$p$-ranks and Representation Theory

Peter Sin

Outline

Introduction

Fundamental examples

Permutation Modules

Point-Hyperplane Incidences

Points versus flats

Application to GQs

Open problems

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The representations will be $p$-modular representations of groups of automorphisms of the geometries.

The incidence matrices lead naturally to interesting modules.

The study of these modules sheds new light on old $p$-rank problems (and solves some of them).
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- Incidence relation for $(S_r, S_s)$ could be inclusion or, more generally, intersection in a set of size $u$.
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- Suppose $V$ has a symplectic, quadratic or hermitian form.
- We can consider incidence as above, but restricted to distinguished subspaces, e.g. totally isotropic subspaces.
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Problem: Compute the $p$-ranks

For some of these examples the $p$-ranks of the incidence matrices have been found. In a few cases we even know integral invariants. For many these problems are open.
Permutation modules

- $G$-sets $X$, $Y$, $G$-invariant relation $I \subseteq X \times Y$.
- $RG$-module homomorphism

$$R[X] \to R[Y], \quad x \mapsto \sum_{(x,y) \in I} y$$

- In this talk, $X$ and $Y$ will come from a classical geometry over a finite field of order $q = p^t$, $G$ will be a classical group and $k$ will be an algebraically closed field of characteristic $p$. 
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- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G = \text{group of linear transformations preserving } b(−, −)$.
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$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
We consider the \( p \)-ranks, where \( q = p^t \).

The \( p \)-rank of \( A \) is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the \( p \)-rank of \( A_1 \) was found by Blokhuis and Moorhouse.

Moorhouse (Linz, 2006): What is the \( p \)-rank of \( A_{11} \)?

The formulae for \( A_1 \) provide a hint. In the orthogonal case:

\[
\text{rank } A_1 = 1 + \left[ \binom{p + n - 1}{n} - \binom{p + n - 3}{n} \right]^t
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The green part is an instance of Weyl’s Dimension Formula.
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Experimental evidence

Computations by Moorhouse indicate some irregularity. For orthogonal case, \( q = 5 \):

<table>
<thead>
<tr>
<th>dim ( V )</th>
<th>rank ( A_1 )</th>
<th>rank ( A_{11} )</th>
<th>difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>26</td>
<td>26</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>56</td>
<td>56</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>106</td>
<td>86</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>183</td>
<td>183</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>295</td>
<td>294</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>451</td>
<td>451</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>661</td>
<td>661</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>936</td>
<td>871</td>
<td>65</td>
</tr>
<tr>
<td>12</td>
<td>1288</td>
<td>1288</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>1730</td>
<td>1729</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>2276</td>
<td>2276</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>2941</td>
<td>2941</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>3741</td>
<td>3606</td>
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The answer is to be found in the representation theory of algebraic groups. The first step is to consider the permutation $kG$-module $k[P]$, where $k$ is an algebraically closed field of characteristic $p$. 
Permutation module structure

- (a) $G$ acts on $P$ with permutation rank 3
- (b) $k[P] \cong k.1 \oplus Y$,
- (c) $\text{head}(Y) \cong \text{soc}(Y)$.
- (a),(b),(c) $\implies$ $\text{head}(Y)$ is a simple $kG$-module. Call it $L$.
- $P$ and $P^*$ are isomorphic $G$-sets, so the incidence map induces

$$\phi \in \text{End}_{kG}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

Possibilities for $\text{Im} \phi$ are very limited.

- Deduce

$$\text{Im} \phi = k.1 \oplus L.$$

- Outcome: $\text{rank}_p A_{11} = 1 + \text{dim } L.$
(a) $G$ acts on $P$ with permutation rank 3
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- $k[P] = \text{ind}^G_{G_x}(k), \ x \in P$, so Frobenius Reciprocity implies that $G_x$ has a fixed point on $L$.
- Every simple $kG$-module can be considered as a simple module for a simply connected semisimple algebraic group $G$.
- The simple rational $G$-modules are parametrized as follows:
  - Let $T \subseteq G$ be a maximal torus and $X(T) \cong \mathbb{Z}^\ell$ the character group of $T$, called the weight lattice.
  - $X(T)$ has a certain basis $\{\omega_1, \ldots, \omega_\ell\}$ and the positive integral combinations are denoted $X_+(T)$.
  - For each $\lambda \in X_+(T)$, there is a simple $G$-module $L(\lambda)$, and this gives all simple $G$-modules.
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- The simple rational $G$-modules are parametrized as follows:
  - Let $T \subseteq G$ be a maximal torus and $X(T) \cong \mathbb{Z}^\ell$ the character group of $T$, called the weight lattice.
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$$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$  \hspace{1cm} (1)

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Points and flats in projective space

Hamada’s Formula

$$\text{rank}_p A(L_1, L_r) = 1 + \sum_{(s_0, \ldots, s_{t-1})} \prod_{j=0}^{t-1} \left( \sum_{i=0}^{ps_{j+1} - s_j \leq n} (-1)^i \binom{n+1}{i} \binom{n + ps_{j+1} - s_j - ip}{n} \right) \text{subject to } r + 1 \leq s_j \leq n, 0 \leq ps_{j+1} - s_j \leq (n+1)(p-1)$$
Module interpretation of Hamada

- $\eta_r : k[L_r] \rightarrow k[L_1]$.
- $\text{Im } \eta_r$ is a $kG$-submodule of $k[L_1]$.
- The $kG$-submodule lattice was completely described by Bardoe-Sin (2000).
- At the level of composition factors,

\[
\text{Im } \eta_r = k + \sum_{(\lambda_0, \ldots, \lambda_{t-1}) \in \Lambda_r} \bigotimes_{j=0}^{t-1} S(\lambda_j)^{(p^j)}
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Let $V$ be a $2m$-dimensional space with a nonsingular alternating form, $G = \text{Sp}(V)$.

$I_r$ the set of $r$-dimensional subspaces which are either totally isotropic or the complements of such.

Assume $p$ is odd. $k[I_1]$ has a special basis $B$ with the following properties.

1. Each $kG$-submodule generated by a single element of $B$ is spanned as a vector space by a subset of the basis.
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Symplectic analogue of Hamada’s formula

**Theorem**

Let $A(I_1, I_r)$ be the $(I_1, I_r)$ incidence matrix of $W(2m - 1, p^t)$. Assume that $p$ is odd. Then

1. If $r \neq m$, then $\text{rank}_p A(I_1, I_r)$ is the same as for all $r$-dimensional subspaces, so is given by Hamada’s formula.

2. If $r = m$,

$$
\text{rank}_p A(I_1, I_m) = 1 + \sum_{(s_0, \ldots, s_{t-1})} \prod_{j=0}^{t-1} d(s_j, s_{j+1}),
$$

where

$$
d(s_j, s_{j+1}) = \begin{cases} 
(d_{m(p-1)} + p^m)/2, & \text{if } s_j = s_{j+1} = m, \\
\lambda_j, & \text{otherwise.}
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$$
Characteristic 2

The characteristic 2 case can also be analyzed using this point of view. The results [C-S-X, 2008] are analogous but more complicated, reflecting greater complexity in the module structure.
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\alpha_1, \alpha_2 = \frac{p(p + 1)^2}{4} \pm \frac{p(p + 1)(p - 1)}{12} \sqrt{17}.
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