Assignment 2 Solutions

2.1.14. Let V and W be vector spaces and $T : V \longrightarrow W$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

Solution. **(a)** Suppose first that T is one-to-one. Let $A \subseteq V$ be linearly independent. We wish to show that $T(A) = \{t(a) : a \in A\}$ is a linearly independent subset of W. Recall that a subset of W is linearly independent if and only if every linear combination equal to 0_W must have only coefficients equal to 0_F. Hence, let $c_1 b_1 + \cdots + c_n b_n = 0_W$ be a linear combination where $b_i \in T(A)$ for each $i \in \{1, \ldots, n\}$. For each i, there is $a_i \in A$ such that $T(a_i) = b_i$. Note now that

$$T(c_1 a_1 + \cdots + c_n a_n) = c_1 T(a_1) + \cdots + c_n T(a_n) = c_1 b_1 + \cdots + c_n b_n = 0_W.$$

As T is one-to-one, $N(T) = \{0_V\}$ so $c_1 a_1 + \cdots + c_n a_n = 0_V$. But $c_1 a_1 + \cdots + c_n a_n$ is a linear combination of elements from A and A is linearly independent! This tells us that $c_i = 0_F$ for each i, so $T(A)$ is linearly independent.

Suppose conversely that for each linearly independent $A \subseteq V$, we have $T(A) \subseteq W$ is linearly independent. We wish to now show that T is one-to-one, for which it suffices to show $N(T) = \{0_V\}$. Before proceeding, we recall some things. Note that for $v \in V$, $\{v\}$ is linearly dependent if and only if $v = 0_V$. If $v = 0_V$, then $1 \cdot v = 0_V$ is a trivial linear combination from elements of $\{v\}$ equal to 0_V, yet has nonzero coefficient. Conversely, if $\{v\}$ is linearly dependent, there is $c \in F$ nonzero such that $cv = 0_V$. But as c is nonzero, it has an inverse so $v = c^{-1}(cv) = c^{-1} 0_V = 0_V$.

Now we continue with the proof. Let $v \in V$ with $T(v) = 0_W$. We want that $v = 0_V$, so what happens if this isn't the case? If $v \neq 0_V$, then $\{v\}$ is linearly independent. By hypothesis, $T(\{v\}) = \{T(v)\}$ is also linearly independent. But $\{T(v)\} = \{0_V\}$ is linearly dependent by the preceding discussion, which is a contradiction! Hence, $v = 0_V$ and we are done.

(b) Suppose that T is one-to-one, let $S \subseteq V$. By **(a)**, we know that T carries linearly independent subsets of V onto linearly independent subsets of W. Then immediately, if S is linearly independent, then $T(S)$ is as well. The converse remains to be proven. Say $T(S)$ is linearly independent, and let $c_1 s_1 + \cdots + c_n s_n = 0_V$ be a linear combination with vectors in S. As T is linear, we have

$$0_W = T(0_V) = T(c_1 s_1 + \cdots + c_n s_n) = c_1 T(s_1) + \cdots + c_n T(s_n)$$

which is a linear transformation from elements of $T(S)$, so $c_i = 0_F$ for each i. Hence, S is linearly independent.

(c) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V, and let T be one-to-one and onto. We wish to show that $T(\beta)$ is a basis, i.e a linearly independent set that spans W. By **(c)** and the fact that β is a linearly independent subset of V, we have that $T(\beta)$ is a linearly independent subset of W. Now suppose that $w \in W$. We wish to show it can be expressed as a linear combination of elements from $T(\beta)$. As T is onto, there is $v \in V$ such that $T(v) = w$. But $v \in \mathrm{Span}(\beta) = \mathrm{Span}(\{v_1, \ldots, v_n\})$ so there are $c_1, \ldots, c_n \in F$ such that $v = c_1 v_1 + \cdots + c_n v_n$. It then follows that

$$w = T(v) = T(c_1 v_1 + \cdots c_n v_n) = c_1 T(v_1) + \cdots + c_n T(v_n) \in \mathrm{Span}(\{T(v_1), \ldots, T(v_n)\}) = \mathrm{Span}(T(\beta)).$$

Thus, $T(\beta)$ is a basis. □

2.1.20. Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $T : V \longrightarrow W$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$ is a subspace of V.

Solution. We first prove that $T(V_1)$ is a subspace of W. As V_1 is a subspace, $0_V \in V_1$, and as T is linear, $0_W = T(0_V) \in T(V_1)$. Suppose that $a, b \in T(V_1)$ with $a = T(c)$, $b = T(d)$ for some $c, d \in V_1$. Then $c + d \in V_1$ and hence

$$a + b = T(c) + T(d) = T(c + d) \in T(V_1).$$

If $f \in F$, then

$$fa = fT(c) = T(fc) \in T(V_1)$$

as $fc \in V_1$ by scalar closure of subspaces. Thus $T(V_1)$ is a subspace of W.

Now we will see that $T^{-1}(W_1) = \{x \in V : T(x) \in W_1\}$ is a subspace of V (recall that $T^{-1}(W_1)$ is called the *inverse image*). As W_1 is a subspace of W, $0_W \in W_1$, and as T is linear, $T(0_V) = 0_W \in W_1$, so $0_V \in W_1$. Let $a, b \in T^{-1}(W_1)$ and $f \in F$. Then $T(a+b) = T(a) + T(b) \in W_1$ as $T(a), T(b) \in W_1$ and W_1 is closed under vector addition, so $a + b \in T^{-1}(W_1)$. Also, $T(fa) = fT(a) \in W_1$ as subspaces are closed under scalar multiplication. Hence, $fa \in T^{-1}(W_1)$ so $T^{-1}(W_1)$ is a subspace of V. □