}{1 + m^2}((1 - m^2)a + 2mb, 2ma + (m^2 - 1)b)$$

(b) We will use a similar technique to that seen in part **(a)**. First recall that if W_1 and W_2 are subspaces of V such that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0_V\}$, then V is the **direct sum** of W_1 and W_2, written $V = W_1 \oplus W_2$. This has the property that each $v \in V$ has a unique split into $v = w_1 + w_2$. In such a case, say that $T : V \longrightarrow V$ is a **projection on** W_1 **along** W_2 if $T(w_1 + w_2) = w_1$ whenever $w_1 \in W_1$ and $w_2 \in W_2$. Consider again $m \neq 0$ and the line L determined by $y = mx$. Let P be the line perpendicular to L, which has equation $y = -1/mx$. Of course $L \cap P = \{(0, 0)\}$, and $\dim(L) = \dim(P) = 1$, so that $\dim(L + P) = \dim(L) + \dim(P) = 2$. As $\dim(\mathbb{R}^2) = 2$, we have $L + P = \mathbb{R}^2$ so $\mathbb{R}^2 = L \oplus P$. Note that a point on L is $(1, m)$, and a point on P is $(-m, 1)$, and that these are linearly independent. Any such points can be used, but we are using these because we already used them for part **(a)**. In fact, we can consider the same ordered basis $\beta' = \{(1, m), (-m, 1)\}$. Such a function T must have that $T(1, m) = T((1, m) + (0, 0)) = (1, m)$, since $(1, m) \in L$, $(0, 0) \in P$ and T is a projection. Likewise, $T(-m, 1) = T((0, 0) + (-m, 1)) = (0, 0)$ since $(0, 0) \in L$ and $(-m, 1) \in P$. Thus,

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

as before so

$$Q^{-1} = \frac{1}{1+m^2} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}.$$

Let $T(a, b) = (c, d)$. As before,

$$\begin{pmatrix} c \\ d \end{pmatrix} = [T]_\beta \begin{pmatrix} a \\ b \end{pmatrix}$$

and after computation, one may verify that

$$[T]_\beta = Q[T]_{\beta'} Q^{-1} = \frac{1}{1+m^2} \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

so that

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{1+m^2} \begin{pmatrix} a + mb \\ am + m^2 b \end{pmatrix}$$

which gives the definition of T. □

2.5.11. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinates matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

Solution. Recall that by Theorem 2.11 of the book, that if V, W and Z are finite dimensional spaces with ordered bases, α, β, and γ, respectively, and if $T : V \longrightarrow W$ and $U : W \longrightarrow Z$ are linear, then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

Also by Theorem 2.18, we have that if T is invertible, then $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$

(a) Recall that if $I_V : V \longrightarrow V$ denotes the identity linear transformation, then by definition, $Q = [I_V]_\alpha^\beta$ and $R = [I_v]_\beta^\gamma$. Hence, $RQ = [I_V]_\beta^\gamma [I_V]_\alpha^\beta = [I_V I_V]_\alpha^\gamma = [I_V]_\alpha^\gamma$. Thus, by definition we have that RQ changes α-coordinates into γ-coordinates.

(b) As $I_V^{-1} = I_V$, we have that $Q^{-1} = ([I_V]_\alpha^\beta)^{-1} = [I_v^{-1}]_\beta^\alpha = [I_V]_\beta^\alpha$ so that by definition, Q^{-1} changes β-coordinates to α-coordinates. □

2.5.14. Prove the converse of Exercise 8: If A and B are each $m \times n$ matrices with entries from a field F, and if there exist invertible $m \times m$ and $n \times n$ matrices P and Q, respectively, such that $B = P^{-1}AQ$, then there exist an n-dimensional vector space V and an m-dimensional vector space W (both over F), ordered bases β and β' for V and γ and γ' for W, and a linear transformation $T : V \longrightarrow W$ such that

$$A = [T]_\beta^\gamma \text{ and } B = [T]_{\beta'}^{\gamma'}.$$

Hints: Let $V = F^n$, $W = F^m$, $T = L_A$, and β and γ be the standard ordered bases for F^n and F^m, respectively. Now apply the results of Exercise 13 to obtain ordered bases β' and γ' from β and γ via Q and P, respectively.

Solution. Let $A, B \in M_{m\times n}(F)$, $P \in M_m(F)$, $Q \in M_n(F)$ with P, Q invertible, such that $B = P^{-1}AQ$. Let $V = F^n$, $W = F^m$, and consider $L_A : V \longrightarrow W$ with $L_A(x) = Ax$. Let $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$ be the standard ordered bases for F^n and F^m, respectively. Then immediately $A = [L_A]_\beta^\gamma$, so it remains to show that $B = [L_A]_{\beta'}^{\gamma'}$.

For $1 \le i \le n$ and $1 \le j \le n$, let $x'_j = \sum_{i=1}^n Q_{i,j} x_i$, so by Exercise 2.5.13, $\beta' = \{x'_1, \ldots, x'_n\}$ is another ordered basis for V for which Q is the change of coordinate matrix changing β'-coordinates into β-coordinates. That is, $Q = [I_V]_{\beta'}^\beta$. Likewise, for $1 \le h \le m$ and $1 \le k \le m$, let $y'_k = \sum_{h=1}^m P_{h,k} y_h$, so that $\gamma' = \{y'_1, \ldots, y'_m\}$ is another ordered basis for W for which P is the change of coordinate matrix that changes γ'-coordinates into γ-coordinates, i.e $P = [I_W]_{\gamma'}^\gamma$. By Exercise 2.5.11 **(b)**, we have P^{-1} changes γ-coordinates into γ'-coordinates. That is, $P^{-1} = [I_W]_\gamma^{\gamma'}$. It follows from double application of Theorem 2.11 that

$$B = P^{-1}AQ = [I_W]_\gamma^{\gamma'} [L_A]_\beta^\gamma [I_V]_{\beta'}^\beta = [I_W \circ L_A \circ I_V]_{\beta'}^{\gamma'} = [L_A]_{\beta'}^{\gamma'}$$

as desired. □