p-ranks and Representation Theory

Peter Sin

BIRS Workshop, March 30th, 2009.

Outline

- This talk is about some connections between incidence matrices and group representations.
- Most of the incidence matrices I will consider come from classical geometries over a finite field $\mathbb{F}_q$, $q = p^t$.
- The representations will be p-modular representations of groups of automorphisms of the geometries.
- The incidence matrices lead naturally to interesting modules.
- The study of these modules sheds new light on old p-rank problems (and solves some of them).

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Subsets of a finite set

- X finite set, S_r the set of r-subsets of X.
- Incidence relation for (S_r, S_s) could be inclusion or, more generally, intersection in a set of size u.
- The symmetric group $Sym(X)$ acts.

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- q-analogue of the above.
- V vector space over $\mathbb{F}_q$
- $L_r =$ set of all r-dimensional subspaces of V.
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Distinguished Subspaces

- Suppose V has a symplectic, quadratic or hermitian form.
- We can consider incidence as above, but restricted to distinguished subspaces, e.g. totally isotropic subpaces.
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Problem: Compute the p-ranks

For some of these examples the p-ranks of the incidence matrices have been found. In a few cases we even know integral invariants. For many these problems are open.

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Permutation modules

- G-sets X, Y, G-invariant relation $I \subset X \times Y$.
- R commutative ring, $R[X]$, $R[Y]$ free modules.
- RG-module homomorphism

$$R[X] \to R[Y], \qquad x \mapsto \sum_{(x,y)\in I} y$$

- In this talk, X and Y will come from a classical geometry over a finite field of order $q = p^t$, G will be a classical group and k will be an algebraically closed field of characteristic p.

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Points and polar hyperplanes

- V vector space over $\mathbb{F}_q$ with nonsingular form $b(-,-)$.
- b may be alternating or symmetric or hermitian.
- $\widehat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\widehat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- G = group of linear transformations preserving $b(-,-)$.
- A = incidence matrix of $(\widehat{P}^*, \widehat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

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$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

p-ranks

- We consider the p-ranks, where $q = p^t$.
- The p-rank of A is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the p-rank of A_1 was found by Blokhuis and Moorhouse.
- Moorhouse (Linz, 2006): What is the p-rank of A_{11}?
- The formulae for A_1 provide a hint. In the orthogonal case:

$$\operatorname{rank} A_1 = 1 + \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^t$$

The green part is an instance of Weyl's Dimension Formula.

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Experimental evidence

Computations by Moorhouse indicate some irregularity. For orthogonal case, $q = 5$:

dim V	rank A_1	rank A_{11}	difference
4	26	26	0
5	56	56	0
6	106	86	20
7	183	183	0
8	295	294	1
9	451	451	0
10	661	661	0
11	936	871	65
12	1288	1288	0
13	1730	1729	1
14	2276	2276	0
15	2941	2941	0
16	3741	3606	135

The answer is to be found in the representation theory of algebraic groups. The first step is to consider the permutation kG-module $k[P]$, where k is an algebraically closed field of characteristic p.

Permutation module structure

- (a) G acts on P with permutation rank 3
- (b) $k[P] \cong k.\mathbf{1} \oplus Y$,
- (c) $\mathrm{head}(Y) \cong \mathrm{soc}(Y)$.
- (a),(b),(c) $\implies$ $\mathrm{head}(Y)$ is a simple kG-module. Call it L.
- P and P^* are isomorphic G-sets, so the incidence map induces

$$\phi \in \mathrm{End}_{kG}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

  Possibilities for $\mathrm{Im}\,\phi$ are very limited.
- Deduce

$$\mathrm{Im}\,\phi = k.\mathbf{1} \oplus L.$$

- Outcome: $\mathrm{rank}_p A_{11} = 1 + \dim L$.

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Identifying the simple module L

- $k[P] = \mathrm{ind}_{G_x}^{G}(k)$, $x \in P$, so Frobenius Reciprocity implies that G_x has a fixed point on L.
- Every simple kG-module can be considered as a simple module for a simply connected semisimple algebraic group $\mathbf{G}$
- The simple rational $\mathbf{G}$-modules are parametrized as follows:
- Let $T \subseteq \mathbf{G}$ be a maximal torus and $X(T) \cong \mathbb{Z}^\ell$ the character group of T, called the weight lattice.
- $X(T)$ has a certain basis $\{\omega_1, \ldots, \omega_\ell\}$ and the positive integral combinations are denoted $X_+(T)$.
- For each $\lambda \in X_+(T)$, there is a simple $\mathbf{G}$-module $L(\lambda)$. and this gives all simple $\mathbf{G}$-modules.

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Tensor Product Theorem

- The fixed point condition characterizes L:

$$L \cong L((q-1)\omega),$$

where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.

- Steinberg's Tensor Product Theorem.

$$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})} \tag{1}$$

- Outcome: $\mathrm{rank}_p A_{11} = 1 + (\dim L((p-1)\omega))^t$.
- Note that it suffices to know the answer for $q = p$.

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- Outcome: $\mathrm{rank}_p A_{11} = 1 + (\dim L((p-1)\omega))^t$.
- Note that it suffices to know the answer for $q = p$.

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- Determining the dimensions of simple $\mathbf{G}$-modules is a central unsolved problem in representation theory.
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Outline

Points and flats in projective space

Hamada's Formula

$$\operatorname{rank}_p A(L_1, L_r) = 1 + \sum_{\substack{(s_0,\ldots,s_{t-1}) \\ r+1 \le s_j \le n \\ 0 \le ps_{j+1} - s_j \le (n+1)(p-1)}} \prod_{j=0}^{t-1} \sum_{i=0}^{\lfloor \frac{ps_{j+1}-s_j}{p} \rfloor} (-1)^i \binom{n+1}{i} \binom{n + ps_{j+1} - s_j - ip}{n}.$$

Module interpretation of Hamada

- $\eta_r : k[L_r] \to k[L_1]$.
- $\operatorname{Im} \eta_r$ is a kG-submodule of $k[L_1]$.
- The kG-submodule lattice was completely described by Bardoe-Sin (2000).
- At the level of composition factors,

$$\operatorname{Im} \eta_r = k + \sum_{(\lambda_0,\ldots,\lambda_{t-1})\in\Lambda_r} \bigotimes_{j=0}^{t-1} S(\lambda_j)^{(p^j)}$$

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Points and symplectic flats

- Chandler-Sin-Xiang (2005-8)
- Let V be a $2m$-dimensional space with a nonsingular alternating form, $G = \mathrm{Sp}(V)$.
- I_r the set of r-dimensional subspaces which are either totally isotropic or the complements of such.
- Assume p is odd. $k[I_1]$ has a special basis $\mathcal{B}$ with the following properties.
  1. Each kG-submodule generated by a single element of $\mathcal{B}$. is spanned as a vector space by a subset of the basis.
  2. Each such module has a unique maximal submodule.
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Symplectic analogue of Hamada's formula

Theorem

Let $A(I_1, I_r)$ be the (I_1, I_r) incidence matrix of $\mathrm{W}(2m-1, p^t)$. Assume that p is odd. Then

1. *If $r \neq m$, then $\mathrm{rank}_p\, A(I_1, I_r)$ is the same as for all r-dimensional subspaces, so is given by Hamada's formula.*
2. *If $r = m$,*

$$\mathrm{rank}_p A(I_1, I_m) = 1 + \sum_{\substack{(s_0,\ldots,s_{t-1}) \\ (\forall j) 1 \le s_j \le m}} \prod_{j=0}^{t-1} d_{(s_j, s_{j+1})},$$

where

$$d_{(s_j, s_{j+1})} = \begin{cases} (d_{m(p-1)} + p^m)/2, & \text{if } s_j = s_{j+1} = m, \\ d_{\lambda_j}, & \text{otherwise.} \end{cases}$$

Characteristic 2

The characteristic 2 case can also be analyzed using this point of view. The results [C-S-X, 2008] are analogous but more complicated, reflecting greater complexity in the module structure.

Outline

p-rank of Symplectic GQs

Theorem

Let $q = p^t$, p odd and $A(I_1, I_2)$ the point-line incidence matrix of the symplectic GQ over $\mathbb{F}_q$. Then

$$\mathrm{rank}_p A(I_1, I_2) = 1 + \alpha_1^t + \alpha_2^t,$$

where

$$\alpha_1, \alpha_2 = \frac{p(p+1)^2}{4} \pm \frac{p(p+1)(p-1)}{12}\sqrt{17}.$$

- Together with earlier results (Bagchi-Brouwer-Wilbrink, Sastry-Sin, de Caen-Moorhouse) this completes the determination of the p-ranks for the symplectic GQs.

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Outline

Open p-rank problems

Problem 1. General case of (L_r, L_s) where I is any relation other than nonzero intersection.

For subsets of a set the relations of inclusion and nonempty intersection lead to equivalent problems. However, for subspaces inclusion and nonzero intersection are different.

Problem 2. Orthogonal and unitary analogues of Hamada's formula.

Problem 3. Analogue of Problem 1 for distinguished subspaces for any I.

The known cases suggest that the easiest cases will be for complementary dimensions and the relation of nonzero intersection.

Problem 4. p-ranks for point-line incidences of classical generalized polygons.

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