Assignment 5 Solutions

4.3.10. A matrix $M \in M_{n\times n}(C)$ is called **nilpotent** if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

Solution. If O is the $n \times n$ zero matrix, then of course $\det(O) = 0$. Recall that if $A, B \in M_{n\times n}(C)$, then $\det(AB) = \det(A)\det(B)$. It follows from an easy extension via induction that if $A_1, \ldots, A_m \in M_{n\times n}(C)$, then $\det(A_1 \cdots A_m) = \det(A_1) \cdots \det(A_m)$.

Suppose now that $M \in M_{n\times n}(C)$ is nilpotent. Say $k \in \mathbb{N}$ is such that $M^k = O$. Then

$$0 = \det(O) = \det(M^k) = \det(M \cdots M) = \det(M) \cdots \det(M) = \det(M)^k$$

so that $\det(M) = \sqrt[k]{0} = 0$. □

4.3.11. A matrix $M \in M_{n\times n}(C)$ is called **skew-symmetric** if $M^t = -M$. Prove that if M if skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

Solution. Recall that if A and B are $n \times n$ matrices such that B is obtained from A by multiplying a row by a nonzero scalar $k \in C$, then $\det(B) = k\det(A)$. Now consider the matrix kA, which is obtained from multiplying **every** row of A, by k. It can be verified then that $\det(kA) = k^n \det(A)$. In particular, $\det(-A) = \det((-1)A) = (-1)^n \det(A)$.

Suppose now that $M \in M_{n\times n}(C)$ is skew-symmetric, and n is odd. Then $\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M)$. As n is odd, $(-1)^n = -1$ so that $\det(M) = -\det(M)$. The only number $x \in \mathbb{N}$ such that $-x = x$ is $x = 0$. Hence, $\det(M) = 0$ so M is not invertible.

We cannot make many such conclusions about the invertibility if n is even. Indeed, there exist even skew-symmetric matrices that are invertible, and those that are not invertible. The 2×2 zero matrix O is clearly skew-symmetric, yet not invertible For any nonzero $a \in C$, $A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$ is also skew-symmetric though will have nonzero determinant and hence be invertible. □

4.3.14. Let $\beta = \{u_1, \ldots, u_n\}$ be a subset of F^n containing n distinct vectors, and let B be the matrix in $M_{n\times n}(F)$ having u_j as column j. Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

Solution. As β consists of n distinct vectors in an n-dimensional space, we have that β is a basis for F^n if and only if β is linearly independent. As B has n linearly independent columns if and only if it has rank n, and these columns are precisely the vectors in β, we have that β is linearly independent if and only if B has rank n. However, B has rank n if and only if it is invertible, which it true if and only if $\det(B) \neq 0$. □